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DIATONICISM, CHROMATICISM, AND ENHARMONICISM: A STUDY IN
COGNITION AND PERCEPTION

City University of New York

Ph.D. 1986

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DIATONICISM, CHROMATICISM, AND ENHARMONICISM:
A STUDY IN COGNITION AND PERCEPTION
by
EYTAN AGMON

A dissertation submitted to the Graduate Faculty
in Music in partial fulfillment of the require-
ments for the degree of Doctor of Philosophy, The
City University of New York.

1986

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This manuscript has been read and accepted for the Graduate Faculty in Music in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy

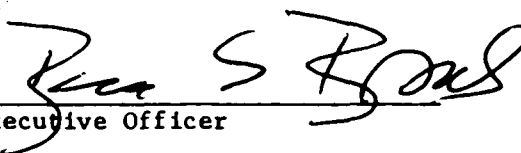
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Abstract

DIATONICISM, CHROMATICISM, AND ENHARMONICISM:

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by

EYTAN AGMON

Adviser: Professor Carl Schachter

The basic position taken in this study is that diatonicism is a mental construct, that is, it is a product of the human mind, in particular, a perceiving mind, a mind interacting with the external, physical world. A distinction is drawn between competence, that is, what it is that the mind constructs, and performance, that is, how this "what" is constructed, given the constraints within which the mind must operate. Another distinction drawn is between perception and cognition, that is, between a level of mental representation maintaining a simple, one-to-one correspondence to some aspect of the physical signal transmitted to the sense organ, and other levels of mental representation, embodying abstract structures which do not relate to this physical signal on a

simple, one-to-one basis. In particular, it is argued that it is senseless to attempt to describe diatonicism in purely perceptual terms; a combined cognitive/perceptual model of diatonicism is offered instead. This model, considered from both the competence and the performance points of view, is extended to embrace chromaticism and enharmonicism as well. Finally, some consideration is given to higher levels of mental representation which constitute the tonal system.

PREFACE AND ACKNOWLEDGEMENT

This dissertation originated in a study of enharmonic relations in Schubert songs. Outwardly, it has little in common with its origin. Yet a feeling that enharmonic equivalence has far-reaching implications with respect to diatonicism in general has underlain my interest in enharmonic relations from the very outset. In the course of working on enharmonic equivalence this feeling grew stronger and stronger. It became apparent to me that the study of enharmonicism must begin with the study of diatonicism.

The theories presented in this study are based on precise mathematical models. Any theory of music that uses numbers as a model must be very clear as to what it is, exactly, that the numbers model. Music theory is not mathematics. Music theory is an empirical science, and to the extent that it uses mathematics, it must use it as a means, not as an end; it simply cannot afford to get carried away by sheer numbers without being very clear about the empiric domain in which these numbers are interpreted. Much of this study is accordingly devoted not to mathematics per se, but to empirical issues. Indeed, the empirical framework within which the mathematical models are developed is an essential aspect of the theory. This point must be emphasized because some of the mathematics

employed here resembles standard twelve-tone theory; empirically, however, it is interpreted very differently.

I wish to thank my adviser, Prof. Carl Schachter, for his helpful guidance, and above all, for inspiring in me his deep commitment to the tonal system, without which this study would not have come into being. I wish to thank my readers, Prof. Richmond Browne, Prof. Charles Burkhart, Prof. John Graziano, Prof. Joseph Straus, and in particular, Prof. Joel Lester, for carefully reading the manuscript and for offering valuable criticisms and suggestions. And I wish to thank my father, Prof. Shmuel Agmon from the Department of Mathematics at the Hebrew University in Jerusalem, for lending me his mathematical expertise.

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Dedicated to the Memory of my Mother

Galia Yardeni-Agmon

1924 - 1968

Chapter I

METHODOLOGICAL PRELIMINARIES

1.1 Music Theory as Cognitive Science

Recently Fred Lerdahl and Ray Jackendoff have claimed for music theory a place among the branches of cognitive science, a composite discipline that has newly emerged out of parts of psychology, linguistics, neurophysiology, philosophy, and computer science.¹ "Cognitive science" as an independent discipline is in its infancy. Yet to the extent that this independence proves justified in the long run, it may well be that a truly appropriate scientific context for research in music theory has finally been established.

"Cognition" means thought, knowledge, or understanding. A science of cognition is "a science of the mind, of intelligence, of thought, a science concerned with knowledge and its uses."² As Lerdahl and Jackendoff point out, "perhaps its most important goal is to understand the nature of mental representation,"³ that is, the manner in which the mind constructs and makes use of an internal model of external reality. For the study of art and artworks, "cognition" is a notion pregnant with implications. In the words of Mary Louise Serafine:

This artwork [the idealized, hypothetical musical composition, the area of overlap among all the individual performances and conceptions of the work] is not a fixed, external object, but an abstract and fluid one that rests on human cognitive construction in all phases of its existence--composing, performing, and listening.⁴

And in the words of Rudolf Arnheim (in reference to the impact of gestalt psychology on the study of perception):

Far from being a mechanical recording of sensory elements, vision proved to be a truly creative apprehension of reality --imaginative, inventive, shrewd, and beautiful. It became apparent that the qualities that dignify the thinker, and the artist distinguish all performances of the mind.⁵

One of the most important realizations that underlie the cognitive paradigm of the mind--a realization that is anticipated in 17th and 18th-century rationalism and ultimately dates back to Plato--is that we construe the external world in terms that do not necessarily correspond on a simple, one-to-one basis to the physical signal it transmits to our senses; our "inner model" of external reality, although related to this signal, is by no means isomorphical to it. This being so, it is apparent that music theory has been implicitly dealing with cognition for some time now. Consider, as an example, Schenkerian analysis. In his review (1978) of Eugene Narmour's Beyond Schenkerism (1977) entitled "The Empiricist Illusion," Allen Keiler evokes the enduring "rationalist vs. empiricist" controversy. Thus, in refuting Narmour's criticism of Schenker's reductive techniques, Keiler points out that analytic reductions in general "distort" the "musical facts" in one way or another since analysis by its very nature involves selection and gene-

ralization, that is, analysis relies on abstraction. Keller notes that similar attacks in the name of "empirical truth" have been launched against Rameau's fundamental bass. At any rate, as far as the subject matter of cognition is concerned, I will point out that it is difficult to attribute empirical content to analytic statements of any sort--be they Schenker's, Rameau's, or Narmour's (and note the ironical twist in the familiar terms of the empiricist vs. rationalist controversy) --except as mental representations. Mary Louise Serafine makes a very strong statement in a similar vein (1985, p. 90):

It is this claim--that the structures and patterns in music are human mental constructions--that forces me to assert that the music theorist's formal analysis is implicitly a description of cognitive activity. What I mean to say is not that music theorists should be concerned with cognitive processes but rather that they are concerned with them whether they wish it so or not. (Original italics)

And further: "I am unable to entertain no notion of what music is that does not construe the art form as a human cognitive act."

1.2 Levels of Mental Representation

Perception and cognition. A complex mental construct such as "tonal music" surely is can be thought of as being composed hierarchically of many levels. I will use the terms "perception," "perceptual level," and "perceptual domain" to denote the lowest-ranking level of mental representation. The perceptual level is the only level of mental representation

that is directly mappable onto some aspect of the physical signal (the physical level or physical domain). With respect to music, "pitch" can be regarded as a mental representation of the perceptual type, for it is directly mappable onto the physical construct "frequency."

Besides being directly mappable onto the physical level, the perceptual level has another important distinguishing characteristic: it is the only level of mental representation that is practically isomorphical to the relevant physical domain. In our case, a simple logarithmic function defines a one-to-one correspondence between "pitch" and "frequency" that holds more or less steady through the audible frequency continuum up to about 5000 Hz (the highest tones of the piano; see section 1.5 of this chapter, and Chapter 3, section 3.2, for further discussion). For any given frequency there exists, in principle, a matching pitch and vice versa, the faster the frequency, the "higher" the pitch (and vice versa). Thus defined, "perception" in general and pitch perception in particular involve a direct and more or less "mechanical" translation of physical into psychic constituents. Since there exists, by definition, a one-to-one correspondence between the perceptual and physical domains, it is often convenient to evoke a composite "perceptual/physical" domain of pitches/frequencies. This practice should not obscure the fact that the physical domain belongs to the external world, while the perceptual domain belongs to the internal world of mental constructs.

I will reserve the terms "cognition," "cognitive level," and "cognitive domain," from now on, to any level of mental representation other than the perceptual level as discussed above. Thus, while there is only one perceptual level, the number of cognitive levels is indeterminate. A cognitive level, by definition, is not isomorphic to the physical level.

1.3 Overview of the Theory

A theory of mental representation must make a clear distinction between "competence," that is, a description of what it is that the mind constructs, and "performance," that is, a description of how this "what" is constructed, given the constraints within which the mind must operate. For further discussion of this important distinction I refer the reader to the "Interlude" that follows Chapter 3. Right now I will only note that for the most part I am concerned in this study with competence. The question of performance in connection with diatonicism proper is raised and discussed in the "Interlude" that follows Chapter 3, while in Chapter 5 I deal with performance in connection with chromaticism and enharmonicism.

The theory proposed here defines "diatonic competence" (henceforth, "diatonicism") as a mental construct composed of two corresponding levels, a perceptual level and a cognitive level, as defined in the previous section. The theory begins (Chapter 2) in defining the cognitive level of diatoni-

cism. This includes definitions for such notions as "octave equivalence," "fifth relationship," "diatonic interval," and "scale step," all in strictly mathematical terms, in particular, a combination of the modulo 12 and modulo 7 arithmetical systems which I term the "mod-(12,7) model."

The definition of diatonicism is completed in Chapter 3 in defining a correspondence between the cognitive level and the perceptual/physical level; this constitutes, in effect, a theory of intonation. The theory of intonation proposed in Chapter 3 rests on two underlying assumptions. The first assumption, logarithmic pitch perception, has been already mentioned. The other assumption, "the principle of pure intonation," states that our auditory system is somehow inherently biased towards small-integer frequency ratios such as $2/1$, $3/2$, etc. (both assumptions are discussed further in section 1.5 of this chapter).

Chapter 4 is a theory of chromatic and enharmonic competence. Chapter 5, as was already mentioned, is concerned with chromatic and enharmonic performance (diatonic performance is dealt with in the "Interlude" that follows Chapter 3). And finally, Chapter 6 offers a glimpse into the higher levels of mental representation which constitute the tonal system; it defines, informally, the notions "triad," "consonance" and "dissonance," and "tonic."

1.4 Interdependence of Levels

As has been already noted, I define the diatonic system in Chapter 2 in terms of such notions as "octave equivalency," "fifth relationship," "diatonic interval," and "scale step"; the notion "tonic" is defined only at the tonal level (Chapter 6). This means that from a purely diatonic point of view none of the seven scale steps has a "tonic" status, or alternatively, each of the seven scale steps is a potential "tonic."

Yet given that diatonicism in this study is conceived within the context of "tonality" in general, it seems redundant to have to consider seven alternative tonics at the diatonic level, when only two, the tonics that define the major and natural-minor modes, are ultimately selected. One would like to define "diatonic" in such a way that excludes all but the "tonal" modes.

This seems impossibly circular. If "tonal" is defined in terms of "diatonic," one cannot, so it seems, define "diatonic" in terms of "tonal" at the same time. However, Chomsky has already distinguished (1957, pp. 56-7) between interdependence of levels and circularity. Interdependent levels exist in any domain of some complexity, and "tonal music" is no exception. Thus, interdependent diatonic and tonal levels can be defined in assuming the notion "tonal" while defining "diatonic," and then showing that "tonal" can be defined independently of this assumption. Surely, as long as the latter condition is fulfilled, there is no circularity. For comparison,

in a genuine circularity one would have "defined" "tonal" on the basis of assuming "tonal." A "classic" example of circular reasoning of this sort is the age-old "overtone theory" (see Chapter 3, section 3.5).

There are several other interdependencies between levels of description in the present theory, in particular, between the cognitive and perceptual levels which jointly define the diatonic system. For example, the perceptual/physical level of diatonic intonation is dependent upon the number 12 (mod 12) at the cognitive level, while at the same time, the number 12 itself is dependent upon the perceptual/physical level (Chapter 3). Here also, the interdependence of levels does not lead to circularity, for it is shown that the number 12 can be defined without assuming it at the cognitive level. Other interdependencies of this sort are pointed out in the course of the discussion. For further discussion of interdependent levels and related methodological issues, see Chapter 6 in Chomsky 1957.

1.5 Two Underlying Assumptions

Logarithmic pitch perception. In defining the perceptual/physical domain in Chapter 3 I am assuming that "pitch" is a logarithmic function of "frequency." While this assumption may seem self-evident to most readers, it is a fact that some psychophysicists have proposed a non-logarithmic scale, the "mel" scale, as representing the true relation of pitch

to frequency. This forces me to give the assumption of logarithmic pitch perception further consideration.

A "mel scale" was first proposed by Stevens, Volkmann, and Newman in 1937; later versions of the same idea include Stevens and Volkmann (1940) and Beck and Shaw (1963). The scale divides the pitch continuum into equal units called "mels" (from "melody"), whose relation to the frequency continuum is not consistent throughout with that of a logarithmic unit such as "cent"; thus, for example, the "distance" between 1000 and 2000 Hz in mels is considerably larger than the distance between 100 and 200 Hz, although logarithmically both intervals represent an "octave" (see Pedersen 1965, Figure 1, for the 1940 version of the scale).

The methodology underlying the derivation of the mel scale has been severely criticized by Pikler (1966). Yet even assuming that a mel scale of one type or another is empirically valid, it should be noted that no one nowadays, music psychologists in particular, seems to be taking seriously Stevens and Volkmann's assertion that their findings ". . . contradict the widespread notion that equal ratios of frequency give rise to equal intervals of pitch" (Stevens and Volkmann 1940, p. 347). Instead, music psychologists seem to be distinguishing between two "modes" of pitch perception, a musical mode, based on the logarithmic scale, and a non-musical mode, based on a "mel" scale or some other more or less ad hoc function. "Measuring pitch in mels, . . . especially if the subject has any

musical training, is analogous to pacing off a room for wall-to-wall carpeting when a steel measuring tape is handy. Why not measure length to a fraction of an inch using the tape instead of merely determining that it is between three and four yards?" (Ward 1970, p. 412; italics added). ". . . The musical experience of the observer influences the performance in the experience" (Noorden 1982, p. 254). ". . . In a musical context there is something unique about a particular musical interval. . . ." (Chopard 1982, p. 346). This distinction has emerged most forcefully in a study by Attneave and Olson (1971), describing two experiments. In one experiment, subjects were asked to transpose sequences of two tones representing intervals of 1 to 12 chromatic steps. While subjects defined as "musical" transposed logarithmically, "unmusical" subjects transposed on a more or less ad hoc basis (not, incidentally, according to any mel scale). In a second experiment all subjects were defined as "unmusical." Yet rather than a musically impoverished sequence of two tones, the task of transposition involved a familiar, melodic sequence of three tones, the NBC chimes. Here the "unmusical" subjects' performance was practically indistinguishable from that of the "musical" subjects in the earlier experiment; they transposed on a logarithmic scale.

To the extent that logarithmic pitch perception is indeed a mode of pitch perception, in particular, a musical mode of pitch perception, I am assuming in this study that the mind is employing this mode in particular. That is to say, I do

not consider the problem of how the mind knows to employ this mode of pitch perception, rather than any other (recall the "unmusical" subjects in Attneave and Olson's second experiment); I simply assume that it does.

The principle of pure intonation. As has been previously noted (section 1.3), the theory of intonation developed in Chapter 3 is based not only on the assumption of logarithmic pitch perception, but also on the assumption that our auditory system is somehow inherently biased towards frequency ratios that correspond to small integer ratios ($2/1$, $3/2$, etc.); I have termed this second assumption "the principle of pure intonation" (Chapter 3, section 3.4).

The principle of pure intonation should be self-evident to any musician who believes that pure ratios of frequencies such as are naturally present in the "overtone series" must play at least some role in the formation of the diatonic system (that this role has been persistently misconstrued is a different matter; see Chapter 3, section 3.5); how could pure ratios play any such role unless our auditory system is somehow biased in their favor? Surely, the mere existence of pure ratios physically within the overtones cannot serve as an explanation, just as the mere existence of frequencies is not the reason why we perceive pitches in the first place.

Many modern theories of pitch perception involve an assumption comparable to the principle of pure intonation. To quote from one study:

The rules of operation in auditory processing for periodicity pitch of complex tones [i.e., perceiving the fundamental frequency, e.a.] boil down to the simple statement that the auditory system makes a best match between an internally stored template comprising a perfect harmonic series of frequencies and the somewhat noisy measurements of the frequencies of all resolvable simple tones in the stimulus and its aural combination tones.⁷

That the principle of pure intonation is, in principle at least, capable of being implemented neurologically, has been shown
8
in studies involving cats and bats.

1.6 The Mod-(12,7) Model and "Psychological Reality"

It should be stressed that the mod-(12,7) model proposed in Chapter 2 is not merely some elegant theoretical artifact; it is claimed that the model actually embodies a mental reality. Given that diatonicism can be only regarded as a mental construct (rather than, say, a physical one), the claim that the mod-(12,7) model which describes it is "psychologically real" seems to me completely uncontroversial, in fact, unexceptional. To some readers, however, such a claim may appear rather audacious and without support. I will therefore attempt to dispel this impression.

The issue of "psychological reality" has already plagued theories of mental representation in domains other than music, notably Chomsky's generative grammar and David Marr's theory of vision. "What is commonly said is that theories of grammar

or universal grammar, whatever their merits, have not been shown to have a mysterious property called 'psychological reality' (Chomsky 1980, p. 106); "sceptics challenge the claim that the nervous system forms explicit representations of visual features, and literally computes further representations from them" (Clark 1984, p. 3). As Chomsky discusses (1980, pp. 108-9), critics seem to be demanding some new and special kind of evidence to establish "psychological reality," a position that makes absolutely no sense. He elaborates on this point later on in the book in an extensive passage, which deserves to be quoted here in full (pp. 189-92):⁹

There is much discussion in the literature of linguistics and psychology of the "psychological reality" of the linguist's constructions. I take it that the question at issue is whether it is legitimate to "impute existence" to the "apparatus," the properties of which are characterized by particular grammars or by universal grammar (which is, of course, not a grammar but rather a system of conditions on the range of possible grammars for possible human languages). The discussion of "psychological reality" sometimes seems to me to be rather misleading. Perhaps I can explain my misgivings by an analogy.

Consider the problem of determining the nature of the thermonuclear reactions that take place deep in the interior of the sun. Suppose that available evidence permits astronomers to study only the light emitted at the outermost layers of the sun. On the basis of the information thereby attained, they construct a theory of the hidden thermomuclear reactions, postulating that light elements are fused into heavier ones, converting mass into energy, thus producing the sun's heat. Suppose that an astronomer presents such a theory, citing the evidence that supports it. Suppose now that someone were to approach this astronomer with the following contention: "True, you have presented a theory that explains the available evidence, but how do you know that the constructions of your theory have physical reality-- in short, how do you know that your theory is true?" The astronomer could respond only by repeating what he had

already presented; here is the evidence available and here is the theory that I offer to explain it. The evidence derives from investigation of light emitted at the periphery. We might want to place a laboratory inside the sun to obtain more direct evidence, but being unable to do so, we must test and confirm our theory indirectly. One might argue that the evidence is inconclusive or that the theory is objectionable on some physical (or, conceivably, methodological) grounds. But it is senseless to ask for some other kind of justification for attributing reality to the constructions of the theory, apart from consideration of their adequacy in explaining the evidence and their conformity to the body of natural science as currently understood. There can be no other grounds for attributing physical reality to the scientist's constructions.

Suppose now that an ingenious experimenter hits upon a more direct method for studying events taking place in the interior of the sun, namely, study of the neutrinos that are released by the assumed thermonuclear reactions in the solar interior and that escape into space. Using this new evidence, he may substantiate the old theory or construct a better one. Has this more "direct" investigation of events in the interior of the sun now answered the original objections? Are we now entitled to attribute a "higher order of physical reality" to the constructions that were only postulated before? Not really. No empirical evidence can be conclusive. Again, we can only say that with our more direct and more conclusive evidence, we may now be more confident that the entities and events postulated are physically real--that the theoretical statements in which reference is made to these entities, processes, and so on are, in fact, true. But again, there is little sense to the contention that we still do not know that what is postulated is physically real, as if there were some further standard that might be achieved in some qualitatively different way. The enterprise in question is empirical science, not mathematics; at best, we can settle on one of indefinitely many possible theories that account for crucial evidence, attributing physical reality to whatever is postulated in that theory.

Our investigation of the apparatus of the language faculty, whether in its initial or final steady state, bears some similarity to the investigation of thermonuclear reactions in the solar interior that is limited to evidence provided by light emitted at the periphery. We observe what people say and do, how they react and respond, often in situations contrived so that this behaviour will provide some evidence (we hope) concerning operative mechanisms. We then try, as best as we can, to devise a theory of some depth and significance with regard to these mechanisms,

testing our theory by its success in providing explanations for selected phenomena. Challenged to show that the constructions postulated in that theory have "psychological reality," we can do no more than repeat the evidence and the proposed explanations that involve these constructions. Or, like the astronomer dissatisfied with study of light emissions from the periphery of the sun, we can search for more conclusive evidence, always aware that in empirical inquiry we can, at best, support a theory against substantive alternatives and empirical challenge; we cannot prove it to be true. It would be quite reasonable to argue against a claim for psychological reality--that is, truth of a certain theory--on the grounds that the evidence is weak and susceptible to explanation in different terms; needless to say, the evidence that supports the linguist's constructions is incomparably less satisfying than that available to the physicist. But in essence the problems are the same, and the question of psychological reality is no more and no less sensible in principle than the question of the physical reality of the physicist's theoretical constructions.

The mod-(12,7) model is part of a theory, which like a physical or linguistic theory attempts to account for certain evidence, namely, the evidence of diatonicism. One may argue that the theory fails to account satisfactorily for the available evidence by showing that this is so or by presenting some new evidence, or one may show that the theory is faulty on some methodological grounds; or one may devise an alternative theory and show that it is at least equally capable of coping with the same facts. Yet it is senseless to argue that the theory is not psychologically real. Naturally, one will constantly seek further evidence in support of the theory, particularly, more "direct" evidence of the type neurological research might yield. For the moment, so it seems, this remains a hope for the future.

To claim that the mod-(12,7) model is psychologically real is to claim that the mind represents diatonicism at the cognitive level in terms which are isomorphical to a number system, in particular, a combined mod-12 and mod-7 integer system. It is interesting that recent theories of visual representation, in particular, the theories of David Marr,¹⁰ also hinge upon the idea of numeric representation. In a penetrating article, Austen Clark has defended the idea of numeric visual representation against sceptical attacks: "Claims for an internal number system and for neural computation should be taken literally."¹¹ I quote some further passages from his article:

Marr's accounts imply that the visual system employs representations which have all the properties of a number system. The visual system literally measures features of light and maps them into numeric codes. Those codes are numeric because they satisfy the postulates of elementary number theory, and are therefore isomorphic to rational numbers. (Pp. 2-3)

The codes must be "numeric" in the sense of exhibiting all structural properties of numbers. The content, format, and implementation of the codes is left open. However, by specifying computations performed on the codes, the theorists presupposes that, whatever the codes are, they satisfy the logical relational structure of a number system (P. 10)

. . . A computational account of vision explicitly posits a sequence of internal representational states. It claims that there are indeed number-like codes in the head. If this seems scandalous, try to account for depth from disparity without postulating codes for quantities! (P. 11)

Note that an implicit claim for numeric musical representation is far from new; numbers are already very much part

of traditional music theory. Traditional theory speaks of "prime," "second," "third," etc. in characterizing diatonic intervals, and in the domain of rhythm, of "half note," "whole note," "quarter note," etc., notions which are difficult to interpret except as numerical relations. To the extent that traditional theory has any descriptive power at all (and I believe it does), its use of numbers must be taken quite seriously. In short, the claim that the mod-(12,7) model is psychologically real is far from unreasonable; certainly there is nothing objectionable about such a claim on any a priori grounds.

1.7 Concluding Remarks

To many readers, no doubt, the theory of diatonicism presented in this study will appear highly innovative. The use of numbers, mathematical notions, special notations, etc., render the theory in appearance very different from traditional elementary music theory. It is therefore important to stress that, innovative as it may indeed be, the present theory is in many ways a highly conservative one. The terms of presentation may be indeed quite novel, yet the content of traditional theory is very much preserved. In fact, the theory can be seen as a formal, point-by-point vindication of the informal statements of conventional music theory. Whenever an impression to the contrary may arise (as with certain aspects of chromaticism and enharmonicism, for example), the matter at hand is one

about which traditional theory is simply not quite clear. In short, there is much substance to the view that the theory presented here is essentially a more articulate version of elementary music theory.¹²

But now the shoe is on the other foot. If, indeed, what is presented here is essentially well known, then "why bother"? Is it really worth the effort involved only to confirm what we more or less already know?

I believe such a stance simply misses the point. The difference between a theory and a non-theory (and by "theory" I mean a formal, or explicit theory) is not that one deals with obscure facts and the other with well-known ones; the subject matter of most of cognitive psychology, for example, are facts of everyday life. Rather, a theory deals with some facts systematically and explicitly, while a non-theory is always to some extent ad hoc, and may leave many crucial matters only hinted at or even completely unstated. For example, one of the most prominent defining properties of the diatonic system, "mod-(12,7) correspondence" (see Chapter 2, section 2.5), has passed so far unobserved. Ironically enough, it is not the obscurity of diatonicism which, I believe, has caused this to occur, but on the contrary, the fact that it is so familiar. Like many familiar, "obvious" realities, mod-(12,7) correspondence was simply taken for granted.

As Chomsky has noted in connection with linguistic theory (1957, p. 5), the quest for rigorous formulation has a much more serious basis than "mere concern for logical niceties"

or methodological purity.

Precisely constructed models. . . can play an important role, both negative and positive, in the process of discovery itself. By pushing a precise but inadequate formulation to an unacceptable conclusion, we can often expose the exact source of this inadequacy and, consequently, gain a deeper understanding of the. . . data. More positively, a formalized theory may automatically provide solutions for many problems other than those for which it was explicitly designed. Obscure and intuition-bound notions can neither lead to absurd conclusions nor provide new and correct ones, and hence they fail to be useful in two important respects.

Both the negative and positive consequences of formalization are illustrated in this study. On the negative side, it is shown (Chapter 2, section 2.2) that the mod-12 model of diatonicism currently in vogue is inadequate. I will cite two examples illustrating the positive side. I believe this study solves a perennial problem in music theory, the role of the overtone series, or better stated, the role of small-integer ratios of frequencies, in the formation of the diatonic system (Chapter 3, section 3.4). It will thus satisfy, I hope, many musicians who believe that pure ratios of frequencies are somehow important for tonal music, yet are unable to accept the "overtone theory" in its familiar terms (Chapter 3, section 3.5). A positive result of a totally different kind involves the study of traditional staff notation (Appendix B). Since staff notation is so intimately associated with the diatonic system, it is imperative that its structure be well understood. To what extent does this structure reflect the structure

of diatonicism, and to what extent is this structure an idiosyncratic one, following rules of its own? Answers to these questions are offered for the first time.

I am sure that most musicians would agree with Edward Aldwell and Carl Schachter that the tonal system ". . . must rank among the great achievements of the human spirit."¹³

I am not so sure that most musicians realize how much of this achievement involves that remarkable structure, so often taken for granted, that stands at the basis of tonality, namely the diatonic system. In bringing the structure of diatonicism to full light I hope to lend further support to Aldwell and Schachter's statement. It is my hope that anyone who truly cares about the tonal system, and who believes in the human spirit which created it, will find that this study does justice to both.

NOTES

1. Lerdahl and Jackendoff 1983, p. 332.
2. Norman 1981, p. 1.
3. 1983, p. 332.
4. Serafine 1983, p. 156.
5. Arnheim 1974, p. 5.
6. The sense in which I am using "perception" here is perhaps analogous to that in which it is customary to use the term "sensation." Regarding the terms "sensation" and "perception" through the history of philosophy and experimental psychology, see Boring 1942, pp. 3-34.
7. Goldstein 1978, pp. 433-4; italics added. See also Booms-liter and Creel (1961), Goldstein (1973), Wilson (1974), and Gerson and Goldstein (1978). Terhardt (1974) cites a number of studies, none of which I have been able to examine at first hand, as embodying a principle according to which "the singularity of small-integer frequency ratios is inherent in the pitch-analyzing mechanism of the auditory system" (p. 1063). These studies are Cremer (1951), Ritsma (1966), Schouten (1969 and 1970), and Roederer (1973).
8. Evans (1974), Suga and Jen (1976).
9. A footnote at the end of the first paragraph of the quoted passage reads: "The analogy is modeled on an account given by John N. Bahcall and Raymond Davis, Jr., 'Solar Neutrinos: A Scientific Puzzle,' Science (1976), 191: 254-57."
10. Marr, 1982.
11. Clark 1984, p. 1 (from the abstract).
12. It is interesting to note that a similar relation exists between modern and traditional theories of syntax. In particular, Chomsky has stressed that traditional grammar is essentially correct, as far as it goes. See Chomsky 1965, p. 5 and note 2.
13. Aldwell and Schachter 1979, p. 250.

Chapter II

A COGNITIVE THEORY OF DIATONICISM

2.1 Diatonicism as a Cognitive System

In the previous chapter I have noted that the definition of diatonicism offered in this study involves two levels of mental representation, a perceptual level of pitches (and one might add, pitch differences, that is, intervals), and a cognitive level that is defined in terms of the mod-(12,7) model. One may wonder whether such a complicated definition is really necessary. After all, is it not possible to define the diatonic system in strictly perceptual terms, say, as a "pitch collection"?¹ I believe one can expose the inadequacy of such an approach by way of breaking down the notion of diatonicism into its underlying components and examining them. At least four such components, I believe, can be isolated, namely "octave equivalence," "fifth relationship," "diatonic interval," and "scale step."

Octave equivalence. Any perceptual approach to diatonicism must come to terms, at the very outset, with the phenomenon of octave equivalence, "the Alpha and Omega of the musical system and the musical ear."² Why should such a phenomenon exist? Certainly, there is nothing in the physical domain

to suggest that two frequencies whose ratio corresponds to a power of 2 are somehow "equivalent."

It has been suggested by several psychologists "that pitch should be regarded as a bidimensional attribute; the first dimension representing overall pitch level, and the second defining the position of a tone within the octave."³ These two attributes of pitch have come to be known as "tone height" and "tone chroma," and are commonly represented graphically by means of a helix.⁴ A "two-component" theory of this sort amounts to a definition of "pitch" which embraces octave equivalence (in addition to, say, logarithmic pitch perception), or in other words, to an extension of what I have termed the "perceptual domain."

There is little motivation, empirically as well as purely theoretically, to adopt a "bidimensional" definition of pitch. It will take me too far afield to review the empirical evidence that has been brought to bear upon the subject,⁵ and to show⁶ that it is either irrelevant or at best, inconclusive. I will only note that the "similarity of the octave" is far from universal in early and primitive music.⁷ Had octave equivalence been a perceptual attribute of the frequency-ratio 2/1 (and its powers), it would have been hard to imagine such a possibility. For comparison, it is inconceivable that even the most "primitive" culture would produce visual artifacts with no awareness of the perceptual attribute of lightwave frequency bands known as "color."

Theoretically speaking, it is not only unnecessary, but downright impeditive, to constrain the notion "pitch" in such a way. Thus, I am able to offer an explanation for the "octave phenomenon" without committing myself to the strong assumption of "bidimensional" pitch perception (see Chapter 3, section 3.4). And how will a "two-component theory" of pitch deal with such intervals as "diminished second" and "augmented seventh"? The ratio involved is a power of 2, yet there is no "equivalence." In short, I agree with the essence of the following remark by Ernst Terhardt (1982, p. 356):

. . . The wide-spread concept which asserts that pitch is a bimodal entity, composed of "tone height" and "tone chroma" is a superfluous conceptual complication, apparently born from confusion of the terms "tone" and "pitch." While tone actually may be characterized by several sensory attributes (i.e. pitch and chroma, among others), this is not true for pitch. Pitch on its own part is an attribute. In fact, psychophysical and musical evidence strongly suggests that the one-dimensional definition of pitch provides the most adequate conceptual approach to understanding musical tone perception. . . .

Fifth relationship. The diatonic interval "perfect fifth," like the "perfect octave," holds a special status within the diatonic system. It is the only diatonic interval that can "generate" through cyclic application all seven scale steps, resulting in the familiar "cycle of fifths"; and two diatonic scale steps a fifth apart are perceived as somehow specially related. Indeed, the fifth and fourth are the only diatonic intervals save the octave (or prime) for which the quality "perfect" (or "pure") is evoked (the semantics of the terms

"perfect" and "pure" is not the issue here, but rather the fact that tonal theory draws a qualitative difference between the octave and fifth and the other intervals).

If it has proven wise to resist the temptation of accounting for "octave equivalence" in perceptual terms, then the case for a perceptual approach to "fifth relationship" is even weaker. Do we have a separate perceptual apparatus for tonal music and, say, twelve-tone music? (Recall the sense in which "perception" is intended in this study--the lowest-ranking level of mental representation). And what about the diminished sixth? Surely, one is better off not assuming that "fifth relationship" is a perceptual attribute of, say, the frequency ratio $3/2$.

Diatonic interval. The notion "diatonic interval" is difficult to understand from a purely perceptual point of view. There are twelve perceptually distinct intervals (that is, pitch differences) within the diatonic "octave," yet there are thirteen distinct "diatonic intervals," since the diatonic system chooses to distinguish between an "augmented fourth" and a "diminished fifth" (I am assuming for the moment twelve-tone equal temperament; I return to the issue of intonation later on). In conjunction with "chromatic alteration," diatonicism distinguishes between other "enharmonically equivalent" intervals as well, such as "diminished third" and "major second," "diminished fourth" and "major third," etc., which are of course indistinguishable (assuming equal temperament) in perceptual/physical terms. Note the relative complexity of traditional

interval nomenclature. Why is it necessary to use a combined system of types ("prime," "second," "third," etc.) plus qualities ("perfect," "major," "diminished," etc.)? A purely perceptual approach will be unable to come to grips with all of these aspects of the notion "diatonic interval."

Scale step. One of the most striking facts about hearing a "diatonic scale" is that we seem to be far less impressed by the inequality of "half steps" and "whole steps" than one might have supposed; our impression is rather that of a uniform progression from one "scale step" to another. In the words of Hermann Pfrogner:

Die erste Eigentümlichkeit unserer Siebenordnung besteht darin, dass wir die Stufenfolge der sieben Töne, von jedem beliebigen Ausgangston aus, als eine in sich geschlossene Ausfüllung des Oktavraumes erleben. Diese Empfindung ist ja bei der Pentatonik für uns keineswegs gegeben, deren beide unvermeidlichen Terzschriffe als Lücken auffallen. Im Gegensatz hiezu vermittelt uns die Siebentonleiter den Eindruck, dass hier der Oktavraum wirklich "diatonisch"--frei übersetzt: von Ton zu Ton fortschreitend --durchmessen wird, wobei das ungleiche Abwechseln der Ganz- und Halbtonschritte uns nicht im geringsten irritiert.⁹

This feature of diatonicism is reflected in its notation. Traditional systems of notation (e.g. staff notation, letter-name notation, "Sol-Fa" notation, Schenker's integer notation) do not seem to be concerned with the implied twelve-fold division of the "octave" (the frequency ratio 2/1) that characterizes diatonicism in perceptual/physical terms (again, I am assuming twelve-tone equal temperament for the moment; see "the problem of intonation" below). Indeed, to someone who was familiar with only the notation of diatonic music, the

"octave" would, in all probability, appear to be divided, if anything, into seven equal parts. He would have no way of knowing that the "diatonic scale" consists of "half steps" and "whole steps."

That "scale step" cannot be equated with "pitch/frequency" becomes blatantly apparent once one extends "scale step" to encompass "chromatically altered scale step" as well. For it is possible for one and the same pitch/frequency (in equal temperament) to represent different scale steps. Indeed, the phenomenon of "enharmonic equivalence" must discourage from the very outset anyone who hopes to capture diatonicism in perceptual/physical terms.

The problem of intonation. In exposing the inadequacy of the perceptual approach to diatonicism it has been convenient to assume equal temperament. After all, diatonicism can (and often does) manifest itself, in perceptual/physical terms, through equal temperament. Yet the perceptual approach to diatonicism is in an equally bad way without assuming equal temperament. There are myriad different diatonic "intonations," all of which are distinct from one another in perceptual/physical terms. Thus, a perceptual definition of diatonicism will ultimately have to constitute an indefinitely large list of "pitch collections." Surely, a more unenlightning kind of definition can hardly be imagined. It should be stressed that the theory of intonation developed in Chapter 3 offers equal temperament as only one possibility out of an infinite number of possible diatonic intonations. Equal temperament holds no conceptually

prior status.

I hope to have demonstrated the need for a cognitive definition of diatonicism. As has been already indicated, the cognitive theory of diatonicism offered in this study is based on a combination of the modulo 12 and modulo 7 arithmetical systems which I term the "mod-(12,7) model." Since a mod-12 model of diatonicism has already been offered by several authors,¹⁰ it will serve as my point of departure. I will show that the mod-12 cognitive model is incapable of expressing the structural content of diatonicism adequately, and will then proceed to show how through a combination with the mod-7¹¹ model its deficiencies can be fully remedied. The discussion up to section 2.5 is, on the whole, rather informal, and serves as a preliminary introduction to the mod-(12,7) model and some of its structural features. In section 2.5 the approach is somewhat more formal, and some missing structural information is filled in. Finally, the outlines of a formal definition of diatonicism as a cognitive system are sketched in section 2.6; it is here that the logical status of the number 12 within the theory is given explicit consideration.

2.2 A Mod-12 Model of Diatonicism

We start by equating "pitch" with an integer, any integer, positive, negative, or zero. We then define "pitch class" (PC) as an integer class, modulo 12; this gives us 12 different pitch classes (PCs), corresponding to the twelve integers between 0 and 11 (PC0, PC1, PC2, . . . PC11). The operation "mod 12"

has established an equivalence relationship between all the integers, or pitches, whose remainder, following a division by some whole multiple of 12, is the same (for example, the remainder of -10, 2, 14, and 26 is the same, namely 2, when divided by $-1 \cdot 12$, $0 \cdot 12$, $1 \cdot 12$, and $2 \cdot 12$, respectively). Thus, "equivalence modulo 12" (or just "mod 12") is a definition of "octave equivalence." We define "interval class" (IC) as a difference, mod 12, between two PCs; this gives us, also, 12 ICs, corresponding to the same 12 integers: IC0, IC1, IC2, . . . IC11. Since any pair of non-identical PCs can be subtracted from each other in two different ways, yielding two ICs, it is sometimes convenient to regard such a pair of "inversionally equivalent" or "octave complementary" ICs as a single entity. The twelve ICs reduce into six pairs of "octave complementary" ICs, or absolute ICs (abs. ICs), which are numbered from 1 to 6. For example, abs. IC1 corresponds to ICs 1 and 11, abs. IC2 corresponds to ICs 2 and 10, etc.; abs. IC6 corresponds to a pair of ICs of the same size, namely IC6.¹²

Before proceeding any further with applying the mod-12 model to the diatonic system, it is important to note that while the present discussion obviously owes a great deal to twelve-tone theory, it also departs from standard twelve-tone theory practice in certain respects, some of which are of a rather fundamental nature. On the more superficial side, I am using the term "interval class" in the sense that standard twelve-tone theory uses the term "interval," and the term "absco-

lute interval class" in the sense that 12-tone theory uses the term "interval class" (see, for example, Forte 1973, pp. 13-15). I believe that the standard 12-tone usage is mathematically inconsistent. The twelve mod-12 integer classes yield twelve mod-12 integer classes as differences; analogously, pitch classes yield interval classes, not intervals, as differences. I borrow the term "absolute IC" from Lansky and Perle, 1980.¹³

A much more substantive difference involves the sense in which the mod-12 model is employed. In 12-tone theory, according to the testimony of its practitioners, the mod-12 model is employed as a perceptual/physical model of 12-tone equal temperament:

If the elements of the twelve-tone pitch system are, indeed, "traditional" ones, both insofar as they are pitch classes with class membership defined by octave equivalence, and as there are twelve such pitch classes--corresponding to the chromatically equal-tempered quantization of the frequency continuum--even here essential deviations must be noted. In the twelve-tone system there is a one-to-one correlation between pitch notation and presented pitch, as opposed to the many-to-one correlation of triadic-tonal music. . . .¹⁴

Indeed, the twelve integer classes of the mod-12 system correspond to the logarithms of twelve equally-tempered frequencies within the "octave" $2/1$ (see Chapter 3, section 3.2). Twelve-tone theory can be thus interpreted as embracing a conception of pitch which combines logarithmic pitch perception and octave equivalence somewhat in the manner of the "two-component" theory discussed in section 2.1.

Unlike 12-tone theory, the ~~mod-12~~ model is employed here in a purely cognitive sense; its relation to the perceptual/physical domain of pitches/frequencies is as yet unspecified. The mod-12 model, in other words, is taken here as an abstract mathematical structure. Accordingly, such terms as "pitch class" and "interval class" are used for the reader's convenience only as substitutes for purely mathematical terms (in this case, "mod-12 integer class" and "mod-12 difference class"), and are devoid of any perceptual/physical content. Throughout this chapter, the reader may find it helpful to enclose the terms "PC" and "IC," mentally, in quotation marks, so as to avoid confusion with 12-tone theory, and particularly, 12-tone equal temperament (or 7-tone equal temperament, in conjunction with mod 7).

The "diatonic set" of seven mod-12 PCs is an absolute IC5 cycle construct. Given an arbitrary PC as a point of departure, one can "generate" a diatonic set by measuring abs. IC5 (i.e., IC5 or IC7) six times. The process is depicted graphically in Figure 2-1. Since I have chosen PC0 to correspond to the "tonic" ($\hat{1}$) of an arbitrary "major mode," PCs 5 or 11 are my points of departure. It should be stressed, however, that the notions "scale step" and "tonic" have not yet been defined, so that the notations " $\hat{7}$," " $\hat{3}$," " $\hat{6}$," etc., that appear in parentheses in Figure 2-1 are to be understood as an aid to the reader only. Moreover, the notion "tonic" (unlike "scale step") is defined only at the tonal level (see Chapter 6); as discussed in Chapter 1, section 1.4, the notion "tonic" is being tentatively

assumed in this chapter (and the following ones), since I have chosen to define "diatonic" interdependently with "tonal."

I could have, of course, chosen PC0 to correspond to the tonic of the natural-minor mode just as well (in which case PCs 8 or 2 would have served as points of departure for the abs. IC5 cycle).

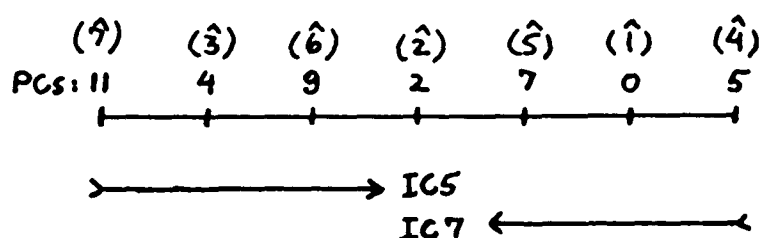


Figure 2-1. The diatonic set as an abs. IC5 cycle.

Since the cyclic application of abs. IC5 (IC5 or IC7) at least six times yields all six abs. ICs (a property shared, trivially, only with abs. IC1), the diatonic set possesses a number of highly interesting properties. 1) Naturally, it is the smallest "cyclic" set within the mod-12 system (except the abs. IC1 set alluded to above) to include each of the six abs. ICs at least once; 2) the number of times that each abs. IC appears within the set is different in each case and varies from once to six times; 3) the set of ICs (not abs. ICs) representing the IC relations of each PC to the other six (the "inter-¹⁵val context" of each PC) is different in each case.

All three properties of the diatonic set are conveniently displayed by means of a seven-by-seven matrix of which the

forty-nine different "slots" are occupied by ICs (Figure 2-2).¹⁶

Each of the seven rows and columns is labeled for one of the seven PCs of the diatonic set, arranged in exactly the same order as in Figure 2-1, namely, as an abs. IC5 cycle starting with PC11 or PC5. Each IC represents a difference, mod 12, between two PCs, particularly, those which label the row and column the intersection of which the IC occupies.

	5	6	1	8	3	10	5	0	ICs
	0	1	8	3	10	5	0	7	
	7	8	3	10	5	0	7	2	
	2	3	10	5	0	7	2	9	
	9	10	5	0	7	2	9	4	
	4	5	0	7	2	9	4	11	
	11	0	7	2	9	4	11	6	
PCs	11	4	9	2	7	0	5		

Figure 2-2. The total IC content of the diatonic set represented in terms of a seven-by-seven matrix.

That the diatonic set contains all six abs. ICs (the first of the three properties enumerated above) is reflected in the fact that all 12 ICs from IC0 to IC11 are represented in the array. Reading diagonally from bottom left to top right,

one sees that each abs. IC occurs a different number of times (the second property). Thus, abs. IC6 (represented by IC6 and its "octave complement," also IC6, at the top left and bottom right corners), appears once, abs. IC1 (ICs 1 and 11) appears twice, abs. IC4 (ICs 8 and 4) appears three times, abs. IC3 (ICs 3 and 9) appears four times, abs. IC2 (ICs 10 and 2) appears five times, and abs. IC5 (ICs 5 and 7) appears six times (IC0 naturally appears seven times; regarding "absolute IC0," see note 12).¹⁷ Reading across each of the seven rows (and for that matter, along the seven columns as well), one is struck by the different "interval context" of each PC (the third and last property). Each seven-IC set is manifestly different from any of the other six. It can be shown that the special properties of the diatonic set reflect certain mathematical relationships among the numbers 12 (as in "mod 12"), 5 (as in "abs. IC5 mod 12"), and 7 (the "cardinal number" of the diatonic set).¹⁸ In particular,

- 1) a) 5 is prime to 12,¹⁹ and further, (2.1)
 b) 5 is different from 1, which we
 exclude from consideration;
- 2) 7 equals the integral part of²⁰
 half 12 (that is, 6), plus 1.

The fact that the two constraints of (2.1) uniquely specify the "diatonic set" among all the possible mod-12 PC sets may mislead many to believe that diatonicism has been defined.²¹ This, however, is not the case. For the mod-12

model, revealing as it may be, is descriptively inadequate. What about the difference between an "augmented fourth" and a "diminished fifth" (and other "enharmonically equivalent" intervals)? Why do we refer to diatonic intervals in terms of "type" plus "quality," rather than, say, "size in half steps" in the manner of the mod-12 model? Why do we perceive the seven diatonic "scale steps" as somehow uniformly spaced within the octave, and not only as an uneven distribution of "whole steps" and "half steps" as the mod-12 model would have it? How come traditional systems of notation can well-nigh ignore the number twelve altogether? These are questions that the mod-12 model fails even to address. Indeed, the special properties of the "diatonic set" that the mod-12 model has revealed are not in themselves defining properties of diatonicism; rather, as we shall see, they reflect a defining property of diatonicism that cannot be captured in terms of the mod-12 model alone.

2.3 Diatonicism and the Mod-7 System

The connection between diatonicism and mod 7 is not difficult to see. As has been already noted, we perceive the seven diatonic scale steps as somehow evenly spaced within the octave, that is, we perceive the seven scale steps as a full set of seven, and not only (or even primarily) as a partial set of twelve, as the mod-12 model would have it. Indeed, conventional systems of notation, whether letter names ("C," "D," "E," etc.), Sol-Fa syllables ("Do," "Re," "Mi," etc.), integers (Schenker's " $\hat{1}$," " $\hat{2}$," " $\hat{3}$," etc.), or, in staff nota-

tion, staff positions, constitute, in effect, mod-7 systems. With diatonic intervals, mod 7 is also clearly involved. The seven interval types ("prime," "second," "third," etc.) clearly represent mod-7 differences between scale steps (thus, a "prime" represents the mod-7 difference 0, a "second" represents 1, etc.). The mod-7 and mod-12 models are strictly analogous. Thus, the mod-7 model defines seven PCs (PC0, PC1, PC2, . . . PC6) and seven ICs, reducible to three absolute ICs; "octave equivalence" is now "mod-7 equivalence."

2.4 A Mod-(12,7) Model of Diatonicism

Let us establish a correspondence between the seven diatonic PCs, mod 12, and the seven mod-7 PCs; we will order each set in sequence, starting with PC0 (Figure 2-3).

mod 12	↔	mod 7 (PCs)
0	↔	0
2	↔	1
4	↔	2
5	↔	3
7	↔	4
9	↔	5
11	↔	6

Figure 2-3 (on p. 36). A correspondence is established between the seven diatonic PCs, mod 12, and the seven mod-7 PCs.

So far, nothing special seems to have been achieved. Indeed, there is nothing obviously special about seven (out of twelve) mod-12 PCs corresponding to seven (out of seven) mod-7 PCs; but what about the ICs? How will the forty-nine mod-12 ICs of twelve different types relate to the forty-nine mod-7 ICs of only seven different types? At this stage we can only guess, although it is clear that some duplication must be involved, that is to say, at least in some cases, mod-12 ICs of different types will correspond to mod-7 ICs of the same type.

In order to determine the exact relation between the mod-7 and mod-12 ICs, let us construct a mod-7 "matrix" analogous to the mod-12 one (Figure 2-2) we have initially constructed. As a first step, let us take Figure 2-2 and replace each of the mod-12 PCs labeling each of the seven rows and columns by its corresponding mod-7 PC following Figure 2-3; see Figure 2-4. (As can be seen, the mod-7 PCs have also arranged themselves, rather surprisingly, in a cyclic order, namely an abs. IC3 cycle; for the time being, I will say no more regarding this remarkable fact.) As before, we execute a subtraction operation of column labels from row labels (this time, however, a subtraction modulo seven), placing each of the forty-nine differences at the appropriate "slot."

3	4	1	5	2	6	3	0
0	1	5	2	6	3	0	4
4	5	2	6	3	0	4	1
1	2	6	3	0	4	1	5
5	6	3	0	4	1	5	2
2	3	0	4	1	5	2	6
6	0	4	1	5	2	6	3
NB [↑] abs.IC3 cycle							
	6	2	5	1	4	0	3

Figure 2-4. The mod-7 IC matrix.

If one compares now the two IC arrays, the original mod-12 array (Figure 2-2), and the derived mod-7 array (Figure 2-4), IC by IC with each other, a remarkable fact emerges. A given mod-12 IC corresponds in each and every instance except one, IC6, to one and only one of the seven mod-7 ICs. This means that all mod-12 ICs of a given category except IC6 correspond to mod-7 ICs of the same category. Thus, leaving the trivial case of IC0 aside, all ICs of size 1, mod 12, and also, all ICs of size 2, mod 12, correspond to a mod-7 IC of size 1; all mod-12 ICs of size 3 and 4 correspond to a mod-7 IC of size 2, and so on. Only in the case of IC6, mod 12, do we

have a double correspondence, to ICs 3 and 4, mod 7; see Figure 2-5. (Note the orderly arrangement of Figure 2-5; I shall say no more about this also, for the time being.)

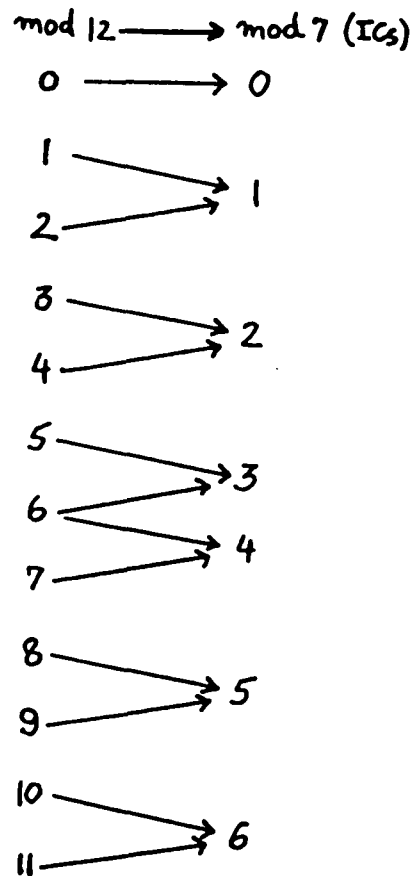


Figure 2-5. The correspondence between the 12 ICs, mod 12, and the 7 mod-7 ICs.

It may be helpful at this point to compare the "diatonic set" with another familiar 7-PC set, namely, the "harmonic-minor set." Within the diatonic set, a mod-12 IC of size 3 or 4 always corresponds to an IC2, mod 7 (a "third," in conventional terms); yet within the harmonic-minor set there is one mod-12 IC of size 3 that corresponds to an IC1, mod 7 (an "augmented second"), and one mod-12 IC of size 4 that corresponds to an IC3, mod 7 (a "diminished fourth"). Note, however, that unlike the diatonic set, the harmonic-minor set is not a "cycle-of-fifths" construct.

Given the special correspondence between the mod-12 and mod-7 ICs, we now proceed to combine the mod-12 and mod-7 arrays (Figures 2-2 and 2-4) into a single mod-(12,7) array (Figure 2-6). Single integers are replaced by integer pairs, of which the first integer represents a value in mod 12, and the second, its corresponding mod-7 value. Naturally, the mod-(12,7) array represents forty-nine subtractions among seven integer pairs (the row and column labels), mod (12,7); first integers are subtracted from first integers, mod 12, while second integers are subtracted from second integers, mod 7.

diatonic
intervals
↙

5,3	6,4	1,1	8,5	3,2	10,6	5,3	0,0
0,0	1,1	8,5	3,2	10,6	5,3	0,0	7,4
7,4	8,5	3,2	10,6	5,3	0,0	7,4	2,1
2,1	3,2	10,6	5,3	0,0	7,4	2,1	9,5
9,5	10,6	5,3	0,0	7,4	2,1	9,5	4,2
4,2	5,3	0,0	7,4	2,1	9,5	4,2	11,6
11,6	0,0	7,4	2,1	9,5	4,2	11,6	6,3

↑
scale
steps ↗

11,6 4,2 9,5 2,1 7,4 0,0 5,3

Figure 2-6. A mod-(12,7) model of diatonicism.

I shall now define scale step, cognitively, as a corresponding PC pair, mod (12,7) (the seven row and column labels), and diatonic interval, cognitively, as a corresponding IC pair, mod (12,7) (the forty-nine differences). Since IC6, mod 12, corresponds to two mod-7 ICs (namely ICs 3 and 4), I have defined thirteen distinct diatonic intervals, as the reader will be able to verify for himself. Naturally, the thirteen diatonic intervals reduce to six absolute diatonic intervals, that is, to six corresponding absolute mod 12 and mod 7 ICs, where each absolute diatonic interval is comprised of a dia-

tonic interval and its "octave complement." Note that the scale-step set is a subset of the diatonic-interval set.

It remains now to be seen whether, and to what extent, our proposed definitions correspond to "scale step" and "diatonic interval" as they are conventionally understood. In the case of scale steps, a glance at Figure 2-3 reveals a one-to-one correspondence between the mod-12 and mod-7 values; each mod-12 value corresponds to only one mod-7 value, and vice versa. As a result of this fact (and in conjunction with another systemic property, "mod-(12,7) coherence," discussed in the following section), the mod-7 values alone suffice, notationally, to specify uniquely a given scale step (that there is more to the mod-7 values alone than mere notational convenience becomes evident in the "Interlude" that follows Chapter 3). Arranged in sequence, the mod-7 values can be easily translated into any number of conventional representations of the notion "diatonic scale," be it in traditional staff notation, letter-name notation, Sol-Fa syllables, or Schenker's integer notation (which simply adds 1 to the mod-7 value of each scale step); indeed, these four systems are essentially isomorphical (Figure 2-7).²² Note that while the mod-12 values do not figure explicitly in any of the above systems, they are implicitly understood. Musicians simply know that the major scale consists of "whole steps" and "half steps."

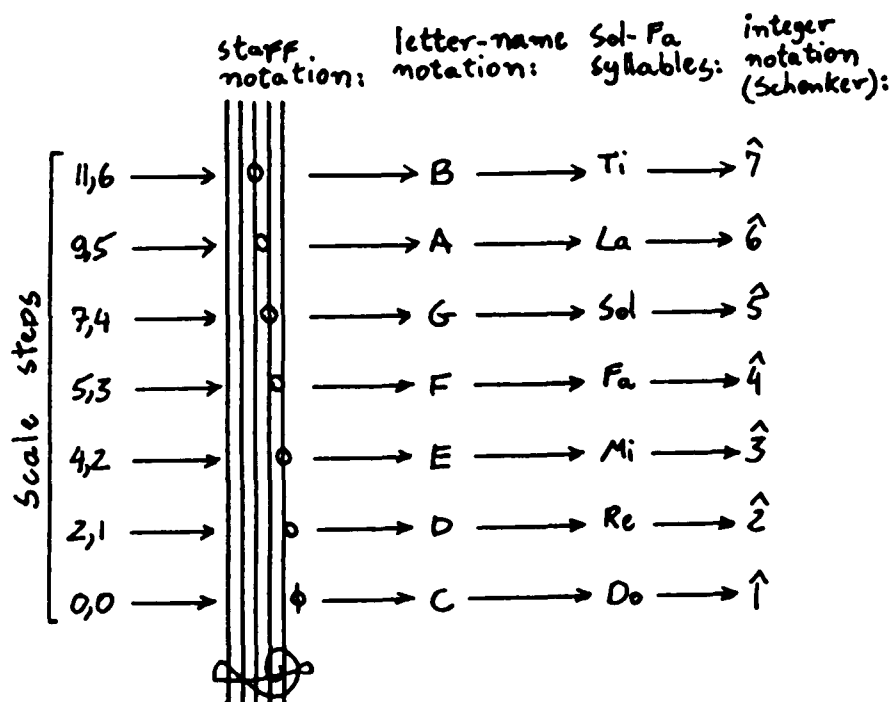


Figure 2-7. The diatonic scale notated in terms of the mod-(12,7) model and in four conventional systems of notation.

In the case of diatonic intervals, the correspondence between the mod-12 and mod-7 values works one way only; each mod-12 value except 6 corresponds to only one mod-7 value, but not vice versa (with one trivial exception, 1C0; see Figure 2-5). As a result, here the mod-7 values alone do not suffice to specify uniquely a given interval, which explains the above-noted two-component conventional system of types plus qualities. The seven mod-7 values are translated into one of seven interval types (which correspond, like Schenker's scale-step integers,

to the mod-7 values plus 1, "prime," "second," "third," etc.), and the mod-12 values, in conjunction with other considerations, are translated into one of five qualities (Figure 2-8). Thus, the "prime" is "perfect" since it conveys octave equivalence (the prime is also the only case with diatonic intervals where the correspondence does work both ways; see Figure 2-5). "Seconds," "thirds," "sixths," and "sevenths," on the other hand, are either "minor" or "major," according to their associated mod-12 values (the interval with the smaller mod-12 value is "minor," etc.). With regard to "fourths" and "fifths," the quality "perfect" for abs. IC(5,3) (that is, ICs 5,3 and 7,4) reflects its special role as the conveyer of "fifth relationship" (see section 2.5), while, at the same time, the qualities "augmented" and "diminished" for ICs (6,3) and (6,4) reflect the fact that their mod-7 "partners" (the perfect fourth and fifth) are special intervals within the system, as well as the fact that (6,3) and (6,4) are the only pair of diatonic intervals with different mod-7 values of which the mod-12 values are the same, a situation ("enharmonic equivalence") that otherwise arises only through chromatic alteration (see Chapter 4, section 4.2). For further discussion of traditional interval nomenclature, see Appendix B.

Diatonic intervals	11,6	→	major seventh (M7)
	10,6	→	minor seventh (m7)
	9,5	→	major sixth (M6)
	8,5	→	minor sixth (m6)
	7,4	→	Perfect fifth (P5)
	6,4	→	diminished fifth (d5)
	6,3	→	Augmented fourth (A4)
	5,3	→	Perfect fourth (P4)
	4,2	→	major third (M3)
	3,2	→	minor third (m3)
	2,1	→	major second (M2)
	1,1	→	minor second (m2)
	0,0	→	Perfect prime (P1)

Figure 2-8. The thirteen diatonic intervals according to the mod-(12,7) model and in conventional nomenclature.

2.5 The Mod-(12,7) Model: Further Systemic Considerations

The property of "mod-(12,7) correspondence." The diatonic system as described informally in the previous section is characterized by a property that I shall term "mod-(12,7)

correspondence" (an additional systemic property, "mod-(12,7) coherence," is revealed later on). This property guarantees, in the case of diatonic intervals, that each of the twelve mod-12 values correspond to only one mod-7 value (with one exception, the mod-12 value 6), and in the case of scale steps (which represent a set of mod-(12,7) integer pairs which yield the diatonic intervals, all and only, through subtraction), that the correspondence between mod 12 and mod 7 work both ways (a one-to-one correspondence); see Figures 2-3 and 2-5.

In Figure 2-9 the thirteen diatonic intervals, represented in integer-pair notation and in conventional nomenclature (right-hand column), are ordered "by fifths," that is, as two coordinated abs. IC cycles, a mod-12 cycle of abs. IC5, and a corresponding mod-7 cycle of abs. IC3. Since cycles of abs. IC(5,3), that is, IC(5,3) or IC(7,4), represent, mathematically, multiples of $\pm(5,3) \bmod (12,7)$,²³ and since the thirteen diatonic intervals embody a symmetrical structure of "octave complementary" pairs (except for 0,0, or P1, which stands alone), the thirteen diatonic intervals represent products, mod (12,7), of (5,3) (alternatively, 7,4) and each of a set of thirteen integers between -6 and 6 (that is, -6, -5, -4, . . . 6, see the left-hand column in Figure 2-9), which clearly, represent a symmetry around 0.

	6	times	(5,3)	mod	(12,7)	=	6,4	(d5)
	5	"	"	"	"	=	1,1	(m2)
	4	"	"	"	"	=	8,5	(m6)
+	3	"	"	"	"	=	3,2	(m3)
= $\pi(5,3)$	2	"	"	"	"	=	10,6	(m7)
	1	"	"	"	"	=	5,3	(P4)
	0	"	"	"	"	=	0,0	(P1)
	-1	"	"	"	"	=	7,4	(P5)
	-2	"	"	"	"	=	2,1	(M2)
-	-3	"	"	"	"	=	9,5	(M6)
= $\pi(7,4)$	-4	"	"	"	"	=	4,2	(M3)
	-5	"	"	"	"	=	11,6	(M7)
	-6	"	"	"	"	=	6,3	(A4)

Figure 2-9. The thirteen diatonic intervals as a "cycle of fifths."

In Figure 2-10 the set of thirteen diatonic intervals, arranged by fifths (left-hand column), is decomposed into seven subsets, where each subset is composed of seven consecutive fifth-related intervals. Each of these seven subsets is a set of scale steps, yielding the diatonic intervals, all and only, as mod-(12,7) differences; as can be seen, each set of seven scale steps embodies a one-to-one correspondence between mod

12 and mod 7 (I have arranged each set as a diatonic scale).²⁴

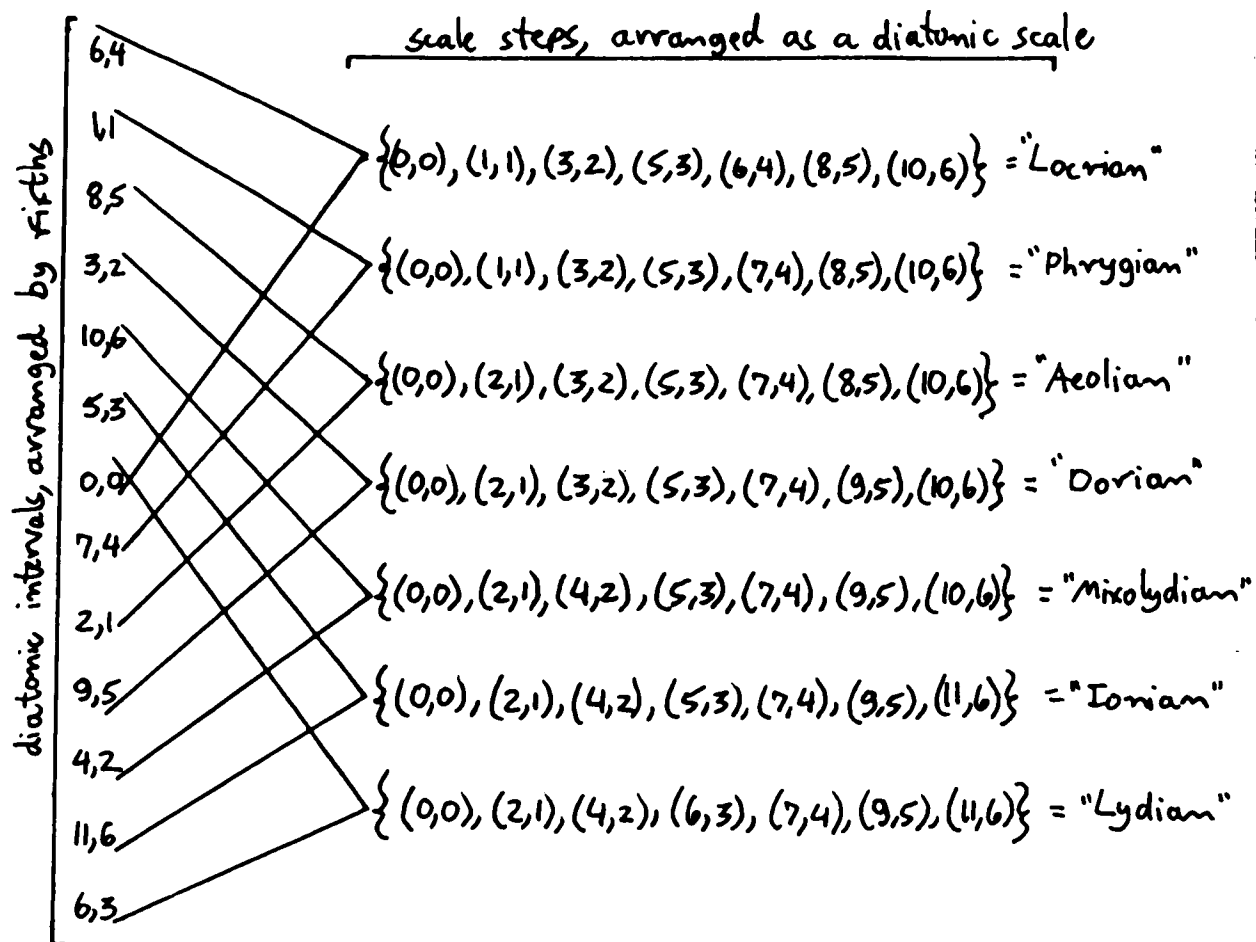


Figure 2-10. The seven diatonic modes as subsets of the diatonic-interval set.

It is not difficult to recognize the seven scale-step sets as the seven diatonic "modes," taking (0,0) as a "tonic" (or "finalis"). Since I have chosen to define "diatonic" interdependently with "tonal" (see Chapter 1, section 1.4), I will ignore all modes except the two "tonal" modes, namely the major (Ionian) mode, with which we have been dealing all the while,

and the natural-minor (Aeolian) mode. For some remarks regarding "tonality" vs. "modality," see Chapter 6.²⁵

It can be shown that the property of mod-(12,7) correspondence reflects certain mathematical relationships among the numbers 12 (as in "mod 12"), 7 (as in "mod 7"), 5 (as in "abs. IC5 mod 12"), 3 (as in "abs. IC3 mod 7"), and 6 (the absolute value of both -6 and 6, which define the thirteen "multipliers" in Figure 2-9). The following relationships, in particular, hold (for a more detailed discussion, see Appendix A):

- 1) a) 5 is prime to 12 and 3 is prime to 7, and further, (2.2)
- b) 5 and 3 are different from 1, which we exclude from consideration;
- 2) a) 7 equals the integral part of half 12 (that is, 6), plus 1, and
- b) 6 equals the integral part of half 12.²⁶

Item (1) of (2.2) is concerned with constraining the notion "fifth" (or "fifth relationship"), in our case, abs. IC(5,3) mod (12,7), so that "cycles of fifths" will potentially yield all the values in mod 12 and mod 7, respectively; since abs. IC1 fulfills this condition trivially, it is excluded from consideration (item 1b). Item (2) of (2.2) guarantees that a certain number of cyclic applications of the "fifth relationship" will indeed yield the correspondence between mod-12 and mod-7 values we are interested in (see Figures 2-9 and 2-10).

Note that (2.2) contains the set of mathematical con-

straints (2.1) (see p. 34), constraints which have been evoked in connection with the special properties of the "diatonic set," mod 12. Since the same constraints are arrived at in (2.2) without evoking any of these properties,²⁷ I am repeating my earlier claim (pp. 34-5) that the special properties of the mod-12 "diatonic set" are not defining properties of the diatonic system, but rather, are consequences of such a defining property, namely the property of mod-(12,7) correspondence, which can only be captured in terms of the mod-(12,7) model. The earlier mod-12 properties, however, can be now translated into mod-(12,7) terms. Thus, each absolute diatonic interval (that is, m2 and M7, M2 and m7, etc.) occurs a different number of times (from once to 6 times) within the system, while the "diatonic-interval context"²⁸ of each scale step is unique.

The set of values (12, 7, 5, 3, 6) we have characterized mathematically in (2.1) (but only partially, as will become evident later on) defines all four components of diatonicism as a cognitive system we have initially identified. Mod (12,7) defines "octave equivalence," $\pm(5,3) \bmod (12,7)$ defines "fifth relationship," the thirteen values between -6 and 6, multiplied by $\pm(5,3) \bmod (12,7)$, define thirteen "diatonic intervals" (Figure 2-9), and any set of seven consecutive values between -6 and 6, multiplied by $\pm(5,3) \bmod (12,7)$, defines seven diatonic "scale steps" (Figure 2-10).

Note the important distinction the system draws between the "perfect fifth" as one of the thirteen diatonic intervals and the systemically prior notion "fifth relationship." "Fifth

relationship," $\pm(5,3) \bmod (12,7)$, is logically defined after "octave equivalence," $\bmod (12,7)$, the highest-ranking systemic component, but before "diatonic interval" (for which the fifth relationship is cyclically applied twelve times). Indeed, just as the "perfect prime" can be said to be the conveyer of "octave equivalence," so can the "perfect fifth" be said to be the conveyer of "fifth relationship."

An additional systemic property: "mod-(12,7) coherence."

The reader may recall that the correspondence between the mod-12 and mod-7 PCs we have initially established (Figure 2-3) was based on a sequential arrangement of both sets, the partial set of PCs mod 12, and the full set of PCs, mod 7. Yet when we have arranged the mod-12 PCs as a cycle of abs. IC5, the corresponding mod-7 PCs have also arranged themselves, rather surprisingly, in a corresponding order, namely as an abs. IC3 cycle (Figure 2-4). Indeed, diatonic scale steps are conventionally ordered in one of these two ways, as a "diatonic scale" or as a "cycle of fifths."

The ability to order the seven scale steps not only as a cycle of fifths, but as a diatonic scale as well, reflects a mathematical relationship among the numbers 12 (as in "mod 12"), 7 (as in "mod 7"), 5 (as in "abs. IC5 mod 12"), 3 (as in

"abs. IC3 mod 7") that is not captured by the constraints of (2.2). With respect to the diatonic scale steps, this mathematical relationship (to be specified shortly) guarantees that given two mod-12 values of any two scale steps, the larger value always corresponds to a larger mod-7 value, and vice versa, given two mod-7 values of any two scale steps. For example, given the mod-12 values 11 and 9 of scale steps (11,6) and (9,5) (from the major mode), the larger value, 11, corresponds to the larger mod-7 value, namely 6. We say that the mod-12 and mod-7 values are monotone increasing functions of each other; a value increase in one set entails a value increase in the other. Surely, it would have been inconceivable to represent diatonic scale steps by their mod-7 values only as noted before (regardless of the one-to-one correspondence between the mod-12 and mod-7 values), in the absence of this property.

Let me demonstrate the significance of this property through a negative example. In Figure 2-11 I have contrived a "diatonic system" which is based on the 5-integer set (12, 7, 5, 2, 6), that is, a set which is identical to that which defines the "real" diatonic system, except that 2 appears in the place of 3. Since 2, like 3, is prime to 7 (7 being a prime number), the set (12, 7, 5, 2, 6), like the set (12, 7, 5, 3, 6), fulfills everything that (2.2) requires; the reader can verify that the contrived system's diatonic intervals and scale steps embody a correspondence between mod 12 and mod 7 analogous to

that with which we are already familiar.

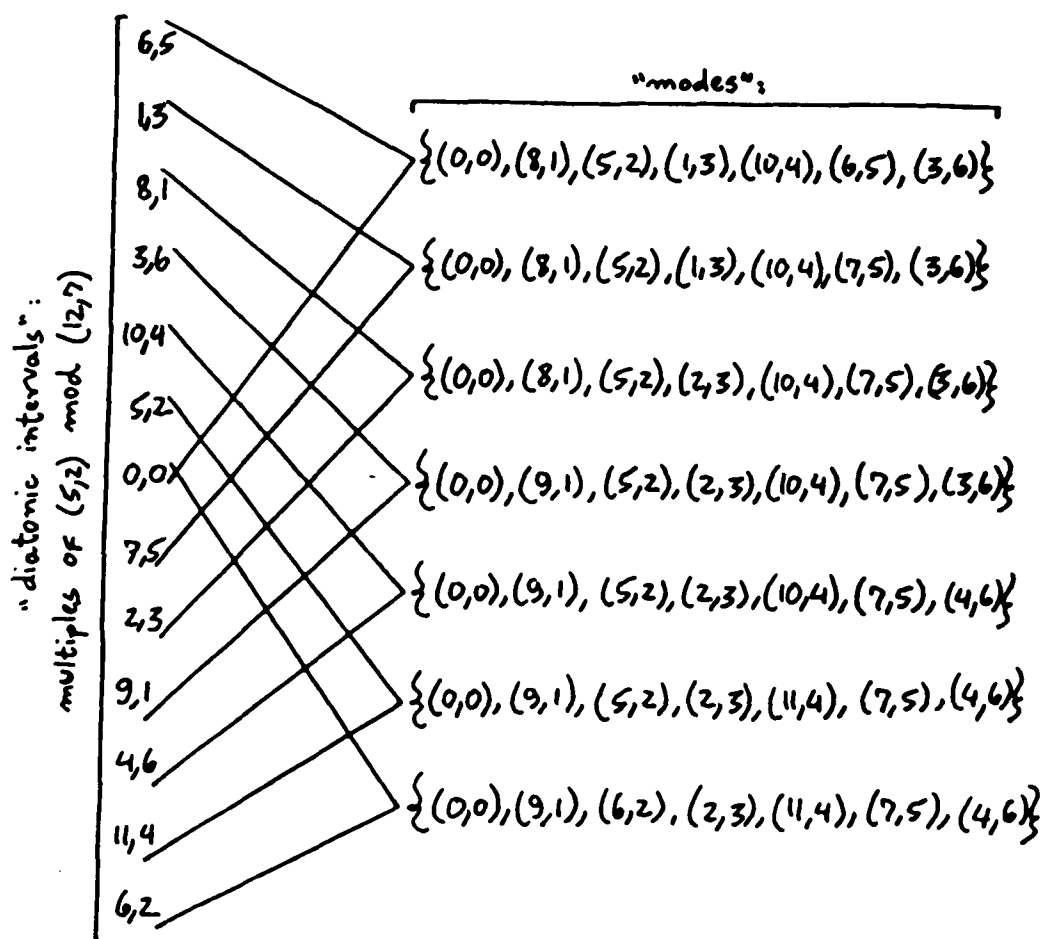


Figure 2-11. A contrived "diatonic system" that exhibits the property of $\bmod-(12,7)$ correspondence, but lacks the additional property of " $\bmod-(12,7)$ coherence."

At the same time, I doubt that the reader will be able to make much musical sense of this "diatonic system." As the reader will note, it is impossible to arrange the system's seven "modes" as "diatonic scales"; I have ordered the $\bmod-7$ values in sequence, yet the corresponding $\bmod-12$ values simply

do not follow. Indeed, the very fact that Figure 2-11 is musically meaningless, only goes to show how much is taken for granted in terms of the additional systemic property with which we are at present concerned; what is lacking in Figure 2-11 is an essential defining property of diatonicism.

The property that is involved here should be described more generally in reference to the set of thirteen diatonic intervals, of which each set of seven scale steps is a subset (recall Figure 2-10). On p. 39 I have noted, parenthetically, the orderly arrangement of Figure 2-5, displaying the correspondence between the mod-12 and mod-7 ICs. Glancing back at Figure 2-5 we now note that the mod-12 values are a monotone increasing function of the mod-7 values (except in the case of A_4 and d_5), but not vice versa. Nonetheless, larger mod-12 values entail, on the whole, larger mod-7 values, and never is it the case that a larger mod-12 value entails a smaller mod-7 value. In our contrived system, for comparison (Figure 2-11), the mod-12 values are not a monotone increasing function of the mod-7 values, and cases abound (e.g. 8,1 and 1,3) where larger mod-12 values correspond to smaller mod-7 values. Balzano (1982) has noted this property of the diatonic system and termed it "scalestep-semitone coherence"; following Balzano, I shall term it "mod-(12,7) coherence." As has been noted earlier, mod-(12,7) coherence reflects a mathematical relationship among the numbers 12 (as in "mod 12"), 7 (as in "mod 7"), 5 (as in "abs. IC5 mod 12"), and 3 (as in "abs. IC3 mod 7") that is not

captured by the constraints of (2.2). In particular, mod-(12,7) coherence reflects the fact that

$$12 \cdot 3 - 7 \cdot 5 = 1. \quad (2.3)$$

I shall return to mod-(12,7) coherence (and mod-(12,7) correspondence) from a different perspective in the "Interlude" that follows Chapter 3, dealing with how the diatonic system is processed.

The combination of both the constraints of (2.2) (mod-(12,7) correspondence) and (2.3) (mod-(12,7) coherence) renders the set of values (12, 7, 5, 3, 6) that defines the diatonic system very special. And although it is not the only set of values that answers both constraints, it is the only set of its type of which the first and largest integer is 12. Assuming mod 12, in other words, one can uniquely specify the diatonic system.

2.6 Diatonicism as a Cognitive System:

A Sketch of a Formal Definition

and Some Concluding Remarks

Let me sketch now a formal definition of diatonicism as a cognitive system. This sketch will include a definition of basis components, and a specification of systemic constraints together with corresponding theorems; specific numerical values that have been used so far for the sake of demonstration are replaced by general numerical values. The reader will find

a full formal definition of diatonicism as a cognitive system, including proofs of theorems and detailed discussions, in Appendix A. As we shall see, the definition proposed is unable to specify the diatonic system uniquely without evoking the number 12 as an independent assumption. I will indicate, however, that this apparent weakness in the theory is temporary only.

I. Basic Components.

1) "Octave Equivalence."

Mod $(\underline{x}, \underline{y})$, where $\underline{x}$ and $\underline{y}$ are natural numbers larger than 1, and (let us say) $\underline{x} > \underline{y}$.

2) "Fifth Relationship."

An integer pair, $\pm(\underline{q}, \underline{r}) \bmod (\underline{x}, \underline{y})$, where $\underline{q}$ is prime to $\underline{x}$ and $\underline{r}$ is prime to $\underline{y}$, yet both are different from ± 1 in mod $\underline{x}$ and mod $\underline{y}$, respectively.

3) "Diatonic Intervals."

A set of integer pairs representing products, mod $(\underline{x}, \underline{y})$, of the "fifth relationship" $\pm(\underline{q}, \underline{r})$ and each of a set of $2\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$ (i.e. $-\underline{N}, -\underline{N}+1, -\underline{N}+2, \dots, \underline{N}$), where $\underline{N}$ is a natural number.

4) "Scale Steps."

A set of integer pairs representing products, mod $(\underline{x}, \underline{y})$, of the "fifth relationship" $\pm(\underline{q}, \underline{r})$ and each of a set of $\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$.

Before proceeding to the systemic constraints and the corresponding theorems, let us note the constraints that are

already built into the definitions of these four basic components (for further discussion, see Appendix A). The definition of "fifth relationship" guarantees that multiples of $\pm q$ and $\pm r$ potentially yield all the values in mod x and mod y , respectively, while the trivial case wherein q and r equal 1 is excluded. The definition of "diatonic intervals" guarantees a set of integer pairs which embody a symmetrical structure of "octave complements" around $(0,0)$, and the definition of "scale steps" guarantees a set of integer pairs which yield the set of diatonic intervals, all and only, through subtraction, and has $(0,0)$ as a member; the scale-step set is a subset of the diatonic-interval set. Nothing more is guaranteed by these definitions.

II. Systemic Constraints and Corresponding Theorems.

1) The Property of "Mod- (x,y) Correspondence."

Theorem.

- a) y equals the integral part of $x/2$, plus 1;
- b) $N = y - 1$.

2) The Property of "Mod- (x,y) Coherence."

Theorem.

$$x \cdot r - y \cdot q = 1.$$

There is an infinite number of sets of values (x, y, q, r, N) which fulfill all of the constraints enumerated above. It can be shown, specifically, that the smallest values for

$\underline{X}$, $\underline{Y}$, $\underline{q}$, $\underline{r}$, and $\underline{N}$ available are 8, 5, 3, 2, and 4 respectively; the remaining sets constitute a simple sequence by which the value of $\underline{X}$ increases each time by 4, the values of $\underline{Y}$, $\underline{q}$, and $\underline{N}$ by 2, and the value of $\underline{r}$ by 1. Thus, the second set is (12, 7, 5, 3, 6) (the diatonic system), the third set is (16, 9, 7, 4, 8), etc. (see also Appendix A). In Figure 2-12 I present, as an example, the "diatonic system" (8, 5, 3, 2, 4); the reader may wish to verify that it exhibits all the systemic properties that have been described in this chapter.

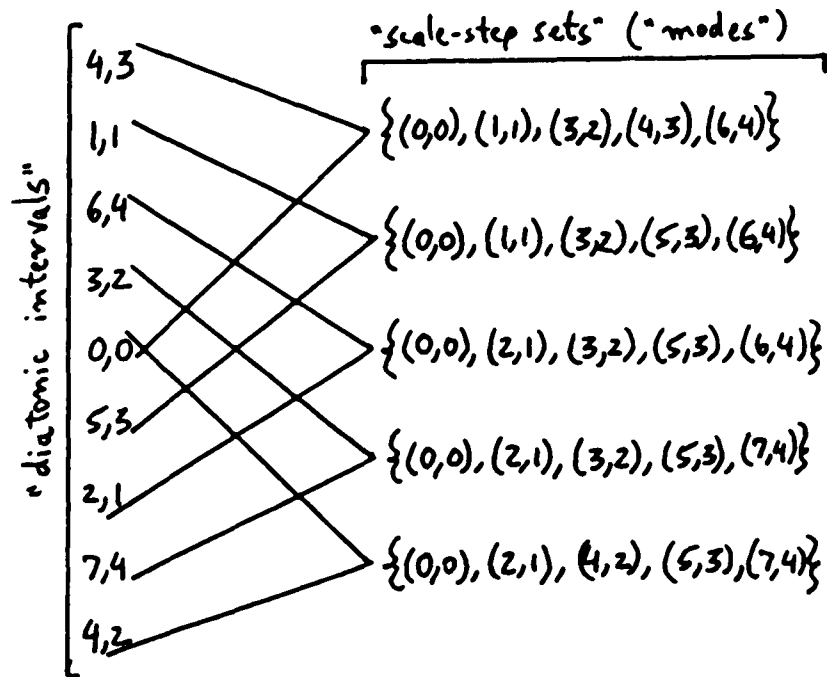


Figure 2-12: A "diatonic system" based on the values 8, 5, 3, 2, 4, for $\underline{X}$, $\underline{Y}$, $\underline{q}$, $\underline{r}$ and $\underline{N}$, respectively.

If one assumes, however, that $\underline{x}$ equals 12, the set of values (12, 7, 5, 3, 4) which uniquely specifies "our" "diatonic system" is the only available solution. Thus, the assumption

$$\underline{x} = 12$$

must be added at the moment to the systemic constraints by which diatonicism was defined above.

The need for an independent assumption $\underline{x} = 12$ may seem a weakness in the theory. Yet as it turns out, quite the opposite is true. We are unable to specify the diatonic system uniquely at the moment because the cognitive structure of diatonicism is not independent of certain perceptual/physical considerations. The details of this remarkable interdependence (see Chapter 1, section 1.4) between the cognitive and perceptual/physical domains are discussed in the following chapter (section 3.4). Right now I will only note that it involves a further constraint on the values $\underline{x}$ and $\underline{q}$, resulting in only one solution to $\underline{x}$, $\underline{y}$, $\underline{q}$, $\underline{z}$, and $\underline{N}$ being selected, namely the diatonic system.

NOTES

1. See, for example, Lerdahl and Jackendoff 1983, p. 294.
2. Révész 1954, p. 61 (originally in italics).
3. Deutsch, 1982a, p. 126.
4. See Deutsch 1982a, p. 126, and 1982b, p. 273, for references.
5. See, for example, Noorden 1982.
6. See, however, Burns and Ward (1982), pp. 262-4, for a discussion along these lines.
7. Nettl (1956), Sachs (1962), cited in Burns and Ward, 1982.
8. In principle, it is of course possible to define "pitch" in such a way so as to embrace any conceivable type of structure. In his "Structural Representations of Musical Pitch" (1982), Roger Shepard proposes in effect a definition of "pitch" which embraces seven structural features, height, chroma circle, whole-tone relations, minor-third relations, major-third relations, circle of fifths, and tritonal relations. Yet it is difficult to see the point in weighing down the notion "pitch" with so much structure. In fact, taking a cue from recent studies in visual representation, any given level of mental representation is highly restricted as to the type of structural information it contains.
9. Pfrogner 1953, p. 20 (italics added).
10. Balzano (1982), Browne (1981), Gamer (1967a and 1967b), Gauldin (1983). See also Babbitt (1965), and Forte (1973).
11. A recent study of the diatonic system from both the mod-12 and mod-7 points of view is Clough and Myerson 1985. The authors find some interesting connections between the mod-12 interpretation of the diatonic scale, which they term "specific," and the mod-7 interpretation, which they term "generic." Clough and Myerson's study is exemplary from the mathematical point of view. What remains unclear, however, is the significance of the properties they discuss, e.g. "cardinality equals variety," the "partitioning" property, and "deepness," from the empirical point of view. To my mind at least, the authors have

not demonstrated that these are important defining properties of diatonicism, and indeed, they do not seem so intuitively. It is much more likely that the Clough and Myerson properties are consequences of the mod-(12,7) definition of diatonicism proposed in this study, and indeed, this is demonstrated to be the case in connection with "deepness" and the "Myhill property" (see Appendix A, corollaries B and C to theorem 1).

Another important study from the mathematical point of view is Regener 1973. Regener is concerned with the structure of staff notation and its relation to equal temperament. Yet his relative lack of concern with the structure of the diatonic system quite apart from its notation ultimately renders the empirical value of much of his work unclear; for most of the time, the reader remains in the dark as to the musical significance of Regener's elaborate mathematics. For a review of Regener, see Chrisman 1975; for an alternative account of staff notation, see Appendix B.

12. There is no "abs. ICO" since the order of subtraction of two identical PCs is irrelevant.
13. The term "absolute IC" derives from the mathematical notion "absolute value." The absolute value of both -1 and 1, for example, is 1. Modulo 12, -1 equals 11, so that 1 and 11, mod 12, have the same absolute value, namely 1.
14. Babbitt 1960, p. 47.
15. The term "interval context" was coined by Richmond Browne (1981); to the best of my knowledge, it was William Mitchell who first observed the property involved (Mitchell, 1939).
16. The idea was derived from Browne 1981, Example 1b.
17. The number of occurrences of each abs. IC within the set can be notated as an "interval vector" $\langle 254361 \rangle$. See Forte 1973, pp. 13-15.
18. See, in this connection, Gamer (1967a and 1967b), and Babbitt, 1962. See also Appendix A, corollary C to theorem 1 and its discussion.
19. "Prime to" means that 12 and 5 have no common divisor, that is, there is no integer other than 1 that divides both 12 and 5 (unlike, let us say, 12 and 8).
20. The "integral part of x " (where x is a real number, not necessarily an integer), is defined as the greatest integer less than or equal to x . Thus, for example, the integral part of both $12/2$ (that is, 6) and $13/2$ (that is, 6.5) is 6.

21. In this respect the two constraints of (2.1) are more stringent than those which Carlton Gamer's notion "deep scale" involves (Gamer, 1967a and 1967b). For Gamer, what counts most is the different number of occurrences of each abs. IC within the set, regardless of whether or not a given abs. IC occurs at all (Gamer, in other words, allows an abs. IC to occur a zero number of times, a possibility which I exclude). This results in "deep scales" of cardinality 6 (that is, the integral part of half twelve) in addition to cardinality 7 (the integral part of half 12 plus 1, as in item 2 of (2.1)). In addition, Gamer allows a deep scale to be generated by abs. IC1 in addition to abs. IC5; I exclude this possibility (item 1b) since 1 is, trivially, prime to any number.
22. In as much as they incorporate the notion "tonic," Sol-Fa syllables and Schenker's integer notation are more tonally oriented than staff or letter-name notation. Regarding staff notation, see Chapter 3, section 3.7, but especially, Appendix B.
23. See, in this connection, note 13. In multiplying an integer pair (a, b) by an integer $N \bmod (12, 7)$ one multiplies a by $N \bmod 12$, and b by $N \bmod 7$. For example, $(5, 3)$ times $3 \bmod (12, 7)$, is an integer pair $(1, 1)$, where 1 equals 3 times 5 mod 12, or 3, and 1 equals 3 times 3, mod 7, or 2.
24. In reality, any set of seven mod-(12, 7) integer pairs that can be arranged as a "cycle of fifths" will yield the diatonic intervals, all and only, through subtraction, and will embody a one-to-one correspondence between the mod-12 and mod-7 values; however, the seven sets of Figure 2-10 are the only such sets with $(0, 0)$ as a member. See also Appendix A.
25. As an equivalent version of Figure 2-10, imagine the thirteen diatonic intervals from $(6, 3)$ to $(6, 4)$ (left-hand column in Figure 2-10) replaced by the set of thirteen tones $F\sharp, B, E, A, D, G, C, F, B\flat, E\flat, A\flat, D\flat, G\flat$. In an intuitive way, $F\sharp$, an augmented fourth above C , replaces the interval $(6, 3)$, or A_4 , B replaces the major seventh $(11, 6)$, etc.. Each set of seven consecutive tones constitutes one of seven modes as in the right-hand column of Figure 2-10, all with C as a "finalis." Thus, the set of seven tones from $F\sharp$ to C constitute a Lydian C , the set of seven tones from B to F constitute an Ionian C , etc.. In each of the seven modes, in other words, the seven scale steps correspond to the set of seven intervals that are measured from the tonic.
26. Regarding the mathematical terms "prime to" and "integral part," see notes 19 and 20.

27. This point is worth emphasizing. In particular, item (1) of (2.2) only guarantees that a cyclic application of the fifth will potentially yield all the mod-12 (and mod-7) values, but does not specify the number of cycles involved. Yet without specifying this number none of the special properties of the "diatonic set" enumerated in section 2.2 can be captured. In item (2) of (2.2) the number 7 has a totally different meaning (namely mod 7) than in item (2) of (2.1).

28. See, in this connection, the discussion of corollary C to theorem 1 in Appendix A.

Chapter III

TOWARDS A THEORY OF INTONATION:

THE COGNITIVE/PERCEPTUAL STRUCTURE OF DIATONICISM

3.1 Preliminary Discussion

In his article "Impact of Computers on Music" (1982), Ernst Terhardt distinguishes between three different domains, or three different "worlds," of musical reality, the acoustic, auditory, and symbolic. As an illustration of his distinction, Terhardt presents ". . . the problem of tone scales, their intonation and perception," and goes on to say:

On the basis of the present concept it becomes clear that a tone scale is just a series of tone symbols which represent corresponding tone categories (for example, C, D, E, F, etc.); thus a tone scale is an entity existing in world 3. Quite another thing is the scale's intonation. As a matter of definition, one may either consider "frequency intonation," thus describing a certain physical aspect of the tones, or "pitch intonation," describing the (musically most relevant) perceptual aspect. In the first case, a certain relationship to world 1 (physical objects) is realized, in the second to world 2 (sensory experience). It may become clear now that a tone scale as such, as well as its intonation, cannot be really understood or even adequately discussed without considering the complex interrelationships between pitch perception (i.e. "frequency-to-pitch transformation") and categorization of musical tones. All the presently existing "theoretical" scale intonations such as, for example, the "just," "Pythagorean," and "equally tempered" intonation are constructed on an ad hoc basis and thus must be considered as preliminary approaches to an appropriate theory of intonation which

as yet does not exist. The enharmonic distinctions (for example, that between E-flat and D-sharp) may have some merits for certain music theoretical approaches, provided that they are strictly confined to the symbolic domain. No ad hoc theory can, however, tell us anything useful about corresponding intonation distinctions.¹

It is clear that Terhardt's three domains of musical reality, the acoustic, auditory, and symbolic, correspond fairly closely to the physical, perceptual, and cognitive domains that have been delineated in this study. It is therefore interesting that, for Terhardt, ". . . an appropriate theory of intonation. . . as yet does not exist." As Terhardt notes, a theory of intonation must interrelate the cognitive (or "symbolic") and perceptual/physical domains. It must, in other words, define the cognitive domain on the one hand, the perceptual/physical domain on the other, and then proceed to define a correspondence between them. Yet for one thing, the cognitive structure of diatonicism has not been so far fully understood. The cognitive theory of diatonicism presented in Chapter 2 has thus removed one of the main obstacles in the way of developing a true theory of diatonic intonation (Terhardt's remark regarding enharmonic equivalence towards the end of the quoted passage is an interesting aside to which I will have a chance to refer later.)

The theory presented in this chapter represents only a fragment of a theory of intonation, even though the fragment is a substantial one. As has been noted above, intonation in this chapter is described in terms of a correspondence bet-

ween the cognitive level of diatonicism as has been defined in Chapter 2, and the perceptual/physical domain. However, the higher levels of tonal structure, which are considered only partially in Chapter 6, are also intonationally relevant. That is to say, there exists an interdependence between the cognitive/perceptual structure of diatonicism and the higher levels of mental representation which constitute the tonal system (regarding interdependence of levels, see Chapter 1, section 1.4). For example, "harmony" is intonationally relevant; but since "harmony" is not actually defined in this study, the role of harmonic factors in connection with intonation cannot be rigorously considered. On the whole, I believe that the role of "context" (that is, higher levels of mental representation) in intonation is relatively marginal; the most characteristic facts of diatonic intonation can be dealt with satisfactorily even without taking context into account. Nonetheless, the role of context is given some consideration in section 3.6, only to give some idea as to what is at stake and how it might be handled within a more comprehensive theory of intonation.

As has been noted in Chapter 1 (see section 1.5), the theory of intonation that is presented here rests on two underlying assumptions, logarithmic pitch perception, and the principle of pure intonation. I start with the assumption of logarithmic pitch perception in section 3.2, where I develop a general formula of intonation, the central theoretical formu-

lation in this chapter; in section 3.3 I present empirical evidence in support of this formulation. The intonational role of the principle of pure intonation is considered in section 3.4. Despite appearances to the contrary, sections 3.2 and 3.4 are interdependent. Thus, the general formula of intonation that is arrived at already in section 3.2 is based in part on assumptions that are rationalized only in section 3.4. As to the rest of the chapter, section 3.5 represents a digression of sorts; it shows why the "overtone theory" in its manifold versions, the historically most persistent "scientific" explanation of diatonicism, is logically defective. This is despite the fact that the overtone series (or rather, the principle of pure intonation), does have an important role in the formation of the diatonic system, as discussed in section 3.4. In section 3.6, as has been noted above, I consider some "contextual" factors and their effect on intonation. Finally, section 3.7 fills in certain formal details of the theory that were left out, to simplify presentation, in sections 3.2 and 3.4; more importantly, it offers a simple definition of the cognitive/perceptual structure of diatonicism on the basis of the perceptual/physical notions "pitch band" and "interval band," and introduces a correspondingly simple notational convention by which this definition can be captured.

3.2 Intonation and Logarithmic Pitch

Perception

"Pitch" as a logarithmic function of frequency. Let us define "frequency," physically, as the rate of recurrence of a sound-wave pattern. A standard unit of measurement is the Hertz (Hz), the rate of recurrence per second. In this study we are concerned with the frequency continuum that stretches from about 20 to 5000 Hz (the standard range of the modern piano, extended further down in the bass), which represents that part of the audible frequency continuum which is musically usable. Intuitively, for any given frequency f_n at the physical level (within the range delimited above) there exists a matching pitch p_n at the perceptual level and vice versa, and for any given frequency ratio fr_n there exists a matching interval i_n and vice versa, where p_n and i_n are monotone increasing functions of f_n and fr_n (and vice versa), respectively; a faster frequency or larger frequency ratio entail a "higher" pitch or larger interval (and vice versa).²

More formally, let us define a "pitch" p_n as a numerical value, in particular, as $\log f_n$ (logarithmic pitch perception); the logarithm base, at the moment, is arbitrary:

$$p_{\underline{n}} = \log f_{\underline{n}}, \text{ where} \quad (3.1)$$

$\underline{n}$ is an integer, positive, negative, or 0.

Clearly, (3.1) defines correspondences between such pitches as p_0, p_1, p_2 , etc., and respectively, such frequencies as f_0, f_1, f_2 , etc., where $f_0 < f_1 < f_2$, etc., and in general, $f_{\underline{n}} < f_{(\underline{n}+1)}$, entails $p_0 < p_1 < p_2$, etc., and in general, $p_{\underline{n}} < p_{(\underline{n}+1)}$, and vice versa.

Let us define an "interval" $i_{(\underline{j}-\underline{k})}$ as the difference between two pitches, specifically, $p_{\underline{j}}$ and $p_{\underline{k}}$ ($\underline{j}$ and $\underline{k}$ are integers, positive, negative, or 0). From (3.1) it follows that

$$\begin{aligned} p_{\underline{j}} - p_{\underline{k}} &= \log f_{\underline{j}} - \log f_{\underline{k}}, \\ &= \log f_{\underline{j}} / f_{\underline{k}}. \end{aligned}$$

Substituting $i_{(\underline{j}-\underline{k})}$ for $(p_{\underline{j}} - p_{\underline{k}})$ and $fr_{(\underline{j}-\underline{k})}$ for $f_{\underline{j}} / f_{\underline{k}}$ one gets:

$$\begin{aligned} i_{(\underline{j}-\underline{k})} &= \log fr_{(\underline{j}-\underline{k})}, \text{ or} \\ i_{\underline{n}} &= \log fr_{\underline{n}}, \end{aligned} \quad (3.2)$$

that is, a correspondence between such intervals as i_0, i_1, i_2 , etc., and respectively, such frequency ratios as fr_0, fr_1, fr_2 , etc., where $fr_0 < fr_1 < fr_2$, etc., and in general, $fr_{\underline{n}} < fr_{(\underline{n}+1)}$ entails $i_0 < i_1 < i_2$, etc., and in general, $i_{\underline{n}} < i_{(\underline{n}+1)}$, and vice versa.

Establishing a correspondence between the perceptual/physical and cognitive domains: general considerations. Having defined "pitch" and "frequency," "interval" and "frequency ratio," we are ready to define a correspondence between the perceptual/physical and cognitive domains. An obvious problem, however, arises. The components of the perceptual/physical domain have been all defined by single numerical values (p_n , f_n , i_n , and fr_n are all real numbers), whereas the components of the cognitive domain have been defined by two, a mod-12 and a mod-7 value. To which of these two values will we ask a value of p_n and i_n (alternatively, f_n and fr_n) to correspond? In order to answer this question we must touch upon the question of how diatonicism is processed, and thus anticipate in some respects the subject matter of the "Interlude" that follows this chapter.

In brief, the mind must construct the cognitive level of diatonicism, that is, the mod-(12,7) system of diatonic intervals and scale steps, on the basis of the available perceptual/physical data. Since these data can correspond only to either the mod-12 or the mod-7 values, the mind must construct one set of values all on its own. Now, we have seen that all the mod-12 values of diatonic intervals except the value 6 correspond to only one mod-7 value, while the reverse is not true (the property of "mod-(12,7) correspondence"). Clearly, the mind will be unable to supply the mod-12 values of diatonic intervals given the mod-7 values (or rather, given a perceptual/physical domain of intervals/frequency-ratios

that corresponds to the mod-7 values), while the reverse is not only possible, but in fact, extremely easy; for each mod-12 value save the value 6, the mind knows exactly which mod-7 value to supply (the "mind" here already knows the diatonic system, and only applies this knowledge in practice). The choice therefore is clear. The perceptual values p_n and i_n must correspond to the mod-12 values of scale steps and diatonic intervals.

Note that one must give up at the very outset any pretense to capture in perceptual/physical terms the difference between A_4 and d_5 , which share the same mod-12 value, namely 6. As we shall see in the following chapter, one is giving up, in fact, any pretense to capture in perceptual/physical terms "enharmonic equivalence" in general (recall, in this connection, Terhardt's comment regarding enharmonic equivalence in the passage cited in section 3.1).

Equal temperament. On the basis of logarithmic pitch perception as defined above I will now establish a correspondence between particular values of p_n and i_n and the mod-12 values of scale steps and diatonic intervals known as "equal temperament."

Many will surely find it objectionable, on historical (and other) grounds, that I am taking equal temperament as my point of departure. It is therefore important to stress that my reason for doing so has to do with presentation only; I believe it is easier to come to terms with the general formula

of intonation presented later on by relating it to equal temperament. As we shall see, there is nothing conceptually prior about equal temperament in this study, that is to say, it is not necessary to have equal temperament defined first in order to arrive at the general formula of intonation.

Throughout this section (and in subsequent sections as well) I will restrict my formulations, for the sake of simplicity, to intonation within the range of one octave only; the lower tone of this octave is arbitrary, so that pitches are defined in relative, not absolute terms. To define the octave pitch itself, I will evoke the cognitive value 12, which is of course not a mod-12 value (the set of cognitive values to which pitches and intervals correspond is therefore a set of thirteen values, 0, 1, 2, . . . 12); one could think of the value twelve as that which defines "octave equivalence," mod (12,7). I am also assuming that an interval i_n is always a positive value (or 0), which is to add the condition $p_j \geq p_k$ on the definition $i_{(j-k)} = p_j - p_k$ (see section 3.2). In section 3.7 these simplifying assumptions (which are of no significant theoretical consequence) are all revoked.

In equal temperament the "octave" $2/1$ is divided logarithmically into twelve equal parts. Defining the value of the perceived "octave" as 1, values of i_n , within any given octave, are fractions of 12, 0/12, 1/12, 2/12, . . . 12/12. Values of p_n are also fractions of twelve from the same set, if one defines the lower frequency of the particular octave

as 1 (this is merely a matter of choosing the unit of measurement; instead of using the Hz, which measures cycles per second, one could use, let us say, a unit that measures cycles per 1/20th second, that is, Hz/20, if the lower frequency is 20 Hz, etc.). Clearly, these values of p_n and i_n correspond to the values 0, 1, 2, . . . 12, respectively, that is, to the mod-12 values of the components of the diatonic system as defined in Chapter 2, and the value 12 in the case of "octave equivalence."

In equal temperament, therefore, p_n and i_n correspond to n , when p_n and i_n equal $n/12$:

$$\begin{aligned} p_n, i_n &\longrightarrow n, \text{ when} \\ p_n, i_n &= n/12, \end{aligned}$$

where n is the mod-12 value of scale steps and diatonic intervals, and the value 12 in the case of "octave equivalence." Pitches correspond to scale steps, and intervals correspond to diatonic intervals; pitches as well as intervals correspond to "octave equivalence":

$$n = 0, 1, 2, \dots 12.$$

In addition, the frequency ratio fr_{12} , which corresponds to i_{12} (which corresponds, in turn, to "octave equivalence" at the cognitive level), is defined as 2/1, or simply 2 (the "octave"):

$$\underline{fr}_{12} = 2.$$

Since $\underline{i}_{12} = \log \underline{fr}_{12}$ (from equation 3.2, $\underline{i}_n = \log \underline{fr}_n$), $12/12$, or 1, equals $\log 2$. This is true, of course, only when the logarithm base is 2; $\log \underline{f}_n$ and $\log \underline{fr}_n$ are therefore base-2 logarithms from now on.

Equal temperament offers a simple, straightforward, and well-defined correspondence between the cognitive and perceptual/physical domains, and thus fits well within the methodology we have initially set forth. At the same time, however, the correspondence is clearly much too narrowly defined to carry any general validity. As has been repeatedly pointed out (see, for example, Lloyd 1940), not even the best piano tuner is able (or, for that matter, willing) to tune in perfect equal temperament. Historically, the older temperaments were more or less completely superseded by equal temperament only during the 19th century. And when fixed intonation is not imposed on the performer, as with some string instruments, for example, or the human voice, intonation is in a constant flux; Moritz Hauptmann is quoted in Lloyd, 1938, as saying: "An animated intonation [on the violin] is just as little mathematically true as an animated time-keeping is strictly according to the metronome." Let us assume, therefore, that the formula of strict equal temperament

$$p_n, i_n = n/12$$

can be rendered more generally useful by way of substituting approximation for equality, thus:

$$p_n, i_n \approx n/12.$$

The range in which p_n and i_n are allowed to deviate from $n/12$, while still corresponding to n , however, must be clearly defined, otherwise any pitch and any interval could correspond to any scale step and any diatonic interval.

A general formula of intonation: first version. I propose, tentatively for the moment, a general formula of intonation according to which p_n and i_n can deviate from $n/12$ by less than $\pm 1/24$ th of an octave, that is, a tempered quarter tone (50 cents) in either direction. Clearly, a deviation of more than $1/24$ (and less than $3/24$, that is, 150 cents) would bring p_n and i_n closer in value to $(n+1)/12$ or $(n-1)/12$, whereas a deviation of $1/24$ exactly would result in ambiguity. In short:

$$p_n, i_n = n/12 \pm \alpha, \quad (3.3)$$

$$0 \leq \alpha < 1/24.$$

Note that p_n and i_n are no longer uniquely defined pitches and intervals, but represent an infinite number of pitches and intervals which occupy one of thirteen bands; of course, the bands are non-overlapping (and therefore, the corresponding

frequency and frequency-ratio bands do not overlap as well).

The definition $\underline{fr}_{12}=2$ is now true only with $\alpha = 0$.

It is not difficult to see that satisfying the formula (3.3) for $\underline{p}_{\underline{n}}$ only does not necessarily produce correct values for $\underline{i}_{\underline{n}}$ as well (imagine two equally tempered pitches, one of which is raised by nearly a quarter tone, while the other is similarly lowered). On the other hand, satisfying the formula for $\underline{i}_{\underline{n}}$ only does necessarily produce correct values for $\underline{p}_{\underline{n}}$ as well (that is, if one defines the value of only one pitch, say, if one defines the value of $\underline{p}_0$ as 0, with $\alpha = 0$); thus,

$$\underline{i}_{\underline{n}} = \underline{n}/12 \pm \alpha \quad (3.3a)$$

is both a necessary and sufficient condition for correct intonation. In Figure 3-1 the interval bands $\underline{i}_{\underline{n}}$ are delimited for all values of $\underline{n}$ according to (3.3a); for comparison, the values of $\underline{i}_{\underline{n}}$ in equal temperament ($\underline{n}/12$) are also given ($\underline{i}_{\underline{n}}$ values are given as base-2 logarithms of frequencies, as well as in cents, that is, base-2 logarithms times 1200).⁵

Figure 3-1 (p. 77). The $\underline{i}_{\underline{n}}$ bands according to (3.3a).

n :	value of j_{12} in equal temperature- ment ($n/12$):	j_{12} -band limits according to (3.30):
0 $\rightarrow$ $0/12 = 0.00$ (0 cents)	$\rightarrow 0/24 = 0.00$ (0 cents)	
1 $\rightarrow$ $1/12 = 0.083333$ (100 cents)	$\rightarrow 1/24 = 0.041666$ (50 cents)	
2 $\rightarrow$ $2/12 = 0.1666$ (200 cents)	$\rightarrow 2/24 = 0.125$ (150 cents)	
3 $\rightarrow$ $3/12 = 0.25$ (300 cents)	$\rightarrow 3/24 = 0.208333$ (250 cents)	
4 $\rightarrow$ $4/12 = 0.3333$ (400 cents)	$\rightarrow 4/24 = 0.291666$ (350 cents)	
5 $\rightarrow$ $5/12 = 0.41666$ (500 cents)	$\rightarrow 5/24 = 0.375$ (450 cents)	
6 $\rightarrow$ $6/12 = 0.5$ (600 cents)	$\rightarrow 6/24 = 0.458333$ (550 cents)	
7 $\rightarrow$ $7/12 = 0.583333$ (700 cents)	$\rightarrow 7/24 = 0.541666$ (650 cents)	
8 $\rightarrow$ $8/12 = 0.666$ (800 cents)	$\rightarrow 8/24 = 0.625$ (750 cents)	
9 $\rightarrow$ $9/12 = 0.75$ (900 cents)	$\rightarrow 9/24 = 0.708333$ (850 cents)	
10 $\rightarrow$ $10/12 = 0.83333$ (1000 cents)	$\rightarrow 10/24 = 0.791666$ (950 cents)	
11 $\rightarrow$ $11/12 = 0.91666$ (1100 cents)	$\rightarrow 11/24 = 0.875$ (1050 cents)	
12 $\rightarrow$ $12/12 = 1.00$ (1200 cents)	$\rightarrow 12/24 = 0.958333$ (1150 cents)	
	$\rightarrow 25/24 = 1.041666$ (1250 cents)	

As has been noted above, I have taken equal temperament as my point of departure only to facilitate presentation. In reality, equal temperament is not an underlying premise of the formula of intonation I have tentatively proposed; rather, the premise that underlies (3.3) has to do, once again, with how diatonicism is processed. Let us, therefore, consider the problem that faces a mind which must construct the diatonic system on the basis of the available perceptual/physical data; let it be clear that the "mind" in question already knows the diatonic system, but must, nonetheless, apply this knowledge in practice.

A mind faced with the task of processing pitches and intervals diatonically must start out by establishing a correspondence between a given pitch or interval within the range of any arbitrary octave and an integer from the set 0, 1, 2, . . . 12 (recall that the perceptual/physical level corresponds, at the cognitive level, to only the mod-12 values of scale steps and diatonic intervals; we have added the value 12 to the set of mod-12 values 0, 1, 2, . . . 11 to simplify presentation in this section). The mind, in other words, must establish a correspondence between a given pitch from the set p_n or a given interval from the set i_n and the integer n :

$$p_n, i_n \longrightarrow n, \text{ where}$$

p_n and i_n are sets of pitches and intervals, and

$$n = 0, 1, 2, \dots 12.$$

Clearly, the mind cannot perform such a task without making certain assumptions. I therefore propose that the mind assumes that pitches and intervals from the set p_n and i_n somehow behave like an integer n . Specifically, I propose that the mind assumes that any pitch from the set p_0 is lower than any set from the set p_1 , etc., and similarly, any interval from the set i_0 is smaller than any interval from the set i_1 , etc.:

$$\begin{aligned} p_0 < p_1 < p_2 < \dots < p_{12}, \\ i_0 < i_1 < i_2 < \dots < i_{12}. \end{aligned} \quad (3.4)$$

Clearly, (3.4) states that p_n and i_n are not just any set of pitches and intervals, but pitch and interval bands.

Continuing along the same line of reasoning, I propose that the mind also assumes that the difference between two pitches that belong to different pitch bands, say the bands p_j and p_k ($j, k = 0, 1, 2, \dots, 12$), equals an interval from a particular interval band, and only this particular interval band, namely, the band $i_{(j-k)}$ (we assume that j is equal to or larger than k , so that j minus k is an integer from the set $0, 1, 2, \dots, 12$):

$$\begin{aligned} p_j - p_k &= i_{(j-k)}, \\ j, k &= 0, 1, 2, \dots, 12, \\ j &\geq k. \end{aligned} \quad (3.5)$$

For example, the mind assumes that just as 4 minus 3 equals 1, and only 1, a pitch from the band p_4 minus a pitch from the band p_3 equals an interval from the band i_1 , and only an interval from this band.

It can be shown that assuming (3.5), p_n and i_n are pitch and interval bands exactly as delimited by (3.3). Note that (3.5) has nothing to do with equal temperament. Equal temperament only emerges, on the basis of (3.5), as a special case of the general definition of diatonic intonation, namely, the case where the value of the "deviation operator" α equals 0.

Let me illustrate the idea involved graphically, and in somewhat simplified terms. Suppose one defines a continuum of real numbers from $-\infty$ to $+\infty$ (Figure 3-2).



Figure 3-2. The real number continuum.

Suppose now that one wishes to define two equal number bands along this continuum, i_0 and i_1 ; arbitrarily, one decides that i_0 center around 0, while i_1 center around 1. One imposes the condition, however, that a number that belongs to one band belong only to this band, and not to the other. Clearly, each

band could extend from 0 and 1, in either direction, by less than 0.5 (Figure 3-3); the number 0.5 (and along with it, the numbers -0.5 and 1.5) must be excluded since it can only belong to both bands at the same time, and not to only one or the other.

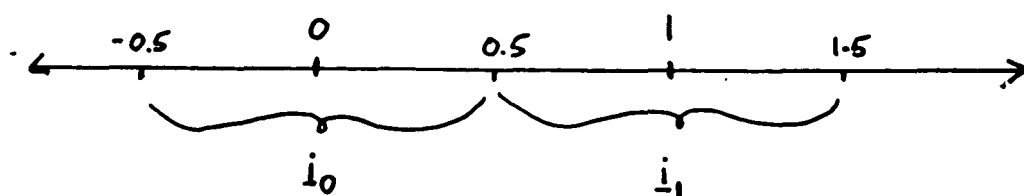


Figure 3-3. Number bands i_0 and i_1 are defined around 0 and 1 so that a given number belongs to only one band.

If one wishes now to define not only two number bands, but let us say, three, i_0 , i_1 , and i_2 (around 0, 1, and some number in between), it is clear that one must define the additional band around 0.5, and that each of the three bands must extend in either direction by less than 0.25 (Figure 3-4).

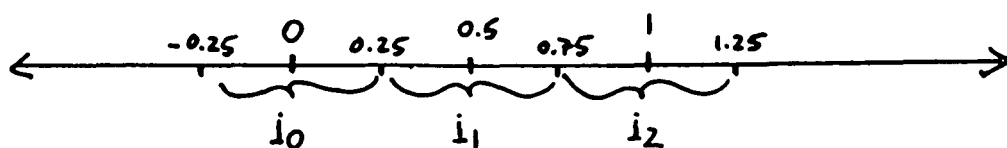


Figure 3-4. Three number bands, i_0 , i_1 , and i_2 , that fulfill the same conditions as in Figure 3-3.

The general case is not difficult to see. If one indexes the bands consecutively, $i_0, i_1, i_2, \dots, i_x$, each band i_n ($0 \leq n \leq x$) centers around n/x , and extends in either direction by less than $1/2x$. Thus, for $x=12$ (our case), each number band i_n centers around $n/12$, and extends in either direction by less than $1/24$ (cf. the formula of intonation). Once again, the set of values n/x (which defines "equal temperament") is not the underlying premise. The underlying premise is that a given number along the number continuum belong to one number band only (assuming that the bands are equal). In musical terms this boils down to the very simple and logical requirement that each interval along the interval continuum belong to one interval band only. That is to say, a mind which processes intervals diatonically must know, for any given interval, to which interval band it belongs.

The difference factor. The formula $i_n = n/12 \pm \alpha$ seems reasonable enough as a general formula of intonation, yet in some respects it is as unrealistic as the formula of strict equal temperament $i_n = n/12$. For the formula predicts that all the intervals from, let us say, 300 cents up to 349.999... cents correspond equally well to the mod-12 value 3 of the diatonic interval (3,2) (minor third) at the cognitive level, that beginning with just over 350 cents we switch from the i_3 to the i_4 band (the band of the major third), while the interval of 350 cents exactly belongs to neither band, or to both the i_3 and i_4 bands at the same time. Yet clearly, an

interval of 349 cents is not as equally useful for intonation as, let us say, an interval of 299 cents, even though both intervals belong to the same band; and it is difficult to believe that such intervals as, let us say, 349, 350, and 351 cents produce such contrasting cognitive effects. Clearly, the formula as it presently stands is defective; but what is the source of this defect, and how can it be remedied?

The answer is quite simple, and has to do, once again, with how intervals are processed diatonically. For it is obvious that as the value of α increases, the intervals of a given band resemble more and more the intervals of the adjacent one; that is to say, it becomes more and more difficult to differentiate between them. Thus, while the intervals around, let us say, 300 cents on the one hand, and 400 cents on the other, are very easy to differentiate, the intervals around 350 cents are very much alike; it makes little sense to draw such a sharp distinction between, let us say, 349 and 351 cents, given that the two intervals are practically identical. Clearly, this difference factor imposes an additional constraint on intonation, namely, the constraint that two intervals that belong to two adjacent bands be different from each other as much as possible. Let us see now how this constraint can be incorporated into the general formula of intonation priorly proposed.

A general formula of intonation: revised version. One can incorporate the difference factor into the general formula of intonation (3.3a) by way of assigning to each value of α a

preference index pr. One will assign maximum preference to $\alpha = 0$ (that is, equal temperament), and 0 preference to $\alpha = 50$ cents; moreover one will allow α to extend up to 100 cents in either direction (that is, by as much as a tempered semi-tone above or below), yet above 50 cents the pr index of α will be negative (which is interpreted as a tendency to reject the particular interval as a member of its band), and will reach a maximum negative value with $\alpha = 100$ cents. If one uses cents (rather than base-2 logarithms) to measure the value of i_n , one can write the revised formula as follows:

$$\begin{aligned} i_n &= 100n \pm \alpha_{pr} \text{ cents,} & (3.6) \\ 0 &\leq \alpha < 100 \\ pr &= 50 - \alpha . \end{aligned}$$

In Figure 3-5 the formula is illustrated graphically for the interval bands i_3 and i_4 . As can be seen, the i_3 and i_4 bands now overlap between 300 and 400 cents. However, for any interval within this range except 350 cents the pr value with respect to one band is always positive, while the other is negative. Thus, for example, the pr value of 351 with respect to the i_3 band would be -1, while the pr value of the same interval with respect to the i_4 band would be 1. The difference between a preference of -1 and a preference of 1 can be considered negligible, which indicates that for all practical purposes, the identity of this interval as a minor third or as a major third is indeterminate ("contextual" factors, however, may help decide the case; see section 3.6 and the "Interlude"

that follows this chapter). If one continues, however, further up along the scale, let us say to 360 cents, the difference between the pr values, -10 as opposed to 10, becomes more meaningful; confusion of identity becomes less and less likely.

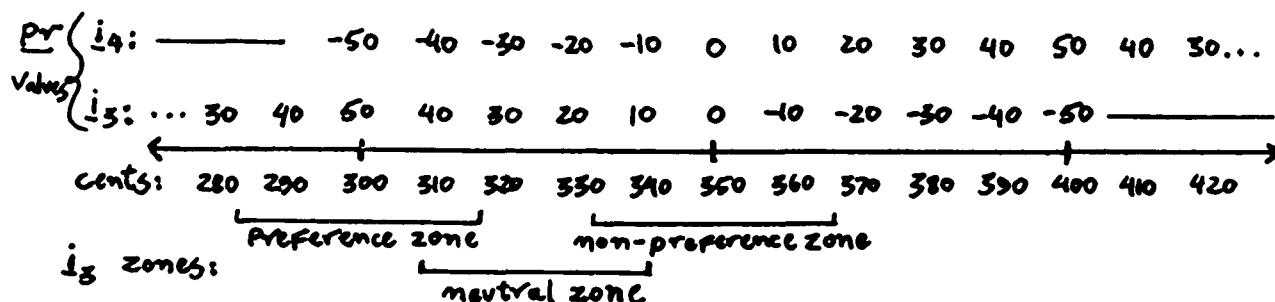


Figure 3-5. The interval bands i_3 and i_4 according to the revised formula of intonation (3.6).

Note that according to (3.6) equal temperament holds a preferred status. Once again, however, equal temperament is not the underlying premise; the underlying premise is that a given interval that belongs to some band be different as much as possible from the intervals of the adjacent bands. Moreover, equal temperament is preferred on a relative basis; it is not the only diatonic intonation available. As can be seen in Figure 3-5, I have labeled the interval zone around equal temperament "preference zone," the zone around $\Delta = 50$ cents "non-preference zone," and the zone in between, "neutral zone" (there is probably no exact borderline between these zones, and indeed, in Figure 3-5 they overlap).

The reader is also reminded that the theory of intonation is as yet incomplete; we have yet to consider the principle of pure intonation as well as "contextual" factors that bear on intonation. Yet even in its present state the theory proposed is of considerable descriptive power, as I hope to demonstrate in the following section.

3.3 Empirical Evidence in Support of the Theory

Historical evidence. Figure 3-6 is taken from the New Grove article "interval" by Mark Lindley ("Table 1," p. 278). It ". . . gives the size in cents of intervals smaller than a tritone, according to various systems of intonation that have been proposed for Western music since the Middle Ages" (p. 278). As can be seen, deviations from equal temperament ("12-part division," fifth row from the bottom) are for the most part relatively small. The largest deviation, almost 37 cents, is that of the "chromatic semitone" of "19-part division," while the septimal ratios $9/7$, $7/6$, and $8/7$ follow with deviations of 35, 33, and 31 cents, respectively (neither system, however, is historically very important). In fact, the great majority of intervals are clearly within the "preference zone" as delimited above (see Figure 3-5); only a few intervals seem to lie within the "neutral zone," and fewer still, within the "non-preference zone."

Systems	dihs		Major 3rds		Minor 3rds		Whole tones		Diatonic semitones		Chromatic semitones	
					pure + schisma	317.60						
					14/53 octave	316.98	8/53 octave	181.13				
							pure 10:9	182.40				
19-part division	8/19 octave	505.26	6/19 octave	378.95	5/19 octave	315.79	3/19 octave	189.47	2/19 octave	126.32	1/19 octave	63.158
					pure 6:5	315.64						
2/7-comma temperament	pure + 2/7 comma	504.19	pure - 1/7 comma	383.24	pure - 1/7 comma	312.57	10:9 + 3/7 comma	191.62	16:15 + 3/7 comma	120.95	25:24	70.672
			pure - schisma	384.36								
			17/53 octave	384.91								
1/4-comma temperament	pure + 1/4 comma	503.42	pure 5:4	386.31	pure - 1:4 comma	310.26	1/2 pure major 3rd	193.16	16:15 + 1/4 comma	117.11	25:24 + 1/4 comma	76.049
31-part division	13/31 octave	503.23	10/31 octave	387.10	8/31 octave	309.68	5/31 octave	193.55	3/31 octave	116.13	2/31 octave	77.419
									apotome	113.69		
									5/53 octave	113.21		
1/5-comma temperament	pure + 1/5 comma	502.35	pure + 1/5 comma	390.61	pure - 2:5 comma	307.04	9:8 - 2/5 comma	195.31	16:15	111.73	25:24 + 3/5 comma	83.576
43-part division	18/43 octave	502.33	14/43 octave	390.70	11/43 octave	306.98	7/43 octave	195.35	4/43 octave	111.63	3/43 octave	83.721
55-part division	23/55 octave	501.82	18/55 octave	392.73	14/55 octave	305.45	9/55 octave	196.36	5/55 octave	109.09	4/55 octave	87.273
1/6-comma temperament	pure + 1/6 comma	501.63	pure + 1/3 comma	393.48	pure - 1/2 comma	304.89	9:8 - 1/3 comma	196.74	16:15 - 1/6 comma	108.15	25:24 + 2/3 comma	88.594
									17:16	104.96		
12-part division	5/12 octave	500.00	1/3 octave	400.00	1:4 octave	300.00	1/6 octave	200.00	1/12 octave	100.00		
									18:17	98.955		
53-part division	22/53 octave	498.11	18/53 octave	407.55	13/53 octave	294.34	9/53 octave	203.77	4/53 octave	90.566		
Pythagorean intonation	pure 4:3	498.05	81:64	407.82	32:27	294.13	pure 9:8	203.91	limma	90.225		
Septimal ratios			9:7	435.08	7:6	266.87	8:7	231.17				

Figure 3-6. Historical table of intonation, from Lindley 1980.

Figure 3-7 presents the deviation from equal temperament of "wolf fifths" of various historical temperaments (this table is based on "Table 1" from the New Grove article "mean-tone" by Mark Lindley).⁷ I suggest that the "wolfish" character of some of these fifths derives from the fact that they occupy the "non-preference zone." Thus, the wolf fifth of 2/7-comma mean-tone has a preference value of only 4 according to formula (3.6), while the wolf fifth of 1/3-comma mean-tone has a negative preference value, -8, or, a positive preference value, 8, as a minor sixth. It is suggestive that Mark Lindley states that the 1/3-comma mean-tone temperament was probably the least used (Lindley, 1980d, p. 662).

System:	Deviation of "Wolf Fifth" from equal temperament:
1/3-comma mean-tone	58 cents
2/7-comma mean-tone	46 cents
1/4-comma mean-tone	38 cents
2/5-comma mean-tone	31 cents
1/5-comma mean-tone	26 cents
1/6-comma mean-tone	21.5 cents

Figure 3-7. The deviation of "Wolf Fifths" from equal temperament in various historical systems of intonation.

Intonation in performance. A number of studies have been carried out with regard to what can be termed "fluid intonation," that is, intonation in which p_n is not constrained to one pitch only, as with some string instruments and the human voice. The findings of these studies are summarized in Ward (1970). Shackford, for example (1961, 1962a, 1962b), has measured the intonation of three professional string trios (violin, viola, cello), performing without open strings four short tonal pieces, composed especially for the study of intonation by the author. Shackford's results are summarized in Figure 3-8 (after Ward, 1970). As can be seen, with only one exception, deviations from equal temperament are never larger than 50 cents; preference values are therefore almost always positive. Only the lower range of the minor second overshoots the 50-cent deviation mark by 6 cents, resulting in a negative preference of ⁸-6. In section 3.6 I consider contextual factors which may account for the tendency to contract minor seconds, to expand major thirds (note the high upper limit of the major third in Figure 3-8, 444 cents), and to expand the augmented fourth relative to the diminished fifth (see Figure 3-8).

Interval:	Range in cents:
Melodic m2	44-122
Melodic M2	187-228
Harmonic M2	179-235
Harmonic m3	275-334
Harmonic M3	385-444
Melodic P4	480-525
Harmonic P4	485-528
Harmonic A4	585-637
Harmonic d5	567-628
Melodic P5	682-723
Harmonic P5	685-734

Figure 3-8. Intonation in performance according to Shackford 1961, 1962a, and 1962b.

All interval ranges as measured by Nickerson (see Ward 1970, Table 3) lie within the positive pr-value range as delimited in (3.6) (no figures for the minor second are given). It is interesting to read Ward's concluding remark regarding intonation in performance (1970, p. 421): ". . . on the average, the internal scale of musical pitch used by musicians corresponds fairly closely to ET [equal temperament, e.a.], but with

a slight stretch that results in a tendency toward sharpening all tones relative to the tonic."

"Categorical perception" of intervals. Additional support to the theory involves the relatively recent notion of "categorical perception" of tonal intervals (see Siegel and Siegel, 1977, and Burns and Ward, 1978).⁹

The idea of categorical perception was first applied in speech-perception research in the late '50's. It was described, in its ideal form, as

. . . a mode by which stimuli are responded to, and can only be responded to, in absolute terms. Successive stimuli drawn from a physical continuum are not perceived as forming a continuum, but as members of discrete categories. They are identified absolutely, that is, independently of the context in which they occur. Subjects asked to discriminate between pairs of such "categorical" stimuli are able to discriminate between stimuli drawn from different categories, but not between stimuli drawn from the same category. In other words, discrimination is limited by identification: subjects can only discriminate between stimuli that they can identify differently.¹⁰

Taking their cue from speech-perception research, researchers in music perception have applied the idea of categorical perception to the perception of musical intervals. They have found that tonal intervals are also perceived categorically, that is, a continuum of intervals within a certain range are perceived as belonging to the same category (let us say, "major third"), and that musicians have considerable difficulty in differentiating between intervals that belong to the same category. Let me present now some representative results.

Figure 3-9 is taken from Siegel and Siegel, 1977. It shows the results of an experiment by which six subjects were asked to assign intervals ranging from a "flat fourth" to a "sharp fifth," in 20-cent increments, to three categories, namely, "fourth," "tritone," and "fifth"; percentage of responses from 0 to 100 percent (ordinate of the graph) are plotted against interval size from 480 to 720 cents in 20-cent increments (abscissa of the graph).

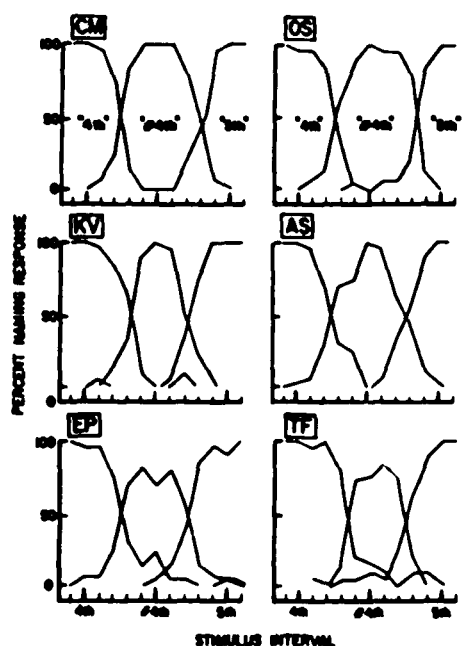


Figure 3-9. Interval identification functions for six musicians with relative pitch. From Siegel and Siegel, 1977.

As Siegel and Siegel note (p. 401), "the naming distributions appear to be regular, symmetrical, with little overlap between adjacent categories," even though variations among the

six subjects evidently exist, and some irregularities, especially with the middle category, the tritone, can be seen (the "cleanest" results are those of "CM"). In particular, note that the "crossing over" point is roughly around 550 and 650 cents, and corresponds to roughly 50% of the responses. Thus, for example, an interval of 550 cents would have been perceived by "CM" as a perfect fourth 50% of the time, and as a tritone, the other 50%. The formula (3.6) would have predicted the same.

A rather startling result from a different experiment involves the subject "CM," who appears to be the most musical. In this experiment subjects were asked to judge the magnitude of intervals numerically, relative to some standard, defined as "100." Subjects were explicitly instructed to use any number that seemed appropriate (this "magnitude estimation" experiment was in fact given first, so that subjects "would not be biased to label, rather than judge, the stimulus intervals in that experiment").¹¹ The intervals to be judged were the same intervals as in the "labeling experiment," namely, from 480 cents ("flat fourth") to 720 cents ("sharp fifth") in 20-cents increments.

Despite the very clear instructions, "CM" used only three numerical responses for the thirteen acoustically different intervals (Figure 3-10). As can be seen, his responses consist of a perfect step-like functions with three steps, where the "leap" from one step to the next seems to take place,

on the average, at the midpoints of the tempered scale (that is, around 550 and 650 cents). The responses of the other subjects were also step-like, although not nearly as clear cut. It should be noted that judgement variability (that is, judging the same interval differently in different blocks of trials) was cyclic (most clearly with "CM" and "OS," but to some degree, with the remaining four subjects as well), "with low variability at the category center and high variability at the category boundary" (p. 403). As Siegel and Siegel note,

this presumably results from the fact that intermediate stimuli are ambiguous, and may fall into one category or another. Our subjects report that the intermediate stimuli seem to "pop" one way or the other; they are not typically heard as halfway between two musical intervals.

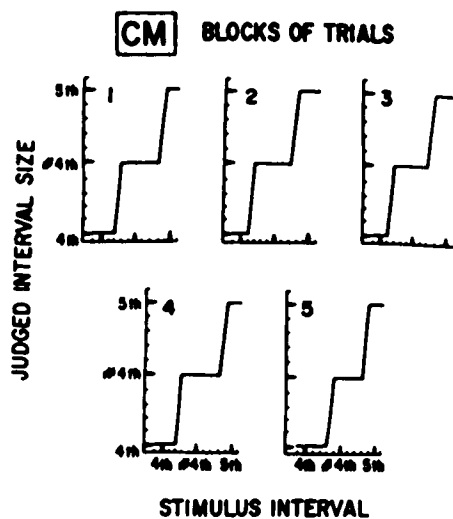


Figure 3-10. Individual judgements of interval size by subject "CM." From Siegel and Siegel, 1977 (Siegel and Siegel have transformed the magnitude estimates into values in cents).

It is interesting to note that no convincing explanation for the phenomenon of "categorical perception" of musical intervals has been so far proposed.¹² I therefore offer formula (3.6), or rather, the rationale that stands behind it, as has been explained, as an explanation of this body of empirical data.

3.4 The Principle of Pure Intonation

The general formula of intonation (3.6) arrived at in section 3.2 may give rise to an impression that logarithmic pitch perception is the only underlying assumption upon which a theory of intonation must be based. This impression is deceptive. In section 3.2 we have merely assumed that "octave equivalence" corresponds to the frequency ratio 2/1 (that is, when α equals 0); we gave no justification for this assumption. Yet from the standpoint of logarithmic pitch perception alone, "octave equivalence" could have corresponded to any arbitrary fr_{12} value. In addition, "pure ratios" such as 3/2 and 5/4 embody intonational preferences which (3.6) seems not to take into account. A theory of intonation, therefore, must rest on two assumptions, logarithmic pitch perception, and the principle of pure intonation. The latter assumption states that frequency ratios from the series 1:2:3:4:5:6:7 etc. are given preference.¹³

It is important to point out that there is more to the principle of pure intonation than just preference for pure

frequency ratios. The series of ratios represents a hierarchy of preferences, where the "octave" $2/1$ embodies the strongest preference, the "fifth" $3/2$ the second strongest preference, the "thirds" $5/4$ and $6/5$ the third strongest preference and so on, the preference for pure intonation diminishes the higher one climbs in the series (specifically, the series of prime numbers, as explained in note 13). Moreover, it is assumed that this hierarchy of preferences corresponds to the hierarchy of diatonic intervals at the cognitive level, in particular, the hierarchy of diatonic intervals according to their special status within the tonal system, if any. Thus, as has been noted in Chapter 2, the perfect octave, or "octave equivalence," is the highest ranking component within the diatonic system, while the perfect fifth, or "fifth relationship," comes next.

The principle of pure intonation will enable me to account in this section for the universal connection, with tonal music at least, between the "octave" $2/1$ and "octave equivalence." I will thus make good an earlier obligation to account for the "octave phenomenon" without evoking the assumption of "bidimensional" pitch perception (see Chapter 2, section 2.1).

I will also make good another earlier obligation, namely, to account for the number 12 at the cognitive level (see Chapter 2, section 2.6). As we shall see, the principle of pure intonation enables me to place additional constraints on the cognitive definition of diatonicism, which results in having the diatonic system uniquely specified. Furthermore, these addi-

tional constraints will lead me to hypothesize as to how the diatonic system originated in the first place. I will consider the pure ratios in hierarchical order, starting with the "octave" $2/1$; the "fifth" $3/2$ comes next, and is followed by the major and minor "thirds" $5/4$ and $6/5$. Other ratios, such as the septimal ratios $9/7$, $8/7$, and $7/6$, are of negligible importance, and are not considered.

The "octave" $2/1$ and its relation to "octave equivalence."

The "octave" $2/1$ participates in all diatonic intonations. It corresponds, as we have seen, to the interval band i_{12} (with $\alpha = 0$), which corresponds, in turn, to "octave equivalence," or mod $(12,7)$, at the cognitive level.

This "universal" relation between the "octave" $2/1$ and "octave equivalence"--universal as far as tonal music is concerned at least--follows directly from our ancillary assumption, namely, that the hierarchy of pure ratios at the perceptual/physical level corresponds to the hierarchy of diatonic intervals at the cognitive level; the "octave" $2/1$ stands at the apex of the perceptual/physical structure much in the same way that "octave equivalence" stands at the apex of the cognitive one. Thus, there is no need to invoke the idea of "bidimensional" pitch perception to account for the "octave phenomenon." The "octave phenomenon" is an expression of corresponding structures at two different levels of mental representation.

Note that the relationship between the "octave" $2/1$ and "octave equivalence" represents a correspondence between the perceptual/physical and cognitive domains that has a logical priority over any such correspondence that one may otherwise define. That is to say, one knows that the "octave" $2/1$ corresponds to "octave equivalence" regardless of how one otherwise defines the correspondence between the perceptual/physical and cognitive domains. There are no special intonational-theoretic assumptions involved in this relationship, such as those which lead to formula (3.6), given the principle of pure intonation (and of course, logarithmic pitch perception).

The "fifth" $3/2$ and its special relation to the maximally preferred value of $\underline{1}_7$ according to (3.6). The "fifth" $3/2$ and the perfect fifth (the conveyor of the "fifth relationship") follow the octave $2/1$ and "octave equivalence" in the corresponding hierarchies of frequency ratios at the perceptual/physical level and diatonic intervals at the cognitive level, which we assume:

"octave" $2/1$ $\longleftrightarrow$ octave equivalence
 "fifth" $3/2$ $\longleftrightarrow$ fifth relationship.

As we have seen, the value of $\underline{1}_7$ in equal temperament, that is, the value of $\underline{1}_7$ according to (3.6) with $\alpha = 0$ or $\underline{pr} = 50$, is $7/12$, or $0.58333\dots$ (see Figure 3-1; $\underline{1}_7$ is of course the interval band that corresponds, at the cognitive level, to the "perfect fifth," or $7/4$). The value of the perceived

frequency ratio $3/2$ (in relation to the perceived "octave" $2/1$, defined as 1), is $\log_2 3/2$, an irrational number, 0.584963 in six-figure accuracy. As can be seen, the two numbers lie close to each other; calculated in cents, the difference between the "tempered" and "pure" fifths is less than 2. This fact is crucial. For it means that there exists a minimal amount of conflict between two intonational preferences with respect to the "perfect fifth" (the "fifth relationship"), namely, that of formula (3.6), which centers around $7/12$, and that of the principle of pure intonation, which centers around $\log_2 3/2$.

It is not difficult to see that what is involved here is an additional constraint on the numbers 12 (as in "mod 12") and 5 (as in "abs. IC5 mod 12"), that is, the values $\underline{x}$ and $\underline{q}$ from the set $(\underline{x}, \underline{y}, \underline{q}, \underline{r}, \underline{N})$ which defines the diatonic system at the cognitive level (see Chapter 2, section 2.6); the fraction $(\underline{x}-\underline{q})/\underline{x}$ must be as close as possible to $\log_2 3/2$:

$$(\underline{x}-\underline{q})/\underline{x} \text{ is as close as possible to } \log_2 3/2. \quad \text{14} \quad (3.7)$$

Given that the smallest values of $\underline{x}$ and $(\underline{x}-\underline{q})$ that answer the constraints that have been specified in Chapter 2 are 8 and 5, and given that the remaining solutions constitute an infinite sequence by which the values of $\underline{x}$ and $(\underline{x}-\underline{q})$ increase each time by 4 and 2, respectively (see Chapter 2, section 2.6, and Appendix A), it is not difficult to see that the higher one climbs in the sequence the closer $(\underline{x}-\underline{q})/\underline{x}$ approaches 0.5 (the sequence tends to 0.5); musically speaking, this means that the larger

the $\underline{x}$ one chooses, the closer the fifth $(\underline{x}-\underline{q})/\underline{x}$ will approach, intonationally, 600 cents (the "tritone"), with $\alpha = 0$. Thus, we begin with 750 cents (5/8ths of an "octave"), and continue to 700 cents (7/12ths of an octave), 675 cents (9/16ths of an octave), 660 cents (11/20ths of an octave), and so on, the fifths diminish gradually in size down to the limit of 600 cents:

750¢, 700¢, 675¢, 660¢, ... limit = 600¢.

Clearly, the best approximation here of the pure fifth is 700 cents, that is, 7/12ths of an octave.

7/12 is not only the nearest number to $\log_2 3/2$ within the given set of ratios $(\underline{x}-\underline{q})/\underline{x}$. It is also the nearest number to $\log_2 3/2$ which any fraction whose denominator is smaller than 29 can yield (the fraction 17/29 offers a slightly better approximation; a substantially better approximation is offered only by 24/41). In other words, the fifth of 12-tone equal temperament is the purest fifth one can get short of dividing the octave into 29 equal parts.

This fact is important as it is remarkable. For given that the sequence of fraction $(\underline{x}-\underline{q})/\underline{x}$ tends to 0.5, it is clear that only one fraction will be closest to $\log_2 3/2$; yet how close to $\log_2 3/2$ it will be is a different matter. Of course, there is no limit as to how close to an irrational number a rational fraction might get, provided that one places no limit on its denominator size (of course, matching an irrational number

with a rational fraction is out of the question). Yet surely, there must be some limit to the size of $\underline{x}$, given that $\underline{x}$ must reflect human cognitive capacities. Suppose one sets this limit to 29. One can now state that $(\underline{x}-q)/\underline{x}$ fulfills in relation to $\log_2 3/2$ a more severe constraint than (3.7), namely,

$$(\underline{x}-q)/\underline{x} \text{ is the nearest fraction to } \log_2 3/2, \quad (3.7a)$$

where $(\underline{x}-q)/\underline{x}$ is any fraction of which

$$\underline{x} < 29.$$

Clearly, (3.7) suffices to specify uniquely the diatonic system; the fact that the diatonic system fulfills the more severe (3.7a) is all the more remarkable. Be that as it may, there is no longer a need to assume that $\underline{x} \leq 12$, as we did in Chapter 2; in conjunction with a theory of intonation based on logarithmic pitch perception and the principle of pure intonation,¹⁵ the value of $\underline{x}$ can be uniquely determined.

Thus, what could have been seen as a weakness in the theory in Chapter 2, ultimately emerges as one of its strengths. That intonational considerations play a part in the cognitive definition of diatonicism only says that the cognitive structure of diatonicism (unlike, perhaps, other musical systems) is not just a self-contained, "pure" structure, but also, a structure which reflects certain aspects of the perceptual/physical domain, namely, the combination of logarithmic pitch perception and the principle of pure intonation. Diatonicism, one might say, thus bridges two realities; the internal, cogni-

tive reality of "pure structure," and the perceptual reality, which faces directly the "real," external world.

The genesis of the diatonic system. The constraint (3.7a) enables me to offer a theory as to how the human mind may have hit upon the structure of diatonicism in the first place, that is, as to the genesis of the diatonic system.

My theory is based on the following assumptions: 1) logarithmic pitch perception; 2) the principle of pure intonation (and its physical counterpart--the overtone series); 3) a cognitive domain of, specifically, rational numbers, which I assume is finite in size. Assuming a cognitive domain of rational numbers smaller, say, than 29, it is clear on the basis of (3.7a) that the "octave" $2/1$ and the "fifth" $3/2$ at the perceptual/physical level would strongly tend to map themselves onto the values 12 and 7, respectively (Figure 3-11); once the number 12 (mod 12) assumes cognitive significance, the number 7 (mod 7), and along with it, the entire structure of¹⁶ diatonicism, is implicit as well (Chapter 2, section 2.6).

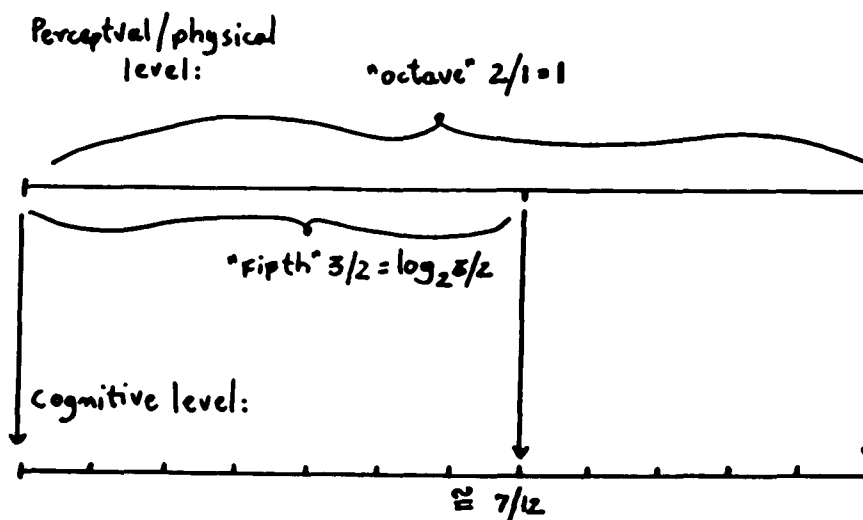


Figure 3-11. The genesis of the diatonic system: the mapping of the irrational number $\log_2 3/2$ representing the relation of the "fifth" to the "octave" onto the rational cognitive domain.

Note that the principle of pure intonation, which embodies an internal preference for pure intonation, is reflected externally by the overtone series. For example, the first two intervals produced by overblowing a simple wind instrument are the "octave" and "fifth." Thus, Nature has had an important (but not exclusive) role in the genesis of the diatonic system; see also section 3.5.

Pythagorean intonation. The proximity of the "pure" fifth to the "tempered" fifth allows for a system of fixed intonation as attractive and as historically important as equal temperament, namely Pythagorean intonation. If equal temperament can be accounted for directly in terms of formula (3.6)

plus the additional constraint of fixed intonation (that is, that p_n can take only one value), Pythagorean intonation can be accounted for directly in terms of the principle of pure intonation, plus the same additional constraint; in Pythagorean intonation all fifths are systematically tuned pure. In Figure 3.12 the Pythagorean i_n values are given for all values of n . If one compares Figure 3-12 with Figure 3-5 (p. 85) one sees that the i_n values in Pythagorean intonation lie well within the "preference zone"; the deviations from equal temperament are relatively small. (The deviation increases by the difference between the tempered and pure fifth, that is, by about 2 cents, as one progresses by fifths along the "cycle of fifths" of intervals, symmetrically in either direction, starting with the perfect prime; the deviation is therefore the largest for the augmented fourth and diminished fifth, and amounts to about 12 cents, or exactly half the "Pythagorean comma").

n :	i_n value:	
0	0.00	(0.00 cents)
1	0.075185	(90.22 cents)
2	0.169926	(203.91 cents)
3	0.245111	(294.13 cents)
4	0.339852	(407.82 cents)
5	0.415037	(498.04 cents)
{ 6 (d5)	0.490222	(588.26 cents)
{ 6 (A4)	0.509778	(611.73 cents)
7	0.584963	(701.95 cents)
8	0.660148	(792.17 cents)
9	0.754889	(905.86 cents)
10	0.830074	(996.08 cents)
11	0.924815	(1109.77 cents)
12	1.00	(1200.00 cents)

Figure 3-12. i_n values in Pythagorean intonation.

Note that there are two values for i_6 in Pythagorean intonation, a smaller one for d5, and a larger one for A4 (the two differ by a Pythagorean comma). This does not mean, how-

ever, that this perceptual/physical difference somehow corresponds to the cognitive difference involved, that is, to the difference between the mod-7 values of these diatonic intervals (4 and 3, respectively). The perceptual/physical difference between A_4 and d_5 in Pythagorean intonation is a necessary result of the particular constraints involved, namely, the principle of pure intonation plus fixed intonation; it carries no further significance. (There is nonetheless a tendency, quite apart from Pythagorean intonation, to contract d_5 relative to A_4 ; see section 3.6.) As the reader will recall, we have given up at the very outset any pretense to capture in perceptual/physical terms the difference between A_4 and d_5 , since the correspondence between the perceptual/physical and cognitive domains involves, in the latter case, the mod-12 values only (see section 3.2). I will return to this important (and all too often misunderstood) point in the subsequent chapter, in connection with "enharmonic equivalency" in general.

Contextual considerations. Although the difference between the "pure" and "tempered" fifths is negligible, its mere existence invites connecting the preference for pure fifths, however weakly, with contextual considerations. That is to say, one may wish to specify that pure fifths are specially preferred when fifths in general are contextually prominent, for example, "harmonically" or "melodically." Pythagorean intonation, as a system of fixed intonation based on pure fifths, indeed seems most suitable to diatonic music of this sort, in

particular, when thirds are not contextually prominent (parallel organum comes to mind). For further discussion of intonation and context, see section 3.6.

The major and minor "thirds" $5/4$ and $6/5$. The major and minor "thirds" $5/4$ and $6/5$ follow the "octave" $2/1$ and "fifth" $3/2$ in the hierarchy of pure-ratio intonational preferences. Assuming the notion "triad," which I define in Chapter 6 in conjunction with the notion "tonic," the major and minor thirds at the cognitive level are intervals of special systemic status, following, hierarchically, the octave and fifth. Thus, the corresponding hierarchies of frequency ratios at the perceptual/physical level and diatonic intervals at the cognitive level embrace the "thirds" in each case as well:

"octave" $2/1$ $\longleftrightarrow$ octave equivalence
 "fifth" $3/2$ $\longleftrightarrow$ fifth relationship
 "thirds" $5/4, 6/5$ $\longleftrightarrow$ major, minor third (triad).

The logarithm of the major "third" $5/4$ ($\log_2 5/4$) is 0.321928, and that of the minor "third" $6/5$ ($\log_2 6/5$) is 0.263034; the values of 1_4 and 1_3 according to (3.6) with $pr = 50$ (that is, in equal temperament) is $4/12$ (0.3333) and $3/12$ (0.25), respectively (see Figure 3-1). Calculated in cents, the pure major third is smaller than the tempered major third by about 14 cents, and the pure minor third is larger than the tempered minor third by about 16 cents. Although these discrepancies between intonational preferences are considerably

larger than the corresponding discrepancies with regard to the fifth, the pure thirds lie pretty much within the "preference zone" according to (3.6) (see Figure 3-5); moreover, a pure major third and a pure minor third add up, of course, to a pure "fifth" $3/2$. The pure thirds, therefore, are eminently suitable for intonation. Given, however, that the preference for pure thirds is not very strong (it is weaker, at any rate, than the preference for pure fifths), contextual considerations play here a much more prominent role. That is to say, the preference for pure thirds is dependent upon thirds in general, or triads, being contextually prominent. This connection between "triadic context" and pure thirds is considered in section 3.6.

Summary. We have seen that there exists a minimal amount of conflict (less than 2 cents) between the "pure" and "tempered" fifths, that is, between the maximally preferred value of 17 according to the principle of pure intonation on the one hand, and formula (3.6), on the other. We have also seen that the preference for purely intoned major and minor thirds is a matter of context. As a result, excluding contextual considerations, the principle of pure intonation does not necessitate any serious revision of (3.6); (3.6) already incorporates, with a negligible error, the preference for pure fifths, and of course, for pure octaves as well.

3.5 The Logical Inadequacy of the "Overtone Theory"

Given that the principle of pure intonation merely restates in psychological terms the underlying premise of what has been, no doubt, the most persistent "scientific" explanation of diatonicism through the history of music theory, namely, the "overtone theory" (and its predecessor, the theory of string division), I find it necessary right now to digress from the main line of inquiry in order to render explicit the terms under which I have employed the premise, as opposed to those of the overtone theory; I will show that the latter are simply logically unacceptable.

The overtone theory, in its earlier guise as the theory of string division, presumably dates back to Pythagoras. It was passed on into the Middle Ages through Boethius, and was embraced in the Renaissance and Baroque by such major figures as Zarlino and Rameau. It gains "scientific respectability" in the 19th century in the theories of Helmholtz (Helmholtz, 1863), and even finds enthusiastic support in the 20th century with such figures as Hindemith (1952) and Leonard Bernstein (1976).

Even though the theory has been criticized time and again, its appeal to many musicians has somehow not been totally exorcised. Musicians realize the shortcomings of the

theory, yet are somehow unable to pass unobserved the connections that obviously exist between the overtones and the tonal system. Thus, for example, Aldwell and Schachter (1978) point out, very carefully, a number of such possible connections, and then go on to say (p. 22): ". . . it is impossible to demonstrate a causal connection between the overtone series and these aspects of tonal music. But if it is a coincidence, it is a most remarkable one." Indeed, it should be clear by now that those musicians who felt that a connection between the overtones and the diatonic system must exist were essentially right. But why is the overtone theory wrong nonetheless? In the light of the present theory a definitive answer to this question can be given.

The overtone theory purports to arrive in a logical manner from phenomenon "A" (the overtone series as a physical manifestation) to phenomenon "B" (the diatonic system as a cognitive/perceptual manifestation, but at any rate, a musical system); yet one phenomenon simply does not follow from the other. At the very outset, for example, there is no justification to consider frequency relationships in terms of ratios (rather than, let us say, differences, from which point of view the overtones would have been perceived as equally spaced), apart from assuming logarithmic pitch perception. And there is nothing in the physical domain to suggest that two frequencies whose ratio corresponds to a power of 2 are somehow "equivalent."

Suppose, however, that one assumes, in addition to the

overtones, logarithmic as well as "bidimensional" pitch perception. The diatonic system still does not follow. There is simply no justification to use such notions as "octave," "fifth," "major third," etc., notions which carry, as we have seen, a heavy baggage of cognitive implications, to describe the frequency ratios $2/1$, $3/2$, and $5/4$, respectively. What does $2/1$ have to do, for example, with the number 8, or $3/2$ with 5? Clearly, the overtone theory simply defines its premises in terms of its conclusions, and thus is guilty of circularity in a very real and serious way. It is no wonder that through circular reasoning of this sort it has been possible to use the overtones to "explain" practically every tonal phenomenon imaginable, from the major triad to parallel fifths.¹⁷

Needless to say, there is no such circular reasoning in the theory that has been here proposed. The "cognitive" and "perceptual/physical" domains are defined independently of each other, and each in its own terms. The principle of pure intonation is relegated, and strictly so, to the perceptual/physical domain; a frequency ratio remains a frequency ratio, and nothing more (note that this does not stand in the way of showing that interdependencies between the levels exist; see section 3.4 of this chapter, and Chapter 1, section 1.4). Indeed, the significance of defining a cognitive domain in Chapter 2 independently of the perceptual/physical domain can be now, perhaps, better appreciated. As the reader will recall, I have insisted in Chapter 2 that the notions "PC" and "IC" be understood in strictly cognitive terms, that is, as devoid

of any perceptual/physical content; I have insisted, in particular, that the number 12 not be understood in the sense of "twelve equally tempered pitches," but rather, in an abstract, mathematical sense.¹⁸

Some authors have had the idea that diatonicism could be explained through the fact that tuning cyclically by pure fifths $3/2$ the octave $2/1$ is approximated at certain points; thus, for example, five, seven, and twelve "fifths," respectively, yield increasingly better "octaves." This idea is incorporated, for example, in Joseph Yasser's elaborate theory of "evolving tonality" (Yasser, 1932), and is echoed in Johnston, 1964; it forms the cornerstone of a discussion by Scott Makeig of the basis of the western harmonic system (Makeig, 1982). As with the overtone theory, however, this undisputable relation between "fifths" and "octaves" at the perceptual/physical level does not entail the cognitive attributes of diatonicism. Incidentally, it is not difficult to recognize in this idea what I have described in terms of rational approximations of $\log_2 3/2$. Clearly, if the perceived "fifth" is approximately (say) $7/12$ ths of an "octave," then twelve "fifths" approximate seven "octaves." The latter formulation, however, is rather dubious, since no one goes through the tiresome process of measuring twelve fifths in any imaginable natural situation.

3.6 Intonation and Context

As we have seen in section 3.3, the formula of intonation (3.6) is of considerable descriptive power. It accounts, at least in general outlines, for the historical facts of intonation, as well as for the facts of intonation in performance and "categorical perception" of diatonic intervals. At the same time, however, the facts of intonation needing explanation have clearly not been exhausted. Of the historical "temperaments," only equal temperament and Pythagorean intonation have been fully accounted for; we have yet to account for "just intonation" as well as the different varieties of "mean-tone" temperament. Clearly, what is involved here is a preference for pure "thirds" which we have so far failed to take into account. Also, we have seen in Shackford's data regarding intonation in performance (Figure 3-8) a tendency to contract the minor second down to a negative preference value of -6, as well as to expand the major third up to a preference value of only 6 (which represents the "non-preference zone"), and to expand the augmented fourth relative to the diminished fifth. To account for intonational facts such as these contextual considerations must be brought into play; a theory of intonation that does not take context into account is simply incomplete.

A "context" is simply a higher level of mental repre-

sentation; thus, for example, "harmony" is a context with respect to intonation. Since the present theory deals primarily with the lower levels of mental representation which constitute "tonality," namely, the diatonic system (see, however, Chapter 6), it obviously cannot deal with the issue of context adequately. Nonetheless, I will consider, however informally, some examples of contextually-dependent intonational data, just to give some idea as to what is at stake, and how it might be handled within the framework of a more comprehensive theory of intonation. I remind the reader that the connection between intonation and context reflects, in principle, an interdependence between the cognitive/perceptual structure of diatonicism and the higher structural attributes of the tonal system; circularity is not necessarily involved.

"Triadic context" and pure thirds. As has been noted in section 3.4, the correspondence between the "thirds" at the perceptual/physical level and the thirds at the cognitive level involves the notion "triad," through which thirds acquire a preferred systemic status (see Chapter 6). But since the preference for pure thirds is weaker than the preference for pure fifths (according to the principle of pure intonation, the preference for pure intonation diminishes the higher one climbs in the series), contextual considerations play here a rather prominent role; thirds, or triads, must be contextually prominent, for example, "harmonically" or "melodically," in order for the preference for pure thirds to manifest itself.

The role of triadic context in connecting the "thirds" at the perceptual/physical level and the thirds at the cognitive level is corroborated by historical evidence. There seems to be a connection between the change into a more triadic style of composition that is noticeable already in some 13th-century music,¹⁹ and the emerging recognition that thirds are consonances,²⁰ beginning in the 13th century, and particularly, the recognition that they correspond to the pure ratios $5/4$ and $6/5$, beginning in the 14th century (Walter Odington). The connection between the "thirds" and the triad, moreover, leads me to hypothesize as to the genesis of the tonal system (see Chapter 6). It can be shown that just as $7/12$ offers the best approximation of $\log_2 3/2$ (using an integer system of an order smaller than 29), given 12 as denominator (that is, given the diatonic system), $4/12$ (that is, $1/3$) and $3/12$ (that is, $1/4$) offer the best rational approximation of $\log_2 5/4$ and $\log_2 6/5$, respectively. That is to say, the major and minor "thirds" $5/4$ and $6/5$ would strongly tend to map themselves onto the diatonic intervals $M3$ and $m3$, respectively, provided that the latter hold a preferred status, which is to say, provided that "triad," and with it, "triadic context," have been defined.

The historical temperaments known as "just intonation" and "mean-tone" temperament are clearly associated with triadic context. They can be accounted for as various attempts to reconcile, within the framework of a fixed intonational system, the conflicts that arise between pure fifths and thirds (for

further detail, see Barbour 1951, and Lindley 1980b and 1980c).

Intonation and the harmonic context of "tendency tones."

We have seen in Shackford's data regarding intonation in performance (Figure 3-8) a tendency to contract the minor second, to expand the major third, and to contract the diminished fifth relative to the augmented fourth. Studies of intonation by Paul Greene (1937a and 1937b) have also shown similar tendencies.²¹ Following Greene's data, Seashore has concluded that ". . . the violinist, when unaccompanied, . . . tends on the whole to conform with the Pythagorean scale. . .,"²² an idea that seems to be rather prevalent.

Dixon Ward (1970) has challenged the notion that Pythagorean intonation is somehow a "natural" scale for string players, and offered instead the formulation that players simply perform a "shade sharper" than equal temperament (see Ward, 1970, pp. 420-2, for details). Indeed, it is difficult to see the relevance of Pythagorean intonation, which involves the additional constraint of fixed intonation, and which is not applicable to triadic music, to violin intonation; are we to believe that a violinist somehow imposes upon himself the constraint to play, as much as possible, only one frequency per scale step, or that he does not recognize the preference for pure "thirds" in a triadic context? Rather, I believe, the violin player (or singer) is working his way between conflicting intonational preferences (i.e., equal temperament, pure "fifths," and, in a triadic context, pure "thirds" as well), trying to

reconcile them as best as possible; the tendency to contract or expand certain intervals has nothing to do with Pythagorean intonation, but is a matter of context.

Consider for a moment Figure 3-13, a V^7-I progression in C major. Both the leading tone and scale degree $\hat{4}$ represent what one might term "tendency tones"; the leading tone pulls strongly in the direction of the tonic, and $\hat{4}$ pulls towards $\hat{3}$. If one assumes that these cognitive tendencies are expressed at the perceptual/physical level in terms of intonation, that is, that $\hat{7}$ is somewhat sharpened and $\hat{4}$ somewhat flatted, one gets a contracted minor second $\hat{7}-\hat{1}$, an expanded major third $\hat{5}-\hat{7}$, and a contracted diminished fifth $\hat{7}-\hat{4}$. A study of violin intonation by Small (1937) has also found "frequent applications of tendencies from the tempered scale," namely, the tendencies for

(a) the fourth degree of the scale to be lowered and the seventh raised, (b) minor and diminished intervals to be contracted, (c) major and augmented intervals to be expanded and (d) chromatically altered notes to over-shoot the alteration in the direction of the chromatic used.²³

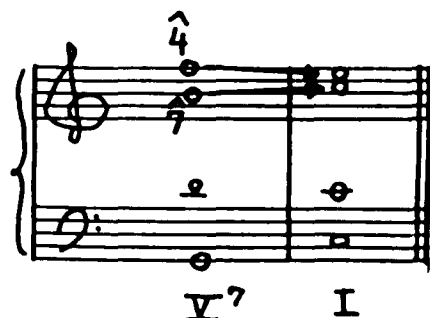


Figure 3-13. The tendency tones $\hat{7}$ and $\hat{4}$ within a V^7 -I progression.

3.7 The Cognitive/Perceptual Structure of Diatonicism

My formulations so far were restricted, for the sake of simplicity, to intonation within the range of one arbitrary octave only, and have involved other simplifying assumptions as well (see p. 72). I will now revoke all earlier simplifications; in particular, I will define "diatonic intonation" through the entire musically usable frequency range, and will pursue the logical consequences that such a definition involves in some detail.

Although the essence of the theory remains unchanged, the "finishing touches" applied in this section are important. In fashioning the perceptual/physical notions "pitch band" and "interval band" to their definitive form, as well as defining such perceptual/physical notions as pitch-band and interval-

band class, I am able to define the cognitive/perceptual structure of diatonicism in relatively simple terms; I am able, moreover, to capture this definition in terms of a correspondingly simple notational convention.

It is not difficult to see that pitch bands (as well as interval bands) whose range is not restricted to one octave only must correspond not only to one of twelve mod-12 integers, but also, to an integer that specifies their register (or, in the case of interval bands, the number of whole octaves that they span). Accordingly, we will define a correspondence between a set of pitch and interval bands p_n and i_n , where n is, in principle, any integer, positive, negative, or 0, and a set of mod-12 values t , as well as a set of "register operators" rg :

$$p_n, i_n \longrightarrow t, rg, \text{ where}$$

n is an integer, positive, negative, or 0,

$t = 0, 1, 2, \dots, 11$, and

rg is an integer, positive, negative, or 0.

Specifically, p_n and i_n correspond to (t, rg) when n equals t plus the "register operator" rg times 12:

$$n = t + 12 \, rg.$$

For example, p_6 corresponds to $t=6$ and $rg=0$, p_{15} corresponds to $t=3$ and $rg=1$, p_{-14} corresponds to $t=10$ and $rg=-2$, etc..

We can now state the formula of intonation as follows:

$$p_n, \frac{1}{n} = (\underline{t} + 12\underline{rg}) / 12 \pm \alpha_{pr}, \quad (3.8)$$

with α_{pr} defined as before. ²⁴ Naturally, one has to define the relation between the perceptual and physical domains that involves the mod-12 0 at the cognitive level, that is to say, the mod-12 value of the diatonic interval (0,0), or "perfect prime," the conveyer of "octave equivalence." In particular, one has to specify that the frequency-ratio band $\underline{fr}(12\underline{rg})$, which corresponds to the interval band $\frac{1}{n}(12\underline{rg})$ (which corresponds, in turn, to the $\underline{t}$ value 0, since $12\underline{rg}$ equals 0, plus $12\underline{rg}$), equals powers of 2 (the "octave") with $\alpha = 0$:

$$\underline{fr}(12\underline{rg}) = 2^{\underline{rg}}, \text{ with } \alpha = 0.$$

For example, $\underline{fr}_0$ (which corresponds to $\frac{1}{n}_0$) equals 2^0 , or 1, $\underline{fr}_{12}$ equals 2^1 , or 2, $\underline{fr}_{24}$ equals 2^2 , or 4, etc.. To give an example with a negative $\underline{rg}$ value, $\underline{fr}_{-12}$ equals 2^{-1} , or 0.5.

Finally, one would like to "anchor" the entire system within an absolute framework. One therefore defines $\underline{f}_0$, the frequency that corresponds to p_0 , as 261.6 Hz (the frequency of "middle C"), with $\alpha = 0$. Since $\underline{f}_0$, with $\alpha = 0$, actually equals 1 (p_0 , with $\alpha = 0$, equals 0, and $p_0 = \log \underline{f}_0$), it is clear that our unit of measurement is not the Hertz, but Hz/261.6 (that is, cycles per 1/261.6 of a second):

$$\begin{aligned}
 f_0 &= 261.6 \text{ Hz,} \\
 &= 1 \text{ Hz}/261.6, \text{ with} \\
 \alpha &= 0.
 \end{aligned}$$

All this seems more formidable than it actually is. I will therefore present a number of concrete examples illustrating how the formula works, starting with the pitch bands p_n .

If p_0 , with $\alpha = 0$, is a tempered "middle C," then p_1 (with $\alpha = 0$) is the tempered semitone above it. Naturally, p_{13} lies a tempered octave above p_1 , while p_{-11} lies a tempered octave below it (all with $\alpha = 0$, of course). That is to say, if one numbers the pitches of equal temperament consecutively, taking the pitch of "middle C" as 0, a pitch band p_n , with $\alpha = 0$, is the equally tempered pitch whose number equals n . A positive n indicates a pitch above middle C, and a negative n , a pitch below middle C.

A set of pitch bands $p_{(t+12rg)}$ of which the value of rg is the same (that is, such sets as all the pitch bands between p_0 and p_{11} , or all the pitch bands between p_{12} and p_{23} , etc.) constitute a registral span rg ($RSrg$):

$$\begin{aligned}
 RSrg &= p_{(t+12rg)}, \text{ with} \\
 &\text{ } rg \text{ fixed to some constant value and} \\
 &\text{ } t \text{ variable.}
 \end{aligned}$$

According to the particular value of rg involved, a registral span RS can be positive, negative, or 0. Clearly, $RS0$ (the set of all pitch bands between p_0 and p_{11}) corresponds, conventionally, to the "one-line octave," $RS1$ corresponds to the two-line octave, etc.; $RS-1$ corresponds to the "small" octave, $RS-2$ corresponds to the "great" octave, etc..

A set of pitch bands $p_{(\underline{t}+12rg)}$ of which the value of $\underline{t}$ is the same (that is, a set of pitch bands $p_{\underline{n}}$ of which $\underline{n}$ is equivalent under the operation mod 12, such as p_1, p_{13}, p_{-11} , etc.), represent a pitch-band class $\underline{t}$ ($PbCt$):

$$PbCt = p_{(\underline{t}+12rg)}, \text{ with} \\ \underline{t} \text{ fixed to some constant value and} \\ \underline{rg} \text{ variable.}$$

Clearly, there are twelve pitch-band classes corresponding to the twelve mod-12 values; $PbC0$ represents the pitch band around any tempered "C," $PbC1$ represents the pitch band around any tempered "C#/D♭," etc.. All pitch bands that belong to the same pitch-band class correspond to the same mod-12 value $\underline{t}$, and therefore, to the same scale step at the cognitive level; in C major, for example, all pitch bands that belong to $PbC0$ correspond to $\hat{1}$. I will return to the notion "pitch-band class," as distinct from the "pitch class" of twelve-tone theory (and of course, the cognitive "PC" of Chapter 2), later on.

Since the mod-12 value $\underline{t}$ that corresponds to $\underline{p}_n$ represents the mod-12 values of scale steps starting with (0,0), specifically, let us say, the mod-12 values of the "major mode" (0,0), (2,1), (4,2), etc., and since FbC0 (which corresponds to $\underline{t}=0$) is a "C," it is clear that we are unable at the moment to define pitch bands other than those of "C major" (recall formula 3.8). In order to overcome this problem, we will revise (3.8) slightly with respect to $\underline{p}_n$. This involves an added mod-12 value $\underline{tr}$, or "transposition operator":

$$\underline{p}_n = \left\{ (\underline{t} + \underline{tr} \bmod 12) + 12\underline{rg} \right\} / 12 \pm \alpha_{\underline{pr}}, \quad (3.8a)$$

$$\underline{t} \text{ (major)} = 0, 2, 4, 5, 7, 9, 11,$$

$$\underline{tr} = 0, 1, 2, \dots, 11.$$

For example, in C# major or D $\flat$ major $\underline{tr}$ equals 1; FbC0 therefore corresponds to $\underline{t}=11$, that is, to the mod-12 value of scale step (11,6), or $\hat{7}$, at the cognitive level, rather than scale step (0,0), or $\hat{1}$, as with C major. In D major $\underline{tr}$ equals 2, etc.. In order to define the pitch bands of minor keys one must use a different set of $\underline{t}$ values:

$$\underline{t} \text{ (minor)} = 0, 2, 3, 5, 7, 8, 10.$$

I shall define the notion "key" in terms of the "transposition operator" $\underline{tr}$ and the set of mod-12 values $\underline{t}$ which is either major or minor. Accordingly, there are twelve major and twelve minor keys, corresponding to the twelve values of $\underline{tr}$.

Let me give now some concrete examples of interval bands,

according to (3.8). First, however, let us remind ourselves that an interval represents the difference between two pitches. Clearly, if the two pitches are non-identical, the interval between them can be either positive or negative, depending upon whether one subtracts the lower pitch from the higher pitch, or vice versa. That is to say, an interval can be either positive or negative (but of the same absolute size), depending upon the direction in which it is measured. I will now establish the convention according to which a positive interval is interpreted as an ascending interval, while a negative interval is interpreted as a descending one.

If i_0 equals 0 cents (with $\alpha = 0$), then i_1 equals a tempered semitone (100 cents), in the ascending direction; i_{13} equals a tempered semitone plus a tempered octave (1300 cents), ascending, while i_{-13} is the descending counterpart of i_{13} (all with $\alpha = 0$, of course). That is to say, if one numbers the tempered intervals consecutively in semitone increments starting with 0 for 0 cents, an interval band i_n , with $\alpha = 0$, is the tempered interval whose number corresponds to n . A positive n indicates an ascending interval, while a negative n indicates a descending one.

A set of interval bands $i_{(t+12rg)}$ of which the value of t is the same (that is, a set of interval bands i_n of which n is equivalent under the operation mod 12, such as i_1, i_{13}, i_{-11} , etc.), represents an interval-band class t (IbCt):

$$\text{IbCt} = \underline{i}(\underline{t} + 12\underline{rg}), \text{ with}$$

$\underline{t}$ fixed to some constant value and
 $\underline{rg}$ variable.

Clearly, there are twelve interval-band classes corresponding to the twelve mod-12 values; IbC0 represents the interval band around any tempered "unison" (that is, 0 cents), and its "octave compound," ascending and descending (that is, an ascending octave, two ascending octaves, etc., as well as a descending octave, two descending octaves, etc.). Similarly, IbC1 represents the interval band around any tempered half step (100 cents), and its "octave compounds," ascending and descending. The "ascending octave compounds" of 100 cents ($\underline{i}_1$ with $\alpha = 0$) are such intervals as 1300 cents, 2500 cents, etc. (that is, such interval bands as $\underline{i}_{13}$, $\underline{i}_{25}$, etc.); the descending octave compounds of 100 cents are such intervals as -1100 cents and -2300 cents (that is, such interval bands as $\underline{i}_{-11}$ and $\underline{i}_{-23}$). Clearly, the absolute size of the descending interval bands within each interval-band class are the "octave complements" (or "inversions") of the ascending interval bands. This means that, for example, both an ascending interval of 700 cents ($\underline{i}_7$ with $\alpha = 0$) and a descending interval of 500 cents ($\underline{i}_{-5}$ with $\alpha = 0$) correspond, at the cognitive level, to the same diatonic interval, namely the "perfect fifth" (7,4). Anyone who has had the experience of evoking the "ascending fifth" and "descending fourth" in the same breath will find

this, I am sure, highly intuitive. For a detailed discussion of conventional interval nomenclature, see Appendix B.

Sometimes the direction of an interval is not a relevant issue, only its absolute size. In such cases it is convenient to use absolute IbCs (or, if one wishes, absolute interval bands). Absolute IbCs (abs. IbCs) correspond to abs. ICs at the cognitive level (review Chapter 2, p. 29), which means that the thirteen diatonic intervals are reduced to six absolute diatonic intervals (Chapter 2, pp. 41-2), namely $m2/M7$, $M2/m7$, $m3/M6$, $M3/m6$, $P4/P5$, and $A4/d5$. For example, $\underline{1}_7$ corresponds to absolute IbC5, which corresponds, at the cognitive level, to the absolute perfect fourth.

With formulas (3.8) and (3.8a) the perceptual/physical notions "pitch band" and "interval band" assume their definitive form; having also defined such perceptual/physical notions as pitch-band and interval-band class, we are now able to define the cognitive/perceptual structure of diatonicism in terms of a correspondence between pitch bands (or pitch-band classes) and interval bands (or interval-band classes), on the one hand, and scale steps and diatonic intervals, on the other. Such a definition can be stated as follows, assuming C major or C minor (regarding the circularity that is apparently involved, see note 25):

$$\begin{aligned} \underline{p}_n, \underline{i}_n &\longrightarrow \underline{t}, \underline{rg}, \text{ with} & (3.9) \\ \underline{n} &= \underline{t} + 12\underline{rg}, \text{ or, more simply,} \end{aligned}$$

$$\text{PbCt}, \text{IbCt} \longrightarrow \underline{t}.$$

($\underline{t}$ is the mod-12 value of scale steps and diatonic intervals, and $\underline{rg}$ is the "register operator.") I will now introduce a notational convention by which this definition can be conveniently represented.

We will notate a correspondence between a pitch band $\underline{p}_n$ and a scale step ($\underline{t}, \underline{s}$), where ($\underline{t}, \underline{s}$) is an integer pair, mod (12, 7), as $\underline{n}:(\underline{t}, \underline{s})$. For example, a correspondence, in C major, between $\underline{p}_0$ and scale step (0,0), or $\hat{1}$, is notated $0:(0,0)$. Figure 3-14 presents some further examples, along with a staff-notational "translation"; for a detailed discussion of staff notation and its relation to the cognitive/perceptual structure of diatonicism, see Appendix B.

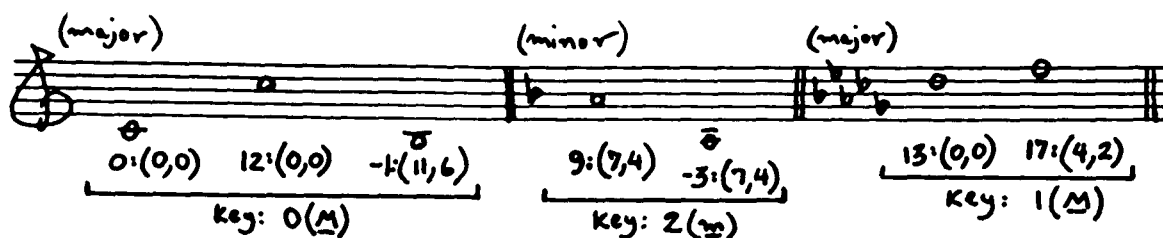


Figure 3-14. The notation $\underline{n}:(\underline{t}, \underline{s})$ for a corresponding pitch-band/scale-step, and its staff-notational translation.

Note that it is necessary to specify, in each case, the "key" according to which the correspondence is defined; that is to say, one has to specify the value of $\underline{tr}$, and whether

the $\underline{t}$ set involved is "major" or "minor" (recall formula 3.8a and its discussion). For this end another notational convention is used in Figure 3-14, namely $\underline{tr}(\underline{M})$ or $\underline{tr}(\underline{m})$. For example, $0(\underline{M})$ is a "key" that is known, conventionally, as "C major," $2(\underline{m})$ is known as "D minor," etc.. Following conventional practice, I will use the term "key," by extension, to denote also the cognitive/perceptual structure it defines, in terms of pitch-band classes and scale steps (and implicitly, IBCs and diatonic intervals). For example, the "key" $1(\underline{M})$ (conventionally, $D\flat$ or $C\sharp$ major) defines correspondences between the set of PBCs $(1, 3, 5, 6, 8, 10, 0)$, and respectively, the set of scale steps $\{(0,0), (2,1), (4,2), (5,3), (7,4), (9,5), (11,6)\}$. For further discussion of conventional "keys," see Appendix B.

We will notate a correspondence between $PbCt$ and scale step $(\underline{t}, \underline{s})$ as $\underline{t}:(\underline{t}, \underline{s})$; the "key" $1(\underline{M})$ ($D\flat$ or $C\sharp$ major) mentioned in the previous paragraph is thus notated as $\{1:(0,0), 3:(2,1), 5:(4,2), 6:(5,3), 8:(7,4), 10:(9,5), 0:(11,6)\}$.

The notation of a corresponding interval $\underline{i}_n$ and a diatonic interval $(\underline{t}, \underline{s})$ is analogous to that of a corresponding pitch band and scale step, namely $\underline{n}:(\underline{t}, \underline{s})$; Figure 3-15 presents a number of examples, along with a translation into conventional terminology. Note that the conventional "octave" and "prime" are distinguished only at the perceptual/physical level; cognitively, the two intervals are indistinguishable. See also Appendix B.

similar, and even identical, to the integer $\underline{t}$ at the cognitive level, should not blur the distinction between the perceptual/physical and cognitive domains. One must keep in mind that $\underline{n}$ and $\underline{t}$ at the perceptual/physical level are indices only. That is to say, they are merely labels of such perceptual/physical entities as pitch bands and pitch-band classes, which in themselves, have none of the properties of integers, or integers mod 12. Thus, the pitch band $\underline{p_n}$ is a band of pitches, or log-frequencies, each with a preference value $\underline{pr}$ attached, as defined by the formula (3.8a). The index $\underline{n}$ is only a convenient means of identifying the pitch band $\underline{p_n}$ with the values $\underline{t}$ and $\underline{rg}$, to which it corresponds (since $\underline{n} = \underline{t} + 12\underline{rg}$); it is not to be confused with what $\underline{p_n}$ is.

It is particularly important to keep in mind that pitch-band classes (as well as interval-band classes) have none of the properties of integers mod 12; the mod-12 value $\underline{t}$ by which pitch-band classes are notationally represented is, again, an index only. Thus, a pitch-band class (or interval-band class), as a perceptual/physical entity, does not involve the notion of octave equivalence; the latter belongs to the cognitive level only. Pitch bands are members of classes by virtue of their locus along the pitch continuum, that is, by virtue of their being "octaves" apart, without implying that "octaves" are "equivalent."

Since there is obviously room for confusion here, let me clarify the difference between the cognitive "PC" of Chapter

2, the perceptual/physical "pitch-band class," and the "pitch class" of 12-tone theory.

The cognitive "PC" of Chapter 2 is a terminological substitute for a purely mathematical notion, namely "mod-12 integer class"; it is devoid of any perceptual/physical content. The notion "pitch-band class," on the other hand, is a purely perceptual/physical notion, and is devoid of any of the cognitive attributes of an integer mod 12; in particular, being a member of the same class of pitch bands does not imply equivalence. Finally, the "pitch class" of 12-tone theory is a hybrid notion. It combines the assumption of 12-tone equal temperament with the assumption of octave equivalence. A "pitch class" in 12-tone theory is of course a much more narrowly defined perceptual/physical entity than the pitch-band class employed here, which²⁵ does not involve the assumption of strict equal temperament.

And one final note regarding terminology. From now on I will use such notions as "diatonicism," "diatonic system," "scale step," "diatonic interval," etc., in both the purely cognitive sense (as in Chapter 2) and the combined cognitive/perceptual sense, that is, with the perceptual level of PbCs and IbCs understood. Unless the ambiguity is harmless, I will make sure that the intended meaning is clear.

NOTES

1. Terhardt 1982, pp. 356-7.
2. For the purposes of this study I shall ignore the phenomenon of "combination tones"; that is, I shall assume that the "fundamental frequency" is always present. I am also ignoring the marginal effect on pitch of such factors as loudness. Since two frequencies which lie close enough together can be experienced as the same pitch, I am assuming that any two frequencies f_n and f_m are separated by more than the discriminatory threshold of our ear at that range. Regarding the relation of pitch to frequency above 5000 Hz, see Attneave and Olson 1971.
3. Logarithmic pitch perception can be seen as a special case of "Fechner's Law," formulated by and named after the 19th-century psychophysicist Gustav Fechner, often considered the founder of psychophysics. Fechner sought to determine the magnitude of sensation quantitatively by way of measuring physical stimuli and responses. He proposed that

$$\underline{S} = \underline{K} \cdot \log \underline{I},$$
 "where $\underline{S}$ is the magnitude of sensation, $\underline{I}$ is a dimension of the stimulus, and $\underline{K}$ is a constant of proportionality that varies with sense modality" (Hirsh 1952, pp. 10-11). For further discussion of Fechner's work and its background, see Boring 1942, pp. 34-45.
4. "The subject of just intonation is fatally fascinating to people whose mathematical insight has not attained to the notion of approximation. In art, as in mathematics, accuracy lies in estimating the relevant degree of approximation rather than in unrolling interminable decimals." Donald Tovey, "Harmony" in Encyclopedia Britannica, 11th ed., cited in Lloyd 1938, p. 443.
5. Since I am excluding negative f_n values in this section, the f_0 band is only half as large as the other bands, and extends from 0 to 50 cents, rather than -50 to 50 cents, as (3.3a) requires. Regarding negative intervals, see section 3.7.
6. The f_0 band thus includes negative values, which are otherwise excluded in this section; see note 5.

7. Lindley presents the deviations of "wolf fifths" in relation to the pure fifth $3/2$ (approximately 702 cents); I have therefore added 2 cents to his figures.
8. Out of 146 minor seconds analyzed, Shackford gives only two instances of minor seconds smaller than 50 cents (44 and 47 cents).
9. See also Winckel 1967, Figure 79 on p. 124 and its accompanying discussion.
10. Studdert-Kennedy et al., 1970.
11. From note 1, p. 407.
12. "It is tentatively concluded that categorical perception in the case of musical intervals is related to the degree of stimulus uncertainty associated with the procedures used, although obviously considerably more research is necessary in this regard" (Burns and Ward 1978, p. 466). Related to categorical perception of intervals is a study by Tanner (1972), cited by Scott Makeig (1982, pp. 238-9), according to which minor sixths were judged best "in tune" between 792 and 814 cents. Tanner interpreted these data in terms of a "commatic zone" between the Pythagorean minor sixth $128/81$ (approximately 792 cents) and the just minor sixth $8/5$ (approximately 814 cents). I believe the same data can be much more reasonably accounted for in terms of a "preference zone" around 800 cents, as has been explained in section 3.2.
13. Given logarithmic pitch perception, we are primarily concerned with the prime numbers within the series 1:2:3:4:5:6:7 etc., that is, with the series 1, 2, 3, 5, 7, 11, etc.; a frequency ratio such as $8/7$ can be expressed as $2 \cdot 2 \cdot (2/7)$, that is, as $2/7$ plus two "octaves." It is important to keep this in mind since the hierarchical aspect of pure-ratio preferences discussed in the following paragraph involves the series of primes, not the series of consecutive integers.
14. Since for us, q is smaller than $-q \bmod x$ (see Appendix A), the fraction that corresponds to $7/12$ is $(x-q)/x$ (where $(x-q)$ equals, of course, $-q \bmod x$), rather than q/x . Clearly, q/x is the "fourth" in equal temperament, and must embody a relation to $\log_2 4/3$ analogous to that of $(x-q)/x$ in relation to $\log_2 3/2$.
15. This means that the theory of intonation must be formulated in general terms, that is, without assuming that $x=12$. But there is no difficulty to do so. In place of (3.6) one would have (using base-2 logarithms of frequencies, rather than cents):

$$\underline{i}_n = n/x \pm \alpha \underline{p}_x,$$

with α and $\underline{p}_x$ defined as follows:

$$\begin{aligned} 0 &\leq \alpha < 1/x, \\ \underline{p}_x &= 1/(2x) - \alpha, \end{aligned}$$

from which it would follow that

$$\underline{i}_{(x-q)} = (x-q)/x$$

with $\alpha = 0$. But according to the principle of pure intonation,

$$\underline{i}_{(x-q)} = \log_2 3/2$$

is maximally preferred. Therefore, in order to avoid conflict, $(x-q)/x$ must be as close as possible to $\log_2 3/2$, exactly as (3.7) and (3.7a) require. As the reader will recall, there is an interdependence between the cognitive and perceptual/physical levels of diatonicism with respect to the number 12 (see Chapter 1, section 1.4, and Chapter 2, section 2.6). Yet there is no circularity since, as I have just shown, one can arrive at the number 12 without assuming it beforehand.

16. It is a striking coincidence that $4/7$ offers the best rational approximation of $\log_2 3/2$ using fractions whose denominator is smaller than 12. Thus, the mod-7 system may have also been suggested by a process analogous to that which is illustrated in Figure 3-11.

17. Critics of the overtone theory have not emphasized enough its fundamental logical inadequacy. Milton Babbitt, for example (1961), notes certain logical problems (for example, the need for an independent assumption of octave equivalence), but reserves his most sarcastic remarks for the far less important numerological fallacy. But surely, had the conclusions of the overtone theory really followed from its premises, there would have been nothing wrong in invoking the numbers 5 or 6 as an independent assumption (after all, twelve-tone theory does so with respect to the number 12); one would have simply disposed of the unnecessary numerological "justification." Other criticisms that I have seen (e.g. Serafine 1983, pp. 163-5, Lerdahl and Jackendoff 1983, pp. 290-3), seem to fall back on certain standard arguments, for example, that the theory fails to account for the minor triad, and either miss its logical inadequacy altogether, or place it on a more or less equal par with its other "weaknesses."

18. I believe that a theory of rhythm must start out in defining such notions as "half note," "whole note," "quarter note," etc. in exactly the same abstract sense, that is, as numerical relations devoid of any durational content. The theory would then proceed to define a correspondence between the "cognitive" domain of "rhythmic relationships" and the durational domain, much in the same way that the theory of intonation here defines a correspondence between the mod-(12,7) model and the perceptual/physical domain of pitches/frequencies and intervals/frequency-ratios.

19. See Reese 1940, Part 3, "Polyphony Based on the Perfect Consonances and its Displacement by Polyphony Based on the Third." See especially pp. 249-51, 294-5, 303, 333-7, 370, and 398-405.

20. Johannes de Garlandia termed the major and minor thirds "imperfect consonances," but termed the sixths dissonant, the major sixth an "imperfect dissonance," and the minor sixth, an "intermediate" one. Anonymous IV has noted that in the west of England thirds were thought to be the best consonances.

21. Cited in Seashore 1938, pp. 218-24. Greene's data (as cited by Seashore, at any rate), do not include the tritone; according to Greene, the minor thirds and perfect fourths are also somewhat contracted relative to equal temperament, and the major second, somewhat expanded.

22. Ibid., p. 224.

23. Cited in Seashore 1938, p. 212. Victor Zuckerkandl (1956, pp. 79-81) cites a study of intonation by Otto Abraham, according to which singers "sang unimaginably off pitch," for example, a minor third of 238 cents, and a perfect fourth of 586 cents. It is interesting that Zuckerkandl notes that "the great discrepancies always appeared where a rise or fall of the melody was clearly marked, and that, in the majority of cases, the direction of the discrepancy followed the upward or downward direction of the melody."

24. One defines α_{pr} here in terms of base-2 logarithms:

$$0 \leq \alpha < 1/12, \\ pr = 1/24 - \alpha;$$

see also note 15.

25. The claim that such notions as "pitch band" and "pitch-band class" are purely perceptual/physical notions may seem fallacious to some readers in view of the fact that it is impossible to define them not knowing their relation to integers,

or integers mod 12 (see formulas 3.8 and 3.8a). While this is of course true, it only indicates another interdependence between levels; that no circularity is involved should be apparent from the fact that we have arrived at the notions "pitch band" and "pitch-band class" without assuming them beforehand (review, in this connection, the discussion of interdependent levels in Chapter 1, section 1.4).

All the same, the situation here, methodologically, is somewhat complicated, since an interdependence between the cognitive and perceptual/physical levels already exists in connection with the number 12 (section 3.4). To dispel any doubts regarding circularity, I will note that there are two levels of interdependence involved here, one embedded within the other. At the higher level of interdependence, notions such as "pitch band" and "pitch-band class" at the perceptual level are assumed; at the embedded level of interdependence, the number 12 at the cognitive level is assumed. As soon as one can show that there is no need to assume the number 12 at the embedded level of interdependence (see note 15), one can show the same regarding "pitch band," "pitch-band class," etc., at the higher level (formulas 3.8 and 3.8a).

Interlude

The Processing of Diatonic Music:

A Study in Diatonic "Performance"

"Competence" and "performance." So far we have concentrated on the task of characterizing diatonicism as a mental construct. We have attempted to describe, in other words, what a mind that can be said to "know" the diatonic system must know. We have seen, for example, that the mind must "know" logarithmic pitch perception and the principle of pure intonation, that it must know the mod-(12,7) model, that it must know the correspondence between pitch-band classes at the perceptual/physical level and particular mod-12 values at the cognitive level, etc.. Yet knowing the diatonic system is clearly not enough. For in listening to tonal music, the mind must construct the diatonic system under specific and highly constrained circumstances, in particular, the mind must construct the diatonic system on the basis of whatever physical data are available. A theory of diatonicism, therefore, must deal not only with what diatonicism is, but with how diatonic music is processed.

This distinction, between the what and the how, can be

seen to underly Chomsky's distinction between "competence" and "performance": "To the extent that we have an explicit theory of competence, we can attempt to devise performance models to show how this knowledge is put to use" (Chomsky 1980, p. 225). The distinction is an essential one for any theory of mental representation. David Marr, who calls the theory of competence "computational theory," is very incisive on this point (1982, pp. 28-9):

Failure to recognize this theoretical distinction between what and how also greatly hampered communication between the fields of artificial intelligence and linguistics. Chomsky's (1965) theory of transformational grammar is a true computational theory in the sense defined earlier. It is concerned solely with specifying what the syntactic decomposition of an English sentence should be, and not at all with how that decomposition should be achieved. Chomsky himself was very clear about this--it is roughly his distinction between competence and performance, though his idea of performance did include other factors, like stopping in midutterance--but the fact that his theory was defined by transformations, which look like computations, seems to have confused many people. Winograd (1972), for example, felt able to criticize Chomsky's theory on the grounds that it cannot be inverted and so cannot be made to run on a computer; I have heard reflections of the same argument made by Chomsky's colleagues in linguistics as they turn their attention to how grammatical structure might actually be computed from a real English sentence.

The explanation is simply that finding algorithms by which Chomsky's theory may be implemented is a completely different endeavor from formulating the theory itself. In our terms, it is a study at a different level, and both tasks have to be done. . . .

And in our terms, having presented a theory of diatonic "competence" in Chapters 2 and 3, we must turn our attention to diatonic "performance"; we must account, in other words, for the process by which the physical level of frequencies is mapped

onto the cognitive/perceptual structure of diatonicism, as follows:

physical level of frequencies $\longrightarrow$ cognitive/perceptual structure of diatonicism.

Two things must be borne here in mind. First, performance can be seriously considered only after competence has been well understood; the what necessarily precedes the how. Correspondingly, the mind whose performance has been schematically depicted above bases its performance on its knowledge of the diatonic system; clearly, a mind which is not already familiar with the diatonic system will have no way of processing frequencies (and frequency ratios) diatonically (as to how the mind comes to "know" the diatonic system in the first place-- this should be the subject matter of a theory of diatonic ¹learning). At the same time, however (and there is no contradiction in this), the fact the competence precedes performance does not mean that a theory of competence does not take performance into account. As the reader will recall, in the theory of intonation developed in the previous chapter (see especially section 3.2) I have referred time and again to the question of performance; for example, I have noted that the perceptual/physical level corresponds to the mod-12 values (rather than the mod-7 values) at the cognitive level because only in this way can the mind process intervals as well as pitches diatonically. This interdependence between the theory of competence

and the theory of performance should come as no surprise. For clearly, one of the most important constraints on the structure of diatonicism is that it be derivable from a physical level of frequencies and frequency ratios. Clearly, one can devise, artificially, a structure incomparably more complex than that of the diatonic system; but whether this structure will be as readily processable as that of the diatonic system is a different matter. Indeed, the greatness of the diatonic system surely lies, among other things, in the exquisite balance inherent in its structure between complexity (the structure of diatonicism, we have seen, is far from trivial) and processability; Richmond Browne may have had this balance in mind when he commented that one ". . . of the facets of tonal music which seem to need explanation and discussion" is "its complexity under investigation vs. the apparent ease of our real-time processing of it."² As we shall see, our theory of diatonic competence has been performance-motivated throughout, so that diatonic performance is easily accounted for on the basis of diatonic competence. For a discussion of interdependent levels vs. circularity, see Chapter 1, section 1.4.

The distinction between competence and performance in the sense I have been discussing can be profitably applied to music-theoretic domains other than that with which we are presently concerned. Consider for example the domain of rhythm, specifically, let us say, meter. A theory of meter gains considerable coherence once a distinction between metric competence

and metric performance is drawn. A theory of metric competence must define the notion "metric structure," that is to say, must specify what meter is; this, I believe, is a relatively simple task, meter being a relatively simple structure. Quite a different matter is the theory of metric performance. Here one must specify how a metric structure is derived from the complicated and often conflicting patterns of attacks and durations with which the mind is actually presented; as studies of meter have shown (see for example Schachter, forthcoming), this aspect of meter, namely metric performance, is far from simple. Note that while a theory of metric competence is formulated in purely rhythmic terms, a theory of metric performances must engage practically every conceivable music-theoretic domain, since the patterns of attacks and durations mentioned before can derive from practically every conceivable aspect of the composition, including "harmony," "motivic structure," "timbre," etc..

Consider also Schenker's theory of tonal structure. I believe a convincing case can be made in support of the claim that Schenker's theory is, by and large, a theory of competence. Schenker is concerned, for the most part, with characterizing what, in his belief, are the structures by which tonal music is conceived. Schenker does not tell us very much about how these structures are derived from actual tonal "surfaces"; this he usually leaves the intelligent reader to figure out for himself from his numerous examples. Yet this second aspect

of Schenkerian theory, namely "performance," is equally important, and, as anyone with a first-hand experience in Schenkerian analysis can testify, quite problematic. Note that the "competence" part of Schenker's theory is purely tonal (cf. the theory of meter); the "performance" part of the theory, on the other hand, must deal with all kinds of music-theoretic matters, including rhythm, dynamics, motivic structure, timbre, etc., since all of these are relevant to how tonal structure is derived. This settles, I believe, the controversy between those authors who have criticized Schenker for ignoring rhythm, and those who have countered that Schenker is in fact very much concerned with rhythm (see, in this connection, Keiler 1978). The "competence" part of Schenkerian theory indeed "ignores" rhythm, but only because its subject matter is something else; the "performance" part of Schenkerian theory, on the other hand, is very much concerned with rhythmic matters, and with other "non-tonal" matters besides.³

Having defined the notions "competence" and "performance," let me attempt now to account for diatonic performance. I propose a two-stage theory of diatonic performance (Figure In-1). At the first stage, frequencies and frequency ratios at the physical level are mapped onto pitch-band classes and interval-band classes at the perceptual level; at the second stage, PbCs and IbCs at the perceptual level are mapped onto scale steps and diatonic intervals at the cognitive level.

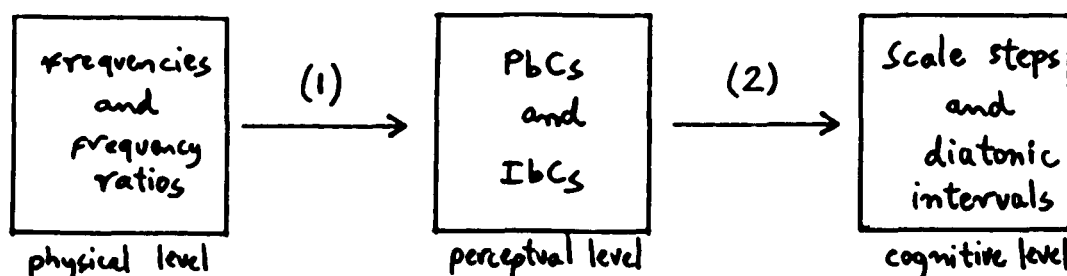


Figure In-1. The two stages of diatonic performance.

From frequencies and frequency ratios to PbCs and IbCs.

We have seen that logarithmic pitch perception defines a one-to-one correspondence between frequency and frequency ratios at the physical level and, respectively, pitches and intervals at the perceptual level. Now, unless a pitch or interval lies exactly at the mid-points of the tempered scale (e.g. 250 cents, 350 cents, etc.), in which case it has the same preference value, namely 0, with respect to two adjacent pitch or interval bands (see Figure 3-5), its preferred pitch-band or interval-band membership (and consequently, its PbC or IbC membership), in principle, is unambiguous. For it has a different pr value with respect to each pitch or interval band, where only one preference value is positive.

In reality, however, the mind cannot rely on small differences between pr values. To the extent that the given pitch or interval occupies the "preference zone" and even the "neutral

zone" (see Figure 3-5), the difference between its positive and negative preference values is large, and there is no difficulty to assign the pitch or interval to only one band. In case, however, the given pitch or interval occupies the "non-preference zone," the difference between its positive and negative preference values is small, and the mind has to consider two alternative possibilities, allowing, ultimately, "contextual" considerations to decide the case. More often than not, an ambiguity of this sort can be settled at the second stage of diatonic processing, without appealing to higher levels of tonal structures.

From PbCs and IbCs to scale steps and diatonic intervals. The next and final stage of diatonic performance, the mapping of PbCs and IbCs at the perceptual level onto scale steps and diatonic intervals, or mod-(12,7) integer pairs, at the cognitive level, is equally straightforward. The twelve PbCs and IbCs correspond, by definition, to twelve mod-12 values; and we have seen that each of the mod-12 values of scale steps and diatonic intervals (except, in the latter case, the value 6) corresponds to only one mod-7 value (the property of mod-(12,7) correspondence). Thus, the mind can pair any given PbC or IbC except IbC6 with only one scale step or diatonic interval.

Note that it is possible to process PbCs as scale steps first, and then derive the diatonic intervals by subtracting scale steps, or alternatively, to process IbCs as diatonic

intervals first, and then derive the scale steps as a set of mod-(12,7) integer pairs which yield the diatonic intervals through subtraction. In the latter case, the mind considers in the case of IbC6 two alternative possibilities, namely A_4 and d5. However, since this is the only such case, and since, furthermore, IbC6 occurs the least number of times within the system of diatonic IbCs, namely twice (where one IbC6 is the "octave complement" of the other), the mind has no difficulty in determining, ultimately, which IbC6 is the A_4 , and which is the d5. It can be shown, in fact, that the mind requires only one pitch-band class in addition to the two which yield IbC6 to decide between the two cognitive alternatives.⁴ I shall refer to diatonic performance that is based on processing the IbCs (rather than the PbCs) as the "strategy of diatonic IbC processing" (Figure In-2); as will be noted later on, the strategy of diatonic IbC processing must ultimately bear the burden of diatonic performance.

IbCs:	diatonic intervals:
0	→ P1
1	→ m2
2	→ M2
3	→ m3
4	→ M3
5	→ P4
6	→ A4; d5
7	→ P5
8	→ m6
9	→ M6
10	→ m7
11	→ M7

Figure In-2. The strategy of diatonic IbC processing: the mapping of IbCs at the perceptual level onto diatonic intervals at the cognitive level.

Note the important role of mod-(12,7) coherence in connection with diatonic IbC processing. As the reader will recall, mod-(12,7) coherence guarantees that given the thirteen diatonic intervals, larger mod-12 values correspond, generally, to larger mod-7 values; at any rate, a larger mod-12 value never corresponds to a smaller mod-7 value. This fact is undoubtedly of

enormous help in pairing mod-7 to mod-12 values while processing IbCs diatonically.

Mod-7 processing of PbCs. Mod-(12,7) coherence allows for a powerful strategy in processing PbCs diatonically. As the reader will recall, given the seven integer pairs that define the seven scale steps, mod-(12,7) coherence guarantees that larger mod-12 values correspond to larger mod-7 values, and vice versa. Since there is only one way to establish a correspondence between the seven "diatonic" mod-12 values and the seven mod-7 values that fulfills this requirement, and since the diatonic PbCs correspond to the mod-12 values,⁵ the mind can derive the mod-7 values of scale steps directly from the perceptual level of PbCs. That is to say, there is no need to process PbCs in terms of mod 12 first, in order to determine their corresponding mod-7 values; the mind simply knows that "higher" PbCs correspond to larger mod-7 values. I shall term this kind of PbC processing based on mod-(12,7) coherence "mod-7 processing of PbCs." Mod-7 PbC processing accounts, I believe, for the often noted impression that diatonic scale steps are somehow evenly spaced within the octave, in contradiction to the perceptual/physical reality of "half steps" and "whole steps" (see the comment by Hermann Pfrogner, cited in Chapter 2, p. 26). It also accounts for the conventional representation of scale steps through their mod-7 values only, noted in Chapter 2, where it was assumed that only notational convenience was involved (for further discussion of traditional no-

tational systems, staff notation in particular, see Appendix B).

Diatonic performance and interdependent levels. As the reader will recall, we have chosen to define the notion "diatonic" interdependently with the notion "tonal"; accordingly, we have assumed, tentatively, that scale step (0,0) corresponds to the "tonic" of the major or natural minor modes, with an understanding that this assumed "tonic" will ultimately correspond to a defined tonic at a different level of description (see Chapter 6). What are the implications of interdependencies of this sort for performance?

Tentative assumptions at any given level of the theory of competence correspond to tentative assumptions in performance. Thus, diatonic performance involves tentatively assuming that (0,0) corresponds to the tonic of the major or natural minor modes; clearly, without such a tentative assumption, the mind must consider seven sets of scale steps as corresponding to the set of diatonic PbCs (see Figure 2-10 in Chapter 2). But just as a theory of competence ultimately replaces tentative assumptions by actual definitions, a theory of performance must ultimately replace tentative performance by real performance; that is to say, the mind must have a way to verify the assumed "tonic" at a later stage of tonal processing. As we shall see in Chapter 6, where the notion "tonic" is defined, tonal performance is based on the strategy of diatonic IbC processing. Strategies

that are based on processing PbCs as scale steps, whether mod-12 or mod-7 processing is involved, must ultimately yield to the strategy of diatonic IbC processing (Figure In-2); the latter, in other words, must ultimately bear the burden of diatonic performance.

Summary. We have seen that competence and performance are two separate, yet interdependent theories. We have seen, in particular, that our theory of diatonic competence was fully performance-motivated, so that there was little difficulty to account for diatonic performance on the basis of diatonic competence. This, we have noted, is not surprising, since the ability to process the physical level diatonically is one of the most important constraints on the structure of diatonicism.

NOTES

1. A theory of diatonic learning would have to take "cultural exposure" into account. It should be clear, however, that while cultural exposure is a necessary condition for acquiring diatonic competence, it is not a sufficient one as well; the innate structure of the human mind is another necessary condition.
2. Browne 1981, p. 3.
3. Lerdahl and Jackendoff (1985, pp. 148-9) state that Schenker's theory, insofar as it "... is formal rather than just a repository of analytic procedures, ... proceeds from underlying axioms (the Klang and Ursatz) through various elaborational rules to the musical surface," unlike their own theory, "... that models the listener's intuition by predicting analyses from musical surfaces," and then conclude that Schenker's "... primary purpose is correspondingly not psychological but aesthetic." Later on they pose a number of questions which, they believe, would have to be answered if Schenker's theory "sought to achieve something like the predicative power to which GTTM /Lerdahl and Jackendoff 1983, e.a./ aspires." It is clear from these questions, as from their earlier comments, that Lerdahl and Jackendoff simply miss the distinction between a theory of competence and a theory of performance, in the sense I have been discussing.
4. For example, given the pitch bands 2, 5, and 11 (the pitch bands of "D," "F," and "B" in the one-line octave), the mind processes the interval band between the pitch bands 2 and 5 as a "minor third," and the interval band between the pitch bands 2 and 11 as a "major sixth"; this specifies the interval band 6 between the pitch bands 5 and 11 as an "augmented fourth" (rather than a diminished fifth). See also Browne 1981, p. 7.
5. In keys other than C major or C minor one has to take into account the "transposition operator" tr in connection with the correspondence between pitch-band classes and the "diatonic" mod-12 values; see formula (3.8a) and its discussion in Chapter 3.

Chapter IV

CHROMATICISM AND ENHARMONICISM

4.1 Preliminary Discussion

The discussion of chromaticism and enharmonicism in this chapter is divided into two parts, paralleling the discussion of diatonicism in the preceding chapters. Section 4.2, "A Cognitive Theory of Chromaticism and Enharmonicism," corresponds to Chapter 2; section 4.3, "The Cognitive/Perceptual Structure of Chromaticism and Enharmonicism," corresponds to Chapter 3.

It is important to keep in mind that sections 4.2 and 4.3 combined amount to a theory of chromatic and enharmonic competence; a theory of chromatic and enharmonic performance is presented in Chapter 5. Unlike the theory of diatonicism proper, there is some discrepancy here between competence (that is, the structure of chromaticism and enharmonicism) and performance (that is, the terms under which this structure manifests itself in processing chromatic music); unlike the theory of diatonicism proper, in other words, the theory of chromatic and

enharmonic competence is not fully performance-motivated. This may sound strange thus stated, but is in fact consistent with musical reality. Consider the notion of "chromatic alteration." Surely, "chromatic alteration" is not constrained to a single sharpening or flattening, that is to say, it includes, theoretically, the possibility for double alterations, triple alterations, etc.. However, in practice one hardly encounters in any given key scale-step alterations that exceed a single sharp or flat, and these, too, are not applied indiscriminately; one would not encounter an F-flat, for example, in a C-major context. As we shall see, this discrepancy between "theory" and "practice" reflects the discrepancy between chromatic and enharmonic "competence" and "performance," and is captured in this chapter and the subsequent one, respectively. Let me also add that the issue of "context" is not raised in this chapter. Contextual considerations belong to the theory of chromatic and enharmonic performance, and therefore enter the discussion only in Chapter 5.

4.2 A Cognitive Theory of Chromaticism and Enharmonicism

The two aspects of the cognitive structure of chromaticism and enharmonicism. Diatonicism, we have seen, is based on the assumption mod (12,7), or "octave equivalence." In particular, we have seen, thirteen mod-(12,7) integer pairs

define thirteen diatonic intervals, while seven out of the same thirteen pairs define seven scale steps. Yet the assumption $\text{mod } (12,7)$ delimits more than just thirteen different ordered integer pairs; the number, in fact, is exactly twelve times seven, or eighty-four (Figure 4-1). I propose that the cognitive domain of chromaticism and enharmonicism is a structure by which the diatonic system, that is, the system of seven scale steps and thirteen diatonic intervals as described in Chapter 2, is projected onto the $\text{mod}-(12,7)$ system "at large."

mod-12 values

11	11,0	11,1	11,2	11,3	11,4	11,5	11,6
10	10,0	10,1	10,2	10,3	10,4	10,5	10,6
9	9,0	9,1	9,2	9,3	9,4	9,5	9,6
8	8,0	8,1	8,2	8,3	8,4	8,5	8,6
7	7,0	7,1	7,2	7,3	7,4	7,5	7,6
6	6,0	6,1	6,2	6,3	6,4	6,5	6,6
5	5,0	5,1	5,2	5,3	5,4	5,5	5,6
4	4,0	4,1	4,2	4,3	4,4	4,5	4,6
3	3,0	3,1	3,2	3,3	3,4	3,5	3,6
2	2,0	2,1	2,2	2,3	2,4	2,5	2,6
1	1,0	1,1	1,2	1,3	1,4	1,5	1,6
0	0,0	0,1	0,2	0,3	0,4	0,5	0,6
	0	1	2	3	4	5	6

mod-7 values

Figure 4-1. The mod-(12,7) system at large.

Given that the mod-(12,7) system at large is a finite system of integer pairs, I am proposing a finite domain of "chromatically altered" scale steps and diatonic intervals. Conventionally, it is often assumed that chromatic alteration is limitless, that is to say, that there is no limit, "theore-

tically" of course, to the number of "sharps" and "flats" by which, say, a scale step can be raised or lowered (see, for example, Pfrogner 1953, pp. 40-1). Yet lacking an explicit model one cannot but assume an infinite domain of altered scale steps and diatonic intervals, since any finite number specified would seem arbitrary. I am proposing here an explicit model of chromatic and enharmonic competence; I believe the reader will find this model not only logical, but ultimately highly intuitive as well.

The mod-(12,7) system at large and the system by which the seven scale steps and thirteen diatonic intervals are projected onto it are the two aspects of the cognitive structure of chromaticism and enharmonicism. The latter system is expressed in terms of the notion "chromatic alteration," to which I will now turn my attention. Since I start out with chromatically altered scale steps, I have highlighted in Figure 4-1 the seven integer pairs that define the seven scale steps of the major mode, namely (0,0), (2,1), (4,2), (5,3), (7,4), (9,5), and (11,6); of course I could have used the natural-minor mode just as well.

Chromatically altered scale steps. Figure 4-1 displays seven diatonic scale steps (the highlighted pairs) and seventy-seven (eighty-four minus seven) non-diatonic scale steps. A non-diatonic scale step (integer pair) is interpreted as chromatically altered relative to a diatonic scale step (integer pair) with which it shares the same mod-7 value. In the case

of scale steps, there is only one such diatonic integer pair for any given non-diatonic pair. Thus, for example, the non-diatonic pair (6,3) (intersection of seventh row with fourth column), is interpreted as chromatically altered relative to the diatonic pair (5,3), or $\hat{4}$, easily located along the same column as a highlighted pair.

What do I mean by "chromatically altered" relative to some diatonic pair? If we compare the non-diatonic pair (6,3) with the diatonic pair (5,3) (using the example from the previous paragraph) we note that the mod-12 value of the former is larger by 1. Thus, (6,3) is chromatically altered relative to (5,3) in the sense that its mod-12 value is larger by 1. We will notate (6,3), therefore, as $(\hat{5}^1, 3)$, that is, as the diatonic scale step (5,3), or $\hat{4}$, whose mod-12 value 5 (the "base") has been chromatically altered by the "alteration operator" +1. Clearly, we have captured the conventional system of scale-step alteration that is notated in terms of "sharps" and "flats"; $(\hat{5}^1, 3)$ corresponds, conventionally, to $\sharp\hat{4}$ (F-sharp in C major). By the same token, an alteration operator of +2 denotes double sharpening, +3 denotes triple sharpening, etc.; a negative operator denotes flatting. In Figure 4-2 the mod-(12,7) system as priorly displayed in Figure 4-1 is interpreted as unaltered and altered scale steps. For the reader's convenience, conventional notation ($\sharp\hat{4}$, $\flat\hat{6}$, etc.) is used in addition to integer notation. For a chromatically altered scale step raised by more than a double sharp (x) or lowered by more than a double flat ($\flat\flat$) I notate

"#3" (triple sharp), "b4" (quadruple flat), etc..

11	0 ¹ ,0 b↑	2 ³ ,1 b ⁵ 2	4 ⁵ ,2 b ⁵ 3	5 ³ ,3 #b ⁴ 4	7 ¹ ,4 #4 ⁵	9 ² ,5 x6	11,6 7
10	0 ² ,0 bb↑	2 ⁴ ,1 b ⁴ 2	4 ³ ,2 #b ⁶ 3	5 ¹ ,3 #5 ⁴	7 ³ ,4 #3 ⁵	9 ⁴ ,5 #6	11,6 b7
9	0 ³ ,0 b ³ ↑	2 ⁵ ,1 b ⁵ 2	4 ¹ ,2 #5 ³	5 ⁴ ,3 #4 ⁴	7 ² ,4 x5	9 ⁵ ,5 6	11 ² ,6 bb7
8	0 ⁴ ,0 b ⁴ ↑	2 ⁶ ,1 #b ⁶ 2	4 ⁴ ,2 #4 ³	5 ² ,3 #3 ⁴	7 ⁴ ,4 #5	9 ¹ ,5 b6	11 ³ ,6 b ³ 7
7	0 ⁵ ,0 b ⁵ ↑	2 ¹ ,1 #5 ²	4 ⁶ ,2 #3 ³	5 ⁵ ,3 x4	7 ⁵ ,4 5	9 ³ ,5 bb6	11 ⁴ ,6 b ⁴ 7
6	0 ⁶ ,0 #b ⁶ ↑	2 ¹ ,1 #4 ²	4 ² ,2 x3	5 ¹ ,3 #4	7 ¹ ,4 b5	9 ³ ,5 b ³ 6	11 ⁵ ,6 b ⁵ 7
5	0 ¹ ,0 # ¹ ↑	2 ³ ,1 #5 ²	4 ¹ ,2 #3	5 ³ ,3 4	7 ² ,4 bb5	9 ⁴ ,5 b ⁴ 6	11 ⁶ ,6 #b ⁶ 7
4	0 ² ,0 # ² ↑	2 ⁴ ,1 x2	4 ² ,2 3	5 ² ,3 b4	7 ³ ,4 b ³ 5	9 ⁵ ,5 b ⁵ 6	11 ¹ ,6 #57
3	0 ³ ,0 # ³ ↑	2 ⁵ ,1 #2	4 ³ ,2 b3	5 ³ ,3 bb4	7 ⁴ ,4 b ⁴ 5	9 ¹ ,5 #b ⁶ 6	11 ² ,6 #47
2	0 ⁴ ,0 x↑	2 ¹ ,1 2	4 ⁴ ,2 bb3	5 ⁴ ,3 b ⁴ 4	7 ⁵ ,4 b ⁵ 5	9 ² ,5 #56	11 ³ ,6 #37
1	0 ⁵ ,0 #↑	2 ¹ ,1 b2	4 ⁵ ,2 b ⁵ 3	5 ⁵ ,3 b ⁵ 4	7 ¹ ,4 #b ⁶ 5	9 ³ ,5 #46	11 ⁴ ,6 x7
0	0,0 ↑	2 ² ,1 bb2	4 ¹ ,2 b ⁴ 3	5 ² ,3 b ⁵ 4	7 ² ,4 #5	9 ³ ,5 #36	11 ⁴ ,6 #7
	0	1	2	3	4	5	6

Figure 4-2 (p. 157). The mod-(12,7) system at large interpreted as altered and unaltered scale steps.

It is not difficult to see that alteration-operator numerals correspond to values in a mod-12 system. Thus, a given scale step (which can be thought of as "altered" by the operator 0) can be raised or lowered only eleven times; chromatically altering a given scale step twelve times is like not altering it at all (twelve equals zero, mod 12). Moreover, the systems of scale-step raising (positive alteration operators) and scale-step lowering (negative alteration operators) are congruent, since modulo 12, 1 equals -11, 2 equals -10, etc.. Thus, to lower a scale step eleven times is equivalent to raising it once, and so on. We will use a system of thirteen negative and positive alteration operators ranging from -6 (six flats) to 6 (six sharps); their correspondence to the twelve mod-12 values is shown in Figure 4-3. Note that two alteration operators, namely -6 (six flats) and 6 (six sharps) correspond to the same mod-12 value, namely 6, and are therefore interchangeable. (The reader is asked to keep in mind that I am describing the cognitive structure of chromaticism and enharmonicism at this stage only. The notions "sharp" and "flat" here do not carry any perceptual/physical content, as opposed to the conventional "sharp" and "flat." The relation between the cognitive and perceptual/physical levels of chromaticism and enharmonicism is discussed in the subsequent section; see

also Appendix B.)

alteration operators:	mod-12 values:
-1 (b) —————→	11
-2 (bb) —————→	10
-3 (b3) —————→	9
-4 (b4) —————→	8
-5 (b5) —————→	7
±6 (b/b) —————→	6
5 (#5) —————→	5
4 (#4) —————→	4
3 (#3) —————→	3
2 (x) —————→	2
1 (#) —————→	1
0 —————→	0

Figure 4-3. The thirteen negative and positive alteration operators and their correspondence to the twelve mod-12 values.

Scale steps and altered scale steps of which the mod-12 values (that is, the base plus alteration operator, mod 12) are the same are termed enharmonically equivalent. The mod-(12,7) system defines seven enharmonically equivalent (altered)

scale steps for each of the twelve mod-12 values. In Figure 4-2 each of the twelve rows corresponds to seven enharmonically equivalent (altered) scale steps.

In Figure 4-4 the eighty-four scale steps and altered scale steps are arranged "by fifths" (the direction of the arrow indicates, specifically, a progression by fourths). As can be seen, the cycle of fifths is a finite cycle. Starting with $\hat{1}$, one passes through single-flatted scale steps, to double-flatted scale steps, and so on, until one reaches six-times flatted scale steps, which one equates with six-times sharped scale steps. From here on the alteration operator's absolute value gradually diminishes, from +6 (six sharps), to 5 (five sharps), and so on, until the operator 0 is reached once again with $\hat{7}$. Five additional fourths bring one back to $\hat{1}$, the point of origin.¹

	to bb2	to b3	to b4	to #5	to #6	to #7	to 1
	↑	↑	↑	↑	↑	↑	↑
	bb6	b4	b5	#2	#3	x4	5
	bb3	b3	b5	#6	#7	x1	2
	bb7	b3	b2	#3	#4	x5	6
	b4	b5	b6	#7	#1	x2	3
	b1	b2	b3	#4	#5	x6	7
	b3	b6	b7	#1	#2	x3	#4
	b2	b3	b4	#5	#6	x7	#1
	b6	b7	b1	#2	#3	#4	#5
	b3	bb4	b5	#6	#7	#1	#2
fourths ↑	b7	bb1	b2	#3	#4	#5	#6
	4	bb5	b6	#7	#1	#2	#3
	1	bb2	b3	b4	#5	#6	#7
	↑	↑	↑	↑	↑	↑	↑
	from 5	from bb6	from b4	from b5	from #2	from #3	from x4

Figure 4-4. The eighty-four scale steps and altered scale steps arranged by fifths.

Note that every seven fifths one returns to the same scale step, only altered by, specifically, one more sharp or flat; that is to say, given the scale step (1,1), a progression

by seven fifths yields the scale step $(\underline{1}, \underline{1})$ or $(\bar{1}, \underline{1})$. Note also that every twelve fifths one returns to an enharmonically equivalent scale step, specifically, an enharmonically equivalent adjacent scale step; that is to say, given the scale step $(\underline{1}, \underline{1})$, a progression by twelve fifths yields the scale step $(\underline{1}, \underline{k})$ where $\underline{k}$ (the mod-7 value) equals $\underline{1}$ plus one or $\underline{1}$ minus one. While we may have learned to accept this systematic aspect of chromaticism and enharmonicism as self-evident, the fact is that it reflects mod-(12,7) coherence (see Chapter 2, pp. 51-5). As the reader will recall, mod-(12,7) coherence reflects the fact that

$$12 \cdot 3 - 7 \cdot 5 = 1,$$

from which it follows that

$$7 \cdot 5 \bmod 12 = -1, \text{ and}$$

$$12 \cdot 3 \bmod 7 = 1,$$

which is exactly the systematic property we have been describing (see, in this connection, the discussion of mod- $(\underline{x}, \underline{y})$ coherence in Appendix A).

Chromatically altered diatonic intervals. In Figure 4-5 the mod-(12,7) system is interpreted as diatonic intervals and chromatically altered diatonic intervals. The principle is the same here as with altered scale steps. A non-diatonic integer pair is interpreted as chromatically altered relative to a diatonic pair with which it shares the same mod-7 value.

Clearly, however, the situation here is somewhat more complicated than was the case with scale steps, since there are two diatonic intervals defined for each of the seven mod-7 values save 0 (see the highlighted intervals in Figure 4-5); and then there is the special case of fourths and fifths.

11	d1	dd2	d ⁴ 3	A ⁴ 4	A ⁴ 5	A ⁴ 6	1 ⁴ 6 A ⁴ 7
10	dd1	d ⁵ 2	d ⁵ 3	A ⁵ 4	A ⁵ 5	A ⁵ 6	1 ⁵ 6 A ⁵ 7
9	d ³ 1	d ⁴ 2	A ⁴ 3	A ⁴ 4	A ⁴ 5	2 ⁴ 6 A ⁴ 6	d ⁴ 7
8	d ⁴ 1	d ⁵ 2	A ⁵ 3	A ⁵ 4	A ⁵ 5	3 ⁵ 6 A ⁵ 6	dd7
7	d ⁵ 1	A ⁵ 2	A ⁵ 3	A ⁵ 4	7 ⁵ 4 A ⁵ 5	d ⁵ 6	d ⁵ 7
6	A ⁴ 1	A ⁴ 2	A ⁴ 3	5 ⁴ 3 A ⁴ 4	7 ⁴ 4 A ⁴ 5	dd6	d ⁴ 7
5	A ⁵ 1	A ⁵ 2	A ⁵ 3	5 ⁵ 3 A ⁵ 4	dd5	d ⁵ 6	d ⁵ 7
4	A ⁴ 1	A ⁴ 2	4 ⁴ 2 A ⁴ 3	d ⁴ 4	d ⁴ 5	d ⁴ 6	A ⁴ 7
3	A ⁵ 1	A ⁵ 2	3 ⁵ 2 A ⁵ 3	dd4	d ⁵ 5	d ⁵ 6	A ⁵ 7
2	A ⁴ 1	2 ⁴ 1 A ⁴ 2	d ⁴ 3	d ⁴ 4	d ⁴ 5	A ⁴ 6	A ⁴ 7
1	A ⁵ 1	1 ⁵ 1 A ⁵ 2	dd3	d ⁵ 4	A ⁵ 5	A ⁵ 6	A ⁵ 7
0	0 ⁰ A ⁰ 1	d ⁵ 2	d ⁵ 3	d ⁵ 4	A ⁵ 5	A ⁵ 6	A ⁵ 7
	0	1	2	3	4	5	6

Figure 4-5. The mod-(12,7) system at large interpreted as diatonic intervals and chromatically altered diatonic intervals.

Let us begin with the chromatically altered diatonic interval (0,0), or perfect prime, which represents the simplest case. Since, as was already indicated, only the perfect prime has 0 as its mod-7 value, the situation here is fully analogous to chromatically altered scale steps, say, the scale step (0,0), or $\hat{1}$ (see Figure 4-2). As with chromatically altered scale steps, we have a system of negative and positive alteration-operator numerals ranging from -6 to 6 which correspond to twelve mod-12 values (see Figure 4-3); positive operators are conventionally identified by the terms "augmented," "doubly augmented," etc., while negative operators are identified by the terms "diminished," "doubly diminished," etc.. Thus, for example, the integer pair (1,0) (second row, first column) is represented as $(0, \overset{+}{1})$, or "augmented prime" (A1), and the integer pair (11,0) (twelfth row, first column) is represented as $(0, \bar{1})$, or "diminished prime" (d1), etc.. I use "AA" for "doubly augmented" and "dd" for "doubly diminished." For altered diatonic intervals of which the alteration operator's absolute value is larger than 2 I notate "A³" (triply augmented), "d⁴" (quadruply diminished), etc., in analogy with altered scale steps. (The reader who experiences at this point difficulties with altered diatonic intervals such as the diminished prime is asked to keep in mind that we are dealing at the moment with the cognitive structure of chromaticism and enharmonicism only. As will become evident in the following section, where the cognitive level of chromaticism and enharmonicism is related

to the perceptual/physical level, there is nothing logically or intuitively problematic about a "diminished prime" or any such altered diatonic interval; see also the discussion of conventional interval nomenclature in Appendix B.)

Chromatic alteration can be seen as a system by which, for each of the seven mod-7 values, each of the twelve mod-12 values is rewritten as a pair of values, the mod-12 value of the given diatonic mod-(12,7) integer pair (the diatonic integer pair whose mod-7 value is the same) I have termed "base," and a value between -6 and 6 (which corresponds to one of twelve mod-12 values) I have termed "alteration operator." Let us see now what happens when we are given two bases, as is the case with seconds, thirds, sixths, and sevenths, as well as fourths and fifths.

The chromatic alteration of the major and minor seconds is presented in Figure 4-6. I have left out the mod-7 value, which is always 1. The mod-12 values (left-hand column) are rewritten as pairs of values, where each pair consists of a "base" and a positive or negative "alteration operator" (right-hand column).

mod-12 value:		base plus alteration operator:
7	=	2^{+5}
6	=	2^{+4}
5	=	2^{+3}
4	=	2^{+2}
3	=	2^{+1}
2	=	2^{+0}
1	=	1^{+0}
0	=	1^{-1}
11	=	1^{-2}
10	=	1^{-3}
9	=	1^{-4}
8	=	1^{-5}

Figure 4-6. The chromatic alteration of the major and minor second.

Clearly, the pairs 1^{+0} and 2^{+0} (enclosed within a box) correspond to the diatonically given intervals, namely (1,1) (major second) and (2,1) (major second). The other pairs fall into one of two categories; pairs of the base 2 and a positive alteration operator from 1 to 5, and pairs of the base 1 and a negative alteration operator from -1 to -5. This peculiar

situation (note that the alteration operators ± 6 are not used) arises from the fact that, as one moves (on the right-hand column) from 1 to 2 or vice versa, the alteration operator remains constant (namely 0), while the base changes (from 1 to 2 or vice versa). Note that since positively altered seconds are always altered major seconds, while negatively altered seconds are always altered minor seconds, it is not really necessary to specify whether an altered major or minor second is meant (see Figure 4-5); "A2" is an altered major second, "dd2" is an altered minor second, etc..

The chromatic alteration of sevenths, thirds, and sixths, is analogous to that of seconds (see Figure 4-5). One would think that the same should hold for fourths and fifths as well, since, here also, there are two bases in each case, 5 and 6, and 6 and 7, respectively. Two things, however, must be borne here in mind. First, the diatonic intervals (5,3) and (7,4) (perfect fourth and fifth) convey the "fifth relationship," and as such hold a superior systemic status relative to their mod-7 "partners." And second, the augmented fourth (6,3) and the diminished fifth (6,4) are the only two diatonic intervals with different mod-7 values of which the mod-12 values are the same, a situation that otherwise involves chromatic alteration. Conventional musical nomenclature reflects both facts. We use the quality "perfect" in connection with the diatonic intervals (5,3) and (7,4) in analogy to the perfect prime (0,0), which conveys "octave equivalence," and we use the qualities "aug-

mented" and "diminished" for the intervals (6,3) and (6,4), in the manner of chromatically altered diatonic intervals. Following conventional practice, we shall treat A_4 and d_5 as if they were chromatically altered diatonic intervals, namely, altered P_4 and P_5 , respectively (see Figure 4-5; needless to say, A_4 and d_5 are nonetheless diatonic intervals, in the sense defined in Chapter 2). As a result, the chromatic alteration of fourths and fifths is analogous to that of the perfect prime. The base, in each case, remains constant (namely 5 and 7, respectively), and the system of alteration operators covers the full scale of values from -6 to 6.

As with scale steps, diatonic intervals and chromatically altered diatonic intervals of which the mod-12 values (that is, the base plus alteration operator, mod 12) are the same are termed enharmonically equivalent; there are seven enharmonically equivalent (altered) diatonic intervals for each of the twelve mod-12 values (see the twelve rows in Figure 4-5). Unlike scale steps, however, where enharmonic equivalence arises only through chromatic alteration, the diatonic system defines one pair of enharmonically equivalent diatonic intervals, namely the augmented fourth and diminished fifth.

In Figure 4-7 I have arranged the eighty-four (altered) diatonic intervals by fifths. As with (altered) scale steps (see Figure 4-4), the alteration operator's absolute value increases up to a point, and then it diminishes gradually back to 0.

d6	d ³ 7	d ⁵ 1	A ⁵ 2	A ⁵ 3	AA ⁴	P5
d3	d ³ 4	d ⁶ 5	A ⁵ 6	A ⁵ 7	AA1	M2
d7	d ³ 1	d ⁴ 2	A ⁵ 3	A ⁴ 4	AA5	M6
d4	d ³ 5	d ⁴ 6	A ⁵ 7	A ⁴ 1	AA2	M3
d1	dd2	d ⁴ 3	AA ⁴ 4	A ⁵ 5	AA6	M7
d5	dd6	d ⁴ 7	AA ⁴ 1	A ⁴ 2	AA3	A4
m2	dd3	d ⁴ 4	AA ⁴ 5	A ⁴ 6	AA7	A1
m6	dd7	d ⁴ 1	d ⁵ 2	A ⁴ 3	A ⁵ 4	A5
m3	dd4	d ⁴ 5	d ⁵ 6	A ⁴ 7	A ⁵ 1	A2
m7	dd1	d ⁵ 2	d ⁵ 3	A ⁵ 4	A ⁵ 5	A6
P4	dd5	d ⁵ 6	d ⁵ 7	A ⁵ 1	A ⁵ 2	A3
P1	d2	d ⁵ 3	d ⁵ 4	A ⁵ 5	A ⁵ 6	A7

Fourth
↑

Figure 4-7. The eighty-four diatonic intervals and altered diatonic intervals arranged by fifths.

4.3 The Cognitive/Perceptual Structure of Chromaticism and Enharmonicism

In the previous section I have described the cognitive structure of chromaticism and enharmonicism in terms of the mod-(12,7) system at large and the system of chromatic alteration. To complete the picture, we must now consider the relation between this cognitive structure and the perceptual/

physical domain.

The relation between the cognitive and perceptual/physical domains in connection with chromaticism and enharmonicism is defined in terms analogous to those of the diatonic system proper. As the reader will recall, the cognitive/perceptual structure of diatonicism was defined in terms of pitch bands and intervals bands $\underline{p}_n$ and $\underline{i}_n$, and the mod-12 value $\underline{t}$ of scale steps and diatonic intervals, as well as a "register operator" $\underline{rg}$. In particular:

$$\underline{p}_n, \underline{i}_n \xrightarrow{\quad} \underline{t}, \underline{rg}, \text{ with} \\ \underline{n} = \underline{t} + 12\underline{rg}.$$

(We are assuming C major or C minor; in other keys one has to take into account the "transposition operator" $\underline{tr}$ in connection with $\underline{p}_n$).

The same definition also applies to chromaticism and enharmonicism, except that the mod-12 value $\underline{t}$ is broken into a sum of a "base" $\underline{b}$ (the mod-12 value of the unaltered scale step or diatonic interval that shares the same mod-7 value) and an "alteration operator" $\underline{a1}$ between -6 and 6:

$$\underline{t} = \underline{b} + \underline{a1} \text{ mod } 12, \text{ where} \\ \underline{b} = 1, 2, 3, \dots, 11, \text{ and} \\ \underline{a1} = -6, -5, -4, \dots, 6.$$

Thus, one can define the cognitive/perceptual structure of chromaticism and enharmonicism as follows:

$$\begin{aligned} p_{\underline{n}}, \frac{1}{\underline{n}} &\longrightarrow (\underline{b} + \underline{a1}), \underline{rg}, \text{ with} \\ \underline{n} &= (\underline{b} + \underline{a1} \bmod 12) + 12\underline{rg}, \end{aligned}$$

and where $(\underline{b} + \underline{a1})$ is the base plus alteration operator of an altered scale step or diatonic interval, and $\underline{rg}$ the register operator. I will use $(\underline{t}, \underline{s})$, from now on, to designate any mod-(12,7) integer pair; $(\underline{b}, \underline{s})$, on the other hand, will be used to designate only a mod-(12,7) integer pair that is an unaltered scale step or diatonic interval. Clearly, if $\underline{a1} = 0$, then $\underline{b} = \underline{t}$.

We will notate a corresponding pitch band $p_{\underline{n}}$ and altered scale step $(\frac{+a1}{\underline{b}}, \underline{s})$ as $\underline{n} : (\frac{+a1}{\underline{b}}, \underline{s})$. For example, a correspondence between the pitch band p_{-1} and the altered scale step $(\frac{-1}{0}, 0)$ ($b\hat{1}$, C-flat in C major) is notated as $-1 : (\frac{-1}{0}, 0)$; the same altered scale step an octave higher is notated: $11 : (\frac{-1}{0}, 0)$.

See Figure 4-8.

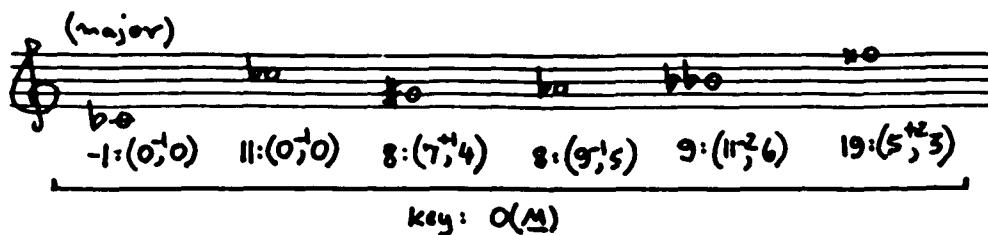


Figure 4-8. The notation $n:(\underline{b}^{\text{al}}, \underline{s})$ for a corresponding pitch band and altered scale step, and its staff notational translation.

A correspondence between an interval band $\underline{i}_n$ and an altered diatonic interval $(\underline{b}^{\text{al}}, \underline{s})$ is notated in exactly the same manner. For example, a correspondence between the interval band $\underline{i}_{11}$ and the altered diatonic interval $(0, 0)^{-1}$ (diminished prime) is notated as $11:(0, 0)^{-1}$; see Figure 4-9. Note that conventionally, this interval is known as a "diminished octave" (recall that the conventional "octave" is $12:(0, 0)$; see Figure 3-15). There are, of course, other interval bands to which $(0, 0)^{-1}$ (diminished prime) at the cognitive level corresponds. In particular, $\underline{i}_{-1}$, a descending tempered half step with $\alpha = 0$, is such an interval band. This is highly intuitive as well as logical. If, say, C-C# (ascending) is an augmented prime, and C-C (stationary) is a perfect prime, then C-Cb (descending!) is a diminished prime. Note, however, that conventionally one could have referred to such an interval as a descending augmented prime; for further discussion of conventional interval nomenclature, see Appendix B.

4: (0 ⁻¹ , 0)	 diminished octave
-1: (0 ¹ , 0)	 diminished prime (desc. aug. prime)
13: (0 ⁴ , 0)	 augmented octave
8: (7 ⁴ , 4)	 A5
9: (10 ¹ , 6)	 d7
16: (2 ² , 1)	 doubly aug. ninth
0: (1 ¹ , 1)	 d2
-1: (1 ⁻² , 1)	 dd2 (see Appen- dix B)

Figure 4-9. The notation $\underline{n}:(\underline{b}, \underline{s})$ for a corresponding interval band and altered diatonic interval, and its translation into conventional musical nomenclature.

Using pitch-band and interval-band classes, the definition of the cognitive/perceptual structure of chromaticism and enharmonicism is once again identical with the corresponding definition of diatonicism:

$$\text{PbC}\underline{t}, \text{IbC}\underline{t} \longrightarrow \underline{t}.$$

Once again, however, the mod-12 value $\underline{t}$ is broken into the sum $(\underline{b} + \underline{a})$. For example, the pitch-band class 11 (PbC11), which includes such pitch bands as $\underline{p}_{-1}$ and $\underline{p}_{11}$, corresponds,

in C major, to $\flat\hat{1}$ (11,0, or $\bar{0}\hat{1}_0$); the interval-band class 11 (IbC11), which includes such interval bands as $\underline{1}_{-1}$ and $\underline{1}_{11}$, corresponds to the diatonic interval "diminished prime" (11,0, or $\bar{0}\hat{1}_0$). Clearly, each of the twelve PbCs and IbCs potentially corresponds to one of seven enharmonically equivalent (altered) scale steps or diatonic intervals. For example, PbC11 corresponds (in C major) not only to $\flat\hat{1}$, but also to $\hat{7}$, $x\hat{6}$, etc., and IbC11 corresponds not only to $d\hat{1}$, but also, to $M\hat{7}$, $AA\hat{6}$, etc. (see Figures 4-2 and 4-5). This relation, indeed, lies at the heart of enharmonicism as a cognitive/perceptual manifestation; there is a clash between "sameness" (or "near sameness") at the perceptual level, and "non-sameness" at the cognitive level.

Note that enharmonically equivalent (altered) scale steps and diatonic intervals can be embodied, at the perceptual/physical level, by both identical and slightly differing pitches and intervals; we have defined the cognitive/perceptual structure of enharmonicism, as with the diatonic system proper, in terms of pitch and interval bands. Thus, for two enharmonically equivalent (altered) scale steps or diatonic intervals to appear as two slightly differing pitches or intervals is consistent with our entire approach; there is, in fact, nothing exceptional about such a possibility.

It is important to bear in mind, at the same time, that a perceptual/physical difference of any kind between enharmonically equivalent (altered) scale steps or diatonic intervals

does not correspond to the cognitive difference involved, that is, to the differing mod-7 values. A "G#" and "Ab" are different entities regardless of whether they are embodied at the perceptual level by identical or slightly differing pitches, and regardless, in the latter case, of whether the G# pitch is higher or lower than the Ab pitch. Enharmonic equivalence cannot be captured in perceptual/physical terms since the relation between the cognitive and perceptual domains involves, in the former case, the mod-12 values only (this does not mean that contextual factors cannot influence intonation in the case of enharmonic equivalence so that, for example, G# may be normatively higher than Ab; see Chapter 3, section 3.6). I believe that any attempt to establish a one-to-one correspondence between intonational nuances and enharmonic equivalence, even such a sophisticated attempt as Regener 1975 (proposing a correspondence between the ratio $7/4$ and the augmented sixth), is, at bottom, ad hoc.

In closing, I wish to remind the reader that the theory of chromaticism and enharmonicism I have presented in this chapter is a theory of competence; in actual performance, only a small fraction of the seventy-seven (eighty-four minus seven) altered scale steps and seventy-one (eighty-four minus thirteen) altered diatonic intervals are in fact used (Chapter 5). If one wishes, one can define chromatic and enharmonic competence and performance interdependently, so as to overcome the obvious

redundancy that chromatic and enharmonic competence as described in this chapter involves; see, in this connection, Chapter 1, section 1.4.

NOTES

1. As to cycles of other diatonic intervals, only the cycle of minor seconds (or major sevenths) covers all eighty-four (altered) scale steps before returning to the point of origin. The cycle of major seconds is a cycle of every other fifth, and is therefore a cycle of $84/2$, or 42 (altered) scale steps. Similarly, the cycle of minor thirds is a cycle of $84/3$, or 28 (altered) scale steps (every third fifth), the cycle of major thirds is a cycle of $84/4$, or 21 (altered) scale steps (every fourth fifth), and the cycle of augmented fourths is a cycle of $84/6$, or 14 (altered) scale steps (every sixth fifth).

2. In particular, one should not confuse perceptual/physical differences between enharmonically equivalent scale steps and diatonic intervals in Pythagorean intonation with cognitive differences, as has been already noted in Chapter 3 in connection with the Pythagorean A_4 and d_5 . In fact, to assume such a correspondence is ultimately nonsensical. Moving twelve fifths in Pythagorean intonation one reaches an adjacent enharmonically equivalent scale step that is a Pythagorean comma higher or lower (see Figure 4-4). But since, as we saw, the cycle of fifth is finite, one would ultimately return in this manner to one's point of origin, at a difference of seven Pythagorean commas. One has not only transgressed the limits of one's original pitch band (seven Pythagorean commas amount to about 165 cents), but has in fact contradicted the idea of a fixed intonational system, and moreover, of intonation altogether. For it is clear that the cyclic process can continue in this manner, so that a given scale step corresponds to some pitch plus any whole multiple of seven Pythagorean commas, which is to say, a given scale step corresponds to any pitch.

Chapter V

CHROMATICALLY EXTENDED KEYS:

A STUDY IN CHROMATIC AND ENHARMONIC PERFORMANCE

In the previous chapter I was concerned with chromatic and enharmonic competence. Using the mod-(12,7) system at large as a model I have defined seventy-seven altered scale steps and seventy-one altered diatonic intervals at the cognitive level; and I have argued that each of the twelve pitch-band classes and interval-band classes potentially corresponds to one of seven enharmonically equivalent (altered) scale steps and diatonic intervals. As has been already noted, however, chromatic and enharmonic competence thus defined does not coincide with performance. In any given key, we do not encounter more than a handful of altered scale steps and diatonic intervals. In the case of scale steps, for example, we hardly encounter alterations that exceed one sharp and flat, and we do not even encounter all such altered scale steps. In this chapter I will account for this discrepancy between chromatic and enharmonic competence and performance.

Let us begin by reminding ourselves that a theory of diatonic performance must account for the two-stage process

by which the physical level of frequencies and frequency ratios is mapped onto the perceptual level of PbCs and IbCs, while the perceptual level of PbCs and IbCs is mapped, in turn, onto the cognitive level of scale steps and diatonic intervals. In our case, the cognitive level consists of altered scale steps and diatonic intervals in addition to unaltered ones; the mapping of the physical level onto the perceptual level, as discussed in the "Interlude" that follows Chapter 3, is of course unaffected by this change, so that it need not concern us here.

How would a mind map pitch-band classes and interval-band classes onto (altered) scale steps and diatonic intervals? That is, how would one account for the following process:

$$\text{PbC}\underline{t}, \text{IbC}\underline{t} \longrightarrow (\underline{b} \overset{+a1}{\underline{s}}),$$

where $\underline{t}$ is a mod-12 value, $(\underline{b}, \underline{s})$ an integer pair mod (12, 7) representing an unaltered scale step or diatonic interval, and $\underline{a1}$ an alteration operator between -6 and 6?

Surely, the mind starts out by establishing a correspondence between the given pitch-band or interval-band class and a mod-12 value (we are assuming C major or C minor):

$$\text{PbC}\underline{t}, \text{IbC}\underline{t} \longrightarrow \underline{t}.$$

This is a simple process since PbCs and IbCs correspond, by definition, to mod-12 values.

The mind must next establish a correspondence between the mod-12 value t and the mod-7 value s:

$$\underline{t} \longrightarrow \underline{s}.$$

This stage is problematic. For unlike the diatonic system, where the mod-12 values of scale steps and diatonic intervals (except, in the latter case, the mod-12 value 6) correspond to only one mod-7 value, here each of the twelve mod-12 values potentially corresponds to one of seven mod-7 values (see Chapter 4, Figure 4-1). That is to say, each of the twelve PbCs and IbCs potentially corresponds to one of seven enharmonically equivalent (altered) scale steps and diatonic intervals; how does the mind know which alternative to choose? I propose that the mind is making two assumptions in performing this task; as we shall see, these two assumptions ultimately reduce the "theoretical" domain of chromatic and enharmonic competence to its familiar, "practical" size. One of the two assumptions, in particular, has important general consequences as far as the correspondence between PbCs and IbCs and (altered) scale steps and diatonic intervals is concerned.

Diatonic and chromatic pitch-band classes. I propose that the mind is assuming that pitch-band classes that correspond to (unaltered) scale steps--and there are seven such PbCs, whichever specific PbCs they may be--correspond only to (unaltered) scale steps:

$$\text{PbC}\underline{t} \longrightarrow (\underline{b}, \underline{s}), \text{ with} \\ \underline{t} = \underline{b},$$

or alternatively stated,

$$\text{PbC}\underline{t} \longrightarrow (\underline{t}, \underline{s}), \text{ with} \\ \underline{a}_1 = 0.$$

(Recall that $(\underline{t}, \underline{s})$ is any mod-(12,7) integer pair, while $(\underline{b}, \underline{s})$ is a mod-(12,7) pair that is an unaltered scale step or diatonic interval; if $\underline{a}_1 = 0$, therefore, then $\underline{b} = \underline{t}$.)

That is to say, a PbC that corresponds to an unaltered scale step does not correspond to any of the other six enharmonically equivalent altered scale steps. In C major, for example, PbC0 corresponds to (0,0), or $\hat{1}$, and only to (0,0); it does not correspond to $\hat{7}$, $\flat\hat{2}$, etc.. Clearly, forty-two altered scale steps (seven times six) are simply eliminated¹ in this way (see Chapter 4, Figure 4-2).

Note that as a result of the assumption specified above the twelve PbCs divide into two groups: a set of seven PbCs that correspond (and only correspond) to unaltered scale steps (henceforth, diatonic PbCs), and a set of five PbCs that correspond (and of course, only correspond) to altered scale steps, as yet unspecified (henceforth, chromatic PbCs). The IbCs divide into three groups: IbCs defined by a pair of diatonic PbCs (henceforth, diatonic IbCs), which correspond (and only correspond) to unaltered diatonic intervals, IbCs defined by

a pair of PbCs, where one PbC is diatonic while the other is chromatic (henceforth, diatonic/chromatic IbCs), and IbCs defined by a pair of chromatic PbCs (henceforth, chromatic IbCs). Diatonic/chromatic and chromatic IbCs correspond to either unaltered or altered diatonic intervals (unlike scale steps, there is a diatonic interval defined for each of the twelve mod-12 values). I will refer to a set of diatonic and chromatic PbCs (and implicitly, diatonic, diatonic/chromatic, and chromatic IbCs) and their corresponding (altered) scale steps (and diatonic intervals) as a "chromatically extended key."

The role of context. Having defined the notion "chromatically extended key" we can now state that a mind set upon the task of processing the twelve PbCs and IbCs chromatically and enharmonically is in fact performing two tasks, separating diatonic from chromatic PbCs, and establishing a correspondence between the chromatic PbCs (and diatonic/chromatic and chromatic IbCs) and altered scale steps (and altered or unaltered diatonic intervals), respectively. (In reality, these two tasks are no doubt inseparable from one another; I am thus assuming here that no harm is done in considering one task apart from the other. That is to say, I am assuming here that the ultimate results are the same, whichever way one proceeds.) But note that there are twelve different ways to divide the twelve PbCs into diatonic and chromatic PbCs, corresponding to twelve different major or minor chromatically extended keys. The mind cannot know which specific PbC is diatonic and which is chroma-

tic on the basis of just the PbCs and IbCs presented; contextual information, in this case, is indispensable.

Figure 5-1 is a contrived tonal composition in a chromatically extended C major. Clearly, in processing such a composition, the mind can benefit from a great deal of information on the basis of which to recognize specific PbCs as either diatonic or chromatic, information that derives from levels of structure higher than those we have been dealing with so far. The nature of this information is a subject that lies outside the scope of the present study; Chapter 6 offers a glimpse into this highly complex theoretical domain. For our purposes here it will suffice to assume that the mind can recognize specific PbCs as either diatonic or chromatic (we will assume, in particular, that the mind recognizes as diatonic the PbCs of C major or C minor). That is to say, I take it that this aspect of chromatic and enharmonic performance could be accounted for within a more comprehensive theory of tonal performance in general. I will return to the role of context in connection with chromatic and enharmonic performance later on.

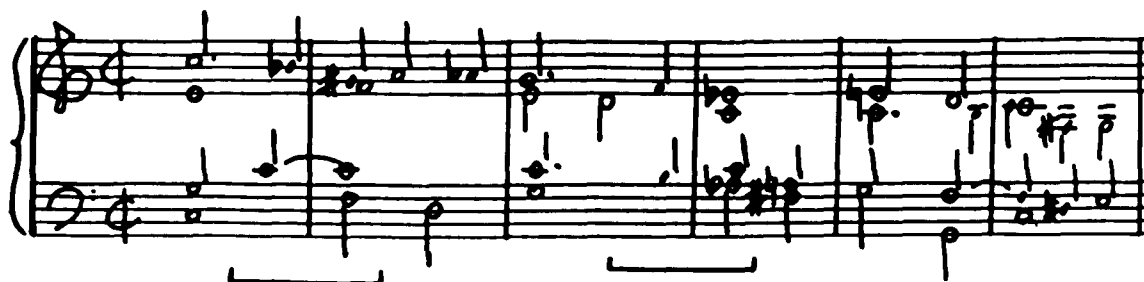


Figure 5-1. A contrived tonal composition in a chromatically extended C major.

Note that we are assuming that there is only one key in terms of which to process a given set of PbCs and IbCs chromatically and enharmonically. A glance at Figure 5-1 will reveal that this assumption does not square with reality; at the bracketed segments, C major is not the only viable "interpretation." But surely, multiple interpretations of this sort (in our case, C major and F major, and C major and C minor) must be treated independently of each other first, that is, as if only one or the other existed, as we are assuming here; only then one would proceed to relate one interpretation to the other. "Modulation" in the latter sense, that is, the relation between two independently defined (chromatically extended) keys, lies outside the scope of this study.

The mind's second assumption: diatonic IbC processing. Assuming diatonic and chromatic PbCs, and in conjunction with contextual information, which we assume exists, the mind can

easily process the set of diatonic PbCs and IbCs as unaltered scale steps and diatonic intervals. The mind, however, must still process the set of chromatic PbCs and chromatic as well as diatonic/chromatic IbCs, and is faced here with the same problem we have initially noted: each PbC and IbC can be processed in terms of seven enharmonically equivalent altered scale steps and altered or unaltered diatonic intervals, respectively. We thus come to the second of the mind's two assumptions noted earlier. To facilitate presentation, I will start with a simplified version of this second assumption; the unsimplified version will suggest itself later on in connection with certain difficulties that we will encounter.

The mind's second assumption is this: diatonic/chromatic IbCs (that is, IbCs defined by pairs of PbCs, where one is diatonic while the other is chromatic), correspond, and only correspond, to (unaltered) diatonic intervals:

$$\text{IbC}\underline{t} \longrightarrow (\underline{b}, \underline{s}), \text{ with} \\ \underline{t} = \underline{b},$$

and where $\text{IbC}\underline{t}$ is a diatonic/chromatic IbC.

It is not difficult to recognize in this second assumption the strategy of diatonic IbC processing that is familiar from diatonic performance (see the "Interlude" that follows Chapter 3): $\text{IbC}0$ is processed as $P1$, $\text{IbC}1$ is processed as $m2$, $\text{IbC}2$ is processed as $M2$, etc. (in conjunction with purely diatonic performance there was no need to specify that IbCs cor-

respond only to diatonic intervals, since only diatonic intervals were available). Let us see now how this second assumption, diatonic IbC processing, finally yields a correspondence between the chromatic PbCs (and diatonic/chromatic and chromatic IbCs) and altered scale steps (and altered or unaltered diatonic intervals).

Suppose the mind is given, in C major, a PbC5 and a PbC6 (a diatonic pitch-band class and a chromatic pitch-band class, respectively), where the direction $\text{PbC5} \longrightarrow \text{PbC6}$ represents an ascending interval band; the mind will determine the corresponding interval-band class, namely, IbC1^2 . Processing this IbC diatonically (the mind's second assumption) yields a correspondence with the diatonic interval "minor second," that is, the mod-(12,7) integer pair (1,1):

$$\text{IbC1} \longrightarrow (1,1) \text{ (m2)}.$$

Now, the mind also assumes that PbC5 is a diatonic PbC, and thus knows that it corresponds to $\hat{4}$, or the mod-(12,7) integer pair (5,3):

$$\text{PbC5} \longrightarrow (5,3) \text{ } (\hat{4}).$$

Clearly, the mind can now add the minor second to $\hat{4}$, and find out the mod-(12,7) integer pair that corresponds to PbC6, namely, (6,4):

$$\begin{aligned}\text{PbC6} &\longrightarrow (5,3) + (1,1) \bmod (12,7), \\ \text{PbC6} &\longrightarrow (6,4).\end{aligned}$$

The mind must now translate the mod-(12,7) integer pair (6,4) into an altered scale step (recall the two aspects of the structure of chromaticism and enharmonicism discussed in Chapter 4, section 4.2). The mind must find the unaltered scale step whose mod-7 value is 4, namely (7,4), and then use its mod-12 value 7 as a "base" to which an "alteration operator" between -6 and 6 is added to equal 6, mod 12; clearly, this alteration operator must be -1. The mod-(12,7) integer pair (6,4) thus becomes $(\bar{7}^1, 4)$, or $\flat 5$, and the mind has completed its task:

$$\text{PbC6} \longrightarrow (\bar{7}^1, 4) (\flat 5).$$

Note that finding the unaltered scale step whose mod-7 value is 4 is greatly facilitated by the fact that the mind assumes diatonic and chromatic PbCs; the mind has eliminated all altered scale steps whose mod-7 value is 4, and therefore has only one integer pair of this type to consider.

In Figure 5-2 the process described above is applied systematically to all diatonic/chromatic IbCs. The seven rows are labeled for the seven diatonic PbCs (C major), arranged cyclically by abs. IbC5; the five columns are labeled for the five chromatic PbC, also arranged cyclically by abs. IbC5. At the intersection of every row with every column I have indicated

the IbC that corresponds to the pair of PbCs that label the particular row and column, assuming that the direction

diatonic PbC $\longrightarrow$ chromatic PbC

represents an ascending interval band (see, in this connection, note 2). Finally, the altered scale step that corresponds to the chromatic PbC involved, as derived on the basis of the mind's two assumptions, is indicated by means of an arrow pointing to the lower left or lower right (in the case of IbC6 there are two alternative altered scale steps, indicated by means of two arrows).

		chromatic PbCs				
		6	1	8	3	10
diatonic PbCs	5	IbC1 ↙ b5^	IbC8 ↙ b2^	IbC3 ↙ b6^	IbC10 ↙ b3^	IbC5 ↙ b7^
	0	IbC6 ↙ b5^	IbC1 ↙ b2^	IbC8 ↙ b6^	IbC3 ↙ b3^	IbC10 ↙ b7^
	7	IbC11 ↙ #4^	IbC6 ↙ b2^	IbC1 ↙ b6^	IbC8 ↙ b3^	IbC3 ↙ b7^
	2	IbC4 ↙ #4^	IbC11 ↙ #1^	IbC6 ↙ b6^	IbC1 ↙ b3^	IbC8 ↙ b7^
	9	IbC9 ↙ #1^	IbC4 ↙ #1^	IbC11 ↙ #5^	IbC6 ↙ b3^	IbC1 ↙ b7^
	4	IbC2 ↙ #4^	IbC9 ↙ #1^	IbC4 ↙ #5^	IbC11 ↙ #2^	IbC6 ↙ b7^
	11	IbC7 ↙ #4^	IbC2 ↙ #1^	IbC9 ↙ #5^	IbC4 ↙ #2^	IbC11 ↙ #6^
		2:6	3:5	4:4	5:3	6:2

Figure 5-2. Chromatic and enharmonic performance on the basis of assuming diatonic and chromatic PbCs and diatonic IbC processing (see text).

As can be seen, the altered scale steps selected on the basis of the two assumptions specified as corresponding to the five chromatic PbCs are those of which the alteration operator's absolute value does not exceed 1 (that is, a single flat or sharp). Thus, PbC1 corresponds to either $\flat^{\hat{2}}$ or $\sharp^{\hat{1}}$, PbC3 to either $\flat^{\hat{3}}$ or $\sharp^{\hat{2}}$, PbC6 to either $\flat^{\hat{5}}$ or $\sharp^{\hat{4}}$, PbC8 to either $\flat^{\hat{6}}$ or $\sharp^{\hat{5}}$, and PbC10 to either $\flat^{\hat{7}}$ or $\sharp^{\hat{6}}$. Scanning for example the first column, representing PbC6, from top to bottom, we find at the intersection with the first row (diatonic PbC5) an IbC1 that yields a corresponding $\flat^{\hat{5}}$ in conjunction with the mind's two assumptions, as has been described in detail before. At the intersection with the second row (diatonic PbC0) we find an IbC6. Since diatonic IbC processing yields in this case two diatonic intervals, A_4 and d_5 , we find here a corresponding $\sharp^{\hat{4}}$ as well as $\flat^{\hat{5}}$. From here on, all IbCs yield a corresponding $\sharp^{\hat{4}}$: an IbC11 in relation to diatonic PbC7, an IbC4 in relation to diatonic PbC2, an IbC9 in relation to diatonic PbC9, an IbC2 in relation to diatonic PbC4, and an IbC7 in relation to diatonic PbC11. In all, we count 2 IbCs that yield a $\flat^{\hat{5}}$, and six IbCs that yield a $\sharp^{\hat{4}}$ (IbC6 is counted twice); these numbers are given as a ratio, 2:6, below the bottom row. Scanning the five columns from left to right, we see that the balance between the IbCs which yield the lowered scale step and those which yield the raised scale step gradually tips over. Thus, with PbC1 the ratio is 3:5, whereas with PbC8 the ratio is balanced, 4:4; the ratio with PbC3, 5:3, is the ratio with PbC1, inverted,

and similarly, with PbC10 and PbC6 (2:6 vs. 6:2).

Obviously, there is more to be said about Figure 5-2, its implications as far as chromaticism and enharmonicism are concerned, as well as its striking systematic aspects. Yet it is impossible to proceed right now with the discussion without noting a logical difficulty that our theory, as it presently stands, involves. Given that there are two sets of IbCs along each of the five columns of Figure 5-2, yielding two "rival" altered scale steps as corresponding to each of the five chromatic PbCs (where IbC6 belongs to both sets), it is obvious that each of the two altered scale steps taken in relation to the seven diatonic PbCs will yield a set of altered diatonic intervals (in addition to a set of unaltered diatonic intervals), corresponding, specifically, to the set of IbCs which yield its rival altered scale step. For example, $b\hat{5}$ in relation to the seven diatonic PbCs yields two unaltered diatonic intervals, a minor second in relation to diatonic PbC5, and a diminished fifth in relation to diatonic PbC0; these correspond to the IbC1 and IbC6 which have yielded $b\hat{5}$ in the first place (see Figure 5-2). But in relation to the diatonic PbCs 7, 2, 9, 4, and 11, $b\hat{5}$ yields altered diatonic intervals, namely, d1, d4, d7, d3 and d6, respectively; these correspond to the set of IbCs (exclusive of IbC6) which yield $\sharp\hat{4}$. The problem in this lies in the fact that we have assumed that the mind assumes that diatonic/chromatic IbCs correspond, and only correspond, to unaltered diatonic intervals (the assumption of diatonic IbC

processing). Thus, a mind that actually operates on the basis of such an assumption is ultimately forced to contradict itself. I have noted at the outset, however, that the mind's second assumption has been formulated in simplified terms; let me now, therefore, present the unsimplified version of this assumption.

Active and passive IbC processing. Let us define active IbC processing as a process by which on the basis of being given the corresponding (altered) scale step of one of two PbCs, and the IbC that they define, the corresponding (altered) scale step of the other PbC is derived:

given (altered) scale step, IbC $\longrightarrow$ derived (altered) scale step.

The given scale step is defined, recursively, in the same terms. That is to say, the given scale step is itself a derived scale step, given another scale step, and so on, until the given scale step can no longer be derived from another scale step, and is simply assumed; the order in which one scale step is derived from another is arbitrary. Let us define passive IbC processing as a process by which on the basis of being given the corresponding (altered) scale steps of two PbCs, the (altered) diatonic interval that corresponds to the IbC they define is derived:

two (altered)
scale steps $\longrightarrow$ (altered) dia-
tonic interval.

In active IbC processing IbCs are processed, and only processed, as unaltered diatonic intervals; in passive IbC processing, on the other hand, IbCs are processed as unaltered or altered diatonic intervals, as the case may be. Assuming that, ultimately, no way other than active IbC processing exists to derive (altered) scale steps, passive IbC processing is clearly dependent upon active IbC processing. On the basis of this definition it is apparent that Figure 5-2 depicts active IbC processing. The corresponding scale steps of the diatonic PbCs are given, by definition, while those of the chromatic PbCs are derived (note, however, that the definition of active IbC processing applies to chromatic as well as diatonic/chromatic IbCs; it should therefore be noted that the former are ignored, for the time being). What has been described earlier in connection with b^2 is passive IbC processing; here the corresponding (altered) scale steps of two PbCs (a diatonic PbC and a chromatic PbC) are given. Note that there is nothing problematic about the possibility that active and passive IbC processing conflict with each other.

It is important to point out that the distinction between active and passive IbC processing has not been introduced here ad hoc. The distinction applies to diatonic as well as chromatic and enharmonic performance, even though its relevance is

felt most strongly in conjunction with chromatic and enharmonic performance. Taking a set of diatonic PbCs, or any subset thereof, it is clear that practically no enharmonic conflicts would arise in conjunction with active IbC processing; this is guaranteed by the cognitive/perceptual definition of diatonicism, upon which the definition of active IbC processing is based. As a result, active and passive IbC processing practically coincide, and there is hardly a need to evoke the distinction between them.³

Returning to chromatic and enharmonic performance, we can now apply passive IbC processing systematically to the altered scale steps that have been derived in Figure 5-2 through active IbC processing in relation to the seven diatonic PbCs. In Figure 5-3 there are ten rows, labeled for each of the five lowered and five raised scale steps, and seven columns, labeled for the seven (unaltered) scale steps. The perceptual/physical level is understood, even though it is not explicitly represented, yet no assumption is made in connection with (implied) IbCs regarding direction; for this reason two "octave complementary" (altered) diatonic intervals appear at each intersection. As can be seen, passive IbC processing yields in this case, in addition to unaltered diatonic intervals, five pairs of altered diatonic intervals (five absolute altered diatonic intervals), all of which are singly altered (see the highlighted areas in Figure 5-3).

	$\hat{4}$	$\hat{1}$	$\hat{5}$	$\hat{2}$	$\hat{6}$	$\hat{3}$	$\hat{7}$
$b\hat{5}$	m2/m7	A4/d5	A1/d1	d4/A5	A2/d7	d3/A6	A3/d6
$b\hat{2}$	M3/m6	m2/M7	A4/d5	A1/d1	d4/A5	A2/d7	d3/A6
$b\hat{6}$	m3/m6	M3/m6	m2/M7	A4/d5	A1/d1	d4/A5	A2/d7
$b\hat{3}$	M2/m7	m3/m6	M3/m6	m2/M7	A4/d5	A1/d1	d4/A5
$b\hat{7}$	P4/P5	M2/m7	m3/m6	M3/m6	m2/M7	A4/d5	A1/d1
$\#4$	A1/d1	A4/d5	m2/M7	M3/m6	m3/m6	M2/m7	P4/P5
$\#1$	d4/A5	A1/d1	A4/d5	m2/M7	M3/m6	m3/m6	M2/m7
$\#5$	A2/d7	d4/A5	A1/d1	A4/d5	m2/M7	M3/m6	m3/m6
$\#2$	d3/A6	A2/d7	d4/A5	A1/d1	A4/d5	m2/M7	M3/m6
$\#6$	A3/d6	d3/A6	A2/d7	d4/A5	A1/d1	A4/d5	m2/M7

Figure 5-3. Passive IbC processing applied to the altered scale steps of Figure 5-2 in relation to the unaltered scale steps.

Once again: the role of context. According to the theory of chromatic and enharmonic performance I have proposed each of the five chromatic PbCs corresponds to two altered scale steps at the same time, where each altered scale step carries a certain weight in relation to its enharmonically equivalent "rival," as expressed in the ratio of IbCs by which each is derived through active IbC processing (see Figure 5-2). Thus, $\sharp 4$ and $\flat 7$ carry the greatest weight, 6:2, in relation to their enharmonics, $\sharp 1$ and $\flat 3$, respectively, the ratio in favor of $\sharp 1$ and $\flat 3$ versus $\flat 2$ and $\sharp 2$, respectively, is somewhat more balanced, 5:3, whereas $\sharp 5$ and $\flat 6$ are equally weighted, 4:4.

The implication of all this is that enharmonicism is inseparable from chromaticism; as soon as one adds chromatic PbCs to the set of diatonic PbCs, enharmonic conflicts necessarily arise. One must keep in mind, however, that the theory of chromatic and enharmonic performance proposed does not take contextual considerations into account. The effect of active IbC processing has been described as if all diatonic/chromatic IbCs are perceived on an equal basis. In reality however, contextual considerations render some IbCs more prominent than others, so that more often than not, one of the two enharmonic alternatives is suppressed, at least to some degree, in favor of the other. A theory of chromatic and enharmonic performance that wishes to take context into account could define the notion "contextually-favored IbC" (or "relatively contextually favored IbC") interdependently with the higher levels of tonal structure,

stating that such IbCs carry a greater weight in active IbC processing. As has been already noted, the issue of "context" (the higher levels of tonal structure) lies outside the scope of this study (for a partial discussion, see Chapter 6). I will, nonetheless, present a few examples of what I believe are contextual considerations of the sort a "context-sensitive" theory of chromatic and enharmonic performance would have to take into account.

Consider once again Figure 5-1 for a moment, in particular, the PbC10 in m. 1. Surely, this PbC10 is even less of an A# ($\hat{6}$) than the ratio 2:6 depicting the "weight" of A# relative to B $\flat$ ($\hat{b7}$) would have it. The point is, however, that not only are the IbCs that yield the Bb contextually preferred (on harmonic and melodic grounds), but also, the only diatonic/chromatic IbC that yields the rival A# unambiguously, the IbC1 in relation to B, is contextually unpreferred; a nearby B $\sharp$ ($\hat{7}$) is simply nowhere to be found. As has been already indicated, the bracketed segment of which the PbC10 is part is also perceived in terms of F major. From an F-major point of view, the PbC10 is of course a diatonic PbC, corresponding to $\hat{4}$. The PbC8 in the second measure is an interesting case. Here the mind may mistake the PbC8 at first for a triadic tone (the third of a minor subdominant), in which case the harmonic IbCs (PbC8 in relation to F and C) are preferred, strengthening the A $\flat$ interpretation; as soon as the A $\sharp$ ($\hat{6}$) enters, however, the PbC8 must be reinterpreted as a melodic element relating to

to the A, in which case the melodic IbC involved is preferred, strengthening the G# interpretation.

Figure 5-4a depicts the opening measures of the third movement of Schumann's *Fantasie*, Op. 17, and Figure 5-4b is a reduction of the first three. Note the PbCl in the second measure. Schumann's notation (C#) expresses its harmonic contextual aspect. At the same time, however, it has a melodic contextual aspect (see the connecting beam in Figure 5-4b) involving the enharmonic rival, namely D $\flat$, which although not notated by Schumann, is quite audible. That is to say, there is a hidden enharmonic conflict here that results from two different kinds of contextual preferences. It is suggestive that in the coda of this movement, based on the opening measures, the "same" C# is enharmonically "transformed" into a D $\flat$ (Figure 5-4c).

d.

Langsam getragen J. 66
Durchweg leise zu halten

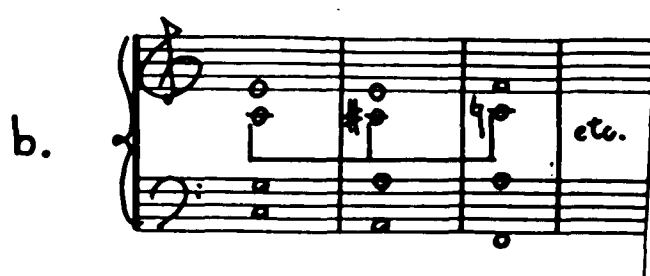


Figure 5-4. Schumann, Fantasie Op. 17, third movement; a) opening measures; b) reduction of mm. 1-3; c) from the coda.

Some residual matters. I have already noted that the definition of active IbC processing applies not only to diatonic/chromatic IbCs, that is, IbCs defined by a pair of PbCs where one is diatonic while the other is chromatic, but also to chromatic IbCs, that is, IbCs defined by a pair of chromatic PbCs (recall the recursive aspect of active IbC processing); yet

I have developed a theory of chromatic and enharmonic performance based on the idea of active IbC processing which essentially ignores chromatic IbCs altogether. I will now show that this omission is of no serious consequence.

The issue that is involved here can be perhaps better appreciated in consulting Figure 5-5. Figure 5-5a corresponds to the process that is depicted in Figure 5-2. There are five chromatic PbCs ("C"), and, for the sake of simplicity, only one diatonic PbC ("D"); the arrows indicate the direction of derivation through active IbC processing. Figure 5-5b depicts derivations of the type we have so far ignored (for the sake of simplicity again, only one of the five derivations of Figure 5-5a is followed, and in only one of many possible "routes"); we must find now how these added derivations effect chromatic and enharmonic performance in general.

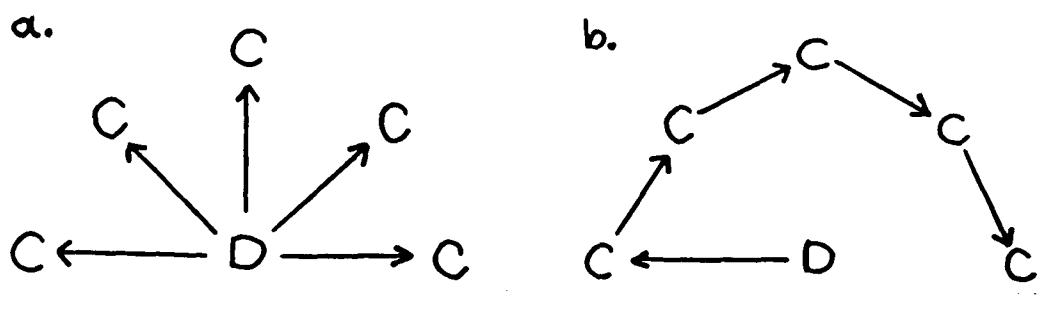


Figure 5-5. a) Active IbC processing that is restricted to the diatonic/chromatic IbCs; b) active IbC processing that includes the chromatic IbCs as well.

Since active IbC processing ultimately depends on some PbC whose corresponding (altered) scale step is given (rather than derived through active IbC processing), it is clear that processing the chromatic IbCs (Figure 5-5b) ultimately depends on processing the diatonic/chromatic IbCs (Figure 5-5a), since only here the corresponding scale step of one PbC (the diatonic PbC) is given, by definition. Thus, derivations of the sort depicted in Figure 5-5a, with which we have been originally exclusively concerned (Figure 5-2), must serve, at the very least, as a point of departure for chromatic and enharmonic performance in general.

Now, it is not difficult to see that for any of the five altered scale steps that belong to either the "sharp group" or the "flat group," as derived in terms of Figure 5-5a (see Figure 5-2), derivations in terms of Figure 5-5b simply yield the remaining four. For example, if the derivation "D" $\longrightarrow$ "C" in Figure 5-5b yields, say, a corresponding $b\hat{5}$, then the derivations "C" $\longrightarrow$ "C" would yield $b\hat{2}$, $b\hat{6}$, $b\hat{3}$, and $b\hat{7}$, in whatever order they are taken; no enharmonic conflicts would arise (and therefore, active and passive IbC processing coincide), since the set of chromatic PbCs corresponds to a subset of a diatonic set. That is to say, for any given flat scale step the other four are derived, and similarly, for any given sharp scale step. No new altered scale steps are derived, and no enharmonic conflicts arise, other than those that are already inherent in active derivations of the type depicted in Figure

5-5a (see Figure 5-2).

If one counts all the flat derivations vs. all the sharp derivations in this way, the two balance one another. Glancing back at Figure 5-2, one can read across the top row all the derivations in relation to $\hat{4}$. As can be seen, there are five derivations, all of which are flat; these are balanced by the five sharp derivations in relation to $\hat{7}$ (bottom row). The five flat derivations and one sharp derivation in relation to $\hat{1}$ are balanced by the five sharp derivations and one flat derivation in relation to $\hat{3}$, and similarly, derivations in relation to $\hat{5}$ balance derivations in relation to $\hat{6}$, while derivations in relation to $\hat{2}$ balance one another. This seems to lead to the unexpected conclusion, in contradiction to an earlier one, that any given pair of enharmonic "rivals" are equally weighed. Thus, for example, the $\flat\hat{5}$ derived in relation to $\hat{4}$ (top left corner in Figure 5-2) is balanced by a $\sharp\hat{4}$ that is derived, say, from the $\sharp\hat{6}$ that is derived, in turn, from $\hat{7}$ (bottom right corner in Figure 5-2), and so on, for any derived $\flat\hat{5}$, whether directly from a diatonic PbC or indirectly from a chromatic one. The key words here, however, are "directly" and "indirectly." Since indirect derivations (from one chromatic PbC to another), as has been already noted, are ultimately dependent upon direct derivations (from a diatonic PbC to a chromatic one), the latter necessary have a preferred status. That is to say, diatonic/chromatic IbCs are inherently preferred

relative to chromatic IbCs since the mind must actively seek the former in order to deal with the latter. As with contextually preferred IbCs, derivations in this case carry a greater weight. We have thus come a full circle back to our point of origin, namely, derivations from a diatonic PbC to a chromatic one (see Figure 5-5a), as systematically pursued in Figure 5-2. In context-free terms at least, derivations from one chromatic PbC to another can be essentially ignored.

We have not yet considered the issue of passively processing the chromatic IbCs, given the altered scale steps of Figure 5-2. Given that we are dealing with relatively unpreferred IbCs, the idea of passive processing is not quite applicable (this is true with contextually unpreferred IbCs as well). How can one apply passive processing to IbCs that are not even "important" enough to be processed actively? Consequently, passive processing of chromatic IbCs is ultimately tied in with contextual considerations, through which chromatic IbCs can attain a preferred status (for an example, see the final measure in Figure 5-1).

It will take me too far afield now to consider the effect of context on active IbC processing in general, that is, actively processing the chromatic as well as diatonic/chromatic IbCs, as has been previously described. I will only note that in conjunction with diatonic/chromatic IbCs, with which the earlier discussion of context was exclusively concerned, this effect can be essentially ignored, as we have seen is the case context-

free. In conjunction with chromatic IbCs as well as diatonic/chromatic IbCs, it is at least clear that whatever the effect of context may be, the altered scale steps as depicted in Figure 5-2 remain unchanged. I can therefore allow myself to present the passive processing of chromatic IbCs without a discussion of context (Figure 5-6), a topic which anyhow, as has been already mentioned, lies outside the scope of this study. Since either the "sharp group" or the "flat group" passively yields, within itself, only unaltered diatonic intervals, I ignore this possibility, and concentrate instead on intervals composed of one altered scale step from each group; as in Figure 5-3, no assumption is made regarding direction, so that two "octave complementary" altered diatonic intervals result in each case. As can be seen, nine pairs of altered diatonic intervals are processed in this way, five of which (d2/A7, dd5/AA4, dd1/AA1, dd4/AA5, and AA2/dd7--see the highlighted area in Figure 5-6), have not appeared in Figure 5-3 (which deals with passively processing diatonic/chromatic IbCs), and include four pairs of doubly altered diatonic intervals.

	$b\hat{7}$	$b\hat{3}$	$b\hat{6}$	$b\hat{2}$	$b\hat{5}$
$\#6^{\wedge}$	d2/A7	AA4/AA5	AA1/AA1	AA4/AA5	AA2/AA7
$\#2^{\wedge}$	A3/d6	d2/A7	AA4/AA5	AA1/AA1	AA4/AA5
$\#3^{\wedge}$	d3/A6	A3/d6	d2/A7	AA4/AA5	AA1/AA1
$\#1^{\wedge}$	A2/d7	d3/A6	A3/d6	d2/A7	AA4/AA5
$\#4^{\wedge}$	d4/AS	A2/d7	d3/A6	A3/d6	d2/A7

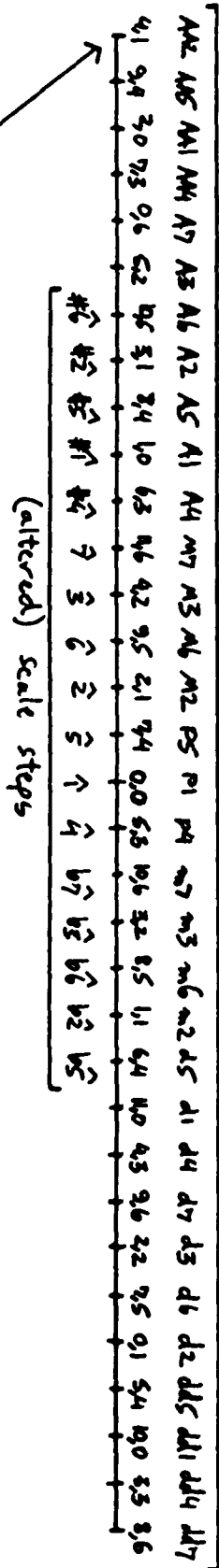
Figure 5-6. Passive IbC processing among the altered scale steps of Figure 5-2.

Finally, Figure 5-7 depicts a segment of the mod-(12,7) system at large, arranged by fifths, and interpreted as the (altered) scale steps and diatonic intervals which constitute a chromatically extended key. As with an unextended key, the (altered) scale steps constitute a subset of the (altered) diatonic intervals. The arrangement by fifths follows from active IbC processing; since altered scale steps derive from (unaltered) scale steps plus (unaltered) diatonic intervals, both of which are structured by fifths, altered scale steps (and consequently, altered diatonic intervals) are structured in the same way. The arrangement by fifths of Figure 5-2 re-

sults in a striking systematic aspect which simply reflects the distance in fifths of altered scale steps from diatonic scale steps, as I believe, in conjunction with Figure 5-7, is readily apparent.

Figure 5-7 (on p. 207). A segment of the mod-(12,7) system at large, arranged by fifths, interpreted as a chromatically extended major key.

(altered) diatonic intervals



multiples of
 $\pm(5,3) \bmod (12,7)$

NOTES

1. That this elimination indeed corresponds to musical reality is, I believe, readily apparent. There is however, apparently at least, one notable discrepancy. In minor, one encounters from time to time a "leading tone to the leading tone" (in D minor, B $\sharp$), an altered scale step that is, of course, enharmonically equivalent to an unaltered scale step, namely 7. But clearly, this situation involves the "harmonic minor scale," which we have not considered so far, and which, in fact, we will only touch upon in Chapter 6; the "harmonic minor" engages, as will be explained, levels of tonal structure which lie beyond the scope of this study.

2. Recall that the PbC5 and PbC6 of our example are, in reality, pitch bands, that is, they exist each in a specific register, so that the interval band between them can be measured in a specific direction, in which case interval-band class membership is unambiguous; recall the definitions of "pitch band," "interval band," "pitch-band class," "interval-band class," and "absolute interval-band class" given in Chapter 3, section 3.7.

3. Note the qualifications "hardly" and "practically." Even within a set of diatonic PbCs an enharmonic conflict arises in conjunction with active IbC processing in the case of IbC6; as a result, active and passive IbC processing do not fully coincide. Actively, an IbC6 is processed as A $_4$ as well as d $_5$, while passively it is processed as either A $_4$ or d $_5$ (see, in this connection, note 4 in the "Interlude" that follows Chapter 3). Thus, strictly speaking, the distinction between active and passive IbC processing is a necessary one even in conjunction with purely diatonic performance.

The main point I am driving at, however, is that chromatic and enharmonic performance, as well as diatonic performance, can be embraced within the framework of a single theory, which seems to me an extremely attractive possibility. Such a combined theory of performance would be based on the assumption of diatonic and chromatic PbCs, that is, the PbCs that correspond to unaltered scale steps correspond only to unaltered scale steps, and the assumption of active and passive IbC processing. Viewed in these terms, purely diatonic performance is simply a special case of performance in general; it is not necessary to assume diatonic and chromatic PbCs since only unaltered scale steps (that is, diatonic PbCs) are given, and it is not necessary to distinguish between active and passive IbC processing since the two practically coincide (moreover, it is not

necessary to specify that, actively, IBCs correspond only to unaltered diatonic intervals, since only unaltered diatonic intervals are given). It is also unnecessary to break the mod-12 value t into the sum (b+a₁) since, by definition, b=t, or a₁=0.

Chapter VI

EPILOGUE:

TOWARDS TRIADIC TONALITY

In this short, concluding chapter I make good a long-standing obligation to define the notion "tonic," a notion that was simply assumed in the previous chapters in connection with defining "diatonic" and "tonal" interdependently (see Chapter 1, section 1.4). I define "tonic" on the basis of defining "triad" and "consonance" and "dissonance" first. These definitions (of "triad," "consonance" and "dissonance" and "tonic"), although not stated formally, constitute a beginning of a theory of tonal structure.

Triad. The notion "triad" is implicit in the structure of the diatonic system. It can be easily shown that the perfect and diminished fifths are the only diatonic intervals that can be recursively subdivided down to the level of the major and minor seconds; the diminished fifth divides into two minor thirds, which divide, in turn, into a major and minor second, while the perfect fifth divides into a major and minor third, which divide, in turn, into two major seconds and a major and minor second, respectively. Note the hierarchical aspect of

the subdivision: fifths, thirds, seconds. In particular, thirds divide fifths, and as such are "harmonic" intervals; this level defines the notion "triad." Seconds, in turn, divide thirds, and as such are "melodic" intervals. There are six "major" and "minor" triads within the tonal system, corresponding to the six perfect fifths, and one "diminished" triad, corresponding to the one diminished fifth. Since the seven scale steps of the diatonic scale can be arranged as a cycle of perfect fifths, one can construct a cycle of alternating major and minor thirds, or interlocking major and minor triads, that exhausts the diatonic scale (Figure 6-1).

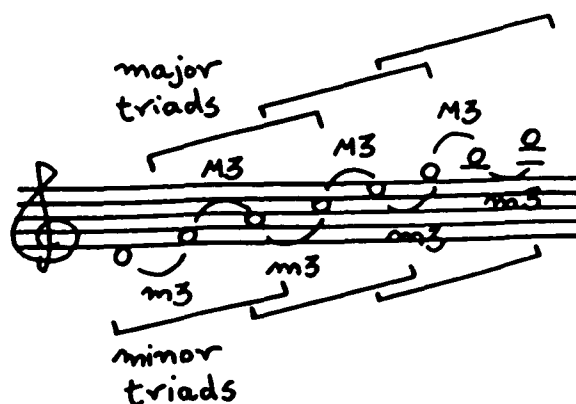


Figure 6-1. The diatonic system as a cycle of alternating major and minor thirds, or interlocking major and minor triads.

That the notion "triad" coincides in all instances but one with a divided perfect fifth is extremely important, given the special status of the perfect fifth--the fifth relationship--within the diatonic system. The triad thus provides a

smooth connection between the diatonic and tonal levels.

Consonance and dissonance. The notion "triad," in conjunction with the special systemic status of the perfect fifth at the diatonic level, leads directly to the notions "consonance" and "dissonance." All the intervals of a major or minor triad, that is, a divided P⁵, are defined as "consonant"; all other intervals are "dissonant" by definition.¹ The hierarchical division of consonances into "perfect" and "imperfect" consonances reflects the special systemic status of the octave and fifth at the diatonic level.

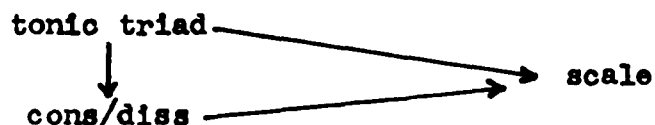
Note that the perfect fourth according to this definition is consonant. Surely, the fourth is dissonant only in conjunction with contextual considerations, that is, in conjunction with higher levels of tonal structure. If one wishes, one can define "consonance" and "dissonance" interdependently with the higher levels of tonal structure, so as to capture the ambiguous nature of the perfect fourth from the very outset. Be that as it may, I believe that taking the fourth as fundamentally consonant, and dissonant only under special circumstances, is empirically a sound position. A phenomenon such as the "consonant six-four chord" would have been difficult to account for had the fourth not been defined as consonant long before such notions as "bass part" and "upper parts" are defined. And no other dissonant interval shares the analogous idiosyncracies; a second, for example, is always dissonant, regardless of contextual considerations.

Tonic. Gregory Proctor offers in Godfrey Winham's name the formulation that

Any major or minor triads's scale contains the notes of the tonic triad and four additional notes which both
 1) form adjacencies to the notes of the tonic triad and
 2) are consonant with those tonic triad notes to which they are not adjacent,

and goes on to say: "Under these conditions the major triad has the ordinary major scale proper to it and the minor triad has the natural minor scale, and no other."²

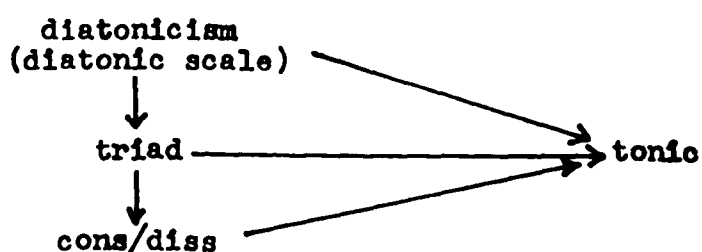
Winham-Proctor's ingenious idea is to derive the major and minor scale from the major and minor tonic triad, as follows:



that is, to derive consonance and dissonance from the major and minor triads (an idea that I have adopted), and then to derive the major and minor scale from both.

But clearly, such a derivation is hopelessly circular; for how would one define the notion "tonic triad," or just "triad," apart from defining first a diatonic system of scale steps and diatonic intervals along the lines proposed in this study? What is a triad if not a subset of the seven scale steps, with the corresponding subset of the thirteen diatonic intervals? And how would one derive "consonance" and "disso-

nance" from "triad" (however the latter is defined), without defining "diatonic interval" first? But we have already defined diatonicism independently of the Winham-Proctor idea; we can therefore use this idea in an inverted form to define the notion "tonic":



Among the six major and minor triads (that is, the triads that correspond to a divided P5) only two triads have the property that their tones form only consonances with the other tones of the diatonic scale, adjacencies excluded (Figure 6-2a); the "roots" of these triads are defined as "tonics," yielding two alternative "tonal" modes, the major mode and the natural minor mode. Since other than adjacencies (that is, seconds and sevenths) the only dissonances that remain to be reckoned with are the augmented fourth and diminished fifth, it is clear that what is special about these two triads is that none of their tones form the dissonance of an A₄ or d₅ in relation to any of the other tones of the scale; in other words, these are the only major and minor triads in the diatonic system that do not contain either of the two tones that form an A₄ or d₅ with each other (the tones "B" and "F" in the "white-key system"). As

can be seen in Figure 6-2b, the augmented fourth or diminished fifth that appear in relation to any of the other major or minor triads are destabilizing elements which ultimately render these triads unfit to serve as "tonics."



Figure 6-2. a) The consonant relationship of the tonic triad in major and minor to the non-adjacent scale members; b) the dissonant A_4/d_5 in relation to the other four major and minor triads.

Note that the notion "tonic" divides the seven diatonic triads, hierarchically, into three groups. The first group includes the diminished triad, which being dissonant is excluded from "tonic-candidacy" from the very outset. The remaining six consonant triads divide into a group of four triads which form a dissonant interval (A_4/d_5) with some non-adjacent scale member, and a group of two triads which form consonances only with all non-adjacent scale members, as discussed above. If the latter group defines the notion "tonal," the former group defines the notion "modal"; "modal" can thus be interpreted as "partial-tonal," in the sense of involving at the very least the requirement for a consonant "tonic." Note that the "Loc-

rian mode," based on the diminished triad, is "non-tonal" (and therefore "non-modal" as well); indeed, it exists in theory only.

In processing some scale step as a tonic the mind must check the intervallic relationships of its corresponding triad (assuming that it is not the diminished triad) to the non-adjacent tones of the scale, verifying that there is no $A\sharp$ or $d\sharp$. Clearly, the mind must employ the strategy of diatonic IbC processing. Thus, as has been noted in the "Interlude" that follows Chapter 3, the strategy of diatonic IbC processing must ultimately bear the burden of diatonic performance.

Higher levels of tonal structure: functional harmony.

The next steps, it seems to me, that a theory of tonal structure must undertake, is to define the notion "functional harmony"; tonal theory must account for the special qualities of such progressions as VII-I, V^7 -I, IV-V-I, and II-V-I. Needless to say, defining "functional harmony" is a major theoretical challenge, for the notion involves the interaction of many different factors, among them rhythm (even in the most elementary sense of "before" and "after"), voice leading, and dissonant treatment; to enter this highly complex theoretical domain requires a study in its own. One comment, however, should be made in the context of the present study. I believe it is in connection with "functional harmony" that the Aeolian mode produces its chromatic variants, in particular, the familiar "harmonic minor" scale. Specifically, it seems to me, the harmonic minor is a product

of a conflict between two levels of mental representation, the "tonal" level as has been previously discussed, where the tonic triad is constrained to participate in only consonant relationships with all non-adjacent scale members, and the level of "functional harmony," which imposes further constraints on the notion "tonic," such as the half-step relationship $\hat{7}-\hat{1}$.

Thus, there is still a great distance to cover along the road towards "triadic tonality." But I hope to have provided in this study at least a secure point of departure in defining the notion "diatonic," and some ideas as to how one might continue beyond the diatonic level in defining such notions as "triad," "consonance" and "dissonance," and "tonic." Further research may bring us closer to the ultimate goal.

NOTES

1. See Proctor 1977, pp. 10-11.
2. Ibid., p. 11.

APPENDIX A

DIATONICISM AS A COGNITIVE SYSTEM:

A FORMAL DEFINITION

I. Basic Components.

1) "Octave Equivalence."

Definition 1. Mod $(\underline{x}, \underline{y})$, where $\underline{x}$ and $\underline{y}$ are natural numbers larger than 1, and (let us say) $\underline{x} > \underline{y}$. For reasons that will become apparent in connection with definition 2 ("fifth relationship"), we ask, in addition, that $\underline{x}$ and $\underline{y}$ be coprime.

Corollary. $\underline{x} \geq 3, \underline{y} \geq 2$.

This definition and its corollary require, I believe, no special comment.

2) "Fifth Relationship."

Definition 2. An integer pair, $\pm(q, r) \bmod (\underline{x}, \underline{y})$, where q is prime to $\underline{x}$ and r is prime to $\underline{y}$, yet both are different from ± 1 in mod $\underline{x}$ and mod $\underline{y}$, respectively.

Corollary. $\underline{x} \geq 7, \underline{y} \geq 5$.

Discussion. Formally speaking, "fifth relationship" is an integer pair, mod $(\underline{x}, \underline{y})$, that generates through multiplication both "diatonic intervals" and "scale steps" (see definitions 3 and 4). In order for the "fifth relationship"

to be a structurally interesting generator, it must be capable of generating all the values in mod $\underline{x}$, on the one hand, and mod $\underline{y}$, on the other. Taking $\underline{x}=12$ as an example, it is clear that certain mod-12 values are incapable of generating all the mod-12 values when multiplied by integers, mod 12; 4, for example, generates only three different mod-12 values, namely 0, 4, and 8, and thus cannot serve as a "fifth" with respect to mod 12. The problem with 4, quite obviously, is that it shares a divisor with 12, namely 4, which is to say, it is not prime to 12 (alternatively, 4 and 12 are not coprime). Hence the requirement that $\underline{q}$ and $\underline{r}$, which define "fifth relationship," be prime to $\underline{x}$ and $\underline{y}$, respectively. We exclude the integer 1 from consideration since it is prime to any integer, and thus is not a structurally interesting case (note also that if $\underline{q}$ and/or $\underline{r}$ equal 1, the scalar arrangement of diatonic intervals and scale steps is the same as the "cycle-of-fifths" arrangement, with respect to the mod- $\underline{x}$ and/or mod $\underline{y}$ values); the corollary to definition 2 derives from this restriction.

Since $\underline{x}$ and $\underline{y}$ are coprime (recall the definition of "octave equivalence"), $\pm(\underline{q}, \underline{r})$ potentially generates not only all the values in mod $\underline{x}$, on the one hand, and mod $\underline{y}$, on the other, but all the $\underline{x} \cdot \underline{y}$ different mod- $(\underline{x}, \underline{y})$ pairs (for example, there are 84, or $12 \cdot 7$, different integer pairs mod-(12,7), as discussed in Chapter 4). This structurally interesting situation is not possible with an $\underline{x}$ and $\underline{y}$ that are not co-

prime, as the reader can easily verify for himself by means of an example (say, 12 and 4 or 12 and 8).

3) "Diatonic Intervals."

Definition 3. A set of integer pairs representing products, mod $(\underline{x}, \underline{y})$, of the "fifth relationship" $\pm(\underline{q}, \underline{r})$ and each of a set of $2\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$ (i.e., $-\underline{N}, -\underline{N}+1, -\underline{N}+2, \dots, \underline{N}$), where $\underline{N}$ is a natural number and equals at most the integral part of half $\underline{x} \cdot \underline{y}$.

Discussion. Since diatonic intervals ultimately represent differences between pairs of "scale steps," or mod- $(\underline{x}, \underline{y})$ integer pairs, they must be generated symmetrically around $(0,0)$; clearly, each interval must have a symmetrically relating "octave complement." We therefore multiply the fifth relationship by a set of consecutive integers (henceforth: "multipliers") between $-\underline{N}$ and $\underline{N}$, representing a symmetry around 0. Since there are no more than $\underline{x} \cdot \underline{y}$ different integer pairs, mod $(\underline{x}, \underline{y})$ (for example, there are no more than twelve times seven, or eighty four, different integer pairs mod $(12,7)$, as discussed in Chapter 4), against $2\underline{N}+1$ multipliers (for example, for $\underline{N}=3$ there are seven multipliers, $-3, -2, -1, 0, 1, 2, 3$), $2\underline{N}+1$ will naturally not be larger than the smallest value that generates all the mod- $(\underline{x}, \underline{y})$ pairs. That is to say, if $\underline{x} \cdot \underline{y}$ is odd, $2\underline{N}+1$ (an odd number) will equal $\underline{x} \cdot \underline{y}$, whereas if $\underline{x} \cdot \underline{y}$ is even, $2\underline{N}+1$ will be larger than $\underline{x} \cdot \underline{y}$ by only 1; from this it directly follows that $\underline{N}$ is not larger than the integral part (see Chapter 2,

note 20) of half $\underline{x} \cdot \underline{y}$.

Note that this definition does not guarantee that the set of diatonic intervals contain all the mod- $\underline{x}$ values (or even all the mod- $\underline{y}$ values), nor does it say anything regarding multiple recurrences of mod- $\underline{x}$ values (and of course, mod- $\underline{y}$ values) within the set.

4) "Scale Steps."

Definition 4. A set of integer pairs representing products, mod $(\underline{x}, \underline{y})$, of the "fifth relationship" $\pm(q, r)$ and each of a set of $\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$.

Discussion. This definition guarantees, for any value of $\underline{N}$, a set of $\underline{N}+1$ "scale steps," or mod- $(\underline{x}, \underline{y})$ integer pairs, of which the set of all differences between ordered pairs is congruent, mod $(\underline{x}, \underline{y})$, with the set of diatonic intervals, not counting multiplicities; the set of scale steps has $(0, 0)$ as a member, and is a subset of the set of diatonic intervals.

Clearly, it is enough to show that the same properties are true with respect to the $\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$ (that is, the set of integers which, multiplied by the "fifth relationship," yield the set of scale steps) in relation to the full set of integers between $-\underline{N}$ and $\underline{N}$ (that is, the set of integers which, multiplied by the "fifth relationship," yield the set of diatonic intervals); one must show, in other words, that by subtracting, in turn, two integers from a set of $\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$, through

all possible combinations, one gets the set of integers $-\underline{N}, -\underline{N}+1, \dots, \underline{N}$, all and only (multiplicities are ignored). Since the demonstration is very easy, I shall leave it out; the reader may wish to convince himself of the validity of the property involved by means of an example.

Note that this definition does not guarantee that the scale-step set contain all the mod- $\underline{y}$ values (and of course, all mod- $\underline{x}$ values), nor does it say anything regarding multiple recurrences of mod- $\underline{y}$ (and even mod- $\underline{x}$) values within the set.

II. Systemic Constraints and Corresponding Theorems.

1) The Property of "Mod-($\underline{x}, \underline{y}$) Correspondence":

The set of diatonic intervals contains all the mod- $\underline{x}$ values, and each mod- $\underline{x}$ value corresponds to only one mod- $\underline{y}$ value; when $\underline{x}$ is even there is necessarily one exception, involving the mod- $\underline{x}$ value $\underline{x}/2$ (see corollary A). The set of scale steps contains all the mod- $\underline{y}$ values, and each mod- $\underline{y}$ value corresponds to only one mod- $\underline{x}$ value; since scale steps are a subset of the diatonic intervals (definition 4), it is clear that the mod- $\underline{x} \longrightarrow$ mod- $\underline{y}$ correspondence applies to them as well, which is to say, scale steps embody a one-to-one correspondence between mod $\underline{x}$ and mod $\underline{y}$.

Theorem 1.

- a) $\underline{y}$ equals the integral part of $\underline{x}/2$, plus 1;
- b) $\underline{N} = \underline{y} - 1$.

Corollary A. When $\underline{x}$ is even, there is one mod- $\underline{x}$ value within the set of diatonic intervals, corresponding to $\pm \underline{N} \cdot \underline{q} \bmod \underline{x}$ and equalling $\underline{x}/2$, that corresponds to two different mod- $\underline{y}$ values, specifically, $\underline{r}$ and $-\underline{r}$.

Corollary B. Within the set of diatonic intervals, all mod- $\underline{y}$ values except 0 correspond to two different mod- $\underline{x}$ values.

Corollary C. From a mod- $\underline{x}$ point of view, the set of scale steps is a "deep scale": each abs. IC (mod $\underline{x}$) occurs a different number of times, from once to $(\underline{y}-1)$ times.

Proofs. Theorem 1. Diatonic intervals are generated by multiplying the "fifth relationship" by $2\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$ (definition 3). This means that $2\underline{N}+1$ must be the smallest value that generates all the values in mod $\underline{x}$; clearly, a larger number of multipliers will cause the mod- $\underline{x}$ values to repeat themselves, resulting in multiple correspondences to the mod- $\underline{y}$ values, which we wish to avoid. Since there are $\underline{x}$ different mod- $\underline{x}$ values, where $\underline{x}$ is either odd or even, against an odd number ($2\underline{N}+1$) of multipliers, $2\underline{N}+1$ will equal $\underline{x}$ when $\underline{x}$ is odd, or will be larger than $\underline{x}$ by $\underline{1}$ when $\underline{x}$ is even; from this it directly follows that $\underline{N}$ equals the integral part of $\underline{x}/2$ (see also the discussion of definition 3).

Scale steps are generated by multiplying the "fifth relationship" by $\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$ (definition 4). Clearly, $\underline{y}$ must equal $\underline{N}+1$, that is (since $\underline{N}$ equals the integral part of half $\underline{x}$), the integral part of half $\underline{x}$ plus 1 (requirement (a) of theorem 1). If $\underline{y} = \underline{N}+1$, then $\underline{N} = \underline{y}-1$ (requirement (b) of theorem 1).

Corollary A. When $\underline{x}$ is even, we have $\underline{x}+1$ multipliers

generating the set of diatonic intervals ($\underline{N} = \underline{x}/2$, and therefore $2\underline{N}+1$ equals $\underline{x}+1$). Clearly, one mod- $\underline{x}$ value is generated twice, by $\underline{q} \cdot \underline{N}$ as well as $\underline{q} \cdot -\underline{N}$. Now, we know that $\underline{q}$ is an odd number, since it is prime to $\underline{x}$ (definition 2), which is even; we also know that $\underline{N} = \underline{x}/2$ (from theorem 1). $\underline{x}/2$ times any odd number, mod $\underline{x}$, equals $\underline{x}/2$, and therefore $\pm \underline{N} \cdot \underline{q}$ equals $\underline{x}/2$. The corresponding mod- $\underline{y}$ values are $\pm \underline{N} \cdot \underline{r}$, or $\pm (\underline{y}-1) \cdot \underline{r}$ (since $\underline{N}$ equals $\underline{y}-1$), or $\pm \underline{y} \cdot \underline{r} \pm \underline{r}$, or $\pm \underline{r}$ mod $\underline{y}$.

As we shall see, mod- $(\underline{x}, \underline{y})$ coherence excludes the possibility of an odd $\underline{x}$, so that an "A₄/d₅"-type ambiguity is a necessary characteristic of diatonicism.

Corollary B. Since $\underline{N} = \underline{y}-1$, the number of multipliers that yield the set of diatonic intervals, $2\underline{N}+1$, is $2 \cdot (\underline{y}-1)+1$, or $2\underline{y}-1$; clearly, these $2\underline{y}-1$ multipliers range from $-(\underline{y}-1)$ to $(\underline{y}-1)$ ($-\underline{N}$ to $\underline{N}$):

$$-(\underline{y}-1), -(\underline{y}-2), \dots, 0, \dots, (\underline{y}-2), (\underline{y}-1).$$

It is not difficult to see that when multiplied by $\underline{r}$ mod $\underline{y}$, these $2\underline{y}-1$ multipliers yield 0 only once ($0 \cdot \underline{x}$ mod $\underline{y}$). Excluding 0, there remains $\underline{y}-1$ different mod- $\underline{y}$ values, against $2\underline{y}-2$ multipliers; that is, each mod- $\underline{y}$ value except 0 is generated exactly twice. We know from theorem 1 that each mod- $\underline{x}$ value is generated only once, except for one mod- $\underline{x}$ value in the case of an even $\underline{x}$, which corresponds, however, to two different mod- $\underline{y}$ values (corollary A). Therefore, each mod- $\underline{y}$ value except 0 corresponds to two different mod- $\underline{x}$ values,

regardless of whether $\underline{x}$ is odd or even ($0 \bmod \underline{x}$ corresponds to $0 \bmod \underline{x}$).

Corollary C. Given a set of $\underline{y} \bmod \underline{x}$ values representing whole consecutive multiples of $\pm \underline{q}$, $\bmod \underline{x}$, one can map all abs. ICs onto multiples of $\pm \underline{q}$, so that each multiple of $\pm \underline{q}$ occurs in a unique multiplicity, as follows:

$(\underline{y}-1)$	multiplicities of	$\pm \underline{q}$,
$(\underline{y}-2)$	"	$\pm 2\underline{q}$,
$(\underline{y}-3)$	"	$\pm 3\underline{q}$,
$\vdots$		$\vdots$
$\vdots$		$\vdots$
$\vdots$		$\vdots$
$\vdots$		$\vdots$
1	multiplicity of	$\pm (\underline{y}-)\underline{q}$.

Assuming that all abs. ICs are represented in the set, it can be shown that each multiple of $\pm \underline{q}$ (i.e. $\pm \underline{q}$, $\pm 2\underline{q}$, $\pm 3\underline{q}$, etc.) corresponds to a different abs. IC, $\bmod \underline{x}$, if and only if $(\underline{y}-1)$ equals the integral part of $\underline{x}/2$, or $\underline{y}$ equals the integral part of $\underline{x}/2$ plus 1, which is exactly what theorem 1 states.

Note that from a $\bmod-(\underline{x}, \underline{y})$ point of view the set of scale steps is a "deep scale" regardless of theorem 1; each multiple of $\pm(\underline{q}, \underline{r})$, i.e. $\pm(\underline{q}, \underline{r})$, $\pm 2(\underline{q}, \underline{r})$, . . . $\pm(\underline{y}-1)(\underline{q}, \underline{r})$ is a different $\bmod-(\underline{x}, \underline{y})$ abs. IC as long as $(\underline{y}-1)$, or $\underline{N}$, is no larger than the integral part of half $\underline{x} \cdot \underline{y}$, which is already guaranteed by definition 3.

2) The Property of "Mod-($\underline{x}, \underline{y}$) Coherence":

Within the diatonic intervals, the mod- $\underline{x}$ values are a monotone increasing function of the mod- $\underline{y}$ values; when $\underline{x}$ is even there is necessarily one exception, involving the mod- $\underline{x}$ value $\underline{x}/2$.

Theorem 2.

$$\underline{x} \cdot \underline{r} - \underline{y} \cdot \underline{q} = 1.$$

Corollary A. Within the diatonic scale steps, the mod- $\underline{x}$ and mod- $\underline{y}$ values are monotone increasing functions of each other.

Corollary B. The smallest values available for $\underline{x}$, $\underline{y}$, $\underline{q}$, $\underline{r}$, and $\underline{N}$ are 8, 5, 3, 2, and 4, respectively; the remaining sets of values constitute an infinite sequence, where the value of $\underline{x}$ increases each time by 4, the values of $\underline{y}$, $\underline{q}$, and $\underline{N}$ by 2, and the value of $\underline{r}$ by 1.

Corollary C. Any "diatonic scale" contains two "minor seconds," or IC (1,1) mod ($\underline{x}, \underline{y}$), interrupting at two point two series of "major seconds," or IC (2,1) mod ($\underline{x}, \underline{y}$), specifically, a series of $\underline{x}/4$ and a series of $(\underline{x}/4)-1$ major seconds.

Proofs. Theorem 2. The proof of theorem 2 can be achieved by proving the following two sub-theorems first:

- (2a) Within the diatonic intervals, the two different mod- $\underline{x}$ values to which each mod- $\underline{y}$ value except 0 corresponds (see corollary B of theorem 1) are consecutive.
- (2b) Within the diatonic intervals, a value increase of 2 within the mod- $\underline{x}$ values entails a value increase of 1 within the mod- $\underline{y}$ values; when $\underline{x}$ is even there are two cases where the corresponding mod- $\underline{y}$ value increase is of 2 as well as 1.

It is not difficult to prove sub-theorem (2a) by assuming that it is not true. We want the mod- $\underline{x}$ values to be monotone increasing functions of the mod- $\underline{y}$ values. Now,

if a given mod- $\underline{y}$ value, say $\underline{Y}_1$, corresponds to two mod- $\underline{x}$ values that are not consecutive, say $\underline{X}_1$ and $\underline{X}_2$, then there exists a mod- $\underline{x}$ value between $\underline{X}_1$ and $\underline{X}_2$, say $\underline{X}_3$, that corresponds to a different mod- $\underline{y}$ value, say $\underline{Y}_2$. It is not difficult to demonstrate that regardless of whether $\underline{Y}_2$ is larger or smaller than $\underline{Y}_1$, mod- $(\underline{x}, \underline{y})$ coherence is violated (the reader may wish to verify that this is so by means of an example).

On the basis of sub-theorem (2a) and theorem 1 (especially, corollary A) one can draw the correspondence between the mod-x and mod-y values of diatonic intervals as follows (Figure Aa-1):

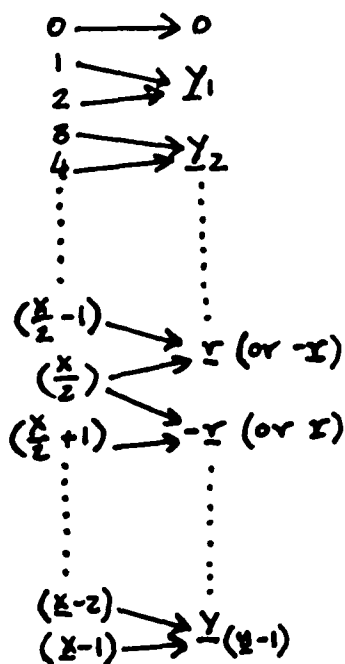
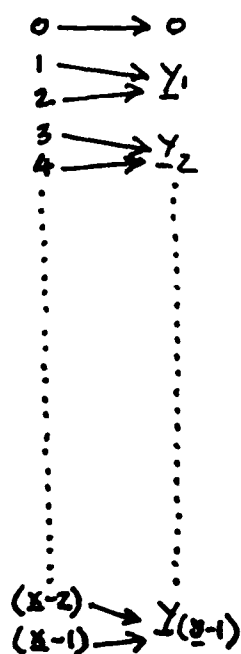


Figure Aa-1 (on p. 228). The correspondence between the mod- $\underline{x}$ and mod- $\underline{y}$ values of diatonic intervals assuming sub-theorem (2a).

Note that, at the moment, the mod- $\underline{y}$ values $0, \underline{y}_1, \underline{y}_2$, etc. are not necessarily consecutive; note also that when $\underline{x}$ is even there is necessarily one case where a value increase within the mod- $\underline{y}$ values does not entail a value increase within the mod- $\underline{x}$ values: the mod- $\underline{y}$ values $\underline{r}$ and $-\underline{r}$ correspond in two cases to the same mod- $\underline{x}$ value, namely $\underline{x}/2$.

Clearly, sub-theorem (2a) does not suffice to ensure mod- $(\underline{x}, \underline{y})$ coherence; one must also require that the mod- $\underline{y}$ values $0, \underline{y}_1, \underline{y}_2$, etc. depicted in Figure Aa-1 be consecutive. It is not difficult to see that this will be the case if one requires, in addition to (2a), that a value increase of two within the mod- $\underline{x}$ values entail a value increase of 1 within the mod- $\underline{y}$ values (sub-theorem 2b). When $\underline{x}$ is even there must be two cases where the corresponding mod- $\underline{y}$ value increase is of 2 as well as 1; see the mod- $\underline{x}$ values $(\underline{x}/2)-2$ and $\underline{x}/2$, and $\underline{x}/2$ and $(\underline{x}/2)+2$, in Figure Aa-2. (Note: it is assumed in Figure Aa-2 that $\underline{r}$ is smaller than $-\underline{r}$, mod $\underline{y}$.)

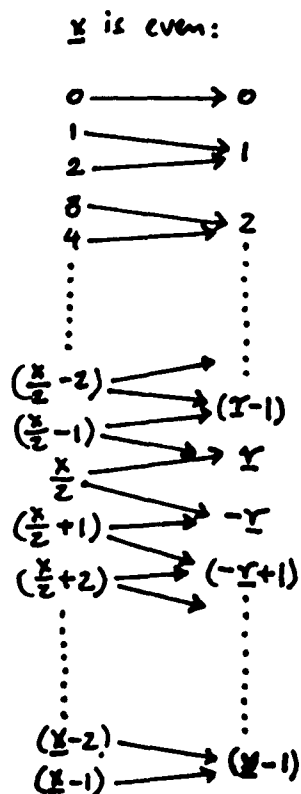


Figure Aa-2. The correspondence between the mod- $\underline{x}$ and mod- $\underline{y}$ values of diatonic intervals assuming sub-theorems (2a) and (2b), and an even $\underline{x}$.

Having proven sub-theorems (2a) and (2b), we can now proceed to prove theorem 2. According to sub-theorem (2a) a given mod- $\underline{y}$ value except 0 corresponds to two consecutive mod- $\underline{x}$ values. Since a progression of $\underline{y}$ fifths brings one back to the same mod- $\underline{y}$ value ($\underline{x} \cdot \underline{y} = 0 \bmod \underline{y}$), it is clear that sub-theorem (2a) is true if and only if

$$\underline{g} \cdot \underline{y} = \pm 1 \bmod \underline{x}.$$

Let us assume, specifically, that

$$q \cdot y \equiv -1 \pmod{x}. \quad (1)$$

From theorem 1 we know that when x is odd,

$$y = \frac{x-1}{2} + 1.$$

Substituting this value of y in (1) we get:

$$\begin{aligned} q \cdot \left(\frac{x-1}{2} + 1 \right) &\equiv -1, \text{ or} \\ \frac{q \cdot (x-1)}{2} + q &\equiv -1, \text{ or} \\ q \cdot (x-1) + 2q &\equiv -2, \text{ or} \\ qx - q + 2q &\equiv -2, \text{ that is,} \\ q &\equiv -2 \pmod{x} \text{ when } x \text{ is odd.} \end{aligned} \quad (2a)$$

We know from sub-theorem (2b) that a value increase of 2, mod x , entails a value increase of 1, mod y . Since $q \pmod{x}$ corresponds to $r \pmod{y}$ (definition of "fifth relationship"), it is clear that

$$r \equiv -1 \pmod{y} \text{ when } x \text{ is odd.} \quad (2b)$$

This conclusion conflicts with our requirement that q and r be different from 1 (definition 2), and therefore an odd x must be ruled out.

We know from theorem 1 that when x is even,

$$y = (x/2) + 1.$$

Substituting this value of $\underline{y}$ in (1) we get:

$$\begin{aligned} \underline{q} \cdot \left(-\frac{\underline{x}}{2} + 1\right) &= -1, \text{ or} \\ \frac{\underline{q}\underline{x}}{2} + \underline{q} &= -1, \text{ or} \\ \underline{q}\underline{x} + 2\underline{q} &= -2, \text{ that is,} \\ 2\underline{q} &= -2 \bmod \underline{x} \text{ when } \underline{x} \text{ is even.} \end{aligned} \quad (3a)$$

On the basis of sub-theorem (2b) we may conclude that, correspondingly,

$$2\underline{r} = -1 \bmod \underline{y} \text{ when } \underline{x} \text{ is even.} \quad (3b)$$

We must be sure, however, that the value decrease of $2 \bmod \underline{x}$ described in (3a) is not the case where, exceptionally, the corresponding $\bmod\text{-}\underline{y}$ value decrease is of 2 rather than 1 (recall Figure Aa-2 and its discussion). Let us, therefore, assume the opposite, that is, that the $\bmod\text{-}\underline{x}$ value decrease described in (3a) is the above mentioned exceptional case. This means that

$$2\underline{r} = -2 \bmod \underline{y} \text{ when } \underline{x} \text{ is even.} \quad (3c)$$

But this leads to a contradiction. The $\bmod\text{-}\underline{y}$ values $\underline{r}$ and $-\underline{r}$ are consecutive (see Figure Aa-2); it can be easily shown that according to (3c) $\underline{r}$ and $-\underline{r}$ are consecutive only when $\underline{y}=3$, a possibility that has been already excluded.

Combining (3a) and (3b) together we get:

$$2(\underline{q}, \underline{r}) = -(2, 1) \bmod (\underline{x}, \underline{y}), \quad (4)$$

that is, a progression by two "fifths" yields a mod-($\underline{x}, \underline{y}$) integer pair smaller (or larger) by a "major second" (from now on it is assumed that $\underline{x}$ is even).

We will now derive theorem 2,

$$\underline{x} \cdot \underline{r} - \underline{y} \cdot \underline{q} = 1$$

from (1) and (3b).

According to (3b)

$$2\underline{r} \equiv -1 \pmod{\underline{y}}.$$

Mod $\underline{y}$, it is also true that

$$\begin{aligned} 2\underline{r} - \underline{y}\underline{r} &\equiv -1, \text{ or} \\ \underline{y}\underline{r} - 2\underline{r} &\equiv 1, \text{ or} \\ (\underline{y}-2)\underline{r} &\equiv 1. \end{aligned} \tag{5}$$

We know from theorem 1 that

$$\underline{y} \equiv (\underline{x}/2) + 1$$

(recall that $\underline{x}$ is even). Substituting this value of $\underline{y}$ in (5) we get:

$$\begin{aligned} \left(\frac{\underline{x}}{2} - 1\right)\underline{r} &\equiv 1, \text{ or} \\ \frac{\underline{r}\underline{x}}{2} - \underline{r} &\equiv 1, \text{ or} \\ \underline{r}\underline{x} - 2\underline{r} &\equiv 2, \text{ or, on the basis of (3b),} \\ \underline{r}\underline{x} + 1 &\equiv 2, \text{ or} \\ \underline{r}\underline{x} &\equiv 1 \pmod{\underline{y}}. \end{aligned} \tag{6}$$

We can write (1) and (6) as

$$k \cdot x - y \cdot q = 1 \text{ and} \quad (1a)$$

$$j \cdot y - x \cdot r = -1. \quad (6a)$$

We can also write (6) as

$$r \cdot x - y \cdot j = 1. \quad (6b)$$

Comparing (6b) with (1a) it is obvious that $r = k$ and $j = q$, and therefore

$$x \cdot r - y \cdot q = 1. \quad (7a)$$

Note that had we assumed in (1) that qy equals 1, mod x , rather than -1, we would have arrived at the formula

$$x \cdot r - y \cdot q = -1. \quad (7b)$$

The difference between (7a) and (7b) is that in (7a) q and r are smaller than $-q$ and $-r$, mod x and mod y , whereas in (7b) they are larger.

Corollary A. We know that within the diatonic intervals the mod- x values are a monotone increasing function of the mod- y values, except for the pair of intervals $\{(x/2), r\}$ and $\{(x/2), -r\}$. We also know that the mod- y values are a monotone increasing function of the mod- x values, except for those pairs of mod- x values that correspond to the same mod- y value (see Figure Aa-2). It is easily shown

that neither of these exceptional cases apply to a set of scale steps, on the basis of the definition of scale steps as $(\underline{q}, \underline{r})$ multiplied by each of $\underline{N}+1$ consecutive integers between $-\underline{N}$ and $\underline{N}$. Therefore, within a set of scale steps, the mod- $\underline{x}$ and mod- $\underline{y}$ values are a monotone increasing function of each other.

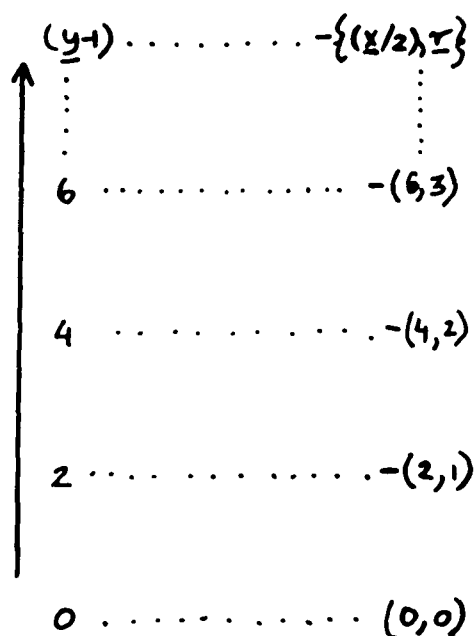
Corollary B. On the basis of theorems 1 and 2 one can draw the following relationships between $\underline{x}$, $\underline{y}$, $\underline{q}$, $\underline{r}$, and $\underline{N}$:

$$\underline{x} = 2\underline{y} - 2 = 2\underline{q} + 2 = 4\underline{r} = 2\underline{N}. \quad (8)$$

Clearly, $\underline{r}$ is the smallest value in (8). Substituting the smallest value available for $\underline{r}$, namely 2, one gets the values 8 for $\underline{x}$, 5 for $\underline{y}$, 3 for $\underline{q}$, and 4 for $\underline{N}$. It is easily shown that increasing the value of $\underline{r}$ by 1, $\underline{x}$ increases by 4, and $\underline{y}$, $\underline{q}$, and $\underline{N}$ by 2. It can be shown, in particular, that $\underline{q}$ is always prime to $\underline{x}$, and $\underline{r}$ is always prime to $\underline{y}$; since $\underline{x} = \underline{y} + \underline{q}$, $\underline{x}$ and $\underline{y}$ are also always coprime.

Corollary C. Let us take, as an example, the diatonic scale that is generated by multiplying the "fifth relationship" by each of the $\underline{y}$ consecutive integers 0, 1, 2, . . . $(\underline{y}-1)$ (Figure Aa-3); since we are concerned here with intervallic structure, it does not matter which of the $\underline{y}$ different "modes" we use.

even multiples of
 $(\underline{q}, \underline{r}) \bmod (\underline{x}, \underline{y})$
 starting with $(0, 0)$:



odd multiples of
 $(\underline{q}, \underline{r}) \bmod (\underline{x}, \underline{y})$
 starting with $(-1, 0)$:

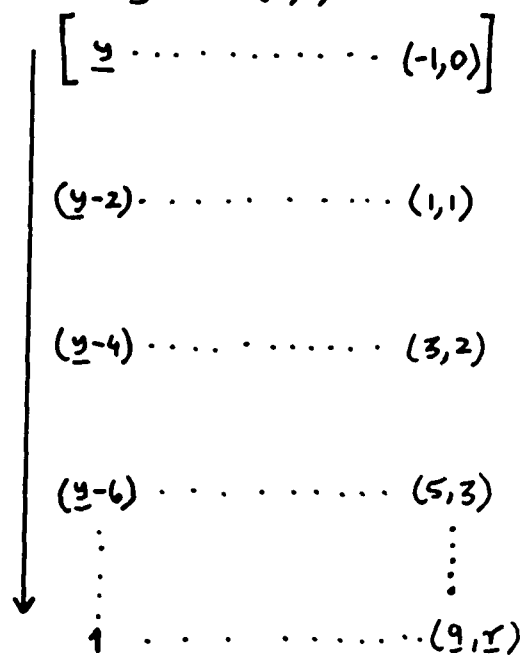


Figure Aa-3. The general structure of a diatonic scale generated by $(\underline{q}, \underline{r})$ multiplied by each of the $\underline{y}$ consecutive integers $0, 1, 2, \dots, (\underline{y}-1)$.

On the left-hand column in Figure Aa-3 we have even multiples of $(\underline{q}, \underline{r}) \bmod (\underline{x}, \underline{y})$ between 0 and $(\underline{y}-1)$ ($\underline{y}$ is an odd number, and therefore $\underline{y}-1$ is even); on the basis of (4) it is clear that the $\bmod-(\underline{x}, \underline{y})$ pairs here constitute a series of "descending" major seconds, starting with $(0, 0)$ and ending with $-\{(\underline{x}/2), \underline{r}\}$. Clearly, there are $(\underline{y}-1)/2$, or $\underline{x}/4$ such major seconds.

On the right-hand column of Figure Aa-3 we have odd multiples of $(q, r) \bmod (\underline{x}, \underline{y})$, taken in the opposite direction as $\underline{y}$ times $(q, r) \bmod (\underline{x}, \underline{y})$, which equals $(-1, 0)$ (from $\bmod-(\underline{x}, \underline{y})$ coherence), plus even multiples of $-(q, r) \bmod (\underline{x}, \underline{y})$; on the basis of (4) it is clear that the $\bmod-(\underline{x}, \underline{y})$ pairs here constitute a series of "ascending" major seconds, starting with $(-1, 0)$ (which, however, is not part of the diatonic scale), and ending in (q, r) . Excluding $(-1, 0)$, there are clearly $(\underline{x}/4)-1$ such major seconds. Now, it is easily shown that $-\{(N/2), r\}$ is larger than (q, r) by $(1, 1)$, that is, by a minor second. And there is another minor second between $(0, 0)$ and $(1, 1)$.

APPENDIX B

CONVENTIONAL INTERVAL NOMENCLATURE,
STAFF NOTATION,
AND CONVENTIONAL KEYS

Conventional interval nomenclature. Conventional interval nomenclature is a logical and convenient system of referring to diatonic intervals and altered diatonic intervals; I will show that it is isomorphic to the notational system $n: (\overset{+a1}{b}, s)$ proposed in this study (where n is an interval-band index), that is, to the cognitive/perceptual structure of (altered) diatonic intervals, except in the case of certain altered diatonic intervals, which conventional interval nomenclature has difficulty to express.

The great appeal of conventional interval nomenclature lies in its conciseness. For example, conventional interval nomenclature will not refer to the diatonic interval $7:(7,4)$ as "a perfect fifth at the cognitive level corresponding to an ascending interval of seven tempered semitones (assuming $\alpha = 0$) at the perceptual level," but simply as "an ascending fifth." Clearly, conventional interval nomenclature takes advantage of the fact that the mod-12 value (7) at the cognitive

level is identical to the interval-band index at the perceptual level. It uses the term "ascending fifth" in a combined cognitive/perceptual sense, meaning the diatonic interval $(7,4)$ of which the mod-12 value 7 performs also as an interval-band index. Clearly, all unaltered diatonic intervals $\underline{n}:(\underline{b},\underline{s})$ of which $\underline{n}$ is positive and smaller than 12 will have the property that $\underline{n} \leq \underline{b}$; conventional interval nomenclature will refer to these intervals in the manner of the "ascending fifth" discussed above ("ascending M2," "ascending m6," etc.). I will refer to diatonic intervals of this type as "natural diatonic intervals."

Let us consider now diatonic intervals of which $\underline{n} \neq \underline{b}$ (assuming unaltered diatonic intervals); specifically, let us consider diatonic intervals with a negative $\underline{n}$ (a descending interval band), but not smaller than -11.

The diatonic interval $-5:(7,4)$ represents a correspondence between the diatonic interval "perfect fifth" at the cognitive level and a descending interval of five semitones (assuming $\alpha = 0$) at the perceptual level. Conventional interval nomenclature, however, refers to such an interval as "a descending fourth." Since $\underline{n}$ and $\underline{b}$ are different from each other here (and moreover, $\underline{n}$ is negative while $\underline{b}$ is positive), conventional interval nomenclature replaces $(7,4)$ ("ascending fifth") at the cognitive level by $-(5,3)$ ("descending fourth"), which is of course identical to $(7,4) \bmod (12,7)$. It can thus refer to the diatonic interval $-5:(7,4)$ in the same concise way noted

before, using the number -5 in both a cognitive and perceptual capacity:

$$-5:(7,4) \longrightarrow \underbrace{-5:-(5,3)}_{N8}.$$

In a similar fashion, conventional interval nomenclature refers to the diatonic interval -2:(10,6) as a "descending M2" -2:-(2,1), to the diatonic interval -8:(4,2) as a "descending m6" -8:-(8,5), etc..

What is important to keep in mind here is that such pairs as (7,4) and -(5,3), (10,6) and -(2,1), etc., are identical mod (12,7); from a purely cognitive point of view, therefore, the distinction is meaningless. It is only meaningful in conjunction with a convention by which a diatonic interval such as (7,4) corresponds, at the perceptual level, to an ascending interval band, while its mod-(12,7) "twin" -(5,3) corresponds to a descending interval band. Such a convention simply makes it possible to refer to diatonic intervals in a conveniently concise way, intertwining the cognitive and perceptual levels. Furthermore, such pairs as (7,4) ("ascending fifth") and -(5,3) ("descending fourth") are not to be confused with pairs of absolute diatonic intervals such as (7,4) and (5,3) (fifth and fourth); while the former are identical mod (12,7), the latter are not.

Conventional interval nomenclature refers to diatonic intervals of which $n > 11$ as natural diatonic intervals plus

a certain number of "octaves." For example, conventional interval nomenclature refers to the diatonic interval $14:(2,1)$ as "an ascending major second plus one octave" or "an ascending major ninth." Note that conventional interval nomenclature uses the term "octave" in a combined cognitive/perceptual sense. That is to say, it does not refer to the diatonic interval $12:(0,0)$ as "a perfect prime at the cognitive level corresponding to an ascending interval of twelve semitones (assuming $\alpha = 0$) at the perceptual level." Instead, it replaces $(0,0)$ ("perfect prime") at the cognitive level by $(12,7)$ ("perfect octave"), which is of course identical to $(0,0) \bmod (12,7)$; it then refers to $12:(0,0)$ in a combined cognitive/perceptual sense as "one octave":

$$12:(0,0) \longrightarrow \underbrace{12:(12,7)}_{N8} \longrightarrow 1 \cdot \{12:(12,7)\};$$

"two octaves" would be

$$2 \cdot \{12:(12,7)\} = \underbrace{24:(24,14)}_{N8},$$

etc.. Since the interval-band index n in the case of the cognitive perfect prime equals, by definition, $12\underline{rg}$ (where $\underline{rg}$ is the "register operator"), it is clear that conventional interval nomenclature simply replaces the notation $(12\underline{rg}):(0,0)$ by the equivalent notation $\underline{rg} \cdot \{12:(12,7)\}$:

$$(12\underline{rg}):(0,0) \longrightarrow \underline{rg} \cdot \{12:(12,7)\}.$$

Here also, such intervals as $(0,0)$ ("perfect prime"), $(12,7)$ ("perfect octave"), $(24,14)$ ("two octaves"), etc., are all identical mod $(12,7)$; the distinction is therefore meaningless from a purely cognitive point of view. It is only meaningful in conjunction with a convention by which $(0,0)$ corresponds, at the perceptual level, to i_0 ($rg = 0$), $(12,7)$ corresponds to i_{12} ($rg = 1$), etc.; such a convention simply makes it possible to use the term "octave" in a concise way, intertwining together the cognitive and perceptual levels.

Since the interval-band index n of any unaltered diatonic interval equals, by definition, $b + 12rg$, we can conclude that conventional interval nomenclature replaces the notation $(b + 12rg):(b,s)$ (for $rg > 0$) by the equivalent notation $b:(b,s)$ plus $rg \cdot \{12:(12,7)\}$, that is, a natural diatonic interval plus a certain number of "octaves," as has been noted before (we are assuming for the moment a positive rg , that is, an ascending interval band):

$$(b + 12rg):(b,s) \longrightarrow b:(b,s) + rg \cdot \{12:(12,7)\}.$$

For example, $14:(2,1)$ becomes $2:(2,1)$ ("ascending major second") plus $1 \cdot \{12:(12,7)\}$ ("one octave"). Clearly, the two systems of notation are isomorphical.

Turning now to chromatic alteration, conventional interval nomenclature intertwines the cognitive and perceptual levels here also. For example, it does not refer to the altered diatonic interval $3:(\overset{+1}{2},1)$ as "an augmented second at the cognitive level corresponding to an ascending interval of three semitones (assuming $\alpha = 0$) at the perceptual level," but simply as "an ascending A2," using the sum $(2+1)$ in both a cognitive and perceptual capacity, thus: $\overset{+1}{2}^1:(\overset{+1}{2},1)$. It is here, however, that conventional interval nomenclature runs in certain cases into difficulty.

Consider the altered diatonic interval $-1:(\bar{1}^2,1)$, representing a correspondence between a descending tempered semitone (with $\alpha = 0$) at the perceptual level and the altered diatonic interval "doubly-diminished second" at the cognitive level. Conventional interval nomenclature has a problem here because on the one hand it is constrained to refer to such an interval, in the manner of descending intervals, as "a descending doubly-augmented seventh," yet on the other, it is constrained to refer to such an interval, in the manner of ascending intervals, as "a doubly-diminished second." This is self-contradictory, and therefore non-existent.

of $13:(\overset{+2}{11},6)$, that is, $\overset{+2}{11}:(\overset{+2}{11},6)$, and not $1:(\overset{+2}{11},6)$. A "des-³ sense of $13:(\overset{+2}{11},6)$, that is, $\overset{+2}{11}^<:(\overset{+2}{11},6)$, and not $1:(\overset{+2}{11},6)$. A "descending doubly-augmented seventh" in conventional interval nomenclature is therefore $-13:(\bar{1}^2,1)$ (in our terms) and not $-1:(\bar{1}^2,1)$; a "doubly-diminished second" in conventional interval nomenclature (but not "doubly-diminished ninth"!) is self-contradictory, and therefore non-existent.

In other words, the intertwining of the cognitive and perceptual levels leads to a contradiction in "altering" natural diatonic intervals, conventionally, below the perceptual unison (that is, 1_0). Needless to say, this contradiction is indigenous to conventional interval nomenclature, and does not reflect any inherent logical (or intuitive) problem about such altered diatonic intervals; once the perceptual and cognitive levels are separated from each other, a "doubly-diminished second" or any such altered diatonic interval poses no problem. The only diatonic interval that can be conventionally "altered" below the perceptual unison without resulting in a contradiction is the perfect prime (the mathematically-minded reader will note that the absolute value of $(\underline{b} + \underline{a1})$ and $-(\underline{b} + \underline{a1})$ here is the same). For example, the altered diatonic interval $-1:(\bar{0}^1_0)$, representing, in our terms, a correspondence between a descending semitone (with $\alpha = 0$) at the perceptual level and the altered diatonic interval "diminished prime" at the cognitive level, can be expressed in conventional terms as "a descending augmented prime" $-\overset{+1}{0}:-(\overset{+1}{0}_0)$.

Staff notation. Staff notation is composed of two main components, a system of staff positions, and a system of accidentals ("sharps" and "flats"). In C major (alternatively, A minor), staff positions correspond to unaltered scale steps; accidentals are used only in conjunction with chromatic alteration. I will restrict the discussion of staff notation in this section to C major only (of course I could have chosen

A minor just as well); other "keys" are considered separately in the following section.

Let us begin with the system of staff positions. Staff positions have two interpretations, an explicit interpretation and an implicit one. Explicitly, staff positions correspond to a sequence of consecutive integers, with (let us say) "middle C" as 0 (Figure Ab-1); taken mod 7, these integers clearly correspond to the mod-7 values of (unaltered) C-major scale steps at the cognitive level.

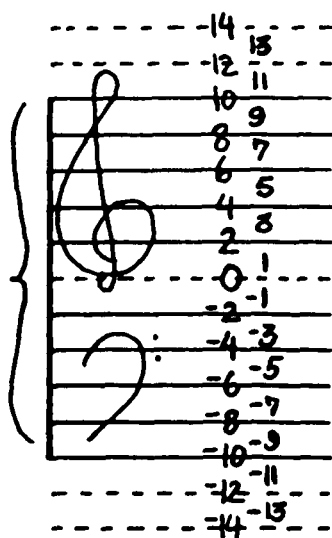


Figure Ab-1. The explicit interpretation of staff positions.

Implicitly, staff positions correspond to a system of integers, which taken mod 12, correspond to the mod-12 values of (unaltered) C-major scale steps at the cognitive level (Figure Ab-2); the same system of integers also corresponds to (unaltered) C-major pitch-band indices at the perceptual level

(with "middle C" once again as 0).

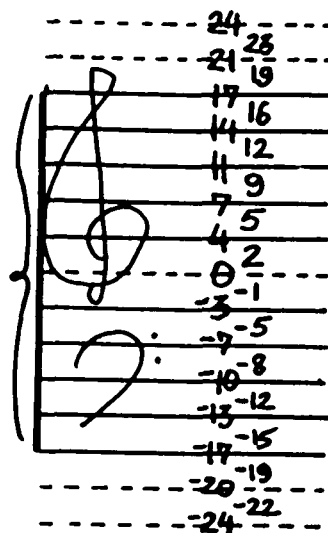


Figure Ab-2. The implicit interpretation of staff positions.

As with conventional interval nomenclature, staff positions "describe" the cognitive and perceptual levels all at once. This means that the same value is used as a pitch-band index at the perceptual level and as a mod-12 value at the cognitive level (see the discussion of conventional interval nomenclature). Scale step 0:(0,0), for example, already has this property, and is notated on the staff on the staff posi-

tion which corresponds, explicitly, to the mod-7 value 0, and implicitly, to the mod-12 value and pitch-band index 0, that is, the position of "middle C" (see Figures Ab-1 and Ab-2). Scale step 12:(0,0), on the other hand, is translated into 12:(12,7); correspondingly, it is notated on the staff on the staff position that corresponds, explicitly, to the mod-7 value 7 (Figure Ab-1), and implicitly, to the mod-12 value and pitch-band index 12 (Figure Ab-2). Note that as with conventional interval nomenclature, (12,7) is cognitively indistinguishable from (0,0); staff notation distinguishes between the two in conjunction with a convention by which (0,0) corresponds to p_0 ($rg=0$), and (12,7) corresponds to p_{12} ($rg=1$). As with conventional interval nomenclature, this makes it possible to describe scale steps concisely with the perceptual and cognitive levels intertwined.

I will give one more example. Scale step -10:(2,1) is translated into -10:-(10,6), and is notated accordingly (see Figures Ab-1 and Ab-2); clearly staff positions are isomorphic to the cognitive/perceptual structure of (unaltered) C-major scale steps.

As with conventional interval nomenclature, chromatic alteration in staff notation applies to both the cognitive and perceptual levels. Thus, a "sharp" applied to "middle C" in staff notation means that both the perceptual and (implicit) cognitive interpretations of the middle-C position (Figure Ab-2)

are "altered" by +1, thus: $0^1:(0^1,0)$. The altered scale step $7:(5^2,3)$, for example, is translated into $5^2:(5^2,3)$, and is notated, correspondingly, on the staff position of an "F" in the "one-line octave" with a double sharp. Note that the same altered intervals that are problematic in conventional interval nomenclature are also problematic on the staff, and for the same reason: the intertwining of the perceptual and cognitive levels. An interval such as $-1:(\bar{1}^2,1)$, a doubly-diminished second, would appear self-contradictory on the staff. The doubly-diminished second B-C $\flat\flat$, for example, is a descending interval, yet is notated on the staff in the ascending direction (B-C).

Conventional keys. As the reader will recall (see Chapter 3, section 3.7), we have defined a "key" in terms of a "transposition operator" tr between 0 and 11, and a set of "diatonic" mod-12 values which is either major (M) or minor (m); accordingly, we have defined 12 major and 12 minor keys. The "key" $1(\underline{M})$ (conventionally, D $\flat$ major or C# major), for example, yields the following cognitive/perceptual structure in terms of a correspondence between pitch-band classes and scale steps: $\{1:(0,0), 3:(2,1), 5:(4,2), 6:(5,3), 8:(7,4), 10:(9,5), 0:(11,6)\}$. Following conventional practice, we have used the term "key," by extension, to denote also the cognitive/perceptual structure it defines.

There are obvious discrepancies between our "keys" and conventional keys. Our system of 12 major and 12 minor keys

fails to capture the distinction between such enharmonically equivalent conventional keys as $D\flat$ major and $C\sharp$ major. Unlike conventional keys, moreover, where all keys except C major and A minor are notated with "sharps" and "flats," our keys are all notated without chromatic alteration.

These discrepancies are easily accounted for once one realizes that conventional keys are conceived in terms of C major or A minor, altered. That is to say, the scale steps of each of the twelve major or minor keys $\text{tr}(\underline{M})$ or $\text{tr}(\underline{m})$ are mapped, conventionally, onto the (altered) scale steps of $0(\underline{M})$ (C major) and $9(\underline{m})$ (A minor), respectively. Thus for example, $D\flat$ major and $C\sharp$ major are simply two different ways to map the scale steps of $1(\underline{M})$ onto (altered) C-major scale steps, as defined in Chapter 4. Naturally, there are five additional ways to map $1(\underline{M})$ onto $0(\underline{M})$ that are available theoretically, corresponding to the theoretical conventional keys $B\sharp$ major, $E\flat^3$ major, $A\sharp^4$ major, $F\flat^4$ major, and $G\sharp/\flat^6$ major. In short, each of the twelve major keys $\text{tr}(\underline{M})$ corresponds, theoretically, to seven enharmonically equivalent conventional keys, bringing the number of conventional major keys, theoretically, to 84 (the same, of course, holds true for minor). Staff notation embodies the same C-major (A-minor) bias; keys on the staff are notated as altered C major or A minor.

Clearly, outside the context of some modulatory scheme, the distinction between such enharmonically equivalent keys as $D\flat$ major and $C\sharp$ major (and theoretically at least, the five

additional enharmonically equivalent keys as well) is a purely notational matter, with no cognitive/perceptual content whatever. Moreover, the sharps and flats with which all major and minor keys other than C major and A minor are conventionally notated are purely notational entities also, as every musician learns early on to understand. An "F#" in G major for example, is properly speaking a "pseudo altered scale step," not a genuine one, since it clearly represents an unaltered scale step ($\hat{7}$), unlike, for example, an F# in C major ($\hat{\#4}$; note, however, that "pseudo accidentals" are usually notated as a "key signature," and are thus distinguished notationally from genuine ones). Since our theory is concerned with cognition and perception, not with notation, it does not distinguish between enharmonically equivalent keys, nor does it recognize pseudo alterations.

Summary. We have seen that there exists a close correspondence between staff notation as well as conventional interval nomenclature and the cognitive/perceptual structure of diatonicism. At the same time, we have seen that staff notation is a C-major biased system, that is to say, keys are notated as altered C major (or A minor); this results in notational distinctions between enharmonically equivalent keys, and in the purely notational entities of "pseudo accidentals." And we have seen in connection with conventional interval nomenclature that the intertwining of the cognitive and perceptual levels leads to problems in connection with certain chromatically altered intervals.

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