

**On the Dynamics of Quasi-Self-Matings of Generalized Starlike
Complex Quadratics and the Structure of the Mated Julia Sets**

by

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Abstract

**On The Dynamics of Quasi-Self-Matings of Generalized Starlike
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Adviser: Professor Linda Keen

It has been shown that, in many cases, Julia sets of complex polynomials can be "glued" together to obtain a new Julia set homeomorphic to a Julia set of a rational map; the dynamics of the two polynomials are reflected in the dynamics of the mated rational map. Here, I investigate the Julia sets of self-matings of generalized starlike quadratic polynomials, which enjoy relatively simple combinatorics. The points in the Julia sets of the mated rational maps are completely classified according to their topology. The presence and location of buried points in these Julia sets are addressed. The interconnections between complex dynamics, combinatorics, symbolic dynamics and Thurston's lamination theory are explored and utilized. The results are then extended to "quasi-self-matings".

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Introduction

Quadratic matings were initially studied by Adrien Douady as a means of describing a certain region of the parameter space of quadratic rational maps. He proposed that certain quadratic rational maps can be understood as topological "gluings" of two quadratic polynomials. The dynamics of the resulting rational map would, therefore, be directly related to the dynamics of the polynomials being mated.

Given two quadratic polynomials, f_1 and f_2 , with filled Julia sets $K(f_1)$ and $K(f_2)$, let $X = (K(f_1) \sqcup K(f_2)) / (\gamma_1(t) \sim \gamma_2(-t))$ be the topological space obtained by gluing the boundaries of the two filled Julia sets along their Caratheodory loops $(\gamma_j(t) := \phi_j^{-1}(e^{2\pi i t}) : \mathbb{R}/\mathbb{Z} \rightarrow J(f_j))$ in reverse order. If X is homeomorphic to \mathbb{S}^2 , then we say that f_1 and f_2 are topologically matable.

The induced map from \mathbb{S}^2 to itself, $f_1 \overset{top}{\sqcup} f_2 = (f_1|_{K_1} \sqcup f_2|_{K_2}) / (\gamma_1(t) \sim \gamma_2(-t))$ is called the **topological mating** of f_1 and f_2 .

Note the "gluing" is a purely topological construction. However, it is not true that any topological mating will be conjugate to a rational map. Tan Lei, Mary Rees and Mitsuhiro Shishikura described certain suitable conditions under which such a construction does result in a branched covering of the sphere that is topologically equivalent to a degree two rational map.

A degree two rational map $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is called a **conformal mating**, or, simply, a **mating** of f_1 and f_2 , denoted by $g = f_1 \amalg f_2$, if it is conjugate to the

topological mating $f_1 \overset{top}{\sqcup} f_2$ by a homeomorphism h , such that, if the interiors of $K(f_j)_{j=1,2}$ are non-empty then $h|_{\text{int}(K(f_j))}$ is conformal.

The work of Michael Yampolsky and Saeed Zakeri focused on mating quadratic polynomials with Siegel disks. Over the last decade John Milnor, Adam Epstein, Peter Haïssinsky, Jiaqi Luo and Magnus Aspenberg, among others, have studied certain specific cases of quadratic matings and their role in understanding the landscape of quadratic rational maps, and suggested numerous interesting questions regarding this topic. Tan Lei and Mitsuhiro Shishikura have also been able to generalize some results to matings of higher degree polynomials.

In this paper, I concentrate on a very special case of the above construction, self-matings, or, more precisely, quasi-self-matings of generalized starlike quadratic polynomials which are defined in section 2.1. The focus will be primarily on the combinatorics of rational maps which arise in this way, and the topological structure of their Julia sets. The techniques used here will involve some symbolic dynamics, a variation of Thurston's lamination theory for quadratic polynomials, as well as the orbit portraits of Milnor and Goldberg.

The Julia sets of generalized starlike quadratics are locally connected. Their filled Julia sets have a natural tree structure, which allows us to construct a well-defined tree of Fatou components. Let F_c be the Fatou component containing the critical point. This component will be the root of the tree. We

may then define *full infinite Fatou chains*, denoted by $\tilde{\mathcal{F}}$, to be infinite sequences of distinct disjoint Fatou components

$\tilde{\mathcal{F}} = \{F_i\}_{i=0}^{\infty} = \{F_0, F_1 \dots F_i, F_{i+1}, \dots\}$ for which

- (i) $F_0 = F_c$, and,
- (ii) $\overline{F_i} \cap \overline{F_{i+1}}$ is a single point for all $i \geq 0$.

We will call any infinite proper subset of a full Fatou chain, which also satisfies condition (ii) above, a *proper infinite Fatou chain*, and denote it by \mathcal{F} . Note that for proper infinite Fatou chains $F_0 \neq F_c$. It is easy to show that for any such chain, $\text{diam}(F_i) \rightarrow 0$ as $i \rightarrow \infty$. Since the Julia set of any generalized starlike quadratic polynomial is locally connected, the chain converges to a well-defined limit point, which is a point on the Julia set and is called the *landing point* of the chain.

For the rational maps we are interested in, that is, those that arise as quasi-self-matings of starlikes, we will give a similar construction that will allow us to define proper infinite Fatou chains with the same properties as above. In particular, these Fatou chains will converge to well-defined limit points. We will say that two proper infinite Fatou chains \mathcal{F}_1 and \mathcal{F}_2 are *distinct* if they have no common elements. We will say that \mathcal{F}_1 and \mathcal{F}_2 are *eventually distinct* if they have distinct "tails". This is addressed in more detail in chapter 6.

Ultimately, one of the things we'll be interested in is how the trees of the quadratics to be mated fit together under the mating construction.

The root r_H of any hyperbolic component H of the Mandelbrot set is known to be the landing point of two external parameter rays, which we will denote by $\mathcal{R}_{t_{\pm}}(r_H)$ (see section 1.3). These rays together with r_H divide the plane into two open sets, one of which contains H , and is called the *wake* of r_H , denoted by W_{r_H} (see section 4.1). In the dynamic plane of the quadratic f_{r_H} , there are two external dynamic rays landing at the same point on the Julia set such that this landing point is the parabolic periodic point on the boundary of the Fatou component which contains the critical value; we will denote these dynamic rays by $R_{t_{\pm}}(r_H)$. The angles of these dynamic rays are the same as the angles of $\mathcal{R}_{t_{\pm}}(r_H)$ (see sections 1.2 and 1.3).

Given two hyperbolic components H_k and H_n such that $H_n \subset W_{r_{H_k}}$, we will say that there exists a *finite chain* $\{H_i\}_{i=k}^n$ from H_k to H_n if $\{H_i\}_{i=k}^n$ is a collection of hyperbolic components with the property that $r_{H_{i+1}} \in \partial H_i$ for $i = k, k + 1, \dots, n - 1$ (see section 1.3).

We are now ready to state the main results.

Main Theorem A. (Keen-Flek) *Let H_k^* be the union of a hyperbolic component H_k and its root r_{H_k} . If $f_c \in H_k^*$, then the rational rays landing on the boundary points of the bounded Fatou components of f_c are precisely all the rays in the union of the sets $\{\bigcup_m f_{r_{H_k}}^{-m}(R_{t_{\pm}}(r_{H_k}))\}$ and $\{\bigcup_m f_{r_{H_n}}^{-m}(R_{t_{\pm}}(r_{H_n}))\}$ for all $H_n \subset W_{r_{H_k}}$ such that there is a finite chain from H_k to H_n .*

Main Theorem B. Let p and q be two relatively prime positive integers with $p < q$ and $\frac{p}{q} \neq \frac{1}{2}$. Let H_q^p be the hyperbolic component of the Mandelbrot set immediately attached to the main cardioid whose rotation number is $\frac{p}{q}$, and let H_q^{p*} be the union of H_q^p and its root. Let $f_1, f_2 \in H_q^{p*}$ and let g be the mating of f_1 and f_2 , denoted by $f_1 \amalg f_2$.

Then g and $g_q^p \equiv \left(\frac{1+e^{2\pi i(\frac{p}{q})}}{2} \right) \left(z + \frac{1}{z} \right)$ are topologically conjugate on their respective Julia sets, and the corresponding Julia sets are homeomorphic. The topological conjugacy is a restriction of a homeomorphism of the sphere.

Main Theorem C. Under the hypotheses of Main Theorem B, the Julia set of g , $J(g \equiv f_1 \amalg f_2)$, consists of the following mutually exclusive sets of points, each dense in $J(g)$:

- (i) a countable set J_α , where each $z_\alpha \in J_\alpha$ is pre-fixed, lies on the boundary of exactly q Fatou components and is the landing point of q distinct proper infinite Fatou chains;
- (ii) a countable set J_β , where each $z_\beta \in J_\beta$ is pre-fixed, does not lie on the boundary of any Fatou component and is the landing point of two distinct proper infinite Fatou chains;
- (iii) an uncountable set $J_{\alpha\text{-type}}$ where each $z_{\alpha\text{-type}}$ is not pre-fixed, lies on the boundary of exactly one Fatou component and is the landing point of one distinct proper infinite Fatou chain;
- (iv) an uncountable set $J_{\beta\text{-type}}$, where each $z_{\beta\text{-type}}$ is not pre-fixed, does not lie

on the boundary of any Fatou component and is the landing point of two distinct proper infinite Fatou chains;

Furthermore, $J_\alpha \cup J_\beta \cup J_{\alpha\text{-type}} \cup J_{\beta\text{-type}} = J(f_1 \amalg f_2)$.

Given a rational map $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, we say that a point x on the Julia set of g , $J(g)$, is **buried** if it does not belong to the boundary of any Fatou component, and the collection of all such points is referred to as the **residual Julia set**, denoted by $J_R(g)$ (see section 1.1).

Main Corollary. *Let p and q be two relatively prime positive integers with $p < q$ and $\frac{p}{q} \neq \frac{1}{2}$. Let $H_{\frac{p}{q}}$ be the hyperbolic component of the Mandelbrot set*

immediately attached to the main cardioid whose rotation number is $\frac{p}{q}$, and let

$H_{\frac{p}{q}}^$ be the union of $H_{\frac{p}{q}}$ and its root. Let $f_1, f_2 \in H_{\frac{p}{q}}^*$. Let $g = f_1 \amalg f_2$.*

Then in $J(g)$ all z_β and $z_{\beta\text{-type}}$ are buried, and all z_α and $z_{\alpha\text{-type}}$ are accessible. In other words, $J_\beta \cup J_{\beta\text{-type}}$ is the residual Julia set, denoted by $J_R(g)$, while $J_\alpha \cup J_{\alpha\text{-type}} = J(g) \setminus J_R(g)$.

Chapter 1. Preliminaries

1.1 Iteration of Rational Maps

Following standard notation, given any rational map $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, letting $g^{\circ n}$ be its n -fold iterate, and fixing some $z_0 \in \overline{\mathbb{C}}$, we obtain the following dichotomy:

Definition. If there exists some neighborhood U of z_0 for which $\{g^{\circ n}\}|_U$ forms a normal family then z_0 belongs to the **Fatou set** of g , denoted by $\Omega(g)$; if no such neighborhood exists then z_0 belongs to the **Julia set** of g , denoted by $J(g)$. Clearly, $J(g)$ is the complement of $\Omega(g)$, and $\Omega(g)$ is open by definition, which implies that $J(g)$ must be closed.

Definition. Let g be such that $\Omega(g)$ is not empty. A **Fatou component** F_i is a connected component of $\Omega(g)$.

Definition. Let $z \in \overline{\mathbb{C}}$; we define the **forward orbit of z under g** as $Orb_g(z) = \bigcup_{n \geq 0} g^{\circ n}(z)$.

Definition. The **grand orbit of z under g** , denoted by $GO_g(z)$, is the set of all z' such that $g^{\circ n}(z') = g^{\circ m}(z)$ for some positive integers n and m .

Definition. With g as above, a point z_0 is **periodic of period q** if q is the smallest positive integer such that $g^{\circ q}(z_0) = z_0$. The orbit of such a point consists of q periodic points, each of period q , and is called a **cycle of period q** .

Definition. Let z_i be any element of a cycle of period q . Define the **multiplier** of the cycle as $\mu = \frac{d}{dz} (g^{\circ q}(z_i))$. Note that this definition is independent of the choice of the z_i in the cycle.

Furthermore, a cycle is:

- | | | | |
|-------|--------------------------------|----|--------------|
| (i) | <i>attracting</i> | if | $ \mu < 1,$ |
| (ii) | <i>supperattracting</i> | if | $\mu = 0,$ |
| (iii) | <i>repelling</i> | if | $ \mu > 1,$ |
| (iv) | <i>neutral</i> | if | $ \mu = 1$ |

Definition. A neutral cycle (or a neutral periodic point) is ***parabolic*** if its multiplier is a root of unity, that is, if $\mu = e^{2\pi i \left(\frac{j}{k}\right)}$.

Definition. Call the set of critical points of g ***Crit(g)***; that is, $Crit(g) = \{c_i \in \bar{\mathbb{C}} \mid g'(c_i) = 0\}$.

Definition. Define the ***post-critical set of g*** as $Post(g) = \bigcup_{c_i \in Crit(g)} \overline{Orb_g(c_i)}$.

Definition. A rational map g is ***post-critically finite*** if $\#(Post(g)) < \infty$. It is ***geometrically finite*** if $\#(Post(g) \cap J(g)) < \infty$, that is, if the post-critical set intersects the Julia set at finitely many points.

The Julia set of a rational map with a non-empty Fatou set can be partitioned further according as to whether a point $x \in J(g)$ belongs to the boundary of some Fatou component.

Definition. With g as above, a point $x \in J(g)$ is ***(internally) accessible*** if there exists a path γ^J in $\Omega(g)$ which converges to x .

Definition. A point $x \in J(g)$ is ***buried*** if it does not belong to the boundary of any Fatou component, and the collection of all such points is referred to as the ***residual Julia set***, denoted by $J_R(g)$.

Remark 1.1.1. *If $x \in J(g)$ is buried then it is not internally accessible.*

However, the converse is not true.

Qiao proved the following conjecture of Makienko, for locally connected Julia sets.

Theorem 1.1.1. (Qiao). [Q] *Let $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be a rational map with degree at least two, such that $J(g)$ is locally connected. Then $J(g)$ has buried points if and only if $\Omega(g)$ has no completely invariant Fatou components.*

■ Properties of $J(g)$ and $\Omega(g)$

For a detailed discussion and proofs of the following propositions see [Keen].

Again, let $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be a rational map.

Proposition 1.1.1. *Both $\Omega(g)$ and $J(g)$ are completely invariant under g . That is, if $z \in J(g)$, or $z \in \Omega(g)$, then $GO_g(z) \subseteq J(g)$, or, respectively, $GO_g(z) \subseteq \Omega_g$.*

Proposition 1.1.2. *$J(g)$ is not empty.*

Proposition 1.1.3. *$J(g)$ is an infinite perfect set.*

Proposition 1.1.4. *If $J(g)$ contains a nonempty open subset of $\overline{\mathbb{C}}$ then $J(g) = \overline{\mathbb{C}}$.*

Proposition 1.1.5. *$J(g)$ is the closure of the repelling periodic points. That is, if $z \in J(g)$, or $z \in \Omega(g)$, then $GO_g(z) \subseteq J(g)$, or, respectively, $GO_g(z) \subseteq \Omega_g$.*

Definition. A Fatou component F_i of $\Omega(g)$ is **periodic** if $g^{\circ q}(F_i) = F_i$ for some positive integer q and is **eventually periodic** if $g^{\circ(n+q)}(F_i) = g^{\circ n}(F_i)$ for some integers $q, n > 0$. The **period of the component** is the smallest q for which its orbit is periodic and the periodic components in its orbit form a **Fatou cycle**.

We may also define the **grand orbit of a Fatou component**,

$GO_g(F_i) = \{F_j \mid g^{\circ n}(F_j) = g^{\circ m}(F_i) \text{ for some positive integers } n \text{ and } m\}$.

Theorem 1.1.2. (Sullivan). *Every Fatou component of $\Omega(g)$ is eventually periodic*

Proposition 1.1.6. *If the number of Fatou components of $\Omega(g)$ is finite, it is at most two.*

■ Holomorphic Index Formula and Its Corollaries

The holomorphic index will prove to be a very useful tool in the techniques used in this paper. Here we explore some of its properties. See [Mil3] for the omitted proofs of some of the theorems below and additional discussions related to the holomorphic index.

Definition. Let z_0 be an isolated fixed point of a holomorphic map $g(z)$, that is, $g(z_0) = z_0$. We define the *holomorphic index of g at z_0* as

$$\text{ind}(g, z_0) = \frac{1}{2\pi i} \oint \frac{dz}{z-g(z)},$$

where the integral is evaluated around a small loop around the fixed point.

Lemma 1.1.1. Let $g : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ be a rational map with a fixed point at z_0 , such that $g'(z_0) \neq 1$. Then

$$\text{ind}(g, z_0) = \frac{1}{1-\mu_0},$$

where μ_0 is the multiplier of g at z_0 .

Lemma 1.1.2. Let $g : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ be a rational map with a fixed point at z_0 and let ϕ be a local holomorphic change of coordinates. Then

$$\text{ind}(g, z_0) = \text{ind}(\phi \circ g \circ \phi^{-1}, \phi(z_0)).$$

Theorem 1.1.3. Let $g : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ be a rational map. Then

$$\sum_{z=g(z)} \text{ind}(g, z) = 1.$$

Lemma 1.1.3. A fixed point z_0 of a holomorphic function g is

- (i) attracting if and only if $\text{Re}\{\text{ind}(g, z_0)\} > \frac{1}{2}$,
- (ii) repelling if and only if $\text{Re}\{\text{ind}(g, z_0)\} < \frac{1}{2}$, and
- (iii) neutral otherwise.

Lemma 1.1.4. Every rational map of degree two must have a repelling fixed point.

Proof. Assume all three fixed points are either attracting or neutral. Then, by lemma 1.1.3, $\text{Re}\{\text{ind}(g, z_k)\} \geq \frac{1}{2}$. This implies that $\text{Re}\left\{\sum_{z=g(z)} \text{ind}(g, z)\right\} \geq \frac{3}{2}$

which contradicts theorem 1.1.1. ■

Lemma 1.1.5. *Let μ_1 and μ_2 be the multipliers of the two finite fixed points of a quadratic polynomial then $\mu_1 + \mu_2 = 2$.*

Lemma 1.1.6. (simple fixed points) *Let z_0 be a fixed point of a holomorphic function g , and denote the multiplicity of z_0 by $\mathbf{mult}(g, z_0)$, then $\mathbf{mult}(g, z_0) = 1$ if and only if $\mu_0 = g'(z_0) \neq 1$.*

Proof. Let $G(z) = z - g(z)$. Then z_0 is a fixed point of g if and only if $G(z_0) = 0$, and, therefore, z_0 is a simple fixed point of g if and only if it is a simple zero of G . Let γ be a simple loop which isolates z_0 . Then, integrating around γ , and using the Argument Principle, we have

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{G'(z)}{G(z)} dz = n - p,$$

where n is the multiplicity of a zero of G if there is one, and p is the multiplicity of a pole of G if there is one. We know that z_0 is a zero of G . Both g and G are analytic inside γ and have no poles there. So, the number n gives the multiplicity of the zero of G , or, equivalently, the multiplicity of the fixed point of g . Furthermore, z_0 is a simple zero of G if and only if $G'(z_0) \neq 0$. Recalling the definition of G , we see that $G'(z) = 1 - g'(z)$. It follows that $G'(z_0) = 0$ if and only if $g'(z_0) = 1$. Moreover, if $\mu_0 = g'(z_0) = 1$ then

$$\mathbf{mult}(g, z_0) = n = \frac{1}{2\pi i} \oint_{\gamma} \frac{G'(z)}{G(z)} dz = \frac{1}{2\pi i} \oint_{\gamma} \frac{1 - g'(z)}{z - g(z)} dz > 1. \blacksquare$$

1.2 Quadratics with Locally Connected Julia Sets

Let us recall some basic theory of dynamics of complex quadratics, the associated definitions and notation, and introduce some new notation as it becomes necessary.

Any quadratic polynomial $f(z) = \alpha z^2 + \beta z + \gamma$, with $\alpha \neq 0$, is affine conjugate to a unique one of type $f_c(z) = z^2 + c$. Using the conjugacy $A(z) = \alpha z + \frac{\beta}{2}$, we obtain $A \circ f \circ A^{-1} = f_c$ where $c = \frac{2\beta - \beta^2 + 4\alpha\gamma}{4}$. The uniqueness is obvious ($f_{c_1} = z^2 + c_1$ is conjugate to f_{c_2} implies that $c_2 = \frac{2(0) - (0)^2 + 4(1)(c_1)}{4} = c_1$).

Another convenient way to normalize quadratics is using the form $f_\lambda(z) = z^2 + \lambda z$, with the relation $c = \frac{2\lambda - \lambda^2}{4}$. Note that here there are two choices of λ . By adding the restriction that $\operatorname{Re}\{\lambda\} \leq 1$, the normalization is well-defined.

■ Properties of f_c and f_λ

$f_c(z)$ has two finite fixed points at $z_1 = \frac{1 - \sqrt{1 - 4c}}{2}$ and $z_2 = \frac{1 + \sqrt{1 - 4c}}{2}$, and a fixed point at ∞ . The corresponding multipliers are $\mu(z_1) = 1 - \sqrt{1 - 4c}$, $\mu(z_2) = 1 + \sqrt{1 - 4c}$, and $\mu(\infty) = 0$. $f_c(z)$ has one finite critical point at $z = 0$ with the corresponding critical value $f_c(0) = c$.

$f_\lambda(z)$ has two finite fixed points at $z_1 = 0$ and $z_2 = 1 - \lambda$, and a fixed point at ∞ . The corresponding multipliers are $\mu(z_1) = \mu(0) = \lambda$, $\mu(z_2) = 2 - \lambda$, and $\mu(\infty) = 0$. The finite critical point is at $z = -\frac{\lambda}{2}$ with the critical value $f_\lambda\left(-\frac{\lambda}{2}\right) = -\frac{\lambda^2}{4}$.

For the remainder of the paper we will mostly use the second normalization with a slight adjustment to the notation. Unless otherwise stated, we will use

the following convention, $f_q^p(z) = e^{2\pi i\left(\frac{p}{q}\right)} z + z^2$.

Definition. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a quadratic polynomial. Define the *filled Julia set of f* to be

$$K(f) = \{z \in \mathbb{C} \text{ such that } |f^n(z)| < \infty \text{ for all } n \in \mathbb{N}\}.$$

The complement of $K(f)$ is called the *basin of infinity* and is denoted by B_∞ .

It is clear that $B_\infty \subseteq \Omega(f)$ is completely invariant under f .

Alternative Definition of $J(f)$ for a quadratic $f : \mathbb{C} \rightarrow \mathbb{C}$:

$$J(f) = \partial K(f) = \partial B_\infty$$

The following is a standard theorem the proof of which can be found in many basic texts on complex dynamics.

Theorem 1.2.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a quadratic polynomial. Then B_∞ is simply connected if and only if the orbit of the critical point is bounded.*

Remark 1.2.1. *It follows that for any such map $J(f)$ is connected.*

■ Geometrically Finite Quadratics and Local Connectivity

Post-critically finite quadratic polynomials come in two types: (*type I*) those with a periodic critical orbit and (*type II*) those with a preperiodic critical orbit. If the critical point is periodic then the quadratic has a superattracting cycle, if it is preperiodic then it is called a *Misiurewicz* quadratic. *Type I* quadratics are centers of the hyperbolic components of the Mandelbrot set, while *type II* lie on the boundary of the Mandelbrot set. Geometrically finite quadratics with connected Julia sets consist of those of *types I* and *II*, together with another, *type III*, for which the orbit of the critical point approaches an attracting or a parabolic cycle. It follows that geometrically finite quadratics are either roots of hyperbolic components, *Misiurewicz* quadratics, or lie in the interiors of the hyperbolic components of the Mandelbrot set (see section 1.3 for a more detailed discussion). Geometrically finite quadratics do not have any Siegel disks or Herman rings, that is, Fatou components on which the dynamics are holomorphically conjugate to a rotation.

We will approach the question of local connectivity of Julia sets following Carleson and Gamelin, as in [CG].

Definition. A rational map g is called *subhyperbolic* if g is expanding on $J(g)$ for some admissible metric $\sigma(z) |dz|$, that is, if there exists $A > 1$ such that the inequality

$$\sigma(g(z)) |g'(z)| \geq A \sigma(z)$$

holds in a neighborhood of $J(g)$.

Note that geometrically finite quadratic polynomials consist of those that are

subhyperbolic and those that are parabolic.

Carleson and Gamelin provide the details of the proofs of the following three theorems, originally stated and proved by Douady and Hubbard, in [CG].

Theorem 1.2.2. (Douady-Hubbard) [CG] *Suppose $J(g) \neq \bar{\mathbb{C}}$. Then g is subhyperbolic if and only if each critical point in $J(g)$ has a finite forward orbit, while each critical point in $\Omega(g)$ is attracted to an attracting cycle.*

Theorem 1.2.3. (Douady-Hubbard) [CG] *If the polynomial p is subhyperbolic on $J(p)$, and if $J(p)$ is connected, then $J(p)$ is locally connected.*

Theorem 1.2.4. (Douady-Hubbard) [CG] *Let p be a polynomial with connected Julia set $J(p)$. If every critical point belonging to the Julia set is preperiodic, then $J(p)$ is locally connected.*

By remark 1.2.1, and, since if a quadratic polynomial f is parabolic then there are no critical points in $J(f)$, we obtain the following.

Corollary 1.2.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a geometrically finite quadratic polynomial. Then $J(f)$ is locally connected.*

Recall that $J(f) = \partial B_\infty$ where ∞ is always the superattracting fixed point of f .

The proofs of the above local connectivity theorems rely on this fact.

Using quasi-conformal surgery to convert attracting fixed points to superattracting fixed points, Carleson and Gamelin prove the following.

Theorem 1.2.5. [CG] *Suppose that each critical point of a rational map g that belongs to $J(g)$ is strictly preperiodic. If F is a simply connected component of $\Omega(g)$ that contains an attracting periodic point, then ∂F is locally connected.*

Corollary 1.2.2. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a geometrically finite quadratic polynomial. Let F_i be any bounded Fatou component of $\Omega(f)$. Then ∂F_i is locally connected.*

Proof. Assume the map f is a geometrically finite quadratic with no parabolic cycles. Then, if f has a critical point on $J(f)$ it must be preperiodic, and the map f must be a Misiurewicz quadratic. This means that $J(f)$ is a dendrite, and, in particular, that $K(f)$ has no interior, and thus no bounded Fatou components.

Now, assume f is still not parabolic but has no critical points on its Julia set. Let \hat{F} be a bounded Fatou component of $\Omega(f)$ which contains an attracting periodic point. By theorem 1.2.6, $\partial \hat{F}$ is locally connected. It follows that the same must be true for all F_i in $\Omega(f)$, since every F_i is eventually periodic and lands on the attracting Fatou cycle, the components of which each have an attracting periodic point.

Next we must consider the case where f has a parabolic cycle. Theorem 2.1.2 along with the discussion immediately following it shows the existence of a conjugacy Φ_c between f_c , where c is in the interior of some hyperbolic component H of the Mandelbrot set, and f_{r_H} , where r_H is the root of that same hyperbolic component and f_{r_H} has a parabolic orbit. The conjugacy Φ_c is a homeomorphism between $J(f_c)$ and $J(f_{r_H})$, and thus preserves the local connectivity of the boundaries of the bounded Fatou components since they are subsets of the corresponding Julia sets ■.

The Böttcher Coordinate Function and External Dynamic Rays

We will need the following two well-known theorems to define the concept of external dynamic rays.

Theorem 1.2.4.(Caratheodory) *Let \mathcal{D} be the open unit disk. Let U be a simply connected domain in $\overline{\mathbb{C}}$ whose boundary has at least two points. Then the Riemann mapping $\psi : \mathcal{D} \rightarrow U$ extends continuously to the closed disk $\overline{\mathcal{D}}$ if and only if ∂U is locally connected.*

Theorem 1.2.5.(Böttcher) *Suppose an analytic function g has a superattracting fixed point at z_0 ,*

$$g(z) = z_0 + a_p(z - z_0)^p + \dots, \quad a_p \neq 0 \text{ and } p \geq 2.$$

Then there is a conformal map $w = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates $g(z)$ to w^p . The conjugating function is unique, up to multiplication by a $(p - 1)$ th root of unity, and is referred to as the Böttcher coordinate function.

Since any polynomial $f(z) = az^d + \dots$ with $d \geq 2$ and $a \neq 0$ has a superattracting fixed point at ∞ , by replacing 0 by ∞ in the above theorem we obtain the corollary below.

Corollary 1.2.1. *Suppose f is a polynomial of degree $d \geq 2$. Then there is a conformal map $w = \phi(z)$ conjugating $f(z)$ to w^d in a neighborhood of ∞ . Furthermore, when the degree $d = 2$, the conjugating map is unique and tangent to the identity at ∞ .*

In particular, since every quadratic f_c has a superattracting fixed point at ∞ ,

the condition that the critical point of f_c is bounded allows us, by theorems 1.2.1, 1.2.4 and corollary 1.2.1, to define the following Riemann mapping:

$$\phi_c : B_\infty \longrightarrow \mathbb{C} \setminus \overline{\mathcal{D}}$$

where ϕ_c is the Böttcher coordinate and conjugates $f_c|_{B_\infty}$ to $z^2|_{\mathbb{C} \setminus \overline{\mathcal{D}}}$.

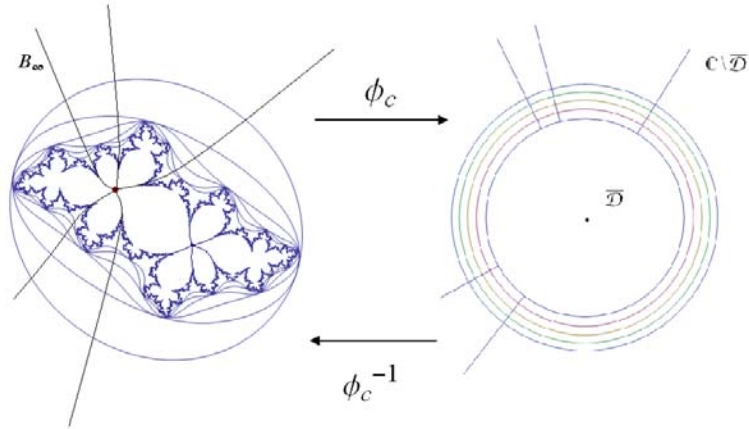


Figure 1.2.1. The Böttcher coordinate and external dynamic rays

Definition. The pre-image of the ray $\rho e^{2\pi i t}$, $\rho > 1$, for a fixed $t \in \mathbb{R}/\mathbb{Z}$, under ϕ_c , is called the **external dynamic ray of angle t** , measured in turns, and is denoted by R_t . More concisely,

$$R_t = \phi_c^{-1}(\rho e^{2\pi i t}), \text{ for all } \rho > 1$$

Note that external dynamic rays can be described as the orthogonal trajectories of the **equipotential curves** $|\phi_c(z)| = \text{constant} > 0$.

Definition. We say that R_t **lands** at $x \in \partial B_\infty = J(f)$ if $\lim_{\rho \rightarrow 1} \phi_c^{-1}(\rho e^{2\pi i t})$ exists and is equal to x . We call x the **landing point** of R_t , and denote it by $x(R_t)$.

Observe that, by the local connectivity of the corresponding Julia sets, guaranteed by theorem 1.2.4 and corollary 1.2.1, for any geometrically finite polynomial, and, in particular, for any geometrically finite quadratic all external rays land.

Definition. When $J(f)$ is locally connected, theorem 1.2.4 allows us to define a continuous map $\gamma_f : \mathbb{R}/\mathbb{Z} \rightarrow J(f)$ called the *Caratheodory semiconjugacy* which satisfies

$$\gamma_f(2t) = f(\gamma_f(t))$$

With one exception, $f(z) = z^2 + \frac{1}{4}$, every quadratic polynomial has two fixed points, and they are distinguished by the following definition.

Definition. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a quadratic polynomial, not conformally conjugate to $f(z) = z^2 + \frac{1}{4}$. Then we say that a fixed point is the ***β -fixed point***, simply denoted by β , if it is the landing point of R_0 , or, equivalently, $\beta = \gamma_f(0)$. The other finite fixed point is the ***α -fixed point***, which we will denote by α .

1.3 The Mandelbrot Set and External Parameter Rays

■ The Mandelbrot Set

Definition. We define the *quadratic connectedness locus*, denoted by \mathbf{M} , as the set of conformal equivalence classes of complex quadratic polynomials with connected Julia sets. At times it will be useful to distinguish between the different normalizations discussed earlier. \mathbf{M} can be embedded in \mathbb{C} in two natural ways. If the normal form being used is $f_c = z^2 + c$, define \mathbf{M}_c as the set of parameters $c \in \mathbb{C}$ for which $J(f_c)$ is connected. This is the usual definition of the *Mandelbrot set*. If the form being used is $f_\lambda = \lambda z + z^2$, define \mathbf{M}_λ as the set of parameters $\lambda \in \mathbb{C}$ with $\operatorname{Re}\{\lambda\} \leq 1$ for which $J(f_\lambda)$ is connected. By a slight abuse of nomenclature, when the choice of an embedding is not be specified, we will refer to \mathbf{M} as the *Mandelbrot set*.

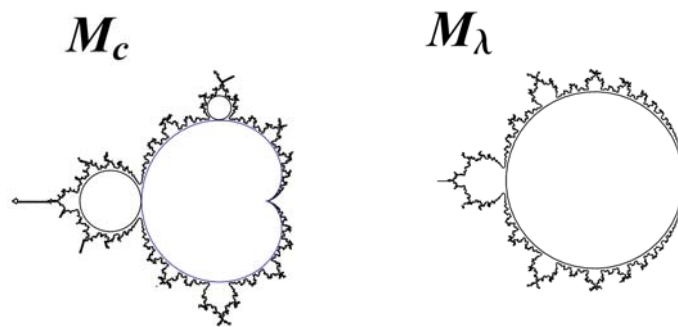


Figure 1.2.2. Different embeddings of the quadratic connectedness locus

The interior of \mathbf{M}_c contains simply connected components H such that for each $c \in H$, f_c has an attracting periodic cycle of the same period. There is a holomorphic map from the parameter c to each of the periodic points in the

attracting cycle of f_c . These components are called **hyperbolic**. The **period** of a hyperbolic component is the period of this attracting cycle.

For each $H_i \subset M_c$ we can define the **multiplier map** $\mu: H_i \rightarrow \mathcal{D}$ which sends the point $c \in H_i$ to the multiplier μ of the attracting cycle of f_c . This map is a holomorphic homeomorphism which extends continuously to ∂H_c , and, in particular to the points where μ is a rational multiple of π .

Definition. Let H be a hyperbolic component of M_c . The **root** r_H of H is the point $\mu^{-1}(1)$ on ∂H . The **center** c_H is the point $\mu^{-1}(0)$.

At each r_{H_i} there is a parabolic cycle and at each c_{H_i} there is a super-attracting cycle.

Hyperbolic components can be divided into two types:

Definition. Suppose the period of H is q and let z be a point in the parabolic cycle of f_{r_H} . Let R_t be an external ray landing at z . If $f_{r_H}^{\circ q}(R_t) = R_t$ the component is **primitive**. Otherwise it is a **satellite**.

The above terminology comes from the fact that root of a satellite component of period q is a boundary point of another hyperbolic component of lower period \hat{q} where $\hat{q} \mid q$. The root of a primitive component does not belong to the boundary of any other hyperbolic component.

Definition. A **chain of hyperbolic components** is a finite collection of components $\{H_i\}_{i=k}^n$, where $0 \leq k < n$, with the property that $r_{H_{i+1}} \in \partial H_i$ for $i = k, k + 1, \dots, n - 1$. We say that the chain goes from H_k to H_n .

The component H_k may or may not be a primitive. If it is not, we can extend

the chain $\{H_i\}_{i=k}^n$ to a chain $\{H_i\}_{i=0}^n$ such that H_0 is primitive.

■ External Parameter Rays

Definition. In the parameter plane there exists a Green's function $G_M : \mathbb{C} \rightarrow [0, \infty)$ which vanishes on M , is harmonic off M and is asymptotic to $\log |z|$ at ∞ . We can use this function as the parameter plane analog of the Böttcher coordinate function ϕ_c in the dynamic plane and define *external parameter rays* as the orthogonal trajectories of the equipotential curves $G_M(z) = \text{constant} > 0$.

An external parameter ray meets the circle at ∞ at a specific angle $2\pi i t$. We call t the angle of the ray, and denote the ray by \mathcal{R}_t . The doubling map $t \mapsto 2t \bmod 1$ on the circle at ∞ extends naturally to the external parameter rays.

Definition. In analogy with external dynamic rays, we say that an external parameter ray \mathcal{R}_t *lands* at a point $c(\mathcal{R}_t) \in \mathbf{M}$ if $\lim_{G_{\mathbf{M}}(z) \rightarrow 0} \mathcal{R}_t(z)$ exists.

Theorem 1.3.1. (Douady, Hubbard) [DH]

- (1) For every rational argument $t = \frac{p}{q}$ the external parameter ray \mathcal{R}_t lands;
- (2) If q is odd then $c(\mathcal{R}_t)$ is a root r_H of a hyperbolic component H whose period is equal to the period of t under angle doubling. Furthermore, r_H is actually the landing point of a total of two external parameter rays, \mathcal{R}_t and \mathcal{R}_s . In addition, there are two external dynamic rays, R_t and R_s , to $K(f_{r_H})$ landing at the parabolic periodic point on the boundary of the Fatou component F_1 containing the critical value. The angles of the dynamic rays are the same as the angles of the parameter rays;
- (3) If q is even then $c(\mathcal{R}_t) = c$ is a Misiurewicz point and is the landing point of exactly one external parameter ray. Furthermore, the external dynamic ray of the same argument, R_t , lands at the critical value of f_c .

Chapter 2. Generalized Starlike Quadratics

2.1 Definitions, Examples and Holomorphic Motions

Definition. We will say that a quadratic polynomial f is *generalized starlike* if it belongs to the interior or is the root of a hyperbolic component H_q^p of the Mandelbrot set which is immediately attached to the main cardioid M_0 .

For the remainder of the paper we will denote $H_q^p \cup r_q^p$ by H_q^{p*} .

Wherever the term "starlike polynomial" appeared in literature, it was defined as a postcritically finite polynomial p with exactly one non-escaping critical point and the property that its associated Hubbard tree has a single branch point located at a single common repelling fixed point α . Starlike quadratics are, therefore, traditionally defined as postcritically finite quadratics whose Hubbard tree \mathbf{Tr}_0 is a union of a finite number of "internal rays", each joining a point in the finite superattracting cycle containing the critical point to a common repelling fixed point α . Thus \mathbf{Tr}_0 is a "star". The tree \mathbf{Tr}_0 is mapped homeomorphically by the quadratic into itself simply permuting the "internal rays". Accordingly, starlike quadratics are simply the centers of $H_{\frac{p}{q}}$. See

Figure 2.1.1 below for an example.

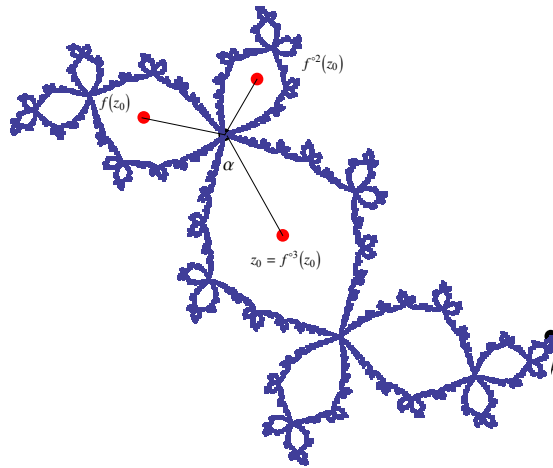


Figure 2.1.1. \mathbf{Tr}_0 and $J(f_\lambda)$ where f_λ is the center of $H_{1/3}$

Our definition is a generalization of this idea, which should clarify the terminology used. Notice that none of the combinatorial information about

such a quadratic is lost by generalizing the definition.

Definition. For the remainder of the paper we will refer to quadratics of the form $f_{\frac{p}{q}}(z) = e^{2\pi i\left(\frac{p}{q}\right)}z + z^2$ with $p < q$ and $(p, q) = 1$ as *parabolic starlike*. In addition, we will let $f_{\frac{p}{q}}^*(z) = a_{\frac{p}{q}}z + z^2$, where $a_{\frac{p}{q}} = \text{center of } H_{\frac{p}{q}}$, and will refer to these as *classically starlike*.

■ Holomorphic Motions of Julia Sets

For a full discussion of the following concepts see [McM].

Definition. Let X be a connected complex manifold, $\lambda \in X$ and $z \in \bar{\mathbb{C}}$. Let $f : X \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ be a *holomorphic family of rational maps* $\{f_\lambda\}$, parametrized by X , where $f_\lambda : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ is a rational map for each $\lambda \in X$ and the map $f : X \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ is itself holomorphic.

Definition. Let x be a basepoint in X . A *holomorphic motion of a set* $E \subset \bar{\mathbb{C}}$ parametrized by (X, x) is a family of injections

$$\Phi_\lambda : E \rightarrow \bar{\mathbb{C}},$$

one for each $\lambda \in X$, such that $\Phi_\lambda(e)$ is a holomorphic function of λ for each fixed $e \in E$, and $\Phi_x = \text{Identity}|_E$.

Theorem 2.1.1. (The λ -Lemma) [MSS] *A holomorphic motion of E has a unique extension to a holomorphic motion of \bar{E} . The extended motion gives a continuous map $\Phi : X \times \bar{E} \rightarrow \bar{\mathbb{C}}$. For each $\lambda \in X$, the map $\Phi_\lambda : E \rightarrow \bar{\mathbb{C}}$ extends to a quasi-conformal map of the sphere to itself.*

Definition. Given a holomorphic family of rational maps, $\{f_\lambda\}$, parametrized by X , and a point $x \in X$, we say that the corresponding Julia sets J_λ **move holomorphically at x** if there exists a neighborhood U of x and a holomorphic motion

$$\Phi_\lambda : J_x \longrightarrow \mathbb{C} \text{ with } \lambda \in U$$

such that

$$\Phi_\lambda(J_x) = J_\lambda$$

and

$$\Phi_\lambda \circ f_x(z) = f_\lambda \circ \Phi_\lambda(z) \text{ for all } z \in J_x.$$

Thus Φ_λ provides a conjugacy between f_x and f_λ on their respective Julia sets.

Among other things, McMullen proves the following.

Theorem 2.1.2. [McM] *Let $\{f_\lambda\}$ be a holomorphic family of rational maps parametrized by X , and let $x \in X$. Then the following conditions are equivalent:*

- (1) *The number of attracting cycles of f_λ is locally constant at x .*
- (2) *The maximum period of an attracting cycle of f_λ is locally bounded at x .*
- (3) *The Julia sets J_λ move holomorphically at x .*
- (4) *The Julia sets J_λ depend continuously on λ (in the Hausdorff topology) in a neighborhood of x .*

Now, let H be a hyperbolic component of the Mandelbrot set, and let c_H be its center. The above theorem establishes the existence of a conjugacy Φ_c , for any

$c \in H$, between the Julia sets $J_c = J(f_c)$ and $J_{c_H} = J(f_{c_H})$. Note that Φ_c preserves the dynamics, that is, $\Phi_c(J_{c_H}) = J_c$ and $f_c \Phi_c(z) = \Phi_c f_{c_H}(z)$ for any $z \in J_{c_H}$.

Another important ingredient is Haïssinsky's parabolic surgery theorem. It is stated below, with slight changes in notation, as it appears in [LH], however, its proof is fully discussed in [H].

Theorem 2.1.3. *Let g_1 be a sub-hyperbolic rational map with an attracting cycle \mathcal{A} of period n and a repelling cycle \mathcal{B} of period less than n on the boundary of the immediate basin of \mathcal{A} . Assume that there exists an invariant access to \mathcal{B} from \mathcal{A} 's immediate basin. Then there are another rational map g_2 and an orientation preserving homeomorphism ϕ , locally quasi-conformal in $\Omega(g_1)$, univalent in the basin of ∞ and tangent to the identity at ∞ such that:*

- (i) $\phi(J(g_1)) = \phi(J(g_2))$, $\phi(\mathcal{B})$ is parabolic and the immediate basin of \mathcal{A} becomes the immediate basin of $\phi(\mathcal{B})$;
- (ii) outside \mathcal{A} 's immediate basin, $\phi \circ f = g \circ \phi$; in particular, $\phi : J(g_1) \rightarrow J(g_2)$ is a homeomorphism which conjugates the dynamics.

Applying the above results to the special case of generalized starlike quadratics we obtain the theorem below.

Theorem 2.1.4. *For a fixed $\frac{p}{q}$ with $p < q$ and $(p, q) = 1$, define $f_{\frac{p}{q}}(z)$ and $f_{\frac{p}{q}}^*(z)$ as above, and let f be any other map in the interior of $H_{\frac{p}{q}}$, then $f_{\frac{p}{q}}(z)$, $f_{\frac{p}{q}}^*(z)$ and f are conjugate on their respective Julia sets, and $J(f)$, $J\left(f_{\frac{p}{q}}^*(z)\right)$*

and $J\left(f_{\frac{p}{q}}^p(z)\right)$ are all homeomorphic. More precisely, there are homeomorphisms of the sphere which conjugate these maps on their respective Julia sets. Furthermore, given two such maps, the homeomorphism can be chosen to be quasi-conformal if neither of the two is $f_{\frac{p}{q}}^p$. If one of the two is $f_{\frac{p}{q}}^p$ the homeomorphism is provided by theorem 2.1.3.

Note that, using \mathbf{M}_λ as the preferred embedding of the Mandelbrot set in parameter space, the roots of starlike components $H_{\frac{p}{q}}^{p*}$ are exactly the points $e^{2\pi i\left(\frac{p}{q}\right)}$ on $\partial\mathcal{D}$.

Remark 2.1.1. Let $f \in H_{\frac{p}{q}}^{p*}$, then $\Omega(f)$ contains exactly one Fatou cycle.

This cycle consists of q Fatou components, $\{F_0, F_1 \dots F_{q-1}\}$, which satisfy the following condition:

$$\overline{F_i} \cap \overline{F_j} = \alpha \text{ for } i \neq j$$

All other Fatou components are eventually periodic and land on this Fatou cycle.

2.2 Symbolic Dynamics

■ Terminology and Basic Lemmas

Let Σ_2 be the space of one-sided infinite binary sequences. Let Σ_2^* be Σ_2 together with the usual identifications. Then Σ_2^* is homeomorphic to \mathbb{R}/\mathbb{Z} , via the map $t: \Sigma_2^* \rightarrow \mathbb{R}/\mathbb{Z}$ defined as $X \mapsto t(X) \in \mathbb{R}/\mathbb{Z}$, where X is the binary expansion of $t(X)$. Let σ be the usual *left shift map* on Σ_2^* , given by $\sigma(x_1 x_2 x_3 \dots) = x_2 x_3 x_4 \dots$.

Definition. Let $X \in \Sigma_2^*$ be a periodic, or, equivalently, a repeating sequence of the form $\overline{x_1 x_2 x_3 \dots x_q}$. To it associate a *finite string* $S_X = x_1 x_2 x_3 \dots x_q$ of length q , where S_X is the repeating block of X . Conversely, given a finite binary string S , to it we may associate a periodic binary sequence $X_S = S S S \dots = \overline{S}$.

Definition. Any finite binary string S of length q is *prime* if none of its cyclic permutations, S_j , $1 \leq j \leq q$, can be written as a string of shorter equal strings. That is, S is prime if there exists no shorter finite string T such that $S_j = T T T \dots T$.

The lemma below follows directly from the two definitions above.

Lemma 2.2.1. *A string S of length q is prime if and only if q is the smallest integer such that $\sigma^q(X_S) = X_S$.*

Definition. Let S and T be two different prime strings of length q , then S is *cyclically equivalent* to T , that is, $S \sim T$, if S is a cyclic permutation of T , and we say that they belong to the same *cyclic equivalence class*, denoted by $[S]$.

Lemma 2.2.2. *Let S and T be two different prime strings of length q , then $S \sim T$ if and only there exists a $p < q$ such that $\sigma^p(X_T) = X_S$.*

Call the corresponding sequences X_S and X_T **cyclically equivalent** as well, and denote the equivalence class of such sequences $[X]$.

Evidently, the length of a prime finite string S_X is equal to the period of the corresponding sequence X_S .

Lemma 2.2.3. *The length of a prime string is equal to the number of elements in its cyclic equivalence class, and each element of the equivalence class has the same number of "1"s.*

Proof. Let S be a prime string of length q , and let $X = X_S \in \Sigma_2^*$ be as defined above. Then, by the previous two lemmas,

$S = S_X \neq S_{\sigma^1(X)} \neq S_{\sigma^2(X)} \neq \dots \neq S_{\sigma^i(X)}$ for all $i < q$ but

$S = S_X \sim S_{\sigma^1(X)} \sim S_{\sigma^2(X)} \sim \dots \sim S_{\sigma^i(X)}$ for all $i < q$. Therefore, $\# [S] = q$, and each

element of $[S]$ has the same number of "1"s, call it p .

Corollary 2.2.3. *Let $X \in \Sigma_2^*$ be a periodic binary sequence. Then its period, say q , is equal to the number of elements in its cyclic equivalence class, and each element of the equivalence class has the same number of "1"s, say p , in its repeating block S_X .*

■ Admissible Equivalence Classes

The validity of the main results in the remainder of this chapter is guaranteed by the corresponding theorems about orbit portraits proved by Milnor in [Mil1]. Although alternative proofs are not discussed here, the theorems can be proved directly using only combinatorics and induction. The statements appear here to demonstrate an algorithm for calculating distinguished ray cycles. This topic, with a heavier emphasis on symbolic dynamics, is addressed in much more detail in [BS] and [FM]. For a detailed discussion of the topic see [V], where the results below are discussed carefully, and strictly within the setting of symbolic dynamics. Another great source is [CAM] available for free on the internet.

Definition. Recall the definition of the map $t: \Sigma_2^* \rightarrow \mathbb{R}/\mathbb{Z}$. Let $[X]$ be a cyclic equivalence class of sequences. Define the *maximum element* of $[X]$ as the unique sequence X_M , for which, $t(X_M) = \max \{t(X_i) \mid X_i \in [X]\}$, and the *minimum element* of $[X]$ as the unique sequence X_m , for which, $t(X_m) = \min \{t(X_i) \mid X_i \in [X]\}$.

Definition. Call $[X]$ *admissible* if $t(X_M) - t(X_m) < \frac{1}{2}$.

Proposition 2.2.1.(Definition of C_q^p). Given two relatively prime positive integers, $q \geq 2$ and $p < q$, there exists a unique admissible equivalence class, call it C_q^p , consisting of q strictly periodic sequences X_i of period q , such that S_{X_i} is prime and contains exactly p "1"s.

Proof. (*General Existence*) Let p and q be two relatively prime positive integers with $p < q$. Let S be a binary string consisting of p "1"s and $q - p$ "0"s. Assume that S is not prime. This implies that S or one of its cyclic permutations, call it S' , can be written as a finite string of k shorter equal strings. That is, there exists a shorter string, call it T , such that $S' = \underbrace{TTT \dots T}_{k \text{ times}}$.

But this implies that p must be divisible by k , $q - p$ must be divisible by k , and q must be divisible by k . But this contradicts the fact that p and q are relatively prime. Hence, S must be a prime string. By lemma 2.2.1, the associated sequence X_S must be periodic of period q , and, by corollary 2.2.3, its cyclic equivalence class, $[X_S]$, must consist of q elements and each element must contain p "1"s. Note that the converse of the above is also true. That is, if a prime string S_{X_i} has length q and contains p "1"s, then p and q must be relatively prime.

(*Existence and Uniqueness of Admissible Strings*) This is a special case of theorem 4.1.3 where C_q^p is the orbit portrait of the α -fixed point of a generalized starlike quadratic $f \in H_{\frac{p}{q}}^*$.

2.3 Distinguished Ray Cycles

Definition. A *ray cycle, RC, of period q* is a collection of q external rays $\{R_{t_i}\}_{i=1}^q$ such that, for each i , q is the smallest number for which $2^q t_i \bmod 1 = t_i$ and $t_i = t_j$ if and only if $i = j$.

Lemma 2.3.1. *Let $q \geq 2$ be any fixed integer. Let k be an integer with $0 < k < 2^q - 1$ such that there exists no $\hat{q} < q$ for which $k(2^{\hat{q}} - 1)$ is an integer multiple of $2^q - 1$. Then we can associate a collection \mathcal{A}_q of external rays R_t with $t = \frac{k}{2^q - 1}$ such that \mathcal{A}_q can be partitioned into ray cycles of period q .*

Recall that $\alpha_{\frac{p}{q}}$ is the α -fixed point of the parabolic starlike quadratic $f_{\frac{p}{q}}$.

Definition. A ray cycle RC_i in \mathcal{A}_q is **distinguished** if, for every $R_t \in RC_i$, $x(R_t) = \alpha_{\frac{p}{q}}$; it is denoted by ${}_{\alpha}RC_{\frac{p}{q}}$.

Note that not all ray cycles in \mathcal{A}_q are distinguished.

Theorem 2.3.1. $(C_q^p \longleftrightarrow H_{\frac{p}{q}}^*)$. *There is a canonical one-to-one correspondence between the set of equivalence classes $\{C_q^p\}$ and the collection of H_i^* , such that $i = \frac{p}{q}$, and, for any $f \in H_{\frac{p}{q}}^*$, the set of external rays landing at the α -fixed point of f is precisely the distinguished ray cycle given by*

$${}_{\alpha}RC_{\frac{p}{q}} = \left\{ R_{t(X_j)} \mid X_j \in C_q^p \right\}.$$

Proof. As was the case for proposition 2.2.1, this is a special case of theorem 4.1.3 where C_q^p is the orbit portrait of the α -fixed point of a generalized starlike quadratic $f \in H_{\frac{p}{q}}^*$, and ${}_{\alpha}RC_{\frac{p}{q}}$ is the associated ray cycle.

Example 2.3.1:

Let $q = 5$, then

$$\mathcal{A}_5 = \left\{ R_{\frac{1}{2^5-1} = \frac{1}{31}}, R_{\frac{2}{31}}, R_{\frac{3}{31}}, R_{\frac{4}{31}}, R_{\frac{5}{31}}, R_{\frac{6}{31}} \dots R_{\frac{29}{31}}, R_{\frac{30}{31}} \right\}$$

with $\sharp(\mathcal{A}_5) = 30$. The number of ray cycles is 6. However, we know that the number of immediately attached hyperbolic components with rotation number $\frac{p}{5}$ is 4. So, \mathcal{A}_5 contains 4 distinguished ray cycles:

$${}_{\alpha}RC_{\frac{1}{5}}, {}_{\alpha}RC_{\frac{2}{5}}, {}_{\alpha}RC_{\frac{3}{5}}, {}_{\alpha}RC_{\frac{4}{5}}.$$

t	x_t	s_{x_t}	t	x_t	s_{x_t}	t	x_t	s_{x_t}
$\frac{1}{31}$	$\overline{.00001}$	00001	$\frac{3}{31}$	$\overline{.00011}$	00011	$\frac{5}{31}$	$\overline{.00101}$	00101
$\frac{2}{31}$	$\overline{.00010}$	00010	$\frac{6}{31}$	$\overline{.00110}$	00110	$\frac{10}{31}$	$\overline{.01010}$	01010
$\frac{4}{31}$	$\overline{.00100}$	00100	$\frac{12}{31}$	$\overline{.01100}$	01100	$\frac{20}{31}$	$\overline{.10100}$	10100
$\frac{8}{31}$	$\overline{.01000}$	01000	$\frac{24}{31}$	$\overline{.11000}$	11000	$\frac{9}{31}$	$\overline{.01001}$	01001
$\frac{16}{31}$	$\overline{.10000}$	10000	$\frac{17}{31}$	$\overline{.10001}$	10001	$\frac{18}{31}$	$\overline{.10010}$	10010
$\frac{16}{31} - \frac{1}{31} < \frac{1}{2}$	\ominus	$[00001] = c_5^1$	$\frac{24}{31} - \frac{3}{31} > \frac{1}{2}$	\ominus	$[00011] \neq c_5^2$	$\frac{20}{31} - \frac{5}{31} < \frac{1}{2}$	\ominus	$[00101] = c_5^2$

t	x_t	s_{x_t}	t	x_t	s_{x_t}	t	x_t	s_{x_t}
$\frac{15}{31}$	$\overline{.01111}$	01111	$\frac{7}{31}$	$\overline{.00111}$	00111	$\frac{11}{31}$	$\overline{.01011}$	01011
$\frac{30}{31}$	$\overline{.11110}$	11110	$\frac{14}{31}$	$\overline{.01110}$	01110	$\frac{22}{31}$	$\overline{.10110}$	10110
$\frac{29}{31}$	$\overline{.11101}$	11101	$\frac{28}{31}$	$\overline{.11100}$	11100	$\frac{13}{31}$	$\overline{.01101}$	01101
$\frac{27}{31}$	$\overline{.11011}$	11011	$\frac{25}{31}$	$\overline{.11001}$	11001	$\frac{26}{31}$	$\overline{.11010}$	11010
$\frac{23}{31}$	$\overline{.10111}$	10111	$\frac{19}{31}$	$\overline{.10011}$	10011	$\frac{21}{31}$	$\overline{.10101}$	10101
$\frac{30}{31} - \frac{15}{31} < \frac{1}{2}$	\ominus	$[01111] = c_5^4$	$\frac{28}{31} - \frac{7}{31} > \frac{1}{2}$	\ominus	$[00111] \neq c_5^3$	$\frac{26}{31} - \frac{11}{31} < \frac{1}{2}$	\ominus	$[01011] = c_5^3$

Chapter 3. Laminations and Accessibility

Most of the statements and theorems in this chapter are consequences of the more general theorems discussed in chapter 5. When necessary, we will refer to the theorems of chapter 5 for the proofs of the claims made here. The main goal of this chapter is to establish the definitions, classifications and constructions needed to properly describe the dynamics and geometry of quasi-self-matings of generalized starlike quadratics which are addressed in chapter 7 and are the main focus of this thesis.

3.1 Accessibility on the Julia Sets of Starlike Quadratics

Let f be any quadratic polynomial with a locally connected Julia set $J(f)$. Let $K(f) = \{z : |f^n(z)| < \infty \text{ for all } n \in \mathbb{N}\}$ be the filled Julia set. Then $J(f) = \partial K(f)$.

Definitions. A point $x \in J(f)$ is

- **externally accessible** if there is a path γ^E in $\mathbb{C} - K(f)$ which converges to x ;
- **q -accessible** if it is the landing point of exactly q external rays;
- **uniasccessible** if it is the landing point of exactly one external ray;
- **internally accessible** if there exists a path γ^I in the interior of $K(f)$ which converges to x .

The following two lemmas are thoroughly discussed and proved in [McM].

Lemma 3.1.1. *A point $x \in J(f)$ is externally accessible if and only if it is a landing point of at least one external dynamic ray R_t .*

Lemma 3.1.2. *A point $x \in J(f)$ is q -accessible if and only if $K(f) - \{x\}$ has q*

components.

Lemma 3.1.3. *With f as above, a point $x \in J(f)$ is internally accessible if and only if x belongs to the boundary of a Fatou component.*

Proof. Recall that f is a quadratic polynomial with $J(f)$ locally connected. Let x belong to the boundary of a Fatou component F_i , which, by definition, is a non-empty open connected set. It is then possible to construct a path in its interior which converges to $x \in \partial F_i \subset J(f)$ since ∂F_i is locally connected by corollary 1.2.2.

Assume that $x \in J(f)$ is internally accessible. Then, by definition, there exists a path $\gamma^I : [0, 1) \rightarrow \text{int}(K(f))$ such that $\gamma^I(s) \rightarrow x$ as $s \rightarrow 1$. If $\text{int}(K(f))$ has more than one component then the entire path must lie in a single non-empty open component F_i and converge to $x \notin F_i$. It follows that x must lie on the boundary of F_i . By definition, F_i is a Fatou component. ■

Proposition 3.1.1. *Let p and q be two relatively prime positive integers with $p < q$. Let $f \in H_{\frac{p}{q}}^*$. Then $J(f)$ is locally connected and consists of landing points of exactly one or q external rays, and we have the following classification:*

- (i) *if t is irrational then $x(R_t)$ is uniaccessible.*
- (ii) *$x(R_{t_i}) = \alpha$ if and only if t_i 's are q -periodic under doubling and their cyclic order is the same as the cyclic order of the Fatou components joined at α ;*
- (iii) *if t is rational then $x(R_t)$ is q -accessible if and only if there exists a*

positive integer n such that $R_{2^n t \pmod{1}} \in {}_\alpha RC_{\frac{p}{q}}$, otherwise it is uniaccessible;

Proof. The local connectivity of $J(f)$ follows from theorem 1.2.2. Statement (i) follows from the fact that if t is irrational then t cannot be periodic under the doubling $(\text{mod } 1)$, which implies that the dynamic ray R_t cannot be periodic under f . This, in turn, implies that the landing point $x(R_t)$ cannot be the landing point of any other dynamic ray.

Statements (ii) and (iii) follow from theorem 4.4.2. Here the α -fixed point of $f \in H_{\frac{p}{q}}^*$ is exactly the dynamic root in the statement of theorem 4.4.2 ■

As a corollary we have

Corollary 3.1.1. *Let p and q be two relatively prime positive integers with $p < q$. Let $f \in H_{\frac{p}{q}}^*$ and $x \in J(f)$. Then x is q -accessible if and only if $x \in GO_f(\alpha)$.*

The following remark is immediate.

Remark 3.1.1. *Let p and q be two relatively prime positive integers with $p < q$. Let $f \in H_{\frac{p}{q}}^*$ and $x \in J(f)$. If $x \in GO_f(\beta)$ then x is uniaccessible.*

We would like to further subdivide the set of uniaccessible points of the Julia set in the following way.

Definition. We will say that a uniaccessible point $x \in J(f) \setminus GO_f(\beta)$ is of α -*type* if it is internally accessible, and β -*type* otherwise.

3.2 α -Laminations for Starlike Quadratics and the Associated

Constructions

■ General Laminations

Identify $S^1 = \mathbb{R}/\mathbb{Z}$ with the boundary of the open unit disk \mathcal{D} via the map $t \mapsto e^{2\pi i t}$. Given an equivalence relation on the unit circle, we will represent the convex hulls of the equivalence classes inside the disk using the hyperbolic metric.

Definition. A *lamination* $\mathcal{L} \subset S^1 \times S^1$ is an equivalence relation on the unit circle such that the convex hulls of distinct equivalence classes are disjoint.

Definition. Sets of equivalent points on $\partial\mathcal{D}$ are then joined by hyperbolic geodesics in \mathcal{D} called *leaves*, denoted by $[[t_1, t_2]]$, where t_1 and t_2 are the arguments of the endpoints of the closure of a given leaf.

If an equivalence class consists of more than two points then its convex hull is the closure of an ideal hyperbolic polygon in \mathcal{D} .

Definition. The *length* of a leaf is defined as $L[[t_1, t_2]] = \min\{|t_1 - t_2|, 1 - |t_1 - t_2|\}$.

Definition. An equivalence class of \mathcal{L} is *trivial* if it consists of a single point t , and will be referred to as a *degenerate leaf* $[[t]]$ in \mathcal{D} , with a single endpoint on $\partial\mathcal{D}$.

Definition. A lamination is *finite* if the union of its nontrivial equivalence classes is finite.

Definition. A lamination is *closed* if the union of its nontrivial equivalence

classes is closed with respect to the product topology on $S^1 \times S^1$.

In terms of the above geometric representation of \mathfrak{L} , \mathfrak{L} is closed if it is finite, or if it is infinite and the union of the endpoints of the closures of its nontrivial leaves is a closed subset of $\partial \mathcal{D}$.

■ **α -Laminations, α -Polygons and Gaps**

Definition. Let $f \in H_{\frac{p}{q}}^*$. We will say that $t_1 \sim_{\alpha} t_2$ if and only if $x(R_{t_1}) = x(R_{t_2}) = x \in J(f)$ and $f^{\circ n}(x) = \alpha$ for some positive integer n . Define the α -**lamination** associated with $H_{\frac{p}{q}}^*$, denoted by $\mathfrak{L}_{\alpha}(\frac{p}{q})$, as the set of equivalence classes defined by the above equivalence relation. Note that this definition is independent of the choice of f in $H_{\frac{p}{q}}^*$.

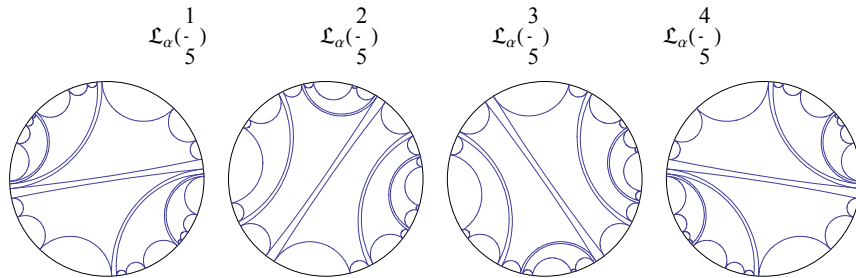


Figure 3.2.1. Finite depth α -laminations for $q = 5$

Note $\mathfrak{L}_{\alpha}(\frac{p}{q})$ is countably infinite but is not closed. In addition, $\mathfrak{L}_{\alpha}(\frac{p}{q})$ is invariant under the **angle doubling map**, $dbl: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$, defined by $dbl(t) = 2t \text{ mod } 1$.

Definition. The α -**polygon** $Q_{\alpha}(\frac{p}{q})$ (ignoring the case when $\frac{p}{q} = \frac{1}{2}$) associated with $f_{\frac{p}{q}}$ is the ideal hyperbolic polygon in \mathcal{D} which is the convex hull of the

unique equivalence class of $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$ consisting of the points t_j for which $x(R_{t_j}) = \alpha$.

As a consequence of proposition 3.1.1, we may define the following collection of q -sided ideal polygons.

Definition. Let $\mathbb{L}_\alpha\left(\frac{p}{q}\right)$ be the set consisting of $Q_\alpha\left(\frac{p}{q}\right)$ and all of the countably many ideal q -gons corresponding to distinct equivalence classes of $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$, one for each $\alpha_i \in GO_{f_q^p}(\alpha)$.

Definition. The action of the angle doubling map on $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$ induces a **lamination map** $\Lambda_{\frac{p}{q}}: \overline{\mathbb{L}_\alpha\left(\frac{p}{q}\right)} \rightarrow \overline{\mathbb{L}_\alpha\left(\frac{p}{q}\right)}$ which in turn induces a map defined on the complementary regions in $\overline{\mathcal{D}}$, as discussed in lemma 3.2.2 below. This extended lamination map, which, by abuse of notation, we still denote by $\Lambda_{\frac{p}{q}}$, will later be shown to be topologically semi-conjugate to $f_q^p: K\left(f_q^p\right) \rightarrow K\left(f_q^p\right)$ (see sections 4.3 and 4.4).

First, we would like to show that the q -gons in $\mathbb{L}_\alpha\left(\frac{p}{q}\right)$ are disjoint and accumulate on $\partial\mathcal{D}$.

Lemma 3.2.1. All q -gons in $\mathbb{L}_\alpha\left(\frac{p}{q}\right)$ are (i) mutually disjoint and (ii) their vertices form a dense subset of $\partial\mathcal{D}$.

Proof. (i) Let $Q_{\alpha_i}\left(\frac{p}{q}\right)$ and $Q_{\alpha_j}\left(\frac{p}{q}\right)$ be two distinct polygons in $\mathbb{L}_\alpha\left(\frac{p}{q}\right)$ corresponding to two distinct points, α_i and α_j , in $GO_{f_q^p}(\alpha)$. Since the external

rays to the Julia set cannot intersect outside the Julia set, the rays landing at α_i cannot intersect those landing at α_j . It follows that the q rays landing at α_i must lie between two rays landing at α_j and, therefore, the q vertices of $\mathcal{Q}_{\alpha_i}(\frac{p}{q})$ must lie between two adjacent vertices of $\mathcal{Q}_{\alpha_j}(\frac{p}{q})$.

(ii) Given $\mathbb{L}_\alpha(\frac{p}{q})$, let C_q^p be the associated equivalence class of periodic binary sequences as defined in proposition 2.2.1, where each element, X_k , of C_q^p corresponds to a unique vertex $\mathbf{t}(X_k) = t_k$ of the α -polygon $\mathcal{Q}_\alpha(\frac{p}{q})$. Note that $t \in \mathbb{Q}/\mathbb{Z}$ is a vertex of some other polygon $\mathcal{Q}_{\alpha_i}(\frac{p}{q})$ if and only if $2^q t \bmod 1 \neq t$ and there exists an integer n such that $2^n t \bmod 1 = \mathbf{t}(X_k)$ for some $X_k \in C_q^p$. It follows that t is a vertex of $\mathcal{Q}_{\alpha_i}(\frac{p}{q})$ if and only if the binary expansion X_t of t is eventually repeating with a repeating block S_{X_k} for some $X_k \in C_q^p$. Now we would like to show that given any $\tau \in \mathbb{R}/\mathbb{Z}$ which is not a vertex of some $\mathcal{Q}_{\alpha_i}(\frac{p}{q})$ and any $\varepsilon > 0$ there exists a $t \in \mathbb{R}/\mathbb{Z}$ such that t is a vertex of some $\mathcal{Q}_{\alpha_i}(\frac{p}{q})$ and $|t - \tau| < \varepsilon$.

Choose any $X_k \in C_q^p$, and let S_{X_k} be the associated finite binary string. Let $X_\tau = .y_1 y_2 y_3 \dots$, $y_l \in \{0, 1\}$, be the binary expansion of τ . Fix an integer N large enough so that $0 < \frac{1}{2^N} < \varepsilon$. Construct a binary sequence X_0 such that the first N entries are equal to the first N entries of X_τ followed by $\overline{S_{X_k}}$. That is,

$X_0 = .y_1 y_2 y_3 \dots y_{N-1} y_N \overline{S_{X_k}}$. It follows that $t(X_0)$ is a vertex of some $\mathcal{Q}_{\alpha_i}(\frac{p}{q})$,

and $\left| t(X_0) - \tau \right| \leq \frac{\frac{1}{2^{N+1}}}{1 - \frac{1}{2}} = \frac{1}{2^N} < \varepsilon$. ■

Note that the collection $\mathbb{L}_\alpha(\frac{p}{q})$ is invariant under the lamination map $\Lambda_{\frac{p}{q}}$.

Definition. A *gap* G of $\mathbb{L}_\alpha(\frac{p}{q})$ is a component of the complement of the closures of elements of $\mathbb{L}_\alpha(\frac{p}{q})$. Note that, by construction, gaps of $\mathbb{L}_\alpha(\frac{p}{q})$ are open infinite sided ideal polygons in \mathcal{D} . Denote the collection of these gaps by $\mathcal{G}_\alpha(\frac{p}{q})$.

Lemma 3.2.2. *The lamination map $\Lambda_{\frac{p}{q}}$ induces a map on $\mathcal{G}_\alpha(\frac{p}{q})$ that sends gaps to gaps.*

Proof. Since $\mathbb{L}_\alpha(\frac{p}{q})$ is completely invariant under $\Lambda_{\frac{p}{q}}$, sides and vertices gaps are mapped to sides and vertices of gaps are mapped to sides and vertices of gaps. We need to show that all sides of a given gap G are mapped to sides of a unique common gap $\Lambda_{\frac{p}{q}}(G)$. Suppose γ and γ' are sides of G but $\Lambda_{\frac{p}{q}}(\gamma)$ and $\Lambda_{\frac{p}{q}}(\gamma')$ belong to the boundaries of two different gaps. Then there is at least one leaf of $\mathbb{L}_\alpha(\frac{p}{q})$ that separates them, say $[[s, t]]$. Since the vertices are mapped by the doubling map their order is preserved. This means that there is a pre-image of s between an endpoint of γ and an endpoint of γ' . Then the pre-images of t either lie between the other endpoints of γ and γ' or are separated from G by γ or γ' . In the first case a leaf s to one of these pre-images divided

G and, thus, G cannot be a gap; in the second, the leaf intersects another boundary leaf of the gap which cannot happen. ■

Note that this extended map is not defined pointwise. As mentioned earlier, by abuse of notation, we will denote the extended lamination map by $\Lambda_{\frac{p}{q}}$ as well.

Lemma 3.2.3. *There is a one-to-one correspondence between the gaps of $\mathcal{L}_{\alpha}\left(\frac{p}{q}\right)$ and the Fatou components of the filled Julia set $K(f)$ for any $f \in H_{\frac{p}{q}}^*$.*

Proof. For simplicity, denote $H_{\frac{p}{q}}$ by H . Note that $f_{\frac{p}{q}}^{\star} = f_{c_H}$ where c_H is the center of H . The correspondence can be constructed by first defining the map $\Psi = \Psi_0 : G_0 \rightarrow F_0$ in the same way as in section 4.3, where G_0 is the critical gap of $\mathcal{L}_{\alpha}\left(\frac{p}{q}\right)$ and F_0 is the Fatou component of $\Omega(f_{c_H})$ containing the critical point. The map Ψ_0 is a semi-conjugacy between $\Lambda_{\frac{p}{q}}^{\circ q}$ and $f_{c_H}^{\circ q}$. More precisely, Ψ_0 commutes with the dynamics. Since every Fatou component F_i in $\Omega(f_{c_H})$ is eventually periodic and maps to F_0 by an appropriate iterate of f_{c_H} , and every gap G_i in $\mathcal{L}_{\alpha}\left(\frac{p}{q}\right)$ is eventually periodic and maps to G_0 by an appropriate iterate of $\Lambda_{\frac{p}{q}}$, we can use a similar construction to define the maps $\Psi_i : G_i \rightarrow F_i$, thus creating the necessary one-to-one correspondence. The generalization to all $f \in H_{\frac{p}{q}}^*$ follows from theorem 2.1.3 ■.

3.3 Invariant Quadratic Laminations

Definition. A *geodesic lamination* \mathcal{GL} , as defined by Thurston, is a collection of hyperbolic geodesics in \mathcal{D} with endpoints z and w on $\partial\mathcal{D}$, denoted by $[[z, w]]$, such that different geodesics do not intersect and their union is closed in \mathcal{D} . [Thu]

Let f be a quadratic polynomial with a locally connected Julia set. Define an equivalence relation on $\partial\mathcal{D}$, \sim_f , in the following way — for $z, w \in \partial\mathcal{D}$ say that $z \sim_f w$, if $x(R_{\text{Arg}[z]}) = x(R_{\text{Arg}[w]}) \in J(f)$, including the cases where $z = w$. Call the corresponding lamination $\mathcal{L}(f)$. Note that, since $J(f)$ is locally connected, every external ray lands and the union of the endpoints of the leaves of $\mathcal{L}(f)$, including the degenerate ones, is equal to $\partial\mathcal{D}$. It is relatively easy to check that $\mathcal{L}(f)$ satisfies the following conditions for any $z, w \in \partial\mathcal{D}$:

- ★ if $[[z, w]] \in \mathcal{L}(f)$, then $[[z^2, w^2]] \in \mathcal{L}(f)$ or $z^2 = w^2$
- ★★ if $[[z, w]] \in \mathcal{L}(f)$, then $[[-z, -w]] \in \mathcal{L}(f)$
- ★★★ if $[[z^2, w^2]] \in \mathcal{L}(f)$, then $[[z, w]] \in \mathcal{L}(f)$ or $[[z, -w]] \in \mathcal{L}(f)$

The above three conditions characterize *quadratic invariance*. Thus, given f as above, we will say that $\mathcal{L}(f)$ is an *invariant quadratic lamination defined by f* . Geometrically, it can be realized as a union of an infinite geodesic lamination and a collection of points on $\partial\mathcal{D}$ which correspond to degenerate leaves, and it is a *closed* geodesic lamination.

In [Thu] and [DH], it is shown that if $J(f)$ is locally connected for a quadratic polynomial f , and $\mathcal{L}(f)$ is the associated lamination, as defined above, then

$J(f)$ is homeomorphic to the quotient space $\mathbf{S}^1 / \sim_{\mathfrak{L}(f)}$.

Since f_q^p is geometrically finite, $J(f_q^p)$ is locally connected, and the lemma below follows immediately.

Lemma 3.3.1: *Let $\mathfrak{L}(\frac{p}{q})$ be the invariant quadratic lamination defined by f_q^p .*

Then $J(f_q^p)$ is homeomorphic to the quotient space $\mathbf{S}^1 / \sim_{\mathfrak{L}(\frac{p}{q})}$.

The following property is unique to starlike quadratics since the polygons of the associated α -lamination $\mathfrak{L}_\alpha(\frac{p}{q})$ accumulate to points on the boundary of \mathcal{D} , that is, $\mathfrak{L}_\alpha(\frac{p}{q})$ does contain any non-degenerate accumulation leaves.

Lemma 3.3.2: *Let p and q be two relatively prime positive integers with $p < q$.*

Let $f_q^p = e^{2\pi i(\frac{p}{q})} z + z^2$, and $\mathfrak{L}(\frac{p}{q})$ be the invariant quadratic lamination

defined by f_q^p . Then $\mathfrak{L}(\frac{p}{q})$ is the closure of $\mathfrak{L}_\alpha(\frac{p}{q})$ in $\overline{\mathcal{D}}$.

Chapter 4. Dynamic Root Portrait Laminations and Boundaries of Fatou Components

The proper way to generalize the idea of the α -lamination to geometrically finite quadratics, which are not necessarily starlike, is by constructing a *dynamic root lamination*. However, we will need the following ideas before we can precisely state the definition.

4.1 Orbit Portraits

In section 3.2, we defined the idea of a formal ray cycle RC of period q for quadratic polynomials. By adding information about the landing patterns of the external rays in RC , we may construct a *formal orbit portrait*. For a detailed discussion see [Mil1].

First, let us construct an orbit portrait for a given quadratic f . Let $O = \{z_0, z_1 = f(z_0), z_2 = f^{\circ 2}(z_0), z_3, \dots, z_{n-1}\}$ be a periodic orbit for f . Let $RC(f)$ be the ray cycle associated with O , that is, the collection of all external rays $\{R_{t_i}\}_{i=1}^q$ which land on the points of O . There is a natural equivalence relation on $RC(O)$, which passes to the angles of the rays, where two rays are identified if they land on the same point of the orbit; the equivalence classes of the ray angles are denoted by A_j .

Definition. The *orbit portrait* associated with O is the collection $\mathcal{P} = \mathcal{P}(O) = \{A_0, \dots, A_{n-1}\}$.

The number of elements is the same in each A_j and is called the *valence* v .

Denote the period of the angles in each A_j under the map $t \mapsto 2^n t \bmod 1$ by r so that the period of each ray in \mathcal{O} is $q = r n$.

If $\nu \geq 2$ or if the only angle in the portrait is zero, $\mathcal{P} = \{0\}$, then the orbit portrait is called *non-trivial*. For a non-trivial orbit with $\nu \geq 2$, the ν rays in each A_j divide the dynamic plane into ν sectors. They divide the full circle of angles at ∞ into ν arcs so that the sum of the arc lengths of the sectors is $+1$.

Under minimal assumptions satisfied in all cases here, the orbit portrait has the following properties:

- ★ Each A_j is mapped onto A_{j+1} under the doubling map;
- ★★ All the angles in $A_0 \cup \dots \cup A_{n-1}$ are periodic under the doubling map with common period $q = r n$;
- ★★★ For each $j \neq k$ the sets A_j and A_k are contained in disjoint sub-intervals of the circle.

Given any rational $t \in \mathbb{Q}/\mathbb{Z}$, we construct its formal orbit portrait by forming the ray cycle using the doubling map and considering the various possible partitions into subsets A_j .

Theorem 4.1.1. (Milnor). [Mil1] *Let \mathcal{O} be an orbit of period $n \geq 1$ for $f = f_c$.*

If there are $\nu \geq 2$ dynamic rays landing at each point of \mathcal{O} , then there is one and only one sector \mathcal{S}_1 based at the same point $z_1 \in \mathcal{O}$ which contains the critical value $c = f(0)$, and whose closure contains no point of \mathcal{O} other than z_1 . Among all of the $n\nu$ sectors based at all points of \mathcal{O} , \mathcal{S}_1 is the unique sector of smallest angular width.

Definition. S_1 is called the **critical value sector**.

Suppose a given f_c admits an orbit with portrait $\mathcal{P}(O)$ whose valence $v \geq 2$.

Let $R_{t_{\pm}}$ be the two dynamic rays defining the critical value sector of f_c ,

$0 < t_- < t_+ < 1$.

Theorem 4.1.2 (Milnor). [Mil1] *The two corresponding parameter rays $\mathcal{R}_{t_{\pm}}$ land at a single point $\mathbf{r}_{\mathcal{P}}$ of the parameter plane. These rays, together with $\mathbf{r}_{\mathcal{P}}$ divide the plane into two open subsets, $W_{\mathcal{P}}$ and $\mathbb{C} \setminus \overline{W_{\mathcal{P}}}$ such that: f_c has a repelling periodic orbit with portrait \mathcal{P} if and only if $c \in W_{\mathcal{P}}$, and has a parabolic orbit with portrait \mathcal{P} if and only if $c = \mathbf{r}_{\mathcal{P}}$.*

Definition. The set $W_{\mathcal{P}}$ is called the **\mathcal{P} -wake** in parameter space and $\mathbf{r}_{\mathcal{P}}$ is the root point of the wake. The set $M_{\mathcal{P}} = M \cap W_{\mathcal{P}}$ is the **\mathcal{P} -limb** of the Mandelbrot set.

Wakes of two roots $\mathbf{r}_{\mathcal{P}}$ and $\mathbf{r}_{\tilde{\mathcal{P}}}$, $\mathcal{P} \neq \tilde{\mathcal{P}}$, are either disjoint or one is contained inside the other.

Definition. We can define a partial order on the root points in \mathbf{M} as follows:

$\mathbf{r}_{\tilde{\mathcal{P}}} > \mathbf{r}_{\mathcal{P}}$ if $W_{\tilde{\mathcal{P}}} \subset W_{\mathcal{P}}$; we will say that $\mathbf{r}_{\tilde{\mathcal{P}}}$ **succeeds** $\mathbf{r}_{\mathcal{P}}$.

Definition. The open arc $\mathcal{I}_{S_1} = (t_-, t_+)$ consisting of all angles of dynamic rays R_t contained in S_1 is the **characteristic arc** $\mathcal{I}_{\mathcal{P}}$ for the orbit portrait \mathcal{P} .

The following corollary is an immediate consequence of theorem 4.1.2.

Corollary 4.1.1. *If \mathcal{P} and \mathcal{Q} are two distinct non-trivial orbit portraits and if $\mathcal{I}_{\mathcal{P}} \subset \mathcal{I}_{\mathcal{Q}}$ then $\overline{W_{\mathcal{P}}} \subset W_{\mathcal{Q}}$.*

The next theorem establishes that any non-trivial formal orbit portrait can be

realized as a root of a hyperbolic component.

Theorem 4.1.3. (Milnor) [Mil1] *There is a one to one correspondence between the set of non-trivial formal orbit portraits and the root points of the Mandelbrot set. If \mathcal{P} is the portrait, denote the corresponding root by $r_{\mathcal{P}}$.*

As part of the proof of the above theorem, it is necessary to distinguish the orbit portraits of primitive and satellite components. For primitive components, there are two distinct ray cycles that land on the parabolic orbit. The valence ν is 2 and the landing rays have the same period as the orbit so that $r = 1$. For satellite components, there are $\nu = r$ rays landing at each point in the orbit and these are permuted by the doubling map. The total number of rays is νn , where n is period of the orbit, and there is only one distinct ray cycle.

Denote the hyperbolic component with root $r_{\mathcal{P}}$ by $H_{r_{\mathcal{P}}}$.

Theorem 4.1.4. (Milnor) [Mil1] *If $c \in H_{r_{\mathcal{P}}}$ then f_c has a repelling periodic cycle with orbit portrait \mathcal{P} and the points in this cycle lie on the boundaries of the Fatou cycle components.*

Figure 4.1.1 shows an example of a Julia set and the orbit portrait of the repelling cycle on the boundary of the Fatou set. Figure 4.1.2 shows the corresponding rays in the parameter plane that define the corresponding wakes.

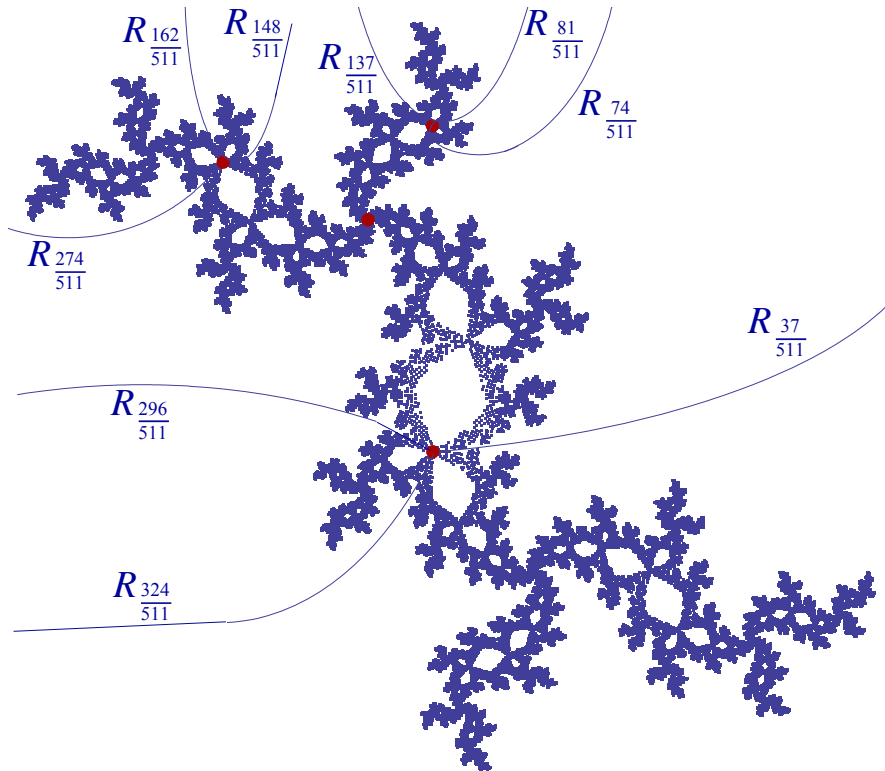


Figure 4.1.1. Dynamic root orbit portrait for f_c with $c \approx -0.03111 + 0.79111 i$

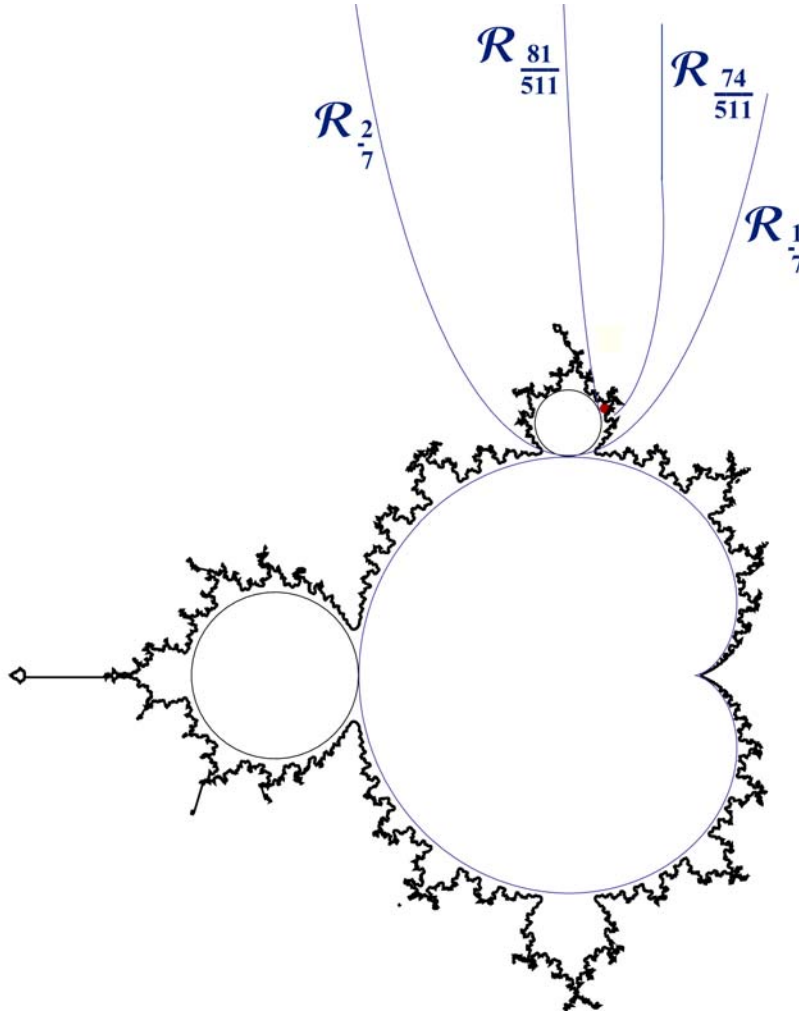


Figure 4.1.2. External parameter rays which define the wakes that contain the parameter $c \approx -0.03111 + 0.79111 i$

We can form chains of hyperbolic components using the following two deformation theorems. Suppose \mathcal{P} is an orbit portrait of period n and ray period $q = r n \geq n$.

Theorem 4.1.5. (Milnor) [Mil1] *Let $H_{\mathbf{r}_{\mathcal{P}}}$ be the hyperbolic component with root $\mathbf{r}_{\mathcal{P}}$. Then we may form a smooth path $c(t)$ in parameter space ending at $\mathbf{r}_{\mathcal{P}}$ such that $f_{c(t)}$ has a repelling orbit of period n with portrait \mathcal{P} whose periodic points lie on the boundaries of the Fatou cycle components of $f_{c(t)}$.*

Furthermore, $f_{c(t)}$ has an attracting orbit of period rn and, as $c(t) \rightarrow \mathbf{r}_\mathcal{P}$, both orbits converge to the parabolic orbit of $f_{\mathbf{r}_\mathcal{P}}$.

Theorem 4.1.6. (Milnor) [Mil1] *Under the same hypothesis as above, there also exists a smooth path $c(t)$ in parameter space ending at $\mathbf{r}_\mathcal{P}$ such that $f_{c(t)}$ has an attracting orbit of period n , and a repelling orbit of period rn whose points lie on the boundaries of the Fatou cycle components of $f_{c(t)}$. Furthermore, the dynamic rays with angles in $\bigcup_{i=0}^{n-1} A_i$, where each $A_i \in \mathcal{P}$, all land on this repelling orbit.*

It follows that given a satellite component H , we can form a chain of components $\{H_i\}$, $i = 0, 1, \dots, n$ as follows: $\mathbf{r}_{H_{i+1}}$ is on the boundary of H_i for $i < n$, $H = H_n$, and H_0 is primitive. We say that each H_i is a satellite of the primitive H_0 for all $i = 1, \dots, n$.

The periods of the attracting cycles of all components in the chain are multiples of the period of the attracting cycle of the primitive H_0 . The wakes $W_{\mathbf{r}_{H_i}} = W_{\mathbf{r}_{\mathcal{P}_i}} = W_{\mathbf{r}_i}$, $i = 0, \dots, n$, based at the roots of the components in the chain are nested; that is, $\overline{W_{\mathbf{r}_{i+1}}} \subset W_{\mathbf{r}_i}$. We call $W_{\mathbf{r}_0}$ the **pre-wake** of any of the wakes $W_{\mathbf{r}_i}$, $i = 1, \dots, n$.

It also follows from this discussion that given any H with root $\mathbf{r}_\mathcal{P}$ of period k_0 , and a set of integers ρ_1, \dots, ρ_n , we can form a chain $\{H_i\}$ such that $H_0 = H_{\mathbf{r}_\mathcal{P}}$, $\mathbf{r}_{H_i} \in \partial H_{i+1}$ and the period k_i of the attracting cycle in H_i is $\rho_i k_{i-1}$. The wakes and characteristic arcs of the portraits are nested.

4.2 Portrait Laminations

We are now ready to generalize the idea of α -laminations to non-starlike geometrically finite quadratics.

Let $\mathcal{P} = \{A_0, A_1 \dots A_{n-1}\}$ be the orbit portrait of a repelling or parabolic cycle of period n such that $r_{\mathcal{P}}$ is not primitive. For the remainder of the paper we assume, unless otherwise stated, that the portraits we consider are non-trivial and are not the $\{\{0\}\}$ portrait.

For each ray R_{t_j} with $t_j \in A_i$, $i = 0, \dots, n-1$, mark the points $e^{2\pi i t_j}$ on the unit circle. Join the points corresponding to the rays in each equivalence class A_i by hyperbolic geodesics so that they form a convex hyperbolic polygon $Q_{A_i}(\mathcal{P})$. The number of sides is the valence v . Since the period of the portrait is n we have n polygons.

Note that if there are only two rays in an equivalence class, the polygon is degenerate and is just a hyperbolic line. This happens, for example, for portraits corresponding to roots of primitive components. Although the discussion below is valid for degenerate polygons, we will assume that our polygons are not degenerate.

We treat the sides of the polygons as leaves of a lamination, and denote each by $[[t_1, t_2]]$, where $e^{2\pi i t_1}$ and $e^{2\pi i t_2}$ are the endpoints. Define their lengths $L[[t_1, t_2]]$ and the associated lamination map, now denoted by $\Lambda_{\mathcal{P}}$, as in section 3.2.

We label the polygons so that the vertices of $Q_{A_{i+1}}$ are obtained from those of

Q_{A_i} by the angle doubling map $dbl(t)$ as defined in section 3.2.

Lemma 4.2.1. *The polygons Q_{A_i} for a given \mathcal{P} are mutually disjoint.*

The proof is identical to the first part of the proof of lemma 3.2.1.

Note that under $\Lambda_{\mathcal{P}}$ the above polygons form a forward invariant cycle.

Definition. Suppose $\mathcal{P} = \{A_0, A_1 \dots A_{n-1}\}$ is a non-trivial orbit portrait of a repelling or parabolic cycle such that $r_{\mathcal{P}}$ is not a primitive root. Define the **portrait lamination** $\mathcal{L}(\mathcal{P})$ as the collection of polygons Q_{A_i} , $i = 0, \dots, n-1$, together with all their pre-images under $\Lambda_{\mathcal{P}}$.

Proposition 4.2.1. *Let $\{H_i\}_{i=0}^n$ be a chain such that H_0 is primitive and $H_n = H_{r_{\mathcal{P}}}$. Suppose $f \in H_{r_{\mathcal{P}}}$. Then f has $n+1$ repelling periodic cycles with non-trivial orbit portraits $\mathcal{P}_0, \dots, \mathcal{P}_n$ and $\mathcal{I}_{\mathcal{P}_j} \subset \mathcal{I}_{\mathcal{P}_{j-1}}$; $\mathcal{P}_n = \mathcal{P}$ may be the portrait of a repelling or parabolic cycle. There is a portrait lamination $\mathcal{L}(\mathcal{P}_j)$ for each $j = 1, \dots, n$, and the polygons in each lamination are mutually disjoint. Moreover, the same is true for the union of the polygons in all of the laminations in the chain.*

Proof. The fact that $f \in H_{r_{\mathcal{P}}}$ has $n+1$ repelling periodic cycles with non-trivial orbit portraits $\mathcal{P}_0, \dots, \mathcal{P}_n$ and $\mathcal{I}_{\mathcal{P}_j} \subset \mathcal{I}_{\mathcal{P}_{j-1}}$ is a consequence of the discussion immediately following theorems 4.1.5 and 4.1.6. It is also clear that $\mathcal{P}_n = \mathcal{P}$ is a portrait of either a repelling or a parabolic cycle. The result of lemma 4.2.1 applied to each of the laminations $\mathcal{L}(\mathcal{P}_j)$, $j = 1, \dots, n$, guarantees that the polygons in each lamination are mutually disjoint. The

final statement of the proposition follows from the fact that external parameter rays to the Mandelbrot set are disjoint. ■

The above proposition also holds if $f = f_{r_{\mathcal{P}}}$ is the root of $H_{r_{\mathcal{P}}}$ but the $n + 1$ -st cycle with portrait \mathcal{P} is parabolic instead of repelling.

There is a repelling or parabolic point corresponding to the orbit portrait \mathcal{P}_n on the boundary of the Fatou component F_0 containing the critical point. This is called the *dynamic root point* by Schleicher in [Sch1]. We will thus call \mathcal{P}_n the *dynamic root portrait* for f , and the associated lamination $\mathcal{L}(\mathcal{P}_n)$ the *dynamic root portrait lamination*, or, simply, the *dynamic root lamination*.

For the remainder of the chapter we will be focusing on this particular portrait lamination and by abuse of notation set $\mathcal{P} = \mathcal{P}_n$.

Define the gaps of $\mathcal{L}(\mathcal{P})$ as in section 3.2, and denote the collection of all gaps of $\mathcal{L}(\mathcal{P})$ by $\mathcal{G}(\mathcal{P})$.

The boundary of a gap consists of geodesic sides of the polygons in a lamination and points on $\partial\mathcal{D}$. Note that since $\mathcal{L}(\mathcal{P})$ is completely invariant under $\Lambda_{\mathcal{P}}$ so is $\mathcal{G}(\mathcal{P})$. Here we extend $\Lambda_{\mathcal{P}}$ to $\mathcal{L}(\mathcal{P}) \cup \mathcal{G}(\mathcal{P})$ in the same way as in lemma 3.2.2.

The endpoints of a given boundary side of a gap subtend two complementary arcs on $\partial\mathcal{D}$; one contains vertices of the gap and the other contains vertices of the polygon the side belongs to. We say the side *bounds* the latter arc.

We want to focus on two particular gaps. One of the gaps has a side γ_v whose endpoints $\{t_-, t_+\}$ bound the characteristic arc $\mathcal{I}_{\mathcal{P}} = (t_-, t_+)$. We call this gap

the *critical value gap* and denote it by G_v . The side γ_v belongs to one of the polygons $Q_{A_i}(\mathcal{P})$. We may label the polygons so this one is Q_{A_1} .

Proposition 4.2.2. *The two pre-images of γ_v under $\Lambda_{\mathcal{P}}$ are symmetric with respect to the origin. Moreover, no side of any polygon in $\mathcal{L}(\mathcal{P})$ separates these pre-images.*

Proof. Assume $0 < t_- < t_+ < 1$. By theorem 4.1.1, the characteristic arc bounds the smallest sector among all sectors determined by the orbit O so that

$$t_+ - t_- < \frac{1}{2}.$$

The two pre-images of this arc under $dbl(t)$ are the arcs, taken counterclockwise,

$$\left(\frac{t_-+1}{2}, \frac{t_+}{2}\right) \text{ and } \left(\frac{t_-}{2}, \frac{t_++1}{2}\right).$$

These are clearly symmetric with respect to the origin and each has length greater than $\frac{1}{2}$.

Denote the sides of the lamination polygons that bound the complements of these arcs by γ_0 and γ_0' . One of these is a side of Q_{A_0} ; call this one γ_0 . The arc bounded by each of these sides is mapped one-to-one onto the complement of the characteristic arc (t_+, t_-) .

Recall that γ_v is a side of a polygon of $\mathcal{L}(\mathcal{P})$ with endpoints $\{t_-, t_+\}$. Suppose there were a side γ of one of the polygons in $\mathcal{L}(\mathcal{P})$ with endpoints $\{s, t\}$ where

$$\frac{t_-}{2} < s < \frac{t_+}{2} \text{ and } \frac{1+t_+}{2} < t < \frac{1+t_-}{2}.$$

Under $\Lambda_{\mathcal{P}}$, the image, $\Lambda_{\mathcal{P}}(\gamma)$, would have one endpoint at $2s$ where $t_- < 2s < t_+$, and the other endpoint at $2t$ where $t_+ < 2t < t_-$. This would imply that $\Lambda_{\mathcal{P}}(\gamma)$ intersects γ_v , however, contradicting the fact the polygons in $\mathcal{L}(\mathcal{P})$ are disjoint. ■

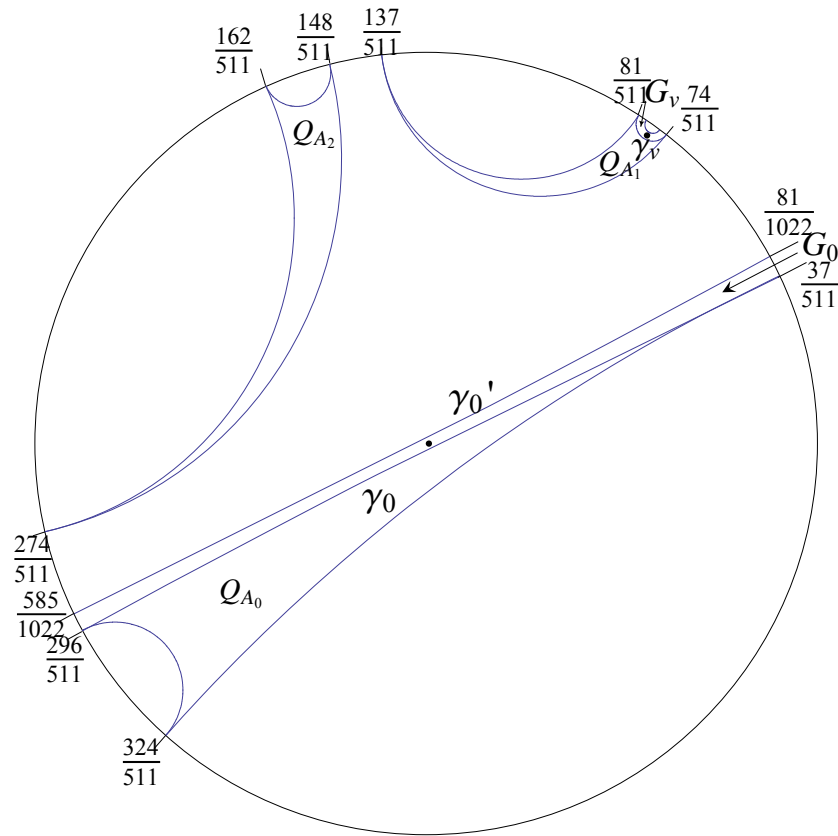


Figure 4.2.1. The construction of the dynamic root portrait lamination and the gaps G_0 and G_v for f_c where $c \approx -0.03111 + 0.79111 i$

By proposition 4.2.2, the two sides γ_0 and γ_0' bound a common gap. This gap is the pre-image of G_v under $\Lambda_{\mathcal{P}}$; we will call it the **critical gap** and denote it by G_0 . Denote the other pre-image of Q_{A_1} under $\Lambda_{\mathcal{P}}$ by Q_{A_0}' . Note that each of the arcs on $\partial\mathcal{D}$ between γ_0 and γ_0' maps one-to-one onto the characteristic arc, and that the grand orbit of G_v under $\Lambda_{\mathcal{P}}$ is $\mathcal{G}(\mathcal{P})$.

As an immediate corollary to the above proposition we obtain

Corollary 4.2.1. *The sides γ_0 and γ_0' are the longest sides among all of the polygons in $\mathcal{L}(\mathcal{P})$.*

It follows from this discussion that if $\gamma = [[s_1, s_2]]$ is a side of a polygon $Q \neq Q_{A_0} \neq Q_{A_0}'$ and, at the same time, is a boundary side of the gap G_0 , then both s_1 and s_2 must lie in one or the other of the intervals $(\frac{t_-}{2}, \frac{t_+}{2})$ and $(\frac{1+t_+}{2}, \frac{1+t_-}{2})$. Any other side of Q must have endpoints in the same interval. Moreover, the side of Q that is on the boundary of G_0 must be the longest side of Q .

Recall that the period of the orbit O of $\mathcal{P} = \mathcal{P}(O)$ is n but the period of a component in the Fatou cycle is vn where v is the valence of \mathcal{P} .

Proposition 4.2.3. *None of the sides $\{\Lambda_{\mathcal{P}}^{-j}(\gamma_0)\}$, $j = 1, \dots, vn - 1$ is a boundary side of G_0 but there is a branch $E = \Lambda_{\mathcal{P}}^{-vn}$ such that $E(\gamma_0)$ is.*

Proof. $\Lambda_{\mathcal{P}}$ maps gaps onto gaps. The vn^{th} iterate fixes the sides and vertices of Q_{A_0} . Any lower iterate either maps Q_{A_0} onto one of the other polygons in $\mathcal{L}(\mathcal{P})$ or permutes its sides.

Since there are two ways to invert the doubling map, we must be careful when defining inverse branches of $\Lambda_{\mathcal{P}}$. The side γ_0 of G_0 is a side of Q_{A_0} , and the lamination map on the polygons extends to a map on the gaps. Thus, we can choose an inverse branch of $\Lambda_{\mathcal{P}}^{vn}$ defined on Q_{A_1} that sends Q_{A_1} to Q_{A_0} and maps γ_v to γ_0 which we denote by E . The extended $\Lambda_{\mathcal{P}}$ sends G_v to G_0 . On

the other hand, we can choose a different inverse branch of $\Lambda_{\mathcal{P}}^{v^n}$ defined on Q_{A_1} that sends Q_{A_1} to Q_{A_0}' and maps γ_v to γ_0' which we denote by E' . This is true because $\Lambda_{\mathcal{P}}$ is defined by a quadratic polynomial whose critical value lies between the external dynamic rays that define γ_v . Then $E(\gamma_0')$ and $E'(\gamma_0')$ will be a pair of symmetric boundary sides of G_0 , distinct from γ_0 and γ_0' , and both endpoints of each will lie in one of the symmetric arcs of $\partial\mathcal{D}$ between γ_0 and γ_0' . ■

Since γ_0 and γ_0' are the longest boundary sides of G_0 , $E(\gamma_0')$ and $E'(\gamma_0')$ are the second longest. It follows that γ_v and $\Lambda_{\mathcal{P}}(E(\gamma_0')) = \Lambda_{\mathcal{P}}(E'(\gamma_0'))$ are the longest and second longest boundary sides of G_v .

Lemma 4.2.2. (Cantor Set Lemma) *Let G_0 be the critical gap of dynamic root portrait lamination $\mathfrak{L}(\mathcal{P})$. Then $\partial G_0 \cap \partial\mathcal{D}$ is a Cantor set.*

Proof. Denote the length of the characteristic arc $\mathcal{I}_{\mathcal{P}} = (t_-, t_+)$ by

$$l = |t_+ - t_-|.$$

Label the closed arcs on $\partial\mathcal{D}$ between γ_0 and γ_0' as I_0 and I_1 . Since $\Lambda_{\mathcal{P}}$ doubles the lengths of sides (*mod* 1), it is straightforward to compute that for $k = 0, 1$ the length of I_k is $l_k = \frac{l}{2}$. Remove the open arcs complementary to I_0 and I_1 .

Let $\tilde{\gamma} = [[s_1, s_2]]$ be a boundary side of G_0 and a side of the polygon $\tilde{Q} = E(Q_{A_0})$ where the branch E of $\Lambda_{\mathcal{P}}^{-v^n}$ is chosen to fix G_0 . Since the other sides of \tilde{Q} are not boundary sides of G_0 , their endpoints must lie in the minor

arc (s_1, s_2) . It follows that $\tilde{\gamma}$ is the longest side of \tilde{Q} . Recall that we denote the length of a leaf of $\mathcal{L}(\mathcal{P})$ by $L[[*]]$. Since $\Lambda_{\mathcal{P}}$ doubles the lengths of sides (*mod* 1), we see that

$$L[[s_1, s_2]] = L[[E(\gamma_0)]] = \frac{1}{2^{v_{n+1}}} l.$$

Both of the endpoints s_1 and s_2 must lie in exactly one of the intervals I_k , $k = 0, 1$. Let $\tilde{\gamma}' = E'(\gamma_0)$ be the pre-image of γ_0 under the other choice E' of $\Lambda_{\mathcal{P}}^{-v_n}$ fixing G_0 . By symmetry, it will have the same length as $\tilde{\gamma}$ and its endpoints will lie in the other I_k .

Now remove the open arcs spanned by $\tilde{\gamma}$ and $\tilde{\gamma}'$ leaving four closed arcs: I_{00} , I_{01} , I_{10} and I_{11} . Computing, we see that each I_{jk} has length

$$l_2 = |I_{jk}| = \frac{2^m - 1}{2^{m+1}} l_1.$$

Iterating this process we obtain the following: at stage m we have 2^m closed arcs of lengths l_m ; we remove an open arc whose length is $\frac{l_m}{2^{v_n}}$ from each of these and obtain 2^{m+1} closed arcs of length

$$l_{m+1} = \frac{1}{2} l_m \left(1 - \frac{1}{2^{v_n}}\right) = \frac{2^{v_n} - 1}{2^{v_n+1}} l_m.$$

This is a standard Cantor set construction and from the formulas above it follows that the sum of the lengths of the removed arcs tends to 1. ■

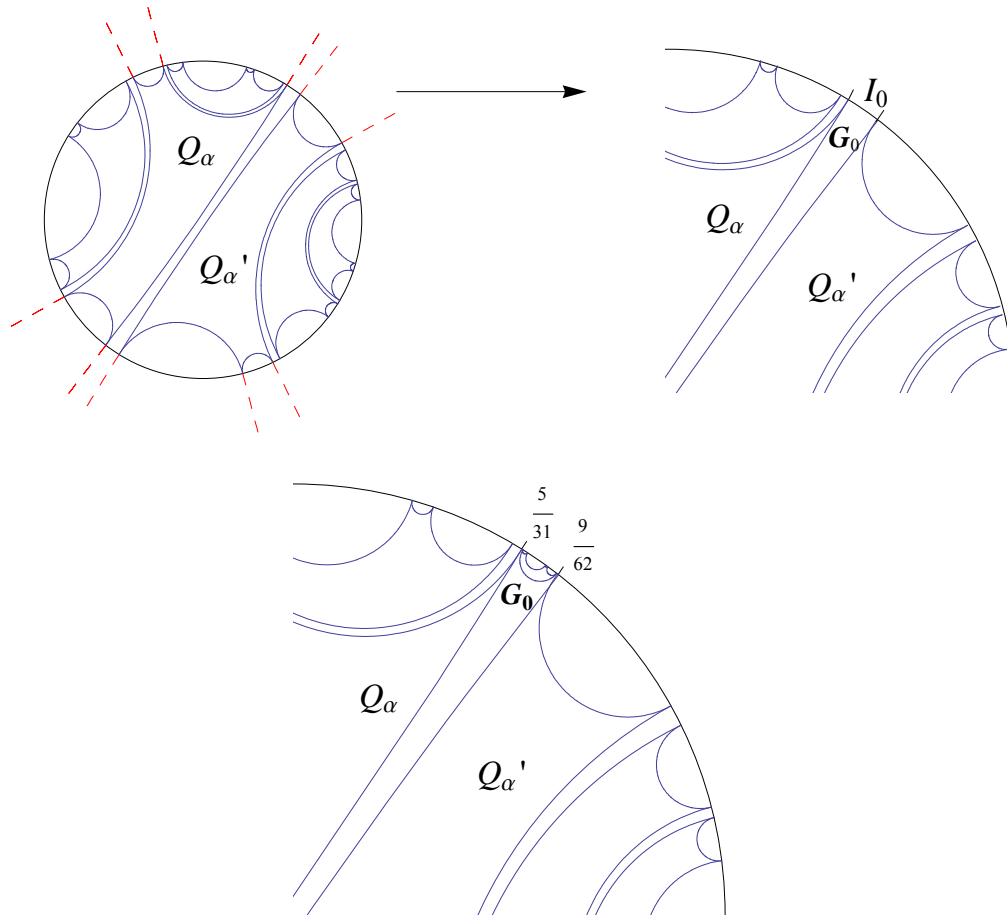


Figure 4.2.2. A zoom on one of the two remaining intervals in the first step of the Cantor set construction for the boundary of G_0 for $\mathcal{L}_\alpha\left(\frac{2}{5}\right)$

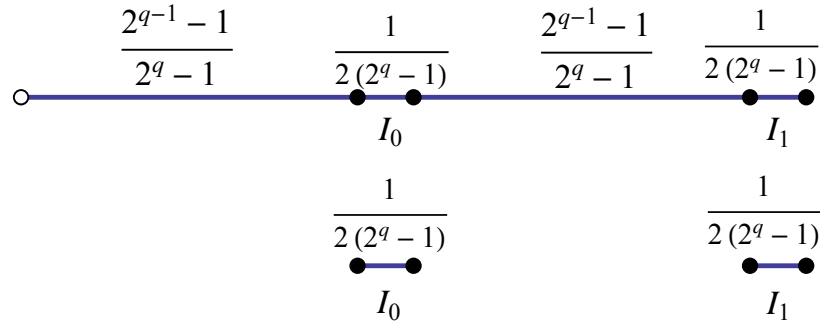


Figure 4.2.3. A different view of the first step of the Cantor set construction for the boundary of G_0 for $\mathcal{L}_\alpha\left(\frac{2}{5}\right)$ where $q = vn = v = 5$

To a given finite chain of hyperbolic components of \mathcal{M} we can associate a collection of dynamic root laminations. The gaps of these dynamic root laminations are related as follows:

Lemma 4.2.3. *Suppose that there is a finite chain $\{H_{r_{\mathcal{P}_i}}\}_{i=0}^n$ from $H_{r_{\mathcal{P}}}$ to $H_{r_{\mathcal{Q}}}$ with $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_n = \mathcal{Q}$ and $r_{\mathcal{Q}} > r_{\mathcal{P}}$. Denote the lamination polygons of $\mathcal{L}(\mathcal{P}_i)$ by $Q_{A_j}(\mathcal{P}_i)$ and the critical gaps and critical value gaps of $\mathcal{L}(\mathcal{P}_i)$ by $G_{0,i}$ and $G_{v,i}$. Then $G_{0,i+1} \subset G_{0,i}$ and $G_{v,i+1} \subset G_{v,i}$. Moreover, if ζ is a boundary point of $G_{0,n}$ on $\partial\mathcal{D}$ then it is a boundary point for all $G_{0,i}$.*

Proof. By proposition 4.2.1, the union of the polygons of all $\mathcal{L}(\mathcal{P}_i)$ are mutually disjoint. It follows that we can fit polygons from each $\mathcal{L}(\mathcal{P}_i)$, $i = 1 \dots n$, into the gap $G_{0,0}$. Because the characteristic arcs of the portraits in the chain are nested, the polygons $Q_{A_0}(\mathcal{P}_1)$ and $Q_{A_0}'(\mathcal{P}_1)$ separate $Q_{A_0}(\mathcal{P}_0)$ and $Q_{A_0}'(\mathcal{P}_0)$ from the origin and the gap $G_{0,1} \subset G_{0,0}$. Continuing in this way, we

obtain a nested sequence of critical gaps. This clearly implies that the collections of boundary points of the critical gaps are nested subsets of $\partial\mathcal{D}$, which proves the last statement of the lemma. In addition, using the appropriate lamination maps, $\Lambda_{\mathcal{P}_i}$, and applying them to the corresponding critical gaps, it follows that the critical value gaps are also nested, thus completing the proof. ■

4.3 The Relationship Between Critical Fatou Components and Critical Gaps

■ Centers of Hyperbolic Components

Fix a hyperbolic component H with $r_H = r_{\mathcal{P}}$ and assume that the period of the parabolic cycle of $f_{r_{\mathcal{P}}}$ is n . Denote the center c_H of H by $c_{\mathcal{P}}$. It follows that $f_{c_H} = f_{c_{\mathcal{P}}}$ has a superattracting cycle of period vn where v is the valence of \mathcal{P} . By theorem 4.1.4, there is a repelling cycle O of period n with portrait $\mathcal{P}(O)$, and the points in this cycle lie on the boundaries of the Fatou cycle components. Let F_0 be the Fatou component containing the critical point. Let z_0 be the point in O which lies on the boundary of F_0 . Then $f_{c_{\mathcal{P}}}^{\circ n}(z_0) = z_0 \in \partial F_0$ is a repelling fixed point of $f_{c_{\mathcal{P}}}^{\circ n}$ and is the dynamic root of $f_{c_{\mathcal{P}}}$. Let $z_k = f_{c_{\mathcal{P}}}(z_{k-1})$, $k = 1, \dots, n$ be the orbit of z_0 . Each z_k is a fixed point of $f_{c_{\mathcal{P}}}^{\circ n}$ and lies on the boundary of v Fatou components.

Since the critical point of $f_{c_{\mathcal{P}}}$ is a superattracting fixed point of $f_{c_{\mathcal{P}}}^{\circ n}$ we can define a Böttcher coordinate function in F_0 , and *internal dynamic rays* in F_0 ,

much in the same way as we defined external dynamic rays in section 1.2.

Define a Böttcher coordinate in F_0 such that the internal ray in F_0 joining the critical point to the dynamic root z_0 has argument 0. Let $z_0' \neq z_0$ be the other point on ∂F_0 such that $f_{c_p}(z_0') = z_1$; it must be the landing point of the internal ray in F_0 with argument $\frac{1}{2}$. Using branches of the inverses of $f_{c_p}^{\circ m \nu n}$ that fix F_0 , we find that for each $m = 1, 2 \dots$ there are m points $\zeta_{k,m} \in \partial F_0$ which are the landing points of internal rays of arguments $\frac{2^k}{2^m}$, $k = 0, \dots, m - 1$. The orbit portrait $\mathcal{P}(O)$ tells us that there are exactly ν external rays landing at z_0 . This clearly implies

Proposition 4.3.1. *Each landing point of an internal ray in F_0 whose argument is $\frac{2^k}{2^m}$, $k = 0, \dots, m - 1$ is a landing point of exactly ν external rays. These points on ∂F_0 are the landing points of the pullbacks of the ν external rays that land at z_0 by the branches of the inverses of $f_{c_p}^{\circ m \nu n}$ that fix F_0 for $m = 1, 2 \dots$*

■ Defining the Map Ψ

Now, let $q = \nu n$ denote the period of F_0 and let g_{\pm} be the two branches of the inverse of $f_{c_p}^{\circ q}$ that fix F_0 . Denote by E_{\pm} the two branches of the inverse of the first return of the lamination map, $\Lambda_{\mathcal{P}}^{\circ q}$, that fix G_0 . We use the fact that, as sets in the plane, both F_0 and G_0 are symmetric with respect to the origin.

Theorem 4.3.1. *There is a continuous map $\Psi : G_0 \rightarrow F_0$ with the following properties:*

★ Ψ preserves the dynamics; that is $\Psi(G_0) = f_{c_p}^{\circ q} \circ \Psi(G_0)$;

★★ Ψ has a continuous extension $\bar{\Psi}: \partial G_0 \cap \partial \mathcal{D} \rightarrow \partial F_0$.

Proof. Draw a ray in G_0 from the origin to each point in $\partial G_0 \cap \partial \mathcal{D}$. These rays remain inside G_0 because by construction, G_0 is hyperbolically convex. Set $\Psi(0) = 0$.

For each boundary side γ_i of G_0 there is a hyperbolic triangle T_i formed by γ_i and the two rays from the origin to the endpoints of γ_i . Since the endpoints of all boundary sides of G_0 are disjoint, the triangles T_i are also disjoint. Define $\Lambda^q(T_i)$, for each i , as the triangle formed by $\Lambda^q(\gamma_i)$ and the two rays from the origin to the endpoints of this image side. The maps E_{\pm} are defined on these triangles in the same way.

If T_0 is the triangle with side $\gamma_0 \in \partial G_0$, parametrize the ray sides of T_0 in \mathcal{D} by the hyperbolic length s ; denote the parametrized ray sides by $t_0(s)$ and $t_0'(s)$. Join the points on these two sides which are at an equal distance from the origin by a geodesic $l_s(t)$, $0 \leq t \leq 1$, where $l_s(0) = t_0(s)$ and $l_s(1) = t_0'(s)$.

The component F_0 admits a hyperbolic metric. Parametrize the internal ray r_0 in F_0 of argument 0 by its hyperbolic length, that is, $r_0 = r_0(s)$.

Define the $\Psi|_{T_0}$ as follows. For all $t \in [0, 1]$ and all $s \in [0, \infty]$ set

$$\Psi(l_s(t)) = r_0(s).$$

Note that the side γ_0 of T_0 and the endpoints of the two rays sides map to a single point on ∂F_0 , specifically, the dynamic root point.

Now use maps E_{\pm} and g_{\pm} to define the map from the remaining triangles to the

internal rays $r_{\frac{2^k}{2^m}}(s)$, $k = 0, \dots, m - 1$. We have

$$E_+(T_0) = T_0, g_+(r_0) = r_0.$$

Set $T_{01} = E_-(T_0)$. If $\mathbf{x}_m = x_0 x_1 \dots x_m$ where $x_i \in \{0, 1\}$ we define inductively,

$$T_{0\mathbf{x}_m} = E_-(T_{\mathbf{x}_m}) \text{ and } T_{1\mathbf{x}_m} = E_+(T_{\mathbf{x}_m}).$$

We can use the same scheme to code the internal rays in F_0 :

$r_{01} = g_-(r_0)$ is the internal ray whose argument is $\frac{1}{2}$

$$r_{10} = g_+(r_0) = r_0$$

$$r_{0\mathbf{x}_m} = g_-(r_{\mathbf{x}_m}) \text{ and } r_{1\mathbf{x}_m} = g_+(r_{\mathbf{x}_m}).$$

It follows from lemma 4.2.2 that the endpoints of the ray sides of the triangles form a dense subset in $\partial G_0 \cap \partial \mathcal{D}$. Therefore, given a point $t \in \partial G_0 \cap \partial \mathcal{D}$ that is not a vertex of a triangle, we can find a sequence $\{T_j\}$ of triangles whose ray sides converge to the ray with endpoint t . The map Ψ , therefore, extends continuously to all of G_0 and to its closure in $\overline{\mathcal{D}}$. Since the images of the triangles are precisely the collection of all internal rays in F_0 whose arguments are of the form $\frac{a}{2^m}$ for some $m \in \mathbb{N}$, and this set is dense in the set of all internal rays in F_0 , the image of the extended map consists of $F_0 \cup \partial F_0$. ■

4.4 Rational Rays Landing on Boundaries of Bounded Fatou Components

We can use theorem 4.3.1 to characterize the external rays that land on ∂F_0 , and, in turn, all those rays that land on ∂F for any bounded Fatou component $F \subset \Omega(f)$ for any geometrically finite quadratic f for which $\text{int}(K(f))$ is not empty.

Theorem 4.4.1. *The external dynamic ray R_t lands on ∂F_0 if and only if $t \in \partial G_0 \cap \partial \mathcal{D}$.*

Proof. The map Ψ assigns the argument of one or two points in $\partial G_0 \cap \partial \mathcal{D}$ to each point of ∂F_0 , depending on whether or not they are endpoints of a boundary side of G_0 . By the construction of the lamination $\mathcal{L}(\mathcal{P})$, if t is not in $\partial G_0 \cap \partial \mathcal{D}$, a ray in \mathcal{D} with this argument t must pass through a side $[[t_1, t_2]]$ of some polygon of $\mathcal{L}(\mathcal{P})$. This means that the external ray R_t is blocked from landing on ∂F_0 because it lies between the two rays, R_{t_1} and R_{t_2} , which land at the same point $x(R_{t_1}) = x(R_{t_2}) \in \partial F_0$.

Let $T_i \subset \overline{G_0}$ be a triangle formed in the same way as in the proof of theorem 4.3.1. If $t \in \partial G_0 \cap \partial \mathcal{D}$, then either t is a boundary vertex of some T_i or a limit of such points, that is, $t = \lim_{n \rightarrow \infty} t_n$, where $\{t_n\}$ are boundary vertices of triangles. Because we have a Böttcher coordinate in F_0 , the boundary of F_0 is locally connected and every internal ray lands at some point on ∂F_0 . The points $\{\Psi(t_n)\}$ lie on ∂F_0 and are endpoints of the external rays $\{R_{t_n}\}$. Hence, each of the rays R_{t_n} does land on ∂F_0 ; it lands at a pre-image of the dynamic root z_0 . Since Ψ is continuous on $\partial G_0 \cap \partial \mathcal{D}$, $\Psi(t_n) \rightarrow \Psi(t)$ and the Hausdorff limit of the external rays R_{t_n} is the ray R_t which lands precisely at $\Psi(t) \in \partial F_0$. ■

As corollaries we have

Corollary 4.4.1. *The rational external ray R_t lands on ∂F for any bounded Fatou component $F \subset \Omega(f_{c_p})$ if and only if $2^n t \bmod 1 \in \partial G_0 \cap \partial \mathcal{D}$ for some n .*

Proof. This follows directly from the fact the map Ψ preserves the dynamics. ■

Corollary 4.4.2. *Suppose that there is a finite chain $\{H_{r_{\mathcal{P}_i}}\}_{i=0}^n$ from $H_{r_{\mathcal{P}}}$ to $H_{r_{\mathcal{Q}}}$ with $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_n = \mathcal{Q}$ and $r_{\mathcal{Q}} > r_{\mathcal{P}}$. Suppose further that $c_{\mathcal{P}}$ is the center of $H_{r_{\mathcal{P}}}$ and $c_{\mathcal{Q}}$ is the center of $H_{r_{\mathcal{Q}}}$. If the rational ray R_t lands on ∂F for some bounded $F \subset \Omega(f_{c_{\mathcal{Q}}})$ then it also lands on ∂F for some bounded $F \subset \Omega(f_{c_{\mathcal{P}_i}})$ for all $i = 0, \dots, n$.*

Proof. Assume, by taking iterates if necessary, that R_t lands on ∂F_0 for $F_0 \subset \Omega(f_{c_{\mathcal{Q}}})$. Denote $x(R_t)$ by ζ_t . Consider the gap G_0 in the associated dynamic root lamination $\mathcal{L}(\mathcal{Q})$; by theorem 4.4.1, $t \in \partial G_0$. It is either an endpoint of a boundary side of G_0 , which implies that ζ_t is the landing point of more than one ray, or not. By lemma 4.2.3, t must be on the boundary of each $G_{0,i}$ corresponding to each $\mathcal{L}(\mathcal{P}_i)$ in the chain and, hence, $\zeta_t \in \partial F_{0,i}$, where $F_{0,i}$ is the critical Fatou component for $f_{c_{\mathcal{P}_i}}$ for all $i = 0, \dots, n$. Applying corollary 4.4.1 we obtain the final statement of the above corollary. ■

We are now ready to characterize all rational rays that land on boundaries of bounded Fatou components.

Theorem 4.4.2. *Assume $\zeta \in \partial F$ for some bounded $F \subset \Omega(f_{c_{\mathcal{P}}})$ where $c_{\mathcal{P}}$ is the center of $H_{r_{\mathcal{P}}}$. Let z_0 be the dynamic root for $f_{c_{\mathcal{P}}}$, v be the valence of $\mathcal{P}(\mathcal{O})$ and n be the period of \mathcal{O} . Then, either*

- (1) ζ is the landing point of exactly $v > 1$ rays and $\zeta \in GO_{f_{c_{\mathcal{P}}}}(z_0)$, or
- (2) there is exactly one ray landing at ζ .

In the latter case, if ζ is the landing point of the single ray R_t and t is rational, then the argument of R_t , or the argument of some iterate of R_t , belongs to some orbit portrait $Q \neq \mathcal{P}$ whose root $r_Q \in W_{\mathcal{P}}$ and can be reached by a finite chain from $H_{r_{\mathcal{P}}}$ to H_{r_Q} . That is, r_Q is the root of a satellite of the same primitive as $H_{r_{\mathcal{P}}}$. The period of Q is a multiple of vn .

Proof. We may assume, by taking iterates if necessary, that R_t lands at a point $\zeta_t \in \partial F_0$ for $f_{c_{\mathcal{P}}}$. Consider the gap G_0 of the associated lamination $\mathcal{L}(\mathcal{P})$. By theorem 4.4.1, $t \in \partial G_0$. It is either a vertex of a lamination polygon of $\mathcal{L}(\mathcal{P})$, in which case it belongs to the grand orbit of the ray cycle of the portrait \mathcal{P} , or not, in which case only one ray lands at ζ_t .

Assume now that only one ray lands at ζ_t . The same is therefore also true for $\zeta_{\hat{t}} = f_{c_{\mathcal{P}}}(\zeta_t) = x(R_{2t}) = x(R_{\hat{t}}) \in \partial F_1$ where \hat{t} lies in the characteristic arc $\mathcal{I}_{\mathcal{P}}$. It follows that $\hat{t} \in \partial G_v$. Denote G_v of $\mathcal{L}(\mathcal{P})$ by $G_{v,0}$, and each G_v for each $\mathcal{L}(\mathcal{P}_i)$ in the chain by $G_{v,i}$. We can find a root $r_{\mathcal{P}_1}$ on the boundary of $H_{r_{\mathcal{P}}}$ such that $\hat{t} \in \mathcal{I}_{\mathcal{P}_1}$. Draw the polygons of $\mathcal{L}(\mathcal{P}_1)$ that lie inside $G_{v,1}$; by proposition 4.2.1, these polygons are disjoint from those bounding $G_{v,0}$. If \hat{t} is a vertex of one of these polygons, it belongs to the ray cycle of the portrait \mathcal{P}_1 , and so, by theorem 4.1.3, the angles in \mathcal{P}_1 form a trivial portrait for $f_{c_{\mathcal{P}}}$. If it is not such a vertex, it belongs to the boundary of $G_{v,1}$ of $\mathcal{L}(\mathcal{P}_1)$ and we repeat this process. We find a root $r_{\mathcal{P}_2}$ such that $\hat{t} \in \mathcal{I}_{\mathcal{P}_2}$ and draw the polygons of $\mathcal{L}(\mathcal{P}_2)$ which lie

inside $G_{v,1}$. We claim that after repeating this process m times, for some finite m , \hat{t} must be a vertex of a polygon of $\mathcal{L}(\mathcal{P}_m)$.

To see this, note first that \hat{t} must be (eventually) periodic under doubling since it is rational. Let its period be k . If the above procedure does not stop, since the periods of the portraits in the chain are increasing multiples of n , the period of \mathcal{P}_m , for some m , will be greater than k . This would mean, however, that there is a point on $\partial G_{v,m}$ of the lamination $\mathcal{L}(\mathcal{P}_m)$ whose period is less than the period of $G_{v,m}$ under the action of $\Lambda_{\mathcal{P}_m}$. This would, in turn, imply that there is a point on $\partial G_{0,m}$ of $\mathcal{L}(\mathcal{P}_m)$ whose period is less than the period of $G_{0,m}$ under the action of $\Lambda_{\mathcal{P}_m}$. Applying theorem 4.3.1, this means that there is a point on ∂F_0 for $f_{c_{\mathcal{P}_m}}$ with lower period than the period of F_0 , which is clearly a contradiction.

The arguments of the rational external rays that land at points on ∂F are thus the grand orbits of the non-trivial portraits obtained from roots reached by finite chains in parameter space. ■

Remark 4.4.1. *Note that the above theorem implies that for any primitive component $H_{r_Q} \subset W_{\mathcal{P}}$, the arguments of the two parameter rays $\mathcal{R}_{t_{\pm}}$ landing at r_Q must be different from those of any dynamic rays that land on ∂F for any bounded $F \subset \Omega(f_{c_{\mathcal{P}}})$.*

Definition. Denote by $\mathbf{Land}(f)$ the set of rational arguments of external dynamic rays that land on the boundaries of the bounded Fatou components of

$\Omega(f)$.

Suppose now that c is either the root $r_\mathcal{P}$ or an arbitrary parameter in $H_{r_\mathcal{P}}$ with center $c_\mathcal{P}$. McMullen's theorem on holomorphic motions of Julia sets, found in [McM], establishes the existence of a conjugacy Φ_c , for any $c \in H_{r_\mathcal{P}}$, between the Julia sets J_c and $J_{c_\mathcal{P}}$ that preserves the dynamics. That is, $\Phi_c : J_{c_\mathcal{P}} \rightarrow J_c$ and $f_c \Phi_c = \Phi_c f_{c_\mathcal{P}}$ (see section 2.1). At the root point $r_\mathcal{P}$, the dynamics of the orbit of the dynamic root point are preserved by theorems 4.1.5 and 4.1.6.

This immediately gives us

Theorem 4.4.3. *The set of rational external rays that land on boundaries of bounded Fatou components of $\Omega(f)$ for geometrically finite quadratics f for which $\text{int}(K(f))$ is not empty depends only on the corresponding dynamic root portrait \mathcal{P} . In other words, let $H_{r_\mathcal{P}}$ be any hyperbolic component and let $c \in H_{r_\mathcal{P}} \cup \{r_\mathcal{P}\}$. Then*

$$\mathbf{Land}(f_c) = \mathbf{Land}(f_{c_\mathcal{P}}) = \mathbf{Land}(f_{r_\mathcal{P}}) = \mathbf{Land}(\mathcal{P})$$

Combining all of the results of section 4.4 we obtain the following theorem for non-Misiurewicz geometrically finite quadratics f_c .

Theorem 4.4.4. *If $c \in H_{r_\mathcal{P}} \cup \{r_\mathcal{P}\}$ and $f_{r_\mathcal{P}}$ has a parabolic orbit of period n , then the rational rays landing on the boundary points of the bounded Fatou components of f_c are precisely all the rays in the union of the sets $\{\bigcup_m f_{r_\mathcal{P}}^{-m}(R_{t_\pm}(r_\mathcal{P}))\}$ and $\{\bigcup_m f_{r_Q}^{-m}(R_{t_\pm}(r_Q))\}$ for all $r_Q > r_\mathcal{P}$ such that there is a finite chain from $H_{r_\mathcal{P}}$ to H_{r_Q} .*

By adjusting the notation and applying theorem 4.3.1 we obtain the statement of *Main Theorem A*.

Chapter 5. Degree Two Rational Maps

5.1 The Moduli Space \mathcal{M}_2 and Its Hyperbolic Components

Definition. Rat_2 is the space consisting of all degree two rational maps $g : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$.

Definition. Let $g_1, g_2 \in \text{Rat}_2$. We say that g_1 and g_2 are *conformally conjugate* if there exists a Möbius transformation m such that $g_2 = m \circ g_1 \circ m^{-1}$.

Definition. Denote by \mathcal{M}_2 , the *moduli space* of holomorphic conjugacy classes, $[g]$, of degree two rational maps g .

Theorem 5.1.1. [Mil2] *Let $g \in \text{Rat}_2$. Let $z_1, z_2, z_3 \in \bar{\mathbb{C}}$ be the fixed points of g , not necessarily distinct, and let μ_1, μ_2, μ_3 be the corresponding multipliers.*

Let

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \sigma_2 = \mu_1 \mu_2 + \mu_2 \mu_3 + \mu_1 \mu_3, \text{ and } \sigma_3 = \mu_1 \mu_2 \mu_3$$

be the elementary symmetric functions of the multipliers. Then $[g]$ is uniquely determined by the three multipliers which are subject only to the condition that

$$\mu_1 \mu_2 \mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0, \quad (\star)$$

or, equivalently,

$$\sigma_3 = \sigma_1 - 2. \quad (\star\star)$$

Hence $\mathcal{M}_2 \cong \mathbb{C}^2$, with coordinates σ_1 and σ_2 .

Proof of (\star) . Assume that the fixed points are distinct. Then by lemma 1.1.6

$\mu_i \neq 1$ for $i = 1, 2, 3$. By theorem 1.1.1, $\frac{1}{1-\mu_1} + \frac{1}{1-\mu_2} + \frac{1}{1-\mu_3} = 1$. Combining

the fractions, clearing the denominators, and rearranging terms, we obtain

$$\mu_1 \mu_2 \mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0$$

as required. If there is a double fixed point, say $z_1 = z_2$, then by lemma 1.1.6 $\mu_1 = \mu_2 = 1$ and (\star) reduces to $\mu_3 - 2 - \mu_3 + 2 = 0$ which is certainly true for any value of μ_3 . $(\star\star)$ follows immediately.

Note that, if $\mu_1 \mu_2 \neq 1$, it follows from (\star) that

$$\mu_3 = \frac{2 - \mu_1 - \mu_2}{1 - \mu_1 \mu_2} \quad (\star\star\star)$$

On the other hand, if $\mu_1 \mu_2 = 1$, it follows from (\star) that $\mu_1 + \mu_2 = 2$, which in turn implies that

$$\begin{aligned} \mu_1 + \frac{1}{\mu_1} &= 2 \\ (\mu_1 - 1)^2 &= 0 \\ \mu_1 &= 1 \end{aligned}$$

Thus, $\mu_2 = 1$. Then, by lemma 1.1.6, $z_1 = z_2$ is a double fixed point, and μ_3 can be arbitrary.

Proof that $[g]$ is uniquely determined by $\{\mu_1, \mu_2, \mu_3\}$:

(Case 1) Assume g has at least two distinct fixed points. Conjugating by an appropriate Möbius transformation, we can place these fixed points at 0 and ∞ .

Then g would have be of the form

$$g(z) = z \left(\frac{a z + b}{c z + d} \right), \text{ with } a \neq 0, d \neq 0 \text{ and } a d - b c \neq 0.$$

Dividing both numerator and denominator by d , we get

$$g(z) = z \left(\frac{\frac{a}{d} z + \frac{b}{d}}{\frac{c}{d} z + 1} \right)$$

Now, let $\hat{g}(z) = \frac{a}{d} g\left(\frac{d}{a} z\right) = z \left(\frac{z + \frac{b}{d}}{\frac{c}{a} z + 1} \right)$. Clearly, $\hat{g} \in [g]$. Furthermore, the

multiplier at 0 is $\mu_1 = \frac{b}{d}$ and the multiplier at ∞ is $\mu_2 = \frac{c}{a}$, and since $ad - bc \neq 0$, or, equivalently, $1 - \left(\frac{b}{d}\right)\left(\frac{c}{a}\right) \neq 0$, it follows that $1 - \mu_1 \mu_2 \neq 0$.

Thus g is uniquely determined, up to holomorphic conjugacy, by the multipliers μ_1 and μ_2 , and μ_3 is given by (★★★).

(Case 2) Assume g has a single triple fixed point, that is, $z_1 = z_2 = z_3$. Then by lemma 1.1.6 $\mu_1 = \mu_2 = \mu_3 = 1$. Conjugating by an appropriate Möbius transformation, we can place the triple fixed point at ∞ , obtaining the normal form

$$\tilde{g}(z) = z + \frac{1}{z}$$

Thus $\mu_i = 1$, for $i = 1, 2, 3$, if and only if $g \in [\tilde{g}]$.

Finally, since σ_1 and σ_2 are functions of the unordered triples of multipliers $\{\mu_1, \mu_2, \mu_3\}$, and σ_3 is given by (★★), the proof is complete. ■

Recall that a rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. In addition, such maps form an open subset of the moduli space, and the connected components of this open set are called hyperbolic components. Mary Rees showed that the hyperbolic components of \mathcal{M}_2 can be separated into four types.

Definition. Using Milnor's convention, the four types are:

Type B: Bitransitive. Each critical point belongs to the immediate basin of a distinct attracting periodic point where both of these periodic points, however,

belong to the same orbit.

Type C: Capture. Only one critical point belongs to the immediate basin of an attracting periodic point while the orbit of the other critical point eventually lands in this immediate basin.

Type D: Disjoint Attractors. The two critical points belong to attracting basins for two disjoint attracting periodic orbits.

Type E: Escape. Both critical orbits converge to the same attracting fixed point. After applying the necessary conjugation, we may assume that this fixed point is at ∞ .

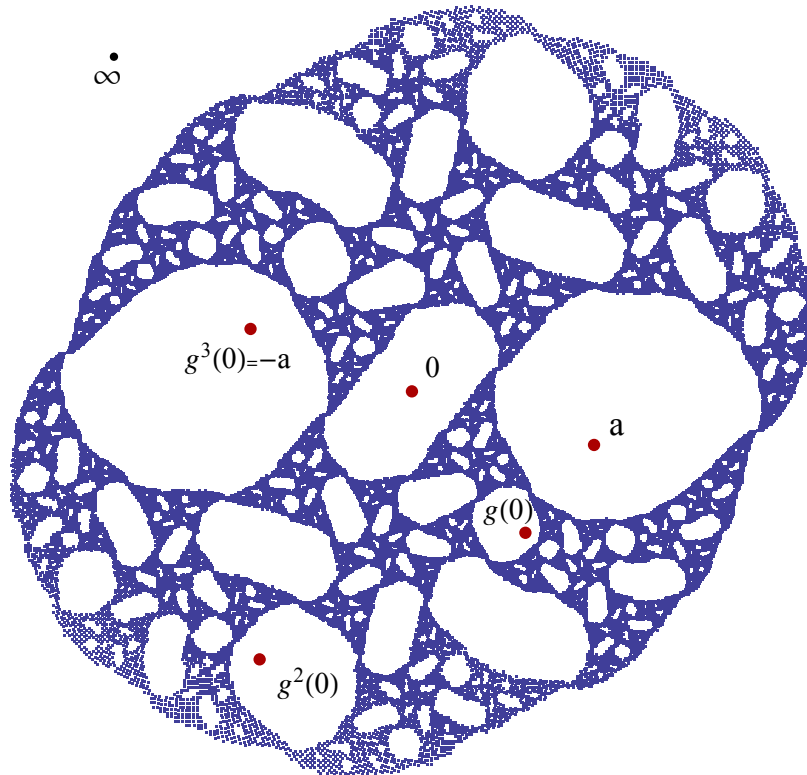


Figure 5.1.1. The Julia set of a **Type C** map $g(z) = a + \frac{1}{z^2 - a^2}$ with $a = \frac{3}{4} - \frac{11}{50}i$.

Here the periodic critical orbit $a \leftrightarrow \infty$ captures the other critical orbit

$$0 \rightarrow g(0) \rightarrow g^2(0) \rightarrow g^3(0) (= -a) \rightarrow \infty$$

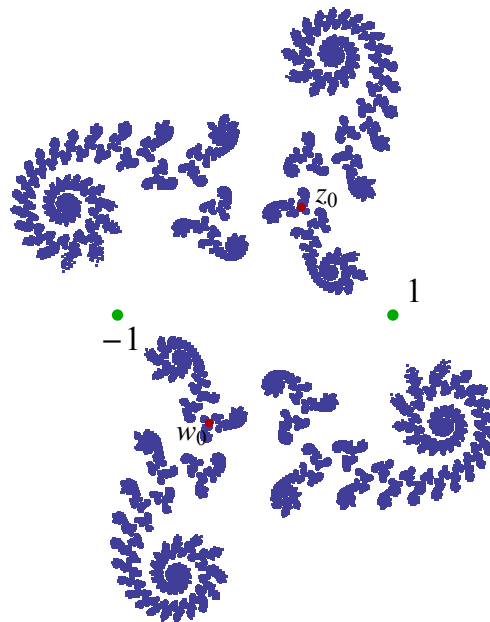


Figure 5.1.2. The Julia set of a **Type E** map $g(z) = \left(\frac{1}{20} + i\right)\left(\frac{z^2+1}{z}\right)$ showing the two repelling fixed points. Here both critical points, 1 and -1 , escape to ∞ , and the Julia set is a Cantor set.

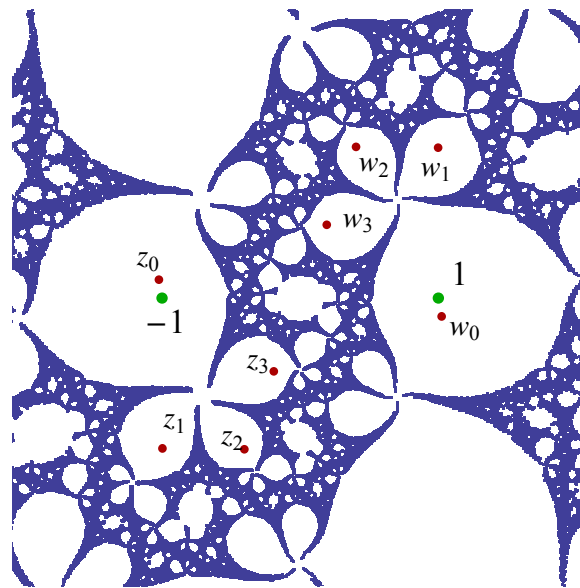


Figure 5.1.3. The Julia set of a **Type D** map $g(z) = \frac{az^2+a}{z}$ with $a = \frac{1}{2} + \frac{11}{20}i$. Here the two distinct critical points are attracted to distinct attracting cycles of period 4, that is, $-1 \rightarrow \dots \rightarrow \{z_0, z_1, z_2, z_3\}$ and $1 \rightarrow \dots \rightarrow \{w_0, w_1, w_2, w_3\}$. This map is an example of a self-mating.

We will be mostly interested in the **Type D** components and their boundaries.

5.2 The Symmetry Locus

Definition. Following Milnor, we will say that a degree two rational map g possesses a non-trivial automorphism m , if m is a non-trivial Möbius transformation which commutes with g , so that $m \circ g \circ m^{-1} = g$. The details of the proofs of the following two results can be found in [Mil2].

Lemma 5.2.1. Any degree two rational map \hat{g} with two distinct critical points, c_1 and c_2 , can be normalized so that $c_i = \pm 1$ and $g(\infty) = \infty$. Moreover, the normalized map must be of the form $g(z) = a\left(z + \frac{1}{z}\right) + b$.

Theorem 5.2.2. [Mil2] A degree two rational map possesses a non-trivial automorphism if and only if it is conjugate to a map of the form

$$g = k\left(z + \frac{1}{z}\right), \quad \text{with } k \in \mathbb{C} \setminus \{0\}$$

Furthermore, any g of the above form commutes with $m(z) = -z$.

Definition. Let $\mathcal{S} \subset \mathcal{M}_2$ be the *symmetry locus*, consisting of all conjugacy classes $[g]$ of degree two rational maps which possess a non-trivial automorphism.

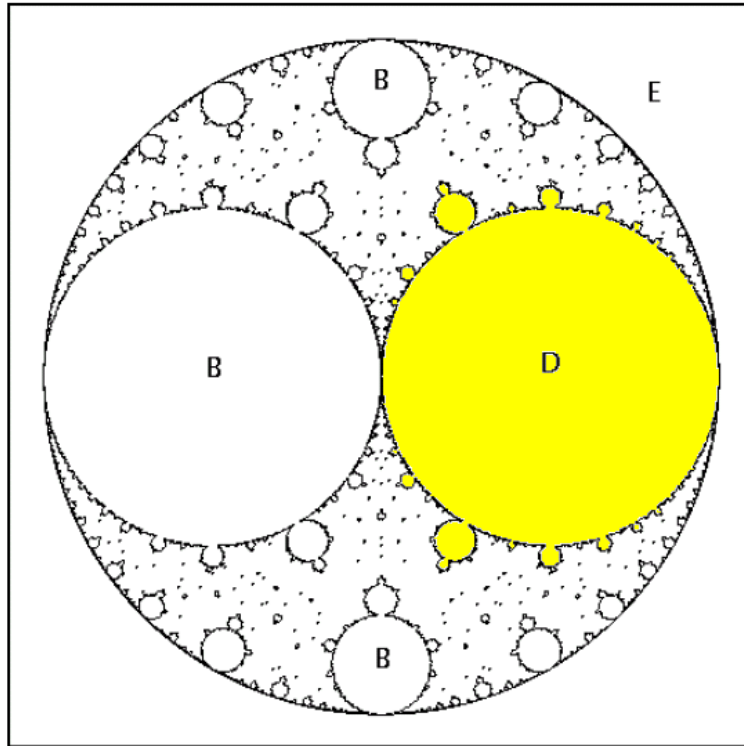


Figure 5.2.1. The symmetry locus \mathcal{S} represented as the k -plane for the family $g_k(z) = k(z + \frac{1}{z})$ with **Type D** components shaded

It is worthwhile to note that for any parameter $k \in \mathcal{S}$, the Julia set $J(g_k)$ is homeomorphic to $J(g_{-k})$, even though the dynamics of g_k may not be equal to that of g_{-k} . In this paper, I address the structure of the Julia sets of self-matings, which lie in **Type D** components of the symmetry locus. However, by the preceding observation, the results can be extended to certain **Type B** components.

Chapter 6. General Matings and Quasi-Self-Matings

6.1 General Matings

Definition. Let $X = (K(f_1) \sqcup K(f_2)) / (\gamma_1(t) \sim \gamma_2(-t))$ be the topological space obtained by gluing the two filled Julia sets along their Caratheodory loops $(\gamma_j(t) := \phi_j^{-1}(e^{2\pi i t}) : \mathbb{R} / \mathbb{Z} \rightarrow J(f_j))$ in reverse order. If X is homeomorphic to \mathbb{S}^2 , then the pair of polynomials (f_1, f_2) is called **topologically matable**. The induced map from \mathbb{S}^2 to itself, $f_1 \overset{top}{\sqcup} f_2 = (f_1|_{K_1} \sqcup f_2|_{K_2}) / (\gamma_1(t) \sim \gamma_2(-t))$ is called the **topological mating** of f_1 and f_2 .

Definition. A degree two rational map $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is called a **conformal mating** of f_1 and f_2 , denoted by $g = f_1 \amalg f_2$, if it is conjugate to the topological mating $f_1 \overset{top}{\sqcup} f_2$ by a homeomorphism h , such that, if the interiors of $K(f_j)_{j=1,2}$ are non-empty then $h|_{\text{int}(K(f_j))}$ is conformal.

Definition. A degree two conformal mating $g = f_1 \amalg f_2$ is a **trivial mating** if either f_1 or f_2 has an attracting fixed point, or, equivalently, if either map lies in the main cardioid of the Mandelbrot set.

Below is an alternative, more illuminating, definition of *non-trivial* matings. Note that our definition is given in terms of quadratics, but it can certainly be generalized to higher degree polynomials.

Recall that, for any quadratic polynomial f , $K(f)$ is symmetric and is completely contained in a disk of radius 2 centered at the critical point. For

certain quadratics a smaller radius would be sufficient. Note that we are using a circle as simply a demonstrative tool, and, for quadratics, a radius of 2 will always work, however, for computing purposes it helps to minimize the radius. Given two maps $f_1 = z^2 + \lambda_1 z$ and $f_2 = z^2 + \lambda_2 z$, neither of which has an attracting fixed point, let f_1 be the one whose filled Julia set has the larger diameter. To it we can associate a Möbius inversion of the form

$$m_{\lambda_1}(z) = \frac{-\lambda_1 z + 2b}{2z + \lambda_1}$$

through the circle C_ρ of radius $\rho = \left| \frac{\sqrt{4b + \lambda_1^2}}{2} \right|$ centered at the critical point of f_1 , $-\frac{\lambda_1}{2}$, where b is chosen such that C_ρ bounds $K(f_1)$. The map m_{λ_1} interchanges the two critical points of f_1 , $-\frac{\lambda_1}{2}$ and ∞ . In addition, it fixes C_ρ , and maps $K(f_1)$ outside C_ρ .

Since $K(f_1)$ is larger, it is guaranteed that $K(f_2)$ lies entirely inside C_ρ . Let D_ρ be the closed disk bounded by C_ρ containing the origin. Define the map $\tilde{f}_1 = m_{\lambda_1} \circ f_1 \circ m_{\lambda_1}$ which is clearly conformally conjugate to f_1 and whose Julia set $J(\tilde{f}_1)$ is conformal to $J(f_1)$. Note that $K(\tilde{f}_1) \amalg K(f_2)$ is a disjoint union of subsets of \mathbb{S}^2 . Now, let $\overline{R}_t = R_t \cap D_\rho$ be the closed truncated external ray to $K(f_2) \subset D_\rho$, and let $\overline{\tilde{R}}_t = m_{\lambda_1}(R_t) \cap \overline{\mathbb{S}^2 \setminus D_\rho}$ be the closed truncated external ray to $K(\tilde{f}_1)$. Define the **ray equivalence relation** to be the smallest equivalence relation, denoted by \sim_{ray} , on \mathbb{S}^2 such that for any t , \overline{R}_t and $\overline{\tilde{R}}_{-t}$ lie in a single

class, and such that given any s , if \overline{R}_t and \overline{R}_s (or \widetilde{R}_t and \widetilde{R}_s) have a common endpoint then \overline{R}_s (or \widetilde{R}_s) lies in this same equivalence class. Then glue the two filled Julia sets together using this equivalence relation on their complement by collapsing each equivalence class to a single point. If the resulting space $\mathbb{S}^2 / \sim_{ray}$ is homeomorphic to \mathbb{S}^2 , and if there exists a rational map g such that $g|_{int(K(\tilde{f}_1))} = \tilde{f}_1$ and $g|_{int(K(f_2))} = f_2$, then we say that $g = f_1 \amalg f_2$. Note that using this definition the point of intersection of \overline{R}_t and C_ρ is identified with the point of intersection of \widetilde{R}_{-t} and C_ρ .

For example, consider a self-mating of $f_{\frac{2}{5}}(z) = z^2 + e^{2\pi i(\frac{2}{5})}z$, denoted by $\Pi f_{\frac{2}{5}}$.

The construction described above is pictured below (along with the corresponding distinguished ray cycles).

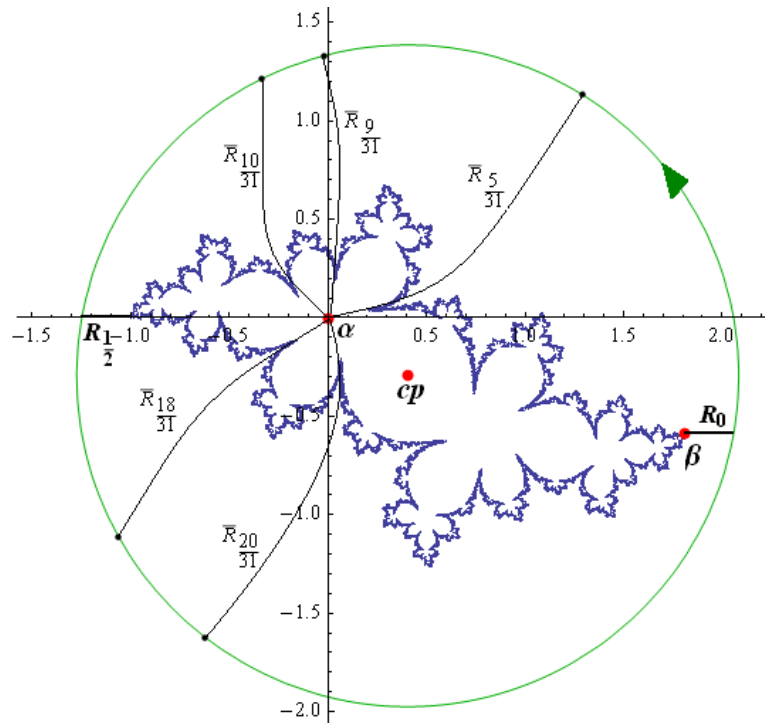


Figure 6.1.1. The Julia set of $f_{\frac{2}{5}}(z) = z^2 + e^{2\pi i(\frac{2}{5})}z$ together with the circle of inversion C_ρ , and the truncated external rays that land at the α -fixed point

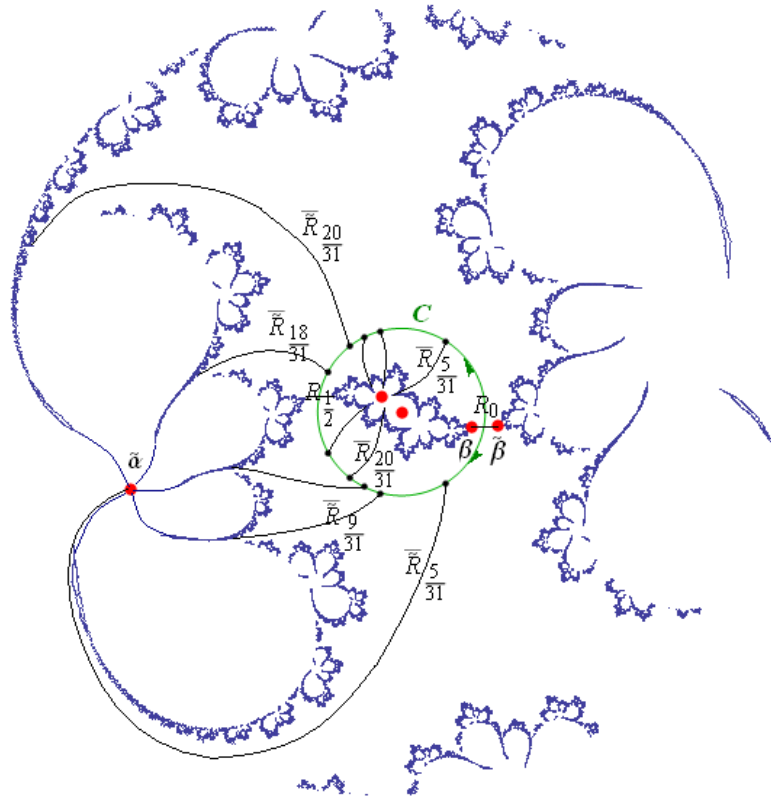


Figure 6.1.2. The construction of Πf_2 where $f_2(z) = z^2 + e^{2\pi i(\frac{2}{5})}$

Note that the truncated rays must be identified in pairs, and the number of rays in a given equivalence class must be at least 2 and at most countably many, since only a finite number of rays can share a common endpoint. Theorem 6.1.3 actually shows that the number of rays in a given equivalence class is finite. Therefore, a given ray equivalence class has an even number of elements. Furthermore, it can be shown that for a fixed mating of two generalized starlike quadratics there exists an even integer k , such that a given ray equivalence class consists of exactly 2 or k truncated rays.

One of the main topological tools in determining matability is a well-known

theorem by Moore stated below.

Theorem 6.1.1. (Moore). *Let \simeq be any equivalence relation on the sphere \mathbb{S}^2 which is topologically closed. Assume also that each equivalence class is connected, but is not the entire sphere. Then the quotient space \mathbb{S}^2 / \simeq is itself homeomorphic to \mathbb{S}^2 if and only if no equivalence class separates the sphere into two or more connected components.*

Using the above theorem along with additional analytic tools, Tan Lei and Peter Haïssinsky proved the following.

Theorem 6.1.2. (Lei, Haïssinskiy). **[LH]** *Two geometrically finite quadratics f_1 and f_2 are matable if and only if they do not belong to conjugate limbs of the Mandelbrot set.*

We now return to the special cases of matings addressed in this thesis, that is, those matings $f_1 \amalg f_2$ for which both f_1 and f_2 lie in the union of the same hyperbolic component $H_{\frac{p}{q}} \subset \mathbf{M}$ and its root $r_{H_{\frac{p}{q}}}$. As before we will denote this union by $H_{\frac{p}{q}}^*$. Recall from section 2.3 that to any $H_{\frac{p}{q}}^*$ we can associate a unique distinguished ray cycle, ${}_{\alpha}RC_{\frac{p}{q}}$, which consists of the q external dynamic rays which land at the α -fixed point of any f in $H_{\frac{p}{q}}^*$. Remember that for these f the α -fixed point is the dynamic root as defined in section 4.2. Hence ${}_{\alpha}RC_{\frac{p}{q}}$ contains the two characteristic dynamic rays $R_{t_{\pm}}$, that is, those

dynamic rays whose arguments are the same as the arguments of the external parameter rays which land at the root of $H_{\frac{p}{q}}$.

Lemma 6.1.1. Fix $\frac{p}{q} \neq \frac{1}{2}$ and let $f \in H_{\frac{p}{q}}^*$. Let $x(R_t), x(R_{-t}) \in J(f)$ denote the landing points of the external dynamic rays R_t and R_{-t} . If $x(R_t)$ is the landing point of more than one external ray then R_{-t} is the only ray that lands at $x(R_{-t})$.

Proof. Essentially, we would like to show that, for $\frac{p}{q} \neq \frac{1}{2}$, it is impossible for both R_t and R_{-t} to eventually land at the distinguished ray cycle ${}_{\alpha}RC_{\frac{p}{q}}$.

Let $f \in H_{\frac{p}{q}}^*$ with $\frac{p}{q} \neq \frac{1}{2}$. Let ${}_{\alpha}RC_{\frac{p}{q}}$ be the associated distinguished ray cycle.

Assume that $x(R_t)$ is the landing point of more than one external dynamic ray.

Then, by proposition 3.1.1 and its corollary, $x(R_t)$ is q -accessible, t is rational,

$x(R_t) \in GO_f(\alpha)$, and there exists a positive integer i such that

$R_{2^i t \pmod{1}} \in {}_{\alpha}RC_{\frac{p}{q}}$. Let n be the smallest positive integer for which

$R_{2^n t \pmod{1}} \in {}_{\alpha}RC_{\frac{p}{q}}$. This implies that $2^n t \pmod{1} = \frac{k}{2^q - 1}$ for some positive

integer $k < 2^q - 1$. Now, assume that $x(R_{-t})$ is also multi-accessible. Arguing

as above, let m be the smallest positive integer for which

$2^m (-t) \pmod{1} = \frac{k}{2^q - 1}$, where k is the integer obtained above. Thus, we obtain

the congruences below:

$$2^n t \pmod{1} \equiv 2^m (-t) \pmod{1} \equiv \frac{k}{2^q - 1}$$

From the first congruence we obtain:

$$2^n t = 2^m (-t) + A, \quad A \in \mathbb{Z}$$

$$(2^n + 2^m) t = A$$

$$t = \frac{A}{2^n + 2^m}$$

Without loss of generality, assume that $n < m$. Then

$$2^n t = \frac{2^n A}{2^n + 2^m} = \frac{A}{1 + 2^{m-n}}$$

Using the second congruence, we get:

$$\frac{A}{1 + 2^{m-n}} \pmod{1} \equiv \frac{k}{2^q - 1}$$

Note the denominators must be equal, so,

$$2^{m-n} + 1 = 2^q - 1$$

$$2^q - 2^{m-n} = 2$$

The only possible solutions are $q = 2$ and $m - n = 1$. However, since $\frac{p}{q} \neq \frac{1}{2}$,

$q \neq 2$ and we arrive at a contradiction. ■

Using the notation presented in the alternative definition of a mating described earlier, theorem 6.1.2 and lemma 6.1.1 easily give us the following.

Theorem 6.1.3. *Let $f_1, f_2 \in H_{\frac{p}{q}}^*$ with $\frac{p}{q} \neq \frac{1}{2}$. Let $f_1 \amalg f_2$ denote the rational map obtained by mating f_1 and f_2 . Let $x \in J(f_1)$ and $y \in J(f_2)$. If, under the mating construction, the two points x and y are identified to a single point $z \in J(f_1 \amalg f_2)$ then at least one of the original two must be uniaccessible.*

Proof. Let $f_1, f_2, f_1 \amalg f_2, x,$ and y be as in the hypothesis. Let \sim_{ray} be the ray equivalence relation, as defined earlier, associated with the construction of $f_1 \amalg f_2$. Let $\tilde{x} = m_{\lambda_1}(x) \in J(\tilde{f}_1)$, where m_{λ_1} is the Möbius inversion defined earlier, and $\tilde{f}_1 = m_{\lambda_1} \circ f_1 \circ m_{\lambda_1}$. Assume that x is q -accessible, and let $\{\overline{\tilde{R}_{t_1}}, \overline{\tilde{R}_{t_2}} \dots \overline{\tilde{R}_{t_q}}\}$ be the collection of truncated external rays landing at \tilde{x} . Since $\tilde{x} \in J(\tilde{f}_1)$ and $y \in J(f_2)$ are identified to a single point, there exists a rational t^* such that there is a truncated external ray $\overline{R_{t^*}}$ landing at y which is ray equivalent to at least one of the rays $\overline{\tilde{R}_{t_i}}$ that land at \tilde{x} . This implies that $t^* = -t_i$ for some i . Since the landing properties asserted in lemma 6.1.1 depend only on the rational $\frac{p}{q}$, and both f_1 and f_2 are elements of the same $H_{\frac{p}{q}}^*$, it follows that y must be uniaaccessible. ■

6.2 Parabolic Self-Matings and Quasi-Self-Matings of Generalized Starlike Quadratics

■ Parabolic Self-Matings

Theorem 6.2.1 (parabolic matings). *Let $p < q$ be two relatively prime positive integers, with $\frac{p}{q} \neq \frac{1}{2}$. Let*

$$f_{\frac{p}{q}}(z) = \lambda_{\frac{p}{q}} z + z^2 = e^{2\pi i \left(\frac{p}{q}\right)} z + z^2,$$

and

$$g_{\frac{p}{q}}(z) = \left(\frac{1 + e^{2\pi i \left(\frac{p}{q}\right)}}{2} \right) \left(z + \frac{1}{z} \right).$$

Then

$$\Pi f_{\frac{p}{q}}(z) = g_{\frac{p}{q}}(z)$$

up to conformal conjugacy.

Proof. Since $\frac{p}{q} \neq \frac{1}{2}$ and $f_{\frac{p}{q}}$ is parabolic, theorem 6.1.2 guarantees the existence of $\Pi f_{\frac{p}{q}}(z)$. Let $\tilde{g} = \Pi f_{\frac{p}{q}}(z)$. It follows, by symmetry of self-matings, that \tilde{g} must be conformally conjugate to a map of the form $g = k\left(z + \frac{1}{z}\right)$, with $k \in \mathbb{C} \setminus \{0\}$. In addition, g must have two distinct parabolic fixed points with multipliers $\lambda_{\frac{p}{q}} = e^{2\pi i \left(\frac{p}{q}\right)}$, and no critical orbit relations.

Using the *Holomorphic Index Formula* we have,

$$\frac{1}{1 - e^{2\pi i \left(\frac{p}{q}\right)}} + \frac{1}{1 - e^{-2\pi i \left(\frac{p}{q}\right)}} + \frac{1}{1 - \frac{1}{k}} = 1$$

Solving for k we get

$$k = \frac{1+e^{2\pi i(\frac{p}{q})}}{2}$$

This implies that

$$g = g_q^p(z) = \left(\frac{1+e^{2\pi i(\frac{p}{q})}}{2} \right) \left(z + \frac{1}{z} \right)$$

Note that g_q^p has two distinct parabolic fixed points, one at $\alpha_1 = \sqrt{\frac{1+\lambda_{\frac{p}{q}}}{1-\lambda_{\frac{p}{q}}}}$ and

one at $\alpha_2 = -\sqrt{\frac{1+\lambda_{\frac{p}{q}}}{1-\lambda_{\frac{p}{q}}}}$, and two critical points at ± 1 . The Fatou set $\Omega(g_q^p)$

contains two distinct parabolic Fatou cycles, $\{F_0, F_1, \dots, F_q\}$ and

$\{V_0, V_1, \dots, V_q\}$, such that $\bigcap_{i=0}^q \overline{F_i} = \alpha_1$ and $\bigcap_{i=0}^q \overline{V_i} = \alpha_2$. It also follows that

$\overline{Orb_g(1)} \subset \bigcup_{i=0}^q \overline{F_i}$ and $\overline{Orb_g(-1)} \subset \bigcup_{i=0}^q \overline{V_i}$, where $g^{on}(1) \rightarrow \alpha_1$ and

$g^{on}(-1) \rightarrow \alpha_2$ as $n \rightarrow \infty$, which completes the proof. ■

■ **Quasi-Self-Matings**

Next, we would like to show that the structure of the Julia set of a self-mating

$\amalg f_c = f_c \amalg f_c$ does not change if we let either or both maps vary but remain

within the same hyperbolic component of the Mandelbrot set as f_c .

Actually, this follows immediately from a much more general result proved by

Lei and Haïssinsky in [LH].

Theorem 6.2.2. (Lei, Haïssinsky). *If P and Q are geometrically finite matable polynomials, then there exist sub-hyperbolic perturbations $(P_t)_t$ and $(Q_t)_t$ such that $J(P_t) \approx J(P)$ and $J(Q_t) \approx J(Q)$, and such that their matings converge to the mating of P and Q .*

The results of theorems 2.1.4, 6.2.1 and 6.2.2 provide the proof of *Main Theorem B*.

■ Julia Sets of Generalized Starlike Quadratics

Note that we only need to investigate the structure of the Julia sets of parabolic starlike quadratics $f_{\frac{p}{q}}(z) = e^{2\pi i\left(\frac{p}{q}\right)}z + z^2$, since, by theorem 2.1.3, the Julia sets of those maps which lie in the interior of $H_{\frac{p}{q}}$ are homeomorphic to $J\left(f_{\frac{p}{q}}\right)$. So, when convenient we will concentrate on the root points rather than any other parameter in $H_{\frac{p}{q}}^*$.

First, we will need the following definition and the subsequent lemmas.

Definition. Let g be a rational map. Let F_{c_j} be a Fatou component containing a critical point. A **full infinite Fatou chain** in $\Omega(g)$, denoted by $\tilde{\mathcal{F}}$ is an infinite sequence of distinct disjoint Fatou components $\{F_0, F_1 \dots F_i, F_{i+1}, \dots\}$ for which

$$(i) \quad F_0 = F_c, \text{ and,}$$

$$(ii) \quad \overline{F_i} \cap \overline{F_{i+1}} \text{ is a single point for all } i \geq 0.$$

A **proper infinite Fatou chain**, denoted by \mathcal{F} , is any proper infinite subset of a full Fatou chain which also satisfies condition (ii) above. We will sometimes refer to condition (ii) as the **chain condition**. In other words, \mathcal{F} is a sequence of adjacent Fatou components $\{F_i\}_{i=0}^{\infty}$ which can be extended to a full infinite Fatou chain. We say that such a chain **lands** if it converges to a well-defined

limit point, and we call this limit point the *landing point* of the chain. Two infinite Fatou chains are *distinct* if they have no common elements.

Remark 6.2.1. *Note that, by construction, if two different proper infinite Fatou chains, $\mathcal{F}_1 = \{F_i\}_{i=0}^\infty$ and $\mathcal{F}_2 = \{G_i\}_{i=0}^\infty$ are not distinct then either (i) one is contained in the other, or (ii) there exist two integers j and k such that the truncated chains $\{F_i\}_{i=j}^\infty$ and $\{G_i\}_{i=k}^\infty$ are distinct. We will refer to those in the latter case as *eventually distinct*.*

Lemma 6.2.1. *If g is a geometrically finite quadratic rational map, then every proper infinite Fatou chain converges.*

Proof. This follows from the local connectivity of Julia sets of geometrically finite quadratic rational maps. ■

Lemma 6.2.2. *Let $f \in H_{q,p}^*$. Every proper infinite Fatou chain has a well-defined limit point, and this limit point is the landing point of exactly one external dynamic ray.*

Proof. Since any $f \in H_{q,p}^*$ is geometrically finite every proper infinite Fatou chain converges by lemma 6.2.1. By proposition 3.1.1, $J(f)$ consists of landing points of exactly one or q external dynamic rays. By corollary 3.1.1, a point $x \in J(f)$ is q -accessible if and only if $x \in GO_f(\alpha)$, where α is the α -fixed point of f . So, since every pre-image of α is the landing point of q external rays it must be the point of intersection of the closures of q Fatou components, and hence, cannot be a limit point of a proper infinite Fatou

chain. This, in turn, implies that a limit point of a proper infinite Fatou chain in $\Omega(f)$ must be uniaccessible. ■

Note that the above lemma is true only for generalized starlike quadratic polynomials.

Lemma 6.2.3. *If x and y are two distinct limit points of two different proper infinite Fatou chains, that is, $\mathcal{F}_1 = \{F_i\}_{i=0}^\infty \rightarrow x$ and $\mathcal{F}_2 = \{G_i\}_{i=0}^\infty \rightarrow y$ and $x \neq y$, then \mathcal{F}_1 and \mathcal{F}_2 are eventually distinct.*

Proof. By remark 6.2.1, if $\{F_i\}_{i=0}^\infty$ and $\{G_i\}_{i=0}^\infty$ are two different proper infinite Fatou chains that are not eventually distinct then one must be contained in the other, and, by lemma 6.2.2, both must converge to the same unique limit point. ■

Lemma 6.2.4. *Let $M_{\frac{1}{2}}$ be the $\frac{1}{2}$ -limb of the Mandelbrot set \mathbf{M} . Suppose f_{r_H} is the quadratic polynomial corresponding to the root r_H of the hyperbolic component $H \subset \mathbf{M} \setminus M_{\frac{1}{2}}$. Let O be the parabolic periodic orbit of f_{r_H} , and let $\mathcal{P}(O)$ be its orbit portrait. If R_t belongs to the grand orbit of the ray cycle with arguments in $\mathcal{P}(O)$ then R_{-t} does not lie on the boundary of any bounded Fatou component in $\Omega(f_{r_H})$.*

Proof. Note that $\mathcal{P}(O)$ is the dynamic root portrait as defined earlier. Taking iterates, if necessary, assume R_t is one of the two characteristic dynamic rays that land at the dynamic root of f_{r_H} . Then there is a parameter ray with the same argument, \mathcal{R}_t , landing at r_H . By the symmetry of \mathbf{M} , it follows that the

landing point of the parameter ray \mathcal{R}_{-t} is the conjugate of r_H . Let us denote it by $\overline{r_H}$. Note that r_H and $\overline{r_H}$ are distinct since we've excluded the $\frac{1}{2}$ -limb of \mathcal{M} . In particular, $\overline{r_H}$ lies in the complement of $\overline{W_{r_H}} = \overline{W_{r_{p(0)}}$. It follows from theorem 4.4.4 that R_{-t} does not land on the boundary of any bounded Fatou component of f_{r_H} . ■

We are now ready to complete the proof of *Main Theorem C*, restated below:

Main Theorem C. *Let p and q be two relatively prime positive integers with $p < q$ and $\frac{p}{q} \neq \frac{1}{2}$. Let $H_{\frac{p}{q}}$ be the hyperbolic component of the Mandelbrot set immediately attached to the main cardioid whose rotation number is $\frac{p}{q}$, and let $H_{\frac{p}{q}}^*$ be the union of $H_{\frac{p}{q}}$ and its root. Let $f_1, f_2 \in H_{\frac{p}{q}}^*$ and let g be the mating of f_1 and f_2 , denoted by $f_1 \amalg f_2$. Then the Julia set of g , $J(g \equiv f_1 \amalg f_2)$, consists of the following mutually exclusive sets of points, each dense in $J(g)$:*

- (i) *a countable set J_α , where each $z_\alpha \in J_\alpha$ is pre-fixed, lies on the boundary of exactly q Fatou components and is the landing point of q distinct proper infinite Fatou chains;*
- (ii) *a countable set J_β , where each $z_\beta \in J_\beta$ is pre-fixed, does not lie on the boundary of any Fatou component and is the landing point of two distinct proper infinite Fatou chains;*
- (iii) *an uncountable set $J_{\alpha\text{-type}}$ where each $z_{\alpha\text{-type}}$ is not pre-fixed, lies on the boundary of exactly one Fatou component and is the landing point of one*

distinct proper infinite Fatou chains;

(iv) an uncountable set $J_{\beta\text{-type}}$, where each $z_{\beta\text{-type}}$ is not pre-fixed, does not lie on the boundary of any Fatou component and is the landing point of two distinct proper infinite Fatou chains;

Furthermore, $J_\alpha \cup J_\beta \cup J_{\alpha\text{-type}} \cup J_{\beta\text{-type}} = J(f_1 \amalg f_2)$.

By lemma 6.1.1 and lemma 6.2.4, we immediately obtain the result below.

Theorem 6.2.3. *Let $f \in H_{\frac{p}{q}}^*$ with (p, q) relatively prime and $\frac{p}{q} \neq \frac{1}{2}$. Let*

$z \in J\left(f_{\frac{p}{q}}\right)$ be a q -accessible point and assume that it is the landing point of a dynamic ray R_t . Then $x(R_{-t})$ is a β -type uniaccessible point.

Note that the above theorem asserts that, for $f \in H_{\frac{p}{q}}^*$, q -accessible points of $J(f)$ are identified with β -type uniaccessible points under the self-mating construction of $g = \amalg f$.

Before proceeding, let us recall and organize the following properties of α -laminations $\mathcal{L}_\alpha\left(\frac{p}{q}\right)$ defined and discussed in sections 3.2 and 3.3. Let p and q be two relatively prime positive integers with $\frac{p}{q} \neq \frac{1}{2}$. Let $\mathcal{L}_\alpha\left(\frac{p}{q}\right)$ be the associated α -lamination and let $f \in H_{\frac{p}{q}}^*$. Let $t \in \partial\mathcal{D}$ (expressed in turns).

Property 1. $\mathcal{L}_\alpha\left(\frac{p}{q}\right)$ has no non-degenerate accumulation leaves;

Property 2. t is either **(i)** a vertex of a lamination polygon and a boundary point of a lamination gap, or **(ii)** t is not a vertex of any lamination polygon but is a boundary point of a lamination gap, or **(iii)** t is neither a vertex of a

polygon nor a boundary point of a gap. In the last case t must be an accumulation point of a nested sequence of lamination polygons, or, equivalently, an accumulation point of a sequence of nested gaps;

Property 3. $\overline{\mathfrak{L}_\alpha\left(\frac{p}{q}\right)}$ is homeomorphic to $K(f)$;

Property 4. *There is a one-to-one correspondence between the gaps of $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$ and bounded Fatou components in $\Omega(f)$.*

Theorem 6.2.4. *Let $f \in H_p^*$. Let $\hat{\beta} \in J(f)$ be of β -type. Then $\hat{\beta}$ is a landing point of some proper infinite Fatou chain.*

Proof. Let R_t be the external dynamic ray landing at $\hat{\beta} \in J(f)$. Let $\partial\mathcal{D}$ be identified with the boundary of $\overline{\mathfrak{L}_\alpha\left(\frac{p}{q}\right)}$. By property 2 above, since $\hat{\beta}$ is β -type, $t \in \partial\mathcal{D}$ must be a degenerate leaf which is an accumulation point of a sequence of nested gaps. By property 1 every non-degenerate leaf of $\overline{\mathfrak{L}_\alpha\left(\frac{p}{q}\right)}$ is a side of a lamination polygon and is a boundary side of a lamination gap. In other words, every non-degenerate leaf separates a lamination polygon and a lamination gap. Let $\gamma_t = [0, t]$ be a radial line segment connecting the origin to the point $t \in \partial\mathcal{D}$. It follows from the preceding discussion that γ_t determines a sequence of nested lamination gaps each separated by a lamination polygon converging to t . In other words, γ_t intersects an alternating sequence of gaps and polygons, where each polygon Q_i corresponds to a point on the Julia set which is common to the boundaries of the two Fatou components corresponding to the two lamination gaps adjacent to Q_i that are also

intersected by γ_t . Property 4 allows us to define a unique full infinite Fatou chain $\tilde{\mathcal{F}}_t \subset \Omega(f)$. Letting \mathcal{F}_t be some proper infinite subset of $\tilde{\mathcal{F}}_t$ which satisfies the chain condition, we obtain a proper infinite Fatou chain landing at $\hat{\beta}$ as required. ■

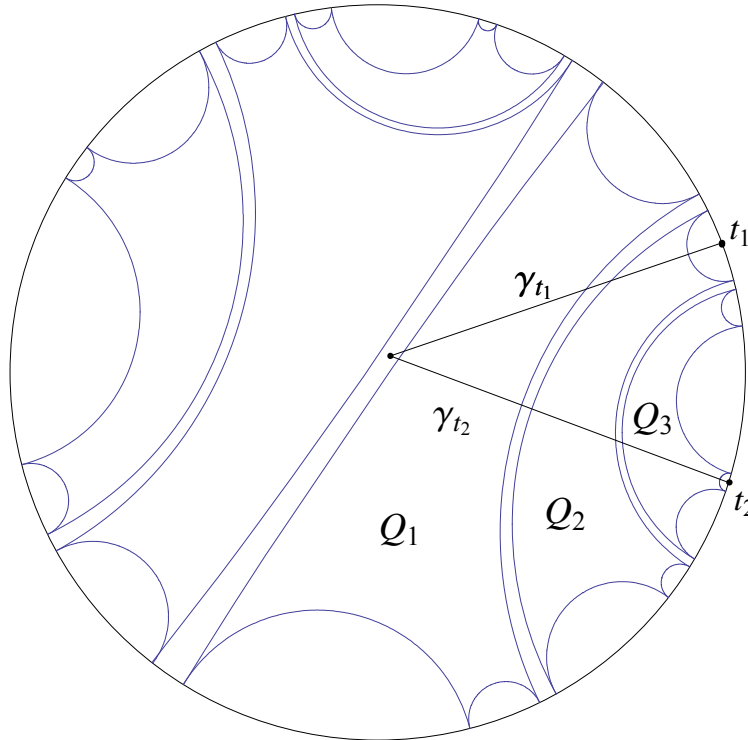


Figure 6.2.1. An example of the construction described in the above proof showing two radial lines which determine two eventually distinct proper infinite Fatou chains

Theorems 6.2.3 and 6.2.4 imply the first statement of *Main Theorem C*. We call this collection of points J_α . Note that $J_\alpha = GO_g(\alpha_1) \cup GO_g(\alpha_2)$, where α_1 and α_2 are the images of the α -fixed points of f_1 and f_2 under the mating construction, and g, f_1 and f_2 are the maps defined in *Main Theorem C*.

Theorem 6.2.5. Let $f \in H_{\frac{p}{q}}^*$ with (p, q) relatively prime and $\frac{p}{q} \neq \frac{1}{2}$. Let $x(R_t)$

be the landing point of R_t . If $x(R_t)$ is α -type then $x(R_{-t})$ is β -type.

Proof. Without loss of generality we can let $f = f_{\frac{p}{q}}$. Let $x(R_t)$ be α -type; that is, it is uniaccessible and lies on the boundary of some bounded Fatou component. By theorem 4.4.4, $R_t \in \bigcup_m f_{r_{H_i}}^{-m}(R_{t_{\pm}}(r_{H_i}))$ for all $r_{H_i} > r_{H_{\frac{p}{q}}}$ such that there is a finite chain from $H_{\frac{p}{q}}$ to H_i . Iterating if necessary, assume R_t is one of the characteristic rays $R_{t_{\pm}}(r_{H_i})$ for some H_i in the chain. The parameter ray \mathcal{R}_{-t} must land at the root point $\overline{r_H}$, the conjugate of r_H . Since $\overline{r_H}$ lies in the complement of $W_{r_{\frac{p}{q}}}$, the dynamic ray R_{-t} does not land on the boundary of the bounded Fatou component $F_1 \subset \Omega(f_{\frac{p}{q}})$, and the same must be true for any ray in the grand orbit of R_{-t} , thus completing the proof. ■

The third statement of *Main Theorem C* follows from theorem 6.2.5 and 6.2.4. By theorem 6.1.3, we are left with one more option; where the points to be identified are both β -type. Note that the set of β -type points on $J(f_{\frac{p}{q}})$ contains those points that are pre-images of the β -fixed point. We will call this subset J_{β} . The β -fixed points of the two copies of $J(f_{\frac{p}{q}})$ are identified and it follows that under the self-mating construction the points in J_{β} are identified with other points in J_{β} , and their images under the self-mating construction are exactly the points $z_{\beta} \in GO_g(\tilde{\beta})$ where $g = \text{II } f_{\frac{p}{q}}$ and $\tilde{\beta}$ is the β -fixed point of g . We will denote the collection of those points which are β -type but are not pre-

images of $\tilde{\beta}$ by $J_{\beta\text{-type}}$. This discussion demonstrates statements (ii) and (iv) of *Main Theorem C*.

6.3 Dual Laminations For Generalized Starlike Quadratics

By abuse of notation we will denote the geometric construction associated with the lamination $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$ by the same symbol, $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$.

Definition. Given $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$, define its **dual lamination**, denoted by $\mathfrak{L}_\alpha^*\left(\frac{p}{q}\right)$, as the image of $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$ under the rotation of π radians about origin. This rotation is equivalent to inverting the lamination in the boundary circle and then reflecting in the real axis so that the labeling of the arguments on the boundary is reversed.

Let ${}_\alpha RC_p^q$ be the distinguished ray cycle associated with $\frac{p}{q}$, and let C_q^p be the associated set of arguments. Let $-C_q^p$ be the collection of negatives of the arguments in C_q^p .

Let $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}^*$ be the corresponding unit disks. As is done in the construction of matings, identify the two points $t \in \overline{\mathcal{D}}$ and $t^* \in \overline{\mathcal{D}}^*$, if $t^* = -t$. Following the discussion of sections 6.1 and 6.2, we have the following classification of such identifications:

★ If $t \in \partial\mathcal{D}$ is a vertex of a polygon of $\mathfrak{L}_\alpha\left(\frac{p}{q}\right)$ then $t^* \in \partial\mathcal{D}^*$ is neither a vertex of a polygon of $\mathfrak{L}_\alpha^*\left(\frac{p}{q}\right)$ nor a boundary point of any gap of $\mathfrak{L}_\alpha^*\left(\frac{p}{q}\right)$, which implies that t^* must be a limit point of an infinite sequence of nested polygons; in addition, t^* must be in the $GO_{dbl}(-C_q^p)$. This also implies that t and t^* are both rational.

★★ If $t \in \partial\mathcal{D}$ is a boundary point of a gap of $\mathfrak{L}_\alpha(\frac{p}{q})$ then $t^* \in \partial\mathcal{D}^*$ is, again, neither a vertex of a polygon of $\mathfrak{L}_\alpha^*(\frac{p}{q})$ nor a boundary point of any gap of $\mathfrak{L}_\alpha^*(\frac{p}{q})$, which implies that t^* must be a limit point of an infinite sequence of nested polygons. Moreover, if t is rational then t is eventually periodic under doubling (*mod* 1) and its eventual period k is a multiple of q ; t^* must also be rational and eventually periodic with the same eventual period k

★★★ If neither $t \in \partial\mathcal{D}$ nor $t^* \in \partial\mathcal{D}^*$ is a boundary point of any gap or a vertex of any polygon in $\mathfrak{L}_\alpha(\frac{p}{q})$, or, respectively, $\mathfrak{L}_\alpha^*(\frac{p}{q})$ then both are limit points of an infinite sequence of nested polygons.

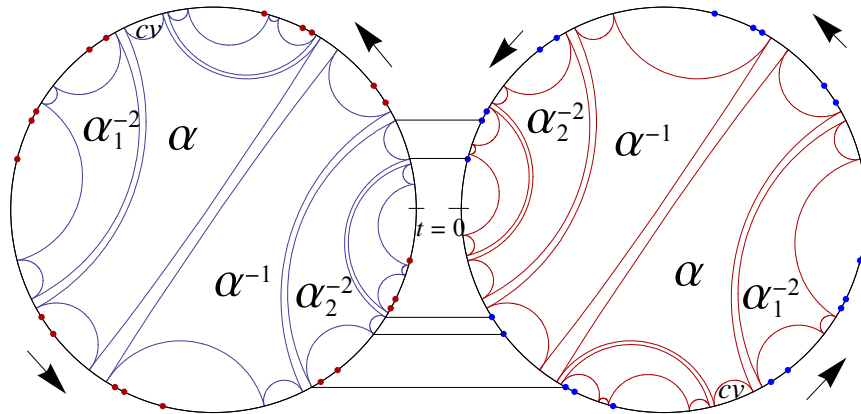


Figure 6.3.1. $\mathfrak{L}_\alpha(\frac{2}{5})$ and its dual $\mathfrak{L}_\alpha^*(\frac{2}{5})$, properly oriented, showing an example of a point-orbit identification

Note that the straight lines identifying the five vertices $\{t_1, t_2 \dots t_5\}$ of the polygon $Q_{\alpha_2^{-2}} \subset \mathfrak{L}_\alpha(\frac{2}{5})$, which is a "blow-up" of one of the pre-images of α , with the five degenerate leaves $\{-t_1, -t_2 \dots -t_5\} \subset \mathfrak{L}_\alpha^*(\frac{2}{5})$ in the preceding figure are for demonstrative purposes only.

The full classification above complements the statement of *Main Theorem C*.

The figure below shows the Julia set $J\left(g_{\frac{2}{5}} = \text{II} f_{\frac{2}{5}}\right)$, as well as how the two copies of the filled Julia set $K\left(f_{\frac{2}{5}}\right)$ fit together, along with the two parabolic fixed points α_1 and α_2 , the two critical points at $z = \pm 1$, the corresponding critical values, and the spines of each copy of $K\left(f_{\frac{2}{5}}\right)$ which spiral toward the origin, which is itself one of the buried points in this Julia set.

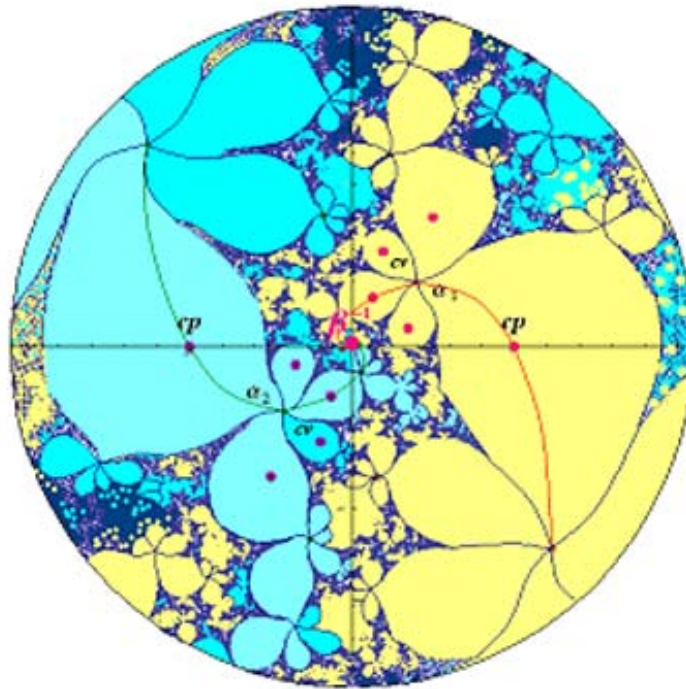


Figure 6.3.2. The Julia set $J\left(g_{\frac{2}{5}} = \text{II} f_{\frac{2}{5}}\right)$

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