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N-EXTENDED SUPERSYMMETRIC QUANTUM MECHANICS

by

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A dissertation submitted to the Graduate Faculty in Physics

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy, The City University of New York

2006

UMI Number: 3205019

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This manuscript has been read and accepted for the
Graduate Faculty in Physics in satisfaction of the
dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

N-extended Supersymmetric Quantum Mechanics

by

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Supersymmetric quantum mechanics is a theory where the Hamiltonian of conventional quantum mechanics forms an algebraic structure: superalgebra, along with other operators: supercharges. This symmetry is responsible for the degeneracy of the spectrum of the resulting partner Hamiltonians. This work extends the number of supersymmetries - number of supercharge operators - to an arbitrary number. The procedure is given in algebraic and matrix formalism. The scheme is related to the backward potential problem of constructing quantum mechanical systems for a wide range of modified potentials. The totally isospectral potentials are a natural result in the extended supersymmetric system. It is shown how to construct families of isospectral potentials for any number of supersymmetries. It is also shown that the supersymmetric Hamiltonians of the many particle system can be extended. The two particle Calogero potential is given for the case of two pairs of supercharges.

Acknowledgments

Thanks to Professor Sultan Catto for giving me invaluable advice at every stage of my study at the GC, supporting me financially and encouraging me to finish the thesis. I thank Professor Vladimir Akulov for helping me with the research topic and a thorough introduction to the subject. I thank the late Professor Anatolii Pashnev for reinvigorating the research and contributing to the study of the multiparticle case. I thank my wife Irina Gladkova for support and help for the last 10 and a half years.

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Chapter 1

Introduction

Despite a long and successful history of the supersymmetric quantum mechanics (SUSY QM), an elegant and simple theory remains largely unknown to the physics community. It is commonly perceived as an auxiliary tool to solve models which can be otherwise completely understood in the terms of the conventional QM. At present SUSY QM must be incorporated in the framework of quantum mechanics not just as a useful calculation procedure but as the foundation of a much broader understanding of the principles of quantum theory.

Among the numerous developments in SUSY QM, this work focuses on the possibilities of extending the number of supersymmetric charges: the generators of the supersymmetric algebra, and the consequences of the resulting theories. In particular, the solutions of exactly solvable models with known potentials for one and many particles will be obtained in the extended case of an arbitrary number of supercharges N. We discuss how the results of the backward potential problems known as ”quantum design” can be understood in terms of the extended SUSY QM.

The fact that factorization is possible and simplifies obtaining the spectrum and eigenstates of the Hamiltonian was known to Schrödinger himself as Darboux’s transformation. For a long time it stood alone as an elegant way to solve some potentials which could be solved anyway by more standard methods. The supersymmetric structure of the Schrödinger equation was discovered not long after the advent of supersymmetric field theory [1]. It was quickly realized that practically all conventional QM can be ”supersymmetrized”. The review of the results can be found in [18].

The dominant view of SUSY QM today is similar, by analogy, to the relation between real and complex analysis. Conventional QM is a satisfactory and complete description of the ”real” physical world (experiment) and SUSY QM is a build-up above it , ”complexification” with respect to imaginary (grassmannian) variables. The situation however could be more complex. This might be a new description and interpretation of experiments in quantum physics.

1.1 Outline of the thesis

First, the general formalism of SUSY QM is presented. The supersymmetric algebra and properties of its generators are given, all in the case of one pair of charges ($N = 1$ SUSY QM). This is used to illustrate the techniques of the backward potential problem using a toy model: infinite well potentials. The most important modification of the spectrum of the potential that can be obtained analytically will be discussed in detail. The Calogero potential - one of the exactly solvable models - has been solved in the case of many particles in conventional QM [3], [4]. It is shown how the spectrum and eigenfunctions can be obtained using the techniques of the backward potential problem. Later in this work a multiparticle Calogero model is extended in the number of supersymmetries. In the next chapter the formalism of the extended SUSY QM is defined. The case of the $N = 2$ extension is given in algebraic and matrix forms. Then the case of arbitrary N is presented. The iteration scheme has been found in [2] that allows us to prove the existence theorem for the extension to arbitrary N. The procedure is discussed in detail for a number of steps of the iterations. It is shown how the family of completely isospectral Hamiltonians which arise in the conventional SUSY QM as the result of the arbitrary constructions [5] are the natural result of the extended SUSY QM. An important conformal QM model is shown to have an algebraic structure in the coefficients of its conformal potential. The final chapter is dedicated to the multiparticle potentials in SUSY QM. This problem might be related to the thermodynamical property of the spectrum of the black hole [21], [8], [17]. The two particle SUSY potential is obtained.

Chapter 2

General formalism

2.1 N=1 extended supersymmetric quantum mechanics.

Supersymmetric quantum mechanics is defined by its generators $Q, \overline{Q}$ called supercharges

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad \overline{Q} = \begin{bmatrix} 0 & \overline{A} \\ 0 & 0 \end{bmatrix} \tag{2.1}$$

where $\overline{A}$ and A are two conjugate differential operators

$$A = \partial + w, \quad \overline{A} = -\partial + w \tag{2.2}$$

where ∂ is the derivative and $\mathrm{Im}(w) = 0$. They are elements of the superalgebra (symbols $\{,\}$ and $[,]$ stand for anti- and commutator)

$$\{Q, \overline{Q}\} = \mathcal{H}, \quad \{Q, Q\} = \{\overline{Q}, \overline{Q}\} = 0, \quad [Q, \mathcal{H}] = [\overline{Q}, \mathcal{H}] = 0 \tag{2.3}$$

together with another generator $\mathcal{H}$, the superhamiltonian

$$\mathcal{H} = \begin{bmatrix} H - \mathcal{E} & 0 \\ 0 & \widehat{H} - \mathcal{E} \end{bmatrix} \tag{2.4}$$

with two partner Hamiltonians H and $\widehat{H}$ written in terms of $\overline{A}$ and A as

$$H = \overline{A}A + \mathcal{E} = -\partial^2 + w^2 - w' + \mathcal{E} \tag{2.5}$$

$$\widehat{H} = A\overline{A} + \mathcal{E} = -\partial^2 + w^2 + w' + \mathcal{E} \tag{2.6}$$

(2.6) appears as the Schrödinger equation with the potentials V and $\widehat{V}$ given by

$$V = w^2 - w' + \mathcal{E} \tag{2.7}$$

$$\widehat{V} = V + 2w' = w^2 + w' + \mathcal{E} \tag{2.8}$$

with $\mathcal{E}$ called the energy of factorization. The Schrödinger equation with potential V is said to be supersymmetric if it can be factorized by means of $\overline{A}$ and A into H or $\widehat{H}$ with its partner. There are a number of exactly solvable potentials and all of them are supersymmetric. In particular, the coulomb potential gives the supersymmetric Schrödinger equation (radial part) of the hydrogen atom.

The consequence of the symmetry in the theory is degeneracy of its spectrum. Now it will be shown that the hamiltonians H and $\widehat{H}$ are isospectral except for the ground state. The following equations reveal the relationship between the eigenfunctions ψ and $\widehat{\psi}$ of the two Hamiltonians H and $\widehat{H}$ with eigenvalues E and $\widehat{E}$

$$H\psi = E\psi, \quad \widehat{H}\widehat{\psi} = \widehat{E}\widehat{\psi} \tag{2.9}$$

$$\widehat{\psi} = A\psi, \quad \psi = \overline{A}\widehat{\psi} \tag{2.10}$$

$$\widehat{H}\widehat{\psi} = A\overline{A}(A\psi) = A(\overline{A}A\psi) = A(E\psi) = E\widehat{\psi} \tag{2.11}$$

$$H\psi = \overline{A}A(\overline{A}\psi) = \overline{A}(A\overline{A}\widehat{\psi}) = \overline{A}(\widehat{E}\widehat{\psi}) = \widehat{E}\psi \tag{2.12}$$

The identity of spectra E and $\widehat{E}$ is not total since the ψ and $\widehat{\psi}$ can not both be ground state wavefunctions, one being the result of action of the differential operator of the first order on the other. This remarkable fact produces the foundation of the backward potential problem, which will be discussed next briefly.

2.2 The backward potential problem

In the backward potential problem (also known as quantum design problem) one can start with a supersymmetric potential V for which the Schrodinger equation can be solved so that its eigenfunctions ψ_n satisfying some boundary conditions and its bound state energy levels E_n are known explicitly. Then one can modify in a number of ways (in principle, arbitrarily) one or some energy levels of E_n keeping others unchanged and group them into a new spectrum $\widehat{E}_n$. Using the relationships of SUSY QM it is possible to find explicitly the potential $\widehat{V}$ and eigenfunctions $\widehat{\psi}_n$ corresponding to the constructed spectrum $\widehat{E}_n$. The resulting new exactly solvable potentials are too complicated for a standard forward methods to obtain. It was shown that some of the results are equivalent to the ones of the inverse scattering problem.It is hard to overstate the significance of this remarkable procedure. And yet it is not in the curriculum of quantum physics courses.

This is all the more regretable as the calculations are simple and accessible. I would like to illustrate some ways to construct a new spectrum as well as the corresponding eigenvalue problems here with the hope of reaching as many readers as the subject deserves. As a simple toy model an infinite well potential

$$V(x) = \left\{ \begin{array}{ll} -1, & |x| \leq \frac{\pi}{2} \\ \infty, & |x| > \frac{\pi}{2} \end{array} \right\} \tag{2.13}$$

will be used. The normalized solutions of the Schrodinger equation (2.9) satisfying the boundary conditions $\psi(\pm\frac{\pi}{2}) = 0$ are

$$\psi_n(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos\left[(n+1)x + n\frac{\pi}{2}\right], \quad E_n = n(n+2), \quad n = 0, 1, 2... \tag{2.14}$$

are shown in Figure 2.1 for several low energy levels (the eigenfunctions are graphed around their levels).

For this system we demonstrate the results of several procedures of the backward potential problem.

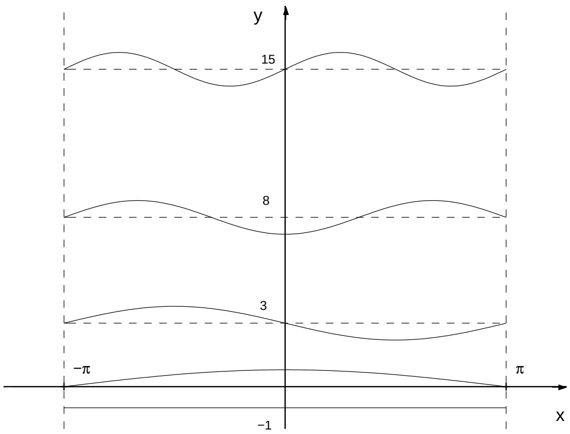

Figure 2.1: Graph of (2.13) and (2.14) for $n = 0,\ 1,\ 2,\ 3$

2.2.1 Deleting a state.

For a given system with Hamiltonian H, potential V, egenfunctions ψ_n and eigenvalues E_n, one can construct a system with Hamiltonian $\widehat{H}$, potential $\widehat{V}$, egenfunctions $\widehat{\psi}_n$ and eigenvalues $\widehat{E}_n$, in which a state (states) corresponding to any energy level is deleted and the rest unchanged. The prepotential w in (2.10) can be written in terms of a new function u as

$$\widehat{\psi}_n = (\partial - w)\psi_n, \quad w = -\frac{u'}{u} \tag{2.15}$$

which, when substituted in (2.7), turns out to be the solution of the Schrodinger equation of H with energy $\mathcal{E}$

$$-u' + Vu = \mathcal{E}u \tag{2.16}$$

Now, if u is chosen as an eigenstate ψ_m of H of energy E_m then $\widehat{\psi}_m$ vanishes (becomes non-normalizable): state m, ground or any excited state, is deleted . For example, when $m = 0$ is deleted in (2.13), (2.14) the new system has

$$\psi_0 = \cos x, \quad w = -\frac{\psi_0'}{\psi_0}, \quad \widehat{\psi}_0 = (\partial + w)\cos x = 0, \quad \widehat{\psi}_n = (\partial + w)\psi_n, \quad n \geq 1 \tag{2.17}$$

$$\widehat{V} = V + 2w' = -1 + \frac{2}{\cos^2 x} \tag{2.18}$$

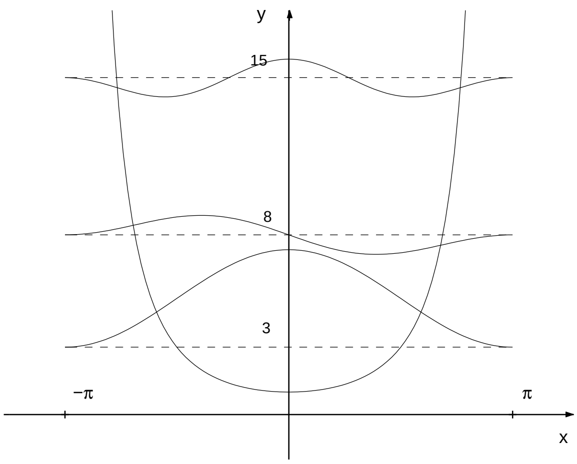

Figure 2.2: Graph of (2.17) and (2.18) for $n = 1,\, 2,\, 3$

The constructed normalized wave functions $\widehat{\psi}_n$ and potential $\widehat{V}$ are shown on the graph (Figure 2.2) around their energy levels.

The ground state of the resulting system can be again deleted. Before repeating (2.15) the new potential must be shifted to make its ground state energy zero. When deleting an excited state the new eigenfunctions and potential in general have singularities, which could be removed by applying another transformation on them (see below).

2.2.2 Inserting a new ground state.

In order to construct a new system of $\widehat{H}$ different from the system of H only in having a new ground state, consider the Schrodinger equation with the factorized supersymmetric Hamiltonian $\widehat{H}$

$$\widehat{H}\widehat{\psi} = (A\overline{A} + \mathcal{E})\widehat{\psi} = \mathcal{E}\widehat{\psi} \tag{2.19}$$

with solution $\widehat{\psi}$ annihilated by $\overline{A}$

$$\overline{A}\widehat{\psi} = (-\partial + w)\widehat{\psi} = 0 \tag{2.20}$$

The last equation is identically satisfied by taking

$$\widehat{\psi} = \frac{1}{u}, \quad w = -\frac{u'}{u} \tag{2.21}$$

The potentials V and $\widehat{V}$ are parametrized with prepotential w according to (2.7). After substitution it can be reduced to

$$-u' + Vu = \mathcal{E}u \tag{2.22}$$

Thus, it follows that the solution of H is the new ground state of $\widehat{H}$ for the same energy $\mathcal{E}$. For the eigenfunction $\widehat{\psi} = \frac{1}{u}$ to be with no singularities u must not have zeros. Such a solution of (2.22) exists for energy below the ground state energy of the bounded states: $\mathcal{E} < E_0$. Thus, this procedure constructs a new system with

$$\widehat{\psi}_{\mathcal{E}} = \frac{1}{u}, \quad w = -\frac{u'}{u} \quad \widehat{\psi}_n = (\partial + w)\psi_n, \quad \widehat{V} = V + 2w', \quad \mathcal{E} < E_n, \quad n \geq 0, \tag{2.23}$$

It should be noticed here that (2.21) can be integrated

$$u = e^{-\int w} \tag{2.24}$$

This shows the sense of the prepotential w: it relates solutions of the Schrödinger equation with its potential; if the latter can be parametrized by w then (2.24) gives the solution.

The insertion of the ground state can be iterated indefinitely along the following scheme:

Start with the systems of H_1 and $\widehat{H}_1$, factorize H_1 by parametrizing V_1 with w_1 then obtain $\widehat{\psi}_{1\mathcal{E}} = \frac{1}{u_1}$ and $\widehat{V}_1 = w_1^2 + w_1' + \mathcal{E}_1 = V_1 + 2w_1'$ as in (2.23).

To construct the systems of H_2 and $\widehat{H}_2$ notice that V_2 is $\widehat{V}_1$. This is equivalent to the equation

$$w_1^2 + w_1' + \mathcal{E}_1 = w_2^2 - w_2' + \mathcal{E}_2 \tag{2.25}$$

For supersymmetric potentials this Riccati equation has simple solutions for w_2 in terms of w_1 (this will be discussed later). The values of $\mathcal{E}_2$ are restricted and related to $\mathcal{E}_1$. With w_2 known construct the system of $\widehat{H}_2$ as in (2.23).

Repeat the above steps for the next system.

To illustrate the construction the two new ground states will be inserted in the system of (2.13), (2.14). A zero-free solution of (2.22) for potential V_1

$$u_1 = \cosh\sqrt{|\mathcal{E}_1| - 1}x \quad , w_1 = -\sqrt{|\mathcal{E}_1| - 1}\tanh\sqrt{|\mathcal{E}_1| - 1}x \tag{2.26}$$

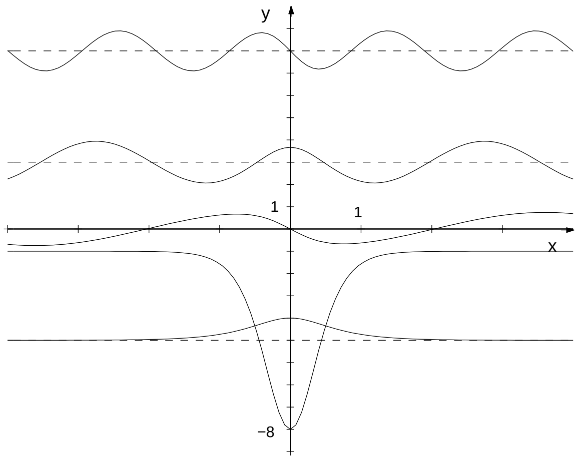

Figure 2.3: Graph of (2.28) and (2.29) for $n = 1,\, 2,\, 3,\, 4$

corresponds to an energy $-|\mathcal{E}_1|$ below the ground state energy $E_0 = 0$. Then the new ground state eigenfunction is

$$\widehat{\psi}_{-|\mathcal{E}_1|} = \frac{1}{\cosh\sqrt{|\mathcal{E}_1| - 1}x}\,, \quad E = -|\mathcal{E}_1|, \tag{2.27}$$

with the rest of the states having energy unchanged in the new potential $\widehat{V}_1$

$$\widehat{\psi}_{n1} = (\partial + w_1)\psi_n, \quad E_n = n(n+2), \quad n \geq 0 \tag{2.28}$$

$$\widehat{V}_1 = V_1 + 2w_1' = -1 - \frac{2(|\mathcal{E}_1| - 1)}{\cosh^2\sqrt{|\mathcal{E}_1| - 1}x} \tag{2.29}$$

The graph (Figure 2.3) shows the new eigenfunctions $\widehat{\psi}_{-|\mathcal{E}_1|}$, $\widehat{\psi}_n$ around their energy levels in the new potential $\widehat{V}_1$ with a new ground state inserted at $\mathcal{E}_1 = -5$.

To insert another state $\widehat{\psi}_{-|\mathcal{E}_2|}$ at $|\mathcal{E}_2| > |\mathcal{E}_1|$ and construct the second system the w_2 has to be obtained from the condition

$$w_1^2 + w_1' - |\mathcal{E}_1| = w_2^2 - w_2' - |\mathcal{E}_2| \tag{2.30}$$

let $w_2 = aw_1$ and using $w_1^2 - w_1' - |\mathcal{E}_1| = -1$ it can be checked that

$$w_2 = 2w_1, \quad |\mathcal{E}_2| = 4(|\mathcal{E}_1| - 1) + 1 \tag{2.31}$$

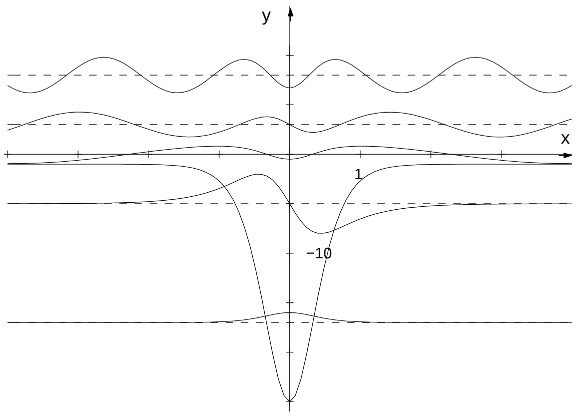

Figure 2.4: Graph of (2.33) - (2.36) for $n = 0,\,1,\,2$

The second system constructed is

$$u_2 = \cosh^2\sqrt{|\mathcal{E}_1| - 1}x, \quad w_2 = -2\sqrt{|\mathcal{E}_1| - 1}\tanh\sqrt{|\mathcal{E}_1| - 1}x \tag{2.32}$$

$$\widehat{\widehat{\psi}}_{-|\mathcal{E}_2|} = \frac{1}{\cosh^2\sqrt{|\mathcal{E}_1| - 1}x}, \quad E = -|\mathcal{E}_2|, \tag{2.33}$$

$$\widehat{\widehat{\psi}}_{-|\mathcal{E}_1|} = (\partial + w_2)\widehat{\psi}_{-|\mathcal{E}_1|}, \quad E = -|\mathcal{E}_1|, \tag{2.34}$$

$$\widehat{\widehat{\psi}}_{n2} = (\partial + w_2)\widehat{\psi}_{n1}, \quad E_n = n(n+2), \quad n \geq 0 \tag{2.35}$$

$$\widehat{V}_2 = \widehat{V}_1 + 2w_2' = -1 - \frac{6(|\mathcal{E}_1| - 1)}{\cosh^2\sqrt{|\mathcal{E}_1| - 1}x} \tag{2.36}$$

The graph of the two constructed ground states $E = -|\mathcal{E}_1| = -5$ and $E = -|\mathcal{E}_2| = -17$ in the potential $\widehat{V}_2$ with extended states $\widehat{\widehat{\psi}}_{n2}$ around their energy levels is shown in Figure 2.4.

2.2.3 Deleting and reinserting states. Isospectral Hamiltonians.

One can, first, delete any number of states starting anywhere in the spectrum of the H. Then, one can reinsert the deleted states at the same energy levels as the ones that were deleted. The result is a new system with the Hamiltonian totally isospectral to H and its eigenfunctions in the constructed potential. To illustrate the construction a single state will be first deleted and then reinserted back. The system of the $\widehat{H}_1$ with a state of the H_1 at energy $\mathcal{E}_1$ deleted is given by (2.15), (2.16). To insert the state at an energy $\mathcal{E}_2$ following the (2.30) one has to solve

$$w_1^2 + w_1' + \mathcal{E}_1 = w_2^2 - w_2' + \mathcal{E}_2 \tag{2.37}$$

for w_2 to insure $V_2 = \widehat{V}_1$. Notice that the two energy $\mathcal{E}_1$and $\mathcal{E}_2$ are the same if the state is inserted back at the energy it was deleted. In general, the two may be different if the procedure results in shifting an arbitrary state (states) in spectrum with the rest remaining unchanged. In the case of $\mathcal{E}_1$= $\mathcal{E}_2$ the equation (2.37) can be reduced, with the substitution

$$w_2 = -w_1 + \frac{1}{z}, \quad w_1 = -\frac{u'}{u_1}, \quad w_2 = -\frac{u'}{u_2} \tag{2.38}$$

to a linear equation

$$z' - 2w_1 z = -1 \tag{2.39}$$

which has a known general solution

$$\frac{1}{z} = -\ln'(\gamma - \int^x u_1^2), \quad u_2 = \frac{1}{u_1}(-\gamma - \int^x u_1^2) \tag{2.40}$$

with a constant parameter $\gamma > 0$ to insure that u_2 is free of zeros. In general, when $\mathcal{E}_1 \neq \mathcal{E}_2$ the general solution can be found in terms of the generalized Legendre functions P_n^μ and Q_n^μ.

Thus, the system is constructed with the same spectrum in the new potential

$$\widehat{V}_2 = -1 - \ln'(-\gamma - \int^x u_1^2), \tag{2.41}$$

which can be readily checked and illustrated below for the system of (2.13), (2.14) with

$$\mathcal{E}_1 = \mathcal{E}_2 = 0, \quad u_1 = \psi_0 = \cos x \tag{2.42}$$

$$\widehat{\psi}_0 = 0, \quad \widehat{\psi}_n = (\partial + w_1)\psi_n, \quad E_n = n(n+2), \quad n > 1 \tag{2.43}$$

$$\widehat{\widehat{\psi}}_0 = \frac{1}{u_2}, \quad E_0 = 0 \tag{2.44}$$

$$\widehat{\widehat{\psi}}_n = (\partial + w_2)\widehat{\psi}_n, \quad E_n = n(n+2), \quad n > 1 \tag{2.45}$$

This procedure can be applied in a number of ways. Several states can be first deleted one followed by the next in the spectrum. Then all are reinserted back starting from the last deleted in reversed order. The resulting potentials are isospectral ones parametrized by many constant one for each state. It was shown that these potentials turn out to be the multi-soliton solutions of the evolution wave equations. The insertion of new states can be performed first followed by the deletion of these states. The isospectral potentials obtained as the result are different from the ones derived above, since the potential is determined by the state function u_1 in (2.41) that is either the deleted or inserted state of the first step.

2.2.4 Finding the spectrum and eigenfunctions of the supersymmetric Hamiltonians by iteration.

The spectrum and eigenfunctions of the supersymmetric Hamiltonians can be obtained by using the relation between the ψ, $\widehat{\psi}$ (2.10) and the potential expression in terms of the function w (2.7). The scheme is similar to the one used in the harmonic oscillator eigenvalue problem where the spectrum is obtained by means of the creation and annihilations operators. One starts with the ground state function of the H_1corresponding to the potential V factorized in terms of w_1. Assuming that the ground state energy is zero, the eigenfunction is found from

$$H_1\psi_0 = \overline{A}_1 A_1 \psi_0 = 0, \quad A_1\psi_0 = (\partial + w_1)\psi_0 = 0, \quad w_1 = -\frac{\psi_0'}{\psi_0}, \quad \psi_0 = e^{-\int w_1} \tag{2.46}$$

The partner Hamiltonian $\widehat{H}_1$ corresponding to the potential $\widehat{V}$ has identical spectrum as that of H_1 except for the ground state (deleted as explained above). In particular, the ground state energy of $\widehat{H}_1$ is the same as the energy of the first excited state of H_1. The eigenfunctions of the two are related by

$$\psi = \overline{A}\widehat{\psi} \tag{2.47}$$

so that one can find the first excited state of H_1 from the ground state of $\widehat{H}_1$. The ground state eigenfunction of $\widehat{H}_1$ can be found in the same way as in (2.46) by rewriting the system of the $\widehat{H}_1$ as an equivalent system of H_2 with the ground state energy zero. Consider

$$(\widehat{H}_1 - E)\widehat{\psi} = H_2\widehat{\psi} = 0 \tag{2.48}$$

so that potentials of the two systems are related by

$$w_1^2 + w_1' - E = w_2^2 - w_2' \tag{2.49}$$

Once the w_2 is known (2.48) can be solved

$$\widehat{\psi} = e^{-\int w_1} \tag{2.50}$$

as well as the excited state of the original system can be found by means of (2.47). One can seek the solution of the Riccati equation (2.49) in the form

$$w_2 = aw_1 \tag{2.51}$$

$$(1 - a^2)w_1^2 + (1 + a)w_1' - E = 0 \tag{2.52}$$

Using the functional properties of the prepotential w (shape invariance) for each specific supersymmetric potential one can satisfy (2.51) for a set of the coefficients a and E. For example, in case of the system of the infinite well potential (2.13), (2.14) using the condition

$$w_1^2 = w_1' - 1 \tag{2.53}$$

results in

$$(2 + a - a^2)w_1' + (a^2 - 1) - E = 0 \tag{2.54}$$

which has two solutions:one

$$a = -1, \quad E = 0 \tag{2.55}$$

corresponds to reinserting the state at the same energy that was deleted (ground state) and was considered above; and another

$$a = 2, \quad E = 3 \tag{2.56}$$

gives the first excited eigenstate of the system (2.13), (2.14). By repeating these steps one can generate the entire spectrum of the supersymmetric system.

2.3 Calogero potential.

The Hamiltonian of $N = 0$ conventional QM of one particle with $\hbar = 2m = 1$ with the Calogero potential is given by

$$H = -\partial^2 + b^2x^2 + \frac{\alpha(\alpha-1)}{x^2} - b(2\alpha - 1), \qquad \alpha > 1 \tag{2.57}$$

It can be supersymmetrized by factorizing in terms of A and $\overline{A}$ as

$$H = \overline{A}A, \quad A = \partial + w, \quad w = bx - \frac{\alpha_1}{x} \tag{2.58}$$

Starting with the ground state on the first iteration

$$H_1\psi_0 = 0, \quad \psi_0 = e^{-\int w_1} = x^\alpha e^{-\frac{1}{2}bx^2}, \quad E_0 = 0 \tag{2.59}$$

one can obtain all the eigenfunctions and the spectrum of H by the method discussed above. The first excited state of H has the ground state energy of the partner Hamiltonian $\widehat{H}_1$. So the Hamiltonian of the second iteration H_2

$$(\widehat{H}_1 \ - E_1)\widehat{\psi}_1 = H_2\widehat{\psi}_1 = 0 \tag{2.60}$$

gives both the energy $E_1$and the eigenfunction ψ_1 of the first excited state through the solution of the

$$w_1^2 + w_1' - E_1 = w_2^2 - w_2' \tag{2.61}$$

Using (2.58) for w_1 and

$$w_2 = bx - \frac{\alpha_2}{x} \tag{2.62}$$

one can find two solutions: one

$$\alpha_2 = -\alpha_1 = -\alpha, \quad E_1 = 2b - 4b\alpha, \quad \psi_1 = (-\partial + w_2)x^{-\alpha}e^{-\frac{1}{2}bx^2} \tag{2.63}$$

is not normalizable but can be used to create an isospectral Hamiltonian and another

$$\alpha_2 = (\alpha + 1), \quad E_1 = 4b, \quad \psi_1 = (-\partial + w_2)x^{\alpha+1}e^{-\frac{1}{2}bx^2} \tag{2.64}$$

is the known solution of the eigenvalue problem with the equidistant spectrum in terms of the Laguerre polynomials

$$E_n = 4bn, \quad \psi_n = x^{\alpha}e^{-\frac{1}{2}bx^2}L_n^{2\alpha-1}(br^2), \quad n = 0, 1, ... \tag{2.65}$$

Chapter 3

N-extended SUSY QM

3.1 Algebra of the extended SUSY QM

The SUSY QM superalgebra (2.3), (2.4) can be generalized to any number N of supercharges Q_i and $\overline{Q}_i$ with N being the order of the extension. First, the $N = 2$ case will be considered in detail, then the case of the arbitrary N will be stated. By analogy with conventional $N = 1$ SUSY QM one can define two sets of supercharges Q_1, Q_2 and their conjugate $\overline{Q}_1$ $\overline{Q}_2$ as

$$Q_1 = \sigma_1^-(\partial + W_0 + \sigma_2 W_1), \quad Q_2 = \sigma_2^-(\partial + W_0 + \sigma_1 W_1) \tag{3.1}$$

$$\overline{Q}_1 = \sigma_1^+(-\partial + W_0 + \sigma_2 W_1), \quad \overline{Q}_2 = \sigma_2^+(-\partial + W_0 + \sigma_1 W_1) \tag{3.2}$$

with W_0 and W_1 being real functions. Here σ - matrices are related to the Pauli matrices as

$$\sigma_1^- = \sigma^- \otimes 1, \quad \sigma_2^- = 1 \otimes \sigma^-, \quad \sigma_1 = \sigma^3 \otimes 1, \quad \sigma_2 = 1 \otimes \sigma^3 \tag{3.3}$$

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.4}$$

which satisfy the usual (anti)commutation relations

$$[\sigma^+, \sigma^-] = \sigma^3, \quad \{\sigma^+, \sigma^-\} = 1 \tag{3.5}$$

The direct product of the matrices is defined here as

$$M_1 = M \otimes 1, \quad M_2 = 1 \otimes M, \quad M_1 M_2 = M_2 M_1 = M_1 \otimes M_2 \tag{3.6}$$

The superalgebra for $N = 2$ is

$$\{Q_1, \overline{Q}_1\} = \{Q_2, \overline{Q}_2\} = \mathcal{H}, \quad \{Q_i, Q_j\} = \{\overline{Q}_i, \overline{Q}_j\} = 0, \quad [Q_i, \mathcal{H}] = [\overline{Q}_i, \mathcal{H}] = 0 \tag{3.7}$$

Both anticommutators evaluated using (3.1), (3.2)

$$\{Q_1, \overline{Q}_1\} = -\partial^2 + W_0^2 + W_1^2 - \sigma_1 W' + \sigma_2 2 W_0 W_1 - \sigma_1 \sigma_2 W_1^{'} \tag{3.8}$$

$$\{Q_2, \overline{Q}_2\} = -\partial^2 + W_0^2 + W_1^2 - \sigma_2 W_0^{'} + \sigma_1 2 W_0 W_1 - \sigma_1 \sigma_2 W_1^{'} \tag{3.9}$$

must give the same expression for the superhamiltonian $\mathcal{H}$, that is

$$\mathcal{H} = -\partial^2 + W_0^2 + W_1^2 - (\sigma_1 + \sigma_2) W_0^{'} - \sigma_1 \sigma_2 W_1^{'} \tag{3.10}$$

with the requirement

$$W_0^{'} = -2 W_0 W_1 \tag{3.11}$$

3.2 Matrix formulation.

In matrix form the supercharges are given by

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 \end{pmatrix} \tag{3.12}$$

with A_1 and A_2 defined as

$$A_1 = \partial + W_0 + W_1, \quad A_2 = \partial + W_0 - W_1 \tag{3.13}$$

W_0, W_1 being same as in (3.1). The anticommutators (3.7) with their conjugates lead to

$$\{Q_1, \overline{Q}_1\} = \begin{pmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & \widehat{H}_1 & 0 \\ 0 & 0 & 0 & \widehat{H}_2 \end{pmatrix} \tag{3.14}$$

$$\{Q_2, \overline{Q}_2\} = \begin{pmatrix} H_1 & 0 & 0 & 0 \\ 0 & \widehat{H}_1 & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & \widehat{H}_2 \end{pmatrix} \tag{3.15}$$

with the hamiltonians factorized by means of 3.13

$$H_1 = \overline{A}_1 A_1, \quad \widehat{H}_1 = A_1 \overline{A}_1, \quad H_2 = \overline{A}_2 A_2, \quad \widehat{H}_2 = A_2 \overline{A}_2 \tag{3.16}$$

The corresponding potentials are

$$V_1 = (W_0 + W_1)^2 - (W_0 + W_1)', \quad V_2 = (W_0 - W_1)^2 - (W_0 - W_1)' \tag{3.17}$$

$$\widehat{V}_1 = (W_0 + W_1)^2 + (W_0 + W_1)', \quad \widehat{V}_2 = (W_0 - W_1)^2 + (W_0 - W_1)' \tag{3.18}$$

In order for the two anticommutators (3.14) and (3.15) to be the same superhamiltonian $\mathcal{H}$ one has to set

$$\widehat{H}_1 = H_2, \quad A_1 \overline{A}_1 = \overline{A}_2 A_2 \tag{3.19}$$

which by using (3.13) is reduced to

$$(W_0 + W_1)^2 + (W_0 + W_1)' = (W_0 - W_1)^2 - (W_0 - W_1)' \tag{3.20}$$

and is simplified giving

$$W_1 = -\frac{1}{2}\frac{W_0{}'}{W_0} \tag{3.21}$$

After introducing new variables w_1 and w_2 as

$$w_1 = W_0 + W_1, \quad w_2 = W_0 - W_1 \tag{3.22}$$

the equation (3.20) turns out to be the condition for $\widehat{V}_1 = V_2$ in the procedure of successive deletion and reinsertion of a state (2.37) with $\mathcal{E}_1$= $\mathcal{E}_2$. Thus it follows that the $N = 2$ extended superhamiltonian $\mathcal{H}$ contains two pairs of completely isospectral hamiltonians H_1, $\widehat{H}_2$ and $\widehat{H}_1$, H_2, the latter being different from the former in only one state deleted.

3.3 N-extended superalgebra.

The N SUSY QM is defined by the generators $Q_i, \overline{Q}_i, \mathcal{H}$ of the superalgebra in the simplest case without the central charges as follows

$$\{Q_i, \overline{Q}_j\} = \mathcal{H}\delta_{ij}, \quad \{Q_i, Q_j\} = \{\overline{Q}_i, \overline{Q}_j\} = 0, \quad [Q_i, \mathcal{H}] = [\overline{Q}_i, \mathcal{H}] = 0 \qquad (3.23)$$

$$Q_i = \sigma_i^- \left[\partial_x + W(x, \sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N)\right], \quad H = -\partial_x^2 + W^2 + \sigma_i W' \qquad (3.24)$$

where σ_i^+, σ_i^- , σ_i^3, $i = 1...N$ are direct products of the Pauli matrices as defined in (3.4) - (3.6). One has to require that the N supercharges are to be transformed by the irreducible representations of the symmetric group. Hence, a superpotential W and its derivative W' must be totally symmetric with respect to its fermionic variables. The form of the expression for N supercharges above insures that the anticommutators of Q_i and $\overline{Q}_j$ vanish for all $i \neq j$. The anticommutators of Q_i and $\overline{Q}_j$ in the (3.23) give the same superhamiltonian $\mathcal{H}$ for all i if $N - 1$ conditions

$$\begin{aligned} & W^{\,2}(x, \sigma_2, \sigma_3, ..., \sigma_N) - \sigma_1 W^{\,\prime}(x, \sigma_2, \sigma_3, ..., \sigma_N) \\ = \;& W^{\,2}(x, \sigma_1, \sigma_3, ..., \sigma_N) - \sigma_2 W^{\,\prime}(x, \sigma_1, \sigma_3, ..., \sigma_N) \end{aligned} \qquad (3.25)$$

are imposed on the coefficients of the prepotential W in (3.24). In fact, since the W is totally symmetric in all σ_i, the equation for only a pair of potentials written above is sufficient. A symmetric form for W with respect to σ_i's is written as

$$W(x, \sigma_2, \sigma_3, ..., \sigma_N) = W_0 + (\sigma_2 + \sigma_3 + ... + \sigma_N)W_1 + ... + \sigma_2\sigma_3...\sigma_N W_{N-1} \qquad (3.26)$$

With this form of W the condition (3.25) turns into a system of $N-1$ first order differential equations for N functions W_0, W_1,..., W_{N-1}. For the supersymmetric potentials parametrized in terms of the W it can be shown that the system (3.25) can be solved for $N-1$ functions W_i, each being a function of only one parameter W_0. The Hamiltonian $\mathcal{H}$ can be parametrized by a single function.

3.3.1 N=2

The superpotential W (3.26) and the compatibility condition (3.25) for the case of $N=2$ are

$$W(x, \sigma_2) = W_0 + \sigma_2 W_1 \tag{3.27}$$

$$W^2(x, \sigma_2) - \sigma_1 W'(x, \sigma_2) = W^2(x, \sigma_1) - \sigma_2 W'(x, \sigma_1) \tag{3.28}$$

This reduces to

$$W_0{}' = -2 W_0 W_1 \tag{3.29}$$

The Hamiltonian $\mathcal{H}$ and the superpotential are parametrized by a single function W_0 as

$$\mathcal{H} = -\partial^2 + W_0^2 + W_1^2 - W_0{}'(\sigma_1 + \sigma_2) - W_1{}'\sigma_1\sigma_2 \tag{3.30}$$

$$W_1 = -\frac{1}{2}\frac{W_0{}'}{W_0} \tag{3.31}$$

3.3.2 N=3

The symmetric form of the W in (3.26) can be written recursively. In the case of $N=3$ it now has the form

$$W(x, \sigma_2, \sigma_3) = W_0 + \sigma_2 W_1 + \sigma_3 \left(W_1 + \sigma_2 W_2\right). \tag{3.32}$$

Comparing this with the $N=2$ case (3.27), one can write the solution (3.29) of the condition (3.28) for the $N=3$ case similarly as

$$[W_0(x) + \sigma_2 W_1(x)]' = -2\,[W_0(x) + \sigma_2 W_1(x)]\,[W_1(x) + \sigma_2 W_2(x)] \tag{3.33}$$

This is equivalent to

$$-\frac{1}{2}{W_0}' = W_0 W_1 + W_1 W_2 \tag{3.34}$$

$$-\frac{1}{2}{W_1}' = W_0 W_2 + W_1^2 \tag{3.35}$$

After substitution of the new variables

$$w_1 = W_0 + W_1, \quad w_2 = W_0 - W_1 \tag{3.36}$$

into (3.33) it is reduced to the equation

$$\frac{1}{2}\frac{w_1'}{w_1} = W_1 + W_2, \quad \frac{1}{2}\frac{w_2'}{w_2} = W_1 - W_2 \tag{3.37}$$

The last two equations can be combined giving

$$w_1 - w_2 = \frac{1}{2}\frac{w_1'}{w_1} + \frac{1}{2}\frac{w_2'}{w_2} \tag{3.38}$$

Setting

$$w_1 = w_2 + \frac{1}{2}\frac{u'}{u} \tag{3.39}$$

with u being an arbitrary function, one obtains the quadratic equation

$$w_1^2 - \frac{1}{2}\frac{u'}{u}w_1 - u = 0 \tag{3.40}$$

which yields two roots

$$w_1 = \frac{1}{4}\frac{u'}{u} \pm \frac{1}{4}\sqrt{\left[\left(\frac{u'}{u}\right)^2 + 16u\right]} \quad w_2 = -\frac{1}{4}\frac{u'}{u} \pm \frac{1}{4}\sqrt{\left[\left(\frac{u'}{u}\right)^2 + 16u\right]} \tag{3.41}$$

Thus, in the case of $N = 3$ the superpotential is parametrized again by means of a single function u through the coefficients

$$W_0 = \frac{1}{4}\sqrt{\left[\left(\frac{u'}{u}\right)^2 + 16u\right]} \quad W_1 = \frac{1}{4}\frac{u'}{u} \quad W_2 = \frac{1}{4}\frac{2u''u - 3(u')^2}{u^2\sqrt{\left[(\frac{u'}{u})^2 + 16u\right]}} \tag{3.42}$$

This is the superpotential that was first obtained by Pashnev [19].

Next, a different parametrization of the superpotential in terms of the function W will be considered with the goal of extending it to the case of an arbitrary N. The equation (3.38) can be written as

$$w_2' = -2w_2^2 + w_2(2w_1 - \frac{w_1'}{w_1}) \tag{3.43}$$

This is a Bernoulli's equation with the general solution

$$w_2 = \frac{1}{2}\ln'(\int \frac{e^{2\int w_1}}{w_1}) \tag{3.44}$$

solved in terms of w_1 and a constant set to zero. Now, one can express the coefficients of W in terms of the w_1 by solving a system of the linear equations (3.36), (3.37). This yields

$$W_0 = \frac{1}{2}(w_1 + w_2), \quad W_1 = \frac{1}{2}(w_1 - w_2), \quad W_2 = \frac{1}{4}\left(\frac{w_1'}{w_1} - \frac{w_2'}{w_2}\right) \tag{3.45}$$

The Hamiltonian $\mathcal{H}$ and the superpotential are parametrized by a single function as

$$\mathcal{H} = -\partial^2 + W_0^2 + 2W_1^2 + W_2^2 - W_0'(\sigma_1 + \sigma_2 + \sigma_3) - W_1'(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - W_2'\sigma_1\sigma_2\sigma_3 \tag{3.46}$$

3.3.3 N=4

A totally symmetric form of W can be now written as

$$\begin{aligned} W_{N=4}(x, \sigma_2, \sigma_3, \sigma_4) &= W_0 + \sigma_4 W_1 + \sigma_3 (W_1 + \sigma_4 W_2) \\ &\quad + \sigma_2 [W_1 + \sigma_4 W_2 + \sigma_3 (W_2 + \sigma_4 W_3)] \end{aligned} \tag{3.47}$$

Following the same steps as between (3.32) and (3.38) we arrive at the two equations, each being analogous to (3.38)

$$w_1 - w_2 = \frac{1}{2}\frac{w_1'}{w_1} + \frac{1}{2}\frac{w_2'}{w_2}, \qquad w_2 - w_3 = \frac{1}{2}\frac{w_2'}{w_2} + \frac{1}{2}\frac{w_3'}{w_3} \tag{3.48}$$

where

$$\begin{aligned} w_1 &= (W_0 + W_1) + (W_1 + W_2), \quad w_2 = (W_0 - W_1) - (W_1 - W_2) \\ w_3 &= (W_0 + W_1) - (W_1 + W_2), \quad \frac{1}{2}\frac{w_1'}{w_1} = (W_1 + W_2) + (W_2 + W_3) \\ \frac{1}{2}\frac{w_2'}{w_2} &= (W_1 - W_2) - (W_2 - W_3), \quad \frac{1}{2}\frac{w_3'}{w_3} = (W_1 + W_2) - (W_2 + W_3) \end{aligned} \tag{3.49}$$

Notice that the right hand side of the equations for w_i can be turned into those of the equations for $\frac{w_{i'}}{w_i}$ by replacing $W_0 \to W_1$, $W_1 \to W_2$, $W_2 \to W_3$. Therefore, in order to invert this system for W_i it is enough to solve only the first three equations, which give W_0, W_1, and W_2, obtaining the last W_3 by replacing in the solution for W_2 all w_i on $\frac{w_{i'}}{w_i}$

$$\begin{aligned} W_0 &= \frac{1}{4}(w_1 + w_2) + \frac{1}{2}w_3, \qquad W_1 = \frac{1}{4}(w_1 - w_2), \\ W_2 &= \frac{1}{4}(w_1 + w_2) - \frac{1}{2}w_3, \qquad W_3 = \frac{1}{4}\left(\frac{w_1'}{w_1} + \frac{w_2'}{w_2}\right) - \frac{1}{2}\frac{w_3'}{w_3} \end{aligned} \tag{3.50}$$

This holds for the case of an arbitrary N. The two equations (3.48) are the same as the equation (3.44) of the previous $N = 3$ case. Hence, they have the same solutions

$$w_2 = \frac{1}{2}\ln'\left[\int \frac{\exp\left(2\int w_1\right)}{w_1}\right], \qquad w_3 = \frac{1}{2}\ln'\left[\int \frac{\exp\left(2\int w_2\right)}{w_2}\right] \tag{3.51}$$

Taking again $w_1$as an arbitrary parameter we can express $W_{N=4}$ trough only this function. The Hamiltonian $\mathcal{H}$ is

$$\begin{aligned} \mathcal{H} &= -\partial^2 + W_0^2 + 3W_1^2 + 3W_2^2 + W_3^2 - W_0{}' \sum_i \sigma_i - W_1{}' \sum_{i\,<\,j} \sigma_i\sigma_j \\ &\quad -W_2{}' \sum_{i\,<\,j\,<\,k} \sigma_i\sigma_j\sigma_k - W_3{}'\sigma_1\sigma_2\sigma_3\sigma_4 \end{aligned} \tag{3.52}$$

3.4 Arbitrary N.

The symmetric form of the $W(x, \sigma_2, \sigma_3, ..., \sigma_N)$ is parametrized recursively by N functions W_0,W_1, ..., W_{N-1}

$$\begin{aligned} W &= W_0 + \sigma_2 W_1 + \sigma_3 (W_1 + \sigma_2 W_2) + ... + \sigma_N(W_1 + \sigma_2 W_2 + \sigma_3 (W_2 + \sigma_2 W_3) \\ &\quad + ... + \sigma_{N-1}(... + \sigma_3 (W_{N-2} + \sigma_2 W_{N-1}))) \end{aligned} \tag{3.53}$$

The new variables w_1, w_2, ... ,w_{N-1} are introduced similarly to the $N = 3$ case (3.36). They are all the distinct combinations of the functions W_0,W_1, ..., W_{N-2} in the first half of the superpotential W as σ_i run over the values ± 1 in turn. We have $N-1$ other relations between W_1,W_2, ..., W_{N-1} from the compatibility condition in a way similar to (3.33). In the new variables w_i the compatibility condition is written as the system of $N-2$ equations

$$\begin{aligned} w_1 - w_2 &= \frac{1}{2}\frac{w_1'}{w_1} + \frac{1}{2}\frac{w_2'}{w_2} \qquad w_2 - w_3 = \frac{1}{2}\frac{w_2'}{w_2} + \frac{1}{2}\frac{w_3'}{w_3} \\ \ldots\, w_{N-2} - w_{N-1} &= \frac{1}{2}\frac{w_{N-2}'}{w_{N-2}} + \frac{1}{2}\frac{w_{N-1}'}{w_{N-1}} \end{aligned} \tag{3.54}$$

The solution of a given equation is expressed in terms of the solution of the previous one according to the iteration scheme

$$w_2 = \frac{1}{2}\ln'\left(\int \frac{e^{2\int w_1}}{w_1}\right)\; w_3 = \frac{1}{2}\ln'\left(\int \frac{e^{2\int w_2}}{w_2}\right) \ldots\; w_{N-1} = \frac{1}{2}\ln'\left(\int \frac{e^{2\int w_{N-2}}}{w_{N-2}}\right) \tag{3.55}$$

One can always invert a system of linear equations relating the new variables in order to express W_i in terms of w_i, so that the Hamiltonian can be parametrized by a single function, say w_1.

3.5 Matrix formalism and isospectral Hamiltonians.

The matrix formalism of the extended SUSY QM was discussed above for the case of $N = 2$. Here it will be presented for the case of arbitrary N. The superhamiltonian $\mathcal{H}$ is given as

$$\mathcal{H} = \begin{pmatrix} H_1 & 0 & \dots & & & & & & 0 \\ 0 & \widehat{H}_1 & & & & & & & \dots \\ \dots & & H_2 & & & & & & \\ & & & \widehat{H}_2 & & & & & \\ & & & & \dots & & & & \\ & & & & & \widehat{H}_{N-1} & & & \\ & & & & & & & H_N & 0 \\ 0 & \dots & & & & & & 0 & \widehat{H}_N \end{pmatrix} \tag{3.56}$$

with the partner hamiltonians H_i and $\widehat{H}_i$ defined as

$$H_i = -\partial^2 + w_i^2 - w', \quad \widehat{H}_i = -\partial^2 + w_i^2 + w' \tag{3.57}$$

and satisfying the $N-1$ conditions on the potentials V_i and $\widehat{V}_i$

$$\widehat{V}_1 = V_2, \quad \widehat{V}_2 = V_3, \quad \dots \quad \widehat{V}_{N-1} = V_N \tag{3.58}$$

This system of first order differential equations can be solved by iterations. After substitution of

$$w_{i+1} = -w_i + \frac{1}{z_i}, \quad i = 1, 2..., N-1 \tag{3.59}$$

into

$$w_i^2 + w' = w_{i+1}^2 - w_{i+1}^{'} \tag{3.60}$$

the system (3.58) is reduced to linear differential equations with general solution obtained by iterations. On the first step the solution is

$$w_2 = -w_1 + \frac{1}{z_1}, \quad \frac{1}{z_1} = -\ln\left[1 + C_1 \int e^{-2\int w_1}\right] \tag{3.61}$$

It is useful to express a prepotential w_1 in terms of the ground state function u_1 of the H_1 (corresponding to $E_0 = 0$) as before

$$w_1 = -\frac{u_1'}{u_1} \tag{3.62}$$

Now (3.61) becomes

$$\frac{1}{z_1} = -\ln'(I_1), \quad I_1 = 1 + C_1 \int u_1^2 \tag{3.63}$$

By continuing iteration one can write the solutions in the form

$$\begin{aligned}
\frac{1}{z_2} &= -\ln'(I_2), \quad I_2 = 1 + C_2 \int \frac{I_1^2}{u_1^2} \\
\frac{1}{z_3} &= -\ln'(I_3), \quad I_3 = 1 + C_3 \int \frac{u_1^2 I_2^2}{I_1^2} \\
&\dots \\
\frac{1}{z_{N-1}} &= -\ln'(I_{N-1}), \quad I_{N-1} = 1 + C_{N-1} \int \frac{u_1^2 I_2^2 \dots I_{N-2}^2}{I_1^2 \dots I_{N-3}^2}
\end{aligned} \tag{3.64}$$

Now using the solutions, one can write the potentials of the superhamiltonian $\mathcal{H}$ explicitly using the definition (3.57) and conditions (3.58)

$$\begin{aligned}
\widehat{V}_{2n} &= V_{2n+1} = V_1 + 2\sum_{i=1}^{n}\left(\frac{1}{z_{2i-1}}\right)', \quad n = 1, 2, ..., \frac{N}{2} \\
\widehat{V}_{2n+1} &= V_{2n+2} = \widehat{V}_1 + 2\sum_{i=1}^{n}\left(\frac{1}{z_{2i}}\right)', \quad n = 1, 2, ..., \frac{N}{2} - 1
\end{aligned} \tag{3.65}$$

These are two families of totally isospectral potentials parametrized in terms of the constants C_i. Notice that the forms I_i are positively definite if the constants C's are positive. This ensures that there are no singularities in the constructed potentials except where they are present in the original potential V_1. Thus, an extended supersymmetric Hamiltonian $\mathcal{H}$of degree N consists of two sets of N isospectral hamiltonians in each. Hamiltonians of one are totally isospectral to H, and of the other to $\widehat{H}$. It follows from the discussion above that the entire system can be constructed just from one function w- the prepotential of H.

Chapter 4

Calogero Potential and Extended SUSY QM

4.1 Conformal Quantum Mechanics.

The conformal quantum mechanics in $d = 1$ is defined [20] by the generators of the conformal group H, K, and D. The Hamiltonian of the theory

$$H = \frac{1}{2}\left(p^2 + \frac{g}{x^2}\right) \tag{4.1}$$

was found to have a non-normalizable ground state. When the isomorphism between the conformal group and the group $SO(2,1)$ was used to construct new generators as the linear combinations of the old, it was suggested to use the generator of the compact rotation of $SO(2,1)$ as a new Hamiltonian (with $\hbar = 1$)

$$H_{new} = \frac{1}{2}\left(-\partial^2 + \frac{g}{x^2} + \frac{x^2}{a}\right) \tag{4.2}$$

which has a normalizable ground state and the spectrum similar to that of a compact operator

$$\begin{aligned} r &= r_0 + n, n = 0, 1, ... \\ r_0 &= \frac{1}{2}(1 + \sqrt{g + \frac{1}{4}}) \end{aligned} \tag{4.3}$$

It has been noted, that this is equivalent to elimination of a coordinate singularity [21] or to a non-linear change of the space-time variables [10]. In the later developments [10], [11] the supersymmetric conformal QM for $N = 1$ and 2 was constructed.

The $N = 1$ Hamiltonian of conformal QM

$$H_{N=1} = -\partial^2 + \frac{\alpha^2}{x^2} - \sigma_1 \frac{\alpha}{x^2} \tag{4.4}$$

can be regularized by adding the term of the harmonic oscillator with the results discussed in the previous section. Here, it should be noticed, that the x-dependence in the conformal potential is factorized, leaving the numerator as a polynomial of the fermionic variable σ. It has been conjectured [19], that the factorization of the bosonic variables in the case of the conformal potential occurs for an arbitrary N. It will be shown next that this is indeed the case. One can writte a whole class of the N supersymmetric potentials with such a property. The ansatz for the prepotential W with fermionic variables factorized and symmetric is

$$\begin{aligned} W(x, \sigma_2, \sigma_3, ..., \sigma_N) &= W_0 P(\sigma_2, \sigma_3, ..., \sigma_N) && (4.5) \\ P(\sigma_2, \sigma_3, ..., \sigma_N) &= 1 + a_1(\sigma_2 + ... + \sigma_N) + ... + a_{N-1}\sigma_2\sigma_3...\sigma_N && (4.6) \end{aligned}$$

where all the coefficients W_i are proportional to $W_0 = \frac{\alpha}{x}$. Substituting it into the condition (3.25) gives

$$\begin{aligned} & W_0^2 P(\sigma_2, \sigma_3, ..., \sigma_N)^2 - \sigma_1 W_0' P(\sigma_2, \sigma_3, ..., \sigma_N) && (4.7) \\ = \;& W_0^2 P(\sigma_1, \sigma_3, ..., \sigma_N)^2 - \sigma_2 W_0' P(\sigma_1, \sigma_3, ..., \sigma_N) \end{aligned}$$

It reduces to a system of $N - 1$ second order algebraic equations for N variables - the coefficients a_i, with α being a parameter. As an example we now consider the $N = 4$ case. The coefficients a_i are the solution of the system

$$
\begin{aligned}
2a_1 + 2a_2a_3 + 4a_1a_2 &= -\frac{1}{\alpha} \\
2a_2^2 + 2a_1^2 + 2a_2 + 2a_1a_3 &= -\frac{a_1}{\alpha} \\
6a_1a_2 + 2a_3 &= -\frac{a_2}{\alpha}
\end{aligned} \tag{4.8}
$$

which can be turned into an equation of the 6th order and gives the roots explicitly. Similarly, $N = 3$ case gives three and $N = 5$ fifteen roots. The solutions could be relevant to the classification of the irreducible representations of S_N. It may turn out, that there exists the correspondence between the n-particle conformal SUSY QM and $SU(n)$ YM in $D = 2$, which was discussed in [22]. Symmetry and classification of the solutions for arbitrary N requires an additional analysis. However, there is one solution that can be found for any N. When x_0 is zero, it corresponds to the known $N = 1$ and 2 conformal SUSY QM of [11], [10] and its generalization for arbitrary N. In order to see that we rewrite the potential as a complete square

$$
W^2 + \sigma_1 W' = \frac{\left[\alpha P(\sigma_2, \sigma_3, ..., \sigma_N) - \frac{\sigma_1}{2}\right]^2 - \left(\frac{\sigma_1}{2}\right)^2}{x^2} \tag{4.9}
$$

The compatibility condition in this case can be written as

$$
\alpha P(\sigma_2, \sigma_3, ..., \sigma_N) - \frac{\sigma_1}{2} = \pm\left[\alpha P(\sigma_1, \sigma_3, ..., \sigma_N) - \frac{\sigma_2}{2}\right]
$$

For any N it has a solution, which corresponds to the $+$ sign in the equation above

$$
\begin{aligned}
a_i &= 0 \qquad i \neq 1 \\
a_1 &= -\frac{1}{2\alpha}
\end{aligned} \tag{4.10}
$$

Thus, the complete Hamiltonian of conformal SUSY QM for any N has identical structure with that of $N = 1$ (4.4), except for the constant

$$
H_N = \frac{1}{2}\left[-\partial^2 + \frac{1}{4}\frac{(2\alpha - \sigma_N - ... - \sigma_2)(2\alpha - \sigma_N - ... - 2\sigma_1)}{x^2}\right] \tag{4.11}
$$

The spectrum is obtained by redefining the constant g in the spectrum of the $N = 0$ theory (4.3) with the result

$$r_0 = \frac{1}{2}\left[1 + \alpha P(s_2, s_3, ..., s_N) - \frac{s_1}{2}\right] \qquad r = r_0 + n \qquad n = 0, 1, ... \tag{4.12}$$

Chapter 5

Toward multiparticle SUSY QM

5.1 Quasiclassical superfield formalism.

In order to write the supersymmetric system for the given Hamiltonian H (supersymmetrize the Hamiltonian H), one can use a superfield approach. In this formulation the quantum mechanical Hamiltonian is obtained by means of canonical quantization. One can start with the classical action written in terms of the fields, which are explicitly invariant under supersymmetric transformations. A theory derived from such an action will be also supersymmetric, that is satisfying the supersymmetric algebra for its generators. This standard approach in the field theory can be used to derive the results of the previous sections. However, in the case of multiparticle systems the process of writing the supersymmetric Hamiltonians is not a straightforward generalization of the one particle case. Here, an attempt will be made to derive a multiparticle extended supersymmetric Hamiltonian from a field theoretical formulation.

5.2 General formalism.

First of all, a superfield parameters of the action must be defined. It was shown that in one particle case the quantization of the classical action written in terms of a scalar superfield defined as

$$\mathbf{\Phi} = \mathbf{x}(t) + \frac{1}{\sqrt{2}}[\overline{\theta}\psi(t) + \overline{\overline{\psi}}(t)\theta] - \overline{\theta}\theta\mathbf{F}(t) \tag{5.1}$$

leads to the correct $N = 1$ supersymmetric system. In order to extend the number of supersymmetries, a second chiral superfield

$$\mathbf{\Lambda} = \lambda(t) + \mathbf{y}(t)\theta + \frac{i}{2}\overline{\theta}\theta\dot{\lambda}(t) \tag{5.2}$$

is introduced. To generalize this standard procedure to the multiparticle case, the bold-faced variables in (5.1) and (5.2) now represent vectors with n components depending only on time. So $\mathbf{x}(t)$ and $\mathbf{F}(t)$ are real n-componet fields, θ and $\overline{\theta}$ are the Grassmann numbers (imaginary time), $\mathbf{y}$ and λ are complex, and $\dot{\lambda}$ is a time derivative. The fermionic degrees ψ (for λ it will be given later) satisfy the usual anticommutation rules

$$\{\psi_l, \overline{\psi}_{l'}\} = \delta_{ll'}, \qquad \{\psi_l, \psi_{l'}\} = \{\overline{\psi}_l, \overline{\psi}_{l'}\} = 0 \tag{5.3}$$

We also define a covariant derivative

$$D = \frac{\partial}{\partial\theta} + \frac{i}{2}\overline{\theta}\frac{\partial}{\partial t}, \qquad \overline{D} = -\frac{\partial}{\partial\overline{\theta}} - \frac{i}{2}\theta\frac{\partial}{\partial t} \tag{5.4}$$

so that the chirality of the superfield $\mathbf{\Lambda}$ is manifest

$$\overline{D}\mathbf{\Lambda} = \left(-\frac{\partial}{\partial\overline{\theta}} - \frac{i}{2}\theta\frac{\partial}{\partial t}\right)\mathbf{\Lambda} = 0 \tag{5.5}$$

The action written in terms of the superfield defined above is invariant under the transformation of the supersymmetry. A proper form of the kinetic term and most general form for the interaction are satisfied action in the form

$$S = \int dt d\theta d\overline{\theta} \left[-2\overline{D}\mathbf{\Phi}D\mathbf{\Phi} + \mathcal{W}(\mathbf{\Phi}, \mathbf{\Lambda}, \overline{\mathbf{\Lambda}})\right] \tag{5.6}$$

The potential $\mathcal{W}$ so far is an arbitrary function of the superfields. It reduces to the action of $N = 2$ and $n = 1$ (one particle) SUSY QM [19] for the potential in the form

$$\mathcal{W} = 2W(\mathbf{\Phi}) + \overline{\mathbf{\Lambda}}\mathbf{V}(\mathbf{\Phi})\mathbf{\Lambda} \tag{5.7}$$

The function W and matrix $\mathbf{V}$ of the scalar superfield $\mathbf{\Phi}$ in the superpotential are understood in terms of the Taylor expansion with respect to the θ and $\overline{\theta}$ as

$$\begin{aligned} W(\mathbf{x}+\mathbf{\Theta}) &= W(\mathbf{x}) + \partial_l W(\mathbf{x})\Theta_l + \frac{1}{2}\partial_l\partial_{l'}W(\mathbf{x})\Theta_l\Theta_{l'} + ... \\ \Theta_l &= \frac{1}{\sqrt{2}}(\overline{\theta}\psi_l + \overline{\psi}_l\theta) - \overline{\theta}\theta F_l \end{aligned} \tag{5.8}$$

In order to write the action of a theory extended to an arbitrary number of supersymmetries N one has to introduce $N-1$ chiral fields $\mathbf{\Lambda}^I$ and further expand the potential $\mathcal{W}$ in the powers of $\mathbf{\Lambda}^I$. The supersymmetry then acts in the superspace of N scalar and chiral superfields.

After integration in (5.6) with respect to the Grassmann variables the action becomes

$$\begin{aligned} S &= \int dtL = \int dt\{\frac{1}{2}\dot{\mathbf{x}}^2 + \frac{i}{2}\dot{\psi}\overline{\psi} - \frac{i}{2}\psi\dot{\overline{\psi}} + \frac{i}{2}\overline{\lambda}\mathbf{V}\dot{\lambda} - \frac{i}{2}\dot{\overline{\lambda}}\mathbf{V}\lambda + 2\mathbf{F}^2 \\ &-\frac{1}{4}\overline{\lambda}\partial_l\partial_{l'}\mathbf{V}[\overline{\psi}_l\psi_{l'} - \psi_l\overline{\psi}_{l'}]\lambda - \frac{1}{2}\partial_l\partial_{l'}W[\overline{\psi}_l\psi_{l'} - \psi_l\overline{\psi}_{l'}] - 2F_l\partial_l W \\ &-\frac{1}{\sqrt{2}}(\overline{\mathbf{y}}\partial_l\mathbf{V}\overline{\psi}_l\lambda + \overline{\lambda}\psi_l\partial_l\mathbf{V}\mathbf{y}) - \overline{\lambda}F_l\partial_l\mathbf{V}\lambda + \overline{\mathbf{y}}\mathbf{V}\mathbf{y}\} \end{aligned} \tag{5.9}$$

Notice, that $\overline{\psi}_l$ and ψ_l are summed over their spatial indices, while in the bold face expressions (for example $\overline{\mathbf{y}}\mathbf{V}\mathbf{y}$) there are matrix summations. To write the kinetic terms in a canonical form new variables are introduced as

$$\lambda = \mathbf{v}^{-1}\chi, \qquad \mathbf{V} = \mathbf{v}^2 \tag{5.10}$$

Next, one can eliminate the auxiliary variables F, y, and $\overline{y}$: minimizing L with respect to a variable, solving the resulting equation and plugging back into the Lagrangian. The result is

$$\begin{aligned} L &= \frac{1}{2}\dot{\mathbf{x}}^2 + \frac{i}{2}\dot{\psi}\overline{\psi} - \frac{i}{2}\psi\dot{\overline{\psi}} + \frac{i}{2}\overline{\chi}\dot{\chi} - \frac{i}{2}\dot{\overline{\chi}}\chi - \frac{1}{2}(\partial_l W)^2 \\ &-\frac{1}{2}\overline{\chi}\partial_l W\mathbf{v}^{-1}\partial_l\mathbf{V}\mathbf{v}^{-\mathbf{1}}\chi - \frac{1}{2}\partial_l\partial_{l'}W[\overline{\psi}_l\psi_{l'} - \psi_l\overline{\psi}_{l'}] \\ &-\frac{1}{4}\overline{\chi}\mathbf{v}^{-1}\partial_l\partial_{l'}\mathbf{V}\mathbf{v}^{-1}\chi[\psi_l\overline{\psi}_{l'} - \overline{\psi}_l\psi_{l'}] + \frac{1}{8}(\overline{\chi}\mathbf{v}^{-1}\partial_l\mathbf{V}\mathbf{v}^{-1}\chi)^2 \\ &-\frac{1}{2}\overline{\chi}\psi_l\mathbf{v}^{-1}\partial_l\mathbf{V}\mathbf{V}^{-1}\partial_{l'}\mathbf{V}\mathbf{v}^{-1}\overline{\psi}_{l'}\chi - \frac{1}{4}\overline{\chi}\mathbf{v}^{-1}\partial_l\mathbf{V}(\overline{\chi}\mathbf{v}^{-1}\partial_{l'}\mathbf{V}\mathbf{v}^{-1}\chi)\mathbf{v}^{-1}\chi \end{aligned} \tag{5.11}$$

Finally, the transformation to the classical Hamiltonian is performed as usual

$$H = \dot{\mathbf{x}}\frac{\delta L}{\delta \dot{\mathbf{x}}} + \dot{\psi}\frac{\delta L}{\delta \dot{\psi}} + \dot{\overline{\psi}}\frac{\delta L}{\delta \dot{\overline{\psi}}} + \dot{\chi}\frac{\delta L}{\delta \dot{\chi}} + \dot{\overline{\chi}}\frac{\delta L}{\delta \dot{\overline{\chi}}} - L \tag{5.12}$$

The Hamiltonian is the result of the involution of the two generators Q_1 and $\overline{Q}_1$

$$\begin{aligned} \{Q_1, \overline{Q}_1\} &= \mathcal{H}, \\ Q_1 &= \psi_l(\partial_l + \partial_l W + \overline{\lambda}\partial_l \mathbf{V}\lambda), \qquad \overline{Q}_1 = \overline{\psi}_l(-\partial_l + \partial_l W + \overline{\lambda}\partial_l \mathbf{V}\lambda) \end{aligned} \tag{5.13}$$

where $\{,\}$ are graded Poisson brackets - Marten brackets. In order for the Hamiltonian of the (5.12) to transform as $N = 2$ supersymmetries one has to introduce another pair of the supercharges Q_2 and $\overline{Q}_2$

$$\begin{aligned} Q_2 &= \lambda_l(\partial_l + \partial_l W + \overline{\psi}\partial_l \mathbf{V}\psi) \\ \overline{Q}_2 &= \overline{\lambda}_l(-\partial_l + \partial_l W + \overline{\psi}\partial_l \mathbf{V}\psi) \end{aligned} \tag{5.14}$$

As before, in order to achieve $N = 2$ supersymmetry one has to require the Hamiltonian H to be totally symmetric with respect to the interchange

$$\psi \leftrightarrow \lambda \tag{5.15}$$

so that the second pair of the supercharges would give the same $\mathcal{H}$

$$\{Q_2, \overline{Q}_2\} = \mathcal{H} \tag{5.16}$$

The potential of the H has to be symmetrized in its combinations of the Grassmann variables: in the second $\overline{\chi}...\chi$, $\overline{\psi}_l...\psi_{l'}$ and fourth $\overline{\chi}\chi$ $...\overline{\chi}\chi$, $\overline{\chi}\psi_{l'}$ $...\overline{\psi}_l\chi$ degree. Using the canonical anticommutation relations (5.3) and

$$\{\chi^i, \overline{\chi}^j\} = \delta^{ij}, \qquad \{\chi^i, \chi^j\} = \{\overline{\chi}^i, \overline{\chi}^j\} = 0 \tag{5.17}$$

one can rearrange all terms in the H to one of the four forms above as in the following

$$H = \frac{1}{2}\dot{\mathbf{x}}^2 + \mathcal{V}_0 + \mathcal{V}_2 + \mathcal{V}_4 \tag{5.18}$$

$$\mathcal{V}_0 = \frac{1}{2}(\partial_l W)^2 \ - \frac{1}{2}\partial_l^2 W \tag{5.19}$$

$$\begin{aligned}\mathcal{V}_2 \ &= \ \overline{\psi}_l \partial_l \partial_{l'} W \psi_{l'} + \frac{1}{2}\overline{\chi}\partial_l W \mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{v}^{-\mathbf{1}}\chi \\ &\quad -\frac{1}{4}\overline{\chi}\mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{V}^{-1}\partial_{l'}\mathbf{V}\mathbf{v}^{-1}\chi + \frac{1}{4}\overline{\chi}\mathbf{v}^{-1}\partial_l^2 \mathbf{V}\mathbf{v}^{-1}\chi\end{aligned} \tag{5.20}$$

$$\begin{aligned}\mathcal{V}_4 \ &= \ \frac{1}{8}(\overline{\chi}\mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{v}^{-1}\chi)(\overline{\chi}\mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{v}^{-1}\chi) \\ &\quad +\frac{1}{2}\overline{\chi}\psi_l \mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{V}^{-1}\partial_{l'}\mathbf{V}\mathbf{v}^{-1}\overline{\psi}_{l'}\chi \\ &\quad -\frac{1}{2}\overline{\chi}\psi_l \mathbf{v}^{-1}\partial_l \partial_{l'}\mathbf{V}\mathbf{v}^{-1}\overline{\psi}_{l'}\chi\end{aligned} \tag{5.21}$$

Upon observation of (5.11) one has to impose a set of conditions on the function W and the matrix $\mathbf{V}$

$$\partial_i \partial_j W = \frac{1}{2}\partial_l W(\mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{v}^{-\mathbf{1}})^{ij} - \frac{1}{4}(\mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{V}^{-1}\partial_l \mathbf{V}\mathbf{v}^{-1})^{ij} + \frac{1}{4}(\mathbf{v}^{-1}\partial_l^2 \mathbf{V}\mathbf{v}^{-1})^{ij} \tag{5.22}$$

$$\frac{1}{4}(\mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{v}^{-\mathbf{1}})^{ij}(\mathbf{v}^{-1}\partial_l \mathbf{V}\mathbf{v}^{-\mathbf{1}})^{i'j'} = (\mathbf{v}^{-1}\partial_j \mathbf{V}\mathbf{V}^{-1}\partial_{i'}\mathbf{V}\mathbf{v}^{-1})^{ij} - (\mathbf{v}^{-1}\partial_j \partial_{i'}\mathbf{V}\mathbf{v}^{-1})^{ij} \tag{5.23}$$

The remaining relationships of the $N = 2$ superalgebra are satisfied automatically for the chosen form of the supercharges. The canonical quantization of the classical Hamiltonian is performed in a standard way: the Poisson brackets are replaced by canonical (anti)commutation relations for the canonical variables.

5.3 Two particle N=2 Calogero potential.

Here, the solution of the conditions (5.22) and (5.23) will be given. The form of W that gives the multiparticle and two particle Calogero potential is

$$\partial_i W = \sum_{j \neq i}^{n} \left[\frac{\alpha}{x_i - x_j}\right] \tag{5.24}$$

$$\partial_1 W = \alpha \phi_{12} = -\partial_2 W, \qquad \phi_{ij} = \frac{1}{x_i - x_j} \tag{5.25}$$

$$\partial_i\partial_j W = -2\phi_{12}^2 A_2^{ij}, \qquad \mathbf{A}_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad i,,j = 1,2 \tag{5.26}$$

Notice that in (5.22) and (5.23) the spatial derivatives match matrix indices. These equations are first order differential equations for the unknown matrix. The inverse of the unknown matrix also enters the equations that makes them very difficult to solve in the multiparticle case.

To solve the equations is to find a two by two matrix $\mathbf{V}$. One can expand the matrix in its projectors A_i as

$$\mathbf{V} = \mathbf{v}^2, \qquad \mathbf{v} = \sum \lambda_i \mathbf{A}_i, \qquad \mathbf{A}_i\mathbf{A}_j = \mathbf{A}_i\delta_{ij}, \qquad \sum_i^{n=2} \mathbf{A}_i = \mathbf{1} \tag{5.27}$$

with λ_i being its eigenvalues. Since the left hand side of the equation (5.22) has matrix $\mathbf{A}_2$- a projector satisfying (5.27) one can use as an ansatz for the solution the following expansion

$$\mathbf{v} = f_1^{\frac{1}{2}}\mathbf{A}_1 + f_2^{\frac{1}{2}}\mathbf{A}_2 \tag{5.28}$$

where $\mathbf{A}_1$ is another projector in (5.27)

$$\mathbf{A}_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{5.29}$$

For the ansatz chosen, the squared and inverted matrices are easy to obtain

$$\mathbf{v}^{-1} = f_1^{-\frac{1}{2}}\mathbf{A}_1 + f_2^{-\frac{1}{2}}\mathbf{A}_2, \qquad \mathbf{V} = \mathbf{v}^2 = f_1\mathbf{A}_1 + f_2\mathbf{A}_2 \tag{5.30}$$

Upon substitution and using the symmetry identities

$$\partial_1 = -\partial_2 \quad \phi_{12} = -\phi_{21} \tag{5.31}$$

the equations (5.22) and (5.23) are found to be

$$-4\phi_{12}^2\mathbf{A}_2 = 2\alpha\left(\frac{\partial f_1}{f_1}\mathbf{A}_1 + \frac{\partial f_2}{f_2}\mathbf{A}_2\right) + \left(\frac{\partial f_1}{f_1}\right)'\mathbf{A}_1 + \left(\frac{\partial f_2}{f_2}\right)'\mathbf{A}_2 \tag{5.32}$$

$$\frac{1}{4}\left(\frac{\partial f_1}{f_1}\mathbf{A}_1+\frac{\partial f_2}{f_2}\mathbf{A}_2\right)^{il}\left(\frac{\partial f_1}{f_1}\mathbf{A}_1+\frac{\partial f_2}{f_2}\mathbf{A}_2\right)^{l'j}=\mp\left[\left(\frac{\partial f_1}{f_1}\right)'\mathbf{A}_1+\left(\frac{\partial f_2}{f_2}\right)'\mathbf{A}_2\right] \tag{5.33}$$

where the right-hand side brackets are taken with minus for $l=l'$ and with plus for $l\neq l'$. The property of the $\mathbf{A}$ matrices

$$\mathbf{A}_1^{il}\mathbf{A}_1^{l'j}=\frac{1}{2}A_1^{ij},\qquad \mathbf{A}_2^{il}\mathbf{A}_2^{l'j}=\pm\frac{1}{2}A_2^{ij} \tag{5.34}$$

(convention on the double sign being the same) forces us to set f_1 as a constant. The second equation is reduced to

$$-\frac{1}{4}\left(\frac{\partial f_2}{f_2}\right)^2=\left(\frac{\partial f_2}{f_2}\right)' \tag{5.35}$$

with the solution

$$f_2=(x_1-x_2)^4 \tag{5.36}$$

It also solves the first one

$$-4\phi_{12}^2=2\alpha\frac{\partial f_2}{f_2}+\left(\frac{\partial f_2}{f_2}\right)' \tag{5.37}$$

when $\alpha=\frac{1}{3}$ and the integration constant γ is zero

$$\frac{\partial f_2}{f_2}=e^{-2\alpha\int\phi_{12}}\left(-4\alpha\int\phi_{12}^2e^{2\alpha\int\phi_{12}}+\gamma\right) \tag{5.38}$$

Thus the unknown matrix V is found to be

$$\mathbf{V}=\mathbf{v}^2=c\mathbf{A}_1+(x_1-x_2)^4\mathbf{A}_2 \tag{5.39}$$

The symmetrized potential of the classical Hamiltonian (5.18) is

$$\mathcal{V}_0=\frac{1}{2}(\partial_l W)^2\ -\frac{1}{2}\partial_l^2W=\frac{2}{9}\frac{1}{(x_1-x_2)^2} \tag{5.40}$$

$$\mathcal{V}_2=-\frac{2}{3}\frac{1}{(x_1-x_2)^2}\left[\overline{\psi}_lA_2^{ll'}\psi_{l'}+\overline{\chi}\mathbf{A}_2\chi\right] \tag{5.41}$$

$$\mathcal{V}_4=-\frac{12}{(x_1-x_2)^2}\overline{\chi}\left[\psi_lA_2^{ll'}\overline{\psi}_{l'}+\overline{\chi}\mathbf{A}_2\chi\right]\mathbf{A}_2\chi \tag{5.42}$$

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