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A SEIFERT-VAN KAMPEN THEOREM FOR THE SECOND HOMOTOPY GROUP

by

STEVEN C. ALTHOEN

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5/25/73

Eldon Dyer

Professor Eldon Dyer
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Professor Alex Heller
Executive Officer

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The City University of New York

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This thesis is dedicated to my wife, Marcia, and son, Michael, in hopes that, in completion, it will in some small way compensate the suffering of the past four years.

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As Administrative Assistant to the Executive Officers, Ione Hutson has been helpful to me throughout the past four years. She also did an excellent job in typing this thesis.

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PREFACE

This paper uses the language of category theory. Prerequisites include the classical Seifert-van Kampen Theorem and covering space theory. Chapters 4 and 5 in Massey, [9], provide the necessary material. All the graph theory that is used is developed within the paper. A reference for the graph theory is [1]. Theorem 1 of Section 3 concerns groupoids. Groupoids occur only briefly in Section 3 and are thence forgotten. To understand the material presented there requires familiarity with Brown's work on groupoids [3] or [8].

The topological setting in which all the constructions are made is, Q , the category of quasi-topological spaces as defined by Spanier [11]. The choice of Q was made for convenience rather than necessity. Very few of the properties of Q are used. In fact, practically any "nice" category of spaces would suffice. The few basic facts and definitions in the theory of Q-spaces which are used may be summarized as follows:

Definition. A quasi-topological space is a set, X , together with a collection of admissible maps $\mathcal{A}(X)$ which satisfies the following five properties

i) each element $a \in \mathcal{A}(X)$ is a set function from some compact Hausdorff space to X ;

ii) each constant function is an element of $\mathcal{A}(X)$;

iii) if $a\colon C \to X$ is admissible and $f\colon C' \to C$ is continuous, then fa is admissible;

iv) if $a_1, a_2 \in \mathcal{A}(X)$, then the disjoint union $a_1 \sqcup a_2 \in \mathcal{A}(X)$;

v) if $a\colon C \to X$ is a function and $p\colon C' \to C$ is an onto continuous map such that pa is admissible, then a is admissible.

If X and X' are Q-spaces and $f\colon X \to X'$ is a set function, then f is a *map* (or *Q-morphism*) provided for each $a \in \mathcal{A}(X)$, $fa \in \mathcal{A}(X')$.

Q denotes the category of Q-spaces and maps.

Definition: An *injection* is a one to one function $f\colon X \to Y$ such that for any space W and function $g\colon W \to X$, fg is a map if and only if f is.

A *projection* is an onto function $f\colon X \to Y$ such that for any space Z and function $g\colon Y \to Z$, gf is a map if and only if f is.

Theorem 1. If $X \in \mathrm{obQ}$ and $i\colon X' \to X$ is a one to one function, then there is a unique Q-topology on X' such that i is an injection.

Proof: The collection $\mathcal{A}(X')$ is defined as follows: $a \in \mathcal{A}(X')$ if and only if $ia \in \mathcal{A}(X)$. □

Theorem 1 gives another way of defining the usual subspace topology.

Theorem 2. If $X \in \mathrm{obQ}$ and $p\colon X \to Y$ is an onto function, then there is a unique Q-topology on Y such that p is a projection.

Proof: The collection $\mathcal{A}(Y)$ is defined as follows: $a\colon C \to Y$ is an element of $\mathcal{A}(Y)$ provided there exists a compact Hausdorff space C' and admissible function $a'\colon C' \to X$ together with an onto map $\varphi\colon C' \to C$ such that the diagram

$$\begin{tikzcd} C' \arrow[r, "a'"] \arrow[d, "\varphi"'] & X \arrow[d, "p"] \\ C \arrow[r, "a"] & Y \end{tikzcd}$$

commutes. □

Theorem 2 gives another way of defining the usual quotient topology.

Definition: A subset A of a Q-space X is open provided for each admissible map $a \in \mathcal{Q}(X)$, $a: C \to X$, $a^{-1}A$ is open in C.

Another fact that is used is that an injection which is onto is a Q-homeomorphism; i.e. an equivalence in Q. One advantage to Q is that function spaces have a natural Q-topology. This fact is not used except perhaps in the definition of the path space functor $P: Q \to Q$: If $X \in obQ$, then

$$PX = \{(f,\ell) \,|\, f: \mathbb{R}^+ \to Q \text{ is a map and } f(t) = f(0),\ t \le 0\ ;\ f(t) = f(\ell), t \ge \ell$$

On morphisms, P is defined via the obvious composition.

If $X \in obQ$ and $x,y \in X$, then

$$P(x,X,y) \subseteq PX$$

denotes the set of all paths (f,ℓ) in X with $f(0) = x$ and $f(\ell) = y$.

$$\Omega(X,x) = P(x,X,x) .$$

If A is a set, then $|A|$ denotes the cardinality of A.

$\cup$ denotes the disjoint union (coproduct) of sets;

$\amalg$ denotes the disjoint union (coproduct) of spaces;

$\oplus$ denotes the direct sum (coproduct) of groups.

The notions of push-out and pull-back are used together with the explicit construction of colimits in Q as a quotient of a certain coproduct.

Gp denotes the category of groups; Ab denotes the category of abelian groups.

INTRODUCTION

The classical Seifert-van Kampen Theorem computes the fundamental group of a space from the fundamental groups of the elements in an open cover of the space. In particular, if $\{X_1, X_2\}$ is an open cover of a space X and $A = X_1 \cap X_2$ is path connected, then X is the push-out

$$\begin{tikzcd} A \arrow[r, "I_1"] \arrow[d, "I_2"'] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

of spaces and $\pi_1(X,*)$ is the push-out

$$\begin{tikzcd} \pi_1(A,*) \arrow[r, "\pi_1 I_1"] \arrow[d, "\pi_1 I_1"'] & \pi_1(X_1,*) \arrow[d] \\ \pi_1(X_2,*) \arrow[r] & \pi_1(X,*) \end{tikzcd}$$

of groups.

A more elegant way to phrase this result is by introducing a category C

$$\begin{tikzcd} a \arrow[r, "i_1"] \arrow[d, "i_2"'] & x_1 \\ x_2 & \end{tikzcd}$$

and a functor $F: C \to Q$ where

$$F(a) = A, \ F(x_1) = X_1, \ F(x_2) = X_2$$

$$F(i_1) = I_1, F(i_2) = I_2 \ .$$

The theorem then asserts that if $F(a)$, $F(x_1)$, and $F(x_2)$ are open and path connected and if $F(i_1)$ and $F(i_2)$ are inclusions (injections), then for any basepoint $* \in F(a)$,

$$\pi_1(\operatorname{colim} F,*) = \operatorname{colim} \pi_1(F(\ ),*) .$$

This paper develops an analagous result for the second homotopy group. The main restriction is that the fundamental group of each open set in the cover inject into the fundamental group of the space. If the space has a two element open cover $\{X_1,X_2\}$ such that $A = X_1 \cap X_2$ is connected and $\pi_1(X_i,*) \to \pi_1(X,*)$, $i=1,2$ and $\pi_1(A,*) \to \pi_1(X,*)$ are monic, then in this special case, the method developed herein is easy to follow.

Coverings, $\hat{A},\hat{X}_1$ and $\hat{X}_2$ of A, X_1 and X_2 are formed by pulling-back (restricting) the universal cover, $\tilde{X}$ of X. For example, $\hat{X}_1$ is the pull-back

$$\begin{tikzcd} \hat{X}_1 \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\ X_1 \arrow[r] & X \end{tikzcd} .$$

It turns out that:

i) there is a push-out

$$\begin{tikzcd} \hat{A} \arrow[r] \arrow[d] & \hat{X}_1 \arrow[d] \\ \hat{X}_2 \arrow[r] & \tilde{X} \end{tikzcd}$$

and ii) $\hat{X}_1$ (resp. $\hat{A}$, $\hat{X}_2$) is the disjoint union of as many copies of the universal cover of X_1 (resp. A, X_2) as cosets in the set

$$\pi_1(X,*)/\pi_1(X_1,*)$$

(resp. $\pi_1(X,*)/\pi_1(A,*)$, $\pi_1(X,*)/\pi_1(X_2,*)$) .

The Mayer-Vietoris Theorem applied to the preceding push-out yields an exact sequence

$$H_2(\hat{A},\mathbb{Z}) \to H_2(\hat{X}_1;\mathbb{Z}) \oplus H_2(\hat{X}_2;\mathbb{Z}) \to H_2(\tilde{X};\mathbb{Z}) \to 0$$

which terminates in zero since $\hat{A}$ is connected and simply connected in each component. Thus $H_2(\tilde{X};\mathbb{Z})$ is the push-out

$$\begin{tikzcd} H_2(\hat{A},\mathbb{Z}) \arrow[r] \arrow[d] & H_2(\hat{X}_1;\mathbb{Z}) \arrow[d] \\ H_2(\hat{X}_2;\mathbb{Z}) \arrow[r] & H_2(\tilde{X};\mathbb{Z}) \end{tikzcd}$$

Since $H_2(\hat{X}_1;\mathbb{Z}) = \oplus H_2(\tilde{X}_1;\mathbb{Z})$ etc., this push-out yields the push-out

$$\begin{tikzcd} \oplus H_2(\tilde{A};\mathbb{Z}) \arrow[r] \arrow[d] & \oplus H_2(\tilde{X}_1;\mathbb{Z}) \arrow[d] \\ \oplus H_2(\tilde{X}_2;\mathbb{Z}) \arrow[r] & H_2(\tilde{X};\mathbb{Z}) \end{tikzcd} .$$

Now the Hurewicz isomorphism applies to yield $\pi_2(X,*)$ as the push-out

$$\begin{tikzcd} \oplus \pi_2(A,*) \arrow[r] \arrow[d] & \oplus \pi_2(X_1,*) \arrow[d] \\ \oplus \pi_2(X_2,*) \arrow[r] & \pi_2(X,*) \end{tikzcd}$$

where $\pi_2(\tilde{A},*) = \pi_2(A,*)$ since $\tilde{A}$ is the universal cover of A , etc.

This paper generalizes the preceding construction to a larger class of open coverings for which the fundamental group of each element in the cover injects into the fundamental group of the space.

Section 1 develops the language necessary to describe the covers to which the construction will apply.

Section 2 treats in detail the case of covers for which the intersection of any three elements is void.

Section 3 constructs a fundamental group functor which is used in Section 4 to generalize the classical Seifert-van Kampen Theorem to the case when not all elements of the cover contain the base point.

Section 5 generalizes the process by which the universal cover of X is pulled-back over each open set in the cover.

Sections 6 and 7 generalize the Mayer-Vietoris Theorem.

Section 8 contains some examples.

Section 9 contains the main result: the coversion of the homology constructions in Section 6 into homotopy constructions.

Section 10 relates these results to the special case considered in Section 2.

Section 11 gives a formal description of the means by which one passes from a cover to the category described in Section 1.

Section 12 contains examples.

§1. s-Categories.

s-Categories are used to model structure associated with a covering of a space. An analagous structure is the nerve of a covering. If X is a space and $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of X indexed by a set Λ, then the nerve of the covering is the simplicial complex described as follows: To each open set in the cover is associated a vertex. If two elements of the cover intersect, then this is recorded by attaching an edge between the vertices representing them. This process continues by induction attaching n-dimensional simplices to represent the intersection of any $n+1$ distinct elements of the cover. In what follows, it will be of interest to record not only the existence of intersections but also the path components of the intersections. That is, whenever $n+1$ elements intersect, an n-simplex will be attached for each path component of that intersection. For various reasons it is convenient to record this information in a category rather than in a complex. s-Categories are a concise means to do this.

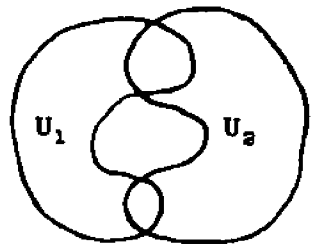
Definition: Let Λ denote a set. An s-category based on Λ is a small category, C, such that

i) each $f \in \text{ob}C$ is an injective function $f\colon X_f \to \Lambda$ with $|X_f| < \infty$;

ii) for each pair $f,g \in \text{ob}C$, $|C(f,g)| \le 1$; and if $C(f,g) \neq \emptyset$, then $\text{im}g \subseteq \text{im}f$;

iii) for each $f \in \text{ob}C$ and non-empty set $Y \subseteq \text{im}f$ there exists a unique $g \in \text{ob}C$ such that $C(f,g) \neq \emptyset$ and $\text{im}g = Y$.

Example 1: X is an annulus with a two piece open cover $\{U_i\}_{i=1,2}$ as illustrated.



The s-category, C , appropriate for this example is based on the set $\{1,2\}$ and has four objects:

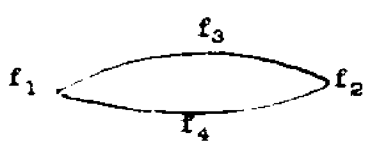
f_1: $\{1\} \longrightarrow \{1,2\}$ $\qquad$ f_3: $\{1,2\}_1 \longrightarrow \{1,2\}$

f_2: $\{2\} \longrightarrow \{1,2\}$ $\qquad$ f_4: $\{1,2\}_2 \longrightarrow \{1,2\}$

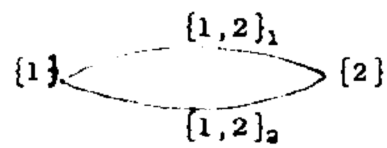
(where $\{1,2\}_i$ is short for $\{1,2\} \times \{i\}$, $i=1,2$). f_i corresponds to U_i , $i=1,2$. f_3 and f_4 each correspond one to each component of the intersection $U_1 \cap U_2$. C has eight non-empty morphism sets: the obvious four containing only the necessary identities and the four

$$C(f_3,f_1) \ , \ C(f_3,f_2) \ , \ C(f_4,f_1) \ , \ C(f_4,f_2)$$

containing morphisms which correspond to the various inclusions of the components of $U_1 \cap U_2$. Composition in C is possible only with identities. This category has a representation as the CW-complex



or



The general procedure for passing from a given open cover of a space to the associated s-category is described in detail in Section 11.

<u>Definition</u>: The dimension function $d: obC \to \mathbb{Z}^+$, the non-negative integers, is defined for $f \in obC$ by $df + 1 = |X_f|$. This function partitions obC into disjoint classes C_n defined by

$C_n = d^{-1}n$ for $n \in \mathbb{Z}^+$.

In Example 1, $df_1 = df_2 = 0$ and $df_3 = df_4 = 1$.

<u>Definition</u>: If C is an s-category, then $f \in \text{ob}C$ is a <u>free face</u> if there exists no $g \in \text{ob}C$, $g \neq f$, such that $C(g,f) \neq \emptyset$. The collection of free faces in C will be denoted $Fr(C)$.

If $f,f' \in \text{ob}C$, the s-category $f \cap f'$ based on $\text{im}f \cap \text{im}f'$ is defined to be the full subcategory of C generated by the set

$$\{g \in \text{ob}C \mid C(f,g) \neq \emptyset \text{ and } C(f',g) \neq \emptyset\} .$$

The fact that $f \cap f'$ has objects will be written $f \cap f' \neq \emptyset$.

<u>Example</u> 2: C is the s-category based on $\Lambda = \{1,2,3\}$ with objects

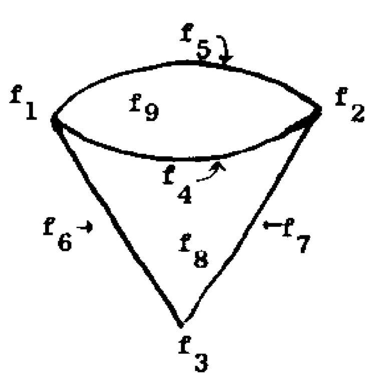
$$\left.\begin{array}{l} f_1\colon \{1\} \longrightarrow \Lambda \\ f_2\colon \{2\} \longrightarrow \Lambda \\ f_3\colon \{3\} \longrightarrow \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_4\colon \{1,2\}_1 \longrightarrow \Lambda \\ f_5\colon \{1,2\}_2 \longrightarrow \Lambda \\ f_6\colon \{1,3\} \longrightarrow \Lambda \\ f_7\colon \{2,3\} \longrightarrow \Lambda \end{array}\right\} = C_1$$

$$\left.\begin{array}{l} f_8\colon \{1,2,3\}_1 \longrightarrow \Lambda \\ f_9\colon \{1,2,3\}_2 \longrightarrow \Lambda \end{array}\right\} = C_2$$

and non-identity morphisms as indicated below:

$$\begin{array}{lll} f_8 \to f_1,f_2,f_3,f_4,f_6,f_7 & f_4 \to f_1,f_2 & f_6 \to f_1,f_3 \\ f_9 \to f_1,f_2,f_3,f_5,f_6,f_7 & f_5 \to f_1,f_2 & f_7 \to f_2,f_3 . \end{array}$$

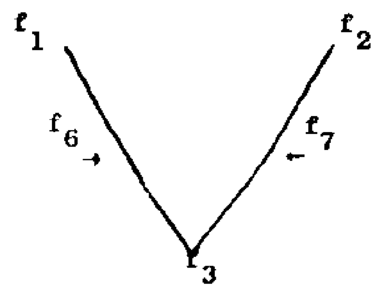
As a complex C is the cone:



$Fr(C) = \{f_8, f_9\}$. $f_8 \cap f_9$ is the s-category with objects

$$\left.\begin{array}{l} f_1\colon \{1\} \longrightarrow \Lambda \\ f_2\colon \{2\} \longrightarrow \Lambda \\ f_3\colon \{3\} \longrightarrow \Lambda \end{array}\right\} = (f_8 \cap f_9)_0 \qquad \left.\begin{array}{l} f_6\colon \{1,3\} \longrightarrow \Lambda \\ f_7\colon \{2,3\} \longrightarrow \Lambda \end{array}\right\} = (f_8 \cap f_9)_1$$

$f_8 \cap f_9$ is the complex



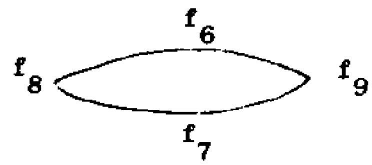
<u>Definition</u>: A <u>graph</u>, Γ , consists of two sets $v\Gamma$ and $e\Gamma$ together with functions $o, t\colon e\Gamma \to v\Gamma$ and a function $\iota\colon e\Gamma \to e\Gamma$. $v\Gamma$ is the set of <u>vertices</u> of Γ and $e\Gamma$ is the set of <u>edges</u> of Γ . The function $o\colon e\Gamma \to v\Gamma$ associates to each edge, $h \in e\Gamma$, its <u>origin</u>, $o(h)$, and the function $t\colon e\Gamma \to v\Gamma$ associates to each edge, $h \in e\Gamma$, its <u>terminal point</u>, $t(h)$. The function ι is the <u>involution</u> and has the property that for each $h \in e\Gamma$, $o(\iota h) = t(h)$ and $t(\iota h) = o(h)$.

Examples of graphs appear in Example 1 and in Example 2. In Example 1, C is a graph with $vC = \{f_1, f_2\}$ and $eC = \{f_3^*, \bar{f}_3, f_4^*, \bar{f}_4\}$ where $o(f_3^*) = o(f_4^*) = t(\bar{f}_3) = t(\bar{f}_4) = f_1, t(f_3^*) = t(f_4^*) = o(\bar{f}_3) = (\bar{f}_4) = f_2$, $\iota f_3^* = \bar{f}_3$, and $\iota f_4^* = \bar{f}_4$. In the illustration the edges f_3^* and $\bar{f}_3$ coincide and are represented by f_3 . Similarly the edges f_4^* and $\bar{f}_4$ coincide and are represented by f_4 . In Example 2, $f_8 \cap f_9$ is a graph with vertices represented by the objects f_1, f_2, f_3 and pairs of edges represented by the objects f_6 and f_7 .

<u>Definition</u>: Let C denote an s-category. The <u>graph</u> of C , $\Gamma(C)$,

has vertices, $v\Gamma(C) = Fr(C)$. If $f, f' \in Fr(C)$ and $f \neq f'$, then the edges of $\Gamma(C)$ with origin f and terminal point f' correspond, under the correspondence $h \to (f,h,f')$ with the free faces of $f \cap f'$. If $h \in Fr(f \cap f')$ and $(f,h,f') \in e\Gamma(C)$, then $\iota(f,h,f') = (f',h,f)$. Since $Fr(f \cap f') = Fr(f' \cap f)$, this defines an involution.

Example 3: In the s-category, C , of Example 2, $v\Gamma(C) = \{f_8, f_9\}$ and $e\Gamma(C) = \{(f_8,f_6,f_9),(f_9,f_6,f_8),(f_8,f_7,f_9),(f_9,f_7,f_8)\}$. $\Gamma(C)$ is represented



where (f_8,f_6,f_9) and (f_9,f_6,f_8) correspond under the involution ι and are represented by f_6 and (f_8,f_7,f_9) and (f_9,f_7,f_8) similarly correspond and are represented by f_7 .

Often it is unnecessary to record the origin and terminal vertices of an edge $(f,h,f') \in e\Gamma(C)$. In these cases such an edge will be denoted simply by the symbol h .

Definition: A generating graph for an s-category C , is any subgraph, Γ , of $\Gamma(C)$ such that

i) $v\Gamma = v\Gamma(C)$

ii) if $(f,g,f') \in e\Gamma(C)$, then there exists a finite sequence of edges of Γ ,

$$\{(f_i,h_i,f_{i+1}) \in e\Gamma \subseteq e\Gamma(C)\}_{i=0}^{n}$$

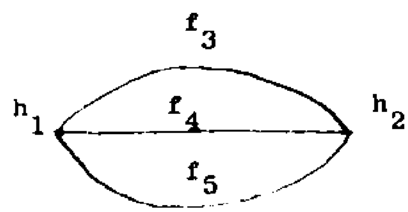
with $f_0 = f$, $f_{n+1} = f'$ and $C(h_i, g) \neq \emptyset$ for $i=0,\dots,n$.

Example 4: Let C be the s-category based on $\Lambda = \{1,2\}$ with objects

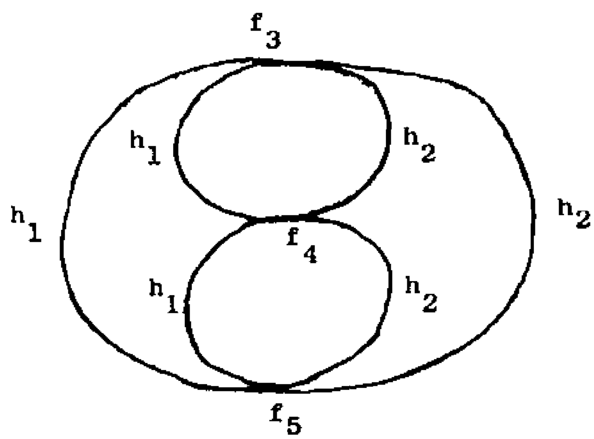
$$\left.\begin{array}{l} h_1 : \{1\} \longrightarrow \Lambda \\ h_2 : \{2\} \longrightarrow \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_3 : \{1,2\}_1 \longrightarrow \Lambda \\ f_4 : \{1,2\}_2 \longrightarrow \Lambda \\ f_5 : \{1,2\}_3 \longrightarrow \Lambda \end{array}\right\} = C_1$$

and with obvious morphisms.

C is represented



and $\Gamma(C)$ is represented

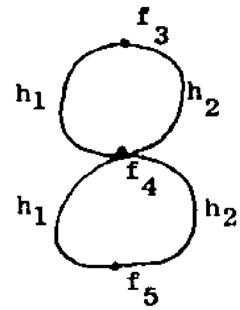


A generating graph for $\Gamma(C)$ is obtained by selecting the edges $\{(f_3,h_2,f_4),(f_4,h_2,f_3),(f_4,h_2,f_5),(f_5,h_2,f_4),(f_3,h_1,f_4),(f_4,h_1,f_3),$ $(f_4,h_1,f_5),(f_5,h_1,f_4)\}$. For example $(f_3,h_2,f_5) \in e\Gamma(C)$ and

$$\{(f_3,h_2,f_4),\ (f_4,h_2,f_5)\}$$

is a sequence of edges of the generating graph of the required type.

This generating graph is represented:

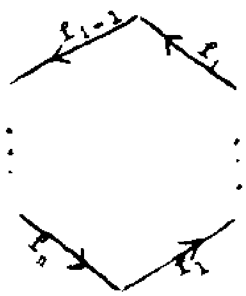


Some definitions from graph theory will be needed in what follows.

<u>Definition</u>: A <u>circuit of length</u> n in a graph, Γ , is a sequence of edges

$$\{f_i\}_{i=1}^{n}$$

such that $o(f_i) = t(f_{i-1})$ $i=2,\ldots,n$, and $o(f_1) = t(f_n)$.



Furthermore, $\iota f_i \neq \iota f_{i-1}$, $i=2,\ldots,n$ and $\iota f_1 \neq \iota f_n$.

A <u>path</u> in a graph, Γ , from a vertex v to a vertex v' is a finite sequence of edges

$$\{f_i\}_{i=1}^{n}$$

with $t(f_i) = o(f_{i+1})$, $i = 1,\ldots,n-1$ and $o(f_1) = v$, $t(f_n) = v'$. n is the <u>length</u> of the path.

A graph is <u>path connected</u> if every pair of vertices may be joined by a path. If a graph is not path connected, then its path components are defined in the obvious way. A graph is a <u>tree</u> if it has no circuits.

A tree is not assumed to be connected unless otherwise stated. For any two vertices in a path component of a tree there is a unique minimal length path between them (or else the tree would contain a circuit). Such a minimal length path is called a <u>geodesic</u>.

<u>Definition</u>: A <u>generating tree</u> for an s-category, C, is any generating graph which is a tree.

<u>Theorem 1</u>. If C is an s-category with a generating tree, then $\Gamma(C)$ has no circuits of length two.

<u>Proof</u>: If $\Gamma(C)$ has a circuit of length two, then there exist distinct vertices $f, f' \in \Gamma(C)$ and two distinct edges $(f,g,f'),(f',g'f)$ with $g,g' \in \mathrm{Fr}(f \cap f')$. Since these are free faces both $C(g,g')$ and $C(g',g)$ are empty.

If every generating graph for $\Gamma(C)$ contains both (f,g,f') and (f,g',f'), then every generating graph has a circuit and is therefore not a tree. Thus we can assume some generating graph, Γ, is a tree and does not contain (f,g,f').

Since Γ is generating there is a sequence of edges

$$\{(f_i,h_i,f_{i+1}) \in e\Gamma\}_{i=0}^{n}$$

of Γ such that $f_0 = f$, $f_{n+1} = f'$ and $C(h_i,g) \neq \emptyset$ for $i=0,\dots,n$. Since $C(g',g) = \emptyset$, no $h_i = g'$. In particular, $h_0 \neq g'$ and $h_n \neq g'$. If $(f,g',f') \in e\Gamma$, then the sequence

$$\{(f',g',f),\ (f_i,h_i,f_{i+1})\}_{i=0}^{n}$$

is a circuit in Γ. Thus $(f,g',f') \notin e\Gamma$ and there exists a sequence

$$\{(f_i',h_i',f_{i+1}') \in e\Gamma\}_{i=0}^{m}$$

with $f_0' = f$, $f_{m+1}' = f'$ and $C(h_i',g') \neq \emptyset$ for $i=0,\ldots,m$.

If these two sequences of paths are unequal, then Γ contains a circuit. Since Γ is a tree, the sequences are equal and there is a sequence $\{(f_i,h_i,f_{i+1}) \in e\Gamma\}_{i=0}^{n}$ of edges with $f_0 = f$, $f_{n+1} = f'$ and $C(h_i,g) \neq \emptyset$, $C(h_i,g') \neq \emptyset$ for $i=0,\ldots,n$.

Then $\mathrm{im}g \subseteq \mathrm{im}h_i$ and $\mathrm{im}g' \subseteq \mathrm{im}h_i$ for $i=0,\ldots,n$. Thus $\mathrm{im}g \cup \mathrm{im}g' \subseteq \mathrm{im}h_i$ for $i=0,\ldots,n$. Since $C(f_i,h_i) \neq \emptyset$ for $i=0,\ldots,n$ and $C(f_{n+1},h_n) \neq \emptyset$, $\mathrm{im}g \cup \mathrm{im}g' \subseteq \mathrm{im}f_i$ for $i=0,\ldots,n+1$. From the definition of an s-category follows the existence of sequences $\{k_i\}_{i=0}^{n+1}$ and $\{k_i'\}_{i=0}^{n}$ such that $C(f_i,k_i) \neq \emptyset$ and $\mathrm{im}k_i = \mathrm{im}g \cup \mathrm{im}g'$, $i=0,\ldots,n+1$ and $C(h_i,k_i') \neq \emptyset$ and $\mathrm{im}k_i'$ $= \mathrm{im}g \cup \mathrm{im}g'$, $i=0,\ldots,n$.

Since $C(f_i,h_i) \neq \emptyset$, $C(f_i,k_i') \neq \emptyset$ and $k_i = k_i'$ for $i=0,\ldots,n$. Also $C(h_n,k_{n+1}) \neq \emptyset$, and $k_{n+1} = k_n$.

For each i , $0 \leq i < n$, $C(f_{i+1},h_i) \neq \emptyset$ and $C(f_{i+1},h_{i+1}) \neq \emptyset$. Thus $C(f_{i+1},k_i) \neq \emptyset$ and $C(f_{i+1},k_{i+1}) \neq \emptyset$, and $k_i = k_{i+1}$ for $i=0,\ldots,n-1$. By induction all the k_i's are equal and in particular, $k_0 = k_{n+1}$. Since $\mathrm{im}g \cup \mathrm{im}g' = \mathrm{im}k_0 = \mathrm{im}k_{n+1}$, there are elements h,h' with $\mathrm{im}h = \mathrm{im}g$ and $\mathrm{im}h' = \mathrm{im}g'$ and

$$C(k_0,h) \neq \emptyset \ , \ C(k_{n+1},h') \neq \emptyset \ .$$

Since $C(f,k_0) = C(f_0,k_0) \neq \emptyset$ and $C(f',k_{n+1}) = C(f_{n+1},k_{n+1}) \neq \emptyset$, $C(f,h) \neq \emptyset$. $C(f',h') \neq \emptyset$. It follows that $h = g$ and $h' = g'$. Thus $C(f,k_0) \neq \emptyset$ and $C(f',k_0) \neq \emptyset$ so that

$$k_0 \in \mathrm{ob}(f \cap f')$$

But $C(k_0,g) \neq \emptyset$ and $C(k_{n+1},g') = C(k_0,g') \neq \emptyset$. Since g and g' are distinct, this shows that at least one of them is not a free face of $f \cap f'$ which is a contradiction. □

<u>Example</u> 5: A 1-dimensional s-category may have a generating tree and a circuit of length three:

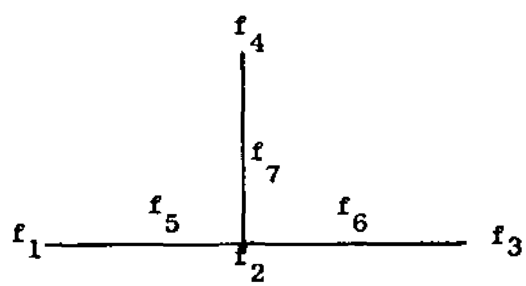
C is the s-category based on $\Lambda = \{1,2,3,4\}$ with objects:

$$\left.\begin{array}{l} f_1\colon \{1\} \longrightarrow \Lambda \\ f_2\colon \{2\} \longrightarrow \Lambda \\ f_3\colon \{3\} \longrightarrow \Lambda \\ f_4\colon \{4\} \longrightarrow \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_5\colon \{1,2\} \longrightarrow \Lambda \\ f_6\colon \{2,3\} \longrightarrow \Lambda \\ f_7\colon \{2,4\} \longrightarrow \Lambda \end{array}\right\} = C_1$$

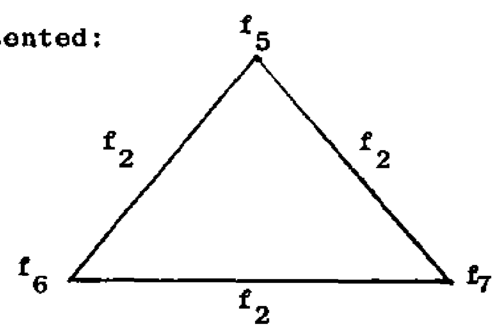
and non-identity morphisms:

$$f_5 \longrightarrow f_1,f_2 \ ; \ f_6 \longrightarrow f_2,f_3 \ ; \ f_7 \longrightarrow f_2,f_4 \ .$$

C is represented:



and $\Gamma(C)$ is represented:



.

It is clear that $\Gamma(C)$ is a circuit of length three and any two edges of $\Gamma(C)$ form a generating tree.

The illustrations in this section have represented s-categories as semi-simplical complexes. In general, they are not simplicial complexes. However, if an s-category has a generating tree, then it frequently is simplicial.

<u>Definition</u>: An s-category C is <u>upward finite</u> if for each object $g \in \mathrm{ob}C$, there is a free face $f \in \mathrm{ob}C$ with $C(f,g) \neq \emptyset$.

<u>Definition</u>: C is an s-category. The elements of C_0 are called vertices of C. A vertex $v \in C_0$ is <u>connected</u> if for any $v' \in C_0$, $\mathrm{im}v' = \mathrm{im}v$ implies $v = v'$.

This definition relates to topology in the following way. If the s-category C arises from an open cover of a space, then the vertices of C correspond to path components of the elements in the cover. Thus if $\{U_\lambda\}_{\lambda\in\Lambda}$ is the cover, then the vertices with image λ correspond to the path components of U_λ, since each U_λ represents the intersection of 1 element in the cover. If each U_λ is path connected then there will be a single vertex with image λ. That is, the vertices will be determined by their images; the vertices will be connected.

Clearly, if a given open cover does not consist solely of path connected spaces, then it can be enlarged to one which does. Without loss of generality it will be assumed that all s-categories have each of their vertices connected.

<u>Definition</u>: C is an s-category and $g \in \mathrm{ob}C$. The <u>vertex set</u> of g is denoted νg and is defined to be the set

$$\nu g = \{v \in C_0 \mid C(g,v) \neq \emptyset\} .$$

The elements of vg are called the <u>vertices</u> of g . Note that
$img = \bigcup_{v \in vg} imv$.

<u>Theorem 2</u>: If C is an upward finite s-category with a generating tree, then C has a representation as a simplicial complex.

<u>Proof</u>: It suffices to show that each object, $g \in obC$, is determined by its vertices. Suppose $g' \in obC$ and $vg = vg'$. Since C is upward finite, there exist free faces $f,f' \in Fr(C)$ with $C(f,g) \neq \emptyset$ and $C(f',g') \neq \emptyset$. If $f = f'$, then since $img = img'$, $g = g'$.

If $f \neq f'$, then each vertex $v \in vg = vg'$ is an object in $f \cap f'$ and there is a free face $h_v \in Fr(f \cap f')$ with $C(h_v,v) \neq \emptyset$. Since C has a generating tree $|Fr(f \cap f')| = 1$ by Theorem 1. Thus all the h_v's are equal to a single element, h . Since $img = img'$ $\subseteq imh$, there is a unique $g'' \in obC$ with $C(h,g'') \neq \emptyset$ and $img'' = img$ $= img'$. Since $C(f,h) \neq \emptyset$, $C(f,g'') \neq \emptyset$ and $g'' = g$. Similarly, $g'' = g'$. Thus $g = g'$. $\square$

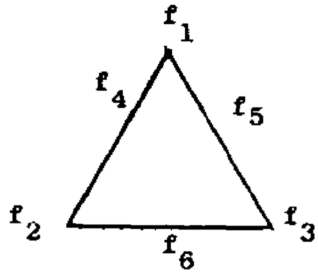
It is a consequence of this theorem that in case the s-category C derives from an open cover, if C has a generating tree, then C represents the nerve of the cover.

The following examples show that the converse of Theorem 2 is false:

<u>Example</u> 6: C is the s-category based on $\Lambda = \{1,2,3\}$ with objects

$$\left.\begin{array}{l} f_1 : \{1\} \longrightarrow \Lambda \\ f_2 : \{2\} \longrightarrow \Lambda \\ f_3 : \{3\} \longrightarrow \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_4 : \{1,2\} \longrightarrow \Lambda \\ f_5 : \{1,3\} \longrightarrow \Lambda \\ f_6 : \{1,4\} \longrightarrow \Lambda \end{array}\right\} = C_1$$

and obvious morphisms. C has a representation as the simplicial complex



but has no generating tree.

<u>Example</u> 7: C is the s-category based on $\Lambda = \{1,2,3,4\}$ with objects:

$$\left.\begin{array}{l} f_1: \{1\} \longrightarrow \Lambda \\ f_2: \{2\} \longrightarrow \Lambda \\ f_3: \{3\} \longrightarrow \Lambda \\ f_4: \{4\} \longrightarrow \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{ll} f_5: \{1,2\} \longrightarrow \Lambda \ ; & f_9: \{2,4\} \longrightarrow \Lambda \\ f_6: \{1,3\} \longrightarrow \Lambda \ ; & f_{10}: \{3,4\} \longrightarrow \Lambda \\ f_7: \{1,4\} \longrightarrow \Lambda & \\ f_8: \{2,3\} \longrightarrow \Lambda & \end{array}\right\} = C_1$$

$$\left.\begin{array}{l} f_{11}: \ \{1,2,3\} \longrightarrow \Lambda \\ f_{12}: \ \{1,2,4\} \longrightarrow \Lambda \\ f_{13}: \ \{1,3,4\} \longrightarrow \Lambda \\ f_{14}: \ \{2,3,4\} \longrightarrow \Lambda \end{array}\right\} = C_2$$

and obvious morphisms. C can be represented as a simplicial complex as the hollow tetrahedron. $\Gamma(C)$ has vertices labeled by $Fr(C) = C_2$. $\Gamma(C)$ has twelve edges, namely

$$\begin{array}{ll} (f_{11}, f_5, f_{12}) & (f_{12}, f_5, f_{11}) \\ (f_{11}, f_6, f_{13}) & (f_{13}, f_6, f_{11}) \\ (f_{12}, f_7, f_{13}) & (f_{13}, f_7, f_{12}) \\ (f_{11}, f_8, f_{14}) & (f_{14}, f_8, f_{11}) \\ (f_{12}, f_9, f_{14}) & (f_{14}, f_9, f_{12}) \\ (f_{13}, f_{10}, f_{14}) & (f_{14}, f_{10}, f_{13}) \ . \end{array}$$

Clearly $C(f_i,f_j) = \emptyset$ if $i,j \in \{5,6,7,8,9,10\}$ and $i \neq j$. It follows that no proper subgraph of $\Gamma(C)$ is a generating graph. Since $\Gamma(C)$ is not a tree, C has no generating tree.

Despite these examples, it may be the case that sufficiently high connectivity or contractibility of the complex guarantees a generating tree (cf. Theorem 1 in Section 2). Since these questions do not arise in what follows, they will not be investigated here.

§2. One Dimensional s-Categories.

Definition: An s-category, C, is n-dimensional if $C_n \neq \emptyset$ but $C_k = \emptyset$ for $k > n$.

This section develops the theory of 1-dimensional s-categories. For the remainder of this section the only categories which will be considered (apart from examples) are 1-dimensional s-categories. A 1-dimensional s-category arises from a cover of a space for which no three elements have a non-empty intersection. A useful fact about such categories is that if C is 1-dimensional and for $g, g' \in \text{ob}C$, $g \neq g'$ but $C(g,g') \neq \emptyset$, then $g \in C_1$ and $g' \in C_0$.

As was indicated earlier if C is 1-dimensional, then the complex it represents has a natural graph structure. This idea is formalized in the following definition.

Definition: C is an s-category. The geometric graph, $\Gamma_g(C)$, of C has as vertex set the set $v\Gamma_g(C) = C_0$. If v, v' are distinct elements of C_0, then the edges of $\Gamma_g(C)$ with origin v and terminal v' are of the form (v,f,v') for $f \in C$, with $C(f,v) \neq \emptyset$ and $C(f,v') \neq \emptyset$. If (v,f,v') is such an edge, $\iota(v,f,v') = (v',f,v)$. In some instances it is possible to ignore the origin and terminal points of an edge, $(v,f,v') \in \Gamma_g(C)$, and to write simply f.

The categories of Example 1 and Example 4 in Section 1 are illustrated with their geometric graphs.

Proposition 1: If C is an s-category, then $\Gamma_g(C)$ is connected if and only if $\Gamma(C)$ is connected.

Proof: If $\Gamma_g(C)$ is connected and f, f' are two vertices of $\Gamma(C)$, then there is a finite length path in $\Gamma_g(C)$ from $o(f)$ to $t(f')$ since these are vertices of $\Gamma_g(C)$ when f is regarded as an edge of $\Gamma_g(C)$. This path has a succession of vertices and edges which are the edges and vertices of a path in $\Gamma(C)$ from f to f'.

If $\Gamma(C)$ is connected, then no vertex, v, of $\Gamma_g(C)$ is a free face of C since $C(v,f) \neq \emptyset$ implies $f = v$ and v is isolated in $\Gamma(C)$. Thus if v, v' are two vertices of $\Gamma_g(C)$, then there exist free faces $f, f' \in Fr(C)$ with $C(f,v) \neq \emptyset$, $C(f',v') \neq \emptyset$ and a path in $\Gamma(C)$ from f to f' which dualizes as before into a path in $\Gamma_g(C)$ from v to v'. □

Example 5 in Section 1 illustrates a 1-dimensional s-category for which $\Gamma_g(C)$ is a tree but $\Gamma(C)$ is not a tree.

Theorem 2. Suppose C is an s-category. i) If $\Gamma_g(C)$ is a tree then every maximal tree in $\Gamma(C)$ is a generating tree. ii) If there is a generating tree in $\Gamma(C)$, then $\Gamma_g(C)$ is a tree.

Proof: i) Suppose that some maximal tree, $\Gamma \subseteq \Gamma(C)$ is not a generating tree. For some edge (f,g,f') in $\Gamma(C)$ there is no sequence.

$$\{(f_i,\ h_i,\ f_{i+1})\}_{i=0}^{n}$$

of edges of Γ with $f_0 = f$, $f_{n+1} = f'$ and $C(h_i,g) \neq \emptyset$ for all $i=0,\ldots,n$. Trivially, each edge in Γ has a one term sequence (itself) which violates this property. It follows that one may assume $(f,g,f') \in e\Gamma(C) - e\Gamma$.

A maximal tree in a graph contains all the vertices of the component in which it lies. Since $f, f' \in v\Gamma(C)$ and $(f,g,f') \in e\Gamma(C)$, f and f' are in the same connected component of $\Gamma(C)$. Since a maximal tree in a connected graph is connected, there is a path in Γ from the vertex f to the vertex f': i.e. a sequence

$$H = \{(f_i,\ h_i,\ f_{i+1})\}_{i=0}^{n}$$

with $f_0 = f$, $f_{n+1} = f'$. Since the elements of H are edges in a tree, it can be assumed that

1) the f_i's are distinct,

(or else H can be simplified to such a path). Since, as was observed earlier, no such sequence has $C(h_i,g) \neq \emptyset$ for all $i=0,\ldots,n$, for at least one k, $0 \le k \le n$,

2) $C(h_k,g) = \emptyset$.

If any $h_i \in C_1$, then since C is one-dimensional, $C(f_i,h_i) \neq \emptyset$, $C(f_{i+1},h_i) \neq \emptyset$ implies $f_i = h_i = f_{i+1}$ which violates condition 1). Similarly g is 0 dimensional or else (f,g,f') does not determine an edge in $\Gamma(C)$. For 0 dimensional objects in an s-category $C(h_k,g) = \emptyset$ is equivalent to the inequality $h_k \neq g$. Thus for some k, $0 \le k \le n$,

3) $h_k \neq g$.

The sequence

$$S = \{g, h_0, \ldots, h_n\}$$

lists the middle terms of the edges of a circuit in $\Gamma(C)$. The subsequence

$$S' = \{g, h_{i_1}, \ldots, h_{i_m}\}$$

is determined by the properties

a) $g \in S'$

b) no consecutive elements are equal

c) $h_{i_j - k} = h_{i_{j-1}}$ for $1 \le k \le i_j - i_{j-1}$, $j=2,\ldots,m$

d) $g = h_{i_1 - k}$ for $1 \le k \le i_1$

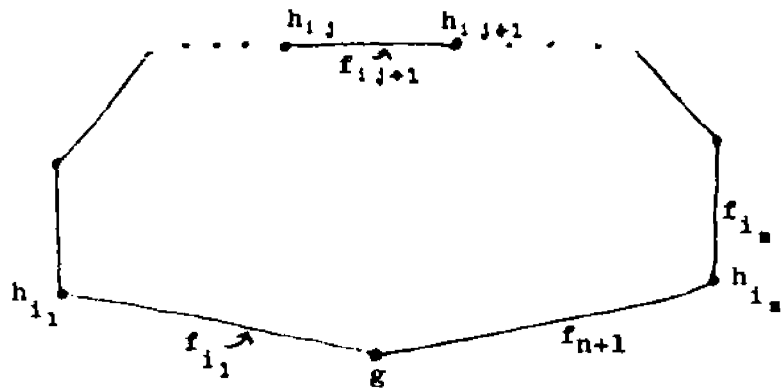
e) $h_{i_m} \ne g$; $h_{i_m + k} = g$ $1 \le k \le n - i_m$

Conditions b) - e) merely affirm that S' is the maximal subsequence of S with properties a) and b). Condition 3) above guarantees that S' has at least two elements.

The subsequence S' will be used to produce a circuit in $\Gamma_g(C)$. By construction $g = h_{i_1 - 1} \in Fr(f_{i_1 - 1} \cap f_{i_1})$ and $h_{i_1} \in Fr(f_{i_1} \cap f_{i_1 + 1})$ if $i_1 > 0$. Thus (g, f_{i_1}, h_{i_1}) is an edge in $\Gamma_g(C)$ from g to h_{i_1}. If $i_1 = 0$, then (g, f_0, h_0) is an edge in $\Gamma_g(C)$ from g to $h_0 = h_{i_1}$.

For $1 \le j < m$, $h_{i_j} = h_{i_{j+1} - 1} \in Fr(f_{i_{j+1} - 1} \cap f_{i_{j+1}})$ and $h_{i_{j+1}} \in Fr(f_{i_{j+1}} \cap f_{i_{j+1} + 1})$, provided $h_{i_j + 1} \ne h_{i_{j+1}}$. Thus $(h_{i_j}, f_{i_{j+1}}, h_{i_{j+1}})$ is an edge in $\Gamma_g(C)$ from h_{i_j} to $h_{i_{j+1}}$. If $h_{i_j + 1} = h_{i_{j+1}}$, then $(h_{i_j}, f_{i_{j+1}}, h_{i_{j+1}})$ is an edge in $\Gamma_g(C)$ from h_{i_j} to $h_{i_{j+1}}$.

Finally, $h_{i_m} = h_{i_n} \in Fr(f_n \cap f_{n+1})$ and $g \in Fr(f_0 \cap f_{n+1})$ so that (h_{i_m}, f_{n+1}, g) is an edge in $\Gamma_g(C)$ from h_{i_m} to g.



If S' has more than two elements, then the sequence

$$\{(g,f_{i_1},h_{i_1}),\ldots,(h_{i_{m-1}},f_{i_m},h_{i_m}),(h_{i_m},f_{n+1},g)\}$$

is a sequence of distinct edges in $\Gamma_g(C)$ which forms a non-trivial circuit in $\Gamma_g(C)$ at g .

If S' has exactly two elements, the sequence becomes

$$\{(g,f_{i_1},h_{i_1})\ ,\ (h_{i_1},f_{n+1},g)\}\ .$$

If $h_1 = h_n$, then $f_{i_1} = f_n$. Otherwise, $h_1 = h_{n'}$ for $n' < n$. In any case, by condition 1) above, $f_{i_1} \neq f_{n+1}$. It follows then that $\Gamma_g(C)$ contains a non-trivial circuit of length two and is therefore not a tree.

ii) If $\Gamma_g(C)$ is not a tree then it contains a non-trivial circuit of minimal finite length with edges

$$\{(g_i,\ f_i\ ,\ g_{i+1})\}_{i=0}^{n}$$

such that $g_0 = g_{n+1}$.

Since the circuit is of minimal length, it may be assumed that the vertices $g_0,\ldots,g_n$ are distinct. If $f_i = f_j$ for $i \neq j$, then since $\nu f_i = \{g_i,g_{i+1}\}$ and $\nu f_j = \{g_j,g_{j+1}\}$, the vertices $g_0,\ldots,g_n$

are not distinct. It follows that the objects $f_0,\dots,f_n$ are also distinct.

Setting $f_{n+1} = f_0$ yields a sequence

$$S = \{(f_i,\ g_i,\ f_{i+1})\}_{i=0}^{n}$$

of edges of $\Gamma(C)$ which determine a circuit. If $\Gamma \subseteq \Gamma(C)$ is a generating graph for C, then at least one edge $(f_k, g_k, f_{k+1}) \in S$ is omitted from Γ. Since Γ is generating there exists in Γ a sequence

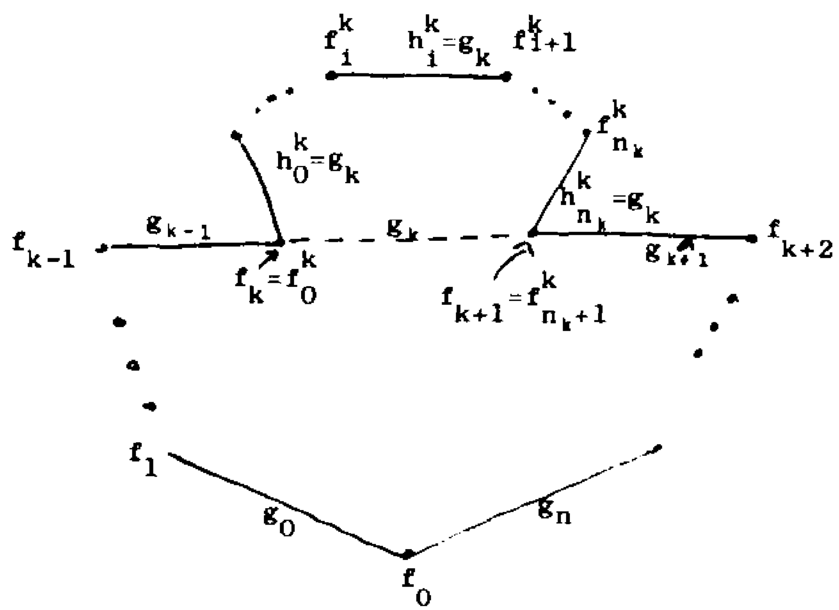
$$\{(f_i^k,\ h_i^k,\ f_{i+1}^k)\}_{i=0}^{n_k}$$

of edges with $f_0^k = f_k$, $f_{n_k+1}^k = f_{k+1}$ and $C(h_i^k, g_k) \neq \emptyset$ for $i=0,\dots,n_k$. As in the proof of part i) of this theorem (cf. remarks after condition 2)), that $C(h_i^k, g_k) \neq \emptyset$ is equivalent to $h_i^k = g_k$, $i=0,\dots,n_k$.

The juxtaposition, $S_1 =$

$$(f_0,g_0,f_1),\dots,(f_{k-1},g_{k-1},f_k),(f_0^k,h_0^k,f_1^k),\dots,$$

$$(f_{n_k}^k,h_{n_k}^k,f_{n_k+1}^k),(f_{k+1},g_{k+1},f_{k+2}),\dots,(f_n,g_n,f_0)\ ,$$



forms a circuit in $\Gamma(C)$. All of the edges

$$\{(f_i^k,\ h_i^k,\ f_{i+1}^k)\}_{i=0}^{n_k}$$

are in Γ and since the circuit S_1 is not all in Γ, some other edge

$$(f_j,\ g_j,\ f_{j+1})$$

for $j \neq k$ must be omitted.

By induction, each edge in S is replaced by a sequence of edges in Γ with middle terms equal to the middle term of the edge they replace. This process yields a circuit in Γ . Thus no generating graph is a tree. $\square$

<u>Corollary</u>: If C is an s-category with a generating tree, then every maximal tree in C is a generating tree.

<u>Proof</u>: By part ii) of Theorem 2, $\Gamma_g(C)$ is a tree. The corollary then follows from part i). $\square$

<u>Example</u> 1: A 2-dimensional s-category, C , may have a generating tree yet not be such that every maximal tree in $\Gamma(C)$ is a generating tree: C is the s-category based on $\Lambda = \{1,2,3,4,5\}$ with objects:

$$\left.\begin{array}{l} f_1\colon \{1\} \longrightarrow \Lambda \\ f_2\colon \{2\} \longrightarrow \Lambda \\ f_3\colon \{3\} \longrightarrow \Lambda \\ f_4\colon \{4\} \longrightarrow \Lambda \\ f_5\colon \{5\} \longrightarrow \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_6\colon \{1,2\} \longrightarrow \Lambda \\ f_7\colon \{1,3\} \longrightarrow \Lambda \\ f_8\colon \{2,3\} \longrightarrow \Lambda \\ f_9\colon \{2,4\} \longrightarrow \Lambda \\ f_{10}\colon \{3,4\} \longrightarrow \Lambda \\ f_{11}\colon \{3,5\} \longrightarrow \Lambda \\ f_{12}\colon \{4,5\} \longrightarrow \Lambda \end{array}\right\} = C_1$$

$$\left.\begin{array}{l} f_{13}\colon \{1,2,3\} \longrightarrow \Lambda \\ f_{14}\colon \{2,3,4\} \longrightarrow \Lambda \\ f_{15}\colon \{3,4,5\} \longrightarrow \Lambda \end{array}\right\} = C_2$$

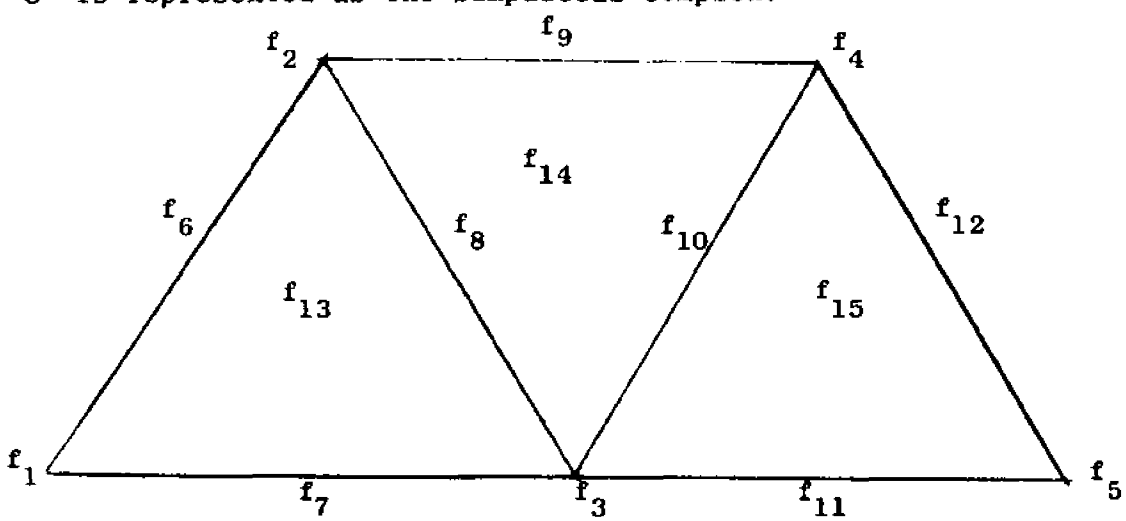
and non-identity morphisms:

$$f_{13} \to f_6, f_7, f_8 \ ; \ f_{14} \to f_8, f_9, f_{10} \ ; \ f_{15} \to f_{10}, f_{11}, f_{12}$$

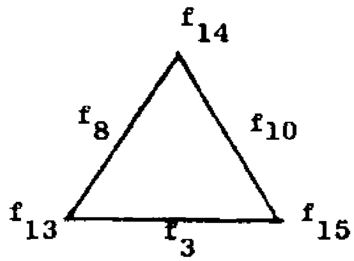
$$f_6 \to f_1, f_2 \ ; \ f_7 \to f_1, f_3 \ ; \ f_8 \to f_2, f_3 \ ; \ f_9 \to f_2, f_4 \ ;$$

$$f_{10} \to f_3, f_4 \ ; \ f_{11} \to f_3, f_5 \ ; \ f_{12} \to f_4, f_5 \ .$$

C is represented as the simplicial complex:



Then $\Gamma(C)$ is represented



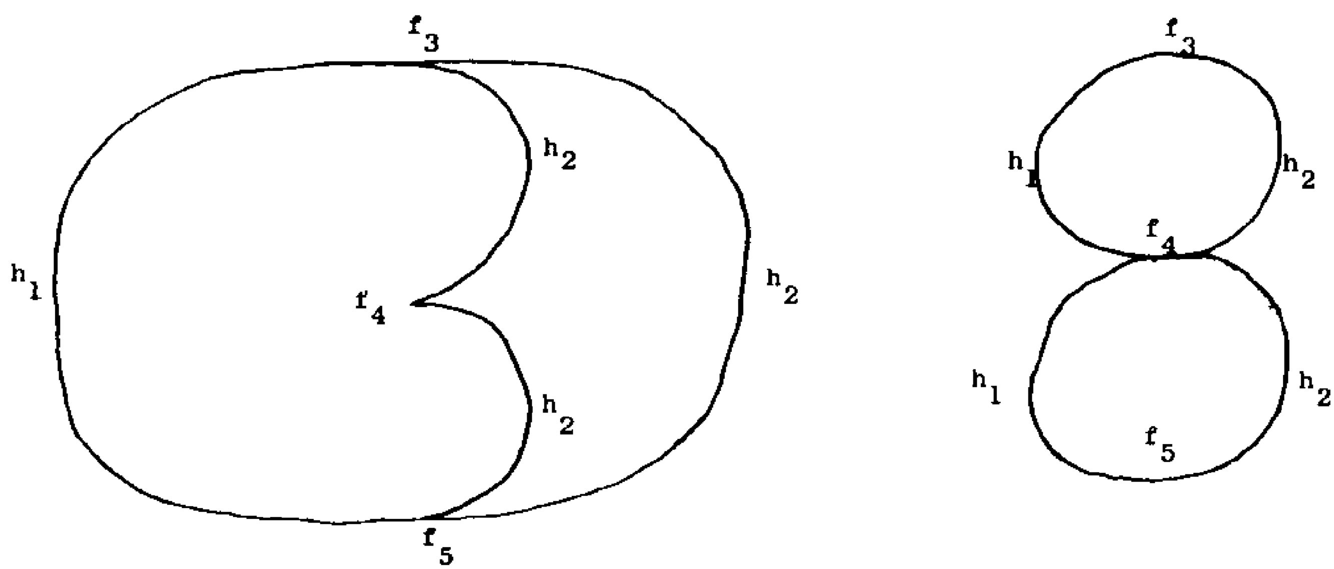
Then $\Gamma(C)$ contains a generating tree, $\{f_8, f_{10}\}$, but the maximal tree $\{f_8, f_3\}$ is not a generating graph since $C(f_3, f_{10}) = \emptyset$.

Even if the following generalization of Theorem 2 is true it will not be needed in what follows:

Conjecture: If C is a 1-dimensional s-category, then $\Gamma(C)$ contains a generating graph whose geometric realization has the same homotopy type as the geometrical realization of $\Gamma_g(C)$.

Example 2 (to illustrate the conjecture): The following comments refer to Example 4 in Section 1.

Since $\Gamma_g(C)$ is a figure eight, $\pi_1(\Gamma_g(C)) = \mathbb{Z} * \mathbb{Z}$. The selected generating graph is also a figure eight. To compute $\pi_1(\Gamma(C))$ the Seifert-van Kampen Theorem is used with the cover



$\pi_1(\Gamma(C))$ is then the push-out

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & \mathbb{Z} * \mathbb{Z} \arrow[d] \\
\mathbb{Z} * \mathbb{Z} \arrow[r] & \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \pi_1(\Gamma(C))
\end{tikzcd}$$

The remainder of this section relates the terminology of this paper to the terminology in the literature.

Definition: I is a set. The full Σ-category based on I , $\Sigma(I)$, has as objects singletons and unordered pairs of I :

$$ob\Sigma(I) = \{\{a,b\} \mid a,b \in I^2\} .$$

The non-identity morphisms of $\Sigma(I)$ are of the form

$$\{a,b\} \xrightarrow{\langle a,b\rangle} \{a\} \ ; \ \{a,b\} \xrightarrow{\langle b,a\rangle} \{b\}$$

for $a \neq b$. There is no composition except with identities.

A Σ-category based on I, C , is any full subcategory of $\Sigma(I)$ such that $\{a,b\} \in obC$ implies that $\{a\},\{b\} \in obC$.

If I is a set and C is a Σ-category based on I , then C can be given the structure of a 1-dimensional s-category based on I (in the obvious way) as follows:

To $\{a,b\} \in obC$, there corresponds the inclusion

$$\overline{\{a,b\}}: \{a,b\} \longrightarrow I$$

(The morphisms of C are the same as s-category and as Σ-category.)

If C is a Σ-category based on I , then for distinct elements $a,b \in I$, there is at most one object $\{a,b\}$ with $C(\{a,b\},\{a\}) \neq \emptyset$ and $C(\{a,b\},\{b\}) \neq \emptyset$. In an arbitrary 1-dimensional s-category there is no such restriction.

Proposition 2: If C is a one-dimensional s-category based on Λ , then C has the structure of a Σ-category based on Λ if and only if $\Gamma_g(C)$ has no circuits of length 2 .

Proof: The previous remark justifies the assertion that if C is a

Σ-category based on Λ then $\Gamma_g(C)$ has no circuits of length 2 .

Conversely, if C as an s-category has no circuits of length 2 , then the objects of C may be regarded as singletons and unordered pairs of elements of Λ . That is, an element $f \in C_1$ is completely determined by $\operatorname{im} f$. If $f \neq f'$ have $\operatorname{im} f = \operatorname{im} f'$, then they induce a circuit of length 2 in $\Gamma_g(C)$. Furthermore, if $f \in C_1$ has $\operatorname{im} f = \{a,b\}$, then there exist $g,g' \in C_0$ with $\operatorname{im} g = \{a\}$, $\operatorname{im} g' = \{b\}$. □

<u>Proposition 3</u>: If C is a 1-dimensional s-category, then $\Gamma(C)$ has no circuits of length two if and only if $\Gamma_g(C)$ has no circuits of length two.

<u>Proof</u>: $\Gamma(C)$ has a circuit of length two if and only if there exist distinct $f,f' \in \operatorname{ob}C_1$ and distinct $g,g' \in \operatorname{ob}C_0$ with $g,g' \in \operatorname{Fr}(f \cap f')$. This happens if and only if $C(f,g) \neq \emptyset$, $C(f,g') \neq \emptyset$, and $C(f',g) \neq \emptyset$, $C(f',g') \neq \emptyset$ with $f \neq f'$, $g \neq g'$. This is equivalent to the existence of the circuit, $\{f,f'\}$ of length two in $\Gamma_g(C)$ from the vertex g to the vertex g' . □

<u>Definition</u>: I is a set. C is a Σ-category based on I . The <u>Σ-graph of</u> C , $\Sigma(C)$, has as set of vertices, $\operatorname{ob}C$, and oriented edges corresponding to morphisms

$$\{a,b\} \longrightarrow \{a\}$$

for $a \neq b$.

It is clear that the Σ-graph of a Σ-category, C , is equivalent (as a graph) to the first barycentric subdivision of its geometric graph, $\Gamma_g(C)$, as a 1-dimensional s-category.

Thus, the Σ-graph of C and its geometric graph $\Gamma_g(C)$ have the same homotopy type and one is a tree if and only if the other is a tree. (Here homotopy type refers to the homotopy type of the geometric realization.)

Theorem 3. If C is a 1-dimensional s-category, then $\Gamma(C)$ has a generating tree if and only if C is a Σ-category and its Σ-graph, $\Sigma(C)$ has a simply connected geometric realization.

Proof: By Theorem 1, if $\Gamma(C)$ has a generating tree, then $\Gamma(C)$ has no circuits of length two. By Propositions 2 and 3, C has the structure of a Σ-category. Since $\Gamma(C)$ has a generating tree it follows from the Corollary to Theorem 2, that $\Gamma_g(C)$ is a tree. Thus $\Gamma_g(C)$ is simply connected. Since $\Sigma(C)$ and $\Gamma_g(C)$ have the same homotopy type, $\Sigma(C)$ is also simply connected.

If $\Sigma(C)$ has a simply connected geometric realization, then since $\Sigma(C)$ has no circuits of length two, its geometric realization is a one-dimensional simplicial complex and is thus contractible and thus $\Gamma_g(C)$ is a tree. Theorem 2 then implies that $\Gamma(C)$ has a generating tree. □

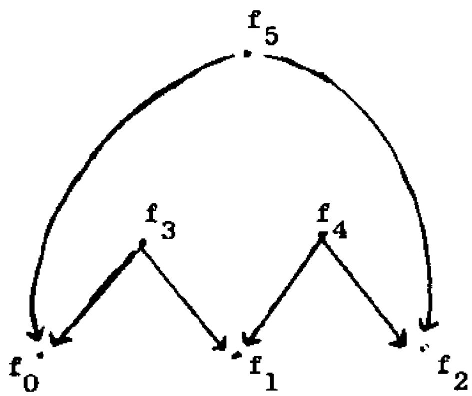
If C is a 1-dimensional s-category with a generating tree and $F: C \to Gp$ is a functor from C into the category of groups such that $F(\alpha)$ is injective for each morphism α in C, then it follows from Theorem 3 and a theorem on amalgamated products [7] that for each $g \in obC$ the colimit induced morphism $\psi_g: F(g) \to \operatorname{colim} F$ is injective.

This result is peculiar to 1-dimensional categories. If C is an n-dimensional s-category, $n \geq 2$, then there is a functor $F: C \to Gp$ such that $F(\alpha)$ is injective for each morphism in C but for some $g \in obC$ the colimit induced morphism $\psi_g: F(g) \to \operatorname{colim} F$ is not injective. This results from the following example:

Example 3: N is the 1-dimensional s-category based on $\Lambda = \{0,1,2\}$ with objects

$$\left.\begin{array}{l} f_0\colon \{0\} \longrightarrow \Lambda \\ f_1\colon \{1\} \longrightarrow \Lambda \\ f_2\colon \{2\} \longrightarrow \Lambda \end{array}\right\} = N_0 \qquad \left.\begin{array}{l} f_3\colon \{0,1\} \longrightarrow \Lambda \\ f_4\colon \{0,2\} \longrightarrow \Lambda \\ f_5\colon \{1,2\} \longrightarrow \Lambda \end{array}\right\} = N_1$$

and obvious morphisms. N is represented as a Σ-category by the drawing

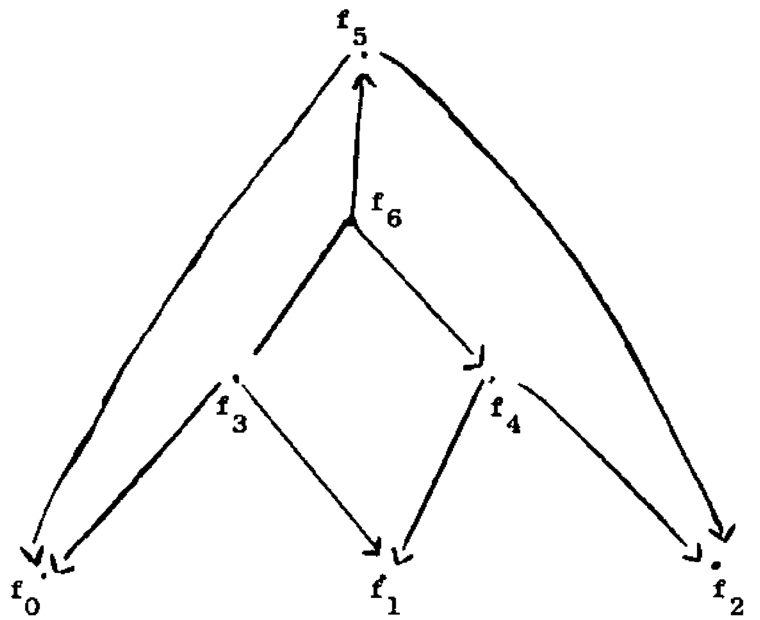


There is an example [D] of a functor, F, from this category into the category of groups such that $F(\alpha)$ is injective for each morphism α in N but some colimit induced morphism $\psi_g\colon F(g) \to \operatorname{colim} F$ is not injective.

The category N is enlarged to the category $\bar{N}$ by the addition of the object

$$f_6\colon \{0,1,2\} \longrightarrow \Lambda$$

and obvious morphisms. The morphisms in $\bar{N}$ are represented as arrows in the diagram



The functor $\bar{F}\colon \bar{N} \to Gp$ is the extension of F defined by setting $\bar{F}(f_6) = \{1\}$ and by sending the additional morphisms to the trivial inclusions. Clearly $\operatorname{colim} \bar{F} = \operatorname{colim} F$ and whichever ψ_g was not injective is still not injective.

If C is any n-dimensional s-category, $n \geq 2$, then $C_2 \neq \emptyset$ and for any $g \in C_2$, $\bar{N}$ is equivalent to $g \cap g$. The functor $\bar{F}\colon \bar{N} \to Gp$ can be extended trivially (as was F to yield $\bar{F}$) to yield $\hat{F}\colon C \to Gp$ which has some colimit morphism non-injective.

§3. Consistent Systems

In this section we restore the generality of Section 1 and consider an arbitrary s-category, C, and a functor $F: C \to Q$. Two questions are considered in this section; both relate to the assignment of base points to the spaces $F(g)$ for each $g \in \mathrm{ob}C$. The first is aimed at making Brown's fundamental groupoid [2] computable. The main result, Theorem 1, is of independent interest and is never referred to in this paper. The second concerns the definition of a fundamental group functor from C to Gp, the category of groups. In the next section the van Kampen Theorem is generalized to enable the computation of certain fundamental groups as colimits without the assumption that each element of the open cover contains the base point. In this section it is shown that the existence of a generating tree allows the consistent labeling of base points required in order to state that generalization.

We begin by defining a system of paths called the graph of F.

Definition: C is an s-category. A functor $F: C \to Q$ is path connected provided $F(g)$ is path connected for each $g \in \mathrm{ob}C$. If F is a path connected functor, then a graph of F is any graph, $\Gamma(F)$, derived by the following process:

i) $v\Gamma(F) = \{x_f\}_{f \in Fr(C)}$ is any collection of points such that $x_f \in F(f)$ for each $f \in Fr(C)$;

ii) if $x_f, x_{f'}$ are two vertices of $\Gamma(F)$ then for each $g \in Fr(f \cap f')$ there is an edge from x_f to $x_{f'}$ which is a path

$$\alpha(f,g,f') \in P(\bar{x}_f, F(g), \bar{x}_{f'})$$

(where $\bar{x}_f = F(\tau)x_f$ and $\bar{x}_{f'} = F(\tau')x_{f'}$, $\tau \in C(f,g)$, $\tau' \in C(f',g)$)

such that i) $\alpha(f,f,f)$ is the trivial path in $F(f)$ at x_f ,

ii) $\alpha(f',g,f) = [\alpha(f,g,f')]^{-1}$.

This last condition provides the required involution in $\Gamma(F)$.

Clearly, $\Gamma(F)$ and $\Gamma(C)$ are isomorphic graphs.

Definition: A groupoid is a small category in which every morphism is an equivalence. If $X \in \mathrm{ob}Q$, then the fundamental groupoid of X , πX , is the groupoid with $\mathrm{ob}\,\pi X = X$. If $x,y \in X$, then the morphism set $\pi X(x,y)$ consists of homotopy classes of paths which begin at x and end at y . Its importance lies in the fact that $\pi X(x,x) = \pi_1(X,x)$. $\mathcal{J}$ denotes the category of groupoids and $\pi\colon Q \to \mathcal{J}$ is the fundamental groupoid functor.

Proposition: Suppose $X \in \mathrm{ob}Q$; $x,y \in X$; and $\gamma \in \pi X(x,y)$ is a fixed path class. Then every element of $\pi X(x,y)$ is of the form $\alpha\gamma$ for some $\alpha \in \pi_1(X,x)$.

Proof: If $\beta \in \pi X(x,y)$, then $\alpha = \beta\gamma^{-1} \in \pi_1(X,x)$ and $\alpha\gamma = \beta$. □

Definition: If $x,y \in X \in \mathrm{ob}Q$, then a generator for $\pi X(x,y)$ is an element $\gamma \in \pi X(x,y)$.

Definition: C is an s-category and $F\colon C \to Q$ is a path connected functor with graph $\Gamma = \Gamma(F)$. The functor $F_\Gamma\colon C \to \mathcal{J}$ is defined as follows:

i) for $g \in \mathrm{ob}C$, $\mathrm{ob}F_\Gamma(g) = \{F(\tau_f)x_f \mid \tau_f \in C(f,g);\ x_f \in v\Gamma\}$

ii) for $\bar{x}_f = F(\tau_f)x_f$, $\bar{x}_{f'} = F(\tau_{f'})x_{f'} \in \mathrm{ob}F_\Gamma(g)$,

$$M(f,f') = \{PF(\tau_h)\alpha(f,h,f') \mid \tau_h \in C(h,g)\ ;\ \alpha(f,h,f') \in e\Gamma\}$$

$F_\Gamma(g)(\bar{x}_f,\bar{x}_{f'}) = \emptyset$ if $M(f,f') = \emptyset$. If $M(f,f') \neq \emptyset$, then

$F_\Gamma(g)(\bar{x}_f, \bar{x}_{f'}) = \pi F(g)(\bar{x}_f, \bar{x}_{f'})$ viewed as a groupoid generated by the path classes of elements in $M(f,f')$.

iii) If $x_{f_i} \in obF_\Gamma(g)$, $i=1,2,3$, and $M(f_1,f_2) \neq \emptyset$, $M(f_2,f_3) \neq \emptyset$ then there exist $h,h' \in obC$ such that $C(f_1,h)$, $C(h,g)$, $C(f_3,h')$, and $C(h',g)$ are non-empty. Thus $g \in ob(f_1 \cap f_3)$ and there exists an $h'' \in Fr(f_1 \cap f_3)$ with $C(h'',g) \neq \emptyset$. It follows that $M(f_1,f_3) \neq \emptyset$. Composition in $F_\Gamma(G)$ is then just composition in $\pi F(g)$.

iv) If $\tau \in C(g,g')$, then $F_\Gamma(\tau)$: $F_\Gamma(g')$ is the obvious functor induced by $F(\tau)$ and $PF(\tau)$.

<u>Definition</u>: A and B are groupoids. A functor $S: A \to B$ is an <u>equivalence of groupoids</u> provided

i) $S(x,y): A(x,y) \to B(Sx,Sy)$ is bijective for all $x,y \in obA$;

ii) For each $b \in obB$ there exists an $a \in obA$ such that either $B(Sa,b) \neq \emptyset$ or $B(b,Sa) \neq \emptyset$.

Two groupoids are <u>equivalent</u> if there exists such an equivalence between them.

<u>Theorem 1</u>. C is an upward finite s-category. $F: C \to Q$ is a path connected functor with graph Γ such that $F(\tau)$ is injective for each morphism τ in C. If $u: F_\Gamma \to \pi F$ is the obvious natural transformation induced by inclusions, then $\operatorname{colim} u: F_\Gamma \to \operatorname{colim} \pi F$ is an equivalence of groupoids.

<u>Proof</u>: By a theorem of Brown (see [2] or [3]) it suffices to show the following:

i) For each $g \in obC$, $u_g: F_\Gamma(g) \to \pi F(g)$ is an equivalence of groupoids.

Proof: a) If $\bar{x}_f, \bar{x}_{f'} \in \mathrm{ob}F_\Gamma(g)$, then $C(f,g)$ and $C(f',g)$ are non-empty. It follows that $g \in \mathrm{ob}(f \cap f')$ and there exists an $h \in \mathrm{Fr}(f \cap f')$ with $\alpha(f,h,f') \in M(f,f') \neq \emptyset$. Then $u_g(f,f')\colon F_\Gamma(g)(\bar{x}_f,\bar{x}_{f'}) \to \pi F(\bar{x}_f,\bar{x}_{f'})$ is the identity.

b) Since there is a free face, f, such that $C(f,g) \neq \emptyset$, there is an element $\bar{x}_f \in F_\Gamma(g)$. Since $F(g)$ is path connected, if $y \in F(g)$ then $\emptyset \neq \pi F(g)(\bar{x}_f,y) = \pi F(g)(u_g\bar{x}_f,y)$.

ii) For each $\tau \in C(g,g')$ the functions $\mathrm{ob}F_\Gamma\tau$ and $\mathrm{ob}\pi F\tau$ are injective and independent of τ.

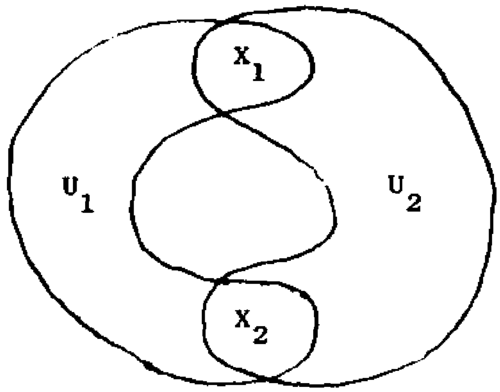
Proof: The first follows by hypothesis; the second follows from the fact that $|C(g,g')| \leq 1$. □

Since by another of Brown's theorems, [3], π colim F and colim πF are equivalent groupoids, this theorem has as corollary the equivalence of π colim F and colim F_Γ. In particular, each of their morphism sets are equivalent.

It is not our purpose to develop a precise characterization of colimits in the category of groupoids. Just as in the category of small categories, morphism sets of a colimit consist of equivalence classes of certain sequences of morphisms from each constituent category. The idea to keep in mind is the trivial case of single object groupoids. Such groupoids are just groups and the colimit reduces to a colimit in groups. In a single object groupoid, the morphisms are the elements of the group it represents. It follows that in this case the colimit consists of equivalence classes of words; i.e., sequences of morphisms.

An example is in order:

<u>Example</u> 1: Suppose X is the annulus

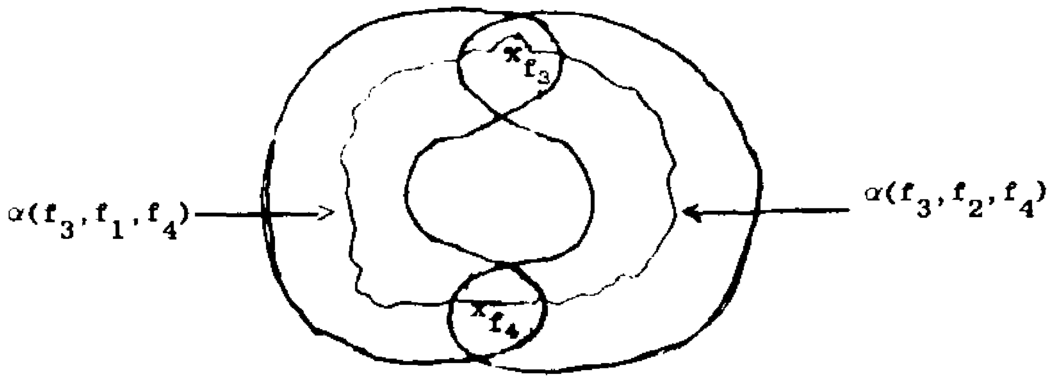


with the two piece open cover indicated. The associated s-category, C, is given in Example 1 of Section 1:

$$\left.\begin{array}{l} f_1 : \{1\} \longrightarrow \{1,2\} \\ f_2 : \{2\} \longrightarrow \{1,2\} \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_3 : \{1,2\}_1 \longrightarrow \{1,2\} \\ f_4 : \{1,2\}_2 \longrightarrow \{1,2\} \end{array}\right\} = C_1$$

$F(f_i) = U_i$, $i=1,2$; $F(f_3) = X_1$; $F(f_4) = X_2$.

A graph $\Gamma(F)$ may be chosen as in the picture



The functor F_Γ is then defined as follows:

$\mathrm{ob}F_\Gamma(f_1) = \{x_{f_3}, x_{f_4}\} = \mathrm{ob}F_\Gamma(f_2)$

$\mathrm{ob}F_\Gamma(f_3) = \{x_{f_3}\}$; $\mathrm{ob}F_\Gamma(f_4) = \{x_{f_4}\}$

$F_\Gamma(f_1)(x_{f_3}, x_{f_4}) = \pi F(f_1)(x_{f_3}, x_{f_4}) = \{[\alpha(f_3, f_1, f_4)]\}$

$F_\Gamma(f_1)(x_{f_3}, x_{f_4}) = \pi F(f_1)(x_{f_3}, x_{f_3})$

$$= \pi_1(F(f_1), x_{f_3}) = 1$$

$$F_\Gamma(f_1)(x_{f_4}, x_{f_4}) = \pi F(f_1)(x_{f_4}, x_{f_4})$$

$$= \pi_1(F(f_1), x_{f_4}) = 1$$

The square brackets denote homotopy class.

$$F_\Gamma(f_2)(x_{f_3}, x_{f_4}) = \pi F(f_2)(x_{f_3}, x_{f_4}) = \{[\alpha(f_3, f_2, f_4)]\}$$

$$F_\Gamma(f_2)(x_{f_3}, x_{f_3}) = 1$$

$$F_\Gamma(f_2)(x_{f_4}, x_{f_4}) = 1$$

$$F_\Gamma(f_3)(x_{f_3}, x_{f_3}) = 1 = F_\Gamma(f_4)(x_{f_4}, x_{f_4}) \ .$$

By the corollary to Theorem 1, $\pi X = \operatorname{colim} F_\Gamma$.

For example, $\pi_1(X, x_{f_3}) = \pi X(x_{f_3}, x_{f_3}) = \operatorname{colim} F_\Gamma(x_{f_3}, x_{f_3})$. This latter group consists of sequences of morphisms. Since there are only two non-identity morphisms, $\pi X(x_{f_3}, x_{f_3})$ consists of finite juxtapositions of those morphisms. In exactly this way, Brown computes $\pi_1(X, x_{f_3})$ to be $\mathbb{Z}$.

The process of determining the graph $\Gamma(F)$ only assigned base points to the spaces $F(f)$ for f a free face of C . To generalize the van-Kampen Theorem it will be necessary to assign base points to each value of F . A $\Gamma(F)$ -system is designed for that purpose.

Definition: C is an s-category and $F\colon C \to Q$ is a path connected functor with graph $\Gamma(F)$. A $\Gamma(F)$ -system is any graph $\Gamma_s(F)$ derived by the following process:

i) $v\Gamma_s(F) = \{y_g\}_{g \in obC}$ is any collection points such that for each $g \in obC$ there exists an $x_f \in v\Gamma(F)$ such that $y_g = F(\tau)x_f$ for $\tau \in C(f,g)$;

ii) if $g' \in obC$, then $B_{g'} = \{f \in Fr(C) \mid C(f,g') \neq \emptyset\}$. An

<u>elementary</u> g'-<u>path</u> is the image $PF(\tau)\alpha(f,h,f')$ of an edge $\alpha(f,h,f') \in e\Gamma(F)$ with $f,f' \in B_{g'}$ and $\tau \in C(h,g')$.

If $y_g, y_{g'} \in v\Gamma_s(F)$ and $C(g,g') \neq \emptyset$, then an edge from y_g to $y_{g'}$ is a finite composition of elementary g'-paths

$$\alpha(g,g') = \sum_{i=0}^{n} \alpha_i$$

such that a) one of the vertices of α_0 is x_f where

$y_g = F(\tau)x_f$, $\tau \in C(f,g)$;

b) one of the vertices of α_n is $x_{f'}$ where

$y_{g'} = F(\tau')x_{f'}$, $\tau \in C(f',g')$; and

c) if $g = g'$, then $\alpha(g',g')$ is the trivial

path at $y_{g'}$.

The inverse of an edge is defined in the obvious way by

$$\alpha(g',g) = \left(\sum_{i=0}^{n} \alpha_i\right)^{-1}$$

to provide the necessary involution in $\Gamma_s(F)$.

<u>Definition</u>: C is an s-category. $F\colon C \to Q$ is a path connected functor with graph, $\Gamma(F)$, and system, $\Gamma_s(F)$. $\{\varphi_g\colon F(g) \to \operatorname{colim} F\}_{g \in obC}$ is the set of morphisms induced by the colimit. A <u>base point</u> for $\Gamma_s(F)$ is the image $z_{g_*} = \varphi_{g_*}(y_{g_*})$ of some vertex $y_{g_*} \in v\Gamma_s(F)$, together with a family of paths

$$\{\alpha_g\}_{g \in obC}$$

such that

i) $\alpha_g \in P(\varphi_g\, y_g, \operatorname{colim} F, z_{g_*})$

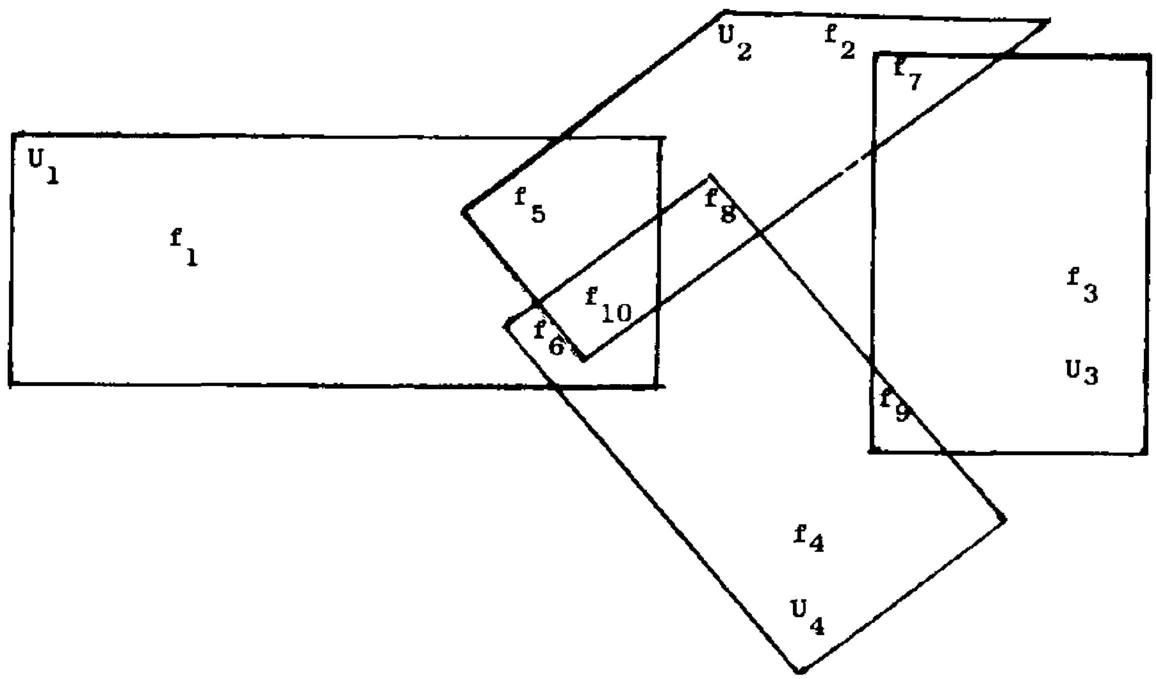
ii) α_{g_*} is the trivial path at z_{g_*}

iii) for each $g \in obC$, α_g is a finite composition

$$\alpha_g = \sum_{i=0}^{n-1} P\varphi_{g_{i+1}}(\alpha(g_i, g_{i+1}))$$

of images of edges in $\Gamma_s(F)$ such that one vertex of $\alpha(g_0, g_1)$ is y_g and one vertex of $\alpha(g_{n-1}, g_n)$ is y_{g_*}.

Example 2: X is the illustrated space with the four piece open cover as shown:

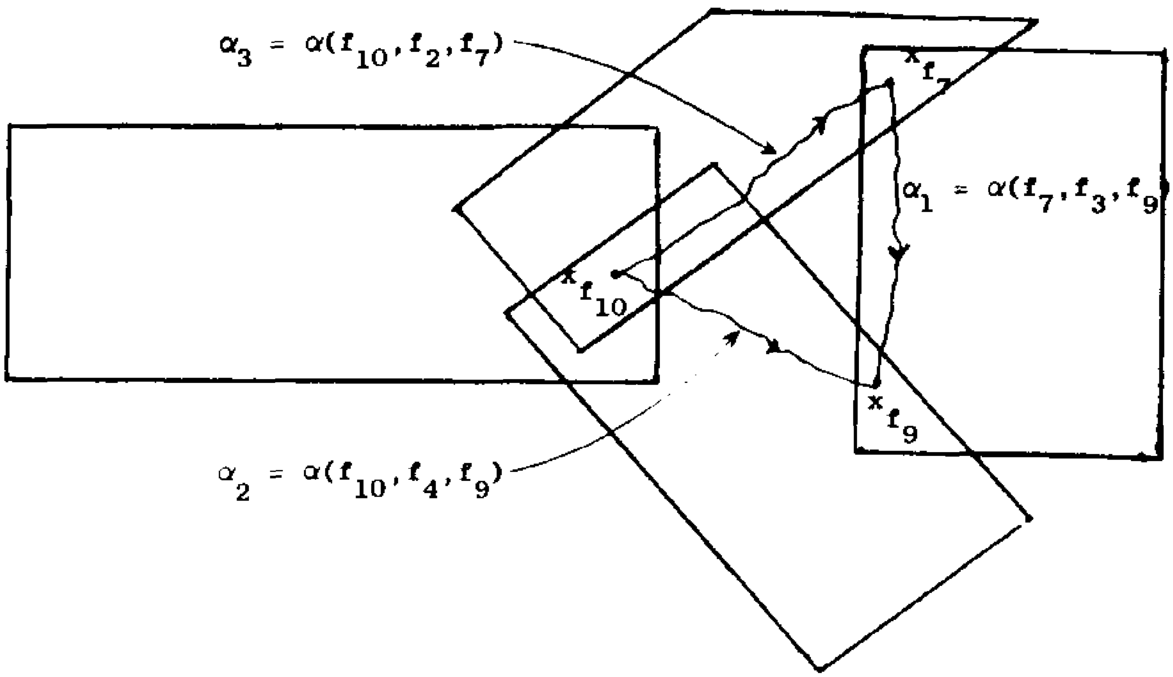


The appropriate s-category is based on $\Lambda = \{1,2,3,4\}$ and has objects

$$\left.\begin{array}{l} f_1: \{1\} \longrightarrow \Lambda \\ f_2: \{2\} \longrightarrow \Lambda \\ f_3: \{3\} \longrightarrow \Lambda \\ f_4: \{4\} \longrightarrow \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_5: \{1,2\} \longrightarrow \Lambda \\ f_6: \{1,4\} \longrightarrow \Lambda \\ f_7: \{2,3\} \longrightarrow \Lambda \\ f_8: \{2,4\} \longrightarrow \Lambda \\ f_9: \{3,4\} \longrightarrow \Lambda \end{array}\right\} = C_1 \qquad f_{10}: \{1,2,4\} \longrightarrow \Lambda\} =$$

The morphisms are obvious.

If F is the appropriate functor, then a graph, $\Gamma(F)$ for F may be represented



There are always many choices for a $\Gamma(F)$-system. One has vertices defined by the indicated $\Gamma(F)$ vertices:

$y_{f_1} = x_{f_{10}}$	$y_{f_6} = x_{f_{10}}$
$y_{f_2} = x_{f_7}$	$y_{f_7} = x_{f_7}$
$y_{f_3} = x_{f_9}$	$y_{f_8} = x_{f_{10}}$
$y_{f_4} = x_{f_9}$	$y_{f_9} = x_{f_9}$
$y_{f_5} = x_{f_{10}}$	$y_{f_{10}} = x_{f_{10}}$

The following table lists a possible choice of the required edges for a $\Gamma(F)$-system based on the previous choice of base points:

origin	terminal point	edge	origin	terminal point	edge
$y_{f_{10}}$	$y_{f_{10}}$	*	y_{f_7}	y_{f_7}	*
$y_{f_{10}}$	y_{f_8}	*	y_{f_7}	y_{f_3}	α_1
$y_{f_{10}}$	y_{f_6}	*	y_{f_7}	y_{f_2}	*
$y_{f_{10}}$	y_{f_5}	*	y_{f_6}	y_{f_6}	*
$y_{f_{10}}$	y_{f_4}	α_2	y_{f_6}	y_{f_4}	α_2
$y_{f_{10}}$	y_{f_2}	α_3	y_{f_6}	y_{f_1}	*
$y_{f_{10}}$	y_{f_1}	*	y_{f_5}	y_{f_5}	*
y_{f_9}	y_{f_9}	*	y_{f_5}	y_{f_2}	α_3
y_{f_9}	y_{f_4}	*	y_{f_5}	y_{f_1}	*
y_{f_9}	y_{f_3}	*	y_{f_4}	y_{f_4}	*
y_{f_8}	y_{f_8}	*	y_{f_3}	y_{f_3}	*
y_{f_8}	y_{f_4}	α_2	y_{f_2}	y_{f_2}	*
y_{f_8}	y_{f_2}	α_3	y_{f_1}	y_{f_1}	*

* denotes the trivial edge.

For a base point one could choose y_{f_1} and the system of paths listed in the following table:

$$\alpha_{f_1} = \alpha_3 \qquad \alpha_{f_6} = \alpha_3$$
$$\alpha_{f_2} = * \qquad \alpha_{f_7} = *$$
$$\alpha_{f_3} = \iota\alpha_1 \qquad \alpha_{f_8} = \alpha_3$$
$$\alpha_{f_4} = \iota\alpha_1 \qquad \alpha_{f_9} = \iota\alpha_1$$
$$\alpha_{f_5} = \alpha_3 \qquad \alpha_{f_{10}} = \alpha_3 .$$

There are two ways to proceed from $y_{f_{10}}$ to the base point. One is directly via $\alpha_{f_{10}} = \alpha_3$; the other is via the path, α_2 , from $y_{f_{10}}$ to y_{f_4} , and then via $\alpha_{f_4} = \iota\alpha_1$. These are homotopically distinct ways. In what follows base points are changed by using the isomorphisms induced by the paths in $\Gamma_s(F)$ and its base point. A system which has unique isomorphisms is called consistent. More precisely we have the following definition:

<u>Definition</u>: C is a small s-category. $F\colon C \to Q$ a functor with $T = (\Gamma_s(F); Z_{g_*}, \{\alpha_g\})$ a pointed $\Gamma(F)$ system. The system T is <u>consistent</u> if

i) for $g_i \in \operatorname{ob}C$ $i=0,1,2$ with $\sigma \in C(g_0,g_1)$, $\tau \in C(g_1,g_2)$ the paths

$$\alpha(g_0,g_2) \quad \text{and} \quad PF(\tau)(\alpha(g_0,g_1)) + \alpha(g_1,g_2)$$

are homotopic rel end points in $F(g_2)$ and

ii) for each pair $g,g' \in \operatorname{ob}C$ such that $C(g,g') \neq \emptyset$, the paths

$$\alpha_g \quad \text{and} \quad \alpha_{g'} + P\varphi_{g'}(\alpha(g,g'))$$

are homotopic rel endpoints in $\operatorname{colim} F$.

Sequences $\{\alpha_i\}_{i=0}^{n}$, $\{\beta_i\}_{i=0}^{n-2} \subseteq e\Gamma(F)$ are $\sim$ related if for some k, $0 \leq k \leq n$,

$$\{\alpha_i\}_{i=0}^{n} = \{\beta_0, \ldots, \beta_{k-1}, \alpha_k, \alpha_k^{-1}, \beta_k, \ldots, \beta_{n-2}\} .$$

The sequences are related if they are related by the equivalence relation generated by $\sim$.

The system T is <u>very consistent</u> if

i) for each $g_i \in \mathrm{ob}C$ $i=0,1,2$ with $\sigma \in C(g_0,g_1)$, $\tau \in C(g_1,g_2)$ the paths

$$\alpha(g_0,g_2) \quad \text{and} \quad PF(\tau)(\alpha(g_0,g_1)) + \alpha(g_1,g_2)$$

are such that some choice of sequences of edges of $\Gamma(F)$ which determine each are related.

ii) for $g,g' \in \mathrm{ob}C$ such that $C(g,g') \neq \emptyset$, the paths

$$\alpha_g \quad \text{and} \quad \alpha_{g'} + P\varphi_{g'}(\alpha(g,g'))$$

are such that some choice of sequences of edges of $\Gamma(F)$ which determine each are related.

<u>Theorem 2</u>. If the upward finite s-category C has a connected graph $\Gamma(C)$ with a generating tree Γ , then any path connected functor $F\colon C \to Q$ has a pointed very consistent system $T = (\Gamma_s(F); z_{g_*}, \{\alpha_g\})$.

<u>Proof</u>: i) $f_* \in Fr(C)$ is an arbitrarily selected base point. For $g \in \mathrm{ob}C$, $f_g \in Fr(C)$ is a free face such that

a) $C(f_g,g) \neq \emptyset$

b) If $f \in Fr(C)$ and $C(f,g) \neq \emptyset$, then the length of the geodesic (in Γ) from f_g to f_* is not greater than the length of the geodesic from f to g .

If $f \in Fr(C)$ also satisfies conditions a) and b), then since $C(f,g) \neq \emptyset$, $g \in ob(f \cap f_g)$ and there is an edge $h \in e\Gamma(C)$ from f to f_g, with $C(h,g) \neq \emptyset$. Since f and f_g are the same distance from f_* if $h \in e\Gamma$ then Γ would contain a non-trivial circuit. Thus $h \notin e\Gamma$, and there exists a finite sequence of edges of Γ

$$\{g_i \in Fr(f_i \cap f_{i+1})\}_{i=0}^{n}$$

such that $f_0 = f$, $f_{n+1} = f_g$ and $C(g_i,h) \neq \emptyset$ for $i=0,\dots,n$. If f_i, $0 < i < n$ is on either the geodesic from f_* to f or the geodesic from f_* to f_g, then since $C(f_i,g_i) \neq \emptyset$, $C(g_i,h) \neq \emptyset$ and $C(h,g) \neq \emptyset$ it follows that $C(f_i,g) \neq \emptyset$ and the distance from f_* to f_i is less than the distance from f_* to f. It follows that $\{g_i\}_{i=0}^{n}$ is a path in Γ from f to f_g distinct from the juxtaposition of the geodesics from them to f_*. Since this is impossible in a tree, conditions a) and b) determine a unique free face f_g for each $g \in obC$.

ii) For $f \in Fr(C)$, $x_f \in F(f)$ is an arbitrary point. For $g \in obC$, $y_g = F(\tau)x_{f_g}$ where $\tau \in C(f_g,g)$.

iii) For $f,f' \in Fr(C)$ and $g \in Fr(f \cap f')$ an edge of Γ, $\alpha(f,g,f')$ is any path in $F(g)$ from $F(\tau)x_f$ to $F(\tau)x_{f'}$ (where $\tau \in C(f,g)$, $\tau' \in C(f',g)$) except $\alpha(f',g,f) = [\alpha(f,g,f')]^{-1}$.

iv) For $g,g' \in obC$ and $\tau \in C(g,g')$, if $f_g = f_{g'}$, then $\alpha(g,g')$ is the trivial path in $F(g')$ at $y_{g'}$. If $f_g \neq f_{g'}$ since $g' \in ob(f_g \cap f_{g'})$ there is an $h \in Fr(f_g \cap f_{g'})$ such that $\sigma \in C(h,g') \neq \emptyset$. If $h \notin \Gamma$, then there is a finite sequence of edges (which may be assumed to be the geodesic):

$$\{g_i \in Fr(f_i \cap f_{i+1})\}_{i=0}^{n}$$

such that $f_0 = f_g$, $f_{n+1} = f_{g'}$ and $\sigma_i \in C(g_i,h) \neq \emptyset$ for $i=0,\ldots,n$. The composition

$$\alpha(g,g') = \sum_{i=0}^{n} PF(\sigma\sigma_i)\alpha(f_i,g_i,f_{i+1})$$

is a path in $F(g')$ from $F(\tau)y_g$ to $y_{g'}$.

v) If $g \in obC$, then there is a unique geodesic from f_g to f_*. The sequence

$$\{g_i \in Fr(f_i \cap f_{i+1})\}_{i=0}^{n}$$

consecutively labels the edges of the geodesic with $f_0 = f_g$ and $f_{n+1} = f_*$. If $\{\varphi_g\colon F(g) \to \operatorname{colim} F\}_{g\in obC}$ is the set of colimit induced morphisms, then the composition

$$\alpha_g = \sum_{i=0}^{n} P\varphi_{g_i}\alpha(f_i,g_i,f_{i+1})$$

is a path in colim F from $\varphi_g y_g$ to $z_{g_*} = \varphi_{f_*}x_{f_*}$.

The very nature of these constructions insures that they form a pointed system $T = (\Gamma_S(F);\ z_{g_*}, \{\alpha_g\})$.

vi) If $g_i \in obC$, $i=0,1,2$, with $\sigma \in C(g_0,g_1)$, $\tau \in C(g_1,g_2)$, then

$$\alpha(g_0,g_2) = \sum_{i=0}^{n_0} PF(\sigma_0\tau_{i0})\alpha(f_{i0},g_{i0},f_{i+1,0})$$

$$PF(\tau)\alpha(g_0,g_1) = \sum_{i=0}^{n_1} PF(\tau\sigma_1\tau_{i1})\alpha(f_{i1},g_{i1},f_{i+1,1})$$

$$\alpha(g_1,g_2) = \sum_{i=0}^{n_2} PF(\sigma_2\tau_{i2})\alpha(f_{i2},g_{i2},f_{i+1,2})$$

for $h_0 \in Fr(f_{g_0} \cap f_{g_2})$, $h_1 \in Fr(f_{g_0} \cap f_{g_1})$, $h_2 \in Fr(f_{g_1}, f_{g_2})$

$$\sigma_0 \in C(h_0, g_2) \ , \ \sigma_1 \in C(h_1, g_1) \ , \ \sigma_2 \in C(h_2, g_2)$$

$$\tau_{ij} \in C(g_{ij}, h_j), \ i=0,\ldots,n_j \quad j=0,1,2 \ .$$

$\alpha(f_{ij}, g_{ij}, f_{i+1,j})$ a path in $F(g_{ij})$ from $F(\gamma_{ij})x_{f_{ij}}$ to $F(\gamma_{i+1,j})x_{f_{i+1,j}}$ for $\gamma_{ij} \in C(f_{ij}, g_{ij})$, $i=0,\ldots,n_j$, $j=0,1,2$.

The existence of these families of paths implies the existence of families of edges of Γ

$$\{g_{ij} \in Fr(f_{ij} \cap f_{i+1,j})\}_{i=0}^{n_j}, \ j=0,1,2$$

with

$$f_{g_0} = f_{00} = f_{1,0}$$

$$f_{g_1} = f_{n_1+1,1} = f_{0,2}$$

$$f_{g_2} = f_{n_0+1,0} = f_{n_2+1,2}$$

Since Γ is a tree, the path

$$\{g_{i1}\}_{i=0}^{n_1} \cup \{g_{i2}\}_{i=0}^{n_2}$$

can be reduced to the geodesic

$$\{g_{i0}\}_{i=0}^{n_0}$$

by elimination of go-returns. (A <u>go-return</u> in a path is a pair of consecutive edges of the form f , ιf for ι the involution in the graph.) That is, for some $k \geq 0$

$$g_{n_1-i,1} = (g_{i2})^{-1}, \quad i=0,\ldots,k-1$$

but

$$g_{n_1-k,1} = (g_{k2})^{-1}$$

(where the inverse refers to the involution in Γ) and

$$\left\{g_{i,0}\right\}_{i=0}^{n_0} = \left\{g_{i1}\right\}_{i=0}^{n_1-k} \cup \left\{g_{i2}\right\}_{i=k}^{n_2} .$$

The construction of the paths $\alpha(f_{ij},g_{ij},f_{i+1,j})$ for $i=0,\ldots,n_j$, $j=0,1,2$ guarantees that the same juxtaposition of inverses occurs in the composition

$$PF(\tau)\alpha(g_0,g_1) + \alpha(g_1,g_2) \sim \alpha(g_0,g_2) .$$

vii) If $g,g' \in \mathrm{ob}C$ and $\tau \in C(g,g')$, then

$$\alpha_g = \sum_{i=0}^{n} P\varphi_{g_i}\alpha(f_i,g_i,f_{i+1})$$

$$\alpha_{g'} = \sum_{i=0}^{n_1} P\varphi_{g'_1}\alpha(f_{i,1},g_{i,1},f_{i+1,1})$$

and

$$P\varphi_{g'}\alpha(g,g') = \sum_{i=0}^{n_2} P\varphi_{g_{i2}}\ \alpha(f_{i,2},g_{i,2},f_{i+1,2}) ;$$

the last equality following from the diagram

$$\begin{tikzcd}
F(g_{i,2}) \arrow[d] \arrow[dr, "\varphi_{g_{i2}}"] & \\
F(h) \arrow[d, "F(\sigma)"] & \mathrm{colim}\ F \\
F(g') \arrow[ur, "\varphi_{g'}"'] &
\end{tikzcd}
\qquad i=0,\ldots,m .$$

As before, the existence of these paths implies the existence of families

$$\{g_i \in Fr(f_i \cap f_{i+1})\}_{i=0}^{n}$$

$$\{g_{ij} \in Fr(f_{ij} \cap f_{i+1,j})\}_{i=0}^{n_j} \quad j=1,2 \ .$$

with

$$f_* = f_{n+1} = f_{n_1+1,1}$$

$$f_g = f_0 = f_{0,2}$$

$$f_{g'} = f_{0,1} = f_{n_2+1,2}$$

The proof concludes as in part vi) above. □

Remark: This theorem indicates that if a category has a generating tree then the existence of very consistent systems is assured independently of any functor. While some categories do not have generating trees, it may be that particular functors defined from them have consistent systems. For example, in a simply connected space any system is consistent. The necessary hypothesis in what follows is merely that the system be consistent.

It is now possible to define the fundamental group functor for s-categories with consistent systems.

Definition: C is a small s-category. $F: C \to Q$ is a functor with a pointed consistent $\Gamma(F)$ -system

$$T = (\Gamma_s(F) \ ; \ z_{g_*} \ , \ \{\alpha_g\}).$$

The functor, F_T: $C \to Gp$, from C into the category of groups is defined as follows:

i) for $g \in obC$,

$$F_T(g) = \pi_1(F(g), y_g)$$

for $y_g \in v(\Gamma_s(F))$.

ii) For $g, g' \in \mathrm{ob}C$ and $\sigma \in C(g,g')$, the homomorphism $F_T(\sigma)$ is the composition

$$\pi_1(F(g), y_g) \xrightarrow{\pi_1 F(\sigma)} \pi_1(F(g'), \bar{y}_g) \xrightarrow{\overline{\alpha(g,g')}} \pi_1(F(g'), y_{g'})$$

where $\bar{y}_g = F(\sigma)y_g$ and $\overline{\alpha(g,g')}$ is the isomorphism induced by the path $\alpha(g,g')$.

Proposition 1: The conditions of the previous definition define a functor F_T: $C \to Gp$.

Proof: For $g_i \in \mathrm{ob}C$, $i=0,1,2$ and $\sigma \in C(g_0,g_1)$, $\tau \in C(g_1,g_2)$, $F_T(\tau)F_T(\sigma)$ is the composition

$$\begin{tikzcd}
\pi_1(F(g_0), y_{g_0}) \arrow[r, "\pi_1 F(\sigma)"] & \pi_1(F(g_1), \bar{y}_{g_0}) \arrow[d, "\overline{\alpha(g_0,g_1)}"] \\
& \pi_1(F(g_1), y_{g_1}) \arrow[d, "\pi_1 F(\tau)"] \\
& \pi_1(F(g_2), \bar{y}_{g_1}) \arrow[d, "\overline{\alpha(g_1,g_2)}"] \\
& \pi_1(F(g_2), y_{g_2}) \ .
\end{tikzcd}$$

Since $\pi_1 F(\tau)$ is a homomorphism, the composition

$$\pi_1(F(g_1), \bar{y}_{g_0}) \xrightarrow{\overline{\alpha(g_0,g_1)}} \pi_1(F(g_1), y_{g_1}) \xrightarrow{\pi_1 F(\tau)} \pi_1(F(g_2), \bar{y}_{g_1})$$

is equal to the composition

$$\pi_1(F(g_1), \bar{y}_{g_0}) \xrightarrow{\pi_1 F(\tau)} \pi_1(F(g_2), \bar{\bar{y}}_{g_0}) \xrightarrow{\overline{PF(\tau)(\alpha(g_0,g_1))}} \pi_1(F(g_2), \bar{y}_{g_1})$$

for $\bar{\bar{y}}_{g_0} = F(\tau)\bar{y}_{g_0}$.

The composition

$$\pi_1(F(g_2),\bar{\bar{y}}_{g_0}) \xrightarrow{PF(\tau)(\alpha(g_0,g_1))} \pi_1(F(g_2),\bar{y}_{g_1}) \xrightarrow{\overline{\alpha(g_1,g_2)}} \pi_1(F(g_2),y_{g_2})$$

is the same as the isomorphism

$$\pi_1(F(g_2),\bar{\bar{y}}_{g_0}) \xrightarrow{\overline{PF(\tau)(\alpha(g_0,g_1))+\alpha(g_1,g_2)}} \pi_1(F(g_2),y_{g_2})$$

which equals the isomorphism

$$\pi_1(F(g_2),\bar{\bar{y}}_{g_0}) \xrightarrow{\overline{\alpha(g_0,g_2)}} \pi_1(F(g_2),y_{g_2})$$

by the consistency of T . The proposition follows. □

<u>Definition</u>: C is a small s-category. $F\colon C \to Q$ is a functor with a consistent system $T = (\Gamma_g(F);\ z_{g_*}, \{\alpha_g\})$. $\{\varphi_g\colon F(g) \to \mathrm{colim}\ F\}$ are the morphisms induced by the colimit. The homomorphisms $h_g\colon F_T(g) \to \pi_1(\mathrm{colim}\ F, z_{g}$ are defined by the composition

$$\pi_1(F(g),y_g) \xrightarrow{\pi_1\varphi_g} \pi_1(\mathrm{colim}\ F,\varphi_g y_g) \xrightarrow{\bar{\alpha}_g} \pi_1(\mathrm{colim}\ F,z_{g_*}) ,$$

where $\bar{\alpha}_g$ is the isomorphism induced by the path α_g .

Of obvious interest is the relationship between $\mathrm{colim}\ F_T$ and $\pi_1(\mathrm{colim}\ F,z_{g_*})$.

<u>Proposition 2</u>: The family $\{h_g\colon F_T(g) \to \pi_1(\mathrm{colim}\ F,z_{g_*})\}_{g\in \mathrm{ob}C}$ as defined above is compatible and thus induces a homomorphism $h\colon \mathrm{colim}\ F_T \to \pi_1(\mathrm{colim}\ F,z$

<u>Proof</u>: For $g,g' \in \mathrm{ob}C$ and $\sigma \in C(g,g')$ the diagram

$$\begin{tikzcd}
F_T(g) \arrow[dr, "h_g"] \arrow[d, "F_T(\sigma)"'] & \\
F_T(g') \arrow[r, "h_{g'}"'] & \pi_1(\mathrm{colim}\ F,z_{g_*})
\end{tikzcd}$$

is the diagram

$$\begin{tikzcd}
\pi_1(F(g),y_g) \arrow[r, "\pi_1\varphi_g"] \arrow[d, "\pi_1(F\sigma)"] & \pi_1(\mathrm{colim}\ F,\ \varphi_g y_g) \arrow[dr, "\bar\alpha_g"] & \\
\pi_1(F(g'),\bar y_g) \arrow[d, "\overline{\alpha(g,g')}"] & & \pi_1(\mathrm{colim}\ F,\ z_{g_*}) \\
\pi_1(F(g'),y_{g'}) \arrow[r, "\pi_1\varphi_{g'}"] & \pi_1(\mathrm{colim}\ F,\ \varphi_{g'} y_{g'}) \arrow[ur, "\bar\alpha_{g'}"'] &
\end{tikzcd}$$

Since $\pi_1\varphi_{g'}$ is a homomorphism the composition

$$\pi_1(F(g'),\bar y_g) \xrightarrow{\overline{\alpha(g,g')}} \pi_1(F(g'),y_{g'}) \xrightarrow{\pi_1\varphi_{g'}} \pi_1(\mathrm{colim}\ F,\ \varphi_{g'}y_{g'})$$

equals the composition

$$\pi_1(F(g'),\bar y_g) \xrightarrow{\pi_1\varphi_{g'}} \pi_1(\mathrm{colim}\ F,\ \varphi_{g'}\bar y_g) \xrightarrow{\overline{P\varphi_{g'}(\alpha(g,g'))}} \pi_1(\mathrm{colim}\ F, \varphi_{g'}y_{g'})$$

and the diagram becomes

$$\begin{tikzcd}
\pi_1(F(g),y_g) \arrow[r, "\pi_1\varphi_g"] \arrow[d, "\pi_1(F\sigma)"] & \pi_1(\mathrm{colim}\ F,\ \varphi_{g'}\bar y_g) \arrow[d, "\overline{P\varphi_{g'}(\alpha(g,g'))}"] \arrow[dd, bend left=60, "\bar\alpha_g"] \\
\pi_1(F(g'),y_{g'}) \arrow[ur, "\pi_1\varphi_{g'}"] & \pi_1(\mathrm{colim}\ F,\ \varphi_{g'}\ y_{g'}) \arrow[d, "\bar\alpha_{g'}"] \\
 & \pi_1(\mathrm{colim}\ F,\ z_{g_*})
\end{tikzcd}$$

which commutes since T is consistent. □

§4. A Seifert-van Kampen Theorem.

We are now ready to generalize the Seifert-van Kampen Theorem to the case when not all the elements in the cover contain the base point. It will be necessary to assume that the category which represents the cover has a generating tree. It will also be necessary to assume that the category accounts for all the intersections of elements in the cover. This condition is expressed in terms of the associated functor.

Definition: C is a small s-category and $F\colon C \to Q$ is a functor with colimit induced morphisms $\{\varphi_g\colon F(g) \to \operatorname{colim} F\}_{g\in \mathrm{ob}C}$. F is closed under finite intersections if for each pair $g, g' \in \mathrm{ob}C$ with

$$\varphi_g F(g) \cap \varphi_{g'} F(g') \neq \emptyset ,$$

there exists a set Δ and a family $\{f_\delta \in \mathrm{ob}C\}_{\delta\in\Delta}$ such that

i) $C(f_\delta, g)$ and $C(f_\delta, g')$ are non-empty for each $\delta \in \Delta$;

ii) $\varphi_{f_\delta} F(f_\delta)$ is a path component of $\varphi_g F(g) \cap \varphi_{g'} F(g')$ for each $\delta \in \Delta$;

iii) $\bigcup_{\delta\in\Delta} \varphi_{f_\delta} F(f_\delta) = \varphi_g F(g) \cap \varphi_{g'} F(g')$.

The following technical lemma will be needed in the proof of the Theorem. Basically it asserts that if the associated category and functor account for the intersection of any two elements in the cover, then they account for the intersection of any four elements.

Lemma 1. C is an s-category and $F\colon C \to Q$ is a functor with colimit induced morphisms $\{\varphi_g\colon F(g) \to \operatorname{colim} F\}$. If F is closed under finite intersections, then for each quadruple $\{g_i \in \mathrm{ob}C\}_{i=0}^{3}$ with

$$\bigcap_{i=0}^{3} \varphi_{g_i} F(g_i) \neq \emptyset$$

there exists a set Δ and a family $\{f_\delta \in \mathrm{ob}C\}_{\delta\in\Delta}$ such that

i) $\varphi_{f_\delta} F(f_\delta) \subseteq \bigcap_{i=0}^{3} \varphi_{g_i} F(g_i)$ for all $\delta \in \Delta$;

ii) $C(f_\delta, g_i) \neq \emptyset$ for all $\delta \in \Delta$, $i=0,1,2,3$;

iii) $\bigcup_{\delta\in\Delta} \varphi_{f_\delta} F(f_\delta) = \bigcap_{i=0}^{3} \varphi_{g_i} F(g_i)$;

iv) if $j,j' \in \{0,1,2,3\}$ then there exists a set $A_{jj'}$ and a family $\{h_\alpha \in \mathrm{ob}C\}_{\alpha\in A_{jj'}}$ such that

a) $C(h_\alpha, g_j) \neq \emptyset$ and $C(h_\alpha, g_{j'}) \neq \emptyset$ for all $\alpha \in A_{jj'}$;

b) for each $\alpha \in A_{jj'}$, $\varphi_{h_\alpha} F(h_\alpha)$ is a path component of

$\varphi_{g_j} F(g_j) \cap \varphi_{g_{j'}} F(g_{j'})$;

c) for each $\delta \in \Delta$ there exists an $\alpha \in A_{jj'}$ such that $C(f_\delta, h_\alpha) \neq \emptyset$.

Proof: The sequence $A = \{a_i\}_{i=0}^{5}$ is the sequence of ordered pairs

$$\{(0,1),\ (0,2),\ (0,3),\ (1,2),\ (1,3),\ (2,3)\} .$$

Since F is closed under finite intersections, there exist sets B_a for $a \in A$ and families

$$\{h_\alpha^a\}_{\alpha\in B_a}$$

that satisfy conditions iva), ivb), and

$$\bigcup_{\alpha\in B_a} \varphi_{h_\alpha^a} F(h_\alpha^a) = \varphi_{g_i} F(g_i) \cap \varphi_{g_j} F(g_j) \quad \text{for } (i,j) = a .$$

$$C_i = \left\{ h_\alpha^{a_i}, h_\alpha^{a_{i+1}} \,\middle|\, \varphi_{h_\alpha}^{a_i} F(h_\alpha^{a_i}) \cap \varphi_{h_\alpha}^{a_{i+1}} F(h_\alpha^{a_{i+1}}) \neq \emptyset \right\} , \; i=0,2,4 .$$

Since F is closed under finite intersections, there exist sets D_c for $c \in C_i$, $i=0,2,4$ and families

$$\left\{ h_\delta^c \right\}_{\delta \in D_c}$$

that satisfy conditions iva), ivb), and

$$\bigcup_{\delta \in D_c} \varphi_\delta^{c_i} F(h_\delta^{c_i}) = \varphi_{h_\alpha}^{a_i} F(_\alpha^{a_i}) \cap \varphi_{h_\alpha}^{a_{i+1}} F(h_\alpha^{a_{i+1}}) .$$

$$E_i = \left\{ h_\delta^{c_i}, h_\delta^{c_{i+2}} \,\middle|\, \varphi_{h_\delta}^{c_i} F(h_\delta^{c_i}) \cap \varphi_{h_\delta}^{c_{i+2}} F(h_\delta^{c_{i+2}}) \neq \emptyset \right\} , \; i=0,2.$$

where

$$h_\delta^{c_i} \in \left\{ h_\delta^{c_i} \right\}_{\delta \in D_{c_i}} \quad \text{for} \quad c_i \in C_i .$$

Since F is closed under finite intersections, there exist sets f_e for $e \in E_i$ $i=0,2$ and families

$$\left\{ h_\varphi^e \right\}_{\varphi \in F_e}$$

that satisfy conditions iva), ivb), and

$$\bigcup_{\varphi \in F_e} \varphi_{h_\varphi^e} F(h_\varphi^e) = \varphi_{h_\delta}^{c_i} F(h_\delta^{c_i}) \cap \varphi_{h_\delta}^{c_{i+2}} F(h_\delta^{c_{i+2}})$$

$$G = \left\{ h_\varphi^{e_0}, h_\varphi^{e_2} \,\middle|\, \varphi_{h_\varphi}^{e_0} F(h_\varphi^{e_0}) \cap \varphi_{h_\varphi}^{e_2} F(h_\varphi^{e_2}) \neq \emptyset \right\}$$

where

$$h_\varphi^{e_i} \in \left\{ h_\varphi^{e_i} \right\}_{\varphi \in F_{e_i}} \quad \text{for} \quad e_i \in E_i \quad i=0,2 .$$

Since F is closed under finite intersections, there exist sets H_g

for $g \in G$ and families

$$\{h^g_\psi\}_{\psi \in H_g}$$

that satisfies conditions iva), and ivb), and

$$\bigcup_{\psi \in H} \varphi_{h^g_\psi} F(h^g_\psi) = \varphi^{e_0}_{h_\varphi} F(h^{e_0}_\varphi) \cap \varphi^{e_2}_{h_\varphi} F(h^{e_2}_\varphi) .$$

The unions $\bigcup_{g \in G} H_g$ and $\bigcup_{g \in G} \{h^g_\psi\}_{\psi \in H_g}$ are a set and a family of objects of C .

i) With each object h^g_ψ of the union is associated a sequence

$$h^g_\psi,\ (h^{e_0}_\varphi, h^{e_2}_\varphi),\ (h^{c_0}_\delta, h^{c_2}_\delta\ ;\ (h^{c_2}_\delta)',\ h^{c_4}_\delta)\ ,$$

$$\left(h^{a_0}_\alpha, h^{a_1}_\alpha\ ;\ h^{a_2}_\alpha, h^{a_3}_\alpha\ ;\ (h^{a_2}_\alpha)',\ (h^{a_3}_\alpha)'\ ;\ h^{a_4}_\alpha, h^{a_5}_\alpha\right)$$

with 14 non-empty morphism sets of the type $C(h^{c_2}_\delta, h^{a_3}_\alpha)$ and 14 inclusions of the type

$$\varphi_{h^{c_0}_\delta} F(h^{c_0}_\delta) \subseteq \varphi^{a_0}_{h_\alpha} F(h^{a_0}_\alpha) \cap \varphi^{a_1}_{h_\alpha} F(h^{a_1}_\alpha) .$$

These yield conditions i), ii), and iv) of the lemma together with the inclusion:

$$\bigcup \varphi_{h^g_\psi} F(h^g_\psi) \subseteq \bigcap_{i=0}^{3} \varphi_{g_i} F(g_i) .$$

If $x \in \bigcap_{i=0}^{3} \varphi_{g_i} F(g_i)$, then since the $h^{a_i}_\alpha$ exhaust their respective intersections there exist six elements

$$h^{a_0}_{\alpha_0},\ h^{a_1}_{\alpha_1},\ h^{a_2}_{\alpha_2},\ h^{a_3}_{\alpha_3},\ h^{a_4}_{\alpha_4},\ h^{a_5}_{\alpha_5}$$

such that $x \in \varphi_{h^{a_i}_{\alpha_i}} F(h^{a_i}_{\alpha_i})$ for $i=0,\ldots,5$. Since the $h^{c_i}_{\delta}$ exhaust their respective intersections, there exist three elements

$$h^{c_0}_{\delta_0},\ h^{c_2}_{\delta_2},\ h^{c_4}_{\delta_4}$$

such that $x \in \varphi_{h^{c_i}_{\delta_i}} F(h^{c_i}_{\delta_i})$ $i=0,2,4$. Similarly there exist two elements

$$h^{e_0}_{\varphi_0},\ h^{e_2}_{\varphi_2}$$

such that $x \in \varphi_{h^{e_0}_{\varphi_0}} F(h^{e_0}_{\varphi_0})$. Finally, there exists an element

$$h^{g}_{\psi}$$

such that $x \in \varphi_{h^{g}_{\psi}} F(h^{g}_{\psi})$. □

The proof of the following theorem is a modified version of a proof in Massey ([9], pages 116-122). His notation and order of presentation are adhered to whenever possible. In particular, the proof is accomplished by two lemmas.

<u>Theorem 1</u>. C is a small s-category. $F: C \to Q$ is a functor which is closed under finite intersections and has a consistent system $T = (\Gamma_s(F); z_{g_*}, \{\alpha_g\})$. $\{\varphi_g : F(g) \to \text{colim } F\}$ are the morphisms induced by the colimit. If for each $g \in \text{ob}C$, φ_g is an injection and if $\{\varphi_g F(g)\}$ forms an open cover of colim F , then

$$h: \text{colim } F_T \approx \pi_1(\text{colim } F, z_{g_*}) \text{ . } \square$$

<u>Lemma 2</u>. [cf. Massey's Lemma 2.3]. The homomorphism, h , is onto.

<u>Proof</u>: For $\alpha \in \pi_1(\text{colim } F, z_{g_*})$, $f: [0,1] \to \text{colim } F$ is a closed

unit length path representing α . ϵ is the Lebesque number of the cover

$$\{f^{-1}\varphi_g F(g)\}_{g\in obC} .$$

$[0,1]$ is subdivided into closed subintervals

$$J_i = [i/n,(i+1)/n] , 0 \le i \le n-1$$

with n so large that $1/n < \epsilon$. Then $f(J_i) \subseteq \varphi_{g_i} F(g_i)$ for some $\{g_i\}_{i=0}^{n-1} \subseteq obC$.

$\alpha_i = P(\varphi_{g_i})^{-1}(f|J_i)$ is a path in $F(g_i)$. For each i , $1 \le i \le n-1$, $f(i/n) \in \varphi_{g_{i-1}} F(g_{i-1}) \cap \varphi_{g_i} F(g_i) \neq \emptyset$ and there is an element $h_i \in obC$ such that

i) $\sigma_i \in C(h_i,g_{i-1})$ and $\tau_i \in C(h_i,g_i)$ are non-empty and

ii) $f(i/n) \in \varphi_{h_i} F(h_i) \subseteq \varphi_{g_{i-1}} F(g_{i-1}) \cap \varphi_{g_i} F(g_i)$

$\gamma_i \in P(y_{h_i}, F(h_i), \varphi_{h_i}^{-1} f(i/n))$ is an arbitrary path except it is trivial if possible. By definition $y_{g_*} = y_{h_0} = y_{h_n}$.

The composition

$$\alpha(h_i,g_i)^{-1} P(\tau_i)\gamma_i \ \alpha_i \ P(\sigma_{i+1}) \ \gamma_{i+1}^{-1} \ \alpha(h_{i+1},g_i)$$

(for $\alpha(h_i,g_i),\alpha(h_{i+1},g_i) \in e\Gamma_s(F)$) is a loop in $\Omega(F(g_i),y_{g_i})$ and the product

$$\prod_{i=0}^{n-1} \alpha(h_i,g_i)^{-1} P(\tau_i)\gamma_i\alpha_i \ P(\sigma_{i+1})\gamma_{i+1}^{-1} \ \alpha(h_{i+1},g_i)$$

is the representative of an element in colim F_T .

$$h\left(\prod_{i=0}^{n-1} \alpha(h_i,g_i)^{-1}\, P(\tau_i)\gamma_i\alpha_i\, P(\sigma_{i+1})\gamma_{i+1}^{-1}\, \alpha(h_{i+1},g_i)\right)$$

$$= \sum_{i=0}^{n-1} h_{g_i}\left(\alpha(h_i,g_i)^{-1}\, P(\tau_i)\gamma_i\alpha_i\, P(\sigma_{i+1})\gamma_{i+1}^{-1}\, \alpha(h_{i+1},g_i)\right)$$

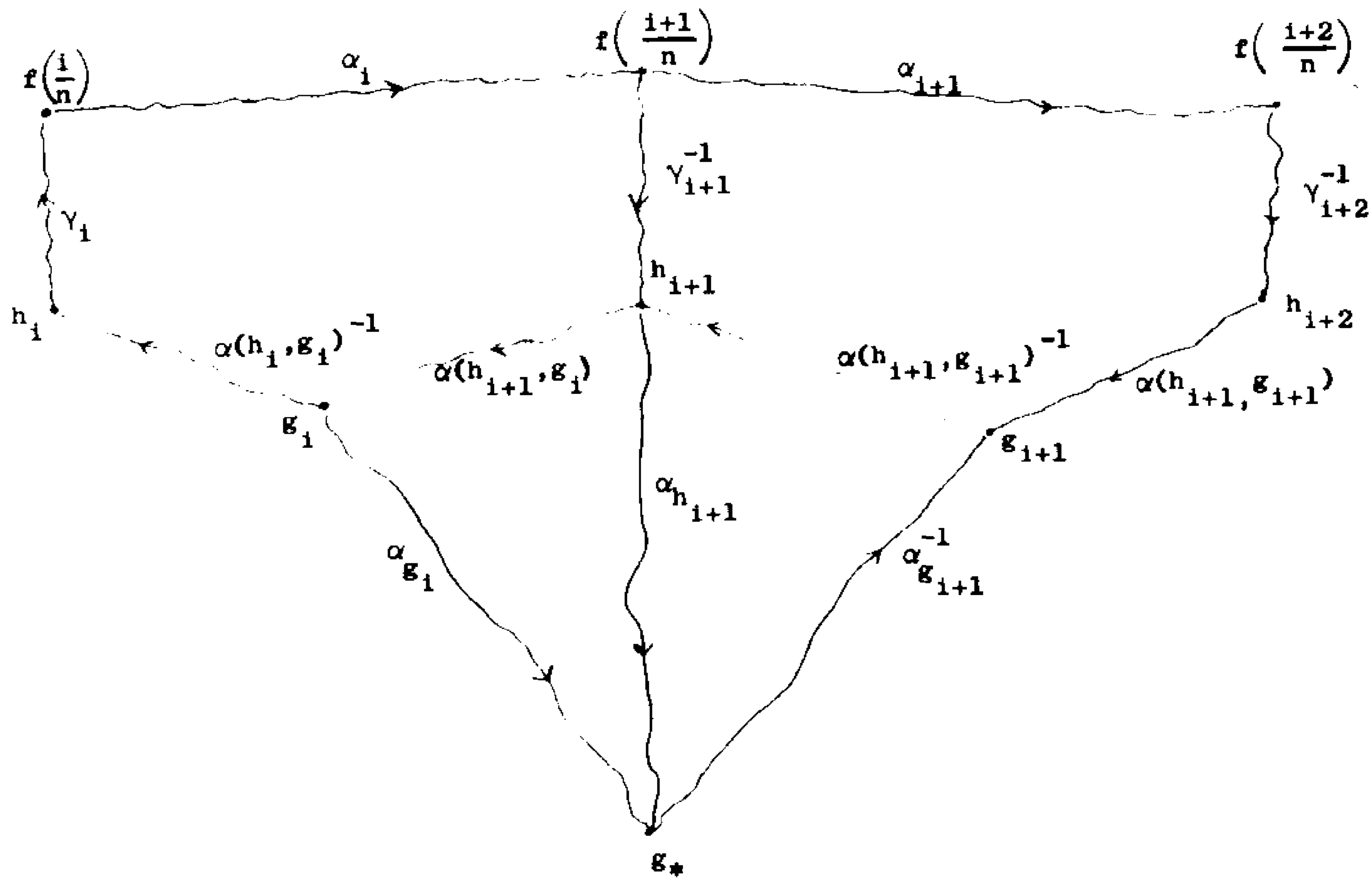
$$= \sum_{i=0}^{n-1} \alpha_{g_i}^{-1}\, P\varphi_{g_i}\left(\alpha(h_i,g_i)^{-1}\, P(\tau_i)\gamma_i\alpha_i\, P(\sigma_{i+1})\gamma_{i+1}^{-1}\, \alpha(h_{i+1},g_i)\right)\alpha_{g_i}$$

$$\cong \sum_{i=0}^{n-1} \alpha_{h_i}^{-1}\, P\varphi_{h_i}\gamma_i\, P\varphi_{g_i}\, \alpha_i\, P\varphi_{h_{i+1}}\gamma_{i+1}^{-1}\, \alpha_{h_{i+1}}$$

$$\cong \sum_{i=0}^{n-1} \alpha_{h_o}^{-1}\, P\varphi_{g_i}\, \alpha_i\alpha_{h_n}$$

$$\cong \alpha_{h_o}^{-1}\, f\, \alpha_{h_n}$$

$$= f\ . \quad \Box$$



Lemma 3. [cf. Massey's Lemma 2.4 pgs. 117-122]. If $H \in obGp$, $\{\rho_g : F_T(g) \to H\}$ is a compatible family of homomorphisms and

$$\left\{\beta_i \in F_T(g_i)\right\}_{i=1}^{q}$$

is a sequence of elements such that

$$h_{g_1}\beta_1 \cdot h_{g_2}\beta_2 \cdot \ldots \cdot h_{g_q}\beta_q = 1 \in \pi_1(\text{colim } F, z_{g_*})$$

then the product

$$\rho_{g_1}\beta_1 \cdot \rho_{g_2}\beta_2 \cdot \ldots \cdot \rho_{g_q}\beta_q = 1 \in H .$$

Proof: The product

$$h_{g_1}\beta_1 \cdot h_{g_2}\beta_2 \cdot \ldots \cdot h_{g_q}\beta_q$$

can be represented by a composition

$$(\alpha_{g_1}^{-1}\ \beta_1'\ \alpha_{g_1})(\alpha_{g_2}^{-1}\ \beta_2'\ \alpha_{g_2}) \cdot \ldots \cdot (\alpha_{g_q}^{-1}\ \beta_q'\ \alpha_{g_q}) = 1$$

for β_i' a loop representing $\pi_1\varphi_{q_i}\beta_i$, $i=1,\ldots,q$.

For each i, α_{g_i} is a finite composition of the form

$$\sum_{j=0}^{k(i)} P\varphi_{g_{j+1}}(\alpha(g_j,g_{j+1}))$$

for $\alpha(g_j,g_{j+1}) \in e\Gamma_s(F)$, a path in $F(g_{j+1})$.

The product

$$h_{g_1}\beta_1 \cdot h_{g_2}\beta_2 \cdot \ldots \cdot h_{g_q}\beta_q$$

can then be represented as the composition

1) $$\sum_{i=1}^{q} \alpha_{g_{i\,k(i)}}^{-1} \cdot \ldots \cdot \alpha_{g_{i1}}^{-1}\ \beta_i'\ \alpha_{g_{i1}} \cdot \ldots \cdot \alpha_{g_{i\,k(i)}}$$

where each $\alpha_{g_{ij}}$ is a path in $\varphi_{g_{ij}}F(g_{ij})$ and β_i' is a loop in $\varphi_{g_i}F(g_i)$.

$$0 = a_0 < a_1 < \ldots < a_{p-1} < a_p = 1$$

is an equipartition of $[0,1]$ and

$$f_{ij}\colon [a_r,a_{r+1}] \longrightarrow \varphi_{g_{ij}}F(g_{ij}) \ , \quad 1 \le j \le k(i),\ 1 \le i \le q$$

$$f_{ij}^{-1}\colon [a_r,a_{r+1}] \longrightarrow \varphi_{g_{ij}}F(g_{ij}) \ , \quad 1 \le j \le k(i),\ 1 \le i \le q$$

$$f_k \ \colon [a_r,a_{r+1}] \longrightarrow \varphi_{g_i}F(g_i) \ , \quad 1 \le i \le q$$

are functions representing respectively

$$\alpha_{ij},\ \alpha_{ij}^{-1} \ \text{ and } \ \beta_i'$$

such that if $f\colon [0,1] \to \operatorname{colim} F$ represents the composition 1), then

$$f\big|[a_r,a_{r+1}] = f_{ij},\ f_{ij}^{-1} \ \text{ or } \ f_i$$

for the appropriate r .

Since f is homotopically trivial, there exists a homotopy

$$G\colon I \times I \longrightarrow \operatorname{colim} F$$

such that if $s,t \in I \times I$,

$$G(s,0) = f(s)$$

$$G(s,1) = G(0,t) = G(1,t) = z_{g_*} \ .$$

ε denotes the Lebesque number of the open covering

$$\left\{G^{-1}(\varphi_g F(g))\right\}_{g \in \mathrm{ob}C} \ .$$

The square $I \times I$ is subdivided as follows:

$$0 = s_0 < s_1 < s_2 < \ldots < s_{m-1} < s_m = 1$$

$$0 = t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m = 1$$

are partitions of I such that

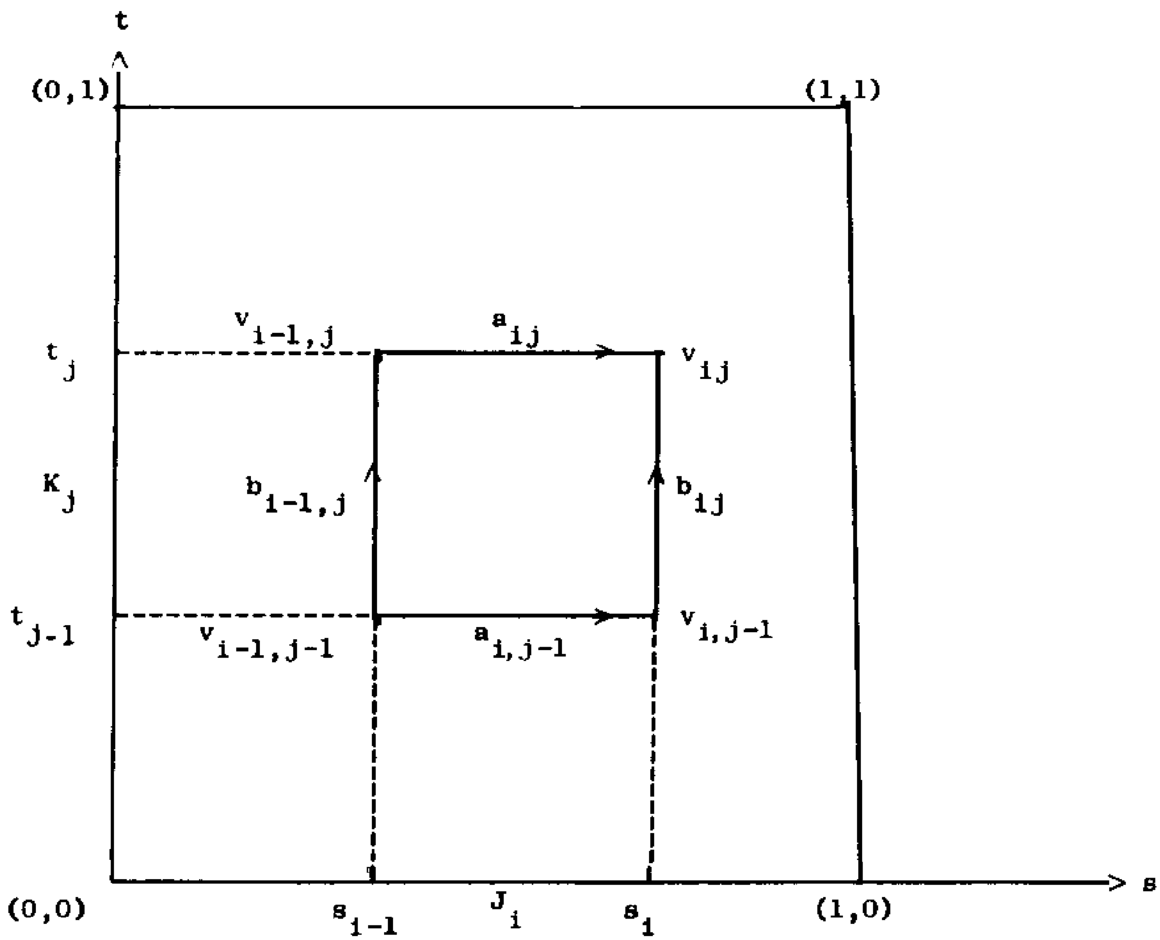
i) $0 = s_0 < s_1 < s_2 < \dots < s_{m-1} < s_m = 1$ refines the partition $0 = a_0 < a_1 < \dots < a_{p-1} < a_p = 1$.

ii) If $I \times I$ is subdivided into rectangles by the vertical and horizontal lines

$$s = s_i\,,\ i=0,\dots,m$$
$$t = t_j\,,\ j=0,\dots,n$$

the length of the diagonal of each rectangle is less than ϵ .

Notation for this subdivision:



Vertices: $v_{ij} = (s_i, t_j)$, $0 \le i \le m$, $0 \le j \le n$.

Subintervals of I :

$$J_i = [s_{i-1}, s_i] \quad , \quad 1 \le i \le m$$

$$K_j = [t_{j-1}, t_i] \quad , \quad 1 \le j \le n \; .$$

Rectangles: $R_{ij} = J_i \times K_j$, $1 \le i \le m$, $1 \le j \le n$

Horizontal edges:

$$a_{ij} = J_i \times \{t_j\} \; , \quad 1 \le i \le m \; , \quad 0 \le j \le n$$

Vertical edges:

$$b_{ij} = \{s_i\} \times K_j \; , \quad 0 \le i \le m \; , \quad 1 \le j \le n$$

Functions:

$$A'_{ij}: J_i \longrightarrow \operatorname{colim} F \quad \text{by} \quad A'_{ij}(s) = G(s, t_j), \; s \in J_i$$

$$B'_{ij}: K_j \longrightarrow \operatorname{colim} F \quad \text{by} \quad B'_{ij}(t) = G(s_i, t), \; t \in K_j$$

i.e. $A'_{ij} = G|a_{ij}$; $B'_{ij} = G|b_{ij}$.

For each rectangle R_{ij} ,

$$U_{ij} = \varphi_{k_{ij}} F(k_{ij})$$

is an element of the cover such that

$$G(R_{ij}) \subseteq U_{ij} \; .$$

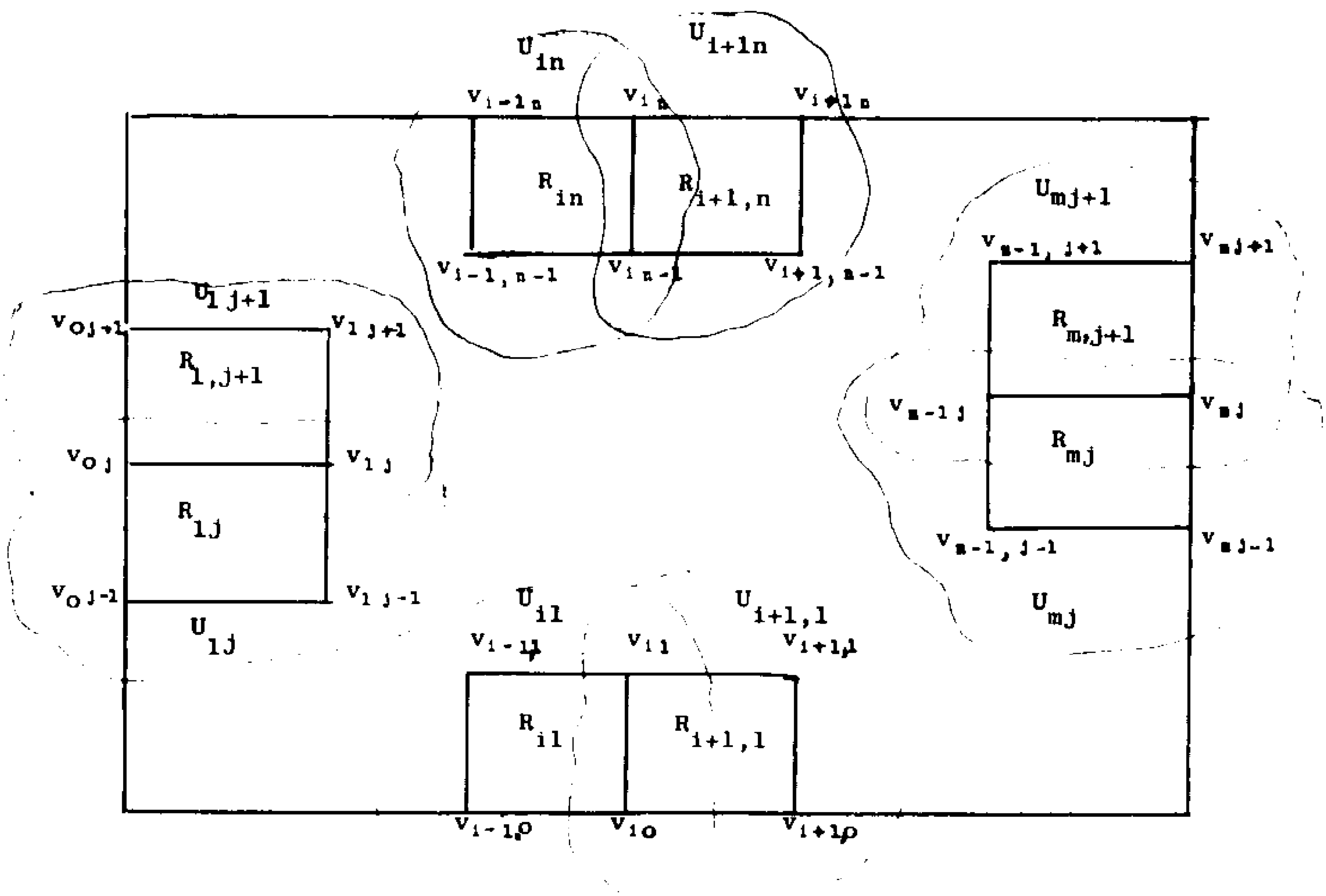
$$A_{ij} = (P\varphi_{k_{ij}})^{-1} A'_{ij} \qquad\qquad B_{ij} = (P\varphi_{k_{ij}})^{-1} B'_{ij} \; .$$

Each vertex v_{ij} is a vertex of 1,2 or 4 rectangles R_{ij} . W_{ij} is the intersection of the corresponding open sets. For each W_{ij} there is an open set

$$\varphi_{h_{ij}} F(h_{ij})$$

such that

$$G(v_{ij}) \in \varphi_{h_{ij}} F(h_{ij}) \subseteq W_{ij} \ . \quad \bar{v}_{ij} = \varphi^{-1}_{h_{ij}} G(v_{ij}) \ .$$



For $1 < i < m$

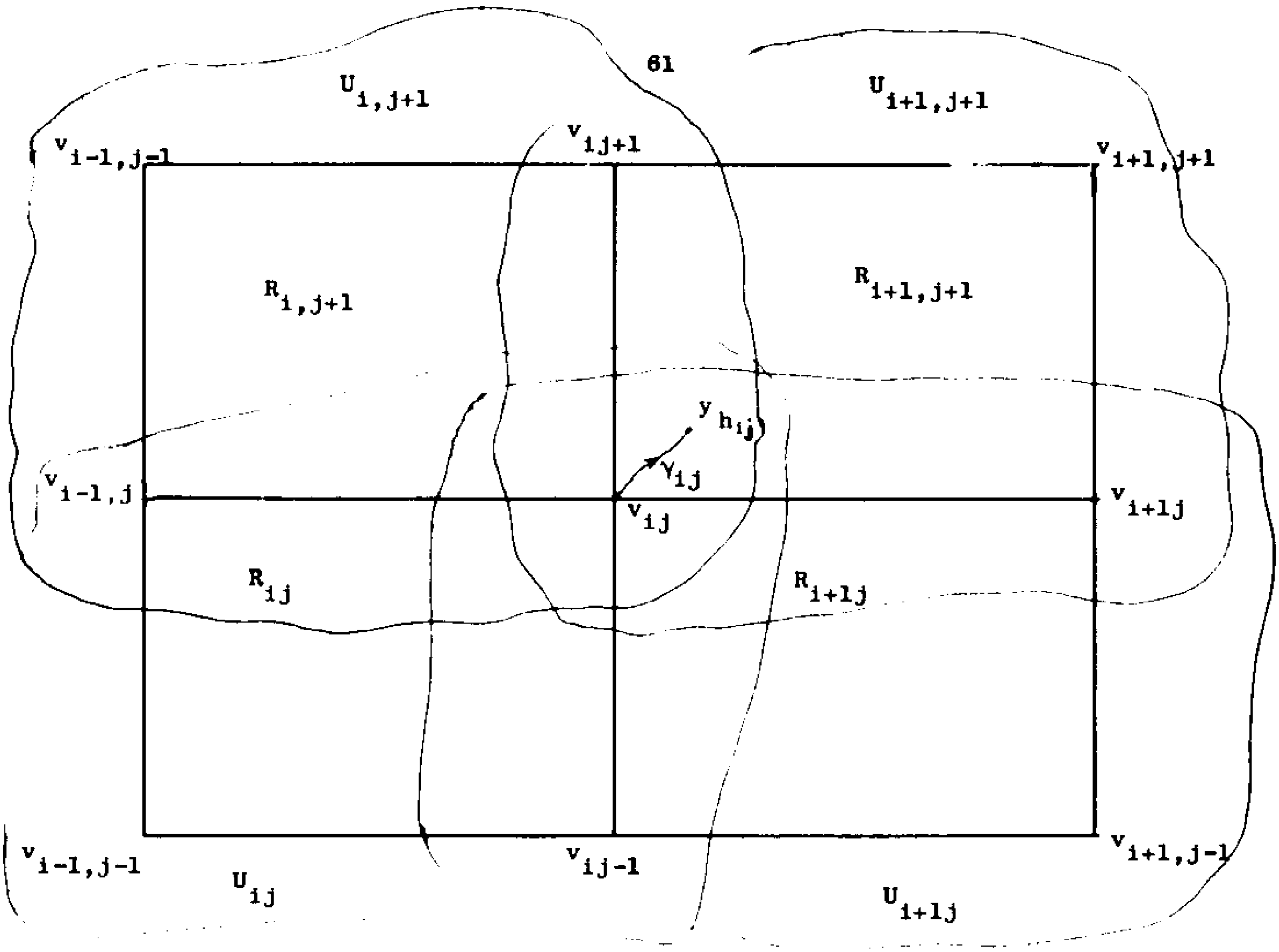
$$\bar{\sigma}^{1}_{i0} \in C(h_{i0}, k_{i1}) \qquad \bar{\sigma}^{2}_{i0} \in C(h_{i0}, k_{i+1,1})$$

$$\bar{\sigma}^{1}_{in} \in C(h_{in}, k_{in}) \qquad \bar{\sigma}^{2}_{in} \in C(h_{in}, k_{i+1,n})$$

For $1 < j < n$

$$\bar{\sigma}^{1}_{0j} \in C(h_{0j}, k_{ij}) \qquad \bar{\sigma}^{2}_{0j} \in C(h_{0j}, k_{i,j+1})$$

$$\bar{\sigma}^{2}_{mj} \in C(h_{mj}, k_{mj}) \qquad \bar{\sigma}^{2}_{mj} \in C(h_{mj}, k_{m,j+1})$$



For $1 < i < m-1$, $1 < j < n-1$,

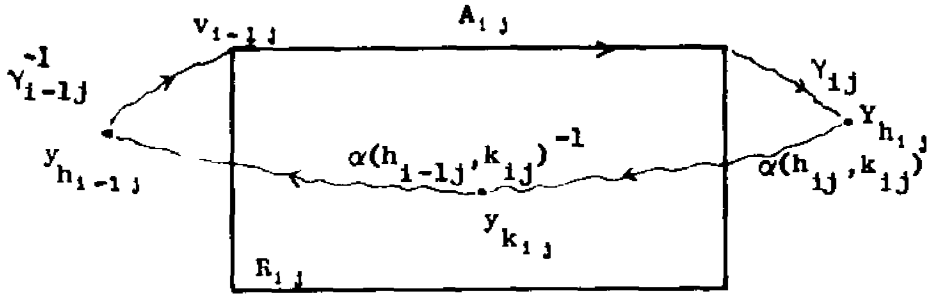
$\bar{\sigma}_{ij}^{1} \in C(h_{ij},k_{i+1,j+1})$; $\bar{\sigma}_{ij}^{2} \in C(h_{ij},k_{ij})$; $\bar{\sigma}_{ij}^{3} \in C(h_{ij},k_{i,j+1})$; $\bar{\sigma}_{ij}^{4} \in C(h_{ij},k_{i+1,j})$. $\sigma_{ij}^{k} = PF(\bar{\sigma}_{ij}^{k})$ for all i, j, and k for which the right side is defined. $\gamma_{ij} \in P(\bar{v}_{ij},F(h_{ij}),y_{h_{ij}})$, $y_{h_{ij}} \in v(\Gamma_c(F))$ is arbitrary except it is of the form $\alpha(g,g')$ if possible.

The loop $\zeta_{ij} \in \Omega(F(k_{ij}),y_{k_{ij}})$, $1 \le i \le m$, $1 \le j \le n$ is defined as follows:

$$\zeta_{ij} = \alpha(h_{i-1,j},k_{ij})^{-1}\sigma_{i-1,j}^{4}(\gamma_{i-1,j})^{-1}A_{ij}\sigma_{ij}^{2}(\gamma_{ij})\alpha(h_{ij},k_{ij}) .$$

If $j = 0$, the loop $\zeta_{i0} \in \Omega(F(k_{i1}),y_{k_{i1}})$, $1 \le i \le m$, is defined as follows:

$$\zeta_{i0} = \alpha(h_{i-1,0},k_{i1})^{-1}\ \sigma_{i-1,0}^{2}(\gamma_{i-1,0})^{-1}\ A_{i0}\ \sigma_{i0}^{1}(\gamma_{i0})\ \alpha(h_{i0},k_{i1})$$



The loop $\eta_{ij} \in \Omega(F(k_{ij}),y_{k_{ij}})$, $1 \le i \le m$, $1 \le j \le n$, is defined as follows:

$$\eta_{ij} = \alpha(h_{i,j-1},k_{ij})^{-1}\ \sigma^3_{i,j-1}(\gamma_{i,j-1})^{-1}\ B_{ij}\ \sigma^2_{ij}(\gamma_{ij})\alpha(h_{ij},k_{ij}) .$$

If $i = 0$, the loop $\eta_{0j} \in \Omega(F(k_{1j}),y_{k_{1j}})$, $1 \le j \le n$, is defined as follows:

$$\eta_{0j} = \alpha(h_{0,j-1},k_{1j})^{-1}\ \sigma^2_{0j-1}(\gamma_{0,j-1})^{-1}\ B_{0j}\sigma^1_{0j}(\gamma_{0j})\alpha(h_{0j},k_{1j}) .$$

The elements α_{ij} and β_{ij} in H are defined by

$$\alpha_{ij} = \rho_{k_{ij}}(\zeta_{ij}) , \quad 1 \le i \le m , \quad 1 \le j \le n$$

$$\beta_{ij} = \rho_{k_{ij}}(\eta_{ij}) , \quad 1 \le i \le m , \quad 1 \le j \le n$$

$$\alpha_{i0} = \rho_{k_{i1}}(\zeta_{i0}) , \quad 1 \le i \le m$$

$$\beta_{0j} = \rho_{k_{1j}}(\eta_{0j}) , \quad 1 \le j \le n .$$

(These correspond to the elements Massey defines on page 120.)

<u>Relation 1</u>. $\alpha_{i,j-1}\beta_{ij} = \beta_{i-1,j}\alpha_{ij}$, $1 \le i \le m$, $1 \le j \le n$.

<u>Proof</u>: <u>Case 1</u>. $1 < i$, $1 < j$.

i) In $F(k_{ij})$, the paths

$$A_{i,j-1}B_{ij} \text{ and } B_{i-1,j}A_{ij} , \quad 1 < i \le m, \ 1 < j \le n$$

are homotopic rel$\{\tilde{v}_{i-1j},\tilde{v}_{i,j-1}\}$ since they are homotopic in colim F by the homotopy G and $\varphi_{k_{ij}}$ is a homeomorphism.

ii) $\alpha_{ij}\beta_{ij}^{-1} =$

$$\rho_{k_{ij}}(\alpha(h_{i-1,j},k_{ij})^{-1}\ \sigma^4_{i-1,j}(\gamma_{i-1,j})^{-1}\ A_{ij}\ \sigma^2_{ij}\ \gamma_{ij}\ \alpha(h_{ij},k_{ij})$$

$$\cdot\ \alpha(h_{ij},k_{ij})^{-1}\ \sigma^2_{ij}(\gamma_{ij})^{-1}B^{-1}_{ij}\ \sigma^3_{i,j-1}(\gamma_{i,j-1})\alpha(h_{i,j-1},k_{ij}))$$

$$= \rho_{k_{ij}}(\alpha(h_{i-1,j},k_{ij})^{-1}\ \sigma^4_{i-1,j}(\gamma_{i-1,j})^{-1}\ A_{ij}B^{-1}_{ij}\ \sigma^3_{i,j-1}(\gamma_{i,j-1})$$

$$\cdot\ \alpha(h_{i,j-1},k_{ij}))$$

$$= \rho_{k_{ij}}(\alpha(h_{i-1,j},k_{ij})^{-1}\ \sigma^4_{i-1,j}(\gamma_{i-1,j})^{-1}\ B^{-1}_{i-1,j}A_{i,j-1}\ \sigma^3_{i,j-1}(\gamma_{i,j-1})$$

$$\cdot\ \alpha(h_{i,j-1},k_{ij}))$$

In colim F,

$$\varphi_{k_{i,j-1}}(\sigma^4_{i-1,j-1}(\gamma_{i-1,j-1})^{-1}\ A_{ij-1}\sigma^2_{i,j-1}(\gamma_{i,j-1}))$$

is a path from

$$\varphi_{h_{i-1,j-1}}\ y_{h_{i-1,j-1}} \quad \text{to} \quad \varphi_{h_{i,j-1}}\ y_{h_{i,j-1}} \quad \text{in } U_{i,j-1}\cap U_{ij}\ .$$

It follows that the points are in the same path component and there exists

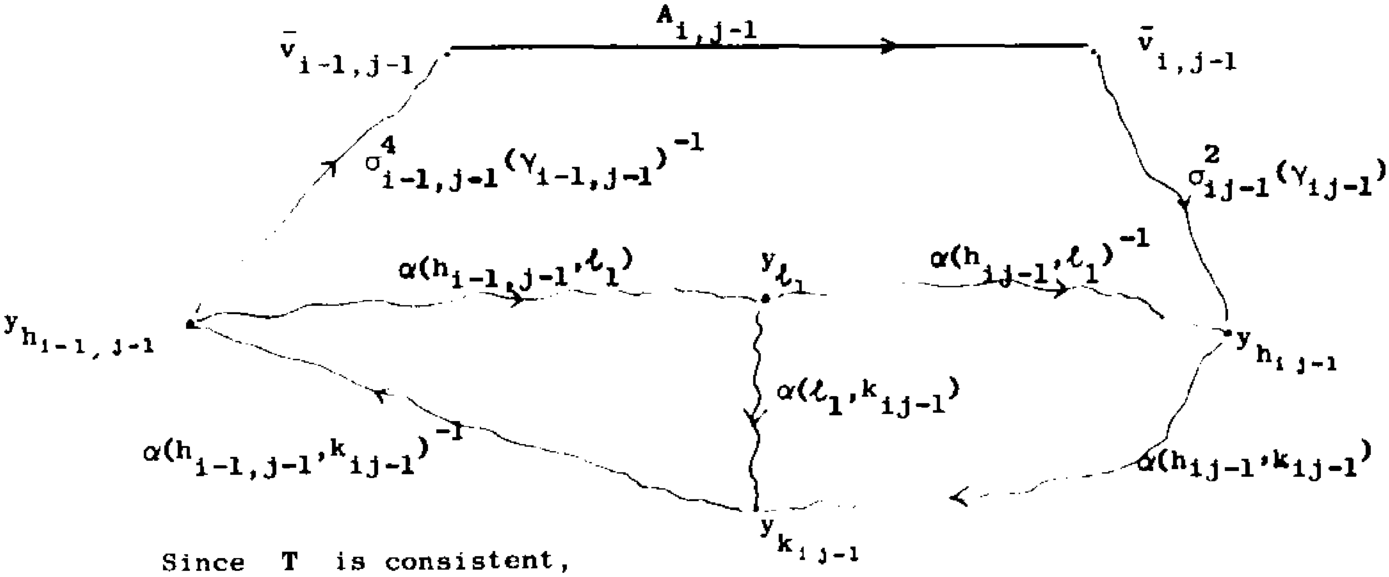
$$V_1 = \varphi_{\ell_1}F(\ell_1)$$

with $\bar{\tau}^1 \in C(\ell_1,k_{i,j-1})$ $\qquad \bar{\tau}^2 \in C(\ell_1,k_{ij})$

$\bar{\tau}^3 \in C(h_{i-1,j-1},\ell_1)$ $\qquad \bar{\tau}^4 \in C(h_{i,j-1},\ell_{i,j-1})$

$\tau^k = F(\bar{\tau}^k)$ for $k = 1,2,3,4$.

In $F(k_{i,j-1})$ there is a graph of paths:



Since T is consistent,

$$P(\tau^1)\alpha(h_{i-1,j-1},\ell_1)\alpha(\ell_1,k_{ij-1}) \simeq \alpha(h_{i-1,j-1},k_{ij-1})$$

and

$$P(\tau^1)\alpha(h_{i,j-1},\ell_1)\alpha(\ell_1,k_{ij-1}) \simeq \alpha(h_{ij-1},k_{ij-1}) \ .$$

Thus

$$\begin{aligned}
\alpha_{ij-1} &= \rho_{k_{ij-1}}(\alpha(h_{i-1,j-1},k_{ij-1})^{-1}\ \sigma^4_{i-1,j-1}(\gamma_{i-1,j-1})^{-1} \\
&\qquad \cdot A_{ij-1}\ \sigma^2_{ij-1}(\gamma_{ij-1})\alpha(h_{i-1,j-1},k_{ij-1})) \\
&= \rho_{k_{ij-1}}(\alpha(\ell_1,k_{ij-1})^{-1}(P(\tau^1)(\alpha(h_{i-1,j-1},\ell_1))^{-1} \\
&\qquad \cdot \sigma^4_{i-1,j-1}(\gamma_{i-1,j-1})^{-1}\ A_{ij-1}\sigma^2_{ij-1}(\gamma_{ij-1}) \\
&\qquad \cdot P(\tau^1)(\alpha(h_{ij-1},\ell_1))\alpha(\ell_1,k_{ij-1}) \\
&= \rho_{k_{ij-1}}(F_T(\bar{\tau}^1)(\alpha(h_{i-1,j-1},\ell_1)^{-1}\ P(\tau^3)(\gamma_{i-1,j-1}) \\
&\qquad \cdot \bar{A}_{ij-1}P(\tau^4)(\gamma_{ij-1})\alpha(h_{ij-1},\ell_1))
\end{aligned}$$

(for $\bar{A}_{ij-1} = (P\varphi_{\ell_1})^{-1}A'_{ij-1})$.

$$\begin{tikzcd} F_T(\ell_1) \arrow[d, "F_T(\bar{\tau}^1)"] \arrow[rd, "\rho_{\ell_1}"] & \\ F_T(k_{ij-1}) \arrow[r, "\rho_{k_{ij-1}}"'] & H \end{tikzcd}$$

Since the family $\{\rho_g\}_{g \in obC}$ is compatible, this element is

$$= \rho_{\ell_1}(\alpha(h_{i-1,j-1},\ell_1)^{-1}P(\tau^3)(\gamma_{i-1,j-1})^{-1}\bar{A}_{ij-1}\,P(\tau^4)(\gamma_{ij-1})$$

$$\cdot\ \alpha(h_{ij-1},\ell_1))$$

$$= \rho_{k_{ij}}(F_T(\bar{\tau}^2)(\alpha(h_{i-1,j-1},\ell_1)^{-1}P(\tau^3)(\gamma_{i-1,j-1})^{-1}\bar{A}_{ij-1}$$

$$\cdot\ P(\tau^4)(\gamma_{ij-1})\alpha(h_{i,j-1},\ell_1)))$$

$$= \rho_{k_{ij}}(\alpha(\ell_1,k_{ij})^{-1}P(\tau^2)(\alpha(h_{i-1,j-1},\ell_1)^{-1})\sigma^1_{i-1,j-1}(\gamma_{i-1,j-1})^{-1}$$

$$\cdot\ A_{ij-1}\sigma^3_{ij-1}(\gamma_{ij-1})P(\tau^2)\alpha(h_{ij-1},\ell_1)\alpha(\ell_1,k_{ij}))\ .$$

Since T is consistent,

$$P(\tau^2)\alpha(h_{i-1,j-1},\ell_1)\alpha(\ell_1,k_{ij}) \simeq \alpha(h_{i-1,j-1},k_{ij})$$

and

$$P(\tau^2)\alpha(h_{ij-1},\ell_1)\alpha(\ell_1,k_{ij}) \simeq \alpha(h_{ij-1},k_{ij})\ .$$

Thus

$$\alpha_{ij}\beta^{-1}_{ij} = \rho_{k_{ij}}(\alpha(h_{i-1,j},k_{ij})^{-1}\sigma^4_{i-1j}(\gamma_{i-1j})^{-1}B^{-1}_{i-1j}\,A_{ij-1}$$

$$\cdot\ \sigma^3_{ij-1}(\gamma_{ij-1})\alpha(h_{ij-1},k_{ij}))$$

$$= \rho_{k_{ij}}(\alpha(h_{i-1j},k_{ij})^{-1}\ \sigma^4_{i-1j}(\gamma_{i-1j})^{-1}\ B^{-1}_{i-1j}\ A_{ij-1}$$

$$\cdot\ \sigma^3_{ij-1}(\gamma_{ij-1})\ P(\tau^2)\alpha(h_{ij-1},\ell_1)\ \alpha(\ell_1,k_{ij}))$$

and

$$\alpha_{ij}\beta^{-1}_{ij}\ \alpha^{-1}_{ij-1} = \rho_{k_{ij}}(\alpha(h_{i-1j},k_{ij})^{-1}\ \sigma^4_{i-1j}(\gamma_{i-1j})^{-1}\ B^{-1}_{i-1j}$$

$$\cdot\ \sigma^1_{i-1,j-1}(\gamma_{i-1,j-1})\ P(\tau^2)(\alpha(h_{i-1,j-1},\ell_1))$$

$$\cdot\ \alpha(\ell_1,k_{ij}))$$

$$= \rho_{k_{ij}}(\alpha(h_{i-1,j},k_{ij})^{-1}\ \sigma^4_{i-1j}(\gamma_{i-1j})^{-1}\ B^{-1}_{i-1j}$$

$$\cdot\ \sigma^1_{i-1,j-1}(\gamma_{i-1,j-1})\alpha(h_{i-1,j-1},k_{ij}))\ .$$

In colim F ,

$$\varphi_{k_{i-1j}}(\sigma^2_{i-1j}(\gamma_{i-1j})^{-1}\ B^{-1}_{i-1j}\ \sigma^3_{i-1,j-1}(\gamma_{i-1,j-1}))$$

is a path from

$$\varphi_{h_{i-1j}}y_{h_{i-1,j}} \quad \text{to} \quad \varphi_{h_{i-1,j-1}}y_{h_{i-1,j-1}} \quad \text{in} \quad U_{i-1j}\cap U_{ij}\ .$$

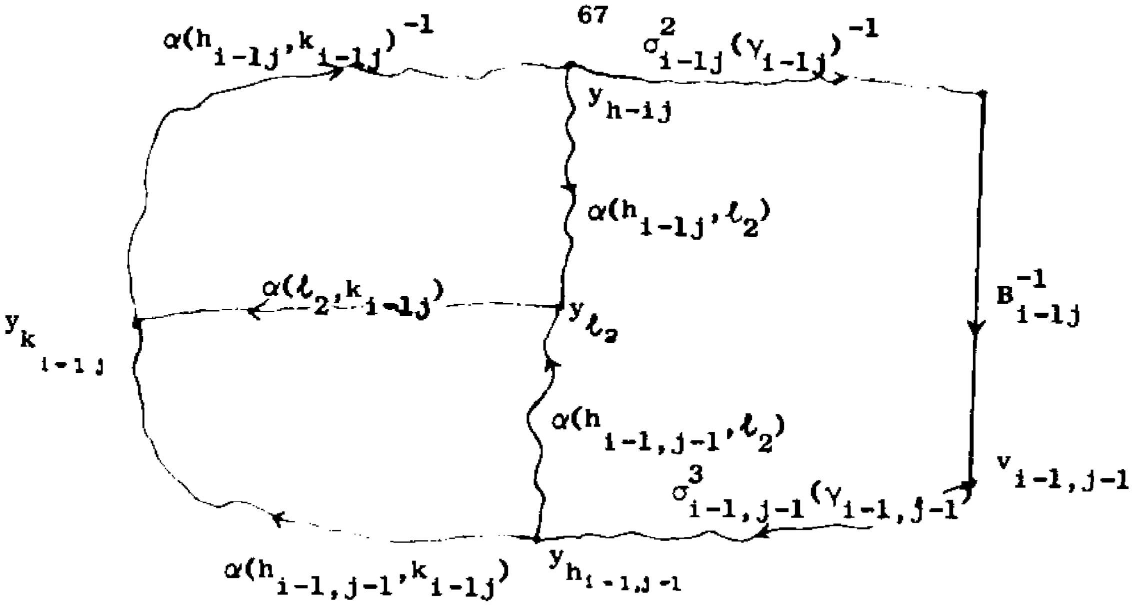
It follows that they are in the same path component and there exists

$$V_2 = \varphi_{\ell_2}F(\ell_2)$$

with $\bar\tau^5 \in C(\ell_2,k_{i-1j})$ $\quad\bar\tau^6 \in C(\ell_2,k_{ij})$ $\quad \tau^k = F(\bar\tau^k)$ for k=5,6,7,8

$\bar\tau^7 \in C(h_{i-1,j-1},\ell_2)$ $\quad\bar\tau^8 \in C(h_{i-1j},\ell_2)$

In $F(k_{i-1j})$ there is a graph of paths:



Since T is consistent,

$$P(\tau^5)\alpha(h_{i-1j},\ell_2)\alpha(\ell_2,k_{i-1j}) \simeq \alpha(h_{i-1j},k_{i-1j})$$

and

$$P(\tau^5)\alpha(h_{i-1,j-1},\ell_2)\alpha(\ell_2,k_{i-1j}) \simeq \alpha(h_{i-1,j-1},k_{i-1j}) \ .$$

Thus

$$\beta_{i-1j} = \rho_{k_{i-1j}}(\alpha(h_{i-1,j-1},k_{i-1j})^{-1}\ \sigma^3_{i-1,j-1}(\gamma_{i-1,j-1})^{-1}$$

$$\cdot\ B_{i-1j}\ \sigma^2_{i-1j}(\gamma_{i-1j})\alpha(h_{i-1j},k_{i-1j}))$$

$$= \rho_{k_{i-1j}}(\alpha(\ell_2,k_{i-1j})^{-1}\ P(\tau^5)\alpha(h_{i-1,j-1},\ell_2)^{-1}$$

$$\cdot\ \sigma^3_{i-1,j-1}(\gamma_{i-1,j-1})^{-1}\ B_{i-1j}\ \sigma^2_{i-1j}(\gamma_{i-1j})$$

$$\cdot\ P(\tau^5)\alpha(h_{i-1j},\ell_2)\alpha(\ell_2,k_{i-1j}))$$

$$= \rho_{k_{i-1j}}(F_T(\tau^{-5})(\alpha(h_{i-1,j-1},\ell_2)^{-1}\ P(\tau^3)(\gamma_{i-1,j-1})^{-1}$$

$$\cdot\ \bar{B}_{i-1j}\ P(\tau^2)(\gamma_{i-1j})\alpha(h_{i-1j},\ell_2))$$

(for $\bar{B}_{i-1j} = (P\varphi_{\ell_2}^{-1})B'_{i-1j}$.)

Since the family $\{\rho_g\}_{g \in obC}$ is compatible, this element is equal to

$$\rho_{\ell_2}(\alpha(h_{i-1,j-1},\ell_2)^{-1}P(\tau^7)(\gamma_{i-1,j-1})^{-1}\ \bar{B}_{i-1j}\ P(\tau^8)(\gamma_{i-1j})\alpha(h_{i-1j},\ell_2))$$

$$= \rho_{k_{ij}}(F_T(\bar{\tau}^6)(\alpha(h_{i-1,j-1},\ell_2)^{-1}\ P(\tau^7)(\gamma_{i-1,j-1})^{-1}\ \bar{B}_{i-1j}\ P(\tau^8)(\gamma_{i-1j})$$

$$\cdot\ \alpha(h_{i-1j},\ell_2)))$$

$$= \rho_{k_{ij}}(\alpha(\ell_2,k_{ij})^{-1}\ P(\tau^6)(\alpha(h_{i-1,j-1},\ell_2)^{-1}\ \sigma^1_{i-1,j-1}(\gamma_{i-1,j-1})^{-1}$$

$$\cdot\ B_{i-1j}\ \sigma^4_{i-1j}(\gamma_{i-1j})\ P(\tau^6)(\alpha(h_{i-1j},\ell_2))\alpha(\ell_2,k_{ij}))\ .$$

Since T is consistent

$$P(\tau^6)\alpha(h_{i-1,j-1},\ell_2)\alpha(\ell_2,k_{ij}) \simeq \alpha(h_{i-1,j-1},k_{ij})$$

and

$$P(\tau^6)\alpha(h_{i-1j},\ell_2)\alpha(\ell_2,k_{ij}) \simeq \alpha(h_{i-1j},k_{ij})\ .$$

Thus

$$\alpha_{ij}\beta_{ij}^{-1}\alpha_{ij-1}^{-1} = \rho_{k_{ij}}(\alpha(\ell_2,k_{ij})^{-1}\ P(\tau^6)\alpha(h_{i-1j},\ell_2)^{-1}\ \sigma^4_{i-1j}(\gamma_{i-1j})^{-1}$$

$$\cdot\ B^{-1}_{i-1j}\ \sigma^1_{i-1,j-1}(\gamma_{i-1,j-1})P(\tau^6)\alpha(h_{i-1,j-1},\ell_2)$$

$$\cdot\ \alpha(\ell_2,k_{ij}))$$

and

$$\beta_{i-1j}\alpha_{ij}\beta_{ij}^{-1}\alpha_{ij-1}^{-1} = 1\ .$$

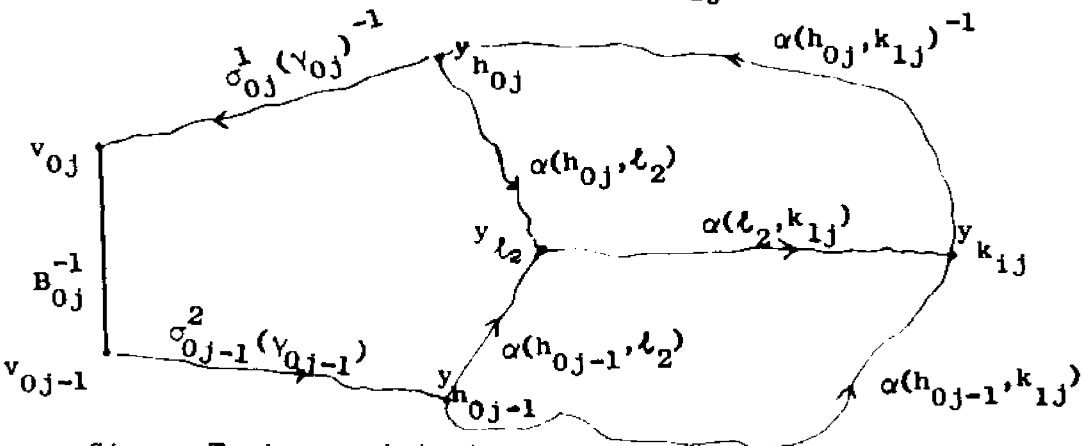
<u>Case 2.</u> $i = 1,\ 1 < j \leq m$.

As in Case 1:

$$\alpha_{1j}\beta_{1j}^{-1}\alpha_{1j-1}^{-1} = \rho_{k_{1j}}(\alpha(\ell_2,k_{1j})^{-1}\ P(\tau^6)\alpha(h_{0j},\ell_2)^{-1}\ \sigma_{0j}^{1}(\gamma_{0j})^{-1}$$

$$\cdot\ B_{0j}^{-1}\ \sigma_{0j-1}^{2}(\gamma_{0j-1})\ P(\tau^6)\alpha(h_{0j-1},\ell_2)\alpha(\ell_2,k_{1j})$$

There is a diagram of paths in $F(k_{1j})$:



Since T is consistent

$$P(\tau^6)\alpha(h_{0j-1},\ell_2)\alpha(\ell_2,k_{1j}) \simeq \alpha(h_{0j-1},k_{1j})$$

and

$$P(\tau^6)\alpha(h_{0j},\ell_2)\alpha(\ell_2,k_{1j}) \simeq \alpha(h_{0j},k_{1j}) \ .$$

$$\beta_{0j} = \rho_{k_{1j}}(\eta_{0j}) = \rho_{k_{1j}}(\alpha(h_{0j-1},k_{1j})^{-1}\ \sigma_{0j-1}^{2}(\gamma_{0j-1})^{-1}\ B_{0j}\ \sigma_{0j}^{1}(\gamma_{0j})$$

$$\cdot\ \alpha(h_{0j},k_{1j}))$$

$$= \rho_{k_{1j}}(\alpha(\ell_2,k_{1j})^{-1}P(\tau^6)\alpha(h_{0j-1},\ell_2)^{-1}\ \sigma_{0j-1}^{2}(\gamma_{0j-1})^{-1}$$

$$\cdot\ B_{0j}\ \sigma_{0j}^{1}(\gamma_{0j})P(\tau^6)(\alpha(h_{0j},\ell_2)\alpha(\ell_2,k_{1j}))$$

and

$$\beta_{0j}\alpha_{1j}\beta_{1j}^{-1}\alpha_{1j-1}^{-1} = 1 \ .$$

Case 3. $j = 1,\ 1 < i \le n$.

As in Case 1,

$$\alpha_{i1}\beta_{i1}^{-1} = \rho_{k_{i1}}(\alpha(h_{i-1,1},k_{i1})^{-1}\ \sigma^4_{i-1,1}(\gamma_{i-1,1})^{-1}\ B^{-1}_{i-1,1}\ A_{i0}\ \sigma^1_{i0}(\gamma_{i0})$$

$$\cdot\ P(\tau^2)\alpha(h_{i0},\ell_1)\ \alpha(\ell_1,k_{i1}))\ .$$

$$\alpha_{i0} = \rho_{k_{i1}}(\zeta_{i0}) = \rho_{k_{i1}}(\alpha(h_{i-1,0},k_{i1})^{-1}\ \sigma^2_{i-1,0}(\gamma_{i-1,0})^{-1}$$

$$\cdot\ A_{i0}\ \sigma^1_{i0}(\gamma_{i0})\alpha(h_{i0},k_{i1}))\ .$$

Since T is consistent,

$$P(\tau^2)\alpha(h_{i0},\ell_1)\cdot\alpha(\ell_1,k_{i1}) \simeq \alpha(h_{i0},k_{i1})$$

and

$$\alpha_{i1}\beta_{i1}^{-1}\alpha_{i0}^{-1} = \rho_{k_{i1}}(\alpha(h_{i-1,1},k_{i1})^{-1}\ \sigma^4_{i-1,1}(\gamma_{i-1,1})^{-1}\ B^{-1}_{i-1,1}$$

$$\cdot\ \sigma^2_{i-1,0}(\gamma_{i-1,0})\alpha(h_{i-1,0},k_{i1}))\ .$$

As in Case 1:

$$\beta_{i-1,1} = \rho_{k_{i1}}(\alpha(\ell_2,k_{i1})^{-1}\ P(\tau^6)(\alpha(h_{i-1,0},\ell_2)^{-1}\ \sigma^2_{i-1,0}(\gamma_{i-1,0})^{-1}$$

$$\cdot\ B_{i-1,1}\ \sigma^4_{i-1,1}(\gamma_{i-1,1})\ P(\tau^6)(\alpha(h_{i-1,1},\ell_2))\alpha(\ell_2,k_{i1}))$$

$$= \rho_{k_{i1}}(\alpha(h_{i-1,0},k_{i1})^{-1}\ \sigma^2_{i-1,0}(\gamma_{i-1,0})^{-1}\ B_{i-1,1}\ \sigma^4_{i-1,1}(\gamma_{i-1,1})$$

$$\cdot\ \alpha(h_{i-1,1},k_{i1}))$$

and

$$\beta_{i-1,1}\alpha_{i1}\beta_{i1}^{-1}\alpha_{i0}^{-1} = 1\ .$$

__Case 4.__ $i = 1,\ j = 1$.

$$\begin{aligned}
\beta_{0,1}\alpha_{1,1}\beta_{1,1}^{-1}\alpha_{1,0}^{-1} &= \rho_{k_{1,1}}(\eta_{0,1})\rho_{k_{1,1}}(\zeta_{1,1})\rho_{k_{1,1}}(\eta_{1,1}^{-1})\rho_{k_{1,1}}(\zeta_{1,0}^{-1}) \\
&= \rho_{k_{1,1}}(\eta_{0,1}\ \zeta_{1,1}\ \eta_{1,1}^{-1}\ \zeta_{1,0}^{-1}) \\
&= \rho_{k_{1,1}}(\alpha(h_{0,0},k_{1,1})^{-1}\ \sigma(\gamma_{0,0})^{-1}\ B_{0,1}\ \sigma_{0,1}^{1}(\gamma_{0,1}) \\
&\quad \cdot\ \alpha(h_{0,1},k_{1,1})\alpha(h_{0,1},k_{1,1})^{-1}\ \sigma_{0,1}^{1}(\gamma_{0,1})^{-1} \\
&\quad \cdot\ A_{1,1}\ \sigma_{1,1}^{2}(\gamma_{1,1})\alpha(h_{1,1},k_{1,1})\alpha(h_{1,1},k_{1,1})^{-1} \\
&\quad \cdot\ \sigma_{1,1}^{2}(\gamma_{1,1})^{-1}\ B_{1,1}^{-1}\ \sigma_{1,0}^{1}(\gamma_{1,0})\alpha(h_{1,0},k_{1,1}) \\
&\quad \cdot\ \alpha(h_{1,0},k_{1,1})^{-1}\ \sigma_{1,0}^{1}(\gamma_{1,0})^{-1}\ A_{1,0}^{-1} \\
&\quad \cdot\ \sigma(\gamma_{0,0})\alpha(h_{0,0},k_{1,1}))
\end{aligned}$$

(for $\sigma \in C(h_{0,0},k_{1,1})$) .

$$\begin{aligned}
&= \rho_{k_{1,1}}(\alpha(h_{0,0},k_{1,1})^{-1}\ \sigma(\gamma_{0,0})^{-1}\ B_{0,1}\ A_{1,1} \\
&\quad \cdot\ B_{1,1}^{-1}\ A_{1,0}^{-1}\ \sigma(\gamma_{0,0})\alpha(h_{0,0},k_{1,1}))
\end{aligned}$$

$$\begin{tikzcd}
v_{0,1} \arrow[r, "A_{1,1}"] & v_{1,1} \\
v_{0,0} \arrow[u, "B_{0,1}"] \arrow[r, "A_{1,0}"'] & v_{1,0} \arrow[u, "B_{1,1}"']
\end{tikzcd}$$

and $B_{0,1}A_{1,1}B^{-1}_{1,1}A^{-1}_{1,0} \simeq 1$ so that $\beta_{0,1}\alpha_{1,1}\beta^{-1}_{1,1}\alpha^{-1}_{1,0} = 1$.

<u>Relation 2.</u> $\prod_{i=0}^{m} \alpha_{i0} = \prod_{k=1}^{q} \rho_{g_k}\beta_k$.

<u>Proof</u>: i) The product

$$h_{g_1}\beta_1 \cdot h_{g_2}\beta_2 \cdot \ldots \cdot h_{g_q}\beta_q$$

is represented as a composition of elements of the types

I) $P\varphi_{g_*}\alpha(g_{r-1},g_r)$ $\qquad * = r-1$ or r

II) β_i' .

For each r , $1 \le r \le p$, if $f|[a_{r-1},a_r]$ represents an element of type I), then $d \in \mathrm{ob}C$ is such that $\alpha(g_{r-1},g_r)$ is in $F(d)$ and ends at y_d . The edge $[a_{r-1},a_r]$ is mapped by G into $\varphi_d F(d)$.

The vertices $\{v_i\}_{i=0}^m$ occur in at most 2 rectangles. If an edge $a_{u0} \subseteq [a_{r-1},a_r]$ for $1 \le r \le p$, then it is possible to choose the elements $\{h_{i0}\}_{i=0}^m$ such that

$$\varphi_{h_{u0}}F(h_{u0}) \subseteq U_{u1} \cap U_{u+1,1} \cap \varphi_d F(d) \quad , \qquad 1 \le u \le m$$

where $U_{m+1,1} = U_{m,1}$.

For $v_{u-1,0}\ v_{u0}$ vertices of a_{k0} , the path

$$\sigma^2_{u-1,0}(\gamma_{u-1,0})^{-1} A_{u0}\ \sigma^1_{u0}(\gamma_{u0})$$

is a path from $y_{h_{u-1,0}}$ to $y_{h_{u0}}$ in $F(h_{u0})$.

Thus there exists $\ell_u \in \mathrm{ob}C$, $1 \le u \le m$ such that

$$\sigma^1_u \in C(\ell_u,k_{i1}) \ , \ \sigma^2_u \in C(\ell_u,d)$$

$$\tau^1_u \in C(h_{u-1,0},\ell_u) \ , \ \tau^2_u \in C(h_{u0},\ell_u)$$

$$\alpha_{u0} = \rho_{k_{u1}}(\zeta_{u0})$$

$$= \rho_{k_{u1}}(\alpha(h_{u-1,0},k_{u1})^{-1}\ \sigma^2_{u-1,0}(\gamma_{u-1,0})^{-1}\ A_{u0}\ \sigma^1_{u0}(\gamma_{u0})\alpha(h_{u0},k_{u1}))\ .$$

Since T is consistent

$$P(\sigma^1_u)\alpha(h_{u-1,0},\ell_u)\alpha(\ell_u,k_{u1}) \simeq \alpha(h_{u-1,0},k_{u1})$$

and

$$P(\sigma^1_u)\alpha(h_{u0},\ell_u)\alpha(\ell_u,k_{u1}) \simeq \alpha(h_{u0},k_{u1})$$

and

$$\alpha_{u0} = \rho_{k_{u1}}(\alpha(\ell_u,k_{u1})^{-1}\ P(\sigma^1_u)\alpha(h_{u-1,0},\ell_u)^{-1}\ \sigma^2_{u-1,0}(\gamma_{u-1,0})^{-1}$$

$$\cdot\ A_{u0}\ \sigma^1_{u0}(\gamma_{u0})P(\sigma^1_u)\alpha(h_{u0},\ell_u)\alpha(\ell_u,k_{u1}))$$

$$= \rho_{\ell_u}(\alpha(h_{u-1,0},\ell_u)^{-1}\ \tau^1_u(\gamma_{u-1,0})^{-1}\ \tilde{A}_{u0}\ \tau^2(\gamma_{u0})\alpha(h_{u0},\ell_u))$$

$$= \rho_d(\alpha(h_{u-1,0},d)^{-1}\ \sigma^2_u\ \tau^1_u(\gamma_{u-1,0})^{-1}\ \bar{A}_{u0}\ \sigma^2_u\ \tau^2_u(\gamma_{u0})\alpha(h_{u0},d_r))$$

where $\bar{A}_{u0} = (P\varphi_d)^{-1}\ A'_{u0}$.

The product $\overset{r_*}{\underset{r-1}{\pi}}\ \alpha_{u0}$ denotes the product of all (consecutive) α_{u0} such that $a_{u0} \subseteq [a_{r-1},a_r]$. $\overset{r}{\underset{r-1}{\pi^*}}\ \alpha_{u0} = \overset{r}{\underset{r-1}{\pi}}\ \rho_d(\alpha(h_{u-1,0}d)^{-1}$

$$\cdot\ \sigma^2_u\ \tau^1_u(\gamma_{u-1,0})^{-1}\ \bar{A}_{u0}\sigma^2_u\ \tau^2_u(\gamma_{u0})\alpha(h_{u0},d))$$

$$= \rho_d(\alpha(h_{r-1,0},d)^{-1}\ \sigma^2_r\ \tau^1_r(\gamma_{r-1,0})^{-1}$$

$$\cdot\ \overset{r}{\underset{r-1}{\pi^*}}\ \bar{A}_{u0}\ \sigma^2_r\ \tau^2_r(\gamma_{r0})\alpha(h_{r0},d))\ .$$

If $y_{h_{r-1,0}} = y_d$, then by construction $\gamma_{r-1,0}$ is trivial and

$\gamma_{r-1,0} = \alpha(h_{r-1,0},d') = \alpha(h_{r-1,0},d')^{-1}$. If $y_{h_{r-1,0}} \neq y_{d'}$, then there exist distinct free faces f, f' such that $C(f,h_{r-1,0}) \neq \emptyset$ and $C(f',d') \neq \emptyset$.

$$\varphi_{f'} y_{d'} \in \varphi_{f'} F(f') \cap \varphi_{h_{r-1,0}} F(h_{r-1,0})$$

Thus there exists an f_δ such that

$$C(f_\delta, f') \neq \emptyset \quad \text{and} \quad C(f_\delta, h_{r-1,0}) \neq \emptyset \ .$$

Since f' is a free face, $f_\delta = f'$ and

$$C(f', h_{r-1,0}) \neq \emptyset$$

Thus there exists an edge of $\Gamma_s(F)$ of the form

$$\alpha(f', h_{r-1,0})$$

in $F(h_{r-1,0})$ and by construction $\gamma_{r-1,0}$ is the path

$$\alpha(h_{r-1,0}, d')^{-1} \ .$$

Similarly $\gamma_{r0} = \alpha(h_{r0},d)^{-1}$.

Also, $\overset{r}{\underset{r-1}{\pi}}{}^{*} \bar{A}_{u0} = \alpha(d',d)$.

Thus,

$$\overset{r}{\underset{r-1}{\pi}}{}^{*} \alpha_{u0} = \rho_d(\alpha(h_{r-1,0},d)^{-1} \ \alpha(h_{r-1,0},d') \ \alpha(d',d)) = 1$$

by the consistency of T .

ii) For r with $1 \leq r \leq p$, if $f|[a_{r-1},a_r]$ represents a loop β' in $F(d)$, then exactly as in part i), the h_{u0} can be rechosen to yield an equation

$$\overset{r}{\underset{r-1}{\pi}}{}^{*} \alpha_{u0} = \rho_d(\alpha(h_{r-1,0},d)^{-1} \ \alpha(h_{r-1,0},d')\beta)$$

since in this case $\overset{r}{\underset{r-1}{\pi}}{}^{*} \bar{A}_{u0} = \beta$.

But since β is a loop, $y_d = y_{d'}$ = the base point of $F(d)$. Thus

$$\pi_{r-1}^{r} * \alpha_{u0} = \rho_d(\beta) .$$

<u>Relation 3</u>. $\alpha_{in} = 1, \ 1 \le i \le m$.

<u>Proof</u>: The vertices $\{v_{in}\}_{i-1}^{n}$ lie in at most 2 rectangles. By construction

$$G(v_{in}) = G(a_{in}) = z_{g_*} \ , \ 1 \le i \le m$$

so that $G(v_{in}) \in \varphi_{g_*} F(g_*)$, $1 \le i \le m$.

It is possible to choose the elements $\{h_{in}\}_{i=1}^{n}$ so that

$$\varphi_{h_{in}} F(h_{in}) \subseteq U_{in} \cap U_{i+1,n} \cap \varphi_{g_*} F(g_*)$$

where $U_{m+1,n} = U_{m,n}$.

Then there are elements $\ell_i \in \text{ob}C$ such that

$$\bar{\sigma}_i^1 \in C(\ell_i, k_{in}) \qquad \bar{\sigma}_i^2 \in C(\ell_i, g_*) \ , \quad 1 \le i \le m$$

$$\bar{\tau}_i^1 \in C(h_{i-1n}, \ell_i) \qquad \bar{\tau}_i^2 \in C(h_{in}, \ell_i) \ , \quad 1 \le i \le m$$

$$\sigma_i^1 = PF(\bar{\sigma}_i^1) \ ; \ \sigma_i^2 = PF(\bar{\sigma}_i^2) \ ; \ \tau_i^1 = PF(\bar{\tau}_i^1) \ ; \ \tau_i^2 = PF(\bar{\tau}_2^{f})$$

As before:

$$\alpha_{in} = \rho_{k_{in}}(\zeta_{in})$$

$$= \rho_{k_{in}}(\alpha(h_{i-1,n}, k_{in})^{-1} \ \sigma_{i-1,n}^{4}(\gamma_{i-1,n})^{-1} \ A_{in} \ \sigma_{in}^{2}(\gamma_{in})\alpha(h_{in}, k_{in}))$$

Since A_{in} is trivial this becomes

$$= \rho_{k_{in}}(\alpha(\ell_i,k_{in})^{-1}\ \tau_i^1\ \alpha(h_{i-1n},\ell_i)^{-1}\ \sigma_{i-1n}^4(\gamma_{i-1n})^{-1}\ \sigma_{in}^2(\gamma_{in})$$

$$\cdot\ \tau_i^2\ \alpha(h_{in},\ell_i)\alpha(\ell_i,k_{in}))$$

$$= \rho_{\ell_i}(\alpha(h_{i-1n},\ell_i)^{-1}\ \tau_i^1(\gamma_{i-1},n)^{-1}\ \tau_i^2(\gamma_{in})\alpha(h_{in},\ell_i))$$

$$= \rho_{g_*}(\alpha(h_{i-1n},g_*)^{-1}\ \sigma_i^2\ \tau_i^1(\gamma_{i-1n})^{-1}\ \sigma_i^2\ \tau_i^2(\gamma_{in})\alpha(h_{in},g_*))\ .$$

As before, since $\bar{v}_{i-1n} = \bar{v}_{in} = y_{g_*}$,

$$\gamma_{i-1n} = \alpha(h_{i-1n},g_*)^{-1}$$

$$\gamma_{in} = \alpha(h_{in},g_*)^{-1}$$

and $\alpha_{in} = 1$.

<u>Relation 4</u>. $\beta_{0j} = 1\ ,\ 1 \le j \le n$.

<u>Proof</u>: The vertices $\{v_{0j}\}_{j=1}^n$ lie in at most 2 rectangles. By construction

$$G(v_{0j}) = G(a_{0j}) = z_{g_*}\ ,\ 1 \le j \le n$$

so that $G(v_{0j}) \in \varphi_{g_*}F(g_*)\ ,\ 1 \le j \le n$.

It is possible to choose the elements $\{h_{0j}\}_{j=1}^n$ so that

$$\varphi(h_{in})F(h_{0j}) \subseteq U_{0j} \cap U_{0j+1} \cap \varphi_{g_*}F(g_*)$$

where $U_{0n+1} = U_{0n}$.

The relation follows as in the previous case by pulling the loop η_{0j} into $F(g_*)$.

<u>Relation 5</u>. $\beta_{mj} = 1\ ,\ 1 \le j \le n$.

<u>Proof</u>: As before.

The rest of the proof of the lemma is exactly as in Massey (pages 121-122): For any j, $1 \le j \le n$,

$$\left(\prod_{i=1}^{m} \alpha_{i,j-1}\right) = \left(\prod_{i=1}^{m} \alpha_{i,j-1}\right)\beta_{mj} \quad \text{by Relation 5}$$

$$= \left(\prod_{i=1}^{m-1} \alpha_{i,j-1}\right)\alpha_{m,j-1}\beta_{mj}$$

$$= \left(\prod_{i=1}^{m-1} \alpha_{i,j-1}\right)\beta_{m-1,j}\alpha_{mj} \quad \text{by Relation 1}$$

$$= \left(\prod_{i=1}^{m-2} \alpha_{i,j-1}\right)\alpha_{m-1,j-1}\beta_{m-1,j}\alpha_{mj}$$

$$= \left(\prod_{i=1}^{m-2} \alpha_{i,j-1}\right)\beta_{m-2j}\alpha_{m-1j}\alpha_{mj} \quad \text{by Relation 1}$$

$$\vdots$$

$$= \alpha_{1,j-1}\beta_{1j} \prod_{i=2}^{m} \alpha_{ij}$$

$$= \beta_{0j}\alpha_{1j} \prod_{i=1}^{m} \alpha_{ij} \quad \text{by Relation 1}$$

$$= \prod_{i=1}^{m} \alpha_{ij} \quad \text{by Relation 4}.$$

Thus $\prod_{i=1}^{m} \alpha_{ij-1} = \prod_{i=1}^{m} \alpha_{ij}$ for $1 \le j \le n$. It follows that

$$\prod_{i=1}^{m} \alpha_{i0} = \prod_{i=1}^{m} \alpha_{in}$$

Since by Relation 3, $\alpha_{in} = 1$, $1 \le i \le m$,

$$1 = \prod_{i=1}^{m} \alpha_{in} = \prod_{i=1}^{m} \alpha_{i0}$$

$$= \prod_{k=1}^{q} \rho_{g_k} \beta_k \quad \text{by Relation 2.} \quad \square$$

§5. Covering Spaces and Connectivity

The Hurewicz isomorphism plays an important role in the theorems which follow. To be able to apply it in dimension two requires our spaces to be connected and simply connected. While we have assumed our spaces to be connected, we do not wish to impose the further restriction that they be simply connected. Rather we will consider the universal cover of the space. This is simply connected and has the same second homotopy group as the original space. The process of passing to the universal cover requires the technical constructions which are made in this section.

Definition: C is an s-category and $F\colon C \to Q$ is a functor which is closed under finite intersections and has colimit induced morphisms $\{\varphi_g\colon F(g) \to \operatorname{colim} F\}_{g \in obC}$. $\{\varphi_g F(g)\}_{g \in obC}$ is an open cover of $\operatorname{colim} F$ and each φ_g is assumed to be an injection. If $Y \in obQ$ and $\rho\colon Y \to \operatorname{colim} F$ is an onto map, then the Y-pull-back of F is denoted $F_Y\colon C \to Q$ and is defined as follows:

i) for $g \in obC$, $F_Y(g)$ is the pull-back

$$\begin{tikzcd} F_Y(g) \arrow[r, "\psi_g"] \arrow[d, "\rho_g"'] & Y \arrow[d, "\rho"] \\ F(g) \arrow[r, "\varphi_g"] & \operatorname{colim} F \end{tikzcd}$$

ii) for $\alpha \in C(g,g')$ the universality of pull-backs induces a map $F_Y(\alpha)\colon F_Y(g) \to F_Y(g')$:

$$\begin{tikzcd}
F_Y(g) \arrow[rr, "\psi_g"] \arrow[dr, "F_Y(\alpha)"'] \arrow[ddd, "\rho_g"'] & & Y \arrow[ddd, "\rho"] \\
& F_Y(g') \arrow[ur, "\psi_{g'}"'] \arrow[d, "\rho_{g'}"'] & \\
& F(g') \arrow[dr, "\varphi_{g'}"] & \\
F(g) \arrow[ur, "F(\alpha)"] \arrow[rr, "\varphi_g"'] & & \operatorname{colim} F \, .
\end{tikzcd}$$

<u>Theorem 1</u>. i) Each ψ_g is monic.

ii) Each ψ_g is an injection.

iii) $\{\psi_g F_Y(g)\}_{g \in obC}$ is an open cover of Y .

iv) F_Y satisfies conditions i) and iii) of the definition of closed under finite intersections.

v) colim $F_Y = Y$.

<u>Proof</u>: i) In general, the pull-back of a monic is monic:

$$\begin{tikzcd}
A \arrow[dr, shift left, "f"] \arrow[dr, shift right, "f'"'] \arrow[drr, bend left, dashed, "\beta f"] \arrow[ddr, bend right, "\gamma f"'] & & \\
& B \arrow[r, "\beta"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\
& C \arrow[r, "\alpha"'] & E
\end{tikzcd}$$

The square is a pull-back and $f, f': A \to B$ are a pair of maps such that $\beta f = \beta f'$. $\alpha\gamma f = \delta\beta f = \delta\beta f' = \alpha\gamma f'$. Since α is monic, $\gamma f = \gamma f'$. The diagram commutes and uniqueness implies that $f = f'$.

ii) In general, the pull-back of an injection is an injection:

If $f: A \to B$ is a morphism in sets such that βf is a Q-morphism

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\beta"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ & C \arrow[r, "\alpha"] & E \end{tikzcd}$$

then $\delta\beta f = \alpha\gamma f$ is a Q-morphism. Since α is an injection, γf is a Q-morphism. The pull-back implies the existence of a unique Q-morphism $f'\colon A \to B$ such that $\gamma f' = \gamma f$, $\beta f' = \beta f$. Since the pull-back in **Q** is the pull-back of the underlying sets, pointwise $f' = f$. It follows that $f = f'$ and β is an injection.

iii) In general, the pull-back of an open set is open: The diagram

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d] & Y \arrow[d, "\rho"] \\ W \arrow[r, "\alpha"] & Z \end{tikzcd}$$

is a pull-back and αW is open in Z. The point x is an element of βX if and only if there exists a pair $(w,y) \in X$ with $\beta(w,y) = y = x$. This occurs if and only if there is a $w \in W$ with $\alpha w = \rho x$. But $\alpha w = \rho x$ if and only if $x \in \rho^{-1}\alpha W$. It follows that $\beta X = \rho^{-1}\alpha W$. If $a\colon C \to Y$ is admissible, then $a^{-1}\beta X = a^{-1}\rho^{-1}\alpha W$ is open in C since αW is open in Z.

If $y \in Y$, then $\rho y \in X$. Since $\{\varphi_g F(g)\}_{g \in obC}$ is a cover, there is an object $g \in obC$ and an element $x \in F(g)$ such that $\varphi_g X = \rho y$. Thus $(x,y) \in F_Y(g)$ and $\psi_g(x,y) = y$. It follows that $\{\psi_g F_Y(g)\}_{g \in obC}$ is a cover of Y.

iv) If $x \in F(g)$ then $\varphi_g x \in \text{colim } F$. Since ρ is onto, there exists a $y \in Y$ such that $\rho y = \varphi_g x$. Thus $(x,y) \in F_Y(g)$ and $\rho_g(x,y) = x$. It follows that ρ_g is onto and as sets $\rho_g F_Y(g) = F(g)$.

If $g, g' \in obC$ and

$$\psi_g F_Y(g) \cap \psi_{g'} F_Y(g') \neq \emptyset ,$$

then $\varphi_g F(g) \cap \varphi_{g'} F(g') = \varphi_g \rho_g F_Y(g) \cap \varphi_{g'} \rho_{g'} F_Y(g)$

$$= \rho \psi_g F_Y(g) \cap \rho \psi_{g'} F_Y(g')$$

$$\neq \emptyset .$$

Since F is closed under finite intersections, there exists a set Δ and a family $\{h_\delta\}_{\delta \in \Delta}$ with

i) $C(h_\delta, g) \neq \emptyset$ and $C(h_\delta, g') \neq \emptyset$

ii) $\bigcup_{\delta \in \Delta} \varphi_{h_\delta} F(h_\delta) = \varphi_g F(g) \cap \varphi_{g'} F(g')$.

If $x \in \psi_g F_Y(g) \cap \psi_{g'} F_Y(g')$, then $\rho x \in \varphi_g F(g) \cap \varphi_{g'} F(g')$ and $\varphi_g \rho_g x = \varphi_{g'} \rho_{g'} x = \rho x$. Since ρx is in the intersection, there exists $\delta \in \Delta, y \in \varphi_{h_\delta} F(h_\delta)$ such that $\varphi_{h_\delta} y = \rho x$. Thus $(y, x) \in F_Y(h_\delta)$ and $\psi_{h_\delta}(y, x) = x$.

v) By construction, the family $\{\psi_g : F_Y(g) \to Y\}$ is compatible and induces a map

$$\psi : \operatorname{colim} F_Y \longrightarrow Y .$$

If $x_1, x_2 \in \operatorname{colim} F_Y$ and $\psi x_1 = \psi x_2$, then for some objects $g, g' \in obC$ and elements $y_1 \in F_Y(g)$, $y_2 \in F_Y(g')$, $i_g y_k = x_1$, $i_{g'} y_2 = x_2$ where $\{i_g : F_Y(g) \to \operatorname{colim} F_Y\}_{g \in obC}$ are the colimit induced morphisms.

Then $\psi_g y_1 = \psi i_g y_1 = \psi x_1 = \psi x_2 = \psi i_{g'} y_2 = \psi_{g'} y_2$. Thus

$$\psi_g y_1 \in \psi_g F_Y(g) \cap \psi_{g'} F_Y(g') \neq \emptyset .$$

Since F_Y is closed under finite intersections, there exists an $h \in \mathrm{ob}C$, $\alpha \in C(h,g)$, $\alpha' \in C(h,g')$ and $z \in F_Y(h)$ with $\psi_g F(\alpha)z = \psi_g y_1$; $\psi_{g'} F(\alpha')z = \psi_{g'} y_2$. Since ψ_g is monic, $F(\alpha)z = y_1$ and $F(\alpha')z = y_2$. The colimit relations yield $x_1 = i_g y_1 = i_{g'} y_2 = x_2$.

Since $\{\psi_g F_Y(g)\}_{g \in \mathrm{ob}C}$ covers Y, ψ is onto.

The rest of this argument is standard:

If $a\colon C \to \mathrm{colim}\, F_Y$ is such that ψa is admissible then since $\psi_g F_Y(g)$ is open in Y, $(\psi a)^{-1}\, \psi_g F_Y(g)$ is open in C. Since C is compact there exists a finite set of indices $g_1, \ldots, g_m$ such that the collection $\{(\psi a)^{-1}\, \psi_{g_i} F_Y(g_i)\}_{i=1}^{m}$ covers C. $V_j = (\psi a)^{-1}\, \psi_{g_i} F_Y(g_i)$ and $C = \bigcup_{j=1}^{m} V_j$. There exist commutative diagrams

$$\begin{tikzcd}
V_j \arrow[r] \arrow[d, "\alpha_j"'] & C \arrow[d, "\psi a"] \\
\psi_{g_i} F_Y(g_i) \arrow[r, "i"] & \mathrm{colim}\, F_Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod V_j \arrow[r, "\coprod \alpha_j"] \arrow[d, "\text{onto}"] & \coprod_{i=1}^{m} \psi_{g_i} F_Y(g_i) \arrow[r] & \coprod_{g \in \mathrm{ob}C} \psi_g F_Y(g) \arrow[d, "q"] \\
C \arrow[rr, "a"] & & \mathrm{colim}\, F_Y
\end{tikzcd}$$

Since i is an injection, α_j is a Q-morphism. q is the projection which topologizes $\mathrm{colim}\, F_Y$. For each $c \in C$, there is a j such that $c \in V_j$. For each $x \in C - V_j$ there are disjoint open sets W_x, S_x in C with $c \in W_x$, $x \in S_x$ and $W_x \subseteq V_j$. The sets $\{S_x\}_{x \in C - V_j}$ together with V_j cover C so there is a finite set of $x_1, \ldots, x_n$ such that $\{S_{x_1}, \ldots, S_{x_n}, V_j\}$ is a cover of C. $W_c = \bigcap_{i=1}^{n} W_{x_i}$ is open in V_j and $\bigcup_{i=1}^{m} S_{x_i} \cap W_c = \emptyset$. $C - \bigcup S_{x_i} = K_c$ is compact and $c \in W_c \subseteq K_c \subseteq V_j$. $\{W_c\}_{c \in C}$ is an open cover of C . Thus there exist $c_1, \ldots, c_\ell$ such that $W_{c_1}, \ldots, W_{c_\ell}$ cover C . $K_{c_1}, \ldots, K_{c_\ell}$ cover C and $\coprod_{i=1}^{\ell} K_{c_i}$ is compact hausdorff and maps onto $\coprod V_j$. Since colim F_Y has the quotient Q-topology, a is admissible and ψ is an injection. □

<u>Definition</u>: C is an s-category and $F: C \to Q$ is a functor which is closed under finite intersections and has a pointed consistent system $T = (\Gamma_s(F); z_{g_*}, \{\alpha_g\})$ such that if $\{\varphi_g : F(g) \to \text{colim } F\}_{g \in \text{ob}C}$ and $\{\psi_g : F_T(g) \to \text{colim } F_T\}$ denote the colimit induced morphisms then

i) φ_g and ψ_g are monic for each $g \in \text{ob}C$;

ii) $\{\varphi_g F(g)\}_{g \in \text{ob}C}$ is an open cover of colim F.

Under these circumstances, the functor $I: C \to Q$ is defined on objects to be the discrete space

$$I(g) = \text{colim } F_T / \psi_g F_T(g) .$$

For $\alpha \in C(g,g')$, there is a commuting diagram of inclusions

$$\begin{tikzcd} F_T(g) \arrow[dr, "\psi_g"] \arrow[dd, "F_T(\alpha)"'] & \\ & \text{colim } F_T \\ F_T(g') \arrow[ur, "\psi_{g'}"'] & \end{tikzcd}$$

and an induced epimorphism $I(\alpha): I(g) \to I(g')$.

The functor $\tilde{F}: C \to Q$ is defined to be the $\widetilde{\text{colim } F}$ pull-back of F for $\widetilde{\text{colim } F}$ the universal cover of colim F . By Theorem 1, colim $\tilde{F} = \widetilde{\text{colim } F}$.

<u>Lemma 1</u>. If the diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "p'"'] & B \arrow[d, "p"] \\ D \arrow[r, "b"] & C \end{tikzcd}$$

is a pull-back and p is a covering projection, then p' is a covering projection.

<u>Proof</u>: Since p is a covering and the preceeding diagram is a pull-back, the following diagrams are pull-backs:

$$\begin{tikzcd} (B,I) \arrow[r, "{(p,I)}"] \arrow[d, "\eta_0"'] & (C,I) \arrow[d, "\eta_0"] \\ B \arrow[r, "p"'] & C \end{tikzcd} \qquad \begin{tikzcd} (A,I) \arrow[r, "{(a,I)}"] \arrow[d, "{(p',I)}"] & (B,I) \arrow[d, "{(p,I)}"] \\ (D,I) \arrow[r, "{(b,I)}"'] & (C,I) \end{tikzcd}$$

where η_0 is the endpoint projection.

If E is a space and α, β are maps such that the diagram

$$\begin{tikzcd} E \arrow[drr, dashed, bend left, "\alpha"] \arrow[dd, dashed, bend right, "\beta"'] & & \\ & (A,I) \arrow[r, "{(p',I)}"] \arrow[d, "\eta_0"'] & (D,I) \arrow[d, "\eta_0"] \\ & A \arrow[r, "p'"'] & D \end{tikzcd}$$

commutes, then $(b,I)\alpha$: $E \to B$, $a\beta$: $E \to B$, and $\eta_0(b,I)\alpha = b\eta_0\alpha = bp'\beta = pa\beta$. Thus there is induced a map γ: $E \to (B,I)$ such that $(p,I)\gamma = (b,I)\alpha$; $\eta_0\gamma = \alpha\beta$. Since $(b,I)\alpha = (p,I)\gamma$, there is induced a map δ: $E \to (A,I)$ such that $(a,I)\delta = \gamma$; $(p',I)\delta = \alpha$.

$\eta_0\gamma: E \to B$ and $\eta_0(p',I)\delta: E \to D$. Since $p\eta_0\gamma = \eta_0(p,I)\gamma = \eta_0(b,I)\alpha = \eta_0(b,I)(p',I)\delta = b\eta_0(p',I)\delta$ there is induced a map $\epsilon: E \to A$ such that $p'\epsilon = \eta_0(p',I)\delta$; $a\epsilon = \eta_0\gamma$. Since $\rho'\beta = \eta_0 x = \eta_0(p',I)\delta$ and $a\beta = \eta_0\gamma$, $\epsilon = \beta$. Since $p'\eta_0\delta = \eta_0(\rho' I)\delta$ and $a\eta_0\delta = \eta_0(a,I)\delta = \eta_0\gamma$, $\epsilon = \eta_0\delta$. It follows that $\beta = \eta_0\delta$ and δ makes diagram 1) commute.

If k also makes diagram 1) commute then since $(\rho,I)(a,I)k = b(p',I)k = (b,I)\alpha$ and $\eta_0(a,I)k = a\eta_0 k = a\beta$, $(a,I)k = \gamma$. Since $(p'I)k = \alpha$, and δ was unique with respect to these properties, $k = \delta$. $\square$

Lemma 2. If the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha'"] \arrow[d, "p'"'] & B \arrow[d, "p"] \\ D \arrow[r, "\alpha"'] & C \end{tikzcd}$$

is a pull-back and p is a covering projection, then the diagram

$$\begin{tikzcd} \pi_1 A \arrow[r, "\pi_1\alpha'"] \arrow[d, "\pi_1 p'"'] & \pi_1 B \arrow[d, "\pi_1 p"] \\ \pi_1 D \arrow[r, "\pi_1\alpha"'] & \pi_1 C \end{tikzcd}$$

is a pull-back in groups for any consistent choice of base points.

Proof: G is a group and f, f' are homomorphisms which make the following diagram commute:

$$\begin{tikzcd}
G \arrow[drr, bend left, "f"] \arrow[dr, bend right, "f'"'] & & \\
& \pi_1 A \arrow[r, "\pi_1\alpha'"] \arrow[d, "\pi_1 p'"'] & \pi_1 B \arrow[d, "\pi_1 p"] \\
& \pi_1 D \arrow[r, "\pi_1\alpha"] & \pi_1 C
\end{tikzcd}$$

For $g \in G$, $f'g = [a]$ is a homotopy class in $\pi_1 D$. If $a \in [a]$, then $a\colon S^1 \to D$. Since p' is a covering, this map can be lifted uniquely to a map $a'\colon S^1 \to A$. Since homotopies may be lifted as well, then entire class $[a]$ lifts to a class $[a']$ in $\pi_1 A$.

If $H\colon G \to \pi_1 A$ is any map which makes the diagram

$$\begin{tikzcd}
G \arrow[r, "H"] \arrow[dr, "f'"'] & \pi_1 A \arrow[d, "\pi_1 p'"] \\
& \pi_1 D
\end{tikzcd}$$

commute, then for each $g \in G$, Hg is a lift of $f'g$. It follows that if there is a homomorphism, it is unique.

If $g, g' \in G$, then since f' is a homomorphism, $f'gg' = f'gf'g'$ has as representative a juxtaposition of loops. Then for the H defined above, $HgHg'$ covers $f'gf'g' = f'gg'$ and thus equals Hgg'.

For $g \in G$, $\pi_1\alpha f'g = [\alpha a]$ is a class in $\pi_1 C$. If $Hg = [a']$, then since the square commutes, $p\alpha'a' = \alpha a$ and $[\alpha'a']$ covers $[\alpha a]$. Since $\pi_1 pf = \pi_1\alpha f'$, fg also covers $[\alpha a]$. Since lifts are unique, $fg = \pi_1\alpha'[a'] = \pi_1\alpha' Hg$. □

Lemma 3. $\pi_1\varphi_g : \pi_1(F(g), y_g) \to \pi_1(\text{colim } F, \varphi_g y_g)$ is monic for each $g \in \text{ob}C$.

Proof: For each $g \in \text{ob}C$, §3 provides a commutative diagram

$$\begin{tikzcd} F_T(g) = \pi_1(F(g), y_g) \arrow[r, "\pi_1\varphi_g"] \arrow[d, "\psi_g"'] & \pi_1(\text{colim } F, \varphi_g y_g) \arrow[d, "\bar{\alpha}_g"] \\ \text{colim } F_T \arrow[r, "h", "\cong"'] & \pi_1(\text{colim } F, z_{g_*}) \end{tikzcd}$$

where $\bar{\alpha}_g$ is an isomorphism, ψ_g is monic by assumption and h is an isomorphism by the Van Kampen Theorem. It follows that $\pi_1\varphi_g$ is monic. □

Theorem 2. For each $g \in \text{ob}C$, $\tilde{F}(g) = \widetilde{F(g)} \times I(g)$ for $\widetilde{F(g)}$ the universal covering of $F(g)$.

Proof: By definition $\tilde{F}(g)$ is the pull-back

$$\begin{tikzcd} \tilde{F}(g) \arrow[r, "\tilde{\psi}_g"] \arrow[d, "\rho_g"'] & \widetilde{\text{colim } F} \arrow[d, "p"] \\ F(g) \arrow[r, "\varphi_g"] & \text{colim } F . \end{tikzcd}$$

By the previous lemmas ρ_g is a covering, the diagram

$$\begin{tikzcd} \pi_1\tilde{F}(g) \arrow[r] \arrow[d] & \pi_1\widetilde{\text{colim } F} = 0 \arrow[d, "\pi_1 p"] \\ \pi_1 F(g) \arrow[r, "\pi_1\varphi_g"] & \pi_1 \text{ colim } F \end{tikzcd}$$

is a pull-back and $\pi_1\varphi_g$ is monic. The construction of pull-back in groups yields $\pi_1\tilde{F}(g) = 0$ for any choice of base point over y_g . It follows that each path component of $\tilde{F}(g)$ is a universal covering of $F(g)$.

The sets $\rho_g^{-1} y_g$, $p^{-1}\varphi_g y_g$ and $\pi_1(\text{colim } F, \varphi_g y_g)$ are isomorphic. $\pi_1(F(g), y_g)$ permutes the elements of each component of $\rho_g^{-1} y_g$ since each component of $\rho_g^{-1} y_g$ is the universal covering space of $F(g)$. $\pi_1\varphi_g\pi_1(F(g), y_g)$ as a subgroup of $\pi_1(\text{colim } F, \varphi_g y_g)$ permutes $p^{-1}\varphi_g y_g$ which is isomorphic to $\pi_1(\text{colim } F, \varphi_g y_g)$. The components of $\rho_g^{-1} y_g$ correspond to the orbits of the action and the orbits are counted by

$$\pi_1(\text{colim } F, \varphi_g y_g)/\pi_1 \varphi_g \pi_1(F(g), y_g)$$

which is isomorphic as a set to colim $F_T/\psi_g F_T(g) = I(g)$. □

<u>Lemma 4</u>. Suppose I is a set and $\{A_\alpha\}_{\alpha\in I}$ is a family of spaces together with maps $\{f_\alpha\colon A_\alpha \to X\}_{\alpha\in I}$ into the same space X . Suppose further that $X' \in \text{ob}Q$ and $f\colon X' \to X$ is a map. The family $\{B_\alpha\}_{\alpha\in I}$ is defined by the pull-backs

$$\begin{tikzcd} B_\alpha \arrow[r, "g_\alpha"] \arrow[d, "h_\alpha"'] & A_\alpha \arrow[d, "f_\alpha"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

for $\alpha \in I$. Under these hypotheses the pull-back

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "h"'] & \sqcup_I A_\alpha \arrow[d, "\sqcup f_\alpha"] \\ X' \arrow[r, "f'"'] & X \end{tikzcd}$$

is homeomorphic to $\underset{I}{\sqcup} B_\alpha$.

<u>Proof</u>: The diagram

$$\begin{tikzcd}
\sqcup B_\alpha \arrow[drr, "\sqcup g_\alpha"] \arrow[ddr, "\sqcup h_\alpha"'] & & \\
& B \arrow[r, "g"] \arrow[d, "h"'] & \sqcup A_\alpha \arrow[d, "\sqcup f_\alpha"] \\
& X' \arrow[r] & X
\end{tikzcd}$$

induces a map $k: \sqcup B_\alpha \to B$.

The definition of pull-back yields an injection

$$B \xrightarrow{i} \sqcup A_\alpha \times X' = \sqcup(A_\alpha \times X')$$

and $B = \sqcup B^\alpha$ where for each α , $B^\alpha = i^{-1}(A_\alpha \times X')$. The diagrams

$$\begin{tikzcd}
B^\alpha \arrow[r, "i"] \arrow[d] & A_\alpha \times X' \arrow[r] & A_\alpha \arrow[dd, "f_\alpha"] \\
A_\alpha \times X' \arrow[d] & & \\
X' \arrow[rr, "f'"] & & X
\end{tikzcd}$$

commute for each $\alpha \in I$, to yield a map $\ell_\alpha: B^\alpha \to B_\alpha$. The composition m

$$B = \sqcup B^\alpha \xrightarrow{\sqcup \ell_\alpha} \sqcup B_\alpha \xrightarrow{k} B$$

has the property that $gm = \sqcup g_\alpha$ and $hm = \sqcup h_\alpha$. It follows that $k \sqcup \ell_\alpha = 1_B$ and the Lemma follows. □

The following generalization of Theorem 2 will be needed in what follows:

<u>Theorem 3</u>. Suppose $X, X' \in \mathrm{ob}Q$ are path connected spaces and $\alpha: X' \to X$ is a map which induces a monomorphism of fundamental groups.

p: $\tilde{X} \to X$ is the universal covering space of X and J is an indexing set. The pull-back Y

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \coprod_J \tilde{X} \arrow[d, "\sqcup p"] \\ X' \arrow[r, "\alpha"] & X \end{tikzcd}$$

has the property that for some indexing set J', $Y = \coprod_{J'} \tilde{X}'$.

<u>Proof</u>: If Y' is the pull-back

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ X' \arrow[r, "\alpha"] & X \end{tikzcd}$$

then by the proof of Theorem 2, Y' is the disjoint union of universal covering spaces of X'. By Lemma 4, $Y = \sqcup Y'$. □

In the next section, a few results about connectivity will be needed. The necessary material consists of the following definition and two propositions.

<u>Definition</u>: C is an s-category. Two vertices, $v, v' \in C_0$ are related if there exists an $f \in \mathrm{ob}C$ such that $C(f,v)$ and $C(f,v')$ are non-empty. $\sim$ denotes the generated equivalence relation. C is <u>connected</u> if $|C_0/\sim| = 1$.

<u>Proposition 1</u>. C is an s-category and F is a path connected functor which is closed under finite intersections and has colimit induced morphisms $\{\varphi_g\colon F(g) \to \operatorname{colim} F\}_{g \in \mathrm{ob}C}$. $\{\varphi_g F(g)\}_{g \in \mathrm{ob}C}$ is an open cover of colim F. C is connected if and only if colim F is path connected.

<u>Proof</u>: =>) If $x, y \in \operatorname{colim} F$, there exist $g, g' \in \mathrm{ob}C$ and $a' \in F(g)$, $b' \in F(g')$ such that $\varphi_g a' = x$ and $\varphi_{g'} b' = y$. If $\lambda \in \mathrm{im}g$ and $\lambda' \in \mathrm{im}g'$ there exist vertices v, v' such that $\mathrm{im}v = \lambda$, $\mathrm{im}v' = \lambda'$ and $\alpha \in C(g,v) \neq \emptyset$, $\alpha' \in C(g',v') \neq \emptyset$. The elements $a \in F(v)$ and $b \in F(v')$ are the images $a = F(\alpha)a'$, $b = F(\alpha')b'$.

Since C is path connected, $v \sim v'$ and there is a finite string of primitively related vertices

$$\{v_i\}_{i=0}^{n}$$

with $v_0 = v$, $v_n = v'$ and a family

$$\{f_i \in \mathrm{ob}C\}_{i=0}^{n-1}$$

with $\alpha_i \in C(f_i, v_i) \neq \emptyset$ and $\beta_i \in C(f_i, v_{i+1}) \neq \emptyset$, $i=0,\dots,n-1$. The points $\{x_i \in F(f_i)\}_{i=0}^{n-1}$ are arbitrary.

By definition of the colimit relations, $F(\alpha_i)x_i = F(\beta_i)x_i$. Since F is path connected, the image $\varphi_{v_i} F(v_i)$ is connected for $i=0,\dots,n$. The paths $\{p_i\}_{i=0}^{n}$ are chosen as follows:

$$p_0 \in P(x, \varphi_v F(v), \varphi_{f_0} x_0)$$

$$p_i \in P(\varphi_{f_{i-1}} x_{i-1}, \varphi_{v_i} F(v_i), \varphi_{f_i} x_i) \quad , \quad i=1,\dots,n-1$$

$$p_n \in P(\varphi_{f_{n-1}} x_{n-1}, \varphi_{v'} F(v'), y) .$$

The composition $\sum_{i=0}^{n} p_i$ is a path in $\operatorname{colim} F$ from x to y.

<=) If $v, v' \in C_0$ are vertices of C, then $\varphi_v F(v)$ and $\varphi_{v'} F(v')$ are subsets of the $\operatorname{colim} F$. $x_0 \in \varphi_v F(v)$ and $x_n \in \varphi_{v'} F(v')$ are arbitrary points. $p\colon I \to \operatorname{colim} F$ is a path from x_0 to x_n.

$\{p^{-1}\varphi_g F(g)\}_{g\in obC}$ is a cover of I and n is chosen such that $1/n$ is less than the Lebesque number of the covering. $\{[i/n,(i+1)/n]\}_{i=0}^{n-1}$ partitions I and $p_i = p|[i/n,(i+1)/n]$, $i=0,\ldots,n-1$. The collection $\{g_i\}_{i=0}^{n-1}$ is chosen such that

$$p_i[i/n,(i+1)/n] \subseteq \varphi_{g_i} F(g_i) \quad , \quad i=0,\ldots,n-1 \ .$$

Since $p_i((i+1)/n) \in \varphi_{g_i} F(g_i) \cap \varphi_{g_{i-1}} F(g_{i+1})$, $i=0,\ldots,n-1$, there exists a sequence $\{h_i \in obC\}_{i=0}^{n-1}$ with

$$C(h_i,g_i) \neq \emptyset \quad , \quad C(h_i,g_{i+1}) \neq \emptyset$$

for $i=0,\ldots,n-2$.

$p_0(0) \in \varphi_v F(v) \cap \varphi_{g_0} F(g_0)$ and there is an $h_{-1} \in obC$ with $C(h_{-1},v) \neq \emptyset$, $C(h_{-1},g_0) \neq \emptyset$. Similarly, $p_{n-1}(n) \in \varphi_{g_{n-1}} F(g_{n-1}) \cap \varphi_{v'} F(v')$ and there exists an h_n with $C(h_n,g_{n-1}) \neq \emptyset$ and $C(h_n,v') \neq \emptyset$. If $\{v_i \in C_0\}_{i=0}^{n-1}$ is any set of vertices with $C(g_i,v_i) \neq \emptyset$, then the sequence $\{v,v_i,v'\}_{i=0}^{n-1}$ relates v to v' . □

<u>Proposition 2</u>. If C is a connected s-category, then $\Gamma(C)$ is connected. If C is an s-category such that for each vertex $v \in C_0$ there is a free face $f \in Fr(C)$ with $C(f,v) \neq \emptyset$, then $\Gamma(C)$ connected implies that C is connected.

<u>Proof</u>: i) C is connected. If $f,f' \in Fr(C)$ are two vertices of $\Gamma(C)$, then there are vertices (of C), $v,v' \in C_0$ with $C(f,v) \neq \emptyset$ and $C(f',v') \neq \emptyset$. Thus there exists a sequence of vertices

$$\{v_i\}_{i=0}^{n}$$

with $v_0 = v$, $v_n = v'$ and a sequence of elements

$$\{f_i\}_{i=0}^{n-1}$$

with

$$C(f_i, v_i) \neq \emptyset \quad \text{and} \quad C(f_i, v_{i+1}) \neq \emptyset \qquad i=0,\ldots,n-1 \ .$$

If $\{h_i \in Fr(C)\}_{i=0}^{n-1}$ is a set of free faces with

$$C(h_i, f_i) \neq \emptyset \quad \text{for} \quad i=0,\ldots,n-1 \ ,$$

then $v_i \in ob(h_{i-1} \cap h_i) \neq \emptyset$ $i=1,\ldots,n-1$. Also, $v_0 \in ob(f \cap h_0)$ and $v_n \in ob(f' \cap h_{n-1})$. These sets contain a sequence of free faces which connects f to f' in $\Gamma(C)$.

ii) $\Gamma(C)$ is connected. If $v, v' \in C_0$, then there are free faces f, f' with $C(f,v) \neq \emptyset$ and $C(f',v') \neq \emptyset$ and a path in $\Gamma(C)$

$$\{h_i \in Fr(f_i \cap f_{i+1})\}_{i=0}^{n}$$

with $f_0 = f$, $f_{n+1} = f'$. If $v_i \in C_0 \cap ob(f_i \cap f_{i+1})$, $i=0,\ldots,n$, then since $C(f_{i+1}, v_i) \neq \emptyset$ and $C(f_{i+1}, v_{i+1}) \neq \emptyset$, $v_i \sim v_{i+1}$, $i=0,\ldots,n-1$. Also, $C(f_{n+1}, v_n) \neq \emptyset$, $C(f_{n+1}, v') \neq \emptyset$; $C(f_0, v_0) \neq \emptyset$, $C(f_0, v) \neq \emptyset$. Thus $v_n \sim v'$ and $v \sim v_0$ so that $v \sim v'$ and C is connected. $\square$

§6. A Mayer Vietoris Theorem.

This section concerns determination of conditions required of an s-category, C, and a functor $F\colon C \to Q$, to guarantee that

$$*)\ H_2(\operatorname{colim} \tilde{F};\mathbb{Z}) = \operatorname{colim}\,(H_2(\ ;\mathbb{Z})\tilde{F}),$$

where $\tilde{F}\colon C \to Q$ is the $\widetilde{\operatorname{colim} F}$ pull-back of F described in Section 5 and $H_2(\ ;\mathbb{Z})\colon Q \to Ab$ denotes two dimensional singular homology with integer coefficients. $H_2(\ ;\mathbb{Z})$ will be denoted H_2. Actually, it turns out that a slightly stronger theorem is easier to prove, namely that

$$**)\ H_2(\operatorname{colim} F_Y) = \operatorname{colim} H_2F_Y$$

for F_Y the Y-pullback of F where Y is the disjoint union of universal covering spaces of $\operatorname{colim} F$:

$$Y = \bigsqcup_I \widetilde{\operatorname{colim} F}$$

for some index set I. Equation $*)$ is equation $**)$ in Case I has a single element.

To make the structure of the proof of equation $**)$ more intelligible, the various technical lemmas needed are only stated here. Their proofs are in Section 7 and a few examples are presented in Section 8. The necessary hypotheses will be discussed and lettered and the proof of $**)$ will be given at the end of this section. The results of this section will apply to the case in which C is 1-dimensional; however, a weaker set of hypotheses than are given here will also imply $**)$ in that case. Since the proofs and related structures are somewhat complicated, the special features of the 1-dimensional case will be considered separately in Section 10.

No attempt will be made to prove this theorem for an arbitrary s-category: it will be assumed that

A) C has at most countably many vertices; and

B) C is upward finite.

Topologically condition A) limits consideration to spaces with countable open covers while condition B) is the assertion that any intersection of elements in the cover involves only finitely many open sets. While entailing no substantial loss of generality, condition A) permits argument by induction.

The only cases of interest are those arising from spaces with open covers. This information is inserted into the hypotheses as the assumptions that

C) F is closed under finite intersections, and

D) $\{\varphi_g F(g)\}_{g \in obC}$ is an open cover of $\operatorname{colim} F$ and each φ_g is an injection for $\{\varphi_g : F(g) \to \operatorname{colim} F\}_{g \in obC}$ the colimit induced morphisms.

D$'$) For each pair $g, g' \in obC$,

$$\varphi_g F(g) = \varphi_{g'} F(g')$$

implies $g = g'$;

D$''$) For each object $g \in obC$, $\varphi_g F(g) \neq \emptyset$.

Condition C) asserts that all the intersections are accounted for. D$'$) is the assertion that each such intersection is accounted for only once and D$''$) asserts that C has no "extra" objects which represent nothing in the cover.

Another assumption on C is

E) C has a generating tree.

If C is 1-dimensional, then the constructions of Section 10 eliminate this requirement. In the higher dimensional cases, Example 1 of Section 8 shows that some hypothesis like that of a generating tree is necessary. Condition E) together with condition B) implies that C has a representation as a simplicial complex (cf. Theorem 2 of Section 1). It has been assumed throughout that the vertices of C are connected (i.e. that each open set in the cover is connected).

The next assumptions are on the connectivity of the space

F) F is path connected

G) colim F is connected.

By Proposition 1 of Section 5, the condition that C is connected follows from conditions C), F), and G).

It turns out that a condition on fundamental group is necessary as well, namely:

H) For each $g \in \mathrm{ob}C$, and some choice of base point the inclusion $\varphi_g\colon F(g) \to \mathrm{colim}\, F$ induces a monomorphism

$$\varphi_g^*\colon \pi_1(F(g),*) \to \pi_1(\mathrm{colim}\, F,*)$$

of fundamental groups.

Conditions F) and G) insure that this condition is independent of the choice of base point. By the remarks following Theorem 3 of Section 2, if C is 1-dimensional, then condition E) enables condition H) to be substituted with the weaker assumption that

> H_1) If C is 1-dimensional, for each pair $g, g' \in obC$ with $\alpha \in C(g,g')$, the inclusion $F(\alpha): F(g) \to F(g')$ induces a monomorphism
>
> $$F^*(\alpha): \pi_1(F(g),*) \to \pi_1(F(g'),*)$$
>
> for some choice of base point.

The importance of condition H) is revealed by Theorem 3 of Section 5 which yields that $F_Y(g)$ is the disjoint union of universal covering spaces of $F(g)$. That is, each component of $F_Y(g)$ is simple connected. Since $H_1(\ ;\mathbf{Z})$ is zero in each component, $H_1(F_Y(g);\mathbf{Z}) = 0$ for each object $g \in obC$.

As was indicated above, the proof of equation **) will be by induction on the number of vertices of C. If C has a single vertex, then it has a single object and equation **) is trivial since it concerns constant functors.

The means of reducing the number of vertices is provided by the following definitions:

Definition: If C is an s-category and $v \in C_0$ is a vertex of C, then the category C_v is the full-subcategory of C generated by the objects

$$obC - \{f \in obC \mid C(f,v) \neq \emptyset\}.$$

Definition: If C is an s-category and $v \in C_0$ is a vertex of C, then the s-category $v \cap C_v$ is the full-subcategory of C generated by the objects

$$\{g \in \text{ob}C \mid C(g,v) \neq \emptyset\} - \{v\} .$$

While both C_v and $v \cap C_v$ are to be regarded as s-categories based on $\Lambda - \{\text{im} v\}$, $v \cap C_v$ must be given an s-category structure which is not compatible with the structure of C. This situation is best understood through the proofs of the following Propositions:

<u>Proposition 1</u>. C_v is an s-category based on $\Lambda - \{\text{im} v\}$. □

<u>Proposition 2</u>. $v \cap C_v$ is an s-category based on $\Lambda - \{\text{im} v\}$ and has connected vertices. □

<u>Proposition 3</u>. If $|C_0| < \infty$, then $|(C_v)_0| < |C_0|$. □

<u>Proposition 4</u>. If $|C_0| < \infty$, then $|(v \cap C_v)_0| < |C_0|$. □

<u>Proposition 5</u>. C_v and $v \cap C_v$ are upward finite. □

The functor F_v is the restriction of F to C_v; it has colimit induced morphisms $\{\varphi_g^v\colon F_v(g) \to \text{colim}\, F_v\}_{g \in \text{ob}C_v}$. The functor $\bar{F}$ is the restriction of F to $v \cap C_v$; it has colimit induced morphisms $\{\bar{\varphi}_g\colon \bar{F}(g) \to \text{colim}\, \bar{F}\}_{g \in \text{ob}(v \cap C_v)}$. These two families induce morphisms

$$\varphi\colon \text{colim}\, F_v \longrightarrow \text{colim}\, F$$

and

$$\bar{\varphi}\colon \text{colim}\, \bar{F} \longrightarrow \text{colim}\, F$$

according to the following commutative diagrams:

$\mathcal{D}_1$:

$$\begin{array}{ccc} F_v(g) & \xrightarrow{\varphi_g^v} & \text{colim}\, F_v \\ \| & & \downarrow{\scriptstyle \varphi} \\ F(g) & \xrightarrow{\varphi_g} & \text{colim}\, F \end{array} \qquad \text{for } g \in \text{ob}C_v$$

$\mathcal{D}_2$:

$$\begin{array}{ccc} \bar{F}(g) & \xrightarrow{\bar{\varphi}_g} & \operatorname{colim} \bar{F} \\ \parallel & & \downarrow \bar{\varphi} \\ F(g) & \longrightarrow & \operatorname{colim} F \quad \text{for} \quad g \in \operatorname{ob}(v \cap C_v) \end{array}$$

The following propositions regarding $\bar{F}$ and F_v will be needed:

Proposition 6. F_v is path connected and closed under finite inter- intersections. $\square$

Proposition 7. $\bar{F}$ is path connected and closed under finite intersections. $\square$

Proposition 7'. F_v and $\bar{F}$ satisfy conditions D' and D''. $\square$

Proposition 8. $\{\varphi_g^v F(g)\}_{g \in \operatorname{ob}C_v}$ is an open cover of $\operatorname{colim} F_v$ and each φ_g^v is an injection. $\square$

Proposition 9. $\{\bar{\varphi}_g \bar{F}(g)\}_{g \in \operatorname{ob}(v \cap C_v)}$ is an open cover of $\operatorname{colim} \bar{F}$ and each $\bar{\varphi}_g$ is an injection. $\square$

Proposition 10. φ is an injection. $\square$

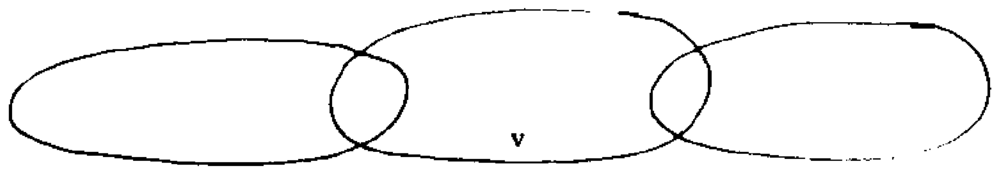
Proposition 11. $\bar{\varphi}$ is an injection. $\square$

Proposition 12. $\bar{\varphi} \operatorname{colim} \bar{F} = \varphi_v F(v) \cap \varphi \operatorname{colim} F_v$. $\square$

Proposition 13. C_v has a generating tree. $\square$

Proposition 14. $v \cap C_v$ has a generating tree. $\square$

So far nothing has been said about the connectivity of C_v . With no restriction at all, C_v may not be connected as seen in the three piece cover:



<u>Proposition 15</u>. If $|C_0| < \infty$, then there exists a vertex $v \in C_0$ such that C_v is connected. □

The vertex v will always be chosen in accordance with Proposition 15. From the connectivity of C_v comes the following facts:

<u>Proposition 16</u>. $\varphi_v F(v) \cap \varphi \operatorname{colim} F_v$ is connected. □

<u>Proposition 17</u>. $\operatorname{colim} \bar{F}$ is connected. □

That C_v and $v \cap C_v$ satisfy condition H) follows trivially from the commutativity of the diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ (together with the fact that C satisfies condition H)).

Thus far C has been broken into three pieces, v, C_v, and $v \cap C_v$, each of which has fewer vertices than C and satisfies the conditions of the Theorem. To be able to use this information, it is necessary to sew C back together again. One step is the following Proposition:

<u>Proposition 18</u>. The diagram

$$\begin{tikzcd}
\varphi_v F(v) \cap \varphi \operatorname{colim} F_v \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \varphi \operatorname{colim} F_v \arrow[d, "\varphi"] \\
\varphi_v F(v) \arrow[r, "\varphi_v"] & \operatorname{colim} F
\end{tikzcd}$$

is a push-out. □

It will be convenient to label the diagram in Proposition 18 with a functor:

<u>Definition</u>. $\bar{C}$ is the 1-dimensional s-category with two vertices $\{v_1, v_2\} = C_0$ and 1 free face $\{f\} = C_1$ together with two non-identity morphisms $a \in \bar{C}(f, v_1)$, $b \in \bar{C}(f, v_2)$. $G\colon \bar{C} \to Q$ is the functor with

$$G(f) = \varphi_v F(v) \cap \varphi \operatorname{colim} F_v$$

$$G(v_1) = \varphi \operatorname{colim} F_v$$

$$G(v_2) = \varphi_v F(v)$$

$$G(a) = \alpha$$

$$B(b) = \beta .$$

Clearly, $\operatorname{colim} G = \operatorname{colim} F$.

In the proof which follows it will be necessary to know that $\bar{C}$ satisfies condition H). If β induces a monomorphism, then $\varphi_v \beta$ does also since φ_v is given by condition H) on C to induce a monomorphism. Thus it would follow that α induced a monomorphism. Since $\varphi_v F(v) \cap \varphi \operatorname{colim} F_v$ is path connected, the classical Seifert-van Kampen Theorem yields $\pi_1(\operatorname{colim} F, *)$ as an amalgam

$$\pi_1(\operatorname{colim} F, *) = \pi_1(\varphi_v F(v), *) \underset{\pi_1(G(f), *)}{*} \pi_1(\varphi \operatorname{colim} F_v, *) .$$

A theorem on amalgamated products then yields that φ induces a monomorphism. The one-dimensional case is easy:

<u>Proposition 19</u>. If C is 1-dimensional, then $\bar{C}$ satisfies condition H). □

Example 2 in Section 8 shows that $\bar{C}$ may not satisfy condition H) if C is 2-dimensional. Unfortunately, just requiring that β induce a monomorphism is not enough since i) the argument will be inductive and the hypothesis must also hold for C_v, ii) the argument will also have to be applied to $v \cap C_v$. One assumption that works requires a short definition.

Definition. The pair (C',w) consisting of a subcategory $C' \subseteq C$ a vertex $w \in C_0 - \mathrm{ob}C'$, is initial provided i) for each $f \in \mathrm{ob}C'$, $C(f,w) \neq \emptyset$ and ii) if F' denotes the restriction of F to C' , then $\operatorname{colim} F'$ is connected.

The last assumption may now be stated:

I) For each initial pair (C',w) , the colimit induced map $c\colon \operatorname{colim} F' \to F(w)$ induces a monomorphism of fundamental groups

$$c^*\colon \pi_1(\operatorname{colim} F',*) \longrightarrow \pi_1(F(w),*)$$

for some choice of base point.

It should be pointed out that in case C' has a single object $\{g\} = \mathrm{ob}C'$, condition I) follows from condition H) and the commutative diagram.

$$\begin{tikzcd}
\operatorname{colim} F = F(g) \arrow[d] \arrow[dr, "\varphi_g"] & \\
 & \operatorname{colim} F \\
F(w) \arrow[ur, "\varphi_w"'] &
\end{tikzcd}$$

This comment is the basis of the proof of the following Proposition.

Proposition 20. If C is a 1-dimensional s-category and $F\colon C \to Q$ is a functor such that C and F satisfy conditions C) and H), then they satisfy condition I). □

With this distinction between the 1 and higher dimensional cases in mind, condition I) will be assumed until the statement of Theorem 1. A weaker version of condition H) together with condition I) implies condition H):

> H$'$) For each vertex $w \in \mathrm{ob}C$ and some choice of base point the inclusion φ_w: $F(w) \to \mathrm{colim}\, F$ induces a monomorphism
>
> $$\varphi_w^*: \ \pi_1(F(w),*) \longrightarrow \pi_1(\mathrm{colim}\, F,*)$$
>
> of fundamental group.

If $g \in \mathrm{ob}C$, and $w \in \nu g$, then $(\{g\},w)$ is initial and the diagram

$$\begin{tikzcd}
\mathrm{colim}\, F' = F(g) \arrow[rr, "\varphi_g"] \arrow[d, "\alpha"'] & & \mathrm{colim}\, F \\
F(w) \arrow[urr, "\varphi_w"'] & &
\end{tikzcd}$$

yields that φ_g induces a monomorphism since φ_w does by condition H$'$) and α does by condition I).

To make use of condition I) requires the following propositions:

Proposition 21. C_v and F_v satisfy condition I). $\square$

Proposition 22. $v \cap C_v$ and $\bar{F}$ satisfy condition I). $\square$

Proposition 23. $\bar{C}$ and G satisfy condition H). $\square$

The last lemma involves converting part of the classical Mayer-Vietoris sequence into a push-out.

Proposition 24. In the category Ab, the sequence

$$R \xrightarrow{\alpha} F_1 \oplus F_2 \xrightarrow{\beta} H \longrightarrow 0$$

is exact if and only if the diagram

$$\begin{tikzcd}
R \arrow[r, "(1\ 0)\alpha"] \arrow[d, "-(0\ 1)\alpha"'] & F_1 \arrow[d, "\beta\binom{1}{0}"] \\
F \arrow[r, "\beta\binom{0}{1}"'] & H
\end{tikzcd}$$

is a push-out. □

The main result may now be stated and proved:

<u>Theorem 1</u>. If C is an s-category and $F\colon C \to Q$ is a functor such that C and F satisfy conditions A) - I), then the map $\alpha\colon \operatorname{colim} H_2F_Y \to H_2 \operatorname{colim} F_Y$ induced by the inclusion induced family of maps

$$\{H_2F_Y(g) \longrightarrow H_2 \operatorname{colim} F_Y\}_{g\in \mathrm{ob}C}$$

is an isomorphism for any disjoint union, $Y = \sqcup\, \widetilde{\operatorname{colim} F}$, of universal covering spaces of $\operatorname{colim} F$.

<u>Proof</u>: The proof is by induction on the number of vertices of C. The case for 1-vertex has been described previously. Since $\bar{C}$ and G satisfy condition H, in the diagram

$$\mathcal{D}_3\colon\quad \begin{tikzcd} \operatorname{colim} \bar{F} = \varphi_v F(v) \cap \varphi \operatorname{colim} F_v \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \varphi \operatorname{colim} F_v \arrow[d, "\varphi"] \\ \varphi_v F(v) \arrow[r, "\varphi_v"'] & \operatorname{colim} F = \operatorname{colim} G \end{tikzcd}$$

$\varphi\beta = \varphi_v\alpha\colon G(f) \to \operatorname{colim} G$ induces a monomorphism of fundamental groups as does $\varphi\colon G(v_1) \to \operatorname{colim} G$.

Y' and Y'' are defined to be the pull-backs

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{colim} F_v \arrow[r, "\varphi"] & \operatorname{colim} F \end{tikzcd} \qquad \begin{tikzcd} Y'' \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{colim} \bar{F} \arrow[r, "\varphi\beta"] & \operatorname{colim} F \end{tikzcd}$$

By Theorem 3 of Section 5, Y' and Y'' are the disjoint union of universal covering spaces. It follows from the induction hypothesis and results of this section that

$$H_2 \operatorname{colim} (F_v)_{Y'} = \operatorname{colim} H_2(F_v)_{Y'}$$

and

$$H_2 \operatorname{colim} \bar{F}_{Y''} = \operatorname{colim} H_2 \bar{F}_{Y''} .$$

By Theorem 1 of Section 5, $\operatorname{colim} G_Y = Y = \operatorname{colim} F_Y$. The classical Mayer-Vietoris sequence for the open cover $\{G_Y(v_1), G_Y(v_2)\}$ of Y is

$$H_2G_Y(f) \longrightarrow H_2G_Y(v_1) \oplus H_2G_Y(v_2) \longrightarrow H_2Y \longrightarrow 0$$

(where $H_1(G_Y(f), \mathbb{Z}) = 0$ since it is simply connected in each component). Proposition 24 coverts this sequence into the push-out $\mathcal{B}_2$:

$$\begin{tikzcd} H_2G_Y(f) \arrow[r] \arrow[d] & H_2G_Y(v_1) \arrow[d] \\ H_2G_Y(v_2) \arrow[r] & H_2Y \end{tikzcd}$$

Clearly, the relation between colimits and push-outs yields the following push-out:

$$\begin{tikzcd} H_2G_Y(f) \arrow[r] \arrow[d] & H_2G_Y(v_1) \arrow[d] \\ H_2G_Y(v_2) \arrow[r] & \operatorname{colim} H_2G_Y . \end{tikzcd}$$

It follows that $H_2Y = \operatorname{colim} H_2G_Y$. Since $G_Y(f) = Y'' = \operatorname{colim} \bar{F}_{Y''}$, $G_Y(v_1) = Y' = \operatorname{colim} (F_v)_{Y'}$, and $G_Y(v_2) = F_Y(v)$, this push-out may be rewritten as the push-out $\mathcal{B}_4$:

$$\begin{tikzcd} H_2 \operatorname{colim} \bar{F}_{Y''} \arrow[r] \arrow[d] & H_2 \operatorname{colim} (F_v)_{Y'} \arrow[d] \\ H_2F_Y(v) \arrow[r] & \operatorname{colim} H_2G_Y . \end{tikzcd}$$

or as $\mathcal{D}_5$:

$$\begin{tikzcd}
\operatorname{colim} H_2\bar{F}_{Y''} \arrow[r] \arrow[d] & \operatorname{colim} H_2(F_v)_{Y'} \arrow[d] \\
H_2F_Y(v) \arrow[r] & \operatorname{colim} H_2G_Y .
\end{tikzcd}$$

For $g \in \mathrm{ob}C_v$, the spaces $(F_v)_{Y'}(g)$ and $F_Y(g)$ are related by the diagram

$$\begin{tikzcd}
 & F_Y(g) \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\
(F_v)_{Y'}(g) \arrow[ur, "\gamma_g"] \arrow[rr] \arrow[dd] & & Y' \arrow[ur] \arrow[dd] & \\
 & F(g) \arrow[rr] & & \operatorname{colim} F \\
F_v(g) \arrow[ur, "\cong"] \arrow[rr] & & \operatorname{colim} F_v \arrow[ur] &
\end{tikzcd}$$

Since the diagrams

$$\begin{tikzcd}
F_Y(g) \arrow[r] \arrow[d] & Y \arrow[d] \\
F(g) \arrow[r] & \operatorname{colim} F
\end{tikzcd}
\qquad
\begin{tikzcd}
(F_v)_{Y'}(g) \arrow[r] \arrow[d] & Y' \arrow[d] \\
F_v(g) \arrow[r] & \operatorname{colim} F_v
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] & Y \arrow[d] \\
\operatorname{colim} F_v \arrow[r] & \operatorname{colim} F
\end{tikzcd}$$

are pull-back diagrams, the diagram

$$\begin{tikzcd}
(F_v)_{Y'}(g) \arrow[r, "\gamma_g"] \arrow[d] & F_Y(g) \arrow[d] \\
F_v(g) \arrow[r, "="] & F(g)
\end{tikzcd}$$

is a pull-back, and $(F_v)_{Y'}(g) = F_Y(g)$.

For $g \in ob(v \cap C_v)$, the spaces $\bar{F}_{Y''}(g), (F_v)_{Y'}(g)$ and $F_Y(g)$ are related by the diagram

$$\begin{tikzcd}
 & & F_Y(g) \arrow[rr] \arrow[ddd] & & Y \arrow[ddd] \\
 & (F_v)_{Y'}(g) \arrow[ur] \arrow[rr] \arrow[ddd] & & Y' \arrow[ur] \arrow[ddd] & \\
\bar{F}_{Y''}(g) \arrow[ur] \arrow[rr] \arrow[ddd] & & Y'' \arrow[ur] \arrow[ddd] & & \\
 & & F(g) \arrow[rr] & & \operatorname{colim} F \\
 & F_v(g) \arrow[ur] \arrow[rr] & & \operatorname{colim} F_v \arrow[ur] & \\
\bar{F}(g) \arrow[ur] \arrow[rr] & & \operatorname{colim} \bar{F} \arrow[ur] & &
\end{tikzcd}$$

and $\bar{F}_{Y''}(g) = (F_v)_{Y'}(g) = F_Y(g)$ as before.

The compositions

$$\{H_2(F_v)_{Y'}(g) \xrightarrow{=} H_2F_Y(g) \longrightarrow \operatorname{colim} H_2F_Y\}_{g \in obC_v}$$

induce a map $j\colon \operatorname{colim} H_2(F_v)_{Y'} \to \operatorname{colim} H_2F_Y$.

The following diagrams commute:

$\mathcal{D}_6$:

$$\begin{tikzcd}
H_2\ \bar{F}_{Y''}(g) \arrow[r, "\alpha_g"] \arrow[dr, "H_2\ \bar{\varphi}_g"'] & \operatorname{colim} H_2\ \bar{F}_{Y''} \arrow[d, "b"] \\
& H_2\ \operatorname{colim} \bar{F}_{Y''}
\end{tikzcd}$$

for α_g the colimit induced map $g \in \operatorname{ob}(v \cap C_v)$;

$\mathcal{D}_7$:

$$\begin{tikzcd}
H_2(F_v)_{Y'}(g) \arrow[r, "\beta_g"] \arrow[dr, "H_2\ \varphi^v_g"'] & \operatorname{colim} H_2(F_v)_{Y'} \arrow[d, "c"] \\
& H_2\ \operatorname{colim} F_v
\end{tikzcd}$$

for β_g the colimit induced map $g \in \operatorname{ob}C_v$;

$\mathcal{D}_8$:

$$\begin{tikzcd}
H_2\ \bar{F}_{Y''}(g) \arrow[r, "h", "="'] \arrow[d, "H_2\ \bar{\varphi}_g"'] & H_2(F_v)_{Y'}(g) \arrow[d, "H_2\ \varphi^v_g"] \\
H_2\ \operatorname{colim} \bar{F}_{Y''} \arrow[r, "d"] & H_2\ \operatorname{colim}\ (F_v)_{Y'}
\end{tikzcd}$$

$\mathcal{D}_9$:

$$\begin{tikzcd}
\operatorname{colim} H_2\ \bar{F}_{Y''} \arrow[r, "e"] \arrow[d, "b"'] & \operatorname{colim} H_2(F_v)_{Y'} \arrow[d, "c"] \\
H_2\ \operatorname{colim} \bar{F}_{Y''} \arrow[r, "d"] & H_2\ \operatorname{colim}\ (F_v)_{Y'}
\end{tikzcd}$$.

Diagram $\mathcal{D}_9$ is the definition of the arrow e . The diagram $\mathcal{D}_{10}$:

$$\begin{tikzcd}
H_2\bar{F}_Y(g) \arrow[r, "h"'] \arrow[d, "\alpha_g"'] & H_2(F_v)_{Y'}(g) \arrow[d, "\beta_g"] \\
\operatorname{colim} H_2\bar{F}_{Y''} \arrow[r, "e"] & \operatorname{colim} H_2(F_v)_{Y'}
\end{tikzcd}$$

commutes since
$$\begin{aligned} c\ e\ \alpha_g &= d\ b\ \alpha_g \\ &= d\ H_2\ \bar{\varphi}_g \\ &= H_2\ \varphi^v_g\ h \\ &= c\ \beta_g\ h \end{aligned}$$

and c is an isomorphism.

The diagram $\mathcal{D}_{11}$:

$$\begin{tikzcd}
H_2\ \bar{F}_{Y''}(g) \arrow[rr, "H_2\ \varphi_g"] \arrow[d, "="'] & & H_2\ \operatorname{colim}\ \bar{F}_{Y''} \arrow[d, "\ell"] \\
H_2\ F_Y(g) \arrow[rr] & & H_2\ F_Y(v)
\end{tikzcd}$$

commutes. Therefore the diagram $\mathcal{D}_{12}$:

$$\begin{tikzcd}
H_2\ \bar{F}_{Y''}(g) \arrow[dr, "\alpha_g"] \arrow[ddr, "H_2\ \varphi'_g"] \arrow[dddr] & \\
& \operatorname{colim}\ H_2\ \bar{F}_{Y''} \arrow[d, "b"] \\
& H_2\ \operatorname{colim}\ \bar{F}_{Y''} \arrow[d, "\ell"] \\
& H_2\ F_Y(v)
\end{tikzcd}$$

commutes for each $g \in ob(v \cap C_v)$.

For each $g \in ob(v \cap C_v)$, the diagrams

$\mathcal{D}_{13}$:

$$\begin{tikzcd}
H_2\ \bar{F}_{Y''}(g) \arrow[r, "h", "="'] \arrow[d, "\alpha_g"'] & H_2(F_v)_{Y'}(g) \arrow[r, "i", "="'] \arrow[d, "\beta_g"] \arrow[dr, phantom, "\mathrm{I}."] & H_2\ F_Y(g) \arrow[d, "\gamma_g"] \\
\operatorname{colim}\ H_2 \bar{F}_{Y''} \arrow[r, "e"] & \operatorname{colim}\ H_2(F_v)_{Y'} \arrow[r, "j"] & \operatorname{colim}\ H_2\ F_Y
\end{tikzcd}$$

$\mathcal{D}_{14}$:

$$\begin{tikzcd}
H_2\ \bar{F}_{Y''}(g) \arrow[rrr, "ih", "="'] \arrow[dr, "ih", "="'] \arrow[dd, "\alpha_g"'] & & & H_2\ F_Y(g) \arrow[dd, "\gamma_g"] \\
& H_2\ F_Y(g) \arrow[d, "H_2F_Y(\alpha)"'] \arrow[rrd, "\alpha_g"] & & \\
\operatorname{colim} H_2\ \bar{F}_{Y''} \arrow[r, "\ell b"] & H_2\ F_Y(v) \arrow[rr, "\gamma_v"] & & \operatorname{colim} H_2\ F_Y
\end{tikzcd}$$

commute for γ_g the colimit induced morphism, j the morphism induced by the family $\{\gamma_g i\}_{g \in obC_v}$, and $\alpha \in C(g,v)$.

By properties of colimit $\gamma_v \ell b = je$ and the diagram $\mathcal{D}_{15}$

$$\begin{tikzcd}
\operatorname{colim} H_2\ \bar{F}_{Y''} \arrow[r, "e"] \arrow[d, "\ell b"'] & \operatorname{colim} H_2(F_v)_{Y'} \arrow[d, "j"] \\
H_2\ F_Y(v) \arrow[r, "\gamma_v"] & \operatorname{colim} H_2\ F_Y
\end{tikzcd}$$

commutes. Since diagram $\mathcal{D}_5$ is a push-out, there is induced a map Φ: $\operatorname{colim} H_2\ G_Y \to \operatorname{colim} H_2\ F_Y$.

Since $\operatorname{colim} H_2\ G_Y = H_2Y$, the family of inclusions induced homomorphisms

$$\{\delta_g :\ H_2\ F_Y(g) \longrightarrow H_2Y\}_{g \in obC}$$

induces a homomorphism

$$\Psi:\ \operatorname{colim} H_2\ F_Y \longrightarrow H_2Y = H_2\ \operatorname{colim} F_Y\ .$$

All that is required to complete the proof is to verify that Ψ is an isomorphism.

i) $\Phi\ \Psi = 1$. By properties of colimit it suffices to show that for each $g \in \mathrm{ob}C$, the diagram $\mathcal{D}_{16}$:

$$\begin{tikzcd}
H_2\, F_Y(g) \arrow[d, "\gamma_g"'] \arrow[dr, "\delta_g"] \arrow[drr, "\gamma_g"] \arrow[d, phantom, shift left=6ex, "\mathrm{I.}"] \arrow[dr, phantom, shift left=6ex, "\mathrm{II.}"] & & \\
\mathrm{colim}\, H_2\, F_Y \arrow[r, "\Psi"] & H_2 Y \arrow[r, "\Phi"] & \mathrm{colim}\, H_2\, F_Y
\end{tikzcd}$$

commutes. In particular, since I. already is known to commute, it suffices to show that the diagram $\mathcal{D}_{17}$:

$$\begin{tikzcd}
H_2\, F_Y(g) \arrow[d, "\delta_g"] \arrow[dr, "\gamma_g"] & \\
H_2 Y \arrow[r, "\Phi"] & \mathrm{colim}\, H_2\, F_Y
\end{tikzcd}$$

commutes for each $g \in \mathrm{ob}C$.

a) If $g \in \mathrm{ob}C_v$, then δ_g is the inclusion induced composition

$$\begin{tikzcd}
H_2\, F_Y(g) \arrow[r, "i", "="'] & H_2(F_v)_{Y'}(g) \arrow[r, "H_2\, \varphi_g^v"] \arrow[dr, "\beta_g"'] & H_2\, \mathrm{colim}\, (F_v)_{Y''} \arrow[d, "c"] \arrow[dl, phantom, "\circlearrowright"] \\
& & \mathrm{colim}\, H_2(F_v)_{Y'} \arrow[d] \\
& & H_2 Y
\end{tikzcd}$$

Previous diagrams yield the commutative diagram $\mathcal{D}_{18}$:

$$\begin{tikzcd}
H_2\, F_Y(g) \arrow[d, "="', "i"] \arrow[dr, "i"] \arrow[ddd, bend right=60, "\delta_g"'] & \\
H_2(F_v)_{Y'}(g) \arrow[r, "i", "="'] \arrow[d, "\beta_g"] & H_2\, F_Y(g) \arrow[d] \\
\mathrm{colim}\, H_2(F_v)_{Y'} \arrow[r, "j"] \arrow[d] & \mathrm{colim}\, H_2\, F_Y \\
\mathrm{colim}\, H_2\, G_Y \arrow[ur, "\Phi"] &
\end{tikzcd}$$

b) If $g \notin obC_v$, then there is a morphism $\alpha \in C(g,v)$ and commutative diagrams

$$\begin{tikzcd}
H_2\,F_Y(g) \arrow[dd, "H_2F_Y(\alpha)"] \arrow[dr, "\gamma_g"] & \\
& \operatorname{colim} H_2\,F_Y \\
H_2\,F_Y(v) \arrow[ur, "\gamma_v"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
H_2\,F_Y(g) \arrow[r, "H_2\,F_Y(\alpha)"] & H_2\,F_Y(v) \arrow[r] \arrow[dr, "\gamma_v"'] & H_2Y \arrow[d, "\Phi"] \arrow[dl, phantom, "\mathrm{I.}"] \\
& & \operatorname{colim} H_2\,F_Y
\end{tikzcd}$$

where I. commutes by the definition of Φ .

Since the composition

$$H_2\,F_Y(g) \xrightarrow{H_2\,F_Y(\alpha)} H_2\,F_Y(v) \longrightarrow H_2Y$$

is inclusion induced, it is δ_g and it follows that diagram $\mathcal{D}_{16}$ commutes for each $g \in obC$. Thus $\Phi\,\Psi = 1$.

ii) $\Psi\,\Phi = 1$. For this it suffices to show that the diagram $\mathcal{D}_{19}$:

$$\begin{tikzcd}
& \operatorname{colim} H_2(F_v)_{Y'} \arrow[d, "m"] \arrow[dr, "j"] \arrow[drr, "m"] & & \\
H_2F_Y(v) \arrow[r, "n"] \arrow[rr, bend right, "\gamma_v"'] \arrow[rrr, bend right=40, "n"'] & \operatorname{colim} H_2\,G_Y \arrow[r, "\Phi"] & \operatorname{colim} H_2F_Y \arrow[r, "\Psi"] \arrow[ur, phantom, "\mathrm{I.}"] \arrow[d, phantom, "\mathrm{II.}"] & \operatorname{colim} H_2G\,Y \\
& & {} &
\end{tikzcd}$$

commutes.

a) Since the diagram

$$\begin{tikzcd}
H_2(F_v)_{Y'}(g) \arrow[rr, "h"', "="] \arrow[d, "\beta_g"] \arrow[drr, phantom, "\mathrm{I.\ from}\ \mathcal{D}_{13}"] & & H_2\,F_Y(g) \arrow[d, "\gamma_g"] \arrow[dr] & \\
\operatorname{colim} H_2(F_v)_{Y'} \arrow[rr, "j"] & & \operatorname{colim} H_2\,F_Y \arrow[r, "\Psi"] & H_2Y
\end{tikzcd}$$

commutes for each $g \in \mathrm{ob}C_v$, and the diagram

$$\begin{tikzcd}
H_2(F_v)_{Y'}(g) \arrow[r, "h"', "="] \arrow[d, "\beta_g"] \arrow[dr] & H_2\,F_Y(g) \arrow[d] \\
\operatorname{colim} H_2(F_v)_{Y'} \arrow[r, "m"] & H_2Y
\end{tikzcd}$$

commutes (since the unlabeled maps are induced by inclusions),

$$\Psi j = m$$

and Triangle I. in diagram $\mathcal{D}_{19}$ commutes.

The diagram

$$\begin{tikzcd}
H_2\,F_Y(v) \arrow[r, "\gamma_v"] \arrow[dr, "n"'] & \operatorname{colim} H_2\,F_Y \arrow[d, "\Psi"] \\
& H_2Y
\end{tikzcd}$$

commutes by definition of Ψ. It follows that diagram $\mathcal{D}_{19}$ commutes and $\Psi\,\phi = 1$. □

As was noted previously, equation *) follows immediately as the case $Y = \operatorname{colim} F$.

Theorem 2. If C is an s-category and $F: C \to Q$ is a functor such that C and F satisfy conditions A) - I), then

$$H_2 \operatorname{colim} \tilde{F} = \operatorname{colim} H_2\tilde{F} . \quad \square$$

Recall from Section 5, that if C is an s-category and $F: C \to Q$ is a functor satisfying conditions D) and H), then the functor $I: C \to Q$ is defined on objects $g \in \text{ob}C$ to be the discrete space

$$I(g) = \pi_1(\operatorname{colim} F;*)/\varphi_g^* \, \pi_1(F(g),*)$$

and on morphisms $\alpha \in C(g,g')$ to be the map induced by the inclusion $F(\alpha): F(g) \to F(g')$.

Definition: If C is an s-category and $F: C \to Q$ is a functor satisfying conditions D) and H), then the functor

$$F_H: C \longrightarrow Ab$$

is defined for objects $g \in \text{ob}C$ by

$$F_H(g) = \bigoplus_{I(g)} H_2\widetilde{F(g)} .$$

If $\alpha \in C(g,g')$ is a morphism in C , then the homomorphism $F_H(\alpha)$ is defined in the obvious way as follows: If $\sum_{i \in A} a_i \in \bigoplus_{I(g)} H_2F(g)$, then

$$F_H(\alpha) \sum_{i \in A} a_i = \sum_{i \in A} [H_2 \, F(\alpha) a_i]_{I(\alpha) i}$$

where the subscript on the bracket indicates the coordinate.

The following theorem relates Theorem 2 to the original space:

Theorem 3. If C is an s-category and $F: C \to Q$ is a functor such that C and F satisfy conditions A) - I), then

$$H_2 \operatorname{colim} \tilde{F} = \operatorname{colim} F_H \ .$$

<u>Proof</u>: By Theorem 2 of Section 5,

$$\tilde{F}(g) = \widetilde{F(g)} \times I(g) = \bigsqcup_{I(g)} \widetilde{F(g)}$$

since $I(g)$ is topologized discretely. Since singular homology is additive, $H_2\, \tilde{F}(g) = \bigoplus_{I(g)} H_2 F(g)$.

For $\alpha \in C(g,g')$ a morphism in C , the map $\widetilde{F(\alpha)}$ is defined by the pull-back

$$\begin{tikzcd}
\widetilde{F(g)} \arrow[r, "\widetilde{F(\alpha)}"] \arrow[d] & \widetilde{F(g')} \arrow[d] \\
F(g) \arrow[r, "F(\alpha)"] & F(g')
\end{tikzcd} \ .$$

The diagram

$$\begin{tikzcd}
\widetilde{F(g)} \times I(g) \arrow[d, equal] \arrow[dr, "\widetilde{F(\alpha)} \times I(\alpha)"] \arrow[drr, bend left] & & \\
\tilde{F}(g) \arrow[ddd] & \widetilde{F(g')} \times I(g') \arrow[r] \arrow[d, equal] & \widetilde{\operatorname{colim} F} \arrow[ddd] \\
& \tilde{F}(g') \arrow[dd] \arrow[r, phantom, "\mathrm{I.}"] & \\
& & \\
F(g) \arrow[r, "F(\alpha)"] & F(g') \arrow[r] & \operatorname{colim} F
\end{tikzcd}$$

commutes since all the non-vertical maps are inclusions and the vertical ones are simply projections and covering maps. Since square I. is a pull-back, $\widetilde{F(\alpha)} \times I(\alpha) = \tilde{F}(\alpha)$.

It follows that there is a natural transformation $A\colon H_2\tilde{F} \to F_H$ by

$$\begin{tikzcd} H_2\tilde{F}(g) \arrow[r, "="] \arrow[d, "H_2\tilde{F}(g)"] & F_H(g) \arrow[d, "F_H(\alpha) = H_2(\widetilde{F(\alpha)} \times I(\alpha))"] \\ H_2\tilde{F}(g') \arrow[r, "="] & F_H(g') \end{tikzcd}$$

Thus $\operatorname{colim} F_H = \operatorname{colim} H_2\tilde{F}$ and Theorem 3 follows from Theorem 2. □

It is possible to use the techniques of this section to prove a weak generalization of Theorem 3 for higher dimensional homology. This requires a definition. As before, $H_n = H_n(\ ;\mathbb{Z})$.

Definition: If C is an s-category and $F: C \to Q$ is a functor satisfying conditions D) and H), then the functor

$$F_H^n: C \longrightarrow Ab$$

is defined for objects $g \in obC$ by

$$F_H^n(g) = \bigoplus_{I(g)} H_n \widetilde{F(g)}$$

If $\alpha \in C(g,g')$ is a morphism in C , then the homomorphism $F_H^n(\alpha)$ is defined in the obvious way for $\sum_{i \in A} a_i \in \oplus H_n \widetilde{F(g)}$ by

$$F_H^n(\alpha) \sum_{i \in A} a_i = \Sigma[H_n \tilde{F}(\alpha) a_i]_{I(\alpha) i}$$

where the subscript on the bracket indicates the coordinate.

Theorem 4. Suppose C is an s-category and $F: C \to Q$ is a functor such that C and F satisfy conditions A) - I). If for each object $g \in obC$, $\pi_i(F(g),*) = 0$ for $i=0,2,\ldots,n$ then

$$H_n \operatorname{colim} \tilde{F} = \operatorname{colim} F_H^n$$

for any $n \geq 2$.

<u>Proof</u>: As before, we first prove a stronger conclusion, namely that for any disjoint union, $Y = \sqcup \widetilde{\operatorname{colim} F}$, of universal covering spaces of colim F ,

$$H_n \operatorname{colim} F_Y = \operatorname{colim} H_n F_Y .$$

The proof is by induction on n . Theorem 1 proves the Theorem in case $n = 2$. It is now assumed to be true for $2 \leq n \leq k-1$. The proof for $n = k$ is by induction on the number of vertices. If C has one vertex, then F_Y is constant and the theorem is trivial. Recall diagram $\mathcal{D}_3$:

$$\begin{tikzcd}
\operatorname{colim} \bar{F} = \varphi_v F(v) \cap \varphi \operatorname{colim} F_V \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \varphi \operatorname{colim} F_V \arrow[d, "\varphi"] \\
\varphi_v F(v) \arrow[r, "\varphi_v"'] & \operatorname{colim} F = \operatorname{colim} G
\end{tikzcd}$$

$\bar{C}$ and $\bar{F}$ satisfy conditions A) – I) and have $\pi_{k-1}\bar{F} = 0$. Thus in the universal cover, $\pi_{k-1}\bar{F}_{Y''} = 0$ in each component. From the Hurewicz theorem we have $H_{k-1}\bar{F}_{Y''} = 0$. Since the theorem is assumed for $k-1$, we have

$$H_{k-1} \operatorname{colim} \bar{F}_{Y''} = \operatorname{colim} H_{k-1}\bar{F}_{Y''} = 0 .$$

The classical Mayer-Vietoris sequence for the open cover $\{G_Y(v_1), G_Y(v_2)\}$ of Y is $H_k\, G_Y(f) \to H_k\, G_Y(v_1) \oplus H_k\, G_Y(v_2) \to H_k\, Y \to 0$ and as before by Proposition 24 we have a push-out

$$\begin{array}{ccc} H_k\ G_Y(f) & \longrightarrow & H_k\ G_Y(v_1) \\ \downarrow & & \downarrow \\ H_k\ G_Y(v_2) & \longrightarrow & \mathrm{colim}\ H_k\ G_Y \end{array}$$

The proof concludes exactly as in Theorem 1.

As in Theorem 2 the Y's are reduced to $\widetilde{\mathrm{colim}\ F}$ to yield the equation

$$H_n\ \mathrm{colim}\ \tilde{F} = \mathrm{colim}\ H_n\tilde{F}\ .$$

The proof of Theorem 3 then yields the result. □

§ 7. Technical Lemmas.

This section contains the proofs of the lemmas stated in Section 6.

Proposition 1. C_v is an s-category based on $\Lambda - \{imv\}$.

Proof: Properties i) and iii) are clear. If $f \in obC_v$ and $\Lambda' \subseteq imf$, then there is a unique $g \in obC$ with $C(f,g) \neq \emptyset$ and $img = \Lambda'$. If $g \notin obC_v$, then $C(g,v) \neq \emptyset$ and $imv \subseteq img \subseteq imf$. Since C has connected vertices, this implies that $C(f,v) \neq \emptyset$. Thus $g \in obC_v$.

To show that C_v is based on $\Lambda - imv$, it suffices to note that if $imv \cap imf \neq \emptyset$ for any $f \in obC$, then since $|imv| = 1$, $imv \subseteq imf$ and $f \notin obC_v$. □

Proposition 2. $v \cap C_v$ is an s-category based on $\Lambda - \{imv\}$ and has connected vertices.

Proof: As was noted in Section 6, while $v \cap C_v$ is a subcategory of the category C , it is not a sub-s-category. If $\iota: v \cap C_v \longrightarrow C$ is the inclusion functor, then for each $g \in ob(v \cap C_v)$, the image ιg will be denoted g_* .

If $g \in ob(v \cap C_v)$, then $g_* \in obC$ and as such maps some set X_{g_*} monomorphically into Λ . Precisely one element $v_g \in X_{g_*}$ is mapped by g_* into imv . As an object of $v \cap C_v$, g is the restriction

$$g = g_* | X_{g_*} - \{v_g\} \longrightarrow \Lambda - \{imv\}$$

of g_* . Clearly $v \cap C_v$ is based on $\Lambda - \{imv\}$.

If $f,g \in ob(v \cap C_v)^2$ and $img \notin imf$, then

$$\text{img}_* = \text{img} \cup \text{imv} \nsubseteq \text{imf} \cup \text{imv} = \text{imf}_*$$

and $\emptyset = C(f_*,g_*) = (v \cap C_v)(f,g)$. That morphism sets have at most one element is trivial.

For $f \in \text{ob}(v \cap C_v)$ and $\Lambda' \subseteq \text{imf}$ a non-empty subset, $\Lambda' \cup \text{imv} \subseteq \text{imf} \cup \text{imv} = \text{imf}_*$. Since C is an s-category, there is a unique $g_* \in \text{ob}C$, with $C(f_*,g_*) \neq \emptyset$ and $\text{img}_* = \Lambda' \cup \text{imv}$. Thus $\text{imv} \subseteq \text{img}_*$ and $C(g_*,v) \neq \emptyset$ so that $g \in \text{ob}(v \cap C_v)$. Since $\text{img} = \text{img}_* - \text{imv} = \Lambda' \cup \text{imv} - \text{imv} = \Lambda'$, condition iii) is satisfied and $v \cap C_v$ is an s-category.

If $w,w' \in (v \cap C_v)_0$, and $\text{imw} = \text{imw}'$, then

$$\text{imw}_* = \text{imw} \cup \text{imv} = \text{imw}' \cup \text{imv} = \text{imw}'_* \ .$$

Since C is upward finite with a generating tree, by Theorem 2 of Section 1, $w_* = w'_*$ and since $\iota\colon v \cap C_v \to C$ is an inclusion, $w = w'$. It follows that $v \cap C_v$ has connected vertices. □

<u>Proposition 3</u>. If $|C_0| < \infty$, then $|(C_v)_0| < |C_0|$.

<u>Proof</u>: Since $(C_v)_0 \subseteq C_0$ and $v \in C_0 - (C_v)_0$, the proposition is trivial. If $v' \in C_0$, then $C(v',v) \neq \emptyset$ implies that $v' = v$. Thus v is the only element in $C_0 - (C_v)_0$ and

$$|C_0| = |(C_v)_0| + 1 \ . \ \square$$

<u>Proposition 4</u>. If $|C_0| < \infty$, then $|(v \cap C_v)_0| < |C_0|$.

<u>Proof</u>: If $g \in (v \cap C_v)_0$ is a vertex of $v \cap C_v$, then $dg = 0$ and $|\text{img}| = 1$. Since $\text{img}_* = \text{img} \cup \text{imv}$, g_* is a 1-dimensional object of C . Since $\text{img} \subseteq \text{img}_*$, there is an object $w \in \text{ob}C$ with

$C(g_*,w) \neq \emptyset$ and $\mathrm{im}w = \mathrm{im}g$. Thus $\vee g_* = \{w,v\}$.

This procedure defines a function

$$\varphi\colon (v \cap C_v)_0 \longrightarrow (C_v)_0$$

by the correspondence $\varphi g = w$. If $g' \in (v \cap C_v)_0$ also corresponds to w , then

$$\vee g'_* = \{w,v\}$$

and $g'_* = g_*$ by Theorem 2 of Section 1. It follows that φ is a monomorphism and $\qquad |(v \cap C_v)_0| \leq |(C_v)_0|$.

This proposition then follows from the previous one. □

<u>Proposition 5</u>. C_v and $v \cap C_v$ are upward finite.

<u>Proof</u>: i) If $g \in \mathrm{ob}C_v$, then $g \in \mathrm{ob}C$. Since C is upward finite, it has a free face $f \in Fr(C)$ with $C(f,g) \neq \emptyset$. $f' \in \mathrm{ob}C_v$ is defined by the properties $\mathrm{im}f' = \mathrm{im}f - \mathrm{im}v$ and $C(f,f') \neq \emptyset$. If $f'' \in \mathrm{ob}C_v$ and $C(f'',f') \neq \emptyset$, then $\mathrm{im}f' \subseteq \mathrm{im}f''$. Since $f'' \in \mathrm{ob}C$, there is a free face $h \in Fr(C)$ with $C(h,f'') \neq \emptyset$. If $h \in \mathrm{ob}C_v$, then clearly $h \in Fr(C_v)$ since if $h' \in \mathrm{ob}C_v$ has the property that $\emptyset \neq C_v(h',h) = C(h',h)$, then $h' = h$. Since $C_v(h,g) \neq \emptyset$, C_v is upward finite in this case. If $h \notin \mathrm{ob}C_v$, then $\mathrm{im}v \subseteq \mathrm{im}h$. Since

$$\mathrm{im}f' \subseteq \mathrm{im}f'' \subseteq \mathrm{im}h ,$$

$$\mathrm{im}f = \mathrm{im}f' \cup \mathrm{im}v \subseteq \mathrm{im}f'' \cup \mathrm{im}v \subseteq \mathrm{im}h \cup \mathrm{im}v = \mathrm{im}h .$$

Thus $C(h,f) \neq \emptyset$, and since $f \in Fr(C)$, $h = f$. Thus $C(f,f'') \neq \emptyset$ and $\mathrm{im}f'' \subseteq \mathrm{im}f$. But

$$\mathrm{im}f - \mathrm{im}v = \mathrm{im}f' \subseteq \mathrm{im}f'' .$$

It follows that $\operatorname{im} f'' = \operatorname{im} f - \operatorname{im} v = \operatorname{im} f'$ and by Theorem 2 of Section 1, $f'' = f'$. Thus f' is a free face and C_v is upward finite.

ii) If $g \in ob(v \cap C_v)$, then there is a free face $f_* \in Fr(C)$ with $C(f_*, g_*) \neq \emptyset$. $C(g,v) \neq \emptyset$; thus $C(f_*, v) \neq \emptyset$, $f \in ob(v \cap C_v)$, and $C(f,g) \neq \emptyset$. Since for no $h \in obC$ is $C(h,f) \neq \emptyset$ and $h \neq f$, it follows that for no object $h \in ob(v \cap C_v)$ is $C(h,f) \neq \emptyset$ and $h \neq f$. It follows that $f \in Fr(v \cap C_v)$ and $v \cap C_v$ is upward finite. □

<u>Proposition 6</u>. F_v is path connected and closed under finite intersections.

<u>Proof</u>: It is trivial that the restriction of a path connected functor is path connected. The proof of the second assertion of the proposition requires a series of Lemmas:

<u>Lemma 1</u>. For each $g \in obC$, $\varphi_g F(g) = \bigcap_{v \in \nu g} \varphi_v F(v)$.

<u>Proof</u>: $\subseteq$: If $x \in \varphi_g F(g)$, then for some $y \in F(g)$, $\varphi_g y = x$. For any $v \in \nu g$, if $\alpha_v \in C(g,v)$, then there is a diagram

$$\begin{tikzcd} & g \arrow[dl, "\alpha_v"'] \arrow[dr, "1_g"] & \\ v & & g \end{tikzcd}$$

in C, with $\alpha_v y = \alpha_v y$ and $1_g y = y$. It follows from the colimit relations that $\varphi_v \alpha_v y = \varphi_g y = x$, and $x \in \varphi_v F(v)$.

$\supseteq$: (By induction on the number of elements in νg.) If νg has two vertices, v_1, v_2 and

$$x \in \bigcap_{v \in \nu g} \varphi_v F(v),$$

then since F is closed under finite intersections, there is an element $h \in \mathrm{ob}C$ with $C(h,v) \neq \emptyset$, $C(h,v') \neq \emptyset$ together with $y \in \varphi_h F(h)$ such that $\varphi_h y = x$. Since $\mathrm{im}v \cup \mathrm{im}v' \subseteq \mathrm{im}h$, there is an element $h' \in \mathrm{ob}C$ with $C(h,h') \neq \emptyset$. Since $\mathrm{im}h' = \mathrm{im}g$ and C has a generating tree, by Theorem 2 in Section 1, $h' = g$. If $\alpha \in C(h,g)$, then $\varphi_g(\alpha\bar{y}) = x$ for $\bar{y} \in F(h)$ with $\varphi_h \bar{y} = y$.

If the inclusion is valid for k vertices, $\{v_1, \ldots, v_k\}$, then for

$$x \in \bigcap_{i=1}^{k+1} \varphi_{v_i} F(v_i)$$

since $x \in \bigcap_{i=1}^{k} \varphi_{v_i} F(v_i)$ there is an element $g' \in \mathrm{ob}C$ with

$\mathrm{im}g' = \bigcup_{i=1}^{k} \mathrm{im}v_i$ and $y \in F(g')$ with $\varphi_{g'} y' = x$.

Since $\varphi_{g'} F(g') \cap \varphi_{v_{k+1}} F(v_{k+1}) \neq \emptyset$, there is an element h with

$C(h,g') \neq \emptyset$ and $C(h,v_{k+1}) \neq \emptyset$ together with an element $\bar{y} \in F(h)$ such that $\varphi_h \bar{y} = x$. Since $\mathrm{im}g = \mathrm{im}g' \cup \mathrm{im}v_{k+1} \subseteq \mathrm{im}h$, $C(h,g) \neq \emptyset$. If $\alpha \in C(h,g)$, then $\alpha\bar{y}$ has the property that $\varphi_g \alpha\bar{y} = x$. □

<u>Lemma 2</u>. For any $g \in \mathrm{ob}C_v$,

$$\varphi_g^v F_v(g) = \bigcap_{\bar{v} \in vg} \varphi_{\bar{v}}^v F_v(\bar{v}) .$$

<u>Proof</u>: $\subseteq$: If $x \in \varphi_g^v F_v(g)$, then for some $y \in F_v(g)$, $\varphi_g^v y = x$. For any $\bar{v} \in vg$ if $\alpha_{\bar{v}} \in C(g,\bar{v})$, then there is a diagram

$$\begin{tikzcd} & g \arrow[dl, "\alpha_{\bar{v}}"'] \arrow[dr, "1_g"] & \\ \bar{v} & & g \end{tikzcd}$$

in C_v with $\alpha_{\bar{v}}y = \alpha_{\bar{v}}y$ and $1_g y = y$. It follows from the colimit relations that $\varphi^v_{\bar{v}}\alpha_{\bar{v}}y = \varphi^v_g y = x$ and $x \in \varphi^v_{\bar{v}}F_v(\bar{v})$.

$\supseteq$: (By induction on the number of elements in $\vee g$.) if $\vee g$ has two vertices, v_1, v_2 and

$$x \in \bigcap_{\bar{v} \in \vee g} \varphi^v_{\bar{v}} F_v(\bar{v})$$

then for some $y_1 \in F_v(v_1)$, $y_2 \in F_v(v_2)$, $\varphi^v_{v_1} y_1 = x = \varphi^v_{v_2} y_2$. Thus there is a finite diagram

$$\begin{tikzcd}
& f_0 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & f_{n+1} \arrow[dl] \arrow[dr] & \\
v_1 & & f_1 & & f_n & & v_2
\end{tikzcd}$$

in C_v which relates y_1 to y_2 in the colimit. But this same diagram is a diagram in C which relates y_1 to y_2 in colim F . Since F is closed under finite intersections, there exists an $h \in \mathrm{ob}C$ together with an element $\bar{y}$ such that $\varphi_h \bar{y} = x$. Since

$$\mathrm{im}g = \mathrm{im}v_1 \cup \mathrm{im}v_2 \subseteq \mathrm{im}h ,$$

$C(h,g) \neq \emptyset$, and if $\alpha \in C(h,g)$, then $\alpha\bar{y} \in F(g) = F_v(g)$.

There is a diagram

$$\begin{tikzcd}
& & h \arrow[d, "\alpha"'] & \bar{y} \arrow[d, mapsto] \arrow[dlll, bend right, mapsto] \arrow[ddrr, bend left, mapsto] & & \\
& & g \arrow[dl] \arrow[drr] & \alpha\bar{y} & & \\
y_1 & v_1 & & & v_2 & y_2
\end{tikzcd}$$

in C . It follows that $\alpha\bar{y} \mapsto y_1$ and $\alpha\bar{y} \mapsto y_2$ so that

$\varphi^v_g \, \alpha\bar{y} = \varphi^v_{v_1} y_1 = x$ and $x \in \varphi^v_g \, F_v(g)$.

If the inclusion is true for $|\nu g| = k$, then if $\nu g = \{v_1, \ldots, v_k, v_{k+1}\}$, the preceding procedure for

$$x \in \bigcap_{i=1}^{k} \varphi^v_{v_i} \, F_v(v_i)$$

yields an element $h \in \mathrm{ob}C_v$ together with a $y \in F_v(h)$ such that $\varphi^v_h y = x$, amd $\mathrm{im}h = \bigcup_{i=1}^{k} \mathrm{im}v_i$. Since $x \in \varphi^v_h \, F_v(h) \cap \varphi^v_{v_{k+1}} F_v(v_{k+1})$,

there exist $y_1 \in F_v(h)$ and $y_2 \in F_v(v_{k+1})$ such that $\varphi^v_h y_1 = x$ and $\varphi^v_{v_{k+1}} y_2 = x$. The same method used in the case $|\nu g| = 2$ is applicable and the Lemma follows. □

<u>Lemma 3</u>. The colimit induced map φ: colim $F_v \to$ colim F defined by the following diagram,

$$\begin{tikzcd}
F_v(g) \arrow[rr, "\varphi^v_g"] \arrow[d, equal] & & \mathrm{colim}\, F_v \arrow[d, "\varphi"] \\
F(g) \arrow[rr, "\varphi_g"'] & & \mathrm{colim}\, F ,
\end{tikzcd}$$

is monic.

<u>Proof</u>: If $x, x' \in$ colim F_v , then there exist objects $g, g' \in \mathrm{ob}C_v$ and elements $y \in F_v(g)$, $y' \in F_v(g')$ such that $\varphi^v_g y = x$, $\varphi^v_{g'} y' = x'$. If $\varphi x = \varphi x'$, then $\varphi \varphi^v_g y = \varphi \varphi^v_{g'} y'$ and $\varphi_g y = \varphi_{g'} y'$. It follows that

$$\varphi_g y \in \varphi_g F(g) \cap \varphi_{g'} F(g')$$

and there exists an object $h \in \mathrm{ob}C$ and an element $z \in F(h)$ such that i) $C(h,g)$ and $C(h,g')$ are non-empty and contain respectively

α and α', ii) $\varphi_h z = \varphi_g y = \varphi_{g'} y'$.

αz has the property that $\varphi_g \alpha z = \varphi_h z = \varphi_g y$. Since φ_g is monic, $\alpha z = y$. Similarly, $\alpha' z = y'$. Since $\mathrm{im} v \nsubseteq \mathrm{im} g \cup \mathrm{im} g' \subseteq \mathrm{im} h$, there exists an $\bar{h} \in \mathrm{ob} C_v$, with $C(h,\bar{h}) \neq \emptyset$. β denotes the morphism in $C(h,\bar{h})$. There is a commutative diagram

$$\begin{tikzcd}
 & h \arrow[d, "\beta"] \arrow[dl, bend right, "\alpha"'] \arrow[dr, bend left, "\alpha'"] & \\
 & \bar{h} \arrow[dl, "\gamma"'] \arrow[dr, "\gamma'"] & \\
g & & g'
\end{tikzcd}$$

in C with $z \in F(h)$ and $\beta z \in F(\bar{h}) = F_v(\bar{h})$. βz has the properties that $\gamma\beta z = \alpha z = y$ and $\gamma'\beta z = \alpha' z = y'$. Since the diagram

$$\begin{tikzcd}
 & \bar{h} \arrow[dl, dashed, "\gamma"'] \arrow[dr, "\gamma'"] & \\
g & & g'
\end{tikzcd}$$

is a diagram in C_v, it follows that

$$x = \varphi_g y = \varphi_{g'} y' = x'. \quad \square$$

<u>Proof of Proposition 6.</u> If $g, g' \in \mathrm{ob} C_v$ and $\varphi_g^v \mathrm{Fr}(g) \cap \varphi_{g'}^v \mathrm{Fr}(g') \neq \emptyset$, then there is some element $x \in \varphi_g^v \mathrm{Fr}(g) \cap \varphi_{g'}^v \mathrm{Fr}(g') = \bigcap_{v \in vg \cup vg'} \varphi_{\bar{v}}^v \mathrm{Fr}(\bar{v})$.

Exactly as in the proof of Lemma 2, induction on the number of vertices in $vg \cup vg'$ yields an element $h_x \in \mathrm{ob} C_v$ with $y \in F_v(h_x)$ such that

$C_v(h_x,g) \neq \emptyset$, $C_v(h_x,g') \neq \emptyset$ and $\overset{v}{\varphi}_{h_x} y = x$. The set

$$\Delta = \overset{v}{\varphi}_g F_v(g) \cap \overset{v}{\varphi}_{g'} F_v(g')$$

and the family

$$\bigcup_{x \in \Delta} \{h_x\}$$

satisfies conditions i) and iii) in the definition of closed under finite intersections.

Let $\Delta = \overset{v}{\varphi}_g F_v(g) \cap \overset{v}{\varphi}_{g'} F_v(g')$, $x \in \Delta$ and $\alpha \in P(x,\Delta,y)$ for any $y \in \Delta$.

$$\varphi\,\Delta = \varphi \bigcap_{\bar{v} \in \nu g \cup \nu g'} \overset{v}{\varphi}_{\bar{v}} F_v(\bar{v}) = \bigcap_{\bar{v} \in \nu g \cup \nu g'} \varphi\,\overset{v}{\varphi}_{\bar{v}} F_v(\bar{v})$$

$$= \bigcap_{\bar{v} \in \nu g \cup \nu g'} \varphi_{\bar{v}} F(\bar{v})$$

$$= \bigcap_{\bar{v} \in \nu g} \varphi_{\bar{v}} F(\bar{v}) \bigcap_{\bar{v} \in \nu g'} \varphi_{\bar{v}} F(\bar{v})$$

$$= \varphi_g F(g) \cap \varphi_{g'} F(g')$$

Since $\varphi\, x \in \varphi_g F(g) \cap \varphi_{g'} F(g')$, there exists an element $f_x \in obC$ with $x_1 \in F(f_x)$ such that $\varphi_{f_x} x_1 = \varphi\, x$ and $\varphi_{f_x} F(f_x)$ is a path component of the intersection $\varphi_g F(g) \cap \varphi_{g'} F(g')$. Since $img \cup img' \subseteq imf_x$ there exists an element $\bar{f}_x \in obC_v$ with $C(f_x,\bar{f}_x) \neq \emptyset$ and $im\bar{f}_x = img \cup img'$. This last condition implies that $\bar{f}_x = h_x$ as previously defined. Clearly any h_x in the family can be arrived at this fashion. a denotes the morphism in $C(f_x,h_x)$. $\varphi\, x = \varphi_{f_x} x_1 = \varphi_{h_x} a x_1 = \varphi\, \overset{v}{\varphi}_{h_x} a x_1$. Since φ is monic, $\overset{v}{\varphi}_{h_x} a x_1 = x$.

$\varphi\,\alpha$ is a path in $\varphi_g F(g) \cap \varphi_{g'} F(g')$ from $\varphi\,x$ to $\varphi\,y$. Thus $\varphi\,y$ and $\varphi\,x$ are in the same path component. Since $\varphi_{f_x} F(f_x)$ is a path component of the intersection, there is an element $y_1 \in F(f_x)$ with $\varphi_{f_x} y_1 = \varphi\,y$. Then $\varphi\,\overset{v}{\varphi}_{h_x}\,a\,y_1 = \varphi_{h_x}\,ay_1 = \varphi_{f_x}\,y_1 = \varphi\,y$. Since φ is monic, $\overset{v}{\varphi}_{h_x}\,ay_1 = y$. It follows that $\overset{v}{\varphi}_{h_x} F_v(h_x)$ is a path component of the intersection for any h_x in the family

$$\bigcup_{x\in\Delta} \{h_x\}$$

provided above. □

Proposition 7. $\bar{F}$ is path connected and closed under finite intersections.

Proof: Again since $\bar{F}$ is the restriction of a path connected functor, it is path connected. The proof of the second assertion needs a lemma:

Lemma 4. $\bar{\varphi}$ is monic.

Proof: If $x, x' \in \operatorname{colim} \bar{F}$, then there exist objects $g, g' \in ob(v \cap C_v)$ and elements $y \in \bar{F}(g)$, $y' \in \bar{F}(g')$ such that $\bar{\varphi}_g y = x$, $\bar{\varphi}_{g'} y' = x'$. If $\bar{\varphi}\,x = \bar{\varphi}\,x'$, then $\bar{\varphi}\,\bar{\varphi}_g y = \bar{\varphi}\,\bar{\varphi}_{g'} y'$ and $\varphi_g y = \varphi_{g'} y'$. It follows that

$$\varphi_g y \in \varphi_g F(g) \cap \varphi_{g'} F(g')$$

and there exists an object $h_* \in obC$ and an element $z \in F(h_*)$ such that i) $C(h_*', g)$ and $C(h_*, g')$ are non-empty and contain respectively α and α'; ii) $\varphi_h z = \varphi_g y = \varphi_{g'} y'$. αz has the property that $\varphi_g \alpha z = \varphi_h z = \varphi_g y$. Since φ_g is monic, $\alpha z = y$. Similarly, $\alpha' z = y'$. Since $\operatorname{im} v \subseteq \operatorname{im} g_* \subseteq \operatorname{im} h_*$, $h \in ob(v \cap C_v)$ and the diagram

$$\begin{tikzcd}
 & h \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\
g & & g
\end{tikzcd}$$

in $v \cap C_v$. It follows that $\bar{\varphi}_g y = \bar{\varphi}_{g'} y'$ and $x = x'$. □

<u>Proof of Proposition 7</u>. If $g, g' \in \mathrm{ob}(v \cap C_v)$ are objects such that

$$\bar{\varphi}_g \bar{F}(g) \cap \bar{\varphi}_{g'} \bar{F}(g) \neq \emptyset ,$$

then $\bar{\varphi}(\bar{\varphi}_g \bar{F}(g) \cap \bar{\varphi}_{g'} \bar{F}(g)) = \varphi_g F(g) \cap \varphi_{g'} F(g') \neq \emptyset$. Since F is closed under finite intersections, there exists a set Δ and a family $\{h_\delta \in \mathrm{ob}C\}_{\delta \in \Delta}$ such that $C(h_\delta, g) \neq \emptyset$, $C(h_\delta, g') \neq \emptyset$ for each $\delta \in \Delta$, $\bigcup_{\delta \in \Delta} \varphi_{h_\delta} F(h_\delta) = \varphi_g F(g) \cap \varphi_{g'} F(g')$ and for each $\delta \in \Delta$, $\varphi_{h_\delta} F(h_\delta)$ is a path component of $\varphi_g F(g) \cap \varphi_{g'} F(g')$. Since $\mathrm{im} v \subseteq \mathrm{im} g \subseteq \mathrm{im} h_\delta$, $C(h_\delta, v) \neq \emptyset$ for each $\delta \in \Delta$ and the family $\{h_\delta\}_{\delta \in \Delta}$ is a subset of $\mathrm{ob}(v \cap C_v)$. Given $x \in \bar{\varphi}_g F(g) \cap \bar{\varphi}_{g'} F(g')$, then $\bar{\varphi}\, x \in \varphi_g F(g) \cap \varphi_{g'} F(g')$ and there exists $h_x \in \mathrm{ob}C$ with $\alpha \in C(h_x, g)$, $z \in \varphi_{h_x} F(h_x)$, $\varphi_{h_x} z = \bar{\varphi}\, x$. Thus $\bar{\varphi}\, \bar{\varphi}_{h_x} z = \bar{\varphi}\, x$. Since $\bar{\varphi}$ is monic, $\bar{\varphi}_{h_x} z = x$ and $\bigcup_{\delta \in \Delta} \bar{\varphi}_{h_x} \bar{F}(h_x) = \bar{\varphi}_g \bar{F}(g) \cap \bar{\varphi}_{g'} \bar{F}(g')$. If $\delta \in \Delta$, $x \in \bar{\varphi}_{h_\delta} \bar{F}(h_\delta)$ and $y \in \bar{\varphi}_g F(g) \cap \bar{\varphi}_{g'} \bar{F}(g')$ with a path $\alpha \in P(x, \bar{\varphi}_g \bar{F}(g) \cap \bar{\varphi}_{g'} \bar{F}(g'), y)$. Then $\bar{\varphi}\, \alpha$ is a path from $\bar{\varphi}\, x$ to $\bar{\varphi}\, y$ in $\varphi_g F(g) \cap \varphi_{g'} F(g')$. Since $\varphi_{h_\delta} F(h_\delta)$ is a path component of $\varphi_g F(g) \cap \varphi_{g'} F(g')$, $\bar{\varphi}\, y \in \varphi_{h_\delta} F(h_\delta)$ and there exists $z \in \varphi_{h_\delta} F(h_\delta)$

with $\varphi_{h_\delta} z = \bar{\varphi}\, y$. Thus $\bar{\varphi}\,\bar{\varphi}_{h_\delta} z = \bar{\varphi}\, y$ and $\bar{\varphi}_{h_\delta} z = y$. It follows that $y \in \bar{\varphi}_{h_\delta} \bar{F}(h_\delta)$ and $\bar{\varphi}_{h_\delta} \bar{F}(h_\delta)$ is a path component of $\bar{\varphi}_g \bar{F}(g) \cap \bar{\varphi}_{g'} \bar{F}(g')$ for each $\delta \in \Delta$. □

<u>Proposition 7′</u>. F_v and $\bar{F}$ satisfy conditions D' and D'' .

<u>Proof</u>: i) If $g, g' \in \mathrm{ob}C_v$ and

$$\varphi_g^v\, F_v(g) = \varphi_{g'}^v F_v(g')$$

then $\varphi_g F(g) = \varphi\, \varphi_g^v\, F_v(g) = \varphi\, \varphi_{g'}^v\, F(g') = \varphi_{g'} F(g')$ and $g = g'$. Similarly for $\bar{F}$.

ii) It is trivial that the restriction of a functor satisfying condition D'' satisfies condition D'' . □

<u>Proposition 8</u>. $\{\varphi_g^v F(g)\}_{g \in \mathrm{ob}C_v}$ is an open cover of $\operatorname{colim} F_v$ and each φ_g^v is an injection.

<u>Proof</u>: That each φ_g^v is an injection follows trivially from diagram $\mathcal{D}_1$ in Section 6 and the hypothesis that each φ_g is an injection. It is also trivial that $\{\varphi_g^v F(g)\}_{g \in \mathrm{ob}C_v}$ covers $\operatorname{colim} F_v$. Since by Lemma 3 of this section, φ is monic, $\varphi^{-1}\varphi = 1_{\operatorname{colim} F_v}$. If $a\colon C \to \operatorname{colim} F_v$ is admissible, then

$$\begin{aligned} a^{-1}\, \varphi_g^v\, F_v(g) &= a^{-1}\, \varphi^{-1}\, \varphi\, \varphi_g^v\, F_v(g) \\ &= (\varphi\, a)^{-1}\, \varphi\, \varphi_g^v\, F_v(g) \\ &= (\varphi\, a)^{-1}\, \varphi_g\, F(g)\ . \end{aligned}$$

Since $\varphi_g F(g)$ is open in $\operatorname{colim} F$, and $\varphi\, a\colon C \to \operatorname{colim} F$ is admissible,

$(\varphi\, a)^{-1}\ \varphi_g F(g)$ is open in C and $a^{-1}\ \varphi^v_g\, F_v(g)$ is open in C. It follows that $\varphi^v_g\, F_v(g)$ is open in colim F_v. □

Proposition 9. $\{\bar\varphi_g \bar F(g)\}_{g \in ob(v \cap C_v)}$ is an open cover of colim $\bar F$ and each $\bar\varphi_g$ is an injection.

Proof: That each φ_g is an injection follows trivially from diagram $\mathcal{D}_2$ in Section 7 and the hypothesis that each φ_g is an injection. It is also trivial that $\{\bar\varphi_g \bar F(g)\}_{g \in ob(v \cap C_v)}$ covers colim $\bar F$. Since by Lemma 4 of this section, $\bar\varphi$ is monic, $\bar\varphi^{-1}\ \bar\varphi = 1_{\text{colim}\ \bar F}$ and the proof concludes precisely as in Proposition 8. □

Proposition 10. φ is an injection.

Proof: By Lemma 3 of this section, φ is monic. Suppose $a\colon C \to \text{colim}\ F_v$ is a function such that the composition

$$C \xrightarrow{a} \text{colim}\ F_v \xrightarrow{\varphi} \text{colim}\ F$$

is admissible. Since colim F has the quotient topology, there is a commutative diagram

$$\begin{tikzcd}
 & \coprod F_v(g) \arrow[r, hook, "b"] \arrow[d, "p'"] & \coprod F(g) \arrow[d, "p"] \\
C' \arrow[ur, dashed, "c"] \arrow[urr, "a'"] \arrow[d, "\alpha", "\text{onto}"'] & & \\
C \arrow[r, "a"] & \text{colim}\ F_v \arrow[r, "\varphi"] & \text{colim}\ F
\end{tikzcd}$$

with a' admissible. If $x \in C'$, then

$$pa'x = \varphi\, a\, \alpha\alpha \in \operatorname{colim} F .$$

Since $a\alpha\alpha \in \operatorname{colim} F_v$ there exists an object $g \in \operatorname{ob}C_v$ with $y \in F_v(g)$ and $p'y = a\alpha\alpha$. This defines a map $c\colon C' \to \coprod F_v(g)$, such that $bc = a'$. Since b is the coproduct of the coproduct injections

$$F_v(g') = F(g') \longrightarrow \coprod F(g)$$

for $g' \in \operatorname{ob}C_v$, b is an injection and c is admissible. Since $\operatorname{colim} F_v$ has the quotient topology, a is admissible and $\operatorname{colim} F_v$ has the injection topology with φ an injection. $\square$

<u>Proposition 11</u>. $\bar{\varphi}$ is an injection.

<u>Proof</u>: By Lemma 4 of this section, $\bar{\varphi}$ is monic. Suppose $a\colon C \to \operatorname{colim} \bar{F}$ is a function such that the composition

$$C \xrightarrow{a} \operatorname{colim} \bar{F} \xrightarrow{\bar{\varphi}} \operatorname{colim} F$$

is admissible. Since $\operatorname{colim} \bar{F}$ has the quotient topology, there is a commutative diagram

$$\begin{tikzcd}
& \coprod \bar{F}(g) \arrow[dr, "b"] \arrow[dd, "p'"] & \\
C' \arrow[ur, dashed, "c"] \arrow[rr, "a'"] \arrow[d, "\alpha"', "\text{onto}"] & & \coprod F(g) \arrow[d, "p"] \\
C \arrow[r, "a"] & \operatorname{colim} \bar{F} \arrow[r, "\bar{\varphi}"] & \operatorname{colim} F
\end{tikzcd}$$

with a' admissible. If $x \in C'$, then

$$pa'x = \varphi a\alpha\alpha \in \text{colim } F .$$

Since $a\alpha\alpha \in \text{colim } F_v$, there exists an object $g \in \text{ob}C_v$ with $y \in F_v(g)$ and $p'y = a\alpha\alpha$. This defines a map $c: C' \to \coprod \bar{F}(g)$ such that $bc = a'$. Since b is the coproduct of the coproduct injections

$$\bar{F}(g') = F(g) \longrightarrow \coprod F(g)$$

for $g' \in \text{ob}(v \cap C_v)$, b is an injection and c is admissible. Since colim $\bar{F}$ has the quotient topology, a is admissible and colim $\bar{F}$ has the injection topology with $\bar{\varphi}$ an injection. □

<u>Proposition 12</u>. $\bar{\varphi}\,\text{colim } \bar{F} = \varphi_v F(v) \cap \varphi \text{ colim } F_v$.

<u>Proof</u>: For each $g \in \text{ob}(v \cap C_v)$,

$$\bar{\varphi}\,\bar{\varphi}_g\,\bar{F}(g) = \varphi_g\,F(g) \subseteq \varphi_v\,F(v) \cap \varphi \text{ colim } F_v .$$

Thus the family

$$\{\bar{\varphi}\,\bar{\varphi}_g : \bar{F}(g) \longrightarrow \varphi_v\,F(v) \cap \varphi \text{ colim } F_v\}_{g \in \text{ob}(v \cap C_v)}$$

induces a φ-morphism Ψ: colim $\bar{F} \to \varphi_v\,F(v) \cap \varphi \text{ colim } F_v$. Any element $x \in \varphi_v F(v) \cap \varphi \text{ colim } F_v$ is an element of

$$\varphi_v\,F(v) \cap \varphi_{h_x} F(h_x)$$

for some $h_x \in \text{ob}C$. Since F is closed under finite intersections, there is some element $g_x \in \text{ob}C$ with $C(g_x, v) \neq \emptyset$ and $x \in \varphi_{g_x} F(g_x)$. Thus there is some element $y \in F(g_x)$ with $\varphi_{g_x} y = x$. Then $\Psi\,\bar{\varphi}_{g_x} y = \bar{\varphi}\,\bar{\varphi}_{g_x} y = \varphi_{g_x} y = x$ and Ψ is onto. Suppose $x, x' \in \text{colim } \bar{F}$ and $\Psi x = \Psi x'$. Then there exist elements $y \in \bar{F}(g)$, $y' \in \bar{F}(g')$ with $\bar{\varphi}_g y = x$, $\bar{\varphi}_{g'} y' = x'$.

$$\Psi\, x = \Psi\, \bar{\varphi}_g y = \bar{\varphi}\, \bar{\varphi}_g y = \bar{\varphi}\, x$$

$$\Psi\, x' = \Psi\, \bar{\varphi}_g y' = \bar{\varphi}\, \bar{\varphi}_g y' = \bar{\varphi}\, x' .$$

Since $\bar{\varphi}$ is monic, $x = x'$ and Ψ is monic.

For each $g \in ob(v \cap C_v)$, the diagram

$$\begin{tikzcd} \bar{F}(g) \arrow[rr, "\bar{\varphi}\,\bar{\varphi}_g"] \arrow[d, equal] & & \varphi_v\, F(v) \cap \varphi \operatorname{colim} F_v \arrow[d, hook] \\ F(g) \arrow[rr, "\varphi_g"'] & & \operatorname{colim} F \end{tikzcd}$$

commutes with φ_g an injection. Thus $\bar{\varphi}\, \bar{\varphi}_g$ is an injection. For

$$C \xrightarrow{a} \varphi_v\, F(v) \cap \varphi \operatorname{colim} F_v$$

admissible, $a^{-1}\, \bar{\varphi}\, \bar{\varphi}_g\, \bar{F}(g) = a^{-1}\, \varphi_g\, F(g)$ is open in C . It follows that $\{\bar{\varphi}\, \bar{\varphi}_g\, \bar{F}(g)\}_{g \in ob(v \cap C_v)}$ forms an open cover of $\varphi_v\, F(v) \cap \varphi \operatorname{colim} F_v$ and previous arguments (cf. Theorem 1v) of Section 5) yield that Ψ is an injection and thus a homeomorphism:

$$\operatorname{colim} \bar{F} = \varphi_v\, F(v) \cap \varphi \operatorname{colim} F_v .$$

It follows from Proposition 11 that $\bar{\varphi}$ is a homeomorphism onto its image. Thus $\bar{\varphi} \operatorname{colim} \bar{F} = \operatorname{colim} \bar{F}$. □

Proposition 13. C_v has a generating tree.

Proof: The following three lemmas are required:

Lemma 5. For each $g \in Fr(C_v)$, there is a unique $f \in Fr(C)$ with $C(f,g) \neq \emptyset$. Furthermore, $\operatorname{im} f - \operatorname{im} v = \operatorname{im} g$.

Proof: If $g \in Fr(C_v)$, then since C is upward finite, there is a free face $f \in Fr(C)$ with $C(f,g) \neq \emptyset$. If $f \in obC_v$, then $f = g$ and since $imv \notin img = imf$, $imf = imf - imv = img$. If $f \notin obC_v$, then $imf - imv \neq \emptyset$ and there is an element $f' \in obC_v$ with $C(f,f') \neq \emptyset$ and $imf' = imf - imv$. But $img \subseteq imf'$, thus since $g \in Fr(C_v)$ $g = f'$ and $img = imf - imv$.

If there were two free faces with these properties, then their images would be the same and by Theorem 2 of Section 1, they would be equal. □

<u>Lemma 6</u>. If $f,f' \in Fr(C_v)$ are distinct free faces and $\bar{f},\bar{f}' \in Fr(C)$ with $C(\bar{f},f) \neq \emptyset$ and $C(\bar{f},f') \neq \emptyset$, then $\bar{f} \neq \bar{f}'$.

<u>Proof</u>: By Lemma 5, $imf = im\bar{f} - imv$ and $imf' = im\bar{f}' - imv$. If $\bar{f} = \bar{f}'$, then $imf = imf'$. Since in this case $C(\bar{f},f) \neq \emptyset$ and $C(\bar{f},f') \neq \emptyset$, it follows from the fact that C is an s-category that $f = f'$. □

<u>Lemma 7</u>. If $\Gamma(C_v)$ has a circuit of length two, then $\Gamma(C)$ has a circuit of length two.

<u>Proof</u>: Since C_v has a circuit of length two, there exist distinct free faces $f,f' \in Fr(C_v)$ and distinct elements $h,h' \in Fr(f \cap f')$. Corresponding to f,f' are distinct free faces $\bar{f},\bar{f}' \in Fr(C)$ with $C(\bar{f},f) \neq \emptyset$, $C(\bar{f},f') \neq \emptyset$ and $h,h' \in ob(\bar{f} \cap \bar{f}')$. There exist free faces $\bar{h},\bar{h}' \in Fr(\bar{f} \cap \bar{f}')$ with $C(\bar{h},h) \neq \emptyset$, $C(\bar{h}',h') \neq \emptyset$. Suppose $\bar{h} = \bar{h}'$; then

$$imh \cup imh' \subseteq im\bar{h} \subseteq im\bar{f} \cap im\bar{f}' .$$

If $imv \notin im\bar{h}$, then $im\bar{h} \subseteq im\bar{f} \cap im\bar{f}' - imv$

$$= imf \cap imf' .$$

Thus $C(f,\bar{h}) \neq \emptyset$ and $C(f',\bar{h}) \neq \emptyset$ so that $h \in ob(f \cap f')$ and since

$C(\bar{h},h) \neq \emptyset$, $C(\bar{h},h') \neq \emptyset$, either h,h' are not distinct or else they are not free faces. Thus $\text{im}v \subseteq \text{im}\bar{h}$.

Since $\bar{h}$ is not 0-dimensional, $\text{im}\bar{h} - \text{im}v \neq \emptyset$. Since $\text{im}\bar{h} - \text{im}v \subseteq \text{im}\bar{h}$, there exists an element $\hat{h} \in \text{ob}C$ with $\text{im}\hat{h} = \text{im}\bar{h} - \text{im}v$. Since $\text{im}v \nsubseteq \text{im}\hat{h}$, $\hat{h} \in \text{ob}C_v$. Since $\text{im}h \subseteq \text{im}\hat{h}$ and $\text{im}h' \subseteq \text{im}\hat{h}$, $C(\hat{h},h) \neq \emptyset$ and $C(\hat{h},h') \neq \emptyset$.

Since $\text{im}\hat{h} = \text{im}\bar{h} - \text{im}v \subseteq \text{im}f \cap \text{im}f'$, $C(f,\hat{h}) \neq \emptyset$ and $C(f',\hat{h}) \neq \emptyset$ and $\hat{h} \in \text{ob}(f \cap f')$. It follows that either h,h' are not distinct or they are not free faces. □

<u>Proof of Proposition 13</u>. $\Gamma \subseteq \Gamma(C)$ is a generating tree for C . The subgraph $\Gamma' \subseteq \Gamma(C_v)$ is defined to have the same vertices as $\Gamma(C_v)$. If $(f,h,f') \in \Gamma(C_v)$, then there exist unique and distinct free faces $\bar{f},\bar{f}' \in \text{Fr}(C)$ with $C(\bar{f},f) \neq \emptyset$ and $C(\bar{f}',f') \neq \emptyset$. Since $h \in \text{ob}(\bar{f} \cap \bar{f}')$, there exists an element $\bar{h} \in \text{Fr}(\bar{f} \cap \bar{f}')$ with $C(\bar{h},h) \neq \emptyset$. Since C has a generating tree, this $\bar{h}$ is unique or else C would have a circuit of length two. The edge (f,h,f') is an edge of Γ' if and only if the edge $(\bar{f},\bar{h},\bar{f}')$ is an edge of Γ .

i) By construction, $v\Gamma' = v\Gamma(C_v)$

ii) If Γ' is not a tree, then it contains a circuit of some finite length n :

$$\{(f_i,\ h_i,\ f_{i+1})\}_{i=0}^{n}$$

with $f_0 = f_{n+1}$ and $h_i \neq h_{i+1}$, $i=0,\ldots,n-1$, $h_n \neq h_0$. By construction, the elements

$$\{(\bar{f}_i,\ \bar{h}_i,\ \bar{f}_{i+1})\}_{i=0}^{n}$$

form a path in Γ . Since Γ is a tree, this path is not a circuit and thus for some i ,

$$(\bar{f}_i, \bar{h}_i, \bar{f}_{i+1}) = \iota(\bar{f}_{i+1}, \bar{h}_{i+1}, \bar{f}_{i+2})$$

That is $\bar{f}_{i+2} = \bar{f}_i$ and $\bar{h}_i = \bar{h}_{i+1}$. It follows from Lemma 5, that $f_i = f_{i+2}$. Thus if Γ' is not a tree, it contains a circuit of length two and thus $\Gamma(C_v)$ has a circuit of length two. By Lemma 7, $\Gamma(C)$ will have a circuit of length two and thus no generating tree.

iii) If $(f,h,f') \in e\Gamma(C_v)$, then the construction above yields a unique edge $(\bar{f},\bar{h},\bar{f}') \in e\Gamma(C)$. Since Γ is a generating tree, there is a sequence of edges

$$\{(\bar{f}_i, \bar{h}_i, \bar{f}'_{i+1}) \in e\Gamma\}_{i=0}^{n}$$

with $\bar{f}_0 = f$, $\bar{f}_{n+1} = f'$, and $C(\bar{h}_i,\bar{h}) \neq \emptyset$, $i=0,\dots,n$.

If for any i , $\bar{h}_i$ is 0-dimensional, $\bar{h}_i = \bar{h} = h$ and $h_i \in obC_v$. Similarly, if $\bar{f}_i$ is 0-dimensional for any i , then $f_i \in obC_v$.

If for some i , $\bar{h}_i \notin obC_v$, then $im\bar{h}_i - imv \neq \emptyset$ and there exists an element $h_i \in obC_v$ with $C(\bar{h}_i,h_i) \neq \emptyset$ and $imh_i = im\bar{h}_i - imv$. If $\bar{h}_i \in obC_v$, then $h_i = \bar{h}_i$. The family $\{f_i \in obC_v\}_{i=0}^{n+1}$ is similarly defined. Since

$$imh_i = im\bar{h}_i - imv \subseteq im\bar{f}_i - imv = imf_i ,$$

$C(f_i,h_i) \neq \emptyset$. Similarly, $C(f_{i+1},h_i) \neq \emptyset$. Thus for each $i=0,\dots,n$,

$$h_i \in ob(f_i \cap f_{i+1}) .$$

If $h'_i \in Fr(f_i \cap f_{i+1}) \cap obC_v$, and $C(h'_i,h_i) \neq \emptyset$, then $h'_i \in ob(\bar{f}_i \cap \bar{f}_{i+1})$ and since $\Gamma(C)$ has no circuits of length two, $C(\bar{h}_i,h'_i) \neq \emptyset$. Since $h'_i \in obC_v$, $imv \subseteq imh'_i$ and

$$\mathrm{imh}_i \subsetneq \mathrm{imh}_i' \subseteq \mathrm{im}\bar{h}_i - \mathrm{imv} = \mathrm{imh}_i$$

and $\mathrm{imh}_i = \mathrm{imh}_i'$. Since $C(\bar{f}_i, h_i)$ and $C(\bar{f}_i, h_i')$ are non-empty, $h_i' = h_i$. It follows that (f_i, h_i, f_{i+1}) is an edge of $\Gamma(C_v)$ and by construction an edge of Γ' . Since

$$\mathrm{imh} \subseteq \mathrm{im}\bar{h} - \mathrm{imv} \subseteq \mathrm{im}\bar{h}_i - \mathrm{imv} = \mathrm{imh}_i \ ,$$

$C(h_i, h) \neq \emptyset$ for each i and the sequence

$$\{(f_i,\ h_i,\ f_{i+1}) \in e\Gamma'\}_{i=0}^{n}$$

is of the required type. □

<u>Proposition 14</u>. $v \cap C_v$ has a generating tree.

<u>Proof</u>: The proof requires a lemma:

<u>Lemma 8</u>. $\Gamma(v \cap C_v) \subsetneq \Gamma(C)$.

<u>Proof</u>: If $f \in Fr(v \cap C_v)$ but $f_* \notin Fr(C)$, then for some $g_* \in Fr(C)$, $C(g_*, f_*) \neq \emptyset$. But since $C(f,v) \neq \emptyset$, $C(g,v) \neq \emptyset$ and $g \in Fr(v \cap C_v)$. Thus

$$v\Gamma(v \cap C_v) = Fr(v \cap C_v) \subseteq Fr(C) = v\Gamma(C) \ .$$

If $f, f' \in Fr(v \cap C_v)$ are distinct free faces and

$$g \in Fr(f \cap f')$$

then $g_* \in Fr(f_* \cap f_*')$ or else there would be an element $h_* \in Fr(f_* \cap f_*')$ with $C(h_*, g_*) \neq \emptyset$. Again this implies that $h \in ob(v \cap C_v)$ and $g \notin Fr(f \cap f')$. Thus

$$e\Gamma(v \cap C_v) \subseteq e\Gamma(C)$$

and the lemma follows. □

<u>Proof of Proposition 14</u>. If $(f,g,f') \in e\Gamma(v \cap C_v)$, then by Lemma 8, $(f_*,g_*,f'_*) \in e\Gamma(C)$ and if $\Gamma \subseteq \Gamma(C)$ is a generating tree for C , then there is a finite sequence

$$\{(f_*^i,\ g_*^i,\ f_*^{i+1}) \in e\Gamma\}_{i=0}^n$$

with $f_*^0 = f_*$, $f_*^{n+1} = f'_*$ and $C(g_*^i, g_*) \neq \emptyset$ for $i=0,\dots,n$. Since $C(g_*,v) \neq \emptyset$, $C(g_*^i,v) \neq \emptyset$ for each i and $g^i \in ob(v \cap C_v)$. Since $C(f_*^i,g^i) \neq \emptyset$ for each i , $f^i \in ob(v \cap C_v)$. It follows that the sequence

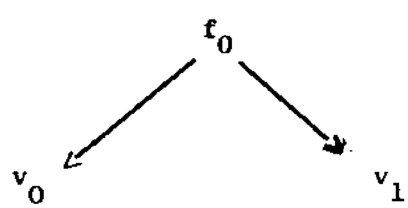
$$\{(f^i,\ g_i,\ f^{i+1}) \in e\Gamma \cap e\Gamma(v \cap C_v)\}_{i=0}^n$$

is of the required type for $\Gamma \cap \Gamma(v \cap C_v)$ to be a generating graph for $\Gamma(v \cap C_v)$. Since Γ is a tree and $\Gamma \cap \Gamma(v \cap C_v) \subseteq \Gamma$, $\Gamma \cap \Gamma(v \cap C_v)$ is a generating tree. □

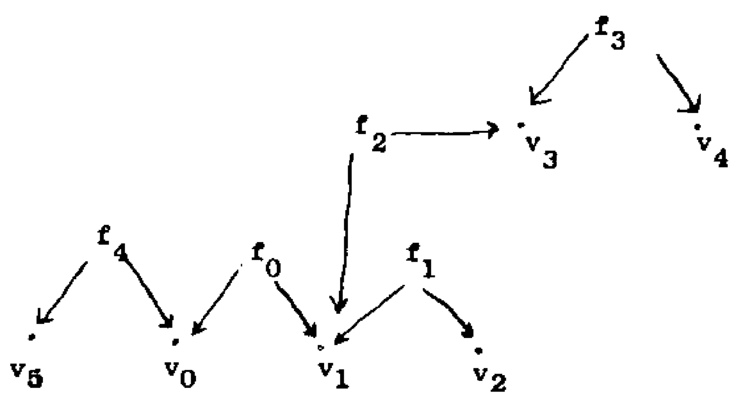
<u>Proposition 15</u>. If $|C_0| < \infty$, then there exists a vertex $v \in C_0$ such that C_v is connected.

<u>Proof</u>: A graph G of the vertices of C_0 is constructed inductively such that $vG = C_0$ and $eG \subseteq Fr(C)$:

i) $v_0 \in C_0$ is arbitrarily chosen. $f_0 \in Fr(C)$ is such that $C(f_0,v_0) \neq \emptyset$. Then f_0 is an edge of the graph from the vertex v_0 to any vertex v_1 with $C(f,v_1) \neq \emptyset$ and $v_1 \neq v_0$. The involution is clear. v_0 and v_1 are the <u>ends</u> of the graph. The graph is represented



ii) If $v_0, \ldots, v_k$ have been selected in this way with ends v_i, v_j then some one of the vertices in $C_0 - \{v_0, \ldots, v_k\}$ has a free face in common with a vertex in $\{v_0, \ldots, v_k\}$ or else C is disconnected and by Proposition 2 of Section 5, $\Gamma(C)$ is disconnected. f_k is such a free face. f_k is an edge from some vertex $v_\ell \in \{v_0, \ldots, v_k\}$ to v_{k+1}. If $\ell = i$, then v_{k+1}, v_j are ends. If $\ell = j$, then v_{k+1}, v_i are ends. If $\ell \neq i$, $\ell \neq j$, then v_i, v_j are ends. The figure illustrates;



v_2 and v_5 are the ends.

This process continues inductively to yield a finite graph (with obvious involution) which contains all the vertices and has two ends. Clearly, removal of an end will not affect the connectivity of the graph. The required vertex v is chosen to be such an end. □

<u>Proposition 16</u>. $\varphi_v F(v) \cap \varphi \operatorname{colim} F_v$ is connected.

<u>Proof</u>: The proof requires three Lemmas:

<u>Lemma 9</u>. For each pair $g, g' \in \mathrm{ob}C$ with

$$\varphi_g F(g) \cap \varphi_{g'} F(g') \neq \emptyset$$

there is an object $h \in \mathrm{ob}C$ with $C(h,g) \neq \emptyset$, $C(h,g') \neq \emptyset$ and

$\varphi_h F(h) = \varphi_g F(g) \cap \varphi_{g'} F(g')$.

Proof: Since $\varphi_g F(g) \cap \varphi_{g'} F(g') \neq \emptyset$ and F is closed under finite intersections, there is a set Δ and a family $\{h_\delta\}_{\delta\in\Delta}$ with $C(h_\delta, g) \neq \emptyset$, $C(h_\delta, g') \neq \emptyset$ for each $\delta \in \Delta$ and

$$\bigcup_{\delta\in\Delta} \varphi_{h_\delta} F(h_\delta) = \varphi_g F(g) \cap \varphi_{g'} F(g')$$

with $\varphi_{h_\delta} F(h_\delta)$ a path component of the intersection for each $\delta \in \Delta$.

If $h, h' \in \{h_\delta\}_{\delta\in\Delta}$, then there are free faces $f, f' \in Fr(C)$ with $C(f,h) \neq \emptyset$ and $C(f',h') \neq \emptyset$. Since $g, g' \in ob(f \cap f')$ and C has a generating tree, there exists a unique $k \in Fr(f \cap f')$ and $C(k,g) \neq \emptyset$, $C(k,g') \neq \emptyset$.

Since $img \subseteq imh \cap imh' \cap imk$, $imh \cap imh' \cap imk \neq \emptyset$ and there is a uniquely determined object $\ell \in obC$ with $im\ell = imh \cap imh' \cap imk$ and $C(\ell, g) \neq \emptyset$, $C(\ell, g') \neq \emptyset$.

Since $C(\ell, g) \neq \emptyset$ and $C(\ell, g') \neq \emptyset$,

$$\varphi_\ell F(\ell) \subseteq \varphi_g F(g) \cap \varphi_{g'} F(g')$$

and since F is path connected, $\varphi_\ell F(\ell)$ is path connected. Since $im\ell \subseteq imh$ and $imh' \subseteq im\ell$, $C(h,\ell) \neq \emptyset$ and $C(h',\ell) \neq \emptyset$. It follows that

$$\varphi_h F(h) \subseteq \varphi_\ell F(\ell) \quad \text{and} \quad \varphi_{h'} F(h') \subseteq \varphi_\ell F(\ell) .$$

Since $\varphi_h F(h)$ and $\varphi_{h'} F(h')$ were path components of the intersection, $\varphi_h F(h) = \varphi_\ell F(\ell) = \varphi_{h'} F(h')$. It follows that $h = \ell = h'$ and $|\{h_\delta\}_{\delta\in\Delta}| = 1$. $\square$

<u>Lemma 10.</u> :

$$\varphi_v\, F(v) \cap \varphi \operatorname{colim} F_v = \bigcup_{\bar{v}\in(C_v)_0} \varphi_v\, F(v) \cap \varphi_{\bar{v}}\, F(\bar{v})$$

<u>Proof</u>: $\supseteq$: If $x \in \bigcup_{\bar{v}\in(C_v)_0} \varphi_v F(v) \cap \varphi_{\bar{v}} F(\bar{v})$, then for some $v' \in (C_v)_0$, $x \in \varphi_{v'} F(v')$ and there is an element $y \in F(v')$ with $\varphi_{v'} y = x$. Then $\varphi\, \varphi^{v}_{v'} y = \varphi_{v'} y = x$ and $x \in \varphi \operatorname{colim} F_v$.

$\subseteq$: If $x \in \varphi_v F(v) \cap \varphi \operatorname{colim} F_v$ then for some $g \in \operatorname{ob}C_v$ and some $y \in F(g)$, $x = \varphi\, \varphi^{v}_{g} y$. If $v' \in \nu g$, then $C(g,v') \neq \emptyset$ and since $g \in \operatorname{ob}C_v$, $v \neq v' \in (C_v)_0$. If $\alpha \in C(g,v')$, then $\varphi_{v'} F(\alpha) y = \varphi_g y = \varphi\, \varphi^{v}_{g} y = x$ and $x \in \varphi_{v'} F(v')$ for some $v' \in (C_v)_0$. □

Lemmas 9 and 10 reduce the proof of the theorem to showing that if $v_1, v_n \in (C_v)_0$ and

$$\varphi_v F(v) \cap \varphi_{v_i} F(v_i) \neq \emptyset \ , \ i=1,n$$

then there is a path in $\varphi_v F(v) \cap \varphi \operatorname{colim} F_v$ from some point in $\varphi_v F(v) \cap \varphi_{v_1} F(v_1)$ to some point in $\varphi_v F(v) \cap \varphi_{v_n} F(v_n)$.

<u>Lemma 11.</u> If $v_i \in C_0$, $i=0,1,2$ are vertices of C such that

$$\varphi_{v_0} F(v_0) \cap \varphi_{v_1} F(v_1) \neq \emptyset \ , \ \varphi_{v_1} F(v_1) \cap \varphi_{v_2} F(v_2) \neq \emptyset$$

and $$\varphi_{v_0} F(v_0) \cap \varphi_{v_2} F(v_2) \neq \emptyset \ ,$$

then

$$\bigcap_{i=0}^{2} \varphi_{v_i} F(v_i) \neq \emptyset \ .$$

<u>Proof</u>: By Lemma 9, there exist objects $h_i \in \operatorname{ob}C$, $i=0,1,2$ such that

$$\varphi_{h_2}F(h_2) = \varphi_{v_0}F(v_0) \cap \varphi_{v_1}F(v_1)$$

$$\varphi_{h_0}F(h_0) = \varphi_{v_1}F(v_1) \cap \varphi_{v_2}F(v_2)$$

$$\varphi_{h_1}F(h_1) = \varphi_{v_0}F(v_0) \cap \varphi_{v_2}F(v_2)$$

and

*) $\qquad C(h_i, v_j) \neq \emptyset$ for $i=0,1,2$ and $j \neq i$.

The free faces $f_i \in Fr(C)$ are chosen so that $C(f_i, h_i) \neq \emptyset$, $i=0,1,2$. Condition *) implies that

$$v_0 \in ob(f_1 \cap f_2)$$

$$v_1 \in ob(f_0 \cap f_2)$$

$$v_2 \in ob(f_0 \cap f_1)$$

Since C has a generating tree, the sets $Fr(f_i \cap f_j)$, $i < j$ are singletons. The objects $g_i \in obC$ are chosen so that

$$g_0 \in Fr(f_1 \cap f_2)$$

$$g_1 \in Fr(f_0 \cap f_2)$$

$$g_2 \in Fr(f_0 \cap f_1) \ .$$

Necessarily, $C(g_i, v_i) \neq \emptyset$ for $i=0,1,2$ or else the sets $Fr(f_i \cap f_j)$, $i < j$ are not singletons.

The sequence

$$S = \left\{(f_0, g_2, f_1), (f_1, g_0, f_2), (f_2, g_1, f_0)\right\}$$

of edges of $\Gamma(C)$ determines a circuit in $\Gamma(C)$. Since Γ is a tree, some element of S is omitted from Γ. By relabeling (if necessary), it is the edge

$$(f_0, g_2, f_1) \ .$$

Since Γ is a generating graph, there is a finite sequence of edges

$$S' = \left\{(f_2^i,\ g_2^i,\ f_2^{i+1}) \in e\Gamma\right\}_{i=0}^{n_2}$$

of Γ with $f_2^0 = f_0$, $f_2^{n_2+1} = f_1$, and $C(g_2^i, g_2) \neq \emptyset$ for $i=0,\ldots,n_2$.

The juxtaposition

$$S', (f_1, g_0, f_2), (f_2, g_1, f_0)$$

contains a circuit unless $\bigcup S' = \{(f_1, g_0, f_2), (f_2, g_1, f_0)\}$ where the union sign means the union of the sequence considered as a set of edges.

If $\bigcup S' = \{(f_1, g_0, f_2), (f_2, g_1, f_0)\}$, then $C(g_0, g_2) \neq \emptyset$ and $C(g_1, g_2) \neq \emptyset$.

Otherwise an edge of the juxtaposition is omitted. Since by construction $S' \subseteq e\Gamma$, either (f_1, g_0, f_2) or (f_2, g_1, f_0) is omitted. By relabeling (if necessary), the omitted edge is

$$(f_1, g_0, f_2) \ .$$

Since Γ is a generating graph, there is a finite sequence of edges

$$S'' = \left\{(f_0^i, g_0^i, f_0^{i+1}) \in e\Gamma\right\}_{i=0}^{n_0}$$

of Γ with $f_0^0 = f_1$, $f_0^{n_0+1} = f_2$ and $C(g_0^i, g_0) \neq \emptyset$ for $i=0,\ldots,n_0$.

As before the juxtaposition

$$S', S'',\ (f_2, g_1, f_0)$$

contains a circuit in $\Gamma(C)$ unless $\bigcup S' = \bigcup S'' \cup \{f_2, g_1, f_0)$, in which case $C(g_1,g_2) \neq \emptyset$ since the middle term of every edge in S' was related to g_2 in this way.

Otherwise (f_2,g_1,f_1) is omitted from Γ. Since Γ is a generating graph, there is a finite sequence of edges

$$S'' = \left\{(f_1^i,g_1^i,f_1^{i+1}) \in e\Gamma\right\}_{i=0}^{n_1}$$

of Γ with $f_1^0 = f_2$, $f_1^{n_1+1} = f_1$ and $C(g_1^i,g_1) \neq \emptyset$ for $i=0,\ldots,n_1$.
Since Γ is a tree and the juxtaposition

$$S', S''$$

is also a path in Γ from f_2 to f_1,

$$\cup S''' = \cup S' \cup \cup S'' .$$

By construction $C(g_1^i,g_1) \neq \emptyset$ for each middle term of each edge of S''.
Thus

$$C(g_0^i,g_1) \neq \emptyset, \; i=0,\ldots,n_0$$

and

$$C(g_2^i,g_1) \neq \emptyset, \; i=0,\ldots,n_2 .$$

In particular, $C(g_0^{n_0},g_1) \neq \emptyset$ and $C(g_2^{n_2},g_1) \neq \emptyset$.

Since $(f_0^{n_0},g_0^{n_0},f_0^{n_0+1}) = (f_0^{n_0},g_0^{n_0},f_2)$ is an edge of $\Gamma(C)$, $C(f_2,g_0^{n_0}) \neq \emptyset$.
Since $C(g_0^{n_0},g_1) \neq \emptyset$, it follows that $C(f_2,g_1) \neq \emptyset$ and

$$g_1 \in ob(f_1 \cap f_2)$$

$$g_1 \in ob(f_0 \cap f_1) .$$

Since g_2 and g_0 are respective free faces in these categories (which have but one free face each) $C(g_0,g_1) \neq \emptyset$ and $C(g_2,g_1) \neq \emptyset$.

All this argument yields that in any case, for the (perhaps relabeled) pair g_0,g_1 ,

$$C(g_0,g_1) \neq \emptyset .$$

It follows that $\varphi_{g_0}F(g_0) \subseteq \varphi_{g_1}F(g_1)$. Since $\varphi_{g_1}F(g_1) \subseteq \varphi_{v_0}F(v_0) \cap \varphi_{v_2}F(v_2)$ and $\varphi_{g_0}F(g_0) \subseteq \varphi_{v_1}F(v_1) \cap \varphi_{v_2}F(v_2)$ any element $x \in \varphi_{g_0}F(g_0)$ is contained in the intersection

$$\bigcap_{i=0}^{2} \varphi_{v_i}F(v_i)$$

which is therefore non-empty. □

<u>Proof of Proposition 16</u>. If $v_1,v_n \in (C_{v_0})$ are vertices such that

$$\varphi_v F(v) \cap \varphi_{v_1}F(v_1) \neq \emptyset \quad \text{and} \quad \varphi_v F(v) \cap \varphi_{v_n}F(v_n) \neq \emptyset ,$$

then there exist elements $h_0,h_n \in \mathrm{ob}C$ with

$$\varphi_{h_0}F(h_0) = \varphi_v F(v) \cap \varphi_{v_1}F(v_1) ,$$

$$\varphi_{h_n}F(h_n) = \varphi_v F(v) \cap \varphi_{v_n}F(v_n) ,$$

and $C(h_0,v) \neq \emptyset$, $C(h_0,v_1) \neq \emptyset$, $C(h_n,v) \neq \emptyset$, $C(h_n,v_n) \neq \emptyset$. The free faces f_0 and $f_n \in Fr(C)$ are chosen so that

$$C(f_i,h_i) \neq \emptyset,\ i=0,n .$$

Since $v \in \mathrm{ob}(f_0 \cap f_n)$ and C has a generating tree, there is a unique $g_{n+1} \in Fr(f_0 \cap f_n)$ with $C(g_{n+1},v) \neq \emptyset$. Then $g_{n+1} \notin \mathrm{ob}C_v$ and

$$(f_0,g_{n+1},f_n) \in e\Gamma(C) .$$

Since C_v is connected and $v_1, v_n \in (C_v)_0$, there is a sequence of vertices

$$\{v_i\}_{i=1}^{n}$$

and elements

$$\{h_i \in \mathrm{ob}C_v\}_{i=1}^{n-1}$$

with $C(h_i, v_i) \neq \emptyset$, $C(h_i, v_{i+1}) \neq \emptyset$ for $i=1,\dots,n-1$.

The sequence

$$\{f_i \in Fr(C)\}_{i=1}^{n-1}$$

is of free faces of C for which $C(f_i, h_i) \neq \emptyset$, $i=1,\dots,n-1$.

Since C has a generating tree and

$$v_i \in \mathrm{ob}(f_{i-1} \cap f_i)$$

for $i=1,\dots,n-1$, there exist unique elements

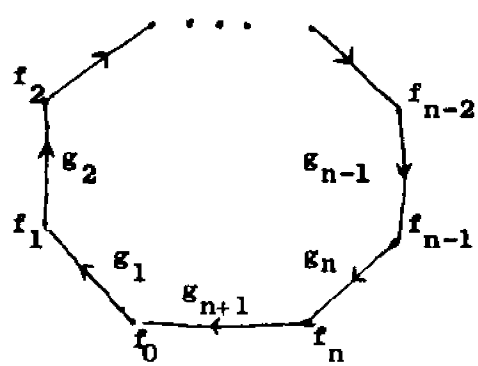
$$\{g_i \in Fr(f_{i-1} \cap f_i)\}_{i=1}^{n}$$

with $C(g_i, v_i) \neq \emptyset$ for $i=1,\dots,n$.

The sequence

$$S = \{(f_{i-1}, g_i, f_i),\ (f_n, g_{n+1}, f_0)\}_{i=1}^{n}$$

is a path in $\Gamma(C)$:



<u>Case 1</u>. For some $j \neq n+1$, $(f_{j-1}, g_j, f_j) = (f_n, g_{n+1}, f_0)$.

i) Since $g_j = g_{n+1}$, $C(g_{n+1}, v_j) \neq \emptyset$. Thus there is a diagram

$$\begin{tikzcd} & g_{n+1} \arrow[dl] \arrow[dr] & \\ v_j & & v \end{tikzcd}$$

in C and $\varphi_{v_j} F(v_j) \cap \varphi_v F(v) \neq \emptyset$.

ii) Since $C(f_0, g_{n+1}) \neq \emptyset$, $C(f_0, g_j) \neq \emptyset$ and there is a diagram

$$\begin{tikzcd} & f_0 \arrow[dl] \arrow[dr] & \\ h_0 \arrow[d] & & g_j \arrow[d] \\ v_1 & & v_j \end{tikzcd}$$

in C and $\varphi_{v_1} F(v_1) \cap \varphi_{v_j} F(v_j) \neq \emptyset$.

iii) By hypothesis $\varphi_v F(v) \cap \varphi_{v_1} F(v_1) \neq \emptyset$. It follows from Lemma 11 that

$$\varphi_v F(v) \cap \varphi_{v_j} F(v_j) \cap \varphi_{v_1} F(v_1) \neq \emptyset .$$

iv) Since $C(f_n, g_{n+1}) \neq \emptyset$, $C(f_n, g_j) \neq \emptyset$ and there is a diagram

$$\begin{tikzcd} & f_n \arrow[dl] \arrow[dr] & \\ h_n \arrow[d] & & g_j \arrow[d] \\ v_n & & v_j \end{tikzcd}$$

in C and $\varphi_{v_n}F(v_n) \cap \varphi_{v_j}F(v_j) \neq \emptyset$. It follows from Lemma 11 that

$$\varphi_v F(v) \cap \varphi_{v_j}F(v_j) \cap \varphi_{v_n}F(v_n) \neq \emptyset .$$

vi) By Lemma 9 there exists an element $k_j \in \mathrm{ob}C$ such that

$$\varphi_{k_j}F(k_j) = \varphi_v F(v) \cap \varphi_{v_j}F(v_j) .$$

Clearly, $\emptyset \neq \varphi_{k_j}F(k_j) \cap \varphi_i F(v_i) = \varphi_v F(v) \cap \varphi_{v_j}F(v_j) \cap \varphi_{v_i}F(v_i)$ for $i=1,n$.

By Lemma 10,

$$\varphi_{k_j}F(k_j) = \varphi_v F(v) \cap \varphi_{v_j}F(v_j) \subseteq \varphi_v F(v) \cap \varphi \operatorname{colim} F_v .$$

If $x_i \in \varphi_{k_j}F(k_j) \cap \varphi_i F(v_i)$, $i=1,n$, then since $\varphi_{k_j}F(k_j)$ is path connected, it is possible to join x_1 to x_n by a path in $\varphi_{k_j}F(k_j) \subseteq \varphi_v F(v) \cap \varphi \operatorname{colim} F_v$. Since $x_1 \in \varphi_v F(v) \cap \varphi_{v_1}F(v_1)$ and $x_n \in \varphi_v F(v) \cap \varphi_{v_n}F(v_n)$, the theorem is proved in this case.

<u>Case 2</u>. For no j , $0 \leq j \leq n$, does $(f_{j-1},g_j,f_j) = (f_n,g_{n+1},f_0)$.

In this case the sequence S contains a circuit which contains (f_n,g_{n+1},f_0) . By relabeling,

$$S' = \{(f_{i-1},g_i,f_i),\ (f_n,g_{n+1},f_0)\}_{i=1}^{m}$$

is a circuit and $S' \subseteq S$. Also (f_0,g_1,f_1) has not been relabeled and $(f_{m-1},g_m,f_m) = (f_{n-1},g_n,f_n)$.

The next step is to prove the following assertion:

<u>Assertion</u>: If $\Gamma \subseteq \Gamma(C)$ is a generating tree, then for some k , $1 \leq k \leq m$ or $k = n+1$, there is a doubly indexed sequence of edges

$$S' = \left\{ [(f_i^j,\ g_i^j,\ f_i^{j+1}) \in e\Gamma]_{j=0}^{n_i} \right\}_{i=1,\ldots,k-1,k+1,\ldots,m,n+1}$$

where for $i = 1, \ldots, k-1, k+1, \ldots, m, n+1$

a) $f_i^0 = f_{i-1}$; $f_i^{n_i+1} = f_i$

b) $C(g_i^j, g_i) \neq \emptyset,\ 0 \leq j \leq n_i$

c) $C(g_i^j, g_k) \neq \emptyset,\ 0 \leq j \leq n_i$.

Proof of Assertion. The proof will only be sketched since it is essentially the same as the first part of the proof of Lemma 11.

Since Γ is a tree, some edge $(f_{k'-1}, g_{k'}, f_{k'})$ in S' is not in $e\Gamma$. As in Lemma 11, there is a sequence S'' of edges of Γ which replaces this edge. If $(f_{k'-1}, g_{k'}, f_{k'})$ is the only edge in S' which is not in $e\Gamma$, then as in Lemma 11, $\bigcup S'' = \bigcup S' - \{(f_{k'-1}, g_{k'}, f_{k'})\}$ and $k' = k$ in the assertion. Otherwise the process continues until all the edges in $S' - e\Gamma$ have been replaced. Then (f_{k-1}, g_k, f_k) is the last edge removed.

Case 2 now breaks down into two subcases:

Subcase A. $k = n+1$. In this case there are diagrams in C :

$$\begin{tikzcd}
g_i^j \arrow[d] \arrow[dr] & \\
g_{n+1} \arrow[d] & g_i \arrow[d] \\
v & v_i
\end{tikzcd}$$

$i=1,\ldots,m$

$$\begin{tikzcd}
 & f_{i+1} \arrow[dl] \arrow[dr] & \\
g_i \arrow[d] & & g_{i+1} \arrow[d] \\
v_i & & v_{i+1}
\end{tikzcd}$$

$i=1,\ldots,m-1$

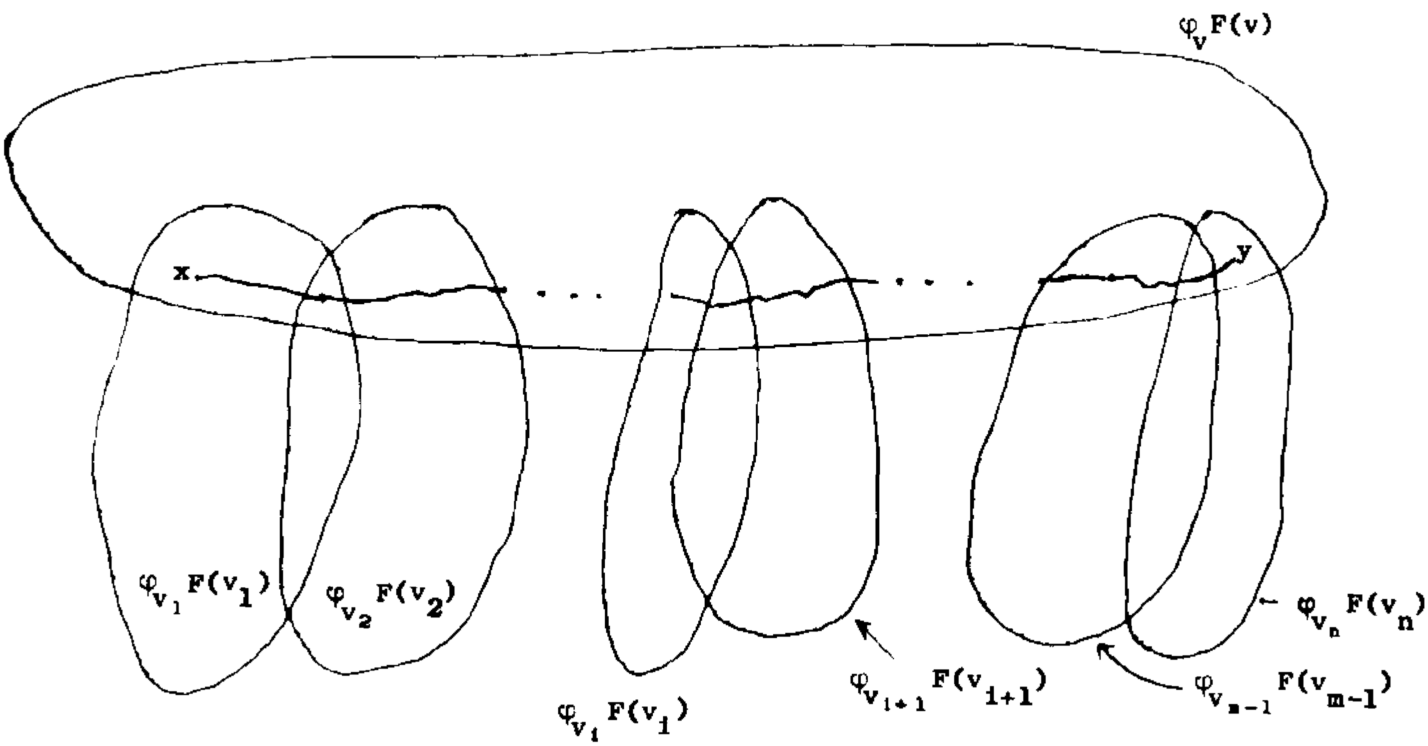
which yield

$$\varphi_v F(v) \cap \varphi_{v_i} F(v_i) \neq \emptyset, \ i=1,\ldots,m$$

$$\varphi_{v_i} F(v_i) \cap \varphi_{i+1} F(v_{i+1}) \neq \emptyset, \ i=1,\ldots,m \ .$$

Lemma 11 now yields a sequence of ineqµalities

$$\varphi_v F(v) \cap \varphi_{v_i} F(v_i) \cap \varphi_{v_{i+1}} F(v_{i+1}) \neq \emptyset \quad \text{for} \quad i=1,\ldots,m-1 \ .$$



Since $\varphi_{v_m}F(v_m) = \varphi_{v_n}F(v_n)$, it is clear from the drawing how to construct a path in $\varphi_v F(v) \cap \varphi \operatorname{colim} F_v$ from the point $x \in \varphi_v F(v) \cap \varphi_{v_1}F(v_1)$ to the point $y \in \varphi_v F(v) \cap \varphi_{v_n}F(v_n)$.

<u>Subcase B</u>. $k \neq n+1$. This case is similar to subcase A but it is a bit easier. As before, there are diagrams

$$\begin{tikzcd}
 & g^i_{n+1} \arrow[dl] \arrow[dr] & & g^i_1 \arrow[dl] \arrow[dr] & \\
g_{n+1} \arrow[d] & & g_k \arrow[d] & & g_1 \arrow[d] \\
v & & v_k & & v_1
\end{tikzcd}$$

$$\begin{tikzcd}
 & g^i_{n+1} \arrow[dl] \arrow[dr] & & g^i_n \arrow[dl] \arrow[dr] & \\
g_{n+1} \arrow[d] & & g_k \arrow[d] & & g_n \arrow[d] \\
v & & v_k & & v_n
\end{tikzcd}$$

in C which together with Lemma 11 yield

$$\varphi_v F(v) \cap \varphi_{v_k}F(v_k) \cap \varphi_{v_1}F(v_1) \neq \emptyset$$

$$\varphi_v F(v) \cap \varphi_{v_k}F(v_k) \cap \varphi_{v_n}F(v_n) \neq \emptyset \ .$$

The proof now concludes exactly as in Case 1 above. $\square$

<u>Proposition 17</u>. $\operatorname{colim} \bar{F}$ is connected.

Proof: Since by Proposition 12

$$\bar{\varphi}\ \mathrm{colim}\ \bar{F} = \varphi_v F(v) \cap \varphi\ \mathrm{colim}\ F_v\ ,$$

it is trivial from Proposition 16, that $\bar{\varphi}$ colim $\bar{F}$ is connected. Since $\bar{\varphi}$ is a homeomorphism, colim $\bar{F}$ is connected. □

Proposition 18. The diagram

$$\begin{tikzcd} \varphi_v F(v) \cap \varphi\ \mathrm{colim}\ F_v \arrow[r] \arrow[d, "\beta"'] & \varphi\ \mathrm{colim}\ F_v \arrow[d] \\ \varphi_v F(v) \arrow[r] & \mathrm{colim}\ F \end{tikzcd}$$

is a push-out.

Proof: i) φ colim F_v is open in colim F : Suppose $a\colon C \to \mathrm{colim}\ F$ is admissible and $x \in a^{-1}\ \varphi\ \mathrm{colim}\ F_v$. Thus $ax \in \varphi\ \mathrm{colim}\ F_v$ and there is an element $y \in \mathrm{colim}\ F_v$ such that $\varphi y = ax$. Thus there is an object $g \in \mathrm{ob}C_v$ and an element $z \in F_v(g)$ such that $\varphi_g^v z = y$, and $\varphi_g z = \varphi\, \varphi_g^v z = \varphi y = ax$. Since $\varphi_g F(g)$ is open in colim F , there is a neighborhood $U \subseteq C$ with $x \in U \subseteq C$ such that $ax \in aU \subseteq \varphi_g F(g) = \varphi\, \varphi_v^g\, F_v(g) \subseteq \varphi\ \mathrm{colim}\ F_v$. It follows that $x \in U \subseteq a^{-1}\varphi\ \mathrm{colim}\ F_v$ and φ colim F_v is open in colim F .

ii) By hypothesis $\varphi_v F(v)$ is open in colim F so that $\{\varphi_v F(v), \varphi\ \mathrm{colim}\ F_v\}$ is an open cover of colim F . An argument similar to the one in Theorem lv) of Section 5 shows that any space is the push-out of a two piece open cover. □

Proposition 19. If C is 1-dimensional, then $\bar{C}$ satisfies condition H).

Proof: The proof requires a lemma:

Lemma 12. There exists an element $h \in obC$ such that

$$\varphi_h F(h) = \varphi_v F(v) \cap \varphi \operatorname{colim} F_v .$$

Proof: By Proposition 16,

$$\varphi_v F(v) \cap \varphi \operatorname{colim} F_v$$

is path connected. By Lemma 10 of this section

$$\varphi_v F(v) \cap \varphi \operatorname{colim} F_v = \bigcup_{\bar{v}\in(C_v)_0} \varphi_v F(v) \cap \varphi_{\bar{v}} F(\bar{v}) .$$

If $|\{\bar{v} \in (C_v)_0 | \varphi_v F(v) \cap \varphi_{\bar{v}} F(\bar{v}) \neq \emptyset\}| > 1$, then since the intersection is path connected for some pair $v_1, v_2 \in (C_v)_0$, of distinct elements

$$\varphi_v F(v) \cap \varphi_{v_1} F(v_1) \cap \varphi_{v_2} F(v_2) \neq \emptyset .$$

By Lemma 1 in Section 4 there is a set Δ and a family $\{f_\delta \in obC\}_{\delta\in\Delta}$ such that

$$C(f_\delta, v) \neq \emptyset , \ C(f_\delta, v_1) \neq \emptyset, \ C(f_\delta, v_2) \neq \emptyset$$

for each $\delta \in \Delta$. Since C is 1-dimensional and $\{v, v_1, v_2\} \subseteq vf_\delta$, $|\{v, v_1, v_2\}| \leq 2$. Since v_1 and v_2 are from C_v, neither equal v. Thus they are not distinct. It follows that for some $v' \in (C_v)_0$

$$\varphi_v F(v) \cap \varphi \operatorname{colim} F_v = \varphi_v F(v) \cap \varphi_{v'} F(v') .$$

Lemma 12 now follows from Lemma 9. $\square$

Proof of Proposition 19. The proposition is trivial since $G(f)$ is representable as $\varphi_h F(h)$ for some $h \in obC$. Thus

$$\beta\colon \varphi_h F(h) \longrightarrow \varphi_v F(v)$$

is just $\varphi_v F(a)$ for some $a \in C(h,v)$:

$$\begin{tikzcd} F(h) \arrow[r, "\varphi_h"] \arrow[d, "F(a)"] & \text{colim } F \\ F(v) \arrow[ur, "\varphi_v"'] & \end{tikzcd}$$

Since by hypothesis φ_h induces a monomorphism, $\beta = \varphi_h$ does also. Proposition 19 now follows from previous remarks in Section 6 . □

<u>Proposition 20</u>. If C is a 1-dimensional s-category and $F\colon C \to Q$ is a functor such that C and F satisfy conditions C) and H) then they satisfy condition I).

<u>Proof</u>: Suppose (C',w) is an initial pair. If $|obC'| \geq 2$, then since colim F' is connected there exist two objects $f, f' \in obC'$ such that

$$*) \qquad \varphi'_f\, F'(f) \cap \varphi'_{f'}\, F'(f') \neq \emptyset$$

for $\{\varphi'_f\colon F'(f) \to \text{colim } F'\}_{f \in obF'}$ the colimit induced morphisms. $\varphi'\colon \text{colim } F' \to \text{colim } F$ is the obvious map which makes the diagrams

$$\begin{tikzcd} F'(f) \arrow[r, "\varphi'_f"] \arrow[d, "="] & \text{colim } F' \arrow[dd, "\varphi'"] \\ F(f) \arrow[dr, "\varphi_f"'] & \\ & \text{colim } F \end{tikzcd}$$

commute for $f \in obC'$.

Equation *) implies that

$$\varphi'(\varphi'_f\, F'(f) \cap \varphi'_{f'}\, F'(f')) \neq \emptyset$$

that is

$$\varphi_f F(f) \cap \varphi_{f'} F(f') \neq \emptyset \ .$$

Since F is closed under finite intersections there exists a set Δ and a family $\{h_\delta\}_{\delta\in\Delta}$ such that

$$C(h_\delta, f) \neq \emptyset \quad \text{and} \quad C(h_\delta, f') \neq \emptyset$$

for each $\delta \in \Delta$.

By assumption w is 0-dimensional and not an object of C' . Thus neither f nor f' equals w . Since $C(f,w) \neq \emptyset$ and $C(f',w) \neq \emptyset$, f and f' are 1-dimensional. It follows that since C is 1-dimensional, $f = h_\delta = f'$ and $|obC'| = 1$. The Proposition now follows from previous remarks in Section 6. □

<u>Proposition 21</u>. C_v and F_v satisfy condition I).

<u>Proof</u>: If (C',w) is an initial pair in C_v , then it is also initial in C . The proposition follows immediately from the hypothesis that C satisfies condition I). □

<u>Proposition 22</u>. $v \cap C_v$ and $\bar{F}$ satisfy condition I).

<u>Proof</u>: Suppose (C',w) is initial in $v \cap C_v$. Since $w \in ob(v \cap C_v)$, $C(w,v) \neq \emptyset$. It follows that (C',v) is initial in C . The diagram

$$\begin{tikzcd}
\operatorname{colim} F' \arrow[r, "\alpha"] \arrow[dr] \arrow[ddr] & F'(w) \arrow[d, equal] \\
& F(w) \arrow[d, "\beta"] \\
& F(v)
\end{tikzcd}$$

commutes, since for each, $f \in \mathrm{ob}C'$, the diagrams

$$\begin{tikzcd} F'(f) \arrow[d] \arrow[rd] \arrow[rrd] & & \\ \operatorname{colim} F' \arrow[r] & F(w) \arrow[r] & F(v) \end{tikzcd}$$

commute. Since (C',v) is initial in C, the composition $\beta\alpha$ induces a monomorphism in fundamental group. Thus α induces a monomorphism and the Proposition follows. $\square$

<u>Proposition 23</u>. $\tilde{C}$ and G satisfy condition H).

<u>Proof</u>: By the remarks prior to Proposition 19 in Section 6, it suffices to show that β induces a monomorphism in fundamental group:

$$\begin{tikzcd} \varphi_v F(v) \cap \varphi \operatorname{colim} F_v \arrow[r] \arrow[d, "\beta"] & \operatorname{colim} F_v \arrow[d] \\ \varphi_v F(v) \arrow[r] & \operatorname{colim} F . \end{tikzcd}$$

Consider the pair $(v \cap C_v, v)$. By construction, $v \cap C_v \subseteq C$ and $v \in C_0 - \mathrm{ob}(v \cap C_v)$. Also by Proposition 17, $\operatorname{colim} \bar{F}$ is connected. Further, for each object $g \in \mathrm{ob}(v \cap C_v)$, $C(g,v) \neq \emptyset$. Thus $(v \cap C_v, v)$ is initial and since C satisfies condition I), $\tilde{C}$ and G satisfy condition H). $\square$

<u>Proposition 24</u>. In the category Ab, the sequence

$$R \xrightarrow{\alpha} F_1 \oplus F_2 \xrightarrow{\beta} H \longrightarrow 0$$

is exact if and only if the diagram

$$\begin{tikzcd} R \arrow[r, "(1\ 0)\alpha"] \arrow[d, "-(0\ 1)\alpha"'] & F_1 \arrow[d, "\beta\binom{1}{0}"] \\ F_2 \arrow[r, "\beta\binom{0}{1}"'] & H \end{tikzcd}$$

is a push-out.

<u>Proof</u>: => : For $r \in R$ if $\alpha(r) = (r_1, r_2)$, then

$$\beta\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\alpha(r) = \beta\begin{pmatrix}1\\0\end{pmatrix}(r_1) = \beta(r_1, 0)$$

$$\beta\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0 & 1\end{pmatrix}\alpha(r) = -\beta\begin{pmatrix}0\\1\end{pmatrix}(r_2) = -\beta(0, r_2)$$

Since $\beta\alpha = 0$, $\beta\alpha(r) = \beta(r_1, r_2) = \beta(r_1, 0) + \beta(0, r_2) = 0$. Thus $\beta(r_1, 0) = -\beta(0, r_2)$ and the diagram commutes.

ii) If F is the push-out

$$\begin{tikzcd}
R \arrow[r, "(1\ 0)\alpha"] \arrow[d, "-(0\ 1)\alpha"'] & F_1 \arrow[d, "p_1"] \arrow[ddr, "\beta\begin{pmatrix}1\\0\end{pmatrix}"] & \\
F_2 \arrow[r, "p_2"] \arrow[drr, "\beta\begin{pmatrix}0\\1\end{pmatrix}"'] & F & \\
& & H ,
\end{tikzcd}$$

then there exists a function $p\colon F \to H$. If $(f_1, f_2)X \in F$ is a coset and

$$0 = p(f_1, f_2)X = \beta\begin{pmatrix}1\\0\end{pmatrix}f_1 + \beta\begin{pmatrix}0\\1\end{pmatrix}f_2$$

$$= \beta(f_1, 0) + \beta(0, f_2)$$

$$= \beta(f_1, f_2),$$

then there exists $r \in R$ such that $\alpha(r) = (f_1, f_2)$.

$(0\ 1)\alpha(r) = (1\ 0)(f_1, f_2) = f_1$ $\qquad$ $(0\ 1)\alpha(r) = (0\ 1)(f_1, f_2) = f_2$

and $(f_1, f_2) = ((1\ 0)\alpha(r), -(-(0\ 1)\alpha(r)))$. Thus $(f_1, f_2) \in X$ and p is 1: 1 .

If $g \in G$, since β is onto, there exists $(f_1, f_2) \in F_1 \oplus F_2$ such that $\beta(f_1, f_2) = g$. Then

$$
\begin{aligned}
p(f_1, f_2)X &= \beta\begin{pmatrix}1\\0\end{pmatrix}f_1 + \beta\begin{pmatrix}0\\1\end{pmatrix}f_2 \\
&= \beta(f_1, 0) + \beta(0, f_2) \\
&= \beta(f_1, f_2) \\
&= g\ .
\end{aligned}
$$

It follows that p is an isomorphism.

$\Leftarrow$: If F is the push-out in the above construction, then there exists a diagram

$$
\begin{tikzcd}
R \arrow[r, "(1\ 0)\alpha"] \arrow[d, "-(0\ 1)\alpha"'] & F_1 \arrow[d, "p_1"] \arrow[dr, "\beta\begin{pmatrix}1\\0\end{pmatrix}"] & \\
F_2 \arrow[r, "p_2"] \arrow[rr, bend right, "\beta\begin{pmatrix}0\\1\end{pmatrix}"'] & F \arrow[r, "p"] & H
\end{tikzcd}
$$

with p an isomorphism.

i) β is onto. If $g \in H$, $(f_1, f_2)X = p^{-1}(g)$, then
$g = p(p^{-1}(g)) = p(f_1, f_2)X = \beta\begin{pmatrix}1\\0\end{pmatrix}f_1 + \beta\begin{pmatrix}0\\1\end{pmatrix}f_2 = \beta(f_1, f_2)$

ii) $\beta\alpha = 0$. For $r \in R$,

$$
\begin{aligned}
\beta\alpha(r) &= \beta\Big((1\ 0)\alpha(r), (0\ 1)\alpha(r)\Big) \\
&= \beta\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)\alpha(r) + \beta\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\alpha(r) \\
&= -\beta\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\alpha(r) + \beta\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\alpha(r) \\
&= 0\ .
\end{aligned}
$$

iii) $\ker\beta \subseteq \operatorname{im}\alpha$. If $0 = \beta(f_1, f_2) = \beta\begin{pmatrix}1\\0\end{pmatrix}f_1 + \beta\begin{pmatrix}0\\1\end{pmatrix}f_2$, then

$\beta\begin{pmatrix}1\\0\end{pmatrix}f_1 = -\beta\begin{pmatrix}0\\1\end{pmatrix}f_2$ and

$$p\ p_1\ f_1 = -\ p\ p_2\ f_2\ ,$$

$$p(p_1\ f_1 + p_2\ f_2) = 0$$

and since p is an isomorphism $p_1f_1 + p_2f_2 = 0$ and $(f_1,f_2) \in X$. By the definition of X there is an $r \in R$ such that

$$\Big((1\ 0)\alpha(r),\ (0\ 1)\alpha(r)\Big) = (f_1,f_2) \quad \text{and} \quad \alpha(r) = (f_1,f_2)\ . \quad \square$$

§8. Two Examples.

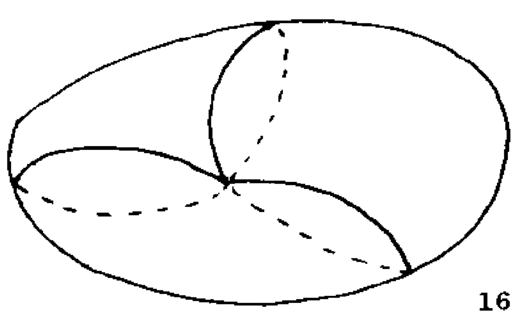
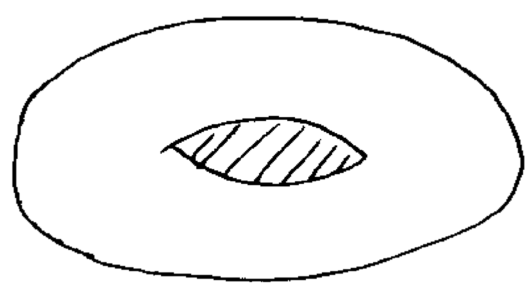
The two examples in this section illustrate the necessity of two of the hypotheses in Theorem 1 of Section 6. The first concerns the necessity of a generating tree in higher dimensional cases.

Example 1. The 2-sphere, S^2, admits a four piece open cover by four overlapping equilateral spherical triangles arranged as in a tetrahedron. The s-category corresponding to this cover is given in Example 7 of Section 1. It was remarked there that this s-category has no generating tree.

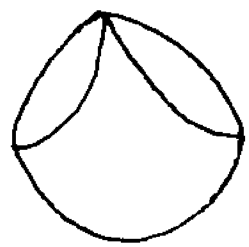
Since the 2-sphere is simply connected, it is its own universal covering space and $H_2S^2 = \mathbb{Z} \neq 0$. However, each element in the cover is contractible, is its own universal cover, and has no second homology in its cover. Since the colimit of the zero functor is zero, the conclusion of Theorem 1 of Section 6 does not hold. It is easy to verify that all the hypotheses of Theorem 1 except that of a generating tree are satisfied by this example.

The second example concerns the necessity of condition I).

Example 2. The space, X, for this example is the quotient space of the 2-sphere determined by the identification of a pair of antipodal points. This space is of the same homotopy type as a torus with a disc sewn across one of the generators:

 $=$ 

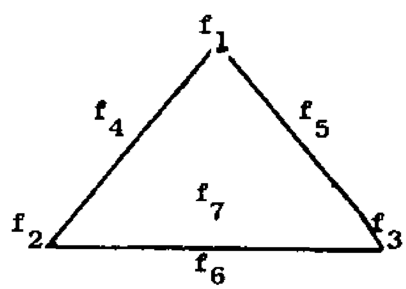
The cover for this space has three pieces as illustrated below: each piece is a strong deformation retract of the closed 2-disc with a pair of boundary points identified:



The cover is obtained from three such pieces by extending them slightly to form open subsets of X. This cover is denoted $\{X_1, X_2, X_3\}$. The s-category associated to this cover is based on $\Lambda = \{1,2,3\}$ and has objects:

$$\left.\begin{array}{l} f_1: \{1\} \to \Lambda \\ f_2: \{2\} \to \Lambda \\ f_3: \{3\} \to \Lambda \end{array}\right\} = C_0 \qquad \left.\begin{array}{l} f_4: \{1,2\} \to \Lambda \\ f_5: \{1,3\} \to \Lambda \\ f_6: \{2,3\} \to \Lambda \end{array}\right\} = C_1 \qquad f_7: \Lambda \to \Lambda \} = C_2$$

and obvious morphisms. C has a representation as the triangle



The functor $F: C \to Q$ which represents the cover is given by

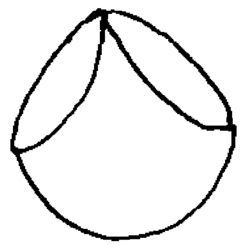
$$F(f_i) = X_i \ , \ i=1,2,3$$

$F(f_4) = X_1 \cap X_2$; $F(f_5) = X_1 \cap X_3$; $F(f_6) = X_2 \cap X_3$

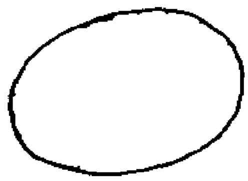
$F(f_7) = X_1 \cap X_2 \cap X_3$

Pictorially, these spaces may be represented as follows:

$F(f_1) = F(f_2) = F(f_3)$:



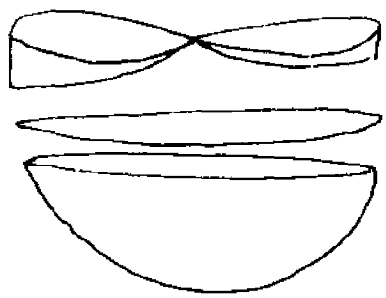
$F(f_4) = F(f_5) = F(f_6)$:



$F(f_7)$:

As for fundamental groups:

$\pi_1 F(f_i) = \mathbf{Z}$ for $i=1,2,3$ as seen through the Van Kampen Theorem and the cover



Then $\pi_1 F(f_i)$ is the push-out

$$\begin{tikzcd} \mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} * \mathbf{Z} \arrow[d] \\ 0 \arrow[r] & \pi_1 F(f_i) \end{tikzcd} \qquad i=1,2,3$$

Clearly

$$\pi_1 F(f_i) = \mathbf{Z} \quad i=4,5,6 \quad \text{and}$$

$$\pi_1 F(f_7) = 0 .$$

It is easy to verify that all the inclusions induce monomorphisms of of fundamental groups. In fact since X is a torus with one killed generator, $\pi_1 X = \mathbb{Z}$ and the inclusion of each piece into X induces a monomorphism of fundamental group.

From the triangular representation of C, one sees that $\Gamma(C) = \{f_7\}$. Thus C has a generating tree. It is now easy to verify that C satisfies conditions A) - H).

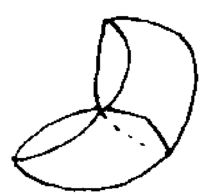
However, C does not satisfy condition I). If $v = f_1$, then C_v has objects

$$f_2,\ f_3,\ \text{and}\ \ f_6$$

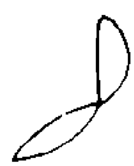
and $v \cap C_v$ has objects

$$f_4,\ f_5,\ f_7\ .$$

colim F_v is the space

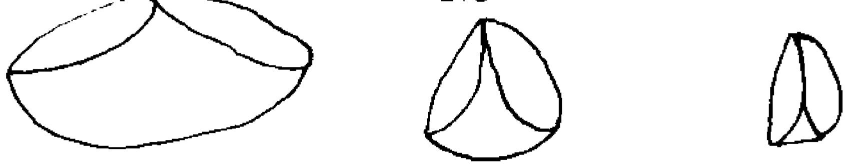


and colim $\bar{F}$ is the space



Clearly, π_1 colim $\bar{F} = \mathbb{Z} * \mathbb{Z}$ but $\pi_1\ F(v) = \mathbb{Z}$ and the inclusion colim $\bar{F} \to F(v)$ does not induce a monomorphism of fundamental groups, so that C and F do not satisfy condition I).

The universal cover of $F(f_i)$, $i=1,2,3$, is most easily pictured by deforming $F(f_i)$, $i=1,2,3$ according to the following sequence of pictures:



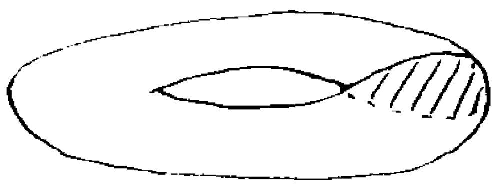
I. II. III.

where $F(f_i)$, $i=1,2,3$ are seen to be the indicated quotient of the space

Thus the universal cover of $F(f_i)$, $i=1,2,3$ is the same as for the circle and is thus contractible. The other cases are easier and yield that the universal cover of $F(f_i)$ is contractible for each $i=1,\ldots,7$. Thus

$$\operatorname{colim} F_H = 0 .$$

Theorem 1 of Section 6 can be used to compute $H_2 \operatorname{colim} \tilde{F}$. Since the generators of a torus are indistinguishable, X may be represented as a torus with a disc sewn across the "inside":



A cover of this space is an infinite circular cylinder with discs sewn in at regular intervals

As this space is clearly simply connected, it is the universal cover of X . It is also a string of 2-spheres.

This space is covered by the string of spheres and this cover is represented by a 1-dimensional s-category $\hat{C}$ based on $\mathbb{Z}$

$$\{f_i : \{i\} \longrightarrow \mathbb{Z}\} = \hat{C}_0$$

$$\{f_{i,i+1} : \{i,i+1\} \longrightarrow \mathbb{Z}\} = \hat{C}_1$$

which has for $\Gamma_g(\hat{C})$ the tree

$$\begin{tikzcd}
\cdots & f_{i-1,i} \arrow[dl] \arrow[dr] & & f_{i,i+1} \arrow[dl] \arrow[dr] & & f_{i+1,i+2} \arrow[dl] \arrow[dr] & \cdots \\
f_{i-1} & & f_i & & f_{i+1} & & f_{i+2}
\end{tikzcd}$$

$\hat{F}$ is the associated functor where $\hat{F}(f_i) = S^2$ and $\hat{F}(f_{i,i+1}) = *$ for $i \in \mathbb{Z}$. Since this space and each piece in the cover is its own universal cover, Theorem 1 of Section 6 can be applied to compute $H_2 \operatorname{colim} \hat{F}$. It is perhaps simpler to use an inductive application of the Mayer-Vietoris Theorem. In any case, clearly

$$H_2 \operatorname{colim} \hat{F} = H_2 \operatorname{colim} \tilde{F} = \bigoplus_{\mathbb{Z}} \mathbb{Z} .$$

Since $\operatorname{colim} F_H = 0$, the conclusion of Theorem 1 of Section 6 fails to hold for this example.

§9. A Seifert-van Kampen Theorem for the Second Homotopy Group.

If C is an s-category and $F: C \to Q$ is a functor such that C and F satisfy conditions A) - I), then it is possible to calculate the second homotopy group of colim F from the first and second homotopy groups of each value, $F(g)$ for $g \in \text{ob}C$, of F and the associated maps. This section makes these constructions explicit.

The hypothesis that C has a generating tree (condition E)) makes it possible to assign a consistent system of base points and to define a second homotopy group functor analagous to the functor F_T defined at the end of Section 3. More precisely we make the following formal definition:

Definition: Suppose C is a small s-category and $F: C \to Q$ is a functor with a pointed consistent $\Gamma(F)$ -system

$$T = (\Gamma_s(F) \; ; \; z_{g_*} \; , \; \{\alpha_g\}) \; .$$

The functor $F^2_T: C \to Ab$, from C into the category of abelian groups is defined as follows:

i) for $g \in \text{ob}C$,

$$F^2_T(g) = \pi_2(F(g), y_g)$$

for $y_g \in v(\Gamma_s(F))$.

ii) For $g, g' \in \text{ob}C$ and $\sigma \in C(g,g')$, the homomorphism $F^2_T(\sigma)$ is the composition

$$\pi_2(F(g), y_g) \xrightarrow{\pi_2 F(\sigma)} \pi_2(F(g'), \bar{y}_g) \xrightarrow{\overline{\alpha(g,g')}} \pi_2(F(g'), y_{g'})$$

where $\bar{y}_g = F(\sigma)y_g$ and $\overline{\alpha(g,g')}$ is the isomorphism induced by the path $\alpha(g,g')$.

Proposition 1. The conditions of the previous definition define a functor $F_T^2\colon C \to Ab$.

Proof: The proof of Proposition 1 of Section 3 applies by changing π_1 to π_2 throughout. □

In view of the results in Section 6, one would not expect that $\operatorname{colim} F_T^2 = \pi_2(\operatorname{colim} F, z_{g_*})$. However, the Hurewicz isomorphism allows us to convert the homology constructions in Theorem 3 of Section 6 into homotopy constructions. First we need a definition:

Definition: Suppose C is an s-category and $F\colon C \to Q$ is a functor such that F and C satisfy conditions E) and H). The functor $F_\pi\colon C \to Ab$ is defined for a pointed consistent $\Gamma(F)$-system T as follows:

i) On objects, $g \in obC$, $F_\pi(g) = \bigoplus_{I(g)} F_T^2(g)$.

ii) On morphisms, $\alpha \in C(g,g')$, if

$$\sum_{i\in A} a_i \in \bigoplus_{I(g)} F_T^2(g)$$

then

$$F_\pi(\alpha)(\sum_{i\in A} a_i) = \sum_{i\in A} [F_T^2(\alpha)a_i]_{I(\alpha)i}$$

where the subscript on the bracket indicates the coordinate.
Note that although F_π depends upon a particular choice of T , the notation does not indicate this.

Theorem 1. Suppose C is an s-category and $F\colon C \to Q$ is a functor such that C and F satisfy conditions A) - I). If $T = (\Gamma_s(F); z_{g_*}, \{\alpha_g\})$

is a pointed consistent $\Gamma(F)$-system, then

$$\pi_2(\mathrm{colim}\ F, z_{g_*}) = \mathrm{colim}\ F_\pi .$$

Proof: The natural isomorphism μ: $F_H \to F_\pi$ is defined for objects $g \in \mathrm{ob}C$ to be the composition

$$F_H(g) = \bigoplus_{I(g)} H_2\widetilde{F(g)} \xrightarrow{h_g^{-1}} \bigoplus_{I(g)} \pi_2(\widetilde{F(g)}, \bar{y}_g) \xrightarrow{p_g^*} \oplus\ \pi_2(F(g), y_g) = F_\pi(g)$$

where

i) h_g is the Hurewicz homomorphism, which in this case is an isomorphism since the universal cover is simply connected;

ii) p_g^* is the isomorphism induced by the universal covering projection p_g: $\widetilde{F(g)} \to F(g)$;

iii) $\bar{y}_g \in p_g^{-1}\ y_g$.

Since the Hurewicz homorphism is natural and the universal covering projection induces a natural transformation, μ is natural and

$$\mathrm{colim}\ F_H = \mathrm{colim}\ F_\pi .$$

It follows from Theorem 3 of Section 6 that

$$\mathrm{colim}\ F_\pi = H_2\ \mathrm{colim}\ F .$$

Again, by use of the Hurewicz isomorphism and the universal covering projection

$$H_2\ \mathrm{colim}\ F = \pi_2(\mathrm{colim}\ F; z_{g_*}) . \quad \square$$

Using Theorem 4 of Section 6, it is possible to prove a weak generalization of Theorem 1 to higher dimensional homotopy groups. First we need two definitions.

Definition: Suppose C is a small category and F: $C \to Q$ is a functor with a pointed consistent $\Gamma(F)$-system

$$T = (\Gamma_s(F);\ z_{g_*},\ \{\alpha_g\}).$$

The functor $F^n_T\colon C \to Ab$, from C into the category of abelian groups is defined as follows:

i) for $g \in obC$,

$$F^n_T(g) = \pi_n(F(g), y_g)$$

for $y_g \in v(\Gamma_s(F))$.

ii) For $g, g' \in obC$ and $\sigma \in C(g,g')$, the homomorphism $F^n_T(\sigma)$ is the composition

$$\pi_n(F(g), y_g) \xrightarrow{\pi_1 F(\sigma)} \pi_n(F(g'), \bar{y}_g) \xrightarrow{\overline{\alpha(g,g')}} \pi_n(F(g'), y_{g'})$$

where $\bar{y}_g = F(\sigma)y_g$ and $\overline{\alpha(g,g')}$ is the isomorphism induced by the path $\alpha(g,g')$.

Proposition 2. The conditions of the previous definition define a functor $F^n_T\colon C \to Ab$.

Proof: As before. □

Definition: Supppose C is an s-category and $F\colon C \to Q$ is a functor such that F and C satisfy conditions E) and H). The functor $F^n_\pi\colon C \to Ab$ is defined for a pointed consistent $\Gamma(F)$-system T as follows:

i) On objects, $g \in obC$, $F^n_\pi(g) = \bigoplus_{I(g)} F^n_T(g)$.

ii) On morphisms, $\alpha \in C(g,g')$ if

$$\sum_{i \in A} a_i \in \bigoplus_{I(g)} F^n_T(g)$$

then

$$F^n_\pi(\alpha)\left(\sum_{i\in A} a_i\right) = \sum_{i\in A}[F^n_T(\alpha)a_i]_{I(\alpha)i}$$

where the subscript on the bracket indicates the coordinate. Note that the T is suppressed.

Theorem 2. Suppose C is an s-category and $F\colon C \to Q$ is a functor such that C and F satisfy conditions A) - I), and suppose also that $T = (\Gamma_s(F);\ z_{g_*},\ \{\alpha_g\})$ is a pointed consistent $\Gamma(F)$-system. If for each object $g \in \mathrm{ob}C$, $\pi_i(F(g);z_{g_*}) = 0$, $i=0,2,\ldots,n-1$, then

$$\pi_n(\mathrm{colim}\ F, z_{g_*}) = \mathrm{colim}\ F^n_\pi$$

for any $n \geq 2$.

Proof: The proof of Theorem 1 suffices here by changing 2 to n throughout. □

As a final corollary we have the following:

Corollary. If for each object $g \in \mathrm{ob}C$, $F(g)$ is $(n-1)$-connected, then

$$\pi_n(\mathrm{colim}\ F; z_{g_*}) = \mathrm{colim}\ F^n_T .$$

Proof: By hypothesis, for each $g \in \mathrm{ob}C$, $\pi_1(F(g),*) = 0$. Since C has a generating tree and F is closed under finite intersections, the van-Kampen Theorem of Section 4 applies and $\pi_1(\mathrm{colim}\ F,*) = \mathrm{colim}\ F_T = 0$. It follows that for each $g \in \mathrm{ob}C$, $I(g) = 1$ and $F^n_\pi = F^n_T$. □

The fundamental group action is an important aspect of the higher dimensional homotopy groups. The theory that we have developed does not in general admit a particularly informative description of this action. In some instances it is easy to give a nice description of a fundamental group action on $F_{\pi}^{n}(g)$ (for each g in the category) which extends to the usual action on the colimit. Usually the action on the colimit will not restrict in any nice way. However, since the group action is most easily seen in the homotopy groups of the universal cover and our construction first computes those groups, in particular cases it is often easy to describe the action on the colimit without referring to the local action. This philosophy is supported by Examples 1, 2 and 3 in Section 12.

§ 10. The One-dimensional Case.

This section develops the analog of Theorem 1 of Section 9 in case C is 1-dimensional and does not have a generating tree. Throughout this section C is assumed to be a 1-dimensional s-category. $F: C \to Q$ is a path connected functor which has colimit induced morphisms $\{\varphi_g : F(g) \to \operatorname{colim} F\}_{g \in obC}$. Furthermore, it is assumed that C and F satisfy conditions A), C), D), D′), D″), F), and H_1) of Section 6 and that $\pi_1(\Gamma_g(C),*)$ is a countable group. Recall that these conditions are that

A) C has at most countably many vertices;

C) F is closed under finite intersections;

D) $\{\varphi_g F(g)\}_{g \in obC}$ is an open cover of $\operatorname{colim} F$ and each φ_g is an injection;

D′) For each pair $g, g' \in obC$;

$$\varphi_g F(g) = \varphi_{g'} F(g')$$

implies $g = g'$,

D″) For each object $g \in obC$, $\varphi_g F(g) \neq \emptyset$;

F) F is path connected;

H_1) For each pair $g, g' \in obC$ with $\alpha \in C(g,g')$ the inclusion $F(\alpha): F(g) \to F(g')$ induces a monomorphism

$$F^*(\alpha): \pi_1(F(g),*) \longrightarrow \pi_1(F(g'),*)$$

for some choice of base point.

The means by which the hypothesis of a generating tree is eliminated is through the construction of a covering category which does have a generating tree.

<u>Definition</u>: $p:\tilde{\Gamma} \to \Gamma_g(C)$ is the universal covering projection. $\tilde{\Gamma}$ is considered to have the structure of a graph. Each edge in $\tilde{\Gamma}$ has distinct vertices since otherwise its projection would be an edge in $\Gamma_g(C)$ with identified vertices. $\tilde{\Gamma}_C$ denotes the s-category based on $v\tilde{\Gamma}$ with

$$\left(\tilde{\Gamma}_C\right)_0 = v\tilde{\Gamma}$$

$$\left(\tilde{\Gamma}_C\right)_1 = e\tilde{\Gamma}$$

with non-identity morphisms of the form

$$h \longrightarrow o(h),\ h \longrightarrow t(h)\ .$$

This construction yields an s-category with geometric graph $\Gamma_g(\tilde{\Gamma}_C) = \tilde{\Gamma}$. Since p was a map of graphs, p induces a functor also denoted $p:\ \tilde{\Gamma}_C \to C$.

<u>Theorem 1</u>. i) Fp is path connected and colim Fp is connected if colim F is.

ii) Fp is closed under finite intersections.

<u>Proof</u>: i) For each $g \in \text{ob}\tilde{\Gamma}_C$, $Fp(g) = F(p(g))$. Since F is path connected, $Fp(g)$ is connected.

If colim F is connected, then by Proposition 1 of Section 5, C is connected. By Proposition 2 of Section 5, $\Gamma(C)$ is connected and by Proposition 1 of Section 2, $\Gamma_g(C)$ is connected. It follows that the universal covering $\tilde{\Gamma}$ is connected. Since $\Gamma_g(\tilde{\Gamma}_C) = \tilde{\Gamma}$, $\Gamma_g(\tilde{\Gamma}_C)$ is connected and by Proposition 1 of Section 2, $\Gamma(\tilde{\Gamma}_C)$ is connected. By Proposition 2 of Section 5, $\tilde{\Gamma}_C$ is connected. Since the first part of the proof of Proposition 1 of Section 5 requires only that F be path connected, colim Fp is connected.

ii) $\{\psi_g\colon Fp(g) \to \text{colim } Fp\}_{g \in \text{ob}(\tilde{\Gamma}_C)}$ denote the colimit induced maps.

If $g, g' \in ob(\tilde{\Gamma}_C)$ are distinct objects such that

$$\psi_g Fp(g) \cap \psi_{g'} Fp(g') \neq \emptyset$$

then for $x \in \psi_g Fp(g) \cap \psi_{g'} Fp(g')$ there exists a finite diagram in $\tilde{\Gamma}_C$ which relates the $a \in Fp(g)$ for which $\psi_g a = x$ to the $a' \in Fp(g')$ for which $\psi_{g'} a' = x$.

<u>Case 1</u>. The diagram is of the form

1)
$$\begin{tikzcd}
 & h_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & h_3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_4"] & \\
g & & h_2 & & g'
\end{tikzcd}$$

where $g, h_2, g' \in (\tilde{\Gamma}_C)_0$; $h_1, h_3 \in (\tilde{\Gamma}_C)_1$ and $a_i \in Fp(h_i)$ with $Fp\alpha_1 a_1 = a$, $Fp\alpha_2 a_1 = a_2$, $Fp\alpha_3 a_3 = a_2$, $Fp\alpha_4 a_3 = a'$. This diagram projects to a diagram

2)
$$\begin{tikzcd}
 & ph_1 \arrow[dl, "p\alpha_1"'] \arrow[dr, "p\alpha_2"] & & ph_3 \arrow[dl, "p\alpha_3"'] \arrow[dr, "p\alpha_4"] & \\
pg & & ph_2 & & pg'
\end{tikzcd}$$

in C for which the existence of the elements $a_i \in Fph_i$ yield

$$\varphi_{ph_1} F(ph_1) \cap \varphi_{ph_3} F(ph_3) \neq \emptyset \ .$$

Since h_1, h_3 are 1-dimensional, ph_1 , ph_3 are 1-dimensional. Since F is closed under finite intersections, there is an element $h \in obC$ with $C(h, ph_1) \neq \emptyset$ and $C(h, ph_3) \neq \emptyset$. It follows that $ph_1 = h = ph_3$. Since C is an s-category, $p\alpha_2 = p\alpha_3$ and diagram 2) reduces to a diagram

$$\begin{tikzcd} & ph_1 \arrow[dl, "p\alpha_1"'] \arrow[d, "p\alpha_2"] \arrow[dr, "p\alpha_4"] & \\ pg & ph_2 & pg' \end{tikzcd} .$$

Since C is an s-category $|\{pg, ph_2, pg'\}| \le 2$. Diagram 1) yields a graph

$$\begin{tikzcd} g \arrow[r, dash, "h_1"] & h_2 \arrow[r, dash, "h_3"] & g' \end{tikzcd}$$

Since $ph_3 = ph_1$, this graph must project to the graph

$$\begin{tikzcd} pg = pg' \arrow[r, dash, "ph_1 = ph_3"] & ph_2 \end{tikzcd}$$

and $pg = pg'$. Since no neighborhood of h_2 can be mapped homeomorphically to a neighborhood of ph_2 and p is a universal covering projection, this case is impossible.

<u>Case 2</u>. The diagram is of the form

$$\begin{tikzcd} & h_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & h_3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_4"] & \\ g & & h_2 & & g' \end{tikzcd}$$

and $g, h_2, g' \in (\tilde{\Gamma}_C)_0$ with $|\{g, h_2, g'\}| = 3$. By the previous case both h_1 and h_3 are not elements of $(\tilde{\Gamma}_C)_1$. If $h_1 \in (\tilde{\Gamma}_C)_0$, then the existence of α_1, α_2 yield $g = h_1 = h_2$. Similarly, if $h_3 \in (\tilde{\Gamma}_C)_0$, then $h_2 = h_3 = g'$.

<u>Case 3</u>. The diagram is of the form

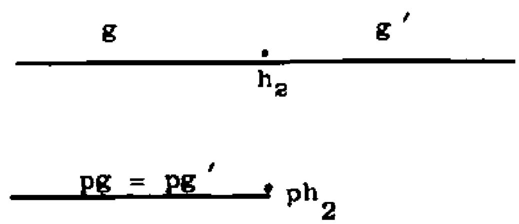
$$\begin{tikzcd} & h_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & h_3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_4"] & \\ g & & h_2 & & g' \end{tikzcd}$$

and $|\{g,h_2,g'\}| = 3$. By previous cases not all $g,h_2,g' \in (\tilde{\Gamma}_C)_0$.

i) If $g \in (\tilde{\Gamma}_C)_1$, then since $\tilde{\Gamma}_C$ is 1-dimensional, $h_1 = g$ and since $|\{g,h_2,g'\}| = 3$, $h_2 \neq h_1$ and thus $h_2 \in (\tilde{\Gamma}_C)_0$. As before, $a_i \in Fph_1$ with $Fp\alpha_1 a_1 = a$, $Fp\alpha_2 a_1 = a_2$, $Fp\alpha_3 a_3 = a_2$, $Fp\alpha_4 a_3 = a'$. The projected diagram is of the form

$$\begin{tikzcd} & pg \arrow[dl, "1"'] \arrow[dr, "p\alpha_2"] & & ph_3 \arrow[dl, "p\alpha_3"'] \arrow[dr, "p\alpha_4"] & \\ pg & & ph_2 & & pg' \end{tikzcd}$$

If $g' \in (\tilde{\Gamma}_C)_1$, then $g' = h_3$ and as before $pg' = pg$. The corresponding graphs are



in $\Gamma_g(\tilde{\Gamma}_C)$

in $\Gamma_g(C)$.

Since p is a covering, this is impossible as in Case 1.

If $g' \in (\tilde{\Gamma}_C)_0$ and $h_3 \in (\tilde{\Gamma}_C)_0$, then $h_2 = h_3 = g'$. Thus $h_3 \in (\tilde{\Gamma}_C)_1$ and as in Case 1, $pg = ph_3$ which contradicts p being a covering.

Since the diagram is symmetric, $g' \notin (\tilde{\Gamma}_C)_1$.

It follows that $h_2 \in (\tilde{\Gamma}_C)_1$ and $h_1 = h_2 = h_3$ so that for some $h \in \mathrm{ob}\tilde{\Gamma}_C$, $\tilde{\Gamma}_C(h,g) \neq \emptyset$ and $\tilde{\Gamma}_C(h,g') \neq \emptyset$.

<u>Case 4</u>. If the diagram is longer, then by induc``on it can be reduced to a diagram in the form of diagram 1).

If $x \in \psi_g Fp(g) \cap \psi_{g'} Fp(g')$, then the previous remarks yield an

$h \in \mathrm{ob}\tilde{\Gamma}_C$ and a diagram

$$\begin{tikzcd} & h \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ g & & g' \end{tikzcd}$$

together with an element $y \in Fp(h)$ such that $Fp(\alpha)y = a$, $Fp(\beta)y = a'$. Thus $\psi_h y = \psi_g Fp(\alpha) y = \psi_g a = x$. Thus

$$\bigcup \psi_h Fp(h) = \psi_g Fp(g) \cap \psi_{g'} Fp(g')$$

where the union is taken over all the $h \in \mathrm{ob}\tilde{\Gamma}_C$ which the previous construction yields.

If $h, h' \in (\tilde{\Gamma}_C)_1$ are distinct elements with $\tilde{\Gamma}_C(h,g)$, $\tilde{\Gamma}_C(h,g')$, $\tilde{\Gamma}_C(h',g)$, $\tilde{\Gamma}_C(h',g')$ non-empty and $g, g' \in (\tilde{\Gamma}_C)_0$, then $\Gamma_g(\tilde{\Gamma}_C) = \tilde{\Gamma}$ contains the circuit

$$\begin{tikzcd} g \arrow[r, bend left, "h"] & g' \arrow[l, bend left, "h'"] \end{tikzcd}$$

This is impossible since $\tilde{\Gamma}$ was the universal covering. If $g \in (\tilde{\Gamma}_C)_1$, then $g = h$ and $\tilde{\Gamma}_C(g,g') \neq \emptyset$. It follows that $\psi_g Fp(g) \subseteq \psi_{g'} Fp(g')$. And if $g \in (\tilde{\Gamma}_C)_1$ or $h \in (\tilde{\Gamma}_C)_0$, then $g = h$ and $\tilde{\Gamma}_C(g,g') \neq \emptyset$. It follows that $\psi_g Fp(g) \subseteq \psi_{g'} Fp(g')$ and

$$\psi_g Fp(g) \cap \psi_{g'} Fp(g') = \psi_g Fp(g) .$$

If $g' \in (\tilde{\Gamma}_C)_1$ or $h' \in (\tilde{\Gamma}_C)_0$, then $g' = h'$ and

$$\psi_g Fp(g) \cap \psi_{g'} Fp(g') = \psi_{g'} Fp(g') .$$

In either case the intersection has a single path component. □

<u>Lemma</u>. If $g, g' \in \mathrm{ob}\tilde{\Gamma}_C$ are distinct elements such that $pg = pg'$, then

$$\psi_g Fp(g) \cap \psi_{g'} Fp(g') = \emptyset \ .$$

<u>Proof</u>: If $\psi_g Fp(g) \cap \psi_{g'} Fp(g') \neq \emptyset$, then there exists an element $h \in \mathrm{ob}\tilde{\Gamma}_C$ such that $\tilde{\Gamma}_C(h,g) \neq \emptyset$, $\tilde{\Gamma}_C(h,g')$. Since $pg = pg'$, g and g' have the same dimension. If their dimension is 1 then they both equal h and are not distinct. If their dimension is 0, and the dimension of h is 0, then they both equal h and are not distinct.

If the dimension of g and g' is 0 and the dimension of h is 1, then the subgraph

$$\begin{tikzcd} g \arrow[d, dash, "h"] \\ g' \end{tikzcd}$$

of $\Gamma_g(\tilde{\Gamma}_C) = \tilde{\Gamma}$ projects onto the vertex pg of $\Gamma_g(C)$. This is impossible since p is a covering. $\square$

<u>Definition</u>. For each $g \in \mathrm{ob}\tilde{\Gamma}_C$, the identity map

$$Fp(g) \longrightarrow F(pg)$$

induces the Q-morphism p_*: colim Fp $\to$ colim F .

<u>Theorem 2</u>. For each $g \in \mathrm{ob}\tilde{\Gamma}_C$, ψ_g: $Fp(g) \to$ colim Fp is an injection and $\{\psi_g Fp(g)\}_{g \in \mathrm{ob}C}$ is an open cover of colim Fp .

<u>Proof</u>: i) For each $g \in \mathrm{ob}\tilde{\Gamma}_C$, there is a commutative diagram

$$\begin{tikzcd} Fp(g) \arrow[r, "\psi_g"] \arrow[d, "="] & \mathrm{colim}\ Fp \arrow[d] \\ F(pg) \arrow[r, "\varphi_{pg}"] & \mathrm{colim}\ F \ . \end{tikzcd}$$

Since φ_{pg} is an injection, ψ_g is also.

ii) $a\colon C \to \operatorname{colim} Fp$ is an admissible map. It follows that the composition

$$C \xrightarrow{a} \operatorname{colim} Fp \xrightarrow{p_*} \operatorname{colim} F$$

is admissible. For $g \in \operatorname{ob}\tilde{\Gamma}_C$, $pg \in \operatorname{ob}C$ and $\varphi_{pg}F(pg)$ is open in $\operatorname{colim} F$. Thus

$$a^{-1}p_*^{-1}\ \varphi_{pg}F(pg)$$

is open in C .

If $g' \in \operatorname{ob}\tilde{\Gamma}_C$ is such that $pg' = pg$, then for each $x \in \psi_{g'}Fp(g')$, $p_* x \in \varphi_{pg'}F(pg')$. Thus

$$\bigcup \psi_{g'}Fp(g') \subseteq p_*^{-1}\ \varphi_{pg}F(pg) \ .$$

If $x \in p_*^{-1}\ \varphi_{pg}F(pg)$, then $p_* x \in \varphi_{pg}F(pg)$. For some $g' \in \operatorname{ob}\tilde{\Gamma}_C$, $x \in \psi_{g'}F(pg')$. If $pg' \neq pg$, then since $p_*\psi_{g'}F(pg') = \varphi_{pg'}F(pg')$,

$$\varphi_{pg}F(pg) \cap \varphi_{pg'}F(pg') \neq \emptyset \ .$$

If the dimensions of pg and pg' are equal, then there is an edge h of $\Gamma_g(C)$ and a diagram

$$\begin{tikzcd} & h \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ pg & & pg' \end{tikzcd}$$

which yields a subgraph

$$\overset{\cdot}{pg} \xrightarrow{\quad h \quad} \overset{\cdot}{pg'}$$

of $\Gamma_g(C)$. This graph can be lifted to g' to yield a subgraph of $\tilde{\Gamma}$:

$$\overset{\cdot}{g'} \xrightarrow{\quad p^{-1}h \quad} \overset{\cdot}{g''}$$

$$\cdot g$$

which yields a diagram

$$\begin{tikzcd}[column sep=small] & p^{-1}h \arrow[dl] \arrow[dr] & \\ g' & & g'' \end{tikzcd}$$

and x is equal in the colimit to an element $y \in \psi_{g''}F(pg'')$ with $pg'' = pg$.

If the dimensions of pg and pg' are unequal, then $C(pg,pg') \neq \emptyset$ [or similarly, $C(pg',pg) \neq \emptyset$]. The diagram

$$\begin{tikzcd}[column sep=small] & pg \arrow[dl] \arrow[dr] & \\ pg & & pg' \end{tikzcd}$$

in C yields a subgraph

$$\overset{pg}{\overline{\qquad\qquad\qquad}}\cdot\, pg'$$

of $\Gamma_g(C)$ which may be lifted to $\tilde{\Gamma}$ at pg'

$$\overset{h}{\overline{\qquad\qquad\qquad}}\cdot\, g'$$

to yield a diagram

$$\begin{tikzcd}[column sep=small] & h \arrow[dl] \arrow[dr] & \\ h & & g' \end{tikzcd}$$

which equates x as an element of $\psi_{g'}F(pg')$ to an element $y \in \psi_h F(ph)$. It follows that if $x \in p_*^{-1}\ \varphi_{pg}F(pg)$, then $x \in \psi_{g'}F(pg')$ for some $g' \in \mathrm{ob}\tilde{\Gamma}_C$ such that $pg' = pg$. Thus

$$\bigcup \psi_{g'} Fp(g') = p_*^{-1}\ \varphi_{pg} F(pg) \ .$$

By the Lemma, the union is disjoint and

$$a^{-1} p_*^{-1}\ \varphi_{pg} F(pg) = a^{-1} \sqcup \psi_{g'} Fp(g') = \sqcup\, a^{-1} \psi_{g'} Fp(g')$$

where the union is taken over all $g' \in \mathrm{ob}\tilde{\Gamma}_C$ such that $pg' = pg$.

Since this disjoint union is open in C , each component of it is open in C . In particular, the component $a^{-1}\psi_g Fp(g)$ is open in C . □

Theorem 3. p_*: colim $Fp \to$ colim F is a covering projection.

Proof: If $x \in$ colim F , then $x \in \varphi_g F(g)$ for some $g \in \mathrm{ob}C$. Since $\varphi_g F(g)$ is an open neighborhood of x which is evenly covered by

$$p_*^{-1}\ \varphi_g F(g) = \bigcup \psi_{g'} Fp(g')$$

for g' such that $pg' = pg$, p_* is a covering projection. □

Remark: Since C satisfies condition H_1) for each morphism α in C , $\pi_1 F(\alpha)$ is a monomorphism. Clearly Fp has the same property.

Theorem 3 implies the following result:

Theorem 4. C satisfies condition H). That is, for each $g \in \mathrm{ob}C$, φ_g induces a monomorphism of fundamental groups.

Proof: For each $g \in \mathrm{ob}C$ there is a commutative diagram

$$\begin{tikzcd}
Fp(g) \arrow[r, "\psi_g"] \arrow[d, "="'] & \mathrm{colim}\ Fp \arrow[d, "p"] \\
F(g) \arrow[r, "\varphi_g"] & \mathrm{colim}\ F \ .
\end{tikzcd}$$

Since $\tilde{\Gamma}_C$ has a generating tree, remarks in Section 6 show that $\tilde{\Gamma}_C$ and Fp satisfy condition H). Since p is a universal covering projection, p induces a monomorphism of fundamental group. Thus φ_g induces a monomorphism for each $g \in \mathrm{ob}C$. □

To apply the results of Section 9 to $\tilde{\Gamma}_C$ and Fp it is necessary to know that they satisfy conditions A) - I). That is, the content of the next theorem.

<u>Theorem 5</u>. If C is a connected 1-dimensional s-category and $F: C \to Q$ is a functor such that C and F satisfy conditions A), C), D), D′), D″), F), and H_1) and $\pi_1(\Gamma_g(C),*)$ is countable, then $\tilde{\Gamma}_C$ and Fp satisfy conditions A, - I).

<u>Proof</u>: A) Since $v\Gamma_g(C) = C_0$, and C_0 is countable, $(\tilde{\Gamma}_C)_0 = v\tilde{\Gamma}_C = \pi_1(\Gamma_g(C),*) \times v\Gamma_g(C)$ is countable.

B) By construction, $\tilde{\Gamma}_C$ is 1-dimensional and it is clear that any 1-dimensional s-category is upward finite.

C) That $\tilde{\Gamma}_C$ satisfies condition C) is verified by Theorem 1ii) of this section.

D) That $\tilde{\Gamma}_C$ satisfies condition D) is verified by Theorem 2 of this section.

D′) Condition D′ is satisfied since in the colimit, colim Fp, no identifications other than the usual colimit identifications are made.

D″) Fp satisfies condition D″ since F does.

E) Since $\Gamma_g(\tilde{\Gamma}_C) = \tilde{\Gamma}$ and $\tilde{\Gamma}$ is a universal covering graph, $\tilde{\Gamma}$ is simply connected and $\Gamma_g(\tilde{\Gamma}_C)$ is a tree. By Theorem 2 of Section 2, every maximal tree in $\Gamma(\tilde{\Gamma}_C)$ is a generating tree.

F) Fp is path connected since F is.

G) Since C is connected, colim F is path connected and by

Theorem 1i) of this section, colim Fp is path connected.

H) It was noted in Section 6 that if a 1-dimensional s-category satisfies condition E) and H_1), then it satisfies condition H). Since C satisfies condition H_1), $\tilde{\Gamma}_C$ does also.

I) By Proposition 20 in Section 6, since $\tilde{\Gamma}_C$ and Fp satisfy conditions C) and H), they satisfy condition I). □

Theorem 1 of Section 9 now applies to yield

$$\pi_2(\text{colim } Fp,*) = \text{colim}(Fp)_\pi .$$

By Theorem 3 of this section p_*: colim $Fp \to$ colim F is a covering projection so that

$$\pi_2(\text{colim } Fp,*) = \pi_2(\text{colim } F,*)$$

and

$$*) \qquad \pi_2(\text{colim } F,*) = \text{colim}(Fp)_\pi$$

By definition, for each $g \in \text{ob}\tilde{\Gamma}_C$,

$$(Fp)_\pi(g) = \bigoplus_{\tilde{I}(g)} (Fp)^2_T(g)$$

$$= \bigoplus_{\tilde{I}(g)} \pi_2(Fp(g),y_g)$$

$$= \bigoplus_{\tilde{I}(g)} \pi_2(F(g),y_g)$$

where $\tilde{I}$: $\tilde{\Gamma}_C \to Q$ is defined as in Section 5 by

$$\tilde{I}(g) = \pi_1(\text{colim } Fp,*)/\pi_1(Fp(g),*)$$

for each object $g \in \text{ob}C$.

$(Fp)_\pi(g)$ is determined by the second homotopy group of $F(g)$. However, to actually compute colim $(Fp)_\pi$ it is necessary to know the effect of $(Fp)_\pi$ on morphisms. If $\alpha \in \tilde{\Gamma}_C(g,g')$, then by definition for $\sum_A a_i \in \bigoplus_{\tilde{I}(g)} \pi_2(F(g),y_g)$,

$$(Fp)_\pi(\alpha)(\sum_A a_i) = \sum_A [(Fp)^2_T(\alpha)a_i]_{\tilde{I}(\alpha)i} .$$

Since the definition of $(Fp)^2_T$ depends upon the consistent $\Gamma(F)$-system T , it is not possible to relate $(Fp)^2_T(\alpha)$ to $F(\alpha)$. In fact, the definition of $(Fp)^2_T(\alpha)$ involves an isomorphism induced by a path which does not exist in colim F . Thus in general it is not possible to relate $(Fp)^2_T(\alpha)$ back to F . It is necessary to have explicit knowledge of colim Fp . The necessary information is determined by F and fundamental groups.

Of interest is the action of $\pi_1(\text{colim } F,*)$ on $\pi_2(\text{colim } F,*)$ which we now proceed to describe. _For the remainder of this section, colim F will be assumed to be locally arcwise-connected._

To begin, there is the usual fixed point free action

$$a: \pi_1(\Gamma_g(C),*) \times \tilde{\Gamma} \to \tilde{\Gamma}$$

of $\pi_1(\Gamma_g(C),*)$ on its universal cover $\tilde{\Gamma}$. This action helps to describe an action

$$b: \pi_1(\Gamma_g(C),*) \times \text{colim } Fp \to \text{colim } Fp$$

of $\pi_1(\Gamma_g(C),*)$ on colim Fp as follows:

i) If $v \in (\tilde{\Gamma}_C)_0$ and $\alpha \in \pi_1(\Gamma_g(C),*)$ is a non-trivial homotopy element, then the action $a(\alpha,v)$ yields another vertex $v' \in (\tilde{\Gamma}_C)_0$ with the properties

a) $v \neq v'$

b) $pv' = pa(\alpha,v) = pv$

c) $\psi_v Fp(v) \cap \psi_{v'} Fp(v') = \emptyset$.

All but the last property follow from covering space theory. Property c) follows from the fact that F is closed under finite intersections: if the intersection were non-empty, then there would be an edge $g \in e\tilde{\Gamma}$ from v to v' which p would project into a circuit of length 1 in $\Gamma_g(C)$ which is impossible in the geometric graph of an s-category.

ii) If $x \in \operatorname{colim} Fp$, then for some vertex $v \in (\tilde{\Gamma}_C)_0$, $x \in \psi_v Fp(v)$. (If $x \in \psi_p Fp(\mu)$ for $g \in (\tilde{\Gamma}_C)_1$, then the morphism from g to a vertex v_g yields the assertion.) For some $y \in Fp(v)$, $\psi_v y = x$. If $\alpha \in \pi_1(\Gamma_g(C),*)$ is a homotopy element, then

$$b(\alpha,x) = \psi_{a(\alpha,v)} y \in \psi_{a(\alpha,v)} Fp(a(\alpha,v)) \ .$$

Since $Fp(v) = F(p(v)) = F(pa(\alpha,v)) = Fp(a(\alpha,v))$, $y \in Fp(a(\alpha,v))$, and the definition makes sense.

iii) If α is a non-trivial homotopy element, then x and $b(\alpha,x)$ occur in distinct and non-intersecting open sets in the cover of $\operatorname{colim} Fp$. It follows that b defines a properly discontinuous action. It is also clear that the quotient of $\operatorname{colim} Fp$ by this action is $\operatorname{colim} F$. Thus by Proposition 8.2 in Massey [9], $\operatorname{colim} Fp$ is a regular covering of $\operatorname{colim} F$.

We now define a graph $\bar{\Gamma} \subseteq \operatorname{colim} F$. If the cover $\{\varphi_v F(v)\}_{v \in C_0}$ is finite, the construction is easy. If the cover is not finite, then it may be written as the colimit of its finite subcovers and the argument is made inductively. Proposition 15 of Section 6 allows a finite sequence of vertices of a connected s-category to be written in such an order that any initial sequence also determines a connected s-category. This ordering is to be used in the induction.

First the graph $\Gamma' \subseteq \operatorname{colim} F$ is defined:

i) The vertex set $v\Gamma' = \{x_g \in \varphi_g F(g)\}_{g \in obC}$ of Γ' is any set of distinct points.

ii) If $x_g, x_{g'} \in v\Gamma'$ and $C(g,g') \neq \emptyset$, $g \neq g'$ then the edge $\alpha(g,g') \in e\Gamma'$ from x_g to $x_{g'}$ is any path

$$\alpha(g,g') \in P(x_g, \varphi_{g'} F(g'), x_{g'}) .$$

The involution is path inversion.

The graph $\bar{\Gamma} \subseteq \Gamma'$ is a subset and has the properties that

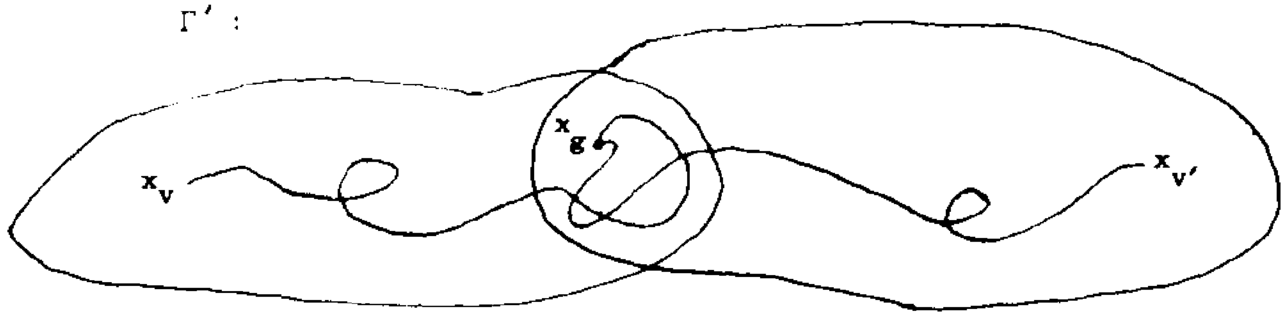
i) $v\bar{\Gamma}$ is a set of distinct points

ii) The paths determining the edges of $\bar{\Gamma}$ are homeomorphs of the unit interval

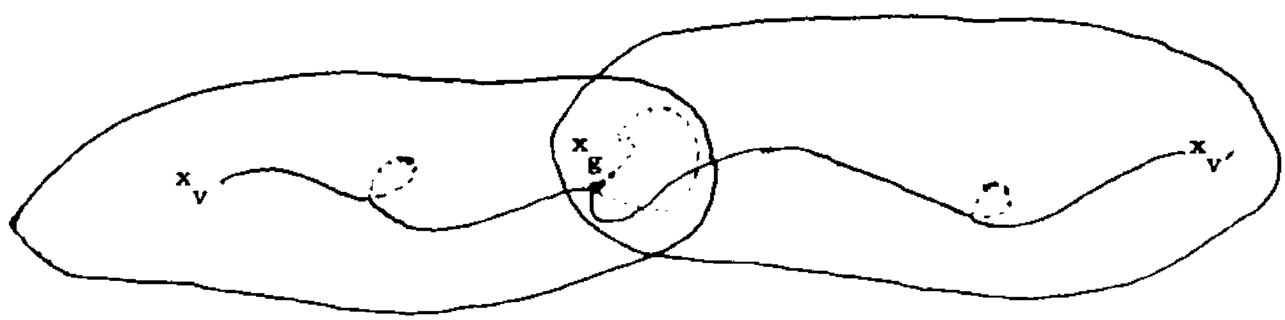
iii) If two edges of $\bar{\Gamma}$ intersect it is only at their endpoints.

Such conditions are easy to guarantee in the finite case according to the following illustration:

Γ' :



$\bar{\Gamma}$:



$\bar{\Gamma}$ is isomorphic to the first barycentric subdivision of $\Gamma_g(C)$ by the map

$$c\colon \Gamma_g(C) \longrightarrow \operatorname{colim} F$$

of $\Gamma_g(C)$ onto $\bar{\Gamma}$ as follows:

i) Each $v \in C_0 = v\Gamma_g(C)$ is mapped to x_v .

ii) If $v \neq v'$ and there is an edge $g \in e\Gamma_g(C)$ with $C(g,v) \neq \emptyset$ and $C(g,v') \neq \emptyset$, then the edge g is mapped homeomorphically onto the composition

$$\iota\alpha(g,v) + \alpha(g,v') \ .$$

iii) If $\alpha(g,v) \in e\bar{\Gamma}$ is an edge, then since g is 1-dimensional, there is another vertex $v_g \in vg$ such that $C(g,v_g) \neq \emptyset$. Then v and v_g are vertices in $\Gamma_g(C)$ and there is an edge g between them. It follows that d is onto. Conditions i)-iii) on $\bar{\Gamma}$ insure that d is a homeomorphism of $\Gamma_g(C)$ onto $\bar{\Gamma}$.

The next step is to lift $\bar{\Gamma}$ to the graph $\bar{\Gamma}'$ in the cover $\operatorname{colim} Fp$. Since $\bar{\Gamma}'$ is a lift, it covers $\bar{\Gamma}$ and is invariant under the action of $\pi_1(\Gamma_g(C),*)$. Thus each component of $\bar{\Gamma}'$ is a regular covering of $\bar{\Gamma}$ and thus of $\Gamma_g(C)$. $\bar{\Gamma}'$ is, however, connected for the following reason: if x_g and $x_{g'}$ are two lifted vertices, then since $\operatorname{colim} Fp$ is connected, there is a path in $\operatorname{colim} Fp$ from x_g to $x_{g'}$. The projection of this path is some path in $\operatorname{colim} F$. Since $\bar{\Gamma}$ has edges in each open set in the cover, there is a path from x_{pg} to $x_{pg'}$ in the graph $\bar{\Gamma}$. The lift of this path is a path in $\bar{\Gamma}'$ from x_g to $x_{g'}$.

Since by construction for any $x \in \operatorname{colim} F$, $p^{-1}x$ has the same cardinality as $\pi_1(\Gamma_g(C),*)$, $\bar{\Gamma}'$ is the universal cover $\tilde{\Gamma}$ of $\Gamma_g(C)$. Thus $\bar{\Gamma}'$ is simply connected.

We now apply Lemma 1.1 of [6] to the space colim Fp , the subspace $\bar{\Gamma}'$ and the group $\pi_1(\Gamma_g(C),*)$. The conclusion of that Lemma states that $\pi_1(\text{colim } F,*)$ is isomorphic to the semi-direct product determined by the split extension

$$1 \to \pi_1(\text{colim Fp},*) \xrightarrow{i} \pi_1(\text{colim } F,*) \underset{j}{\overset{\to}{\leftarrow}} \pi_1(\Gamma_g(C),*) \to 1$$

and the action of $\pi_1(\Gamma_g(C),*)$ on $\pi_1(\text{colim Fp},*)$ is the topologically induced action.

It is possible to describe the action of $\pi_1(\text{colim } F,*)$ on $\pi_2(\text{colim } F,*)$ in terms of this semi-direct sum decomposition. The usual action of $\pi_1(\text{colim Fp},*)$ on $\pi_2(\text{colim } F,*)$ is denoted

$$d\colon \pi_1(\text{colim Fp},*) \times \pi_2(\text{colim } F,*) \longrightarrow \pi_2(\text{colim } F,*) \ .$$

Our first claim is that the diagram $\mathcal{D}_1$:

$$\begin{tikzcd}
\pi_1(\text{colim Fp},*) \times \pi_2(\text{colim } F,*) \arrow[r, "d"] \arrow[d, "i \times 1"] & \pi_2(\text{colim } F,*) \arrow[d, "1"] \\
\pi_1(\text{colim } F,*) \times \pi_2(\text{colim } F,*) \arrow[r, "e"] & \pi_2(\text{colim } F,*)
\end{tikzcd}$$

commutes for e the usual action. This is more easily seen in terms of the universal cover. In the diagram $\mathcal{D}_2$:

$$\begin{tikzcd}
\pi_1(\text{colim Fp},*) \times \widetilde{\text{colim } F} \arrow[r, "f"] \arrow[d, "i \times 1"] & \widetilde{\text{colim } F} \arrow[d, "1"] \\
\pi_1(\text{colim } F,*) \times \widetilde{\text{colim } F} \arrow[r, "g"] & \widetilde{\text{colim } F}
\end{tikzcd}$$

the maps f and g denote the usual action. The way these actions are defined is geometrically by the lifting of paths. If α is a representative of a class in $\pi_1(\text{colim Fp},*)$ and $x \in \widetilde{\text{colim } F}$ is a point over the base point then the action of α on x is determined by a lift $\hat{\alpha}$ of α. If we now include α in $\pi_1(\text{colim } F,*)$, then $\hat{\alpha}$ is clearly a lift of $i\alpha$. Since it is a lift to the universal cover, it is the lift of $i\alpha$. It follows that a class $[\alpha] \in \pi_1(\text{colim } Fp,*)$ determines the same action as its image $i[\alpha] \in \pi_1(\text{colim } F,*)$ and diagram $\mathcal{D}_2$ commutes. Since the action of the fundamental group on the higher homotopy groups is determined by its action on the universal cover diagram $\mathcal{D}_1$ commutes.

We now use the inclusion

$$j\colon \pi_1(\Gamma_g(C),*) \longrightarrow \pi_1(\text{colim } F,*)$$

of the previous split sequence to define an action

$$h\colon \pi_1(\Gamma_g(C),*) \times \pi_2(\text{colim } F,*) \longrightarrow \pi_2(\text{colim } F,*)$$

by the composition

$$\pi_1(\Gamma_g(C),*) \times \pi_2(\text{colim } F,*) \xrightarrow{j\times 1} \pi_1(\text{colim } F,*) \times \pi_2(\text{colim } F,*) \xrightarrow{g} \pi_2(\text{colim } F,*)$$

By construction the diagram

$$\begin{array}{ccc}
\pi_1(\text{colim } Fp,*) \times \pi_2(\text{colim } F,*) & \xrightarrow{d} & \pi_2(\text{colim } F,*) \\
\downarrow i \times 1 & & \updownarrow = \\
\pi_1(\text{colim } F,*) \times \pi_2(\text{colim } F,*) & \xrightarrow{g} & \pi_2(\text{colim } F,*) \\
\uparrow j \times 1 & & \updownarrow = \\
\pi_1(\Gamma_g(C),*) \times \pi_2(\text{colim } F,*) & \xrightarrow{h} & \pi_2(\text{colim } F,*)
\end{array}$$

commutes.

Each element of the semi-direct product, $\pi_1(\text{colim } F,*)$, has a representation as a pair

$$(\alpha, \beta)$$

for $\alpha \in i\pi_1(\Gamma_g(C),*)$ and $\beta \in j\pi_1(\text{colim } Fp,*)$. Furthermore, $(\alpha,\beta) = (\alpha,1) \cdot (1,\beta)$. The action of (α,β) on x is then given by the action of $(\alpha,1)$ on the image of x under the action of $(1,\beta)$. This is, by construction, the usual action.

It is obvious that the constructions of this section generalize to the higher dimensional groups exactly as in Section 9.

§11. Simple Coverings.

Throughout this paper are references to the fact that an open cover of a space can be represented by an s-category and a functor. This section gives an explicit construction which realizes that relationship. This section uses some of the definitions but none of the results of previous sections.

Not every cover is so representable. Those which are, are the simple covers defined as follows:

Definition: Suppose Λ is a set and $X \in \mathrm{ob}Q$ is a space with an open cover $\mathcal{C} = \{X_\lambda\}_{\lambda\in\Lambda}$. $\mathcal{C}$ is a simple cover of X provided only finite subcovers of $\mathcal{C}$ have a non-void intersection.

The s-category corresponding to $\mathcal{C}$ is the s-category, $S(\mathcal{C})$, described as follows:

i) A subset $A \subseteq \Lambda$ is admissible provided

$$X_A = \bigcap_{\lambda\in A} X_\lambda \neq \emptyset .$$

ii) For an admissible subset $A \subseteq \Lambda$, Γ_A indexes the (maximal, non-empty) path components of X_A:

$$X_A = \bigsqcup_{\gamma\in\Gamma_A} {}_\gamma X_A$$

where for each $\gamma \in \Gamma$, ${}_\gamma X_A$ is a (non-empty) path component of X_A..

iii) For an admissible subset $A \subseteq \Lambda$ and element $\gamma \in \Gamma_A$, ${}_\gamma S_A$: $\{\gamma\} \times A \to \Lambda$ is the composition

$$\{\gamma\} \times A \xrightarrow{pr} A \longrightarrow \Lambda .$$

iv) The objects of $S(\mathcal{C})$ consist of all of the functions ${}_\alpha S_A$ for A admissible and $\alpha \in \Gamma_A$. The dimension function, d, is given for such a function by $d({}_\alpha S_A) = |A| - 1$. In terms of this function $ob(S(\mathcal{C}))$ is described

$$(S(\mathcal{C}))_n = \{{}_\alpha S_A | A \text{ is admissible, } \alpha \in \Gamma_A, \ |A| = n + 1\}$$

$$obS(\mathcal{C}) = \bigcup_{n \in \mathbb{Z}^+} (S(\mathcal{C}))_n .$$

v) For ${}_\alpha S_A$, ${}_\beta S_B \in obS(\mathcal{C})$, if $B \subseteq A$ and if there exists an inclusion

$$(\alpha, A; \beta, B): {}_\alpha X_A \longrightarrow {}_\beta X_B .$$

Then $S(\mathcal{C})({}_\alpha S_A, {}_\beta S_B) = \{(\alpha, A; \beta, B)\}$. Otherwise, $S(\mathcal{C})({}_\alpha S_A, {}_\beta S_B) = \emptyset$. Composition is composition of inclusions.

Proposition 1. The procedure outlined above gives the category $S(\mathcal{C})$ the structure of an s-category based on Λ.

Proof: That $S(\mathcal{C})$ is a category is clear. The objects of $S(\mathcal{C})$ are monomorphisms by definition. Since $\mathcal{C}$ is a simple cover, the admissible subsets are finite. Also, for ${}_\alpha S_A \in ob(S(\mathcal{C}))$ $im({}_\alpha S_A) = A$, $|S(\mathcal{C})({}_\alpha S_A, {}_\beta S_B)| \leq 1$ for all ${}_\alpha S_A, {}_\beta S_B \in obS(\mathcal{C})$, and $S(\mathcal{C})({}_\alpha S_A, {}_\beta S_B) = \emptyset$ if $im({}_\beta S_B) = B \not\subseteq A = im({}_\alpha S_A)$.

If ${}_\alpha S_A \in obS(\mathcal{C})$ and $B \subseteq A$, then

$$X_A = \bigcap_{\lambda \in A} X_\lambda \subseteq \bigcap_{\lambda \in B} X_\lambda = X_B$$

and the component ${}_\alpha X_A$ is contained in a unique path component ${}_\beta X_B$ for some $\beta \in \Gamma_B$. It follows that $S(\mathcal{C})({}_\alpha S_A, {}_\beta S_B) \neq \emptyset$ for this and only this object ${}_{\beta}S_B$. Thus $S(\mathcal{C})$ is an s-category. □

The appropriate functor $F\colon S(\mathcal{C}) \to Q$ is defined in the obvious way:

i) On objects ${}_{\alpha}S_A \in \mathrm{obS}(\mathcal{C})$, $F({}_{\alpha}S_A) = X_A$.

ii) On morphisms, F is the identity.

<u>Proposition 2</u>. With the previous conventions, colim $F = X$.

<u>Proof</u>: It is a general theorem that any space is the colimit of its open cover [5]. The argument for Theorem Iv) of Section 5 contains the basic ideas needed for the general proof. □

Some of the conditions in Section 6 are automatically satisfied by F and $S(\mathcal{C})$.

<u>Theorem 1</u>. If $\{\varphi_g\colon F(g) \to \operatorname{colim} F\}_{g \in \mathrm{obC}}$ are the colimit induced morphisms, then

i) Condition C): φ_g is an injection for each $g \in \mathrm{obS}(\mathcal{C})$ and $\{\varphi_g F(g)\}_{g \in \mathrm{obS}(\mathcal{C})}$ is an open cover of colim F ;

ii) Condition D): F is closed under finite intersections;

iii) Condition D′): $\varphi_g F(g) = \varphi_{g'} F(g')$, then $g = g'$;

iv) Condition D″): $\varphi_g F(g) \neq \emptyset$ for all $g \in \mathrm{obS}(\mathcal{C})$;

v) Condition F): F is path connected.

<u>Proof</u>: i) In the diagram $\mathcal{D}$:

$$\begin{tikzcd} F(g) \arrow[d, "\varphi_g"] \arrow[dr, "i_g"] & \\ \operatorname{colim} F \arrow[r, "i"] & X \end{tikzcd}$$

the map i_g is the inclusion of an open subset of X into X . Since $F(g)$ has the subspace topology, i_g is an injection and thus φ_g is.

ii) If ${}_{\alpha}S_A$ and ${}_{\beta}S_B$ are objects of $S(\mathcal{O})$ and

$$\emptyset \neq \varphi_{{}_{\alpha}S_A} F({}_{\alpha}S_A) \cap \varphi_{{}_{\beta}S_B} F({}_{\beta}S_B) = X_A \cap X_B ,$$

then $\emptyset \neq \bigcap_{\lambda \in A \cup B} X_{\lambda} = X_A \cap X_B$, and there is a family $\Gamma_{A \cup B}$ which indexes the path components of $X_A \cap X_B$. Clearly, for each $\gamma \in \Gamma_{A \cup B}$,

$$F({}_{\gamma}S_{A \cup B}) = {}_{\gamma}X_{A \cup B}$$

is a path component of the intersection and

$$\bigcup_{\gamma \in \Gamma_{A \cup B}} F({}_{\gamma}S_{A \cup B}) = X_{A \cup B} .$$

Also, $S(\mathcal{O})({}_{\gamma}S_{A \cup B}, {}_{\alpha}S_A) \neq \emptyset$ and $S(\mathcal{O})({}_{\gamma}S_{A \cup B}, {}_{\beta}S_B) \neq \emptyset$ for each $\gamma \in \Gamma_{A \cup B}$.

iii) If ${}_{\alpha}S_A, {}_{\beta}S_B \in \mathrm{ob}S(\mathcal{O})$ and

$$i\,{}_{\alpha}X_A \cong i\varphi_{{}_{\alpha}S_A} F({}_{\alpha}S_A) = i\varphi_{{}_{\beta}S_B} F({}_{\beta}S_B) \cong i\,{}_{\beta}X_B$$

for i the injection in diagram $\mathcal{D}$, then ${}_{\alpha}X_A = {}_{\beta}X_B$. It follows from the construction that $A = B$ and $\alpha = \beta$.

iv) $\varphi_g F(g) \neq \emptyset$ for all $g \in \mathrm{ob}S(\mathcal{O})$ by definition of admissible subset of Λ .

v) F is path connected since for each object ${}_{\alpha}S_A \in \mathrm{ob}S(\mathcal{O})$, $F({}_{\alpha}S_A) = {}_{\alpha}X_A$ is a path component of X_A . □

§12. Examples.

Example 1. Suppose Y is a finite graph with no circuits of length 1 and n is an integer greater than or equal to 2 . The space (Y,n) is the graph of n-spheres defined as follows:

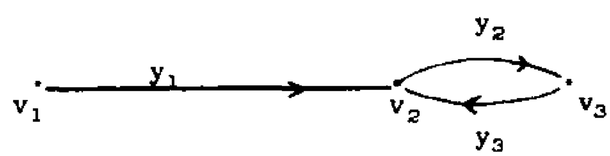
i) For each vertex $w \in vY$ an n-sphere, S_w^n , is associated. For each edge $y \in eY$, with $o(y) = w$, a point $x_y^w \in S_w^n$ is selected at random except distinct points are associated to each such edge.

ii) The space (Y,n) is the quotient

$$\coprod_{vY} S_w^n \,/ \sim$$

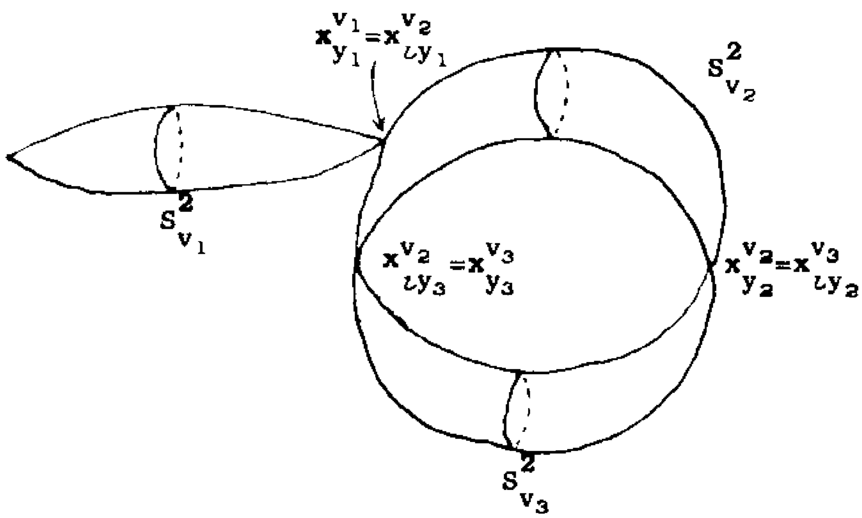
where $x_y^w \sim x_{\iota y}^{w'}$ for y an edge with $o(y) = w$ and $t(y) = o(\iota y) = w'$.

For example, if Y is the graph



,

then $(Y,2)$ is the space



The n-spheres form a cover of (Y,n) and if they are fattened in the usual way they will form an open cover in which no three ever intersect and in which all non-empty intersections are contractible.

The construction of (Y,n) is such that if C is the s-category associated to the described cover, then $\Gamma_g(C) = Y$. We now use the results of Section 10 to compute $\pi_m(Y,n)$ for $m \le n$. Clearly, if $m < n$, then since each entry in the colimit is zero, $\pi_m(Y,n) = 0$. $\pi_n(Y,n)$ is the only interesting case.

If Y is a tree, then the Seifert van-Kampen Theorem of Section 4 yields $\pi_1(Y,n) = 0$ and the corollary to Theorem 2 of Section 9 yields

$$\pi_n(Y,n) = \bigoplus_{vY} \mathbb{Z}$$

with the trivial action.

If Y is not a tree, then by Section 10,

$$\pi_n(Y,n) = \bigoplus_{v\tilde{Y}} \mathbb{Z}$$

for $\tilde{Y}$ the universal cover of Y. In determining the fundamental group action for this case one notes that the comments on the group action in Section 10 yield that $\pi_1(Y,n)$ occurs in the split exact sequence

$$0 \longrightarrow \pi_1\widetilde{(Y,n)} \longrightarrow \pi_1(Y,n) \rightleftarrows \pi_1(Y) \longrightarrow 0 .$$

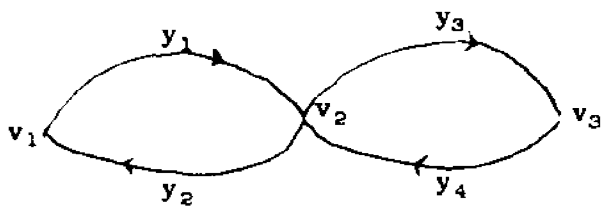
Since $\widetilde{(Y,n)} = (\tilde{Y},n)$ and the category associated with $(\tilde{Y},n)$ has a tree as its geometric graph, the Van Kampen Theorem yields as before that $\pi_1\widetilde{(Y,n)} = 0$. It follows that $\pi_1(Y,n) = \pi_1(Y)$ and the action of $\pi_1(Y,n)$ reduces to the action of $\pi_1(Y)$. This action is determined by the action of $\pi_1(Y)$ on $\tilde{Y}$ as the coordinate shift in $\bigoplus_{v\tilde{Y}} \mathbb{Z}$.

For any finite graph Y with $\pi_1(Y) \neq 0$, the space (Y,n) has

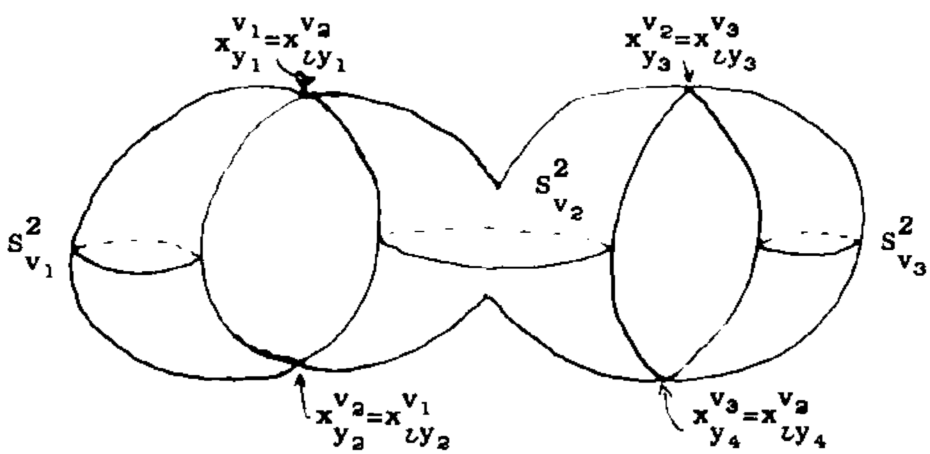
$$\pi_n(Y,n) = \bigoplus_{v\tilde{Y}} \mathbb{Z} .$$

As a group this is just the direct sum of countably many copies of $\mathbb{Z}$. For such graphs, the n^{th} homotopy groups are isomorphic as groups. However, as modules over the fundamental group they are distinct.

We will consider the particular case $n = 2$ for the graph Y :

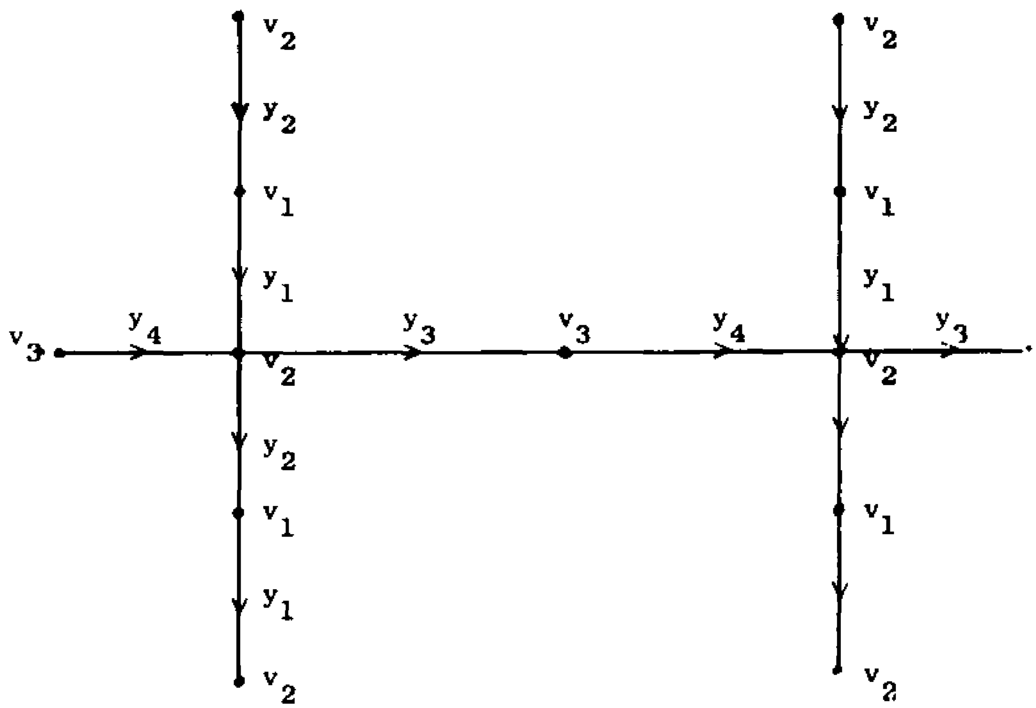


The space $(Y,2)$ may be drawn



Since $\pi_1(Y) = \mathbb{Z} * \mathbb{Z} \neq 0$, $\pi_2(Y,2) = \bigoplus_{v\tilde{Y}} \mathbb{Z}$.

To describe the action we pick a base point at random: say v_2 . Some of the universal cover of Y looks like the graph



where the labels indicate the image of the edge under the covering map. $\pi_1(Y)$ is a free group on two generators which we label a and b . a corresponds to the path $\{y_2, y_1\}$ and b corresponds to the path $\{y_3, y_4\}$. The action only affects the generators which arise from spheres which cover $S^2_{v_2}$. The action of a maps such a generator to a generator corresponding to a sphere one level down in the previous picture. The action of b moves it one unit to the right. A word in a and b then maps the generator of $\pi_2(S^2_{v_2})$ in the obvious way according to the individual action of each consecutive letter. For example, $a^2b^3a^{-1}b^2$ moves the generator 2 to the right, 1 up, 3 more to the right and then 2 down. The action on an arbitrary element of $\pi_2(Y,n)$ is induced in the obvious way by this action on the generators of the direct sum.

<u>Example 2.</u> The previous example can be extended in the following way:

<u>Definition</u>: If Y is a graph, then a Y-family of groups $\mathcal{J}(Y)$ is a

collection of abelian groups $\mathcal{G} = \{G_*\}$ and monomorphisms $\{\alpha_{w,y}\}$ such that

i) to each vertex $w \in vY$. corresponds a group G_w ;

ii) to each edge $y \in eY$, corresponds a group G_y with $G_y = G_{\bar{y}}$ for each $y \in eY$;

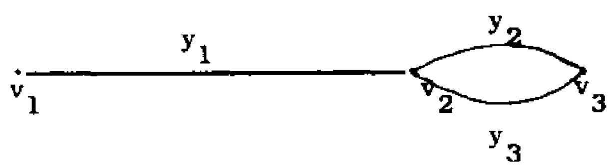
iii) for each G_y there is an inclusion $\alpha_{y,o(y)}: G_y \longrightarrow G_{o(y)}$ from the group corresponding to y to the group corresponding to its initial point.

For $n \geq 2$ such a family of groups is represented geometrically as the space $\mathcal{G}(Y,n)$ as follows:

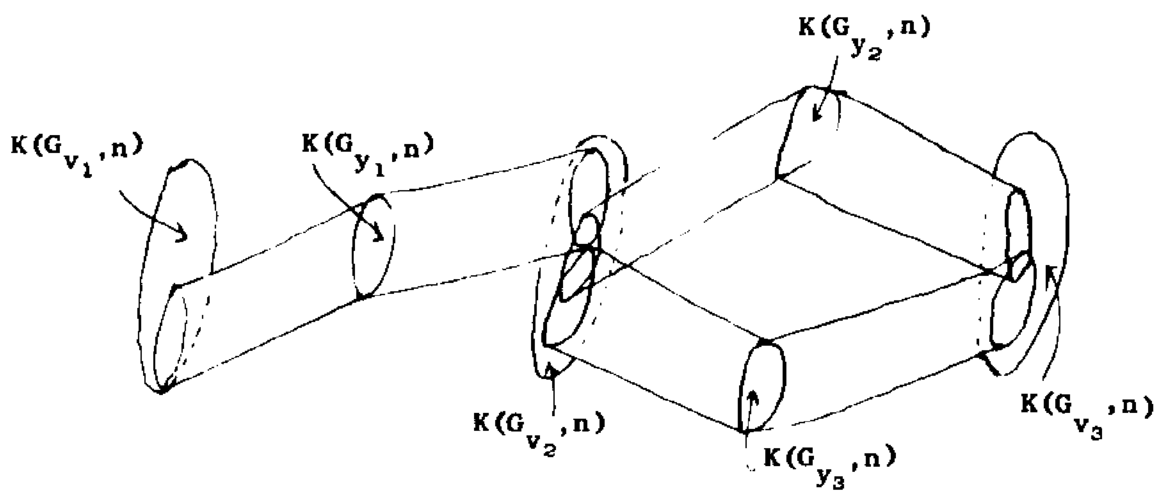
i) to each G_x is associated a $K(G_x,n)$ for some x ,

ii) for each edge y , $K(G_y,n)$ is adjoined to $K(G_{o(y)},n)$ via the mappping cylinder of the inclusion $\alpha_{y,o(y)}$.

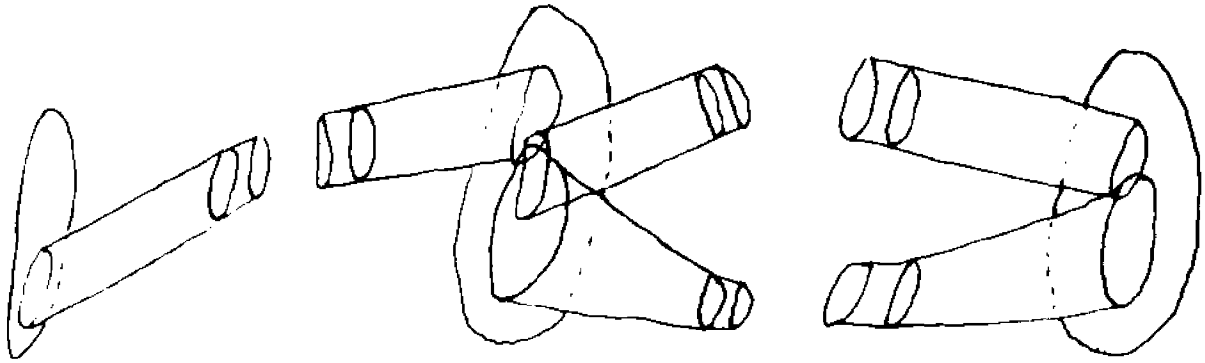
For example if Y is the graph



then $G(Y,n)$ is the space



The appropriate cover for this example is



Again the associated s-category has geometric graph Y .

If Y is a tree then, since $n \geq 2$, all the elements in the cover are simply connected and as before $\pi_1(\mathscr{K}(Y,n)) = 0$. In this case

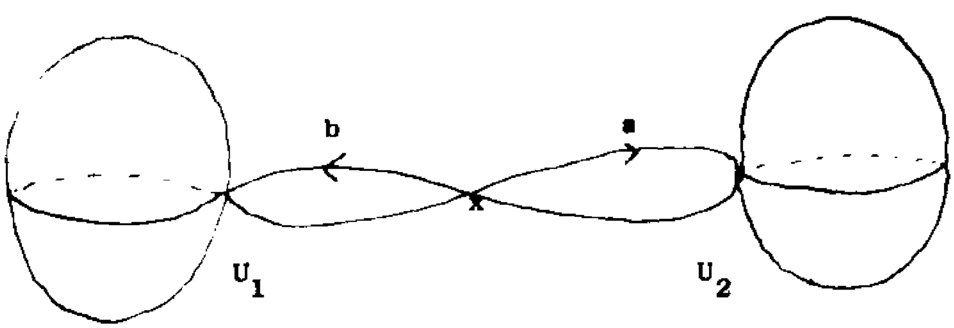
$$\pi_n(\mathscr{K}(Y,n)) = \bigoplus_{w \in vY} G_w$$

with the trivial action.

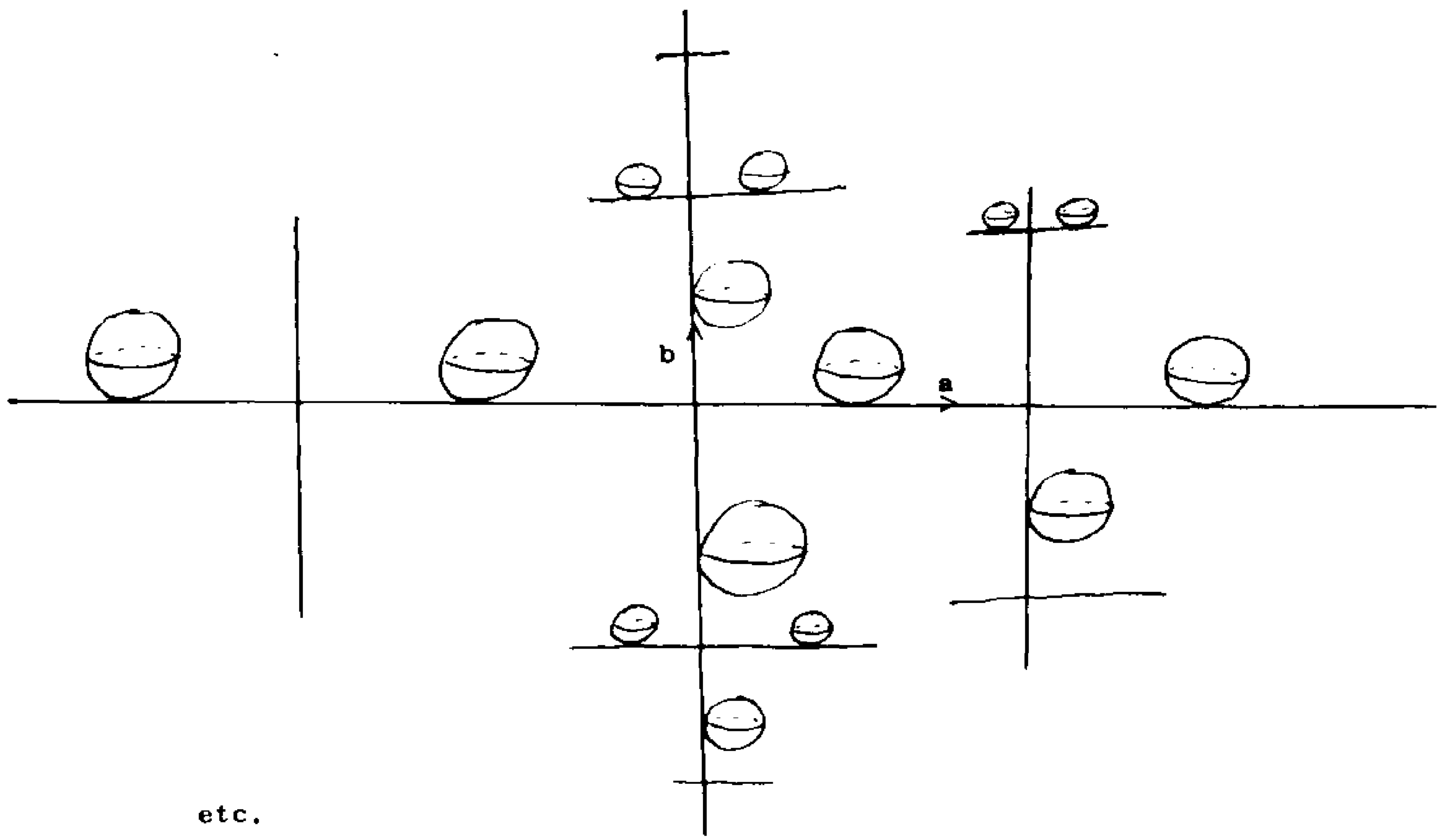
The case when Y is not a tree will yield the direct sum of countably many copies of each G_w for $w \in vY$. The action is described as in Example 1.

If we delete the condition that the groups be abelian and set $n = 1$, then the space $\mathscr{K}(Y,1)$ gives a geometric picture of a group of graphs as studied by Serre [1].

<u>Example 3</u>. In this example the space, X , is the 1 point union of two copies of the 1-point union of a circle and a 2-sphere:

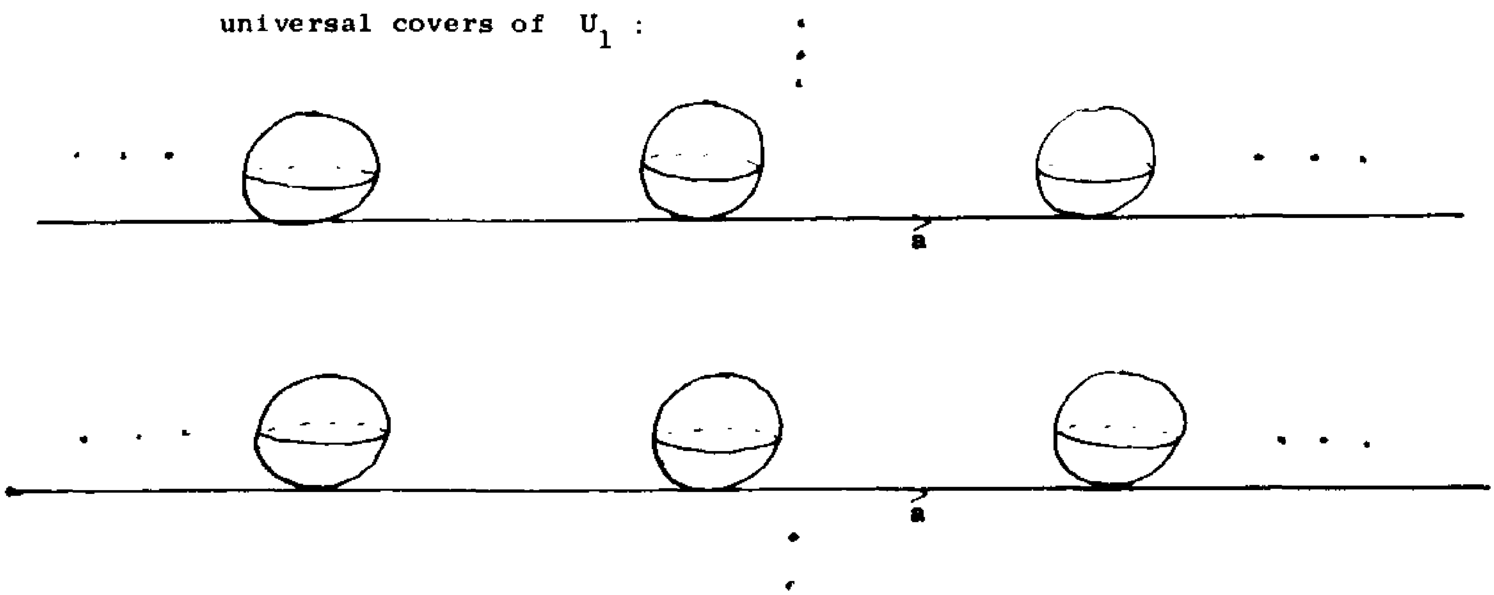


with the indicated two piece open cover. The universal cover of X is the universal cover of the figure 8 with 2-spheres attached between vertices:



etc.

Since the index of $\pi_1(U_1,x)$ in $\pi_1(X,x)$ is countably infinite, the restriction of this cover to U_1 is a countably infinite sequence of universal covers of U_1 :



The action of a on this cover moves the generator one unit to the right. The action of b interchanges sheets and moves the generator one unit up.

The point here is that the description of the fundamental group action on this part of the cover is no easier than its description in general.

Example 4. (due to E. Dyer) This example illustrates the necessity of the hypothesis that the inclusions in the cover induce monomorphisms of the fundamental groups. We will present two spaces with different second homotopy groups but with covers which are indistinguishable by their homotopy groups or the fundamental group action.

Let $A = K(Z_2,1) = P_\infty(R)$ and $X = K(Z_2,2)$. The cohomology of $P_\infty(R)$ is a polynomial ring in a single generator, η , in dimension 1. η^2 is a cohomology class in $H^2(A;\mathbb{Z}_2) = [K(Z_2,1),K(\mathbb{Z}_2,2)]$. Thus there is a map $f\colon A \to X$ such that $f^*(i) = \eta^2$ for i the fundamental class. The mapping cylinder construction replaces f by a cofibration.

The long exact homotopy sequence of the pair (X,A) is

$$0 = \pi_2(A,*) \to \pi_2(X,*) \to \pi_2(X,A,*) \to \pi_1(A,*) \to \pi_1(X,*)$$

$$\to \pi_1(X,A,*) \to \pi_0(A) = 0 \ .$$

Since $\pi_1(X,*) = 0$, this yields the short exact sequence

1) $0 \to Z_2 \to \pi_2(X,A,*) \to Z_2 \to 0$

and the fact that $\pi_1(X,A,*) = 0$. Since X and A are connected the relative Hurewicz isomorphism theorem yields

2) $H_1(X,A;Z) = 0$

and

3) $h\colon \pi_2(X,A,*) \to H_2(X,A;\mathbb{Z})$ is onto.

The universal coefficient theorem for homology yields the split short exact sequence

$$0 \to H_2(X,A:\mathbb{Z}) \otimes \mathbb{Z}_2 \to H_2(X,A:\mathbb{Z}_2) \to \mathrm{Tor}(H_1(X,A:\mathbb{Z}),\mathbb{Z}_2) \to 0$$

which in view of 2) implies

4) $H_2(X,A:\mathbb{Z}) \otimes \mathbb{Z}_2 \cong H_2(X,A;\mathbb{Z}_2)$.

The long exact cohomology sequence of the pair (X,A) is

5) $\leftarrow H^2(A;\mathbb{Z}_2) \xleftarrow{f^*} H^2(X;\mathbb{Z}_2) \leftarrow H^2(X,A;\mathbb{Z}_2) \leftarrow H^1(A;\mathbb{Z}_2) \leftarrow 0$

$H^2(A;\mathbb{Z}_2)$ is $\mathbb{Z}_2$ and is generated by η^2 . $H^2(X;\mathbb{Z}_2)$ is $\mathbb{Z}_2$ and is generated by i . Since $f^*(i) = \eta^2$, f^* is an isomorphism and $\ker f = 0$. Thus $H^2(X,A;\mathbb{Z}_2) \cong H^1(A;\mathbb{Z}_2)$. But $H^1(A;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Since (X,A) is a finite cell complex, $H_2(X,A;\mathbb{Z}_2)$ is a finite vector space and the universal coefficient theorem yields $H^2(X,A;\mathbb{Z}_2) \cong H_2(X,A;Z_2)$. Thus $H_2(X,A;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

If $H_2(X,A;\mathbb{Z})$ were not cyclic then it would be the direct sum of cyclic groups and since tensoring commutes with direct sum, $H_2(X,A;\mathbb{Z}) \otimes \mathbb{Z}_2$ would be a direct sum. Thus $H_2(X,A;\mathbb{Z})$ is cyclic.

The homology sequence of the pair (X,A) is

6) $H_2(A;\mathbb{Z}) \to H_2(X;\mathbb{Z}) \to H_2(X,A;\mathbb{Z}) \to H_1(A;\mathbb{Z}) \to H_1(X;\mathbb{Z})$

Since A is projective space, $H_2(A;\mathbb{Z}) = 0$. The Hurewicz isomorphism theorem yields $H_2(X;\mathbb{Z}) \cong \pi_2(X;*) \cong \mathbb{Z}_2 \cong \pi_1(A,*) \cong H_1(A;\mathbb{Z})$. Thus the long exact sequence 6) yields the short exact sequence

7) $0 \to \mathbb{Z}_2 \to H_2(X,A;\mathbb{Z}) \to \mathbb{Z}_2 \to 0$.

Since $H_2(X,A;\mathbb{Z})$ is cyclic, it is $\mathbb{Z}_4$. The short exact sequence 1) gives $\pi_2(X,A,*)$ to be either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$. By 3) the Hurewicz map is onto so that $\pi_2(X,A,*) \cong \mathbb{Z}_4$.

The kernel of h is given by

$$\ker(h) = \{x-\alpha x \mid x \in \pi_2(X,A,*),\ \alpha \in \pi_1(A,*)\} .$$

Since $\pi_2(X,A,*) = H_2(X,A;\mathbb{Z})$, $\ker h = 0$ and the action of $\pi_1(A)$ on $\pi_2(X,A,*)$ is trivial. By 1) $\pi_2(X,*) \longrightarrow \pi_2(X,A,*)$ is monic. Since the action commutes with this map, the action of $\pi_1(A,*)$ on $\pi_2(X,*)$ is trivial.

From [4] we have that the relative Hurewicz isomorphism may be written as the composition

8) $$\pi_2(X,A,*) \xrightarrow{\text{onto}} \pi_2(X \cup CA,*) \longrightarrow H_2(X \cup CA;\mathbb{Z}) \overset{\cong}{\leftarrow} H_2(X,A;\mathbb{Z})$$

(for CA the cone on A) .

That the first map is onto is a consequence of the triad connectivity theorem and the exact sequence

$$\pi_3(X \cup CA;X,CA,*) \longrightarrow \pi_2(X,A,*) \longrightarrow \pi_2(X \cup CA,CA) = \pi_2(X \cup CA,*) .$$

(Again, see [4]).

The composition 8) may be rewritten

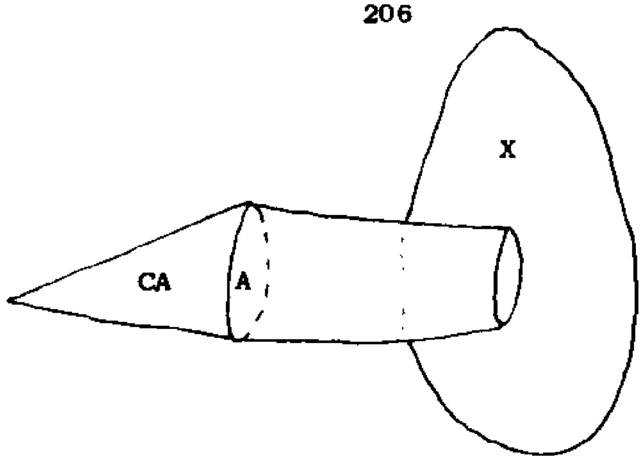
$$\mathbb{Z}_4 \xrightarrow{\text{onto}} \pi_2(X \cup CA,*) \longrightarrow \mathbb{Z}_4 .$$

Since this is an isomorphism

$$\pi_2(X \cup CA,*) = \mathbb{Z}_4 .$$

We are now ready to present the two spaces.

<u>Space 1</u>: This is formed by attaching a cone to A on the mapping cylinder of f :



Then $X \cup CA$ is the push-out

$$\begin{tikzcd} A \arrow[r] \arrow[d] & CA \arrow[d] \\ X \arrow[r] & X \cup CA \end{tikzcd}$$

By hypothesis,

i) $\pi_1(A,*) = \mathbf{Z}_2$ and $\pi_i(A,*) = 0,\ i \neq 1$;

ii) $\pi_i(CA,*) = 0$;

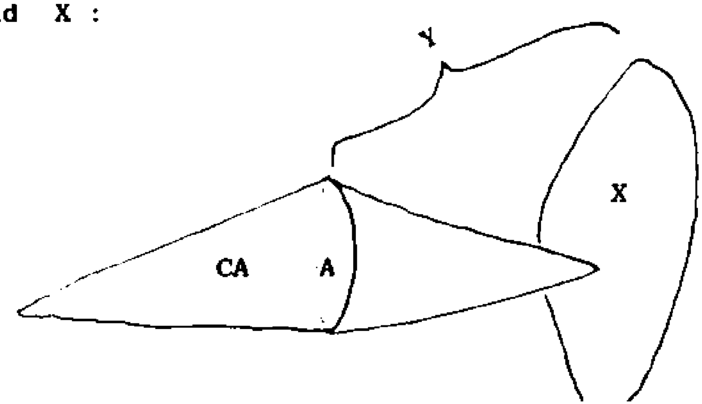
iii) $\pi_2(X,*) = \mathbf{Z}_2$ and $\pi_i(X,*) = 0,\ i \neq 2$;

iv) $\pi_1(A,*)$ acts trivially on $\pi_2(X,*)$ (and on $\pi_i(A,*),\ i \geq 2$, $\pi_i(X,*),\ i > 2$ and $\pi_*(CA,*)$ since all these groups are trivial.)

Our previous result was that in this case:

$$\pi_2(X \cup CA,*) \cong \mathbf{Z}_4 .$$

<u>Space 2</u>: This space is the one point union of the unreduced suspension of A, SA, and X :



This space is the push-out

$$\begin{array}{ccc} A & \longrightarrow & CA \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee SA \end{array}$$

The space A is the same as before and Y has the same homotopy type as X . Since $\pi_1(A,*)$ acts on $\pi_*(X,*)$ through the inclusion $\pi_1(A,*) \to \pi_1(X,*)$, the action of $\pi_1(A,*)$ on $\pi_*(X,*)$ is trivial. However,

$$\pi_2(X \vee SA,*) = \pi_2(K(\mathbb{Z}_2,2) \vee SK(\mathbb{Z}_2,1)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 .$$

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BIBLIOGRAPHY

1. Bass, Hyman and Serre, Jean-Pierre. Groupes Discrets (unpublished notes, 1968.)

2. Brown, Ronald. Elements of Modern Topology. London: McGraw-Hill, 1968.

3. Brown, Ronald. "Groupoids and van Kampen's Theorem." Proceedings of the London Mathematical Society (3), Volume 17 (1967), 385-401.

4. Dyer, E. "The Functors of Algebraic Topology." Studies in Modern Topology. Edited by P.J. Hilton. Englewood Cliffs, N.J.: Mathematical Association of America, 1968, 134-164.

5. Dyer, E. and Eilenberg, S. (book), to appear.

6. Dyer, E. and Vasquez, A.T. "Some Small Aspherical Spaces." Journal of the Australian Mathematical Society. To appear.

7. Dyer, J.L. "A Five Lemma For Free Products of Groups with Amalgamations." Bulletin of the Australian Mathematical Society, Volume 3 (1970), 85-96.

8. Higgins, Philip J. Notes on Categories and Groupoids. London: van Nostrand Reinhold, 1971.

9. Massey, William S. Algebraic Topology: An Introduction. New York: Harcourt, Brace and World, 1967.

10. Neuman, B.H. "An Essay on Free Products of Groups with Amalgamations." Philosophical Transactions of the Royal Society of London, Series A, Volume 246, (1954), 503-554.

11. Spanier, E. "Quasi-topologies." Duke Mathematical Journal, Volume 30, (1963), 1-14.

AUTOBIOGRAPHICAL STATEMENT

Steven C. Althoen was born in Dayton, Ohio in 1946. He received his B.A. from Kenyon College in 1969 and immediately began graduate work at the City University of New York Graduate School. In 1966 he married Marcia Warrick. They have a fifteen month old son, Michael.