Endomorphisms of n-dimensional projective space over function fields

by

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Abstract

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Let $K = k(C)$ be the function field of a complete nonsingular curve C over an arbitrary field k. The main result states a morphism $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ is isotrivial if and only if it has potential good reduction at all places v of K. This generalizes results of Benedetto for polynomial maps on $\mathbb{P}^1_K$ and Baker for arbitrary rational maps on $\mathbb{P}^1_K$. There are two proofs given. The first uses algebraic geometry and more specifically, geometric invariant theory. It is new even in the case of $\mathbb{P}^1_K$. The second proof, using non-archimedean analysis and dynamics, more directly generalizes proofs of Benedetto and Baker for the $N = 1$ case. In addition, two applications for the result are given.

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Contents

Chapter 1

Introduction

Let $K = k(C)$ be a function field of a complete nonsingular curve C over an arbitrary field k, and let $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be a morphism. We say that φ is *isotrivial* if it induces a trivial morphism $\mathbb{P}^N_{K'} \to \mathbb{P}^N_{K'}$ for some finite extension K'/K.

The set of M_K places of K can be naturally identified with the set of closed points on a curve C as above. Given a place $v \in M_K$, denote by $\mathcal{O}_v \subset K$, the ring of regular function at v. We say that φ has *good reduction* at v if there exists a choice of homogeneous coordinates $(x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_K$ such that φ extends to an endomorphism of the associated integral model $\mathbb{P}^N_{\mathcal{O}_v}$ of $\mathbb{P}^N_K$. We say that φ has *potential good reduction* at v if there exists a finite extension K'/K and a place v' of K' over v such that φ has good reduction at v'.

The following theorem which will be proved states that the purely local condition, that φ has potential good reduction at all places v of K, is equivalent to the global condition, that φ is isotrivial.

Theorem 1. *Let* $K = k(C)$ *be a function field, and let* $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ *be a morphism with* $\deg(\varphi) \geq 1$*. Then* φ *is isotrivial if and only if* φ *has potential good reduction at all places* v *of* K*.*

The first remark is that the "only if" direction of the theorem is easy and will be proved in §2.6. When $\deg(\varphi) = 0$ there is nothing to prove. All constant morphisms are trivial and have good reduction at all places. The $\deg(\varphi) = 1$ case is an exercise in linear algebra. A proof will be given in §2.7. This leaves the "if" direction when $\deg(\varphi) \geq 2$ as the interesting part of Theorem 1, and this will be the focus of §3 and §4.

Reflecting the various techniques used to study algebraic dynamics over global fields, there will be two, very different, proofs of Theorem 1 given. The first proof uses algebraic geometry and standard facts from geometric invariant theory. It is new even in the case of $\mathbb{P}^1_K$. Given integers $N \geq 1$ and $d \geq 2$, we first show that the space $\mathcal{M}_{N,d}$ parameterizing endomorphisms morphisms $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ with $\varphi^*\mathcal{O}(1) \simeq \mathcal{O}(d)$ exists as an affine variety. (When $N = 1$, this follows immediately from Silverman [20], Theorem 1.1.) A morphism $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ over the function field $K = k(C)$ with everywhere potential good reduction induces a regular map $C \to \mathcal{M}_{N,d}$, which must be constant since $\mathcal{M}_{N,d}$ is affine. It will follow from this fact that φ is isotrivial.

The second proof of Theorem 1 uses non-archimedean analysis and dynamics. It more directly generalizes proofs given by Benedetto [4] and Baker [2] in the $N = 1$ case. We consider the local homogeneous filled Julia set, $F_{\Phi,v}$, associated to each place v of K and each model Φ for φ. This is a certain dynamical invariant of Φ which detects good reduction at v. The key step is to define a notion of homogeneous transfinite diameter in order to measure the size of the set $F_{\Phi,v}$ and show that these numbers satisfy a product formula over all places v of K. Selecting a globally defined model Φ for φ with certain favorable properties, we use the hypothesis of everywhere potential good reduction along with the product formula to show that Φ must be defined over the constant field k.

Despite the differences of the two approaches, they share several ingredients. For example, both proofs use facts about the resultant of homogeneous polynomials which will be reviewed in §2.3. Both proofs use the fact that the curve C has no non-constant regular functions. Moreover, both proofs make an essential use of a basic result in algebraic dynamics on the Zariski-density of preperiodic points which is discussed in §2.1.

In §5 we give two applications of Theorem 1. The first result states that an endomorphism of $\mathbb{P}^N_K$ of degree at least two is isotrivial if and only if it has an isotrivial iterate. The second gives a dynamical criterion for whether, after base change, a locally free coherent sheaf $\mathcal{E}$ of rank $N+1$ on C decomposes as a direct sum $\mathcal{L} \oplus \cdots \oplus \mathcal{L}$ of $N+1$ copies of the same invertible sheaf $\mathcal{L}$ on C. When $k = \mathbb{C}$, Amerik [1] has obtained a simlar result with the curve C replaced by a smooth projective base B of arbitrary dimension.

Our motivation comes from the study of dynamical systems arising from the iteration of morphisms such as $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ of degree at least two. The isotrivial case and the non-isotrivial case exhibit very different behavior. For example, in the one-dimensional case, Baker [2] showed that if $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a non-isotrivial morphism of degree at least two, then $\mathbb{P}^1(K)$ has only finitely many small points with respect to the Call-Silverman canonical height function $\widehat{h}_\varphi$ associated to φ. In particular, it follows that $\mathbb{P}^1(K)$ contains only finitely many φ-preperiodic points, and that a point $P \in \mathbb{P}^1(\bar{K})$ is preperiodic if and only if $\widehat{h}_\varphi(P) = 0$. These results had been previously established in the special case of polynomial endomorphisms of $\mathbb{P}^1_K$ - that is, endomorphisms with a totally ramified fixed point - by Benedetto [4]. Using entirely different techniques from model theory, Chatzidakis- Hrushovski [8] have recently generalized Baker's results to endomorphisms of $\mathbb{P}^N_K$.

The situation is quite different in the case where φ is isotrivial. Suppose for example that the constant field k is algebraically closed and that $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a morphism defined over k with $\deg(\varphi) \geq 2$; then all of the (infinitely many) φ-preperiodic points in $\mathbb{P}^1(\bar{K})$ are defined over k, and so they are K-rational. Moreover, since φ is isotrivial, the canonical height $\widehat{h}_\varphi$ coincides with the naive height, so unless k is an algebraic closure of a finite field, $\mathbb{P}^1(K)$ may contain non-preperiodic points having canonical height zero.

A key ingredient in the results of Baker and Benedetto for non-isotrivial endomorphisms of $\mathbb{P}^1_K$ is the characterization of isotriviality in terms of everywhere potential good reduction; see Baker [2], Theorem 1.9 and Benedetto [4], Proposition 6.1. The main result here, Theorem 1, generalizes this criterion to endomorphisms of $\mathbb{P}^N_K$.

It would be interesting to investigate to what extent Theorem 1 can be extended to more generalized polarized algebraic dynamical systems as in §2.1. However, a naive restatement of Theorem 1 in this general setting is false, as there exists non-isotrivial endomorphisms $\varphi : X \to X$ of projective K-varieties X with good reduction at every place v of K. That is, φ extends to a projective endomorphism $\varphi_v : X_{\mathcal{O}_v} \to X_{\mathcal{O}_v}$ of an integral model $X_{\mathcal{O}_v}$ for X. For example, let k be an algebraically closed field of characteristic zero, let g and n be large positive integers and let $\mathcal{A}_{g,n}$ denote the (fine) moduli space of principally polarized abelian varieties over k of dimension g with level-n

structure. It is well known that $\mathcal{A}_{g,n}$ contains a complete nonsingular curve C. (This follows from the fact that the boundary $\overline{\mathcal{A}}_{g,n} \backslash \mathcal{A}_{g,n}$ of $\mathcal{A}_{g,n}$ inside the Satake compactification, $\overline{\mathcal{A}}_{g,n}$, has codimension strictly greater than one when g is large enough; see [6].) The resulting abelian scheme $A \to C$ has a generic fiber of an abelian variety A_K over $K = k(C)$, and the doubling map $[2] : A_K \to A_K$ gives rise to a non-isotrivial polarized dynamical system over K with everywhere good reduction.

Chapter 2

Preliminaries

2.1 Polarized algebraic dynamical systems

Let k be a field. A polarized algebraic dynamical system over k is a triple $(X, \varphi, \mathcal{L})$, where X is a projective k-variety, $\varphi : X \to X$ is a morphism, and $\mathcal{L}$ is an ample invertible sheaf on X such that $\varphi^*\mathcal{L} \simeq \mathcal{L}^{\otimes d}$ for some $d \geq 2$. We will now recall several standard definitions and facts about polarized algebraic dynamical systems; for more background see the surveys of Zhang [23] and Chambert-Loir [7].

Given a point $x \in X(\bar{k})$, we say that x is *fixed* if $\phi(x) = x$, *periodic* if $\varphi^n(x) = x$ for some $n \geq 1$, and *preperiodic* if $\varphi^m(x)$ is periodic for some $m \geq 0$. Let $\mathrm{Fix}(\varphi)$, $\mathrm{Per}(\varphi)$ and $\mathrm{PrePer}(\varphi)$ denote the sets of fixed, periodic and preperiodic points in $X(\bar{k})$, respectively. Given integers $n \geq 1$ and $m \geq 0$

we denote by

$$\mathrm{Per}_n(\varphi) = \{x \in X(\bar{k}) \mid \varphi^n(x) = x\}$$

$$\mathrm{Per}_{n,m}(\varphi) = \{x \in X(\bar{k}) \mid \varphi^{n+m}(x) = \varphi^m(x)\}$$

the sets of periodic points of period n, and preperiodic points of type (n, m), respectively.

Proposition 2. *Let* $(X, \varphi, \mathcal{L})$ *be a polarized algebraic dynamical system where* $\varphi^*\mathcal{L} \simeq \mathcal{L}^{\otimes d}$*, and assume that* X *is geometrically integral. Then:*

- **(a)** *The morphism* φ *is finite and* $\deg(\varphi) = d^{\dim(X)}$.
- **(b)** *The sets* $\mathrm{Fix}(\phi)$*,* $\mathrm{Per}_n(\phi)$ *and* $\mathrm{PrePer}_{n,m}(\varphi)$ *are finite.*
- **(c)** *The set* $\mathrm{PrePer}(\varphi)$ *is Zariski-dense in* X*.*

Remark. We will not include a proof of Proposition 2 **(c)**. Fakhruddin, ([12], Theorem 5.1) has shown that Proposition 2 **(c)** is true when $\mathrm{PrePer}(\varphi)$ is replaced with the much smaller set $\mathrm{Per}(\varphi)$. This proof uses a result of Hrushovski ([15], Theorem 1.1) in model theory. However, the proof of Theorem 5.1 in [12] on page 11 can show that the set of preperiodic points, $\mathrm{PrePer}(\varphi)$, is Zariski-dense without the result of Hrushovski.

Proof. **(a)** Suppose on the contrary that φ is not finite. Since a projective morphism is finite if and only if it has finite fibers, this means there exists a point $x \in X(\bar{k})$ such that $\varphi^{-1}(x)$ contains an irreducible curve Z. Pushing foward the intersection product of Z with the first Chern class $c_1(\varphi^*\mathcal{L})$ of $\varphi^*\mathcal{L}$, we have

$$\varphi_*(Z.c_1(\varphi^*\mathcal{L})) = \varphi_*(Z).c_1(\mathcal{L}) \tag{2.1}$$

by the projection formula. We have $Z.c_1(\varphi^*\mathcal{L}) = Z.c_1(\mathcal{L}^{\otimes d}) = d(Z.c_1(\mathcal{L})) > 0$ since $\mathcal{L}$ is ample, and we conclude that the left-hand-side of (2.1) is nonzero. But the right-hand-side of (2.1) vanishes since $\varphi_*(Z)$ is supported on the point x. This contradition shows that φ is finite.

To prove the degree formula, recall that the Euler-Poincaré characteristic $\chi(X, \cdot)$ of (tensor) powers of $\mathcal{L}$ satisfies

$$\chi(X, \mathcal{L}^{\otimes \nu}) = \frac{e(\mathcal{L})}{\dim(X)!}\nu^{\dim(X)} + \text{ lower order terms}$$

for some $e(\mathcal{L}) > 0$ and all sufficiently large positive integers ν, where the right-hand-side is the Hilbert-Samuel polynomial. Since

$$\chi(X, \mathcal{L}^{\otimes d\nu}) = \chi(X, \varphi^*\mathcal{L}^{\otimes \nu}) = \deg(\varphi)\chi(X, \mathcal{L}^{\otimes \nu}),$$

we may compare leading terms to deduce that

$$\frac{e(\mathcal{L})}{\dim(X)!}(d\nu)^{\dim(X)} = \frac{\deg(\varphi)e(\mathcal{L})}{\dim(X)!}\nu^{\dim(X)}$$

and it follows that $\deg(\varphi) = d^{\dim(X)}$.

(b) Let Y be an irreducible component of the closed subvariety $\mathrm{Fix}(\varphi)$ of X. Note that Y is closed and that $\varphi(Y) = Y$, so $(Y, \varphi|_Y, \iota^*\mathcal{L})$ is a polarized dynamical system, where $\iota : Y \to X$ is the inclusion morphism. Moreover, $\deg(\varphi|_Y) = d^{\dim(Y)}$ by part **(a)**. But since φ is restricted to Y is the identity, we have $\deg(\varphi|_Y) = 1$. It follows that $\dim(Y) = 0$ and so the set $\mathrm{Fix}(\varphi)$ is finite. Since $\mathrm{Per}_n(\varphi) = \mathrm{Fix}(\varphi^n)$, replacing φ with φ^n we deduce that $\mathrm{Per}_n(\varphi)$ is finite. Finally, since φ is a finite morphism and $\mathrm{PrePer}_{n,m}(\varphi) = \varphi^{-m}(\mathrm{Per}_n(\varphi))$, we conclude that $\mathrm{PrePer}_{n,m}(\varphi)$ is finite. $\square$

2.2 Endomorphisms of $\mathbb{P}^N_k$

Let $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ be a surjective morphism; thus $\varphi^*\mathcal{O}(1) \simeq \mathcal{O}(d)$ for some integer $d \geq 1$. If $d = 1$ then φ is an automorphism, and thus $\deg(\varphi) = 1$. On the other hand, if $d \geq 2$ then the triple $(\mathbb{P}^N_k, \varphi, \mathcal{O}(1))$ is a polarized algebraic dynamical system in the sense of §2.1, and $\deg(\varphi) = d^N$ by Proposition 2 **(a)**.

Concretely, choose homogeneous coordinates $\mathbf{x} = (x_0, x_1, \dots, x_n)$ on $\mathbb{P}^N_k$, and let

$$\Phi : k^{N+1} \to k^{N+1} \qquad \Phi(\mathbf{x}) = (\Phi_0(\mathbf{x}), \Phi_1(\mathbf{x}), \dots, \Phi_N(\mathbf{x})) \tag{2.2}$$

be a map defined by $N + 1$ homogeneous forms $\Phi_n(\mathbf{x}) \in k[\mathbf{x}]$ of degree d.

We say Φ is *nonsingular* if $\Phi(\mathbf{x}) \neq \mathbf{0}$ for all nonzero $\mathbf{x} \in \bar{k}^{N+1}$. Such a map Φ determines a morphism $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ with $\varphi^*\mathcal{O}(1) = \mathcal{O}(d)$. We call Φ a *model* for φ with respect to $\mathbf{x}$ and we will sometimes write this map as $\Phi(\mathbf{x})$ to indicate the dependence on our choice of coordinates $\mathbf{x}$ on $\mathbb{P}^N_k$. Any surjective morphism $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ has such a model $\Phi(\mathbf{x})$ with respect to $\mathbf{x}$ and if $\Psi(\mathbf{x})$ and $\Phi(\mathbf{x})$ are two models for φ with respect to $\mathbf{x}$, then $\Psi(\mathbf{x}) = c\Phi(\mathbf{x})$ for some nonzero constant $c \in k$.

If $\mathbf{y} = (y_0, y_1, \ldots, y_N)$ is another choice of coordinates on $\mathbb{P}^N_k$, then $\Gamma(\mathbf{x}) = \mathbf{y}$ for some $\Gamma \in \mathrm{GL}_{N+1}(k)$. If $\Psi(\mathbf{y})$ is a model for a morphism φ with respect to the coordinates $\mathbf{y}$, then $\Gamma^{-1} \circ \Psi \circ \Gamma(\mathbf{x})$ is a model for φ with respect to $\mathbf{x}$.

2.3 The Resultant

Fix integers $N \geq 1$ and $d \geq 1$. Let $\Phi : k^{N+1} \to k^{N+1}$ be a map defined as in §2.2 by $N+1$ homogeneous forms $\Phi_n(\mathbf{x}) \in k[\mathbf{x}]$ of common degree $d \geq 1$ in the variables $\mathbf{x} = (x_0, x_1, \ldots, x_N)$. The resultant, $\mathrm{Res}(\Phi)$, of the map Φ is a certain homogeneous integral polynomial in the coefficients of the forms Φ_n; for the definition see [22] §82 or [16]. For example, when $d = 1$ we may view Φ as an $(N+1) \times (N+1)$ matrix, and $\mathrm{Res}(\Phi) = \det(\Phi)$. The following proposition states the most basic property of the resultant.

Proposition 3. $\mathrm{Res}(\Phi) = 0$ *if and only if* $\Phi(\mathbf{x}) = 0$ *for some nonzero*

$\mathbf{x} \in \bar{k}^{N+1}$

Proof. See [22], §82. □

2.4 Non-archimedean fields and reduction

Let $\mathbb{K}$ denote a field which is endowed with a nontrivial, non-archimedean absolute value $|\cdot|$. We denote by $\mathbb{K}^\circ = \{\alpha \in \mathbb{K}| \ |\alpha| \leq 1\}$ the valuation ring of $\mathbb{K}$, by $\mathbb{K}^{\circ\circ} = \{\alpha \in \mathbb{K}| \ \|\alpha| < 1\}$ the maximal idea of $\mathbb{K}^\circ$ and by $\widetilde{\mathbb{K}} = \mathbb{K}^\circ / \mathbb{K}^{\circ\circ}$ the residue field of $\mathbb{K}$. The most important example here is when $\mathbb{K}$ is the function field, $K = k(C)$, of the curve C over an algebraically closed constant field k, with the absolute value $|\cdot|_v = e^{-\mathrm{ord}_v(\cdot)}$ associated to a (closed) point $v \in C$. In this case, $\mathbb{K}^\circ$ coincides with the ring, $\mathcal{O}_v$, of regular functions at v and the residue field $\widetilde{\mathbb{K}}$ is isomorphic to the constant field k via the evaluation map $\mathcal{O}_v \to k$. Here we will assume $\widetilde{\mathbb{K}}$ is algebraically closed because we are concerned with the case that $\mathbb{K} = K = k(C)$ where k is algebraically closed.

Let $N \geq 1$ be an integer and let $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ denote $N + 1$ variables in $\mathbb{K}$. We define the norm $\|\cdot\|$ on $\mathbb{K}^{N+1}$ by

$$\|\mathbf{x}\| = \max\{|x_0|, |x_1|, \ldots, |x_N|\}$$

and the unit ball in $\mathbb{K}^{N+1}$ by $B(0,1) = \{\mathbf{x} \in \mathbb{K}^{N+1}| \ \|\mathbf{x}\| \leq 1\}$. Given a map $F : \mathbb{K}^{N+1} \to \mathbb{K}^M$ defined by M polynomials $F_m(\mathbf{x}) \in \mathbb{K}[x]$, denote by $H(F)$

the maximum absolute value of the coefficients of F. Thus, $H(F) \leq 1$, if and only if F has coefficients in the valuation ring $\mathbb{K}^\circ$.

Proposition 4. *Let $\mathbb{K}$ be an algebraically closed non-archimedean field and let $F : \mathbb{K}^{N+1} \to \mathbb{K}$ be a map defined by a polynomial $F(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$. Then $H(F) = \max\{|F(\mathbf{x})| \mid \mathbf{x} \in B(0,1)\}$.*

Proof. By normalizing F, we may assume without loss of generality that $H(F) = 1$ and then $|F(\mathbf{x})| \leq 1$ for all $\mathbf{x} \in B(0,1)$ by the ultrametric inequality. Since F has coefficients in $\mathbb{K}^\circ$ it reduces to a polynomial $\widetilde{F}(\mathbf{x}) \in \widetilde{\mathbb{K}}[\mathbf{x}]$ over the residue field $\widetilde{\mathbb{K}}$. Since $H(F) = 1$, the reduced polynomial $\widetilde{F}(\mathbf{x})$ is nonzero and therefore nonvanishing on a nonempty Zariski-open subset of $\widetilde{\mathbb{K}}^{N+1}$ (note that $\widetilde{\mathbb{K}}$ is algebraically closed). Select some $\widetilde{\mathbf{x}}_0 \in \widetilde{\mathbb{K}}^{N+1}$ such that $\widetilde{F}(\widetilde{\mathbf{x}}_0) \neq 0$ and let $\mathbf{x}_0 \in B(0,1)$ be a point which reduces to $\widetilde{x}_0$. Thus $|F(\mathbf{x}_0)| = 1$. $\square$

Let $\Phi : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}$ be a homogeneous map of degree $d \geq 1$. Note that by Proposition 3, the map Φ is nonsingular if and only if $\operatorname{Res}(\Phi) \neq 0$. We say that the map Φ has *nonsingular reduction* over $\mathbb{K}$ if Φ is defined over $\mathbb{K}^\circ$ and if the induced map $\widetilde{\Phi} : \widetilde{\mathbb{K}}^{N+1} \to \widetilde{\mathbb{K}}^{N+1}$ over the residue field, $\widetilde{\mathbb{K}}$, is nonsingular. By Proposition 3, the map Φ has nonsingular reduction if and only if Φ has coefficients $\mathbb{K}^\circ$ and $|\operatorname{Res}(\Phi)| = 1$.

Lemma 5. *Let* $\Phi : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}$ *be a nonsingular homogeneous polynomial map of degree* $d \geq 1$*. Then there exists positive constants* $C_1,\ C_2$*, depending on* Φ*, such that*

$$C_1 \|\mathbf{x}\|^d \leq \|\Phi(\mathbf{x})\| \leq C_2 \|\mathbf{x}\|^d$$

for all $\mathbf{x} \in \mathbb{K}^{N+1}$*. If* Φ *has coefficients in* $\mathbb{K}^\circ$ *then we may take* $C_1 = |\mathrm{Res}(\Phi)|$ *and* $C_2 = 1$*. In particular, if* Φ *has nonsingular reduction then* $\|\Phi(\mathbf{x})\| = \|\mathbf{x}\|^d$ *for all* $\mathbf{x} \in \mathbb{K}^{N+1}$*.*

Proof. The upper bound follows immediately from the ultrametric inequality. The lower bound follows from the basic properties of the resultant; see Proposition 8 of Kawaguchi-Silverman [17]. □

Let $\varphi : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ be a morphism of degree at least one. We say φ has *good reduction* over $\mathbb{K}$ if there exists of choice of coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_{\mathbb{K}}$ such that φ extends to an endomorphism of the associated integral model $\mathbb{P}^N_{\mathbb{K}^\circ}$ of $\mathbb{P}^N_{\mathbb{K}}$. Equivalently, φ has good reduction over $\mathbb{K}$ if there exists a choice of coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_{\mathbb{K}}$, and a model $\Phi(\mathbf{x})$ for φ with respect to $\mathbf{x}$, such that $\Phi(\mathbf{x})$ has nonsingular reduction as defined above. Such a model determines a reduced morphism $\widetilde{\varphi} : \mathbb{P}^N_{\widetilde{\mathbb{K}}} \to \mathbb{P}^N_{\widetilde{\mathbb{K}}}$ over the residue field $\widetilde{\mathbb{K}}$.

Lemma 6. *Let* $\varphi : \mathbb{P}^N_{\mathbb{K}} \to \mathbb{P}^N_{\mathbb{K}}$ *be a morphism of degree at least two. Let* $\Phi(\mathbf{x})$ *and* $\Psi(\mathbf{y})$ *be models for* φ *respect to the coordinates* $\mathbf{x}$ *and* $\mathbf{y}$ *on* $\mathbb{P}^N_{\mathbb{K}}$ *respectively, where* $\Gamma(\mathbf{x}) = \mathbf{y}$ *and* $\Phi(\mathbf{x}) = \Gamma^{-1} \circ \Psi \circ \Gamma(\mathbf{x})$ *for some* $\Gamma \in \mathrm{GL}_{N+1}(\mathbb{K})$*. If both* $\Phi(\mathbf{x})$ *and* $\Psi(\mathbf{y})$ *have nonsingular reduction, then* $\Gamma \in \mathrm{GL}_{N+1}(\mathbb{K}^\circ)$*.*

Proof. Replacing $\mathbb{K}$ with $\overline{\mathbb{K}}$ (and extending the absolute value $|\cdot|$ to $\overline{\mathbb{K}}$), we may assume without loss of generality that $\mathbb{K}$ is algebraically closed. Note that by Lemma 5 we have $\|\Phi(\mathbf{x})\| = \|\mathbf{x}\|^d$ for all $\mathbf{x} \in \mathbb{K}^{N+1}$, since $\Phi(\mathbf{x})$ has nonsingular reduction. The same holds true for $\Psi(\mathbf{y})$. By Proposition 4, we may select a point $\mathbf{x}_0 \in B(0,1)$ where the maximum $H(\Gamma) = \max\{\|\Gamma(\mathbf{x})\| \mid \mathbf{x} \in B(0,1)\}$ is achieved. Therefore

$$H(\Gamma)^d = \|\Gamma(\mathbf{x}_0)\|^d = \|\Psi(\Gamma(\mathbf{x}_0))\| = \|\Gamma(\Phi(\mathbf{x}_0))\| \leq H(\Gamma).$$

The last inequality follows from Proposition 4 and the fact that $\Phi(\mathbf{x}_0) \in B(0,1)$. Since $d \geq 2$, we conclude that $H(\Gamma) \leq 1$, which means that Γ has coefficients in $\mathbb{K}^\circ$, by symmetry Γ^{-1} has coefficients in $\mathbb{K}^\circ$ as well, and therefore, $\Gamma \in GL_{N+1}(\mathbb{K}^\circ)$. $\square$

2.5 Extending K

Let $K = k(C)$ and $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be as in the statement of Theorem 1. By their definitions, the properties of isotriviality and everywhere good reduction are invariant when the function field $K = k(C)$ is replaced by a finite extention $K' = k'(C')$, where k'/k is an extension of the constant field and C' is a curve with a finite map $C' \to C$. Therefore, during the proof of Theorem 1 we may replace K with such an extension K' at any time with no loss of generality.

2.6 The "only if" Direction

Let $K = k(C)$ and let $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be a morphism with $\deg(\varphi) \geq 1$ as in the statement of Theorem 1. Assuming that φ is isotrivial, it is easy to see that it must have potential good reduction at each place $v \in M_K$. Extending K if necessary, as described in §2.5, we may assume there exists coordinates $x = (x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_K$ such that φ has a model $\Phi(\mathbf{x})$ with respect to $\mathbf{x}$ with coefficients in k; thus $\operatorname{Res}(\Phi)$ is a nonzero element of k. Given a place v of K, note that $\Phi(\mathbf{x})$ is defined over $\mathcal{O}_v$, since $k \subset \mathcal{O}_v$, and $\operatorname{Res}(\Phi) \in k^\times \subset \mathcal{O}_v^\times$. If follows that φ has good reduction at v.

2.7 Automorphisms of $\mathbb{P}^N_K$

Let $K = k(C)$ be a function field and let $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be an automorphism; thus $\varphi^*\mathcal{O}(1) \simeq \mathcal{O}(1)$ and $\deg(\varphi) = 1$. Note that $(\mathbb{P}^N_K, \varphi, \mathcal{O}(1))$ is not a polarized algebraic dynamical system in the nomenclature of §2.1, since it fails the requirement that $d \geq 2$. However, Theorem 1 still holds in this case. To prove it we use only basic facts of linear algebra.

Proof of Theorem 1 for automorphisms. In view of §2.6 it suffices to prove the "if" part of the theorem. Let $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be an automorphism with potential good reduction at all places $v \in M_K$. Let $\Phi(\mathbf{x}) = (\Phi_0(\mathbf{x}), \Phi_1(\mathbf{x}), \ldots, \Phi_N(\mathbf{x}))$ be a model for φ with respect to the choice of coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_K$. Thus each $\Phi_n(\mathbf{x}) \in K[\mathbf{x}]$ is a linear form, and we may view $\Phi : K^{N+1} \to K^{N+1}$ as a nonsingular $(N+1)\times(N+1)$ matrix over K. Recall the remarks of §2.5. The notions of isotriviality and potential good reduction are invariant under replacing K by a finite extension of K. Therefore, we may assume without loss of generality that K contains an $(N+1)$-th root of $\det(\Phi)$, and re-normalizing Φ we may further assume that $\det(\Phi) = 1$. Again extending K if necessary, we may assume that K contains all the eigenvalues of Φ. Finally, by changing coordinates we may assume that Φ is in Jordan cononical form. We are going to show that the eigenvalues of Φ are in the constant field k of K, showing that Φ is defined over k, and completing the proof that φ is isotrivial.

Let $v \in M_K$ be a place of K and let K'/K be a finite extension such that φ has good reduction at a place v' of K' over v. It follows that there exists a model $\Psi(\mathbf{y})$ for φ, with respect to some choice of coordinates $\mathbf{y} = (y_0, y_1, \ldots, y_N)$ on $\mathbb{P}^N_{K'}$, such that Ψ has coefficients in $\mathcal{O}_{v'}$ and $\det(\Psi) \in \mathcal{O}_{v'}^\times$. Let $\Gamma \in \mathrm{GL}_{N+1}(K')$ be the change-of-coordinate matrix satisfying $\Gamma(\mathbf{x}) = \mathbf{y}$. Thus $\Theta(\mathbf{x}) := \Gamma^{-1} \circ \Psi \circ \Gamma(\mathbf{x})$ is another model for φ with respect to the coordinates $\mathbf{x}$, whereby $\Phi(\mathbf{x}) = c\Theta(\mathbf{x})$ for some nonzero $c \in K'$. We have

$$1 = \det(\Phi) = c^{N+1}\det(\Theta) = c^{N+1}\det(\Psi),$$

and since $\det(\Psi) \in \mathcal{O}_{v'}^\times$ we conclude that $c \in \mathcal{O}_{v'}^\times$ as well. Letting $P_\Phi(T) = \det(TI - \Phi)$ denote the characteristic polynomial of Φ, and similarly for Ψ and Θ, we have

$$P_\Phi(T) = P_{c\Theta}(T) = c^{N+1}P_\Theta(T/c) = c^{N+1}P_\Psi(T/c).$$

Since $P_\Psi(T) \in \mathcal{O}_{v'}[T]$ we deduce that $P_\Phi(T) \in \mathcal{O}_{v'}[T]$ as well, and so in fact $P_\Phi(T) \in \mathcal{O}_v[T]$ since $P_\Phi(T)$ is defined over the smaller field K. Since $P_\Psi(T) \in \mathcal{O}_v[T]$ is monic and splits over K we conclude that the eigenvalues of Φ are in $\mathcal{O}_v$. As $v \in M_K$ is arbitrary, the eigenvalues of Φ are in $\mathcal{O}_v$ at all $v \in M_K$, so they must be in the constant field k as desired.

Chapter 3

Geometric Proof of Theorem 1

3.1 Overview

Throughout this section we let k denote an algebraically closed field, and we fix integers $N \geq 1$ and $d \geq 2$. In this section we sill study the space $\mathrm{End}_{N,d}(k)$ of endomorphisms φ of $\mathbb{P}_k^N$ with $\varphi^*\mathcal{O}(1) \simeq \mathcal{O}(d)$. Generalizing a result of Silverman [20], we show that in §3.3 that the quotient $\mathcal{M}_{N,d}(k) = \mathrm{End}_{N,d}/\mathrm{PGL}_{N+1}(k)$ of this space by the automorphism group of $\mathbb{P}_k^N$ is an affine k-variety. In §3.4 we will give the geometric proof of Theorem 1.

3.2 The space of endomorphisms

Let $\mathrm{Sym}^d(k^{N+1})^{N+1}$ be the space of homogeneous maps $k^{N+1} \to k^{N+1}$ of degree d. Explicitly, an element of this space is given by an $(N+1)$-tuple $\Phi(\mathbf{x}) = (\Phi_0(\mathbf{x}), \Phi_1(\mathbf{x}), \ldots, \Phi_N(\mathbf{x}))$, where each $\Phi_n(\mathbf{x}) \in k[\mathbf{x}]$ is a homogenous form of degree d in the variables $\mathbf{x} = (x_0, x_1, \ldots, x_N)$.

Given a choice of coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_k$, recall from §2.2 that each endomorphism $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ with $\varphi^*\mathcal{O}(1) \simeq \mathcal{O}(d)$ has a model $\Phi(\mathbf{x}) \in \mathrm{Sym}^d(k^{N+1})^{N+1}$, which is unique up to scaling by a constant $c \in k^\times$ and, moreover, $\mathrm{Res}(\Phi) \neq 0$. Thus a model of φ corresponds to a unique point in the projective space $\mathbb{P}(\mathrm{Sym}^d(k^{N+1})^{N+1})$. Note from §2.3, $\mathrm{Res}(\Phi)$ itself, is a homogeneous form in the coefficients of Φ. Thus the condition $\mathrm{Res}(\Phi) = 0$ defines a closed hypersurface, $\mathrm{Res}_{N,d}(k)$, in $\mathbb{P}(\mathrm{Sym}^d(k^{N+1})^{N+1})$ (see [13] §3.3 or [16] §3).

Due to these remarks and Proposition 2 **(a)**, we define the *space of endomorphisms* of $\mathbb{P}^N_k$ of degree d^N by

$$\mathrm{End}_{N,d}(k) := \mathbb{P}(\mathrm{Sym}^d(k^{N+1})^{N+1}) \backslash \mathrm{Res}_{N,d}(k).$$

Therefore, $\mathrm{End}_{N,d}(k)$ is an affine open subvariety of $\mathbb{P}(\mathrm{Sym}^d(k^{N+1})^{N+1})$.

Note that the correspondence between morphisms $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree d^N and points of $\mathrm{End}_{N,d}(k)$ depend on our initial choice of coordinates, $\mathbf{x}$. Moreover, changing coordinates on $\mathbb{P}^N_k$ corresponds to conjugating φ by an element γ of $\mathrm{PGL}_{N+1}(k)$. This leads us to consider the action of $\mathrm{PGL}_{N+1}(k)$ on $\mathrm{End}_{N,d}(k)$ by $(\gamma, \varphi) \mapsto \gamma^{-1}\varphi\gamma$. Therefore, the “correct” coordinate-independent space parameterizing morphisms $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree d^N

is the quotient

$$\mathcal{M}_{N,d} := \mathrm{End}_{N,d}(k)/\mathrm{PGL}_{N,d}(k) \tag{3.1}$$

of this action. While this quotient can be defined set-theoretically, it is not guaranteed that $\mathcal{M}_{N,d}(k)$ is a variety over k or, furthermore, that the fibers of the quotient map $\mathrm{End}_{N,d}(k) \to \mathcal{M}_{N,d}(k)$ are closed. In the next section we will show that the quotient $\mathcal{M}_{N,d}(k)$ does in fact naturally carry the structure of a k-variety, and the associated quotient map is a morphism.

3.3 Existence of a geometric quotient

We recall the definition from geometric invariant theory. Let $\alpha : G \times X \to X$ be an action of an algebraic group G over k on a k-variety X. A pair (Y, π) consisting of a k-variety Y and a morphism $\pi : X \to Y$ is called a *geometric quotient* of X by the action of G if it satisfies the following properties:

(i) The diagram

$$\begin{tikzcd} G \times X \arrow[r, "\alpha"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

commutes.

(ii) π is surjective, and the image of $(\alpha, p_2) : G \times X \to X \times X$ is $X \times_Y X$.

(iii) A subset $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open in X.

(iv) The fundemental sheaf $\mathcal{O}_Y$ is the subsheaf of $\pi_*(\mathcal{O}_X)$ consisting of invariant functions.

Refer to [11] and [19] for additional definitions and background material.

The purpose of this section is to show that the action of $\mathrm{PGL}_{N+1}(k)$ on $\mathrm{End}_{N,d}(k)$ has a geometric quotient. We begin by recording several preliminary results. Recall that the stabilizer of a point $\varphi \in \mathrm{End}_{N,d}$ by the action of $\mathrm{PGL}_{N+1}(k)$ is the subgroup $\mathrm{St}(\varphi) = \{\gamma \in \mathrm{PGL}_{N+1}(k) | \gamma^{-1}\varphi\gamma = \varphi\}$. Given a subset S of $\mathbb{P}^N_k$, we say, S is in general position if every nonempty finite subset T of S with $|T| \leq N+1$ is linearly independent. The following lemma is well known.

Lemma 7. *Let S be a subset of $\mathbb{P}^N_k$ in general position with $|S| = N+2$ points. If $\gamma \in \mathrm{PGL}_{N+1}(k)$ is an automorphism of $\mathbb{P}^N_k$ and $\gamma(P) = P$ for all $P \in S$, then γ is the identity automorphism.*

Proposition 8. *Let $\varphi \in \mathrm{End}_{N+1}(k)$. The stabilizer $\mathrm{St}(\varphi)$ of φ by the action of PGL_{N+1} is finite.*

Proof. By Proposition 2 **(c)** the set $\mathrm{PrePer}(\varphi)$ of φ-preperiodic points in $\mathbb{P}^N_k$ is Zariski-dense. If follows that there exists a set $S = \{P_0, P_1, \ldots, P_{N+1}\}$ of $N+2$ preperiodic points in general position. Explicitly, if r is at most $N-1$

and if $P_0, ..P_r$ are $r+1$ linearly independent preperiodic points, we choose a point P_{r+1} in the projective N-space, preperiodic and not in the linear space of dimension r genarated by the P_i for i less than r. This is possible by Proposition 2 **(c)**. Then let $P_0, P_1, \ldots, P_N$ be any $N+1$ linearly independent preperiodic points and let P_{N+1} be a preperiodic point which is not on any of the $N+1$ hyperplanes generated by N-point subsets of $\{P_0, P_1, \ldots, P_N\}$.

Each preperiodic point P_i lies in $\mathrm{PrePer}_{n_i,m_i}(\varphi)$ for some integers $n_i \geq 1$ and $m_i \geq 0$. Note that given $\gamma \in \mathrm{St}(\varphi)$, we have $\gamma^{-1}\varphi\gamma = \varphi$, which implies $\gamma^{-1}\varphi^r\gamma = \varphi^r$ and $\gamma\varphi^r = \varphi^r\gamma$ for any positive integer r. If P is in $\mathrm{PrePer}_{n_i,m_i}(\varphi)$ then

$$\varphi^{m_i}\gamma(P) = \gamma\varphi^{m_i}(P) = \gamma\varphi^{n_i+m_i}(P) = \varphi^{n_i+m_i}\gamma(P),$$

so we have $\gamma(P) \in \mathrm{PrePer}_{n_i,m_i}(\varphi)$ as well. Thus $\mathrm{St}(\varphi)$ acts on each finite set, $\mathrm{PrePer}_{n_i,m_i}(\varphi)$ and we obtain a group homomorphism.

$$\mathrm{St}(\varphi) \to \prod_{i}^{N+1} \mathrm{Perm}(\mathrm{PrePer}_{n_i,m_i}(\varphi)), \tag{3.2}$$

where $\mathrm{Perm}(\mathrm{PrePer}_{n_i,m_i}(\varphi))$ denotes the group of permutations of the set $\mathrm{PrePer}_{n_i,m_i}(\varphi)$. If γ is the kernel of the map (3.2), then in particular it fixes each point in the set $S = \{P_0, P_1, \ldots, P_{N+1}\}$, whereby γ is the identity by Lemma 7. Thus the map (3.2) is injective, and since each set $\mathrm{PrePer}_{n_i,m_i}(\varphi)$, it follows that $\mathrm{St}(\varphi)$ is finite. □

Corollary 9. *The action of* $\mathrm{PGL}_{N+1}(k)$ *on* $\mathrm{End}_{N,d}(k)$ *is closed.*

Proof. By an argument on p. 10 of [19], if for each $\varphi \in \mathrm{End}_{N,d}(k)$ there exists an open neighborhood U of φ where the dimension of the stabilizer $\mathrm{St}(\psi)$ is constant for all $\psi \in U$, then the action by $\mathrm{PGL}_{N+1}(k)$ is closed. Since $\mathrm{St}(\varphi)$ is zero-dimensional for all $\varphi \in \mathrm{End}_{N,d}(k)$ by Proposition 8, the action by $\mathrm{PGL}_{N+1}(k)$ is closed. □

Proposition 10. *A geometric quotient of* $\mathrm{End}_{N,d}(k)$ *by* $\mathrm{PGL}_{N+1}(k)$ *exists, and moreover it is affine.*

Proof. The proof is just an application of Amplification 1.3 of [19]. Recall $\mathrm{End}_{N,d}(k)$ is affine and $\mathrm{PGL}_{N+1}(k)$ is reductive. Therefore an affine geometric quotient exists if and only if the action of $\mathrm{PGL}_{N+1}(k)$ is closed, which is the case by Corollary 9. □

We let $(\mathcal{M}_{N,d}, \pi)$ denote the geometric quotient of $\mathrm{End}_{N,d}(k)$ by the action of $\mathrm{PGL}_{N+1}(k)$. Note that the set of points $\mathcal{M}_{N,d}(k)$ on this quotient coincide with the set-theoretic quotient defined in (3.1). Note that the case of Proposition 10 where $N = 1$ follows immediately from Silverman [20], Theorem 1.1.

3.4 The geometric proof of Theorem 1

In view of §2.6 and §2.7, it suffices to consider the "if" direction of the statement for morphisms $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ with $\varphi^*\mathcal{O}(1) \simeq \mathcal{O}(d)$ for $d \geq 2$. Moreover, by the remarks of §2.5 we may assume without loss of generality that the constant field k of $K = k(C)$ is algebraically closed.

Suppose that φ has potential good reduction at all places v of K. Since φ has good reduction at all but finitely many places of K, by extending K we may assume without loss of generality, that φ has good reduction at all places v of K.

Let v be a place of K; in other words $v \in C$ is a (closed) point. Recall that the surjective map $\mathcal{O}_v \to k$ given by evaluation at v induces a canonical isomorphism between the residue field of $\mathcal{O}_v$ and the constant field k. Since φ has good reduction at v, there exists a model $\Phi_v(\mathbf{x})$ for φ with respect to a choice of coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_k$ such that the coefficients of $\Phi_v(\mathbf{x})$ are in $\mathcal{O}_v$ and $\mathrm{Res}(\Phi_v(\mathbf{x})) \in \mathcal{O}_v^\times$. Reduction modulo the maximal idea of $\mathcal{O}_v$ defines a map $\widetilde{\Phi}_v : k^{N+1} \to k^{N+1}$ and an associated morphism $\widetilde{\varphi}_v : \mathbb{P}^N_k \to \mathbb{P}^N_k$.

Moreover, the model $\Phi_v(\mathbf{x})$ has nonsingular reduction at all but finitely many points $u \in C$. Therefore, we can find an affine open neighborhood,

$U_v = \mathrm{Spec}(A_v)$ of $v \in C$ such that $\Phi_v(\mathbf{x})$ has coefficients in $A_v \subset K$, where $\Phi_v(\mathbf{x})$ has nonsingular reduction at all points $u \in U_v$. As at the point v, reduction at each point $u \in U_v$ defines a morphism $\widetilde{\varphi}_u : \mathbb{P}^N_k \to \mathbb{P}^N_k$. We obtain a morphism $U_v \to \mathrm{End}_{N,d}$ defined by $u \mapsto \widetilde{\varphi}_u$. The open sets, $\{U_v\}_{v \in C}$, define an open cover of C, and since the curve C is quasi-compact, we can find a finite subcover $\{U_i\}$. Thus for each open set U_i in our finite cover of C, we have the morphisms

$$U_i \to \mathrm{End}_{N,d} \xrightarrow{\pi} \mathcal{M}_{N,d} \tag{3.3}$$

Let $U_{ij} = U_i \cap U_j$ be an intersection between two open sets in the finite cover, $\{U_i\}$, and let $\Phi_i(\mathbf{x})$ and $\Phi_j(\mathbf{y})$ be the models of φ with respect to the open neighborhoods U_i and U_j as described above, respectively. Let $\Gamma \in \mathrm{GL}_{N+1}(K)$ denote the change of coordinate elements satisfying $\Gamma(\mathbf{x}) = \mathbf{y}$. Thus $\Phi_i(\mathbf{x}) = c\Gamma^{-1} \circ \Phi_j \circ \Gamma(\mathbf{x})$ for some $c \in K^\times$. Extending K if necessary, we may assume there exists some $a \in K$ such that $a^{d-1} = c$. Letting $\Gamma' = a\Gamma$, we have $\Phi_i(\mathbf{x}) = (\Gamma')^{-1} \circ \Phi_j \circ \Gamma'(\mathbf{x})$. Therefore, replacing Γ with Γ' and replacing the coordinates $\mathbf{y}$ with $\mathbf{y}' = \Gamma'(\mathbf{x}) = a\mathbf{y}$, we may assume without loss of generality, that $\Phi_i(\mathbf{x}) = \Gamma^{-1} \circ \Phi_j \circ \Gamma(\mathbf{x})$.

Given an arbitrary point $u \in U_{ij} = U_i \cap U_j$, both $\Phi_i(\mathbf{x})$ and $\Phi_j(\mathbf{y})$ have nonsingular reduction at u. This means that $\Gamma \in \mathrm{GL}_{N+1}(\mathcal{O}_u)$ by Lemma 6. Denote by $\widetilde{\Gamma}_u \in \mathrm{GL}_{N+1}(k)$ the reduction of Γ at u and by $\widetilde{\gamma}_u \in \mathrm{PGL}_{N+1}(k)$

the associated automorphism of $\mathbb{P}_k^N$; thus $\widetilde{\Phi}_{u,i} = \widetilde{\Gamma}_u^{-1} \circ \widetilde{\Phi}_{u,j} \circ \widetilde{\Gamma}_u(\mathbf{x})$. Therefore, let $\widetilde{\varphi}_u \in \mathrm{End}_{N,d}(k)$ denote the endomorphism of $\mathbb{P}_k^N$ obtained by reduction of the model $\Phi_i(\mathbf{x})$ at u and likewise define $\widetilde{\varphi}_{u,j} \in \mathrm{End}_{N,d}(k)$ by using the model $\Phi_j(\mathbf{y})$ at u. We deduce that $\widetilde{\varphi}_{u,i} = \widetilde{\gamma}^{-1} \circ \widetilde{\varphi}_{u,j} \circ \widetilde{\gamma}$. We have shown that the image of $u \in U_{ij}$ in $\mathrm{End}_{N,d}$ is well-defined up to $\mathrm{PGL}_{N+1}(k)$-conjugation. That is, it is contained in a unique fiber of the quotient map π. From this, we obtain a morphism $C \to \mathcal{M}_{N,d}$ by the inclusion $U_i \hookrightarrow C$ and (3.3).

The quotient $\mathcal{M}_{N,d}$ is affine by Proposition 10 and C is complete. Hence, the image of $C \to \mathcal{M}_{N,d}$ is a point. By [11] Corollary 6.1, the fiber of this point in $\mathcal{M}_{N,d}$ contains a unique closed $\mathrm{PGL}_{N+1}(k)$-conjugacy class. It follows there exists some $\psi : \mathbb{P}_k^N \to \mathbb{P}_k^N$ in this class, such that φ coincides with the base extension $\psi_K : \mathbb{P}_K^N \to \mathbb{P}_K^N$. Hence, φ is isotrivial.

Chapter 4

Analytic Proof of Theorem 1

4.1 Homogeneous transfinite diameter

Let $\mathbb{K}$ be a non-archimedean field as discussed in §2.4 and let E be a bounded, infinite subset of $\mathbb{K}^{N+1}$. In this section we define the (homogeneous) transfinite diameter $d_\infty(E)$ of the set E. It is a nonnegative number which in in a certain sense is a measure of the size of E. When $N = 1$, this variation on the classical notion of transfinite diameter was introduced and studied by Baker-Rumely [3].

To define $d_\infty(E)$, let $M \geq N + 1$ be an integer, and let $\mathcal{S}_M(E)$ denote the set of subsets of E with exactly $|S| = M$ elements. Given $S \in \mathcal{S}_M(E)$, enumerate by $S_1, S_2, \ldots, S_{J_M}$ the subsets of S with exactly $|S_j| = N + 1$ elements; thus $J_M = \binom{M}{N+1}$. Define

$$\Delta(S) = \prod_{1 \leq j \leq J_M} \det(S_j), \tag{4.1}$$

where $\det(S_j)$ is the determinant of the $(N+1)\times(N+1)$ matrix whose column-vectors are the elements of S_j. Thus each $\det(S_j)$ and the product (4.1) are defined only up to sign. Define the M-diameter of E by

$$d_M(E) = \sup_{S\in\mathcal{S}_M(E)} |\Delta(S)|^{1/J_M}. \tag{4.2}$$

In the following proposition we will show that the sequence $d_M(E)$ is monotone decreasing as $M\to+\infty$, and therefore the limit

$$d_\infty(E) = \lim_{M\to+\infty} d_M(E) \tag{4.3}$$

exists. This limit, $d_\infty(E)$, is called the (homogeneous) transfinite diameter of E. In addition to showing this limit exists, the proposition below summarizes some of the basic properties of the transfinite diameter. Given a set $E\subset\mathbb{K}^{N+1}$, we say E is an *ellipsoid* if it is of the form $E=\Gamma(B(0,1))$ for some $\Gamma\in\mathrm{GL}_{N+1}(\mathbb{K})$.

Proposition 11. *Let $\mathbb{K}$ be a non-archimedean field and let E be a bounded infinite subset of $\mathbb{K}^{N+1}$.*

- **(a)** *The sequence $d_M(E)$ is monotone decreasing, and thus the limit (4.3) exists.*
- **(b)** *If $\Gamma\in\mathrm{GL}_{N+1}(\mathbb{K})$, then $d_\infty(\Gamma(E)) = |\det(\Gamma)|d_\infty(E)$.*

For the remainder of this proposition, assume that $\mathbb{K}$ is algebraically closed.

(c) *If* E *contains the unit ball* $B(0,1)$*, then* $d_\infty(E) \geq 1$*; moreover* $d_\infty(B(0,1)) = 1$.

(d) *If* E *is an ellipsoid such that* $B(0,1) \subseteq E$ *and* $d_\infty(E) = 1$*, then* $E = B(0,1)$.

(e) *If* E *is an ellipsoid which contains the standard unit basis elements* $\mathbf{e}_0 = (1,0,\ldots,0)$*,* $\mathbf{e}_1 = (0,1,\ldots,0)$*,* $\ldots$*,* $\mathbf{e}_N = (0,\ldots,0,1)$*, then* $B(0,1) \subseteq E$.

Proof. **(a)** The following is a variation on the standard argument for the existence of the transfinite diameter which generalizes the proof given for $N = 1$ in [3], Lemma 3.10. Fix $M \geq N+1$ and $\epsilon > 0$. By the definition (4.2) we may choose a set $S = \{\mathbf{x}(1), \ldots, \mathbf{x}(M+1)\}$ of $M+1$ elements in E such that $|\Delta(S)| \geq (d_{M+1}(E) - \epsilon)^{J_M}$. For each $1 \leq m \leq M+1$ denote by $T_m = S \backslash \{\mathbf{x}(m)\}$; thus $|T_m| = M$ and $|\Delta(T_m)| \leq d_M(E)^{J_M}$ by (4.2). Observe that

$$\prod_{1 \leq m \leq M+1} |\Delta(T_m)| = |\Delta(S)|^{M-N},$$

since the left-hand-side is the product of $|\det(S_j)|$ over each $(N+1)$-element subset S_j of S exactly $(M+1)-(N+1) = M-N$ times, which is precisely the same as the right-hand side. Thus

$$d_{M+1}(E) - \epsilon \leq |\Delta(S)|^{1/J_{M+1}} \leq d_M(E)^{J_M(M+1)/J_{M+1}(M-N)} = d_M(E).$$

Since $\epsilon > 0$ is arbitrary, we conclude that $d_{M+1}(E) \leq d_M(E)$.

(b) We have $\Delta(\Gamma(S)) = \pm \det(\Gamma)^{J_M} \Delta(S)$, hence

$$d_M(\Gamma(E)) = |\det(\Gamma)| d_M(E)$$

for all M, and the claim follows.

(c) Suppose that $B(0,1) \subseteq E$. Let $\{\widetilde{\mathbf{x}}(m)\}_{m=1}^{\infty}$ be an infinite sequence of points in $\widetilde{\mathbb{K}}^{N+1}$ such that any $N+1$ terms of the sequence are linearly independent over $\widetilde{\mathbb{K}}$. [To see that such a sequence exists, let $\widetilde{\mathbf{x}}(1), \widetilde{\mathbf{x}}(2), \dots, \widetilde{\mathbf{x}}(N+1)$ be any basis for $\widetilde{\mathbb{K}}^{N+1}$; we define the rest of the sequence by induction. Suppose that $m \geq N+1$ and that the first m terms $\widetilde{\mathbf{x}}(1), \widetilde{\mathbf{x}}(2), \dots, \widetilde{\mathbf{x}}(m)$ of the sequence have been constructed with the desired linear-independence property. Each choice of N elements in the set $\{\widetilde{\mathbf{x}}(1), \widetilde{\mathbf{x}}(2), \dots, \widetilde{\mathbf{x}}(m)\}$ spans a hyperplane in $\widetilde{\mathbb{K}}^{N+1}$, since $\mathbb{K}$ (and therefore $\widetilde{\mathbb{K}}$), is algebraically closed. We let $\widetilde{\mathbf{x}}(m+1)$ be any element in the complement of the union of these hyperplanes. Clearly any $N+1$ elements of the set $\{\widetilde{\mathbf{x}}(1), \widetilde{\mathbf{x}}(2), \dots, \widetilde{\mathbf{x}}(m+1)\}$ are linearly independent, and by induction on m the sequence exists as claimed.]

For each $m \geq 1$ let $\mathbf{x}(m) \in B(0,1)$ be a point reducing to $\widetilde{\mathbf{x}}(m)$, and fix $M \geq N+1$. Let $S = \{\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(M)\} \in \mathcal{S}_M(E)$, and note by the $\widetilde{\mathbb{K}}$-linear-independence property of the sequence $\{\widetilde{\mathbf{x}}(m)\}_{m=1}^{\infty}$ we have $|\Delta(S)| = 1$. It follows from (4.2) that $d_M(E) \geq 1$ for all $M \geq N+1$, and the inequality

$d_\infty(E) \geq 1$ follows from (4.3).

If $E = B(0,1)$, then the opposite inequality $d_\infty(E) \leq 1$ follows at once from (4.2), (4.3) and the ultrametric inequality.

(d) Since E is an ellipsoid we have $E = \Gamma(B(0,1))$ for some $\mathrm{GL}_{N+1}(\mathbb{K})$, and since $B(0,1) \subseteq E$, we conclude that $\Gamma^{-1}(B(0,1)) \subseteq B(0,1)$. Thus Γ^{-1} maps $B(0,1)$ into itself, and it follows from Proposition 4 that Γ^{-1} has coefficients in $\mathbb{K}^\circ$. In addition,

$$1 = d_\infty(E) = d_\infty(\Gamma(B(0,1))) = |\det(\Gamma)| d_\infty(B(0,1)) = |\det(\Gamma)|$$

by part **(b)** of this Proposition. We conclude that Γ^{-1}, and therefore Γ, is an element of $\mathrm{GL}_{N+1}(\mathbb{K}^\circ)$. It follows that $E = \Gamma(B(0,1)) = B(0,1)$.

(e) Again $E = \Gamma(B(0,1))$ for some $\Gamma \in \mathrm{GL}_{N+1}(\mathbb{K})$, and thus $\Gamma^{-1}(\mathbf{e}_n) \in B(0,1)$ for all n. Thus given an arbitrary $\mathbf{x} \in B(0,1)$ we have

$$\mathbf{x} = \sum_{n=0}^{N} x_n \mathbf{e}_n = \Gamma\left(\sum_{n=0}^{N} x_n \Gamma^{-1}(\mathbf{e}_n)\right) \in \Gamma(B(0,1)) = E;$$

here the containment follows from the ultrametric inequality and the fact that $x_n \in \mathbb{K}^\circ$ and $\Gamma^{-1}(\mathbf{e}_n) \in B(0,1)$. Thus $B(0,1) \subseteq E$. $\square$

4.2 Homogeneous local height functions and filled Julia Sets

We now define two dynamical objects associated to each homogeneous map $\Phi : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}$ of degree $d \geq 2$. The *homogeneous local height function* $\hat{H}_\Phi : \mathbb{K}^{N+1}/\{\mathbf{0}\} \to \mathbb{R}$ associated to Φ is defined by

$$\hat{H}_\Phi(\mathbf{x}) = \lim_{\ell \to +\infty} \frac{1}{d^\ell} \log \|\Phi^\ell(\mathbf{x})\|, \tag{4.4}$$

where $\Phi^\ell = \Phi \circ \cdots \circ \Phi$ denotes the map Φ composed with itself ℓ times. It is easy to show, using Lemma 5, that $\hat{H}_\Phi$ defines a continuous real-valued function on $\mathbb{K}^{N+1}/\{\mathbf{0}\}$, and we may extend the definition of $\hat{H}_\Phi$ by setting $\hat{H}_\Phi(\mathbf{0}) = -\infty$. In the case, $N = 1$, the definition and basic properties of these function were studied in [3] and were generalized and further studied by Kawaguchi-Silverman [17] and [18]. The *homogeneous filled Julia set* associated to Φ is the set

$$F_\Phi = \{\mathbf{x} \in \mathbb{K}^{N+1} \mid \sup_{\ell \geq 1} \|\Phi^\ell(\mathbf{x})\| < +\infty\} \tag{4.5}$$

of points whose forward iterates remain bounded. We summarize the basic properties of $\hat{H}_\Phi$ and F_Φ in the following proposition.

Proposition 12. *Let $\mathbb{K}$ be a non-archimedean field, let $\Phi : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}$ be a nonsingular homogeneous map of degree $d \geq 2$, let $c \in \mathbb{K}^\times$ and let $\Gamma \in \mathrm{GL}_{N+1}(\mathbb{K})$. Then the following identities hold:*

(a) $\hat{H}_\Phi(c\mathbf{x}) = \hat{H}_\Phi(\mathbf{x}) + \log|c|$*;*

(b) $\hat{H}_{c\Phi}(\mathbf{x}) = \hat{H}_\Phi(\mathbf{x}) + \frac{1}{d-1}\log|c|$*;*

(c) $\hat{H}_{\Gamma^{-1}\circ\Phi\circ\Gamma}(\mathbf{x}) = \hat{H}_\Phi(\Gamma(\mathbf{x}))$*;*

(d) $F_\Phi = \{\mathbf{x}\in\mathbb{K}^{N+1} \mid \hat{H}_\Phi(\mathbf{x}) \le 0\}$*;*

(e) $F_{c\Phi} = c^{-1/(d-1)}F_\Phi$*, assuming* $c^{-1/(d-1)}\in\mathbb{K}$*;*

(f) $F_{\Gamma^{-1}\circ\Phi\circ\Gamma} = \Gamma^{-1}(F_\Phi)$.

Proof. **(a)** This follows immediately from the definition (4.4).

(b) Note that

$$(c\Phi)^\ell(\mathbf{x}) = c^{d^{\ell-1}+d^{\ell-2}+\cdots+d+1}\Phi^\ell(\mathbf{x}) = c^{(d^\ell-1)/(d-1)}\Phi^\ell(\mathbf{x}),$$

and therefore

$$\frac{1}{d^\ell}\log\|(c\Phi)^\ell(\mathbf{x})\| = \frac{1}{d^\ell}\log\|(\Phi)^\ell(\mathbf{x})\| + \frac{d^\ell-1}{d^\ell(d-1)}\log|c|.$$

Letting $\ell\to+\infty$ gives the desired identity.

(c) By Lemma 5 applied to the map $\Gamma^{-1}:\mathbb{K}^{N+1}\to\mathbb{K}^{N+1}$ we have

$$\frac{1}{d^\ell}\log\|(\Gamma^{-1}\circ\Phi\circ\Gamma)^\ell(\mathbf{x})\| = \frac{1}{d^\ell}\log\|\Gamma^{-1}(\Phi^\ell(\mathbf{y}))\| = \frac{1}{d^\ell}(\log\|\Phi^\ell(\mathbf{y})\| + O(1))$$

where $\mathbf{y}=\Gamma(\mathbf{x})$ and $O(1)$ denotes a function which is bounded as $\ell\to+\infty$. Letting $\ell\to+\infty$ establishes the desired identity.

(d) If $\mathbf{x}\in F_\Phi$ then the iterates $\Phi^\ell(\mathbf{x})$ are bounded and $\hat{H}_\Phi(\mathbf{x})\le 0$ follows immediately from the definition (4.4). Conversely, suppose that $\mathbf{x}\notin F_\Phi$. Let

$T > 0$ be a parameter, and by assumption we have $\|\Phi^{\ell_0}(\mathbf{x})\| > T$ for some ℓ_0 depending on T. For each $\ell > \ell_0$ we then have

$$\begin{aligned}
\|\Phi^{\ell}(\mathbf{x})\| &\geq C_1^{1+d+d^2+\cdots+d^{\ell-\ell_0-1}}\|\Phi^{\ell_0}(\mathbf{x})\|^{d^{\ell-\ell_0}} \\
&> C_1^{1+d+d^2+\cdots+d^{\ell-\ell_0-1}}T^{d^{\ell-\ell_0}} \\
&= C_1^{(d^{\ell-\ell_0}-1)/(d-1)}T^{d^{\ell-\ell_0}}
\end{aligned}$$

by iterating the lower bound in Lemma 5. Thus

$$\begin{aligned}
\hat{H}_{\Phi}(\mathbf{x}) &= \lim_{\ell\to+\infty}\frac{1}{d^{\ell}}\log\|\Phi^{\ell}(\mathbf{x})\| \\
&\geq \lim_{\ell\to+\infty}\frac{1}{d^{\ell}}\log(C_1^{(d^{\ell-\ell_0}-1)/(d-1)}T^{d^{\ell-\ell_0}}) \\
&= d^{-\ell_0}(d^{-1}\log C_1+\log T).
\end{aligned}$$

Selecting any $T > C_1^{1/d}$, we deduce that $\hat{H}_{\Phi}(\mathbf{x}) > 0$.

(e) This follows at once from **(d)** along with **(a)** and **(b)**.

(f) This follows at once from **(d)** and **(c)**. □

The following proposition characterizes the property of nonsingular reduction for a homogeneous map in terms of its homogeneous local height function and its homogeneous filled Julia set.

Proposition 13. *Let* $\Phi : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}$ *be a nonsingular homogeneous map of degree* ≥ 2*. The following conditions are equivalent:*

(a) Φ *has nonsingular reduction;*

(b) $\|\Phi(\mathbf{x})\| = \|\mathbf{x}\|^d$ *for all* $\mathbf{x} \in \mathbb{K}^{N+1}$*;*

(c) $\hat{H}_\Phi(\mathbf{x}) = \log \|\mathbf{x}\|$ *for all* $\mathbf{x} \in \mathbb{K}^{N+1}$*;*

(d) $F_\Phi = B(0,1)$.

Remarks. This proposition generalizes Lemma 3.9 of [3] from the $N = 1$ case to arbitrary $N \geq 1$. The equivalent of **(a)** and **(c)** was proved by Kawaguchi-Silverman in [17], Proposition 14; the proof that **(d)** implies **(a)** utilizes a key argument from their paper.

Proof. If **(a)** holds then **(b)** follows immediately from Lemma 5. It is clear from the definition (4.4) that **(b)** implies **(c)**. If **(c)** holds, then **(d)** follows immediately from Proposition 12. It remains to show that **(d)** implies **(a)**.

Suppose that **(d)** holds. The filled Julia set always satisfies $\Phi(F_\Phi) \subseteq F_\Phi$, which in this case implies that $\|\Phi(\mathbf{x})\| \leq 1$ for all $\mathbf{x} \in B(0,1)$. Thus by Proposition 4 we have $H(\Phi) \leq 1$, which means that Φ is defined over $\mathbb{K}^\circ$. In particular, it follows that $|\mathrm{Res}(\Phi)| \leq 1$ by the ultrametric inequality.

Suppose that $|\mathrm{Res}(\Phi)| < 1$. Then by Proposition 3 (taking k to be the residue field $\widetilde{\mathbb{K}} = \mathbb{K}^\circ/\mathbb{K}^{\circ\circ}$) there exists $\mathbf{x}_0 \in B(0,1)$ such that $\widetilde{\mathbf{x}}_0 \not\equiv 0(\mod \mathbb{K}^{\circ\circ})$ but $\Phi(\widetilde{\mathbf{x}}_0) \equiv 0(\mod \mathbb{K}^{\circ\circ})$. In particular, we have $\|\mathbf{x}_0\| = 1$ and $\|\Phi(\mathbf{x}_0)\| < 1$. By iterating the upper bound in Lemma 5 we have $\|\Phi^k(x_0)\| \leq$

$\|\Phi(\mathbf{x}_0)\|^{d^{k-1}}$, and thus

$$\hat{H}_\Phi(\mathbf{x}_0) = \lim_{k\to\infty} \frac{1}{d^k} \log \|\Phi^k(x_0)\| \leq \frac{1}{d} \log \|\Phi(x_0)\| < 0.$$

Since $\mathbb{K}$ is algebraically closed we can find a (nonzero) scalar $c \in \mathbb{K}$ such that $\hat{H}_\Phi(\mathbf{x}_0) < \log |c| < 0$. In particular $|c| < 1$, whereby $\|c^{-1}\mathbf{x}_0\| > 1$ and thus $c^{-1}\mathbf{x}_0 \notin B(0,1)$. On the other hand by Proposition 12 **(a)** we have

$$\hat{H}_\Phi(c^{-1}\mathbf{x}_0)\hat{H}_\Phi(\mathbf{x}_0) - \log |c| < 0,$$

which according to Proposition 12 **(d)** implies that $c^{-1}\mathbf{x}_0$ is in the filled Julia set F_Φ. These two properties of $c^{-1}\mathbf{x}_0$ contradict the assumption **(d)** that $F_\Phi = B(0,1)$. We conclude that $|\mathrm{Res}(\Phi)| = 1$, and we have shown that **(d)** implies **(a)**. $\square$

The following lemma calculates the transfinite diameter of the filled Julia sets of certain homogeneous maps.

Lemma 14. *Let $\mathbb{K}$ be an algebraically closed non-archimedean local field, and let $\Phi : \mathbb{K}^{N+1} \to \mathbb{K}^{N+1}$ be a nonsingular homogeneous map of degree $d \geq 2$. Suppose that a conjugate $\Psi = \Gamma^{-1} \circ \Phi \circ \Gamma$ of Φ by some $\Gamma \in \mathrm{GL}_{N+1}(\mathbb{K})$ has nonsingular reduction. Then*

$$d_\infty(F_\Phi) = |\mathrm{Res}(\Phi)|^{C(N,d)},$$

where $C(N,d)$ is a constant depending only on N and d.

Proof. In order to prove the lemma, we need the following composition law (or "chain rule") for the resultant: if Φ and Φ' are of $N+1$ systems of homogeneous forms in $N+1$ variables, of degrees d and d' respectively, then

$$\mathrm{Res}(\Phi \circ \Phi') = \mathrm{Res}(\Phi)^a \mathrm{Res}(\Phi')^b, \tag{4.6}$$

where a and b are constants depending on N, d and d'. For a proof see [9], Corollary 5.

Continuing with the proof of the Lemma, since $\Psi = \Gamma^{-1} \circ \Phi \circ \Gamma$ has nonsingular reduction, we have $F_\Psi = B(0,1)$ and it follows

$$1 = d_\infty(F_\Psi) = d_\infty(\Gamma^{-1}(F_\Phi)) = |\det(\Gamma)|^{-1} d_\infty(F_\Phi)$$

from Proposition 11 **(b)** and **(c)**. On the other hand, by (4.6) and the fact that $\mathrm{Res}(\Gamma) = \det(\Gamma)$,

$$1 = |\mathrm{Res}(\Psi)| = |\mathrm{Res}(\Gamma^{-1} \circ \Phi \circ \Gamma)| = |\det(\Gamma)|^{A(N,d)} |\mathrm{Res}(\Phi)|^{B(N,d)},$$

where $A(N,d)$ and $B(N,d)$ are constants depending only on N and d. Thus $d_\infty(F_\Phi) = |\det(\Gamma)| = |\mathrm{Res}(\Phi)|^{-B(N,d)/A(N,d)}$. □

Remarks. The hypothesis in Lemma 14 that some conjugate of Φ has nonsingular reduction is probably unnecessary, but a general proof for general Φ is not known. A proof of the identity for general Φ is given in the case

$\mathbb{K} = \mathbb{C}_p$ by DeMarco-Rumely [10], although it is not known whether their proof generalizes to the equal-characteristic case. Another note is that using an explicit expression for the composition law (4.6), as in [9], it is possible to give an explicit expression for the exponent $C(N, d)$, however, this is not necessary here.

4.3 The analytic proof of Theorem 1

We now give an analytic proof of Theorem 1 in a slightly stronger form. The result and its proof generalizes Theorem 1.9 of Baker [2] and Proposition 6.1 of Benedetto [4].

Theorem 15. *Let $K = k(C)$ be a function field, and let $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be a morphism of degree at least two. The following are equivalent:*

- **(a)** *φ is isotrivial;*
- **(b)** *φ has potential good reduction at all places $v \in M_K$;*
- **(c)** *φ has good reduction over $\mathbb{K}_v$ for all $v \in M_K$.*

Proof. The proof that **(a)** implies **(b)** is in §2.6. It is trivial that **(b)** implies **(c)**. If v is any place of K, K'/K and v' is any place of K' lying over v, then there exists an embedding $K' \hookrightarrow \mathbb{K}_v$ with $\mathcal{O}_{v'} \hookrightarrow \mathbb{K}_v^\circ$.

Finally we show that **(c)** implies **(a)** to finish the proof. Suppose that φ

has good reduction over $\mathbb{K}_v$ at all places v of K. Note that both the conditions of **(a)** and **(c)** are invariant under replacing K by a finite extension of K as in §2.5. By Proposition 2 **(c)**, the φ-preperiodic points are Zariski-dense in $\mathbb{P}^N(\bar{K})$. Therefore, by extending K if necessary, we may assume that there exists at least $N+1$ linearly independent φ-preperiodic points in $\mathbb{P}^N(K)$. Choose coordinates $\mathbf{x} = (x_0, x_1, \dots, x_N)$ on $\mathbb{P}^N_K$ such that these $N+1$ φ-preperiodic points are the points $P_0, P_1, \dots, P_N \in \mathbb{P}^N(K)$ which lift to the standard basis elements $\mathbf{e}_0 = (1, 0, \dots, 0)$, $\mathbf{e}_1 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_N = (0, \dots, 0, 1)$.

Without loss of generality, we may assume there exists a model $\Phi(\mathbf{x})$ of φ with respect to the coordinates $\mathbf{x}$ such that the standard basis elements $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_N \in K^{N+1}$ are Φ-preperiodic. To see this, let $\Psi : K^{N+1} \to K^{N+1}$ be *any* model of φ with respect to $\mathbf{x}$. Note that for each n, P_n is φ-preperiodic, hence, we have $\varphi^{i_n}(P_n) = \varphi^{j_n}(P_n)$ for some integers $1 \leq i_n < j_n$. Thus $\Psi^{i_n}(\mathbf{e}_n) = c_n \Psi^{j_n}(\mathbf{e}_n)$ for some nonzero constant $c_n \in K$. For each n, select an element $\alpha_n \in \bar{K}$ such that $\alpha_n^{d^{i_n}} c_n = \alpha_n^{d^{j_n}}$; thus

$$\begin{aligned} \Psi^{i_n}(\alpha_n \mathbf{e}_n) &= \alpha_n^{d^{i_n}} \Psi^{i_n}(\mathbf{e}_n) \\ &= \alpha_n^{d^{i_n}} c_n \Psi^{j_n}(\mathbf{e}_n) = \alpha^{d^{j_n}} \Psi^{j_n}(\mathbf{e}_n) = \Psi^{j_n}(\alpha_n \mathbf{e}_n) \end{aligned}$$

Therefore each $\alpha_n \mathbf{e}_n$ is Ψ-preperiodic. Replace K by a finite extension containing the α_n, and let $\Phi(\mathbf{x}') = \Gamma^{-1} \circ \Psi \circ \Gamma(\mathbf{x}')$, where $\Gamma \in \mathrm{GL}_{N+1}(K)$ is selected to take $\mathbf{e}_n$ to $\alpha_n \mathbf{e}_n$ for each n. Now the standard basis elements $\mathbf{e}_n$ are Φ-preperiodic. Replacing the coordinates $\mathbf{x}$ with $\mathbf{x}' = \Gamma^{-1}(\mathbf{x})$, the above claim is justified.

To summarize, we have an endomorphism φ of $\mathbb{P}^N_K$ which has good reduction over $\mathbb{K}_v$ for all places v of K, a choice of coordinates $\mathbf{x} = (x_0, x_1, ..., x_N)$ on $\mathbb{P}^N_K$ and a model $\Phi(\mathbf{x})$ of φ with respect to $\mathbf{x}$ such that the standard basis elements $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_N$ of K^{N+1} are Φ-preperiodic. We are going to show that

$$F_{\Phi,v} = B_v(0,1) \text{ for all } v \in M_K \tag{4.7}$$

where $F_{\Phi,v}$ denotes the homogeneous filled Julia set in $\mathbb{K}_v^{N+1}$ associated to $\Phi(\mathbf{x})$, and $B_v(0,1)$ denotes the unit ball in $\mathbb{K}_v^{N+1}$. With this claim, it follows from Proposition 13 that $\Phi(\mathbf{x})$ has nonsingular reduction at all places $v \in M_K$, which means in particular that the coefficients of $\Phi(\mathbf{x})$ are in $\mathbb{K}^\circ$ for all $v \in M_K$. This implies that the coefficients of $\Phi(\mathbf{x})$ are elements of the constant field k of K, since a rational function on C with no poles must be constant. Therefore φ is defined over k, whereby it is isotrivial, completing the proof that **(c)** implies **(a)**.

It now remains to prove (4.7). Fix a place $v \in M_K$. Since φ has a good reduction over $\mathbb{K}_v$ there exists a choice of coordinates $\mathbf{y} = (y_0, y_1, \dots, y_N)$ on

$\mathbb{P}^N_{\mathbb{K}_v}$ and a model $\Psi(\mathbf{y})$ for φ with respect to $\mathbf{y}$ such that $\Psi(\mathbf{y})$ has nonsingular reduction over $\mathbb{K}_v$; thus $\Psi(\mathbf{y})$ has coefficients in $\mathbb{K}_v^\circ$ and $|\mathrm{Res}(\Psi)|_v = 1$. Moreover $F_{\Psi,v} = B_v(0,1)$ by Proposition 13.

Choose $\Gamma \in \mathrm{GL}_{N+1}(\mathbb{K}_v)$ so that $\mathbf{y} = \Gamma(\mathbf{x})$. Thus $\Phi'(x) = \Gamma^{-1} \circ \Psi \circ \Gamma(\mathbf{x})$ is another model for φ with respect to the coordinates $\mathbf{x}$, hence $\Phi'(\mathbf{x}) = c\Phi(\mathbf{x})$ for some $c \in \mathbb{K}_v^\times$. Therefore

$$\Gamma(c^{-1/(d-1)}F_{\Phi,v}) = \Gamma(F_{\Phi',c}) = F_{\Psi,v} = B_v(0,1)$$

by Proposition 12 **(e)** and **(f)**; thus $F_{\Phi,v} = c^{1/(d-1)}\Gamma^{-1}(B_v(0,1))$. In particular, $F_{\Phi,v}$ is an ellipsoid, as defined in §4.1. Note that the standard basis elements are elements of $F_{\Phi,v}$ since they are Φ-preperiodic. By Proposition 11 **(e)** we conclude that $B_v(0,1) \subseteq F_{\Phi,v}$, which implies by Proposition 11 **(c)** that $d_\infty(F_{\Phi,v}) \geq 1$.

On the other hand, by Lemma 14 and the product formula we have

$$\prod_{v\in M_K} d_\infty(F_{\Phi,v}) = \prod_{v\in M_K} |\mathrm{Res}(\Phi)|_v^{C(N,d)} = 1,$$

and since we have already shown that $d_\infty(F_{\Phi,v}) \geq 1$ for all $v \in M_K$, we must have $d_\infty(F_{\Phi,v}) = 1$ for all $v \in M_K$. Since each $F_{\Phi,v}$ is an ellipsoid containing $B_v(0,1)$, we deduce from Proposition 11 **(d)** that $F_{\Phi,v} = B_v(0,1)$ for all $v \in M_K$. Thus we have proved (4.7), which completes the proof that **(c)** implies **(a)**. □

Chapter 5

Two Applications

5.1 Endomorphisms with an isotrivial iterate

The following corollary of Theorem 1 states that an endomorphism is isotrivial if and only if it has an isotrivial iterate.

Corollary 16. *Let* $K = k(C)$ *be a function field, let* $\varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$ *be a morphism of degree at least two, and let* $r \geq 1$ *be an integer. Then* φ *is isotrivial if and only if* φ^r *is isotrivial.*

Proof. By the equivalence of **(a)** and **(c)** in Theorem 15, it suffices to show that given any place $v \in M_K$, φ has good reduction over $\mathbb{K}_v$ if and only if φ^r has good reduction over $\mathbb{K}_v$. As in Theorem 1, the "only if" direction of this statement is trivial. To show the "if" direction, suppose that φ^r has good reduction over $\mathbb{K}_v$. Thus, there exists coordinates $\mathbf{x} = (x_0, x_1, \dots, x_N)$ on $\mathbb{P}^N_{\mathbb{K}_v}$, and a model $\Psi(\mathbf{x})$ for φ^r with respect to $\mathbf{x}$ such that $\Psi(\mathbf{x})$ has nonsingular reduction over $\mathbb{K}_v$; thus $F_{\Psi,v} = B_v(0,1)$ by Proposition 13. Let Φ be a model for φ with respect to the same coordinates $\mathbf{x}$; thus $\Phi^r(\mathbf{x})$ is a model for φ^r, so $\Phi^r(\mathbf{x}) = c\Psi(\mathbf{x})$ for some $c \in \mathbb{K}_v^\times$. It follows from Proposition 12 **(e)** that

$$F_{\Phi,v} = F_{\Phi^r,v} = F_{c\Psi,v} = c^{-1/(d-1)}F_{\Psi,v} = c^{-1/(d-1)}B_v(0,1).$$

Letting $\Phi' = c^{-1}\Phi$, we have $F_{\Phi',v} = c^{-1/(d-1)}F_{\Phi,v} = B_v(0,1)$ by Proposition 12 **(e)**, where $\Phi'(\mathbf{x})$ is a model for φ with nonsingular reduction by Proposition 13. Therefore φ has good reduction as desired. □

5.2 A dynamical criterion for decomposability of locally free coherent sheaves

Let C be a complete nonsingular curve over an algebraically closed field k. Let $N \geq 1$ be an integer, let $\mathcal{E}$ be a locally free coherent sheaf of rank $N+1$ on C, and denote by $\pi : \mathbb{P}(\mathcal{E}) \to C$ the associated projective bundle. The following corollary of Theorem 1 states that, after possibly replacing C with a base extension $p : C' \to C$ and replacing $\mathcal{E}$ with $\mathcal{E}' = p^*\mathcal{E}$, the sheaf $\mathcal{E}$ decomposes as a direct sum of $N+1$ copies of the same invertible sheaf on C if and only if there exists an endomorphism of $\mathbb{P}(\mathcal{E})$ of degree at least two. A similar result was obtained by Amerik [1] in the case $k = \mathbb{C}$.

Corollary 17. *Let $\mathcal{E}$ be a locally free coherent sheaf of rank $N+1$ on a complete nonsingular curve C over an algebraically closed field k. Then the following two conditions are equivalent:*

> **(a)** *There exists a base extension $p : C' \to C$ and an endomorphism $\varphi : \mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E}')$ of degree at least two;*
>
> **(b)** *There exists a base extension $p : C' \to C$ and an invertible sheaf $\mathcal{L}$ on C' such that $\mathcal{E}' \simeq \mathcal{L} \oplus \cdots \oplus \mathcal{L}$.*

Moreover, if **(a)** *and* **(b)** *hold then the two extensions C' can be chosen to coincide, and $\mathbb{P}(\mathcal{E}') \simeq \mathbb{P}^N_k \times C'$ with $\varphi = \varphi_0 \times \mathrm{Id}_{C'}$, where $\varphi_0 : \mathbb{P}^N_k \to \mathbb{P}^N_k$ is a morphism and $\mathrm{Id}_{C'} : C' \to C'$ is the identity.*

Proof. Both conditions **(a)** and **(b)** are invariant under replacing C with a finite extension $p : C' \to C$ (and replacing $\mathcal{E}$ with $\mathcal{E}' = p^*\mathcal{E}$), and therefore we may do this at any time without loss of generality. Moreover, $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E} \otimes \mathcal{B})$ for any $\mathcal{B} \in \mathrm{Pic}(C)$, so we may replace $\mathcal{E}$ with $\mathcal{E} \otimes \mathcal{B}$ at any time without loss of generality as well. We identify the generic fiber of $\pi : \mathbb{P}(\mathcal{E}) \to C$ with $\mathbb{P}^N_K$, where $K = k(C)$ denotes the function field of C, and given a morphism $\varphi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ we denote by $\varphi_K : \mathbb{P}^N_K \to \mathbb{P}^N_K$ the restriction of φ to $\mathbb{P}^N_K$.

Assume that **(b)** holds. Replacing C with a suitable extension $p : C' \to C$ we may assume that $\mathcal{E} \simeq \mathcal{L} \oplus \cdots \oplus \mathcal{L}$ for some invertible sheaf $\mathcal{L}$ on

C. Moreover, replacing $\mathcal{E} \otimes \mathcal{L}^\vee$ we may assume that $\mathcal{O}_C \oplus \cdots \oplus \mathcal{O}_C$ is isomorphic to the trivial vector bundle. Therefore $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}^N_k \times C$, and any endomorphism $\varphi : \mathbb{P}^N_k \to \mathbb{P}^N_k$ of degree at least two induces such an endomorphism $\varphi = \varphi_0 \times \mathrm{Id}_C$ of $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}^N_k \times C$. This completes the proof that **(b)** implies **(a)**.

Conversely, assume that **(a)** holds. We have $\varphi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \otimes \pi^*\mathcal{A}$ for some $d \geq 2$ and $\mathcal{A} \in \mathrm{Pic}(C)$. Replacing C with a suitable extension, $p : C' \to C$, we may assume there exists $\mathcal{B} \in \mathrm{Pic}(C)$ such that $\mathcal{B}^{\otimes(1-d)} \simeq \mathcal{A}$. It follows that $\varphi^*\mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathcal{B})}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathcal{B})}(d)$. Replacing $\mathcal{E}$ with $\mathcal{E} \otimes \mathcal{B}$, we may assume without loss of generality that $\varphi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$.

Since $\mathbb{P}(\mathcal{E})$ is locally isomorphic to $\mathbb{P}^N_k \times U$ for open sets $U \subset C$, the morphism φ_K has everywhere good reduction. Therefore by Theorem 1 it is isotrival. This means, after replacing C with a suitable extension $p : C' \to C$ if necessary, φ_K is induced by an endomorphism $\varphi_0 : \mathbb{P}^N_k \to \mathbb{P}^N_k$. In particular, there exists coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ on $\mathbb{P}^N_K$ and a model $\Phi(\mathbf{x})$ for $\varphi_K : \mathbb{P}^N_K \to \mathbb{P}^N_K$ with coefficients in the constant field k.

Given a point $P \in \mathbb{P}^N(K) \subset \mathbb{P}^N_K \subset \mathbb{P}(\mathcal{E})$ and a closed point $v \in C$, the valuative criterion for properness ([14] Theorem II.4.7) determines a unique point $s_P(v) \in \pi^{-1}(v)$ specializing P. This defines a section $s_P : C \to \mathbb{P}(\mathcal{E})$ of π, along with a surjective morphism $\mathcal{E} \to s_P^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of sheaves on C.

Moreover, if P is a φ-preperiodic point then, after perhaps replacing C with an extension $p : C' \to C$, we have

$$s_P^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq \mathcal{O}_C \tag{5.1}$$

To see this, note that

$$s_{\varphi(P)}^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq s_P^* \varphi^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq s_P^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d).$$

Thus if P is φ-preperiodic with $\varphi^{n+m}(P) = \varphi^m(P)$ for $n \geq 1$ and $m \geq 0$, then $s_P^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d^{n+m}) \simeq s_P^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d^m)$. This implies that $s_P^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d^{n+m} - d^m) \simeq \mathcal{O}_C$, which means that $s_P^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a torsion element of $\operatorname{Pic}(C)$. After replacing C with a suitable extension $p : C' \to C$ we deduce (5.1) as desired.

Now let $P_0, P_1, \ldots, P_N \in \mathbb{P}^N(K) \subset \mathbb{P}^N_K$ be a linearly independent set of k-rational φ-preperiodic points; such a set exists, since φ_K is defined over k and since Proposition 2 **(c)** ensures that the preperiodic points are Zariski-dense in $\mathbb{P}^N(k)$. Extending C if necessary, we may assume that (5.1) holds for each $P \in \{P_0, P_1, \ldots, P_N\}$, and we obtain a morphism

$$\mathcal{E} \to \bigoplus_{j=0}^{N} s_{P_j}^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq \mathcal{O}_C \oplus \cdots \oplus \mathcal{O}_C \tag{5.2}$$

of sheaves on C. In fact, we are going to show that (5.2) is an isomorphism; for it suffices to show that the set $\{s_{P_0}(v), s_{P_1}(v), \ldots, s_{P_N}(v)\}$ is linearly independent on each closed fiber $\pi^{-1}(v) \simeq \mathbb{P}^N_k$ of $\pi : \mathbb{P}(\mathcal{E}) \to C$.

Given a point $v \in C$, there exists a neighborhood $U \subset C$ of v such that $\pi^{-1}(U) \simeq \mathbb{P}_k^N \times U$ and a model $\Psi(\mathbf{y})$ for φ_K which coincides with the morphism $\varphi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ when restricted to $\pi^{-1}(U)$. In particular, $\Psi(\mathbf{y})$ has nonsingular reduction at v. Let $\Gamma \in \mathrm{GL}_{N+1}(K)$ denote the change in coordinate element satisfying $\Gamma(\mathbf{x}) = \mathbf{y}$. Thus $\Phi(\mathbf{x}) = c\Gamma^{-1} \circ \Psi \circ \Gamma(\mathbf{x})$ for some $c \in K^{\times}$. Extending the curve C if necessary, we may assume there exists some $a \in K$ such that $a^{d-1} = c$. Letting $\Gamma' = a\Gamma$, we have $\Phi(\mathbf{x}) = (\Gamma')^{-1} \circ \Psi \circ \Gamma'(\mathbf{x})$. Therefore, replacing Γ with Γ' and replacing the coordinates $\mathbf{y}$ with $\mathbf{y}' = \Gamma'(\mathbf{x}) = a\mathbf{y}$, we may assume without loss of generality that $\Phi(\mathbf{x}) = \Gamma^{-1} \circ \Phi \circ \Gamma(\mathbf{x})$. Both $\Phi(\mathbf{x})$ and $\Psi(\mathbf{y})$ have nonsingular reduction and by Lemma 6, Γ must be an element of $\mathrm{GL}_{N+1}(\mathcal{O}_v)$. This means, it reduces to an automorphism $\gamma_v : \mathbb{P}_k^N \to \mathbb{P}_k^N$ over the residue field k at v. Moreover, $\gamma_v(P_j) = s_{P_j}(v)$ for all $0 \leq j \leq N$ and since the set $\{P_0, P_1, \ldots, P_N\}$ is linearly independent in $\mathbb{P}^N(k)$ as well it follows that the set, $\{s_{P_0}(v), s_{P_1}(v), \ldots, s_{P_N}(v)\}$ is linearly independent in $\mathbb{P}^N(k)$ as well. This holds for all $v \in C$, hence we deduce that (5.2) is an isomorphism. Thus $\mathcal{E} \simeq \mathcal{O}_C \oplus \cdots \oplus \mathcal{O}_C$, completing the proof that **(a)** implies **(b)**. Since $\mathcal{E}$ is the trivial sheaf we have $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}_k^N \times C$ with $\varphi = \varphi_0 \times \mathrm{Id}_C$. □

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