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MATROIDS, GRAPHS AND RESISTANCE NETWORKS

by

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CHAPTER 1. INTRODUCTION

1.1 INTRODUCTION

An area of mathematics which is becoming increasingly useful in combinatorial problems is matroid theory [Wh 1; Ed 3; Tu 7]. In 1935 matroid theory was introduced by Hassler Whitney [Wh 1]. Besides Whitney's work little else was done in matroid theory until 1958 at which time W.T. Tutte made some fundamental contributions to the theory [Tu 1; Tu 2; Tu 3]. At the present time we find that matroid theory has been applied successfully to many areas [Ed 1; Ed 2; Le 1; Tu 7] and in particular to electrical network theory [Mi 1]. This last application is not too surprising a development since graph theory has been applied to electrical network theory and a matroid can be viewed as a generalization of a graph. Furthermore, one can assert that matroid theory is applicable to all lumped physical systems. To see this we need merely recall that a lumped physical system has two aspects, a metrical aspect and a topological one [Tr 1]. The topological nature of a lumped physical system is invariably modeled by a graph and therefore, due to the correspondence between matroids and graphs, the topology can be described by a matroid. This generalization, we believe, can lead to new results and new insights in lumped physical system theory.

In this dissertation we study and obtain new results on structural aspects of matroids, and we apply matroid theory to the p-port resistance network problem. A p-port resistance network (containing only positive resistances and no ideal transformers) is one of the simplest of lumped physical systems and yet we have not yet developed a complete theory for this class of networks. The results of our research into this problem are presented in Chapter 3. In Chapter 4 we shift our emphasis and study certain structural aspects of matroids. We derive unique decompositions of a matroid, that is, we find a set of invariants for a given matroid. Chapters 3 and 4 are independent except for the fact that they are bridged by matroid theory.

In order to make this dissertation self contained and also to translate some results of matroid theory into forms which we can use directly, we offer an introduction to matroid theory in Chapter 2. It is hoped that the reader, not familiar with the kinds of arguments used in this area of mathematics, will find the material sufficient to gain some insight and assurance in matroid theory.

In the following sections of this chapter, we outline the contents of each of the succeeding chapters.

1.2 INTRODUCTION TO MATROID THEORY (Chapter 2)

As mentioned previously, matroid theory was introduced by Hassler Whitney [Wh 1] in 1935. In his pioneering paper, in which he axiomatized certain properties of a finite set of vectors, he brought into being the problem of characterizing those matroids which correspond to graphs. Naturally he was able to exhibit an abstract matroid which corresponds to no graph. Whitney also gave four different axiom systems which he showed were all equivalent ways of defining an abstract matroid. However, the problem of characterizing those matroids which correspond to graphs remained unsolved until Tutte [Tu 1; 2; 3], in 1959, gave the complete theoretical solution as well as an efficient algorithmic solution [Tu 4].

In Chapter 2 we give an introduction to matroid theory which is based largely on the works of Whitney and Tutte [Wh 1; Tu 6]. In section 2.1 the four axiom systems introduced by Whitney are presented and some elementary theorems and proofs are given in order to introduce the reader to the flavor of matroid theory. Vector spaces on a finite set E over a field F are introduced and related to matroids; the terms regular (binary) vector spaces and regular (binary) matroids are then defined. In Chapter 2 we also treat the vector spaces associated with graphs and introduce the polygon and bond matroids of a graph. Duality in matroids and graphs is considered and some of the basic results in this area are presented. In section 2.6 we give Tutte's results which completely characterize the matroids associated with graphs.

Chapter 2 is therefore intended to introduce the ideas and methods of matroid theory and relate matroid theory to graph theory and vector spaces. The results presented in Chapter 2 should be sufficient to make this dissertation self-contained.

1.3 RESISTANCE NETWORKS AND GENERALIZED NETWORKS (Chapter 3)

In this section we give some previously derived properties of the matrices associated with p-port resistance networks, point out some other results previously obtained in this area and indicate the contributions made in Chapter 3 of this dissertation.

The dominant, paramount and totally unimodular matrices relate in one way or another to the p-port resistance network [We 2]. Probably the most far-reaching condition we know is that the immittance (o.c. impedance or s.c. admittance) matrices of any p-port resistance network are necessarily paramount matrices [Ce 4]. This statement deserves some further comment since Boesch [Bo 1] pointed out that the paramountcy condition is an open question with respect to singular immittance matrices. This is not true, however, since M. Iri [Ir 1] recently gave such a proof. However, Boesch's objections were pointed toward certain nonunique augmentations which must be performed even in the case of Iri's proof. In Chapter 3 we give the first direct proof of the paramountcy condition in a more general framework than p-port resistance networks.

It is also known [We 2] that paramountcy is a necessary and sufficient condition for a $p \times p$ matrix to be realizable as the immittance matrix of some resistive p-port network for $p \leq 3$. When $p > 3$, however, paramountcy is known to be no longer sufficient [Ce 3]; there are paramount matrices of order 4 which are not realizable as the immittance matrix of any 4-port network. An even more striking result is that there are paramount matrices which are realizable as o.c. impedance matrices of resistive p-port networks but not as s.c. admittance matrices of any resistive p-port network and vice versa [Ce 3]. It is this last result which appears to threaten our ideas about duality in electrical networks. It is believed that the results on duality in this thesis clarify this situation.

The p-port resistance network is thus understood for $p \leq 3$ and the synthesis procedures, mainly for the case $p = 3$, can be found in Weinberg's book [We 2].

Cederbaum [Ce 1] obtained some significant results by restricting the allowable interconnections of the resistance and port elements. Two of these special cases are: 1) the realization of a given matrix as the s.c. admittance matrix of a network whose ports form a forest of the network graph; 2) the realization of the given matrix as the o.c. impedance matrix of a network whose ports form a coforest of the network graph.

Cederbaum gave his conditions in terms of an algorithm which first requires the factorization of the given matrix and then a test for the topological realizability of a resulting matrix. A matrix is realizable in one of the two restricted forms if and only if we can successfully complete the algorithm. Although others [Gu 1; Bo 2; Bi 1] have treated this special case (only the case of the s.c. admittance matrix), the work of Cederbaum is still the most complete treatment, even though he has not given necessary and sufficient conditions short of the actual realization procedure.

We also know that a $p \times p$ dominant matrix can be realized as the s.c. admittance matrix of a network whose network graph has 2p vertices [We 2], and we are faced with the upsetting fact that no such statement can be made for the. impedance case . The concept of duality might lead one to believe that a corresponding (a dual) statement can be made for impedances [Na 2].

For an interesting discussion of the p-port resistance problem as well as an extensive bibliography the reader is referred to a panel discussion [Pa 1] on the problem of transformerless realizations of resistance networks.

In Chapter 3 the notion of a generalized resistance network is put forth. The generalized network focuses one's attention on the essentials of the p-port resistance network. Matroid concepts provide one with an "interconnection" model for the generalized resistance network and allows basic results to be phrased in terms of matroid structure.

Two special kinds of generalized networks are defined (N_Y and N_Z) which provide the specialization of our general results to the admittance and impedance formulations of ordinary resistance p-port networks. Most of the previous partial results are fitted into the more general structure provided by matroid theory and the generalized resistance networks; that is, one of the contributions made in Chapter 3 is the unification of much of the previous results and, moreover, an extension of these results to generalized resistance networks.

Some of the results in generalized resistance networks specialize to new results for p-port resistance networks. A general proof of the paramountcy condition is given in Chapter 3. This proof relies on the demonstration of a matrix identity which relates the minors of a matrix product of the form

$$B_2^t [B_1 D B_1^t]^{-1} B_2 \quad ,$$

where $B_1 = r \times (n-p)$, $B_2 = r \times p$ and D is a $(n-p) \times (n-p)$ diagonal matrix, to a summation whose terms depend only on the minors of D and the $r \times r$ minors of $[B_1 \vdots B_2]$. As a byproduct of this proof we obtain topological formulas for the generalized network, thus extending the notion of topological formulas [We 2; Se 2] for p-port resistance networks to generalized networks.

Also in Chapter 3 we give an additional necessary condition for a matrix Q to be put in the form ADA^t, where A is a totally unimodular matrix and D is a diagonal matrix with positive diagonal terms. Not only must Q be a paramount matrix (see 3.4-5) but Q must also satisfy the conditions of (3.4-6). These two conditions are, unfortunately, not sufficient for Q to be decomposeable as ADA^t. The modified topological matrices, introduced by Cederbaum [Ce 2], are extended to generalized networks.

The singular impedance matrices have been shown in certain cases to lead to synthesis procedures[Na 1; Bo 1]. In section 3.5, we consider the case of singular immittance matrices for generalized networks and a new converse result on singular paramount matrices is obtained. Specifically, it is shown that the null space of a paramount matrix (considered as an operator) must be of a special nature, namely, a regular subspace. This is reflected in the following result which is proved in section 3.5.

Let Q be a $p \times p$ paramount matrix of rank s and suppose the submatrix Q_s formed by the first s rows and columns of Q is nonsingular. Then Q can be expressed as

$$Q = B Q_s B^t ,$$

where B is a totally unimodular matrix.

In the final section of Chapter 3 we use the duality of matroids and graphs to obtain a foundation for duality in electrical networks. Also the fundamental results of Tutte presented in section 2.6 are used to clarify the duality concepts of p-port resistance networks.

1.4 PRINCIPAL MINORS OF A MATROID (Chapter 4)

In Chapter 4 we study a structural property of matroids which we designate as the principal minors of a matroid. The principal minors of a matroid are shown to constitute a basic part of the structure of a matroid. Since a matroid is a generalization of a graph, we obtain basic properties of the structure of a graph. Also the notion of r-maximally distant bases and r-minors are introduced and related to the principal minors. In section 4.1 we will give some previously obtained results which have bearing on the research presented here. Below we outline the contributions made in Chapter 4 of this dissertation.

In section 4.2 we introduce two special functions g_k and h_k (for $k = 1, 2, \ldots$). Using g_k we define the k^{th} principal minor of the first kind (k-PM1) and the k^{th} augmented principal of the first kind (k-APM1), and using h_k we define the principal minors of the second kind (k-PM2 and k-APM2). Also in section 4.2 we show that the (k-PM1), (k-APM1), (k-PM2) and (k-APM2) exist and are unique and, moreover, satisfy a certain nesting property. Section 4.3 treats the case $k = 1$.

In section 4.4 the concepts of r-maximally distant bases and r-minors are introduced and in Theorem (4.4-2) these two concepts are related. The proof of this theorem is the basis of Algorithm 1 which can be used to efficiently construct r-maximally distant bases. Example 4-2 illustrates the use of Algorithm 1 and also indicates how the results on matroids are specialized to graphs through the bond matroid.

Also in section 4.4, the principal minors are related to r-minors and r-maximally distant bases. Theorem (4.4-3) treats the k-PM1 and Theorem (4.4-7) treats the k-APM1. Algorithms 2 and 3 are provided for the construction of principal minors. In Example 4-3 we consider a non-trivial graph G and show, through the polygon matroid of G, how Algorithms 2 and 3 turn out in graph-theoretic terms.

Section 4.5 is devoted to the case k = 2 and one of the main results is the extension of the principal partition of a graph [Ki 1] to the principal partition of a matroid.

In section 4.6 we show how the concept of the principal partition of a matroid is related to the two-person switching game [Le 1]. Moreover, we use duality to obtain a completely graph-theoretic solution to the two-person switching game. We also show the connection between the k-APM1 and the cospanning sets theorem of Edmonds [Ed 2]. Finally, the hybrid rank of a graph, which is a useful concept in network analysis, is extended to matroids and vector spaces.

CHAPTER 2. INTRODUCTION TO MATROID THEORY

2.1 MATROIDS

A matroid can be viewed as a generalization of the polygon concept of a graph and in fact much valuable insight into abstract matroid theory can be obtained through its graph-theoretic counterparts. For this reason we begin this section with some definitions from graph theory.

A graph G is defined by

(i) E(G), a finite set of edges,

(ii) V(G), a finite set of vertices, and

(iii) a relation of incidence which associates with each edge a pair of vertices, not necessarily distinct, called its ends. An edge with coincident ends is called a loop.

A graph H is called a subgraph of G if $E(H) \subseteq E(G)$, $V(H) \subseteq V(G)$ and the ends of the edges in H are the same as in G.

If $S \subseteq E(G)$, we denote by $G \cdot S$ that subgraph of G whose edges are the members of S and whose vertices are the ends in G of the members of S. $G \cdot S$ is called the reduction of G to S.

Let $v \in V(G)$. The valence of v is equal to the number of edges incident to v, where loops are counted twice.

A connected graph with each vertex having valence 2 is called a polygon graph.

A set $S \subseteq E(G)$ is called a polygon of G if $G \cdot S$ is a polygon graph.

Example 2-1. Consider the graph in Fig. 2-1, where

$$E(G) = \{e_1, \ldots, e_9\}$$

and

$$V(G) = \{v_1, \ldots, v_8\} .$$

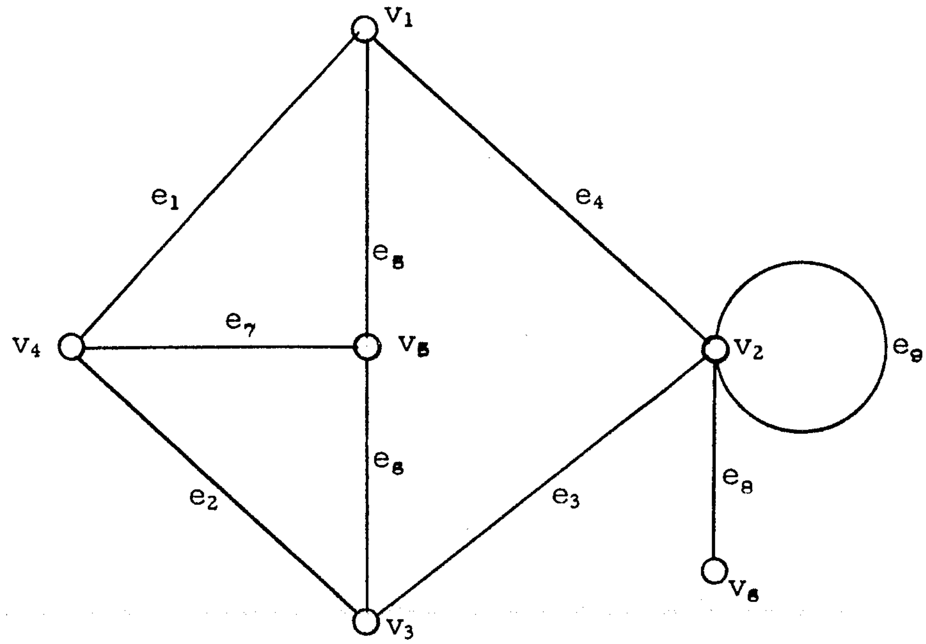


Figure 2-1. Graph of Example 2-1

Let $S = \{e_5, e_3, e_4, e_6\}$. Then $G \cdot S$, the reduction of G to S, is a connected graph and each vertex has valence 2. Consequently, S is a polygon of G. It is not too difficult to see that there are exactly eight polygons of G. If we let $\mathcal{P}(G)$ denote the class of polygons of G, then

$$\mathcal{P}(G) = \{C_1, \ldots, C_8\},$$

where

$$C_1 = \{e_1, e_5, e_7\},$$

$$C_2 = \{e_7, e_6, e_2\},$$

$$C_3 = \{e_1, e_2, e_6, e_5\},$$

$$C_4 = \{e_3, e_5, e_4, e_6\},$$

$$C_5 = \{e_1, e_3, e_4, e_2\},$$

$$C_6 = \{e_9\},$$

$$C_7 = \{e_1, e_3, e_6, e_4, e_7\}$$

and

$$C_8 = \{e_7, e_3, e_2, e_4, e_5\}.$$

In Axiom System I, presented below, certain properties of the polygons of a graph are set down as axioms for an abstract matroid system. The connection between Axiom System I and the polygons of a graph is useful for visualizing matroid concepts.

Axiom System I. Let E be a finite set and $\mathcal{C}$ be a class of non-null subsets of E. The members of $\mathcal{C}$ are called circuits and they define a matroid $\mathcal{M} = (\mathcal{C}, E)$ on the set E if the following conditions are satisfied:

(C1) No member of $\mathcal{C}$ contains any member of $\mathcal{C}$ as a proper subset.

(C2) If C_1, $C_2 \in \mathcal{C}$, $e \in (C_1 \cap C_2)$ and $e' \in (C_1 - C_2)$, then there exists a circuit $C_3 \in \mathcal{C}$ satisfying $e' \in C_3 \subseteq [(C_1 \cup C_2) - \{e\}]$.

If $E = \phi$, then $\mathcal{C} = \phi$ and we denote (ϕ, ϕ), the null matroid by Ω.

The circuits of a matroid constitute a generalization of the polygons of a graph. A precise relationship between matroids and graphs is put forth in the following theorem due to Whitney.

> (2.1-1) Let G be any graph and let $\mathcal{P}(G)$ denote the class of polygons of G. Then $\mathcal{P}(G)$ satisfies (C1) and (C2) and thus defines a matroid on the set E(G).

$\mathcal{P}(G)$ is called the <u>polygon matroid</u> of G. This is not in agreement with the previous notation which would require that $(\mathcal{P}(G), E(G))$ be the polygon matroid. It is, however, at times more convenient to denote the matroid by its set of circuits when the underlying set (in this case E(G)) is understood.

The polygons of a graph G form a matroid which is easily "visualized". However, there is another matroid we can associate with G which is not as obvious but which is at least as important as the polygon matroid. In order to introduce this new matroid we need some definitions.

Let G be a graph and $S \subseteq E(G)$. Define H to be the subgraph of G with vertices V(G) and edges $(E(G) - S)$. Let H_i for $i = 1, \ldots, p$ denote the connected components of H. The graph G ctr S has the vertex set $\{H_1, \ldots, H_p\}$ and edge set S. The ends of a member $e \in S$ in G ctr S are those components H_{i_1} and H_{i_2} which contain the ends of e in G.

We define $G \times S = (G \text{ ctr } S) \cdot S$. $G \times S$ is called the contraction of G to S and differs from G ctr S in that $G \times S$ has no isolated vertices.

Example 2-2. Let G be the graph in Fig.2-2 and take $S = \{e_1, e_2, e_3, e_6\}$. Then G ctr S and $G \times S$ are shown in Fig.2-3.

A graph G is called a bond graph if $V(G) = \{v_1, v_2\}$, $E(G) \neq \phi$ and the ends of each member of E(G) are v_1 and v_2. The graph in Fig.2-4 is a bond graph.

Let G be any graph and $S \subseteq E(G)$. S is called a bond of G if $G \times S$ is a bond graph. Let $\mathcal{B}(G)$ denote the class of bonds of G. We will show later that the members of $\mathcal{B}(G)$ satisfy conditions (C1) and (C2) of Axiom System I and thus define a matroid on the set E(G). $\mathcal{B}(G)$ is called the bond matroid of G. We will also show later that the matroids $\mathcal{P}(G)$ and $\mathcal{B}(G)$ are, in a certain sense, "duals" and that this duality enables us to define a rigorous duality theory for graphs.

The concepts of an independent set and a base are important ones in matroid theory and we now proceed to define them and to derive their basic properties.

Let $\mathcal{M} = (\mathcal{C}, E)$ be a matroid on a finite set E. A set $T \subseteq E$ is called independent if no member of $\mathcal{C}$ is contained in T.

A maximal independent set of a matroid $\mathcal{M}$ is called a base. A set is said to be maximal with a certain property P if it is not properly contained in a set which possesses the same property P.

The following sequence of theorems is concerned with some of the properties of the independent sets and the bases of a matroid. One of our

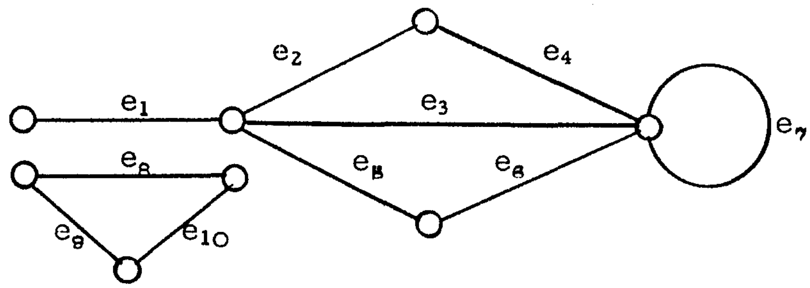


Figure 2-2. Graph of Example 2-2

Figure 2-3. Graphs of Example 2-2

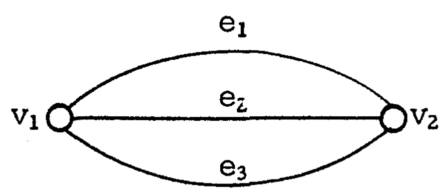


Figure 2-4. A Bond Graph

goals in this section is to prove theorem (2.1-6) which states that the number of elements in every base of a matroid is the same.

(2.1-2) Let $\mathcal{M} = (\mathcal{C}, E)$ be a matroid and $S \subseteq E$ an independent set. If $e \notin S$ and $S \cup \{e\}$ is no longer independent, then there is a unique circuit $C \in \mathcal{C}$ satisfying $e \in C \subseteq (S \cup \{e\})$.

Proof: There is at least one circuit in $S \cup \{e\}$ by hypothesis. Suppose C_1 and C_2 are two circuits contained in $S \cup \{e\}$. We have that

$$e \in (C_1 \cap C_2)$$

since S is independent.

Assume

$$e' \in (C_1 - C_2) \ .$$

Then by (C2), the second property of circuits, there exists a circuit C_3 satisfying

$$e' \in C_3 \subseteq [\,(C_1 \cup C_2) - \{e\}\,] \subseteq S \ . \tag{1}$$

But (1) contradicts the hypothesis. Therefore $(C_1 - C_2) = \phi$. Similarly, $(C_2 - C_1) = \phi$. Consequently, $C_1 = C_2$. ▌

To illustrate theorem (2.2-1), consider the graph in Fig.2-1. The set $S = \{e_1, e_5, e_6, e_8\}$ is independent (with respect to the matroid $\mathcal{P}(G)$), but the set $S \cup \{e_2\}$ is not independent. According to (2.1-2) there exists a unique polygon of G contained in $S \cup \{e_2\}$ and this polygon is C_3 (see Example 2-1).

The next result follows directly from (2.1-2).

(2.1-3) Let $\mathcal{M} = (\mathcal{C}, E)$ be a matroid and b a base of $\mathcal{M}$. If $e \notin b$, then there is a unique circuit $C \in \mathcal{C}$ satisfying $e \in C \subseteq (b \cup \{e\})$.

If b is a base of $\mathcal{M}$ and e a member of E not contained in b, we call the unique circuit C satisfying

$$e \in C \subseteq (b \cup \{e\})$$

the circuit <u>formed by e in b.</u> We denote this circuit by J(b,e), that is, J(b,e) = C, where C is the circuit formed by e in b.

In the graph of Fig.2-1 the set

$$t = \{e_1, e_2, e_3, e_8, e_8\}$$

is a maximal independent set in $\mathcal{P}(G)$ (see Fig.2-5) and the element e_4 not contained in t forms a unique polygon in t. $J(t, e_4) = C_5$ (see Example 2-1).

Let S be any set contained in a finite set E and let $e \in E$. If e' is any element in S, then we introduce the useful notation

$$(e/e')S = (S - \{e'\}) \cup \{e\}.$$

(2.1-4) If b and b' are bases of $\mathcal{M}$ and e is an element of b, then there exists an element e' in $(b' - (b - \{e\}))$ such that (e'/e)b is an independent set.

<u>Proof</u>: We prove (2.1-4) by contradiction. Assume there is an element $e \in b$ such that for every $e_i' \in (b' - (b - \{e\}))$ the set $(e_i'/e)b$ is not independent. Enumerate the elements of E such that

$$b' = \{e_1', e_2', \ldots, e_p'\}.$$

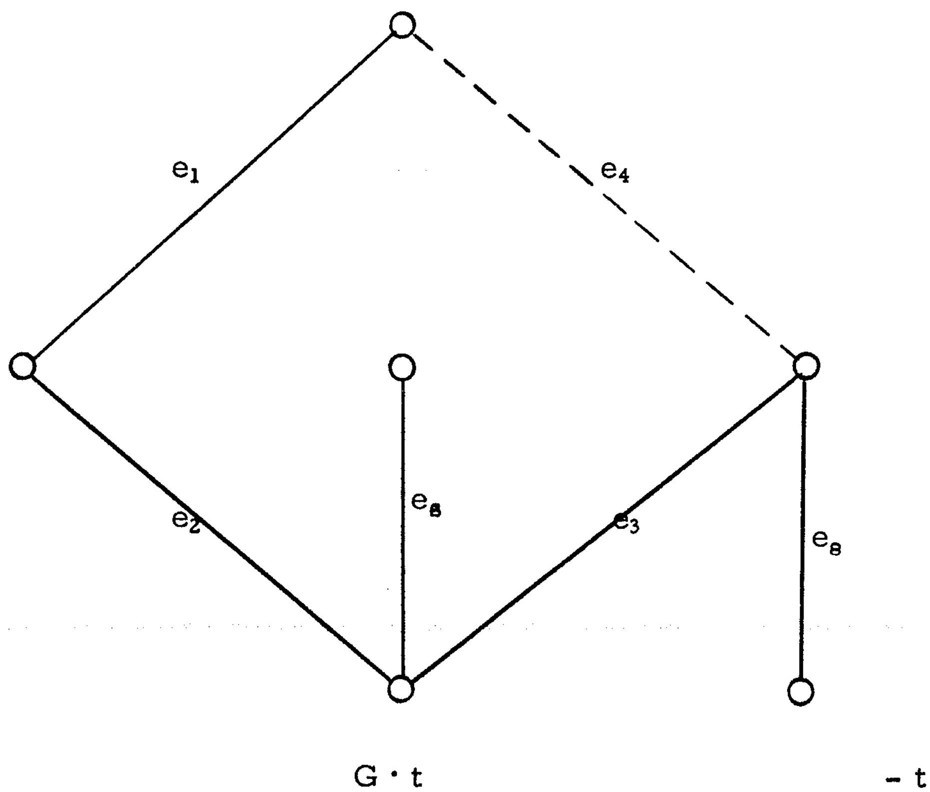


Figure 2-5. A Maximal Independent Set of G

Then by (2.1-2) there is a circuit C_i' satisfying

$$e_i' \in C_i' \subseteq (e_i'/e)b \quad ,$$

for $i = 1, \ldots, p$.

It is also true that $e \notin b'$ since otherwise $(e/e)b$ would be independent contrary to our assumption. Therefore by (2.1-3) there is a unique circuit C satisfying

$$e \notin C \subseteq b' \cup \{e\} \; .$$

Set $C_{(0)} = C$ and define $C_{(k)}$ recursively by

$$e \in C_{(k)} \subseteq [(C_{(k-1)} \cup C_k') - \{e_k'\}] \; , \tag{1}$$

for $k = 1, \ldots, p$.

We will show that $C_{(k)}$ exists for $k = 1, \ldots, p$ by using mathematical induction.

When $k = 1$, (1) reads

$$e \in C_{(1)} \subseteq [C_{(0)} \cup C_1') - \{e_1'\}] \; .$$

If $e_1' \in (C_{(0)} \cap C_1')$, then by the property (C2) of circuits $C_{(1)}$ exists. If $C_{(0)} \cap C_1'$ does not contain e_1', then setting $C_{(1)} = C_{(0)}$ gives us a $C_{(1)}$ satisfying (1).

Assume $C_{(k)}$ exists for $k = n < p$ and consider

$$e \in C_{(n+1)} \subseteq [(C_{(n)} \cup C_{n+1}') - \{e_{n+1}'\}] \; .$$

If $e_{n+1}' \in (C_{(n)} \cap C_{n+1}')$, then by the property (C2) of circuits and the induction hypothesis, $C_{(n+1)}$ exists. If $e_{n+1}' \notin (C_{(n)} \cap C_{n+1}')$, then by the induction hypothesis $C_{(n+1)} = C_{(n)}$ satisfies (1). Thus $C_{(k)}$ exists for $k = 1, \ldots, p$.

Moreover, $C_{(k)}$ satisfies

$$C_{(k)} \subseteq b \cup \{e'_p, e'_{p-1}, \ldots, e'_{k+1}\} \tag{2}$$

for $k = 0, 1, \ldots, p-1$ and

$$e \in C_{(p)} \subseteq b \ . \tag{3}$$

But (3) contradicts the hypothesis and accordingly the theorem is proved. ▌

We can sharpen (2.1-4) as follows.

> (2.1-5) If b and b' are bases of $\mathfrak{M}$ and e is an element of b, then there exists an element e' of b' such that (e'/e)b is a base.

Proof: Let b and b' be any pair of bases of $\mathfrak{M}$ for which the theorem fails. Thus there exists an $e \in b$ such that for every $e' \in b'$ the set (e'/e)b is not a base. Clearly, $e \notin b'$. According to (2.1-4) there is an element $e'' \in (b' - (b - \{e\}))$ such that $b'' = (e''/e)b$ is independent. The element e forms a unique circuit C in b" and $e'' \in C$. However since b" is not a base there exists an element $e''' \in E$ such that $b''' = b'' \cup \{e'''\}$ is an independent set. Therefore, e forms a unique circuit C in b''' by (2.1-2). Since $e'' \in C$, the set (e/e")b''' is an independent set. However $(e/e'')b''' = b \cup \{e'''\}$ and since b is a base, (e/e")b''' is not independent. Thus we have a contradiction and the theorem follows. ▌

Let S be any subset of E. We let $\alpha(S)$ denote the number of elements in S. We use this notation throughout the dissertation.

The following theorem is a basic result.

(2.1-6) If b and b' are any bases of $\mathfrak{M}$, then $\alpha(b) = \alpha(b')$.

Proof: Let b, b' be some pair of distinct bases of $\mathfrak{M}$. Without loss of generality assume that $\alpha(b) \geq \alpha(b')$.

Let $A = (b \cap b')$, $B = (b' - b)$ and $C = (b - b')$. Let the elements of E be enumerated such that

$$A = \{e_1, \ldots, e_p\} ,$$

$$B = \{e'_{p+1}, \ldots, e'_{p+s'}\}$$

and

$$C = \{e_{p+1}, \ldots, e_{p+s'}, e_{p+s'+1}, \ldots, e_{p+s'+s}\} ,$$

where $s \geq 0$. The integer s' cannot equal zero since $b \neq b'$ and, by the definition of a base, b' is not contained in b. By (2.1-5) there is an element e'_{k_1} in b' such that

$$b_1 = (e'_{k_1}/e_{p+1})b$$

is a base. Also $k_1 > p$. Therefore

$$b_1 = \{e_1, \ldots, e_p, e'_{k_1}, e_{p+2}, \ldots, e_{p+s'}, e_{p+s'+1}, \ldots, e_{p+s'+s}\} .$$

Applying (2.1-5) to b_1 and b', there exists an element $e'_{k_2} \in b'$ such that

$$b_2 = (e'_{k_2}/e_{p+2})b_1$$

is a base, $k_2 > p$ and $k_2 \neq k_1$. Continuing this process we can construct a base

$$b_{s'} = b' \cup \{e_{p+s'+1}, \ldots, e_{p+s'+s}\} .$$

But since b' is a base we conclude that $s = 0$. Therefore $\alpha(b) = \alpha(b')$. ▌

Theorem (2.1-6) has two familiar graph-theoretic interpretations. Let G be a graph. A forest of G is a maximal set $S \subseteq E(G)$ which contains no polygon of G. Obviously, a forest of G is a base of the matroid $\mathcal{P}(G)$ and vice versa. Theorem (2.1-6) implies that every forest of G has the same number of edges. In a dual manner we define a coforest of G to be a maximal set $T \subseteq E(G)$ which contains no bond of G. As before, we see that the coforests of G are in 1-1 correspondence with the bases of $\mathcal{B}(G)$ and accordingly, every coforest of G has the same number of edges.

We call a set $t \subseteq E(G)$ a tree if t contains no polygons of G and we call a set $c \subseteq E(G)$ a cotree if c contains no bonds of G. These last two quantities are the graph-theoretic counterparts of an independent set. Clearly, a forest is a maximal tree and a coforest is a maximal cotree.

From graph theory we know that the forests of G are the complements in E(G) of the coforests of G and accordingly there is a 1-1 correspondence between the bases of $\mathcal{P}(G)$ and $\mathcal{B}(G)$. The number of edges in a forest of G is called the rank of G and is denoted by r(G). The number of edges in a coforest of G is called the nullity of G and is denoted by $\mu(G)$; we then have

$$\mu(G) + r(G) = \alpha(E(G)) \quad .$$

Let $\mathcal{M}$ be a matroid on the set E. Then $A \subseteq E$ is called a cobase of $\mathcal{M}$ if $\bar{A}$ is a base of $\mathcal{M}$. We define the rank and the nullity of a matroid as:

(i) $r(\mathcal{M})$ is a non-negative integer equal to the number of elements in a base of $\mathcal{M}$. $r(\mathcal{M})$ is called the rank of the matroid $\mathcal{M}$.

(ii) $\mu(\mathcal{M})$ is a non-negative integer called the nullity of the matroid $\mathcal{M}$:

$$\mu(\mathcal{M}) = \alpha(E) - r(\mathcal{M}) \quad .$$

$\mu(\mathcal{M})$ is equal to the number of elements in a cobase of $\mathcal{M}$.

The rank and nullity of Ω, the null matroid, are zero:

$$r(\Omega) = \mu(\Omega) = 0 \quad .$$

The reductions and contractions of a graph G are operations on G which yield new graphs. There are similar operations we can perform on a matroid and we introduce these next.

Let $\mathcal{M} = (\mathcal{C}, E)$ be a matroid and $S \subseteq E$. Let $\mathcal{C} \times S$ be the set of those members of $\mathcal{C}$ which are contained in S. The set $\mathcal{C} \times S$ obviously satisfies (C1) and (C2) and consequently $(\mathcal{C} \times S, S)$ defines a matroid on S. We denote the matroid $(\mathcal{C} \times S, S)$ by $\mathcal{M} \times S$. The matroid $\mathcal{M} \times S$ is called the contraction of $\mathcal{M}$ to S.

Let $\mathcal{L}_S$ be the class of non-null intersections of the members of $\mathcal{C}$ with S. Let $\mathcal{C} \cdot S$ be the class of minimal members of $\mathcal{L}_S$. Tutte [Tu 12] has shown that $\mathcal{C} \cdot S$ satisfies the conditions of Axiom system I and therefore $(\mathcal{C} \cdot S, S)$ is a matroid on the set S. We denote $(\mathcal{C} \cdot S, S)$ by $\mathcal{M} \cdot S$. $\mathcal{M} \cdot S$ is called the reduction of $\mathcal{M}$ to S.

Tutte [Tu 12] has given the precise relationships between the graph-theoretic reductions and contractions and the matroid-theoretic reductions and contractions. We will state these results without proof; however, the reader should have no difficulty in seeing that the following statements are at least plausible.

Let G be a graph and $S \subseteq E(G)$.

$$(2.1\text{-}7) \quad \mathcal{P}(G) \times S = \mathcal{P}(G \cdot S) \quad .$$

$$(2.1\text{-}8) \quad \mathcal{P}(G) \cdot S = \mathcal{P}(G \times S) \quad .$$

$$(2.1\text{-}9) \quad \mathcal{B}(G) \times S = \mathcal{B}(G \times S) \quad .$$

$$(2.1\text{-}10) \quad \mathcal{B}(G) \cdot S = \mathcal{B}(G \cdot S) \quad .$$

<u>Example 2-3</u>. Let G be the graph of Fig.2-1 and $S = \{e_1, e_2, e_5, e_8\}$. $\mathcal{P}(G) \times S$ and $G \cdot S$ are shown in Fig.2-6 and it is easily verified that $\mathcal{P}(G) \times S = \mathcal{P}(G \cdot S)$.

In Fig.2-7 we show $\mathcal{P}(G) \cdot S$ and $G \times S$ and clearly $\mathcal{P}(G) \cdot S = \mathcal{P}(G \times S)$. Notice that the set S is a polygon as well as a bond of G.

The reader is encouraged to construct some simple examples in order to illustrate (2.1-9) and (2.1-10).

In this section we have taken the point of view of defining a matroid in terms of its class $\mathcal{C}$ of circuits. Whitney [Wh 1], in his pioneering paper, exhibited four axiom systems for defining a matroid on a finite set and showed them to be equivalent to one another. Below we present the three additional axiom systems for a matroid.

With regard to Axiom System I it has been shown [Ro 1; Le 2] that condition (C2) can be replaced by (C2)'.

(C2)' If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in (C_1 \cap C_2)$, then there exists a $C_3 \in \mathcal{C}$ satisfying $C_3 \subseteq [(C_1 \cup C_2) - \{e\}]$.

$$\wp(G) \times S = \{ \{ e_1, e_2, e_5, e_6 \} \}$$

G · S

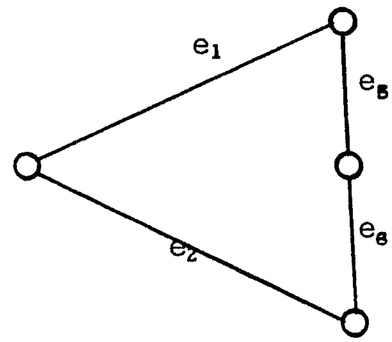


Figure 2-6. Matroid Contraction and Graphical Reduction

$$\wp(G) \cdot S = \{ \{ e_1, e_5 \}, \{ e_1, e_6 \}, \{ e_1, e_2 \}, \{ e_5, e_6 \}, \{ e_5, e_2 \}, \{ e_6, e_2 \} \}$$

G × S

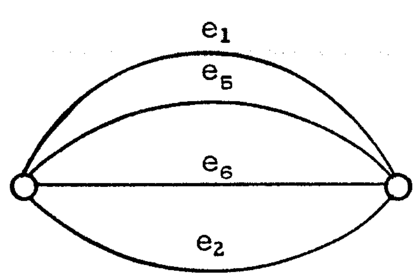


Figure 2-7. Matroid Reduction and Graphical Contraction

Thus properties (C1) and (C2)' are completely equivalent to (C1) and (C2). It is important to have equivalent characterizations of circuits since there are times when one characterization is more useful than the other and vice versa as in the proof of (2.2-1).

Axiom System II. Let E be a finite set and $\mathcal{J}$ a class of subsets of E. The members of $\mathcal{J}$ are called independent sets and define a matroid $\mathcal{M} = (\mathcal{J}, E)$ on the set E if the following conditions are satisfied:

(I1) Any subset of an independent set is also an independent set.

(I2) If S, S' $\in \mathcal{J}$ and $\alpha(S') = \alpha(S) + 1$, then there is an element $e' \in S'$ not belonging to S such that $S \cup \{e'\}$ is an independent set.

Axiom System III. Let E be a finite set and let $\mathcal{B}$ be a class of subsets of E. The members of $\mathcal{B}$ are called bases and define a matroid $\mathcal{M} = (\mathcal{B}, E)$ on the set E if the following conditions are satisfied:

(B1) No proper subset of a base is a base.

(B2) If b and b' are bases and $e \in b$, then there is an element $e' \in b'$ such that $(e'/e)b$ is a base.

Axiom System IV. Let E be a finite set and let ρ be a function from 2^E, the power set of E, to the non-negative integers. The number $\rho(S)$ is called the rank of S where $S \subseteq E$. The function ρ defines the matroid $\mathcal{M} = (\rho, E)$ on the set E if the following conditions are satisfied:

(R1) $\rho(\emptyset) = 0$.

(R2) For any $S \subseteq E$ and $e \notin S$, $\rho(S \cup \{e\}) = \rho(S) + k$, where

$k = 0$ or 1.

(R3) For any $S \subseteq E$ and e_1 and e_2 not in S, if

$\rho(S \cup \{e_1\}) = \rho(S \cup \{e_2\}) = \rho(S)$, then $\rho(S \cup \{e_1, e_2\}) = \rho(S)$.

If the matroid $\mathcal{M}$ is defined by its class $\mathcal{C}$ of circuits, then we have defined what we mean by an independent set and a base. Therefore the classes $\mathcal{I}$ and $\mathcal{B}$ are well defined. The rank function ρ defined in the Axiom System IV is related to the rank of a matroid by

$$\rho(S) = r(\mathcal{M} \times S)$$

for any $S \subseteq E$. Therefore given $\mathcal{M} = (\mathcal{C}, E)$ we can obtain the equivalent descriptions $\mathcal{M} = (\mathcal{I}, E)$, $\mathcal{M} = (\mathcal{B}, E)$ and $\mathcal{M} = (\rho, E)$.

Next we will outline how one gets from any one of the other three descriptions of a matroid to any other description.

Suppose we are given $\mathcal{M} = (\mathcal{I}, E)$, that is, $\mathcal{M}$ is described in terms of its class of independent sets. Any set $S \subseteq E$ which is not a member of $\mathcal{I}$ is called a dependent set. Let $\mathcal{D}$ be the class of dependent sets. A member of $\mathcal{D}$ is called minimal if it does not properly contain any member of $\mathcal{D}$. The class of minimal members of $\mathcal{D}$ is the class of circuits of $\mathcal{M}$. Once we have $\mathcal{C}$, the class of circuits of $\mathcal{M}$, we can obtain any one of the equivalent descriptions of $\mathcal{M}$.

Next assume we have $\mathcal{B}$, the class of bases of $\mathcal{M}$. Then a subset $S \subseteq E$ is a member of $\mathcal{I}$ if either S is a member of $\mathcal{B}$ or S is a subset of some member of $\mathcal{B}$. Once we have $\mathcal{I}$, we can then obtain any descript-ion of $\mathcal{M}$.

Lastly suppose $\mathcal{M}$ is defined in terms of ρ. The class $\mathcal{Q}$ consists of those subsets of E which satisfy $\alpha(S) = \rho(S)$.

Whitney's remarkable contribution was to recognize the four axiom systems and show that they were all equivalent. In Table 2-1 we summarize the relationships between various matroid quantities.

TABLE 2-1

b - Base	- b is a maximal independent set - b is a maximal set which contains no circuits - b is a maximal set satisfying $\alpha(b)=\rho(b)$
I - Independent set	- I is a subset of a base - I contains no circuit - $\alpha(I) = \rho(I)$
C - Circuit	- C is a minimal dependent set - $\alpha(C-\{e\}) = \rho(C-\{e\})$ for every $e \in C$

2.2 VECTOR SPACES AND MATROIDS

In this section we show how to associate a matroid with a vector space. The connection between vector spaces and matroids provides a way to define certain special classes of matroids. We also introduce the representative matrix of a vector space and study some of its properties.

Let E be a finite set $E = \{e_1, \ldots, e_n\}$ and F a field.

By a vector on E over F we mean a mapping f of E into F. The number $f(e_i)$ is called the value of f at e_i. The support $\|f\|$ of the vector f is the set of all members of E whose values under f are nonzero. If $\|f\| = \phi$, then f is the zero vector and it is denoted by 0.

The sum of two vectors f and g on E over F is a vector f+g defined as

$$(f+g)(e_i) = f(e_i) + g(e_i)$$

for $i = 1, \ldots, n$.

The product of a number $\lambda \in F$ and a vector f on E over F is a vector λf defined as

$$(\lambda f)(e_i) = \lambda f(e_i)$$

for $i = 1, \ldots, n$.

Let $\mathcal{V}$ be a collection of vectors in E over F which is closed under the operations of addition and multiplication by elements of F. Then $\mathcal{V}$ is called a vector space on E over F.

If $\mathcal{V}$ is a vector space on E over F, then a vector $f \in \mathcal{V}$ is called elementary if it is nonzero and there is no nonzero vector $g \in \mathcal{V}$ which satisfies $\|g\| \subset \|f\|$. We use $\subset$ to denote proper inclusion, that is, $A \subset B$ implies $A \subseteq B$ and $A \neq B$.

(2.2-1) Let $\mathcal{V}$ be a vector space on E over F and $\mathcal{C}_{\mathcal{V}}$ the class of supports of elementary vectors in $\mathcal{V}$. Then $\mathcal{C}_{\mathcal{V}}$ satisfies conditions (C1) and (C2) of Axiom System I.

Proof: By the definition of an elementary vector, (C1) is satisfied.

To show that (C2) is also satisfied by the members of $\mathcal{C}_{\mathcal{V}}$, pick $C_1, C_2 \in \mathcal{C}_{\mathcal{V}}$ and suppose that $e_k \in C_1 \cap C_2$, where $1 \leq k \leq n$. By the definition of $\mathcal{C}_{\mathcal{V}}$, there exists vectors $f_1, f_2 \in \mathcal{V}$ such that $\|f_1\| = C_1$ and $\|f_2\| = C_2$. Define the vector $h = f_1(e_k)f_2 + (-1)f_2(e_k)f_1$. It is clear from the definition of h that

$$\|h\| \subseteq [(\|f_1\| \cup \|f_2\|) - \{e_k\}] \quad .$$

Suppose $C_1 \neq C_2$ and, without loss of generality, that $(C_1 - C_2) \neq \phi$. Accordingly, $\|h\| \neq \phi$. Therefore there exists an elementary vector $g \in \mathcal{V}$ satisfying $\|g\| \subseteq \|h\|$. Consequently, there exists a $C \in \mathcal{C}_{\mathcal{V}}$ satisfying

$$C = \|g\| \subseteq \|h\| \subseteq [(C_1 \cup C_2) - \{e_k\}] \quad .$$

Therefore the class $\mathcal{C}_{\mathcal{V}}$ satisfies (C1) and (C2)' and by the equivalence of (C1) and (C2)' with (C1) and (C2), the theorem follows. ▌

By (2.2-1), $(\mathcal{C}_{\mathcal{V}}, E)$ defines a matroid on the set E. We denote $(\mathcal{C}_{\mathcal{V}}, E)$ by $\mathcal{M}_{\mathcal{V}}$. $\mathcal{M}_{\mathcal{V}}$ is called the matroid associated with the vector space $\mathcal{V}$. Also, throughout this dissertation $\mathcal{C}_{\mathcal{V}}$, where the subscript $\mathcal{V}$ is any vector space on a set E over F, represents the class of supports of the elementary vectors of $\mathcal{V}$. We are now in a position to define certain classes of matroids which are associated with corresponding classes of vector spaces.

If $\mathcal{V}$ is a vector space on E over F, where F is the field of integers modulo 2, then $\mathcal{V}$ is called a binary vector space. The matroid $\mathcal{M}_{\mathcal{V}} = (\mathcal{C}_{\mathcal{V}}, E)$, associated with a binary vector space $\mathcal{V}$, is called a binary matroid.

Let F be the real number field and $\mathcal{V}$ a vector space on E over F. A vector $g \in \mathcal{V}$ is called a primitive vector if it is an elementary vector all of whose values are ± 1 or 0.

A vector space $\mathcal{V}$ on E over F, where F is the field of real numbers, is called regular if corresponding to each elementary vector $f \in \mathcal{V}$ there is a primitive vector $g \in \mathcal{V}$ satisfying $\|f\| = \|g\|$. If $\mathcal{V}$ is a regular vector space, then we call $\mathcal{M}_{\mathcal{V}} = (\mathcal{C}_{\mathcal{V}}, E)$ a regular matroid.

Tutte [Tu 6] has solved the difficult converse problem. Thus he was able to characterize those abstract matroids $\mathcal{M} = (\mathcal{C}, E)$ which can be shown to be isomorphic to a regular matroid $\mathcal{M}_{\mathcal{V}} = (\mathcal{C}_{\mathcal{V}}, E)$.

The remainder of this section is devoted to the consideration of representative matrices for vector spaces. We will see how the matroid and vector space structures are reflected in the properties of the representative matrix and vice versa.

Let $\mathcal{V}$ be a vector space on E over F. Then for any $f \in \mathcal{V}$ we define R_f, the representative vector of f, as the 1-rowed matrix

$$R_f = [\, f(e_1), \ldots, f(e_n) \,] \;.$$

A matrix R with elements in F is called a representative matrix of $\mathcal{V}$ if it satisfies the following:

(i) The rows of R are linearly independent representative vectors.

(ii) Every nonzero vector of $\mathcal{V}$ has a representative vector which is a linear combination of the rows of R.

A matrix R satisfying (i) and (ii) completely determines the vector space $\mathcal{V}$ on E over F. Conversely, any matrix R with linearly independent rows can be interpreted as a representative matrix of a vector space. Certain submatrices of R have special properties and so we introduce a useful notation.

If $S \subseteq E$, then by R(S) we mean the submatrix of R consisting of those columns of R which correspond to the members of S.

From linear algebra we know, due to the finiteness of E, that any representative matrix for $\mathcal{V}$ will have the same number of rows and that the number of rows cannot exceed the number of columns. It is also a consequence of linear algebra that if R is a representative matrix for $\mathcal{V}$, then any other representative matrix R' for $\mathcal{V}$ can be obtained from R by

$$R' = TR \quad ,$$

where T is a non-singular matrix. Consequently if R is $\mu \times n$, the columns of R which form nonzero μ^{th}-order minors will also form nonzero μ^{th}-order minors in R' and vice versa.

Theorem (2.2-2) relates the nonzero μ^{th}-order minors of R to the structure of the matroid.

(2.2-2) Let the $\mu \times n$ matrix R be a representative matrix for $\mathcal{V}$, a vector space on E over F. Let S be a subset of E. Then $\det[R(S)] \neq 0$ if and only if S is a cobase of $\mathcal{M}_{\mathcal{V}}$. The dimension of the vector space is equal to μ, the number of elements in a cobase of $\mathcal{M}_{\mathcal{V}}$.

Proof: Let b be a base of $\mathcal{M}_{\mathcal{V}} = (\mathcal{C}_{\mathcal{V}}, E)$ and enumerate the elements of E such that

$$\bar{b} = \{e_1, \ldots, e_s\}$$

and permute the columns of R accordingly.

Let f_i be an elementary vector in $\mathcal{V}$ satisfying $\|f_i\| = J(b, e_i)$ for $i = 1, \ldots, s$. Let R_{f_i} be the representative vector of f_i for $i = 1, \ldots, s$ and form the $s \times n$ matrix

$$H = \begin{bmatrix} R_{f_1} \\ \vdots \\ R_{f_s} \end{bmatrix} .$$

We claim that H is a representative matrix for $\mathcal{V}$.

The rows of H are linearly independent, since

$$\det[H(\bar{b})] = f_1(e_1)\, f_2(e_2) \ldots f_s(e_s) \neq 0 .$$

Assume there exists a vector $g \in \mathcal{V}$ whose representative vector R_g is not a linear combination of the rows of H.

Set $$Q = R_g - \left(\frac{g(e_1)}{f_1(e_1)} R_{f_1} + \ldots + \frac{g(e_s)}{f_s(e_s)} R_{f_s}\right) .$$

Consequently there is a nonzero vector $q \in \mathcal{V}$ such that

$$Q = [q(e_1), \ldots, q(e_s), q(e_{s+1}), \ldots, q(e_n)] \ .$$

According to the definition of Q,

$$\|q\| \subseteq b \ . \qquad (1)$$

But (1) is a contradiction since b is a base. Therefore H is a representative matrix for $\mathcal{V}$. Since any representative (specifically R) matrix for can be obtained from H by multiplication on the left by a non-singular matrix, it follows that $s = \mu$ and $\det [R(\bar{b})] \neq 0$.

To show necessity, let R be a $\mu \times n$ representative matrix for $\mathcal{V}$ and $S \subseteq E$ satisfying $\det [R(S)] \neq 0$. Enumerate the elements of E such that $S = \{e_1, \ldots, e_\mu\}$ ($\alpha(S) = \mu$ since $\det [R(S)]$ is defined only when R(S) is a square matrix) and permute the columns of R accordingly.

Let $R' = R(S)^{-1} R$. R' is a representative matrix for $\mathcal{V}$ and can be partitioned as follows:

$$R' = [\, 1_\mu \mid R'_{12} \,] \ ,$$

where 1_μ is a $\mu \times \mu$ unit matrix. According to the form of R', we can characterize the set S as a minimal set which meets every circuit of $\mathcal{C}_\mathcal{V}$. $\bar{S}$, the complement of S in E, is accordingly a maximal set which contains no member of $\mathcal{C}_\mathcal{V}$. Therefore $\bar{S}$ is a base and S is a cobase. ∎

Let R be a $\mu \times n$ representative matrix for the vector space $\mathcal{V}$ and let $\bar{b}$ be a cobase of $\mathcal{M}_\mathcal{V}$. Then by (2.2-2), $\det [R(\bar{b})] \neq 0$ and we can set

$$R' = R(\bar{b})^{-1} R \ .$$

R' is called the <u>standard representative matrix</u> of $\mathcal{V}$ with respect to the cobase $\bar{b}$. It follows that $R'(\bar{b}) = 1_{\mu}$.

(2.2-3) Let $\mathcal{V}$ be a vector space on E over F and R a $\mu \times n$ standard representative matrix of $\mathcal{V}$ with respect to the cobase $\bar{b}$ of $\mathcal{M}_{\mathcal{V}}$. Then the rows of R are the representative vectors of elementary vectors in $\mathcal{V}$.

<u>Proof</u>: By hypothesis $R(\bar{b}) = 1_{\mu}$.

Assume that the s^{th} row of R is a representative vector of a non-elementary vector $f \in \mathcal{V}$. Consequently there exists an elementary vector $g \in \mathcal{V}$ satisfying $\|g\| \subset \|f\|$. Since b is a base $e_{i_s} \in \|g\| \cap \|f\|$, where $\bar{b} = \{e_{i_1}, e_{i_2}, \ldots, e_{i_\mu}\}$ and $1 \leq s \leq \mu$. Then by the property (C2)' of the members of $\mathcal{C}_{\mathcal{V}}$, there exists an elementary vector h satisfying

$$\|h\| \subseteq [(\|g\| \cup \|f\|) - \{e_{i_s}\}] \subseteq b \quad . \qquad (1)$$

But (1) is impossible since b is a base. Accordingly the theorem is proved. ∎

(2.2-4) Let $\mathcal{V}$ be a vector space on E over F. Let h and g be two elementary vectors in $\mathcal{V}$ satisfying $\|h\| = \|g\|$. Then $h = \lambda g$, where λ is some number in F.

<u>Proof</u>: Assume the theorem is false, that is, there exist elementary vectors h and g in $\mathcal{V}$ satisfying $\|h\| = \|g\|$ and for which no $\lambda \in F$ does $h = \lambda g$. Pick some $e_k \in \|h\|$. Set

$$f = \left[\frac{g(e_k)}{h(e_k)}\right] h + (-1)g \quad .$$

It is clear that f satisfies $\|f\| \subset \|h\|$ and by assumption $\|f\| \neq \phi$. Thus we have a contradiction and accordingly the theorem follows. ∎

It is important to keep in mind that theorems (2.2-1) through (2.2-4) are independent of the field F and specifically apply to the field of real numbers and to the field of integers modulo 2. These two fields are of primary interest to us.

We conclude this section with two important theorems on regular vector spaces and their associated representative matrices.

(2.2-5) Let R be a $\mu \times n$ matrix over the field of real numbers whose rows are linearly independent. Then R is a representative matrix of a regular vector space $\mathcal{V}$ if and only if the determinants of its square submatrices of order μ are restricted to the values 0, k and -k, where k is some positive number.

Proof: Let R be a representative matrix for the regular vector space $\mathcal{V}$. By (2.2-2) det $[R(S)] \neq 0$ if and only if S is a cobase of $\mathcal{M}_{\mathcal{V}}$. Set $R' = R(S)^{-1}R$ and $R'' = R(T)^{-1}R$, where S and T are cobases of $\mathcal{M}_{\mathcal{V}}$.

By Thereoms (2.2-3), (2.2-4) and the hypothesis that $\mathcal{V}$ is regular, it follows that R' and R" have entries equal to ± 1 or 0 and that

$$R' = QR'' \quad ,$$

where Q is a nonsingular matrix with entries equal to ± 1 or 0. Consequently, since $\det[R'(S)] = \det[Q] \cdot \det[R''(S)] = 1$, we get that

$$\det[Q] = \det[R''(S)] = \pm 1 \quad .$$

Therefore

$$\det [R(S)] = \pm \det [R(T)] .$$

Consequently, the μ^{th}-order minors of R are equal to $\pm$ k or 0, where $k = |\det [R(S)]|$.

To show sufficiency now let R be a $\mu \times n$ matrix with linearly independent rows and whose μ^{th}-order minors equal $\pm k$ or 0, where k is some positive number. Let $\mathcal{V}$ be the vector space on E over the field of real numbers, which has R as its representative matrix.

Let f be an elementary vector in $\mathcal{V}$ and e_{i_1} any element in $\|f\|$. The set $(\|f\| - \{e_{i_1}\})$ is independent in $\mathcal{M}_{\mathcal{V}}$ and consequently can be extended into a base b such that $e_{i_1} \in \overline{b} = \{e_{i_1}, \ldots, e_{i_\mu}\}$. Let R' be a standard representative matrix for $\mathcal{V}$ with respect to the cobase $\overline{b}$. Therefore R' = TR, where T is some $\mu \times \mu$ nonsingular matrix. By (2.2-3) and (2.1-3) the i_1^{th} row of R' is the representative vector of a $g \in \mathcal{V}$ which satisfies $\|g\| = \|f\|$. Consider the matrices

$$K_p = R'\left((e_{i_p}/e_{i_1})\overline{b}\right) ,$$

where $e_{i_p} \in b$ for $p = \mu + 1, \ldots, n$. By hypothesis and the construction of R', it follows that $\det [K_p] = \pm 1$ or 0, for $p = \mu + 1, \ldots, n$. Accordingly the i_1^{th} row of R' has entries equal to $\pm$ 1 or 0 and thus g is a primitive vector corresponding to the elementary vector f. ∎

A $\mu \times n$ matrix R, over the real field, is called a <u>regular matrix</u> if (i) the rank of R equals μ and (ii) the μ^{th}-order minors of R are restricted to the values $\pm$ k or 0, where k is some positive number. Thus we can restate (2.2-5) as: $\mathcal{V}$ is a regular vector space on E over F, the field of real numbers, if and only if R, any representative matrix for $\mathcal{V}$, is a regular matrix.

A matrix which is related to a regular matrix is the totally unimodular matrix. A matrix K of real numbers is said to be <u>totally unimodular</u> if <u>every</u> minor of K has value $\pm$ 1 or 0.

> (2.2-6) [Tu 1] Let $\mathcal{V}$ be a vector space on E over F, the field of real numbers, and R be a standard representative matrix for $\mathcal{V}$. Then $\mathcal{V}$ is a regular vector space if and only if R is a totally unimodular matrix.

It is not difficult to show that a standard representative matrix of a regular vector space must be a totally unimodular matrix and thus (2.2-6) follows from (2.2-5).

2.3 VECTOR SPACES, GRAPHS AND MATROIDS

In Chapter 3 we will be concerned with electrical networks and the constraints imposed on the currents and voltages by the interconnections of the network. Therefore in this section we will present results which relate the vector spaces associated with a graph to the structure of the graph and its associated matroids. The vector spaces associated with a network graph are the constraint spaces for currents and voltages in an electrical network.

Let G be an oriented graph, that is, we orient the graph by assigning a positive end and a negative end to each member of E(G). We do this by defining the integer $\eta(e, v)$ for each $e \in E(G)$ and $v \in V(G)$ as follows:

$$\eta(e, v) = \begin{cases} 0 \text{ if } v \text{ is not an end of } e \text{ or } e \text{ has coincident ends.} \\ 1 \text{ if } v \text{ has the positive end of } e. \\ -1 \text{ if } v \text{ is the negative end of } e. \end{cases}$$

In Fig.2-8 we have used arrows on the edges to designate the orientation. The arrow points from the positive to the negative end of an edge. Therefore

$$\eta(e_1, v_1) = -1$$

$$\eta(e_1, v_2) = +1 \quad .$$

One must be careful to interpret -1 according to the field F. For example, if F is the field of integers modulo 2, then -1, the additive inverse of 1, is equal to 1, that is, $1 + 1 = 0$ and accordingly η takes on values in $\{0, 1\}$.

Suppose $E(G) = \{e_1, \ldots, e_n\}$ and let f be a vector on E(G) over F. We call f a 1-cycle of G over F if

$$\sum_{i=1}^{n} \eta(e_i, v)\ f(e_i) = 0$$

for all $v \in V(G)$.

Example 2-4. Let G be the graph in Fig.2-8 and F the field of real numbers. Let f be a vector on E(G) over F and R_f the representative matrix for f, where

$$R_f = [1, 1, -2, 1, 1, -4, 0] .$$

In order to check if f is a 1-cycle we must evaluate S_j, where

$$S_j = \sum_{i=1}^{7} \eta(e_i, v_j)\ f(e_i)$$

for $j = 1, \ldots, 5$. When $j = 1$,

$$\eta(e_i, v_1) = \begin{cases} -1 & i = 1, 2, 3 \ . \\ 0 & \text{otherwise}. \end{cases}$$

Consequently,

$$S_1 = -1 -1 +2 = 0 \ .$$

In fact, $S_j = 0$ for $j = 1, \ldots, 5$ and therefore f is a 1-cycle of G.

Now let F be the field of integers modulo 2, g a vector on E(G) over F and R_g the representative matrix for g:

$$R_g = [1,1,0,1,1,1,0] \ .$$

In order to check if g is a 1-cycle we must evaluate S_j for $j = 1, \ldots, 5$. When $j = 4$,

$$\eta(e_i, v_4) = \begin{cases} 1 & i = 2, 5 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

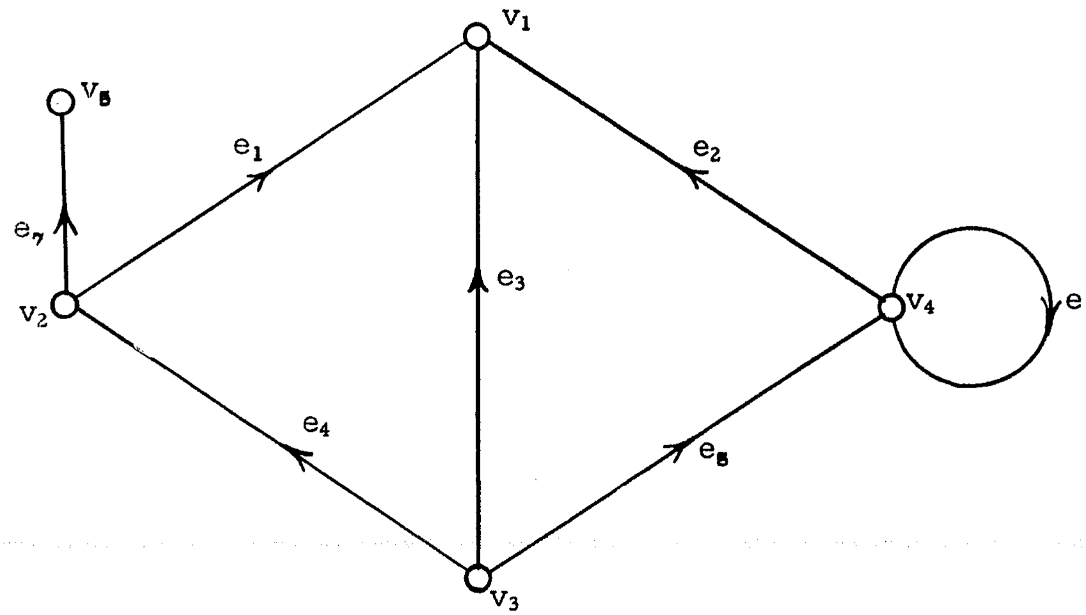


Figure 2-8. Graph G of Example 2-2

Therefore

$$S_4 = 1 + 1 = 0 \quad .$$

Again $S_j = 0$ for $j = 1, \ldots, 5$ and therefore g is a 1-cycle of G.

Let G be a graph and I the class of 1-cycles of G over F. It is clear from the definition of a 1-cycle that I is a vector space on E(G) over F. I is called the <u>1-cycle space</u> of G.

(2.3-1) Let G be an oriented graph and I the 1-cycle space of G over F. Then $\mathcal{C}_I = \mathcal{P}(G)$.

<u>Proof</u>: Suppose S is a polygon of G. We can easily construct a 1-cycle f_S of G such that $\| f_S \| = S$. It is clear from the definition of a 1-cycle that f_S is elementary. (Also note that if F is the field of real numbers we can construct f_S to be a primitive vector.) Therefore

$$\mathcal{P}(G) \subseteq \mathcal{C}_I \quad . \tag{1}$$

Conversely, suppose $C \in \mathcal{C}_I$. Then there exists an elementary 1-cycle f in I satisfying $\|f\| = C$. It is clear from the definition of a 1-cycle that the valence of every vertex in $G \cdot \|f\|$ is greater than one. Consequently $G \cdot \|f\|$ contains a polygon T. Therefore, by the first part of the proof, there exists an elementary vector $g_T \in I$ satisfying $\|g_T\| = T$. Since f is elementary, $\|g_T\| = \|f\|$ and therefore $C = T$. Accordingly

$$\mathcal{C}_I \subseteq \mathcal{P}(G) \quad . \tag{2}$$

Combining (1) and (2), we get $\mathcal{C}_I = \mathcal{P}(G)$. ∎

The next theorem is a simple corollary of (2.3-1).

(2.3-2) The polygon matroid of a graph is a binary (regular) matroid.

In what follows we relate $\mathcal{B}(G)$ to a vector space associated with G.

Let g be a vector defined on V(G) over F, where $V(G) = \{v_1, \ldots, v_m\}$. We define a vector f on E(G) over F by

$$f(e_i) = \sum_{j=1}^{m} \eta(e_i, v_j)\, g(v_j) \quad ,$$

for all $e_i \in E(G)$. f is called the <u>coboundary</u> of g. We denote by V the collection of coboundaries of all vectors g on V(G) over F. It is clear from the definition of a coboundary on G that V is a vector space on E(G) over F. V is called the <u>coboundary space</u> of G.

(2.3-3) Let G be an oriented graph and V the coboundary space of G. Then $\mathcal{C}_V = \mathcal{B}(G)$.

<u>Proof</u>: Let S be a bond of G and H_1 and H_2 the two vertices of $G \times S$. Define a vector g on V(G) over F which assigns to all the vertices of G in H_1 the value +1 and 0 to all other vertices of G. Let f be the coboundary of g. Clearly $\|f\| = S$. A simple graph-theoretic argument shows that f is elementary. Therefore

$$\mathcal{B}(G) \subseteq \mathcal{C}_V \quad . \tag{1}$$

Conversely, let C be a member of $\mathcal{C}_V$ and $f \in V$ an elementary vector satisfying $\|f\| = C$. Pick some $e \in \|f\|$ and let v be one of the distinct ends of e. Let $T \subseteq V(G)$ correspond to the set of vertices connected to v along paths whose every edge has value zero under f. Let

$Q \subseteq E(G)$ be the set of edges which have one end in T and the other not in T. Clearly Q contains a bond S of G and $Q \subseteq \|f\|$. Therefore, by the first part of this proof, there exists an elementary vector $g_S \in V$ satisfying $\|g_S\| = S$. By hypothesis $\|f\| = \|g_S\|$ and therefore $S = C$. Consequently,

$$\mathcal{C}_V \subseteq \mathcal{B}(G) \quad . \tag{2}$$

Combining (1) and (2) we get $\mathcal{C}_V = \mathcal{B}(G)$. ▌

> (2.3-4) The bond matroid of G is a binary (regular) matroid.

From (2.3-2) and (2.3-4) we obtain the following:

> (2.3-5) Let G be an oriented graph and V and I the coboundary space and the 1-cycle space, respectively, of G over the field F of real numbers. Then V and I are both regular vector spaces.

The vector spaces V and I have an additional property which we now present. We say that two vectors f and g on E over F are <u>orthogonal</u> if

$$\sum_{i=1}^{n} g(e_i)\, f(e_i) = 0 \quad .$$

Two vector spaces $\mathcal{V}$ and $\mathcal{W}$ are said to be <u>orthogonal</u> if for every $v \in \mathcal{V}$ and $w \in \mathcal{W}$, v and w are orthogonal vectors.

> (2.3-6) Let G be an oriented graph and V and I the coboundary space and 1-cycle space, respectively, of G over the field F. Then V is orthogonal to I.

<u>Proof</u>: If $f \in I$, then

$$\sum_{i=1}^{n} \eta(e_i, v_j)\, f(e_i) = 0 \quad , \tag{1}$$

for $j = 1, \ldots, m$.

If $f' \in V$, then

$$f'(e_i) = \sum_{j=1}^{m} \eta(e_i, v_j)\, g(v_j) \quad , \tag{2}$$

for $i = 1, \ldots, n$, where $g(v_j)$ is a vector on V(G) over F.

$$\sum_{i=1}^{n} f(e_i)\, f'(e_i) = \sum_{i=1}^{n} f(e_i) \sum_{j=1}^{m} \eta(e_i, v_j)\, g(v_j) = \sum_{j=1}^{m} g(v_j) \sum_{i=1}^{n} \eta(e_i, v_j)\, f(e_i)$$

$$= \sum_{j=1}^{m} g(v_j) \cdot 0 = 0 \quad . \blacksquare$$

2.4 DUALITY IN MATROIDS

An important concept in matroid theory is that of duality and in this section we present, mainly without proof, some of the important definitions and results on dual matroids.

Two sets S and T are said to be <u>orthogonal</u> if $\alpha(S \cap T) \neq 1$.

Let $\mathcal{M} = (\mathcal{C}, E)$ be a matroid on a finite set E and $\mathcal{O}$ the class of subsets of E which are orthogonal to every member of $\mathcal{C}$:

$$\mathcal{O} = \{ S \mid S \subseteq E \text{ and } S \text{ is orthogonal to every member of } \mathcal{C} \} .$$

Let $\mathcal{C}^*$ bt the class of minimal members of $\mathcal{O}$. Tutte [Tu 12] has shown that $\mathcal{C}^*$ satisfies the conditions of Axiom System I and therefore

$$\mathcal{M}^* = (\mathcal{C}^*, E)$$

is a matroid on the set E. $\mathcal{M}^*$ is called the <u>dual</u> matroid of $\mathcal{M}$. Thus every matroid has a dual matroid.

We present next some of the important properties of dual matroids.

(2.4-1) $(\mathcal{M}^*)^* = \mathcal{M}$.

(2.4-2) The bases of $\mathcal{M}^*$ are the complements in E of the bases of $\mathcal{M}$.

(2.4-3) $r(\mathcal{M}) + r(\mathcal{M}^*) = \alpha(E)$.

(2.4-3) is a simple consequence of (2.4-2).

(2.4-4) If $\mathcal{M}$ is a regular (binary) matroid, then $\mathcal{M}^*$ is a regular (binary) matroid.

(2.4-5) Let b be a base of $\mathcal{M}$, $e \notin b$ and $e' \in J(b,e) \cap b$. Then $e \in J^* (\bar{b}, e')$,where J^* is defined with respect to $\mathcal{M}^*$.

Let $\mathcal{V}$ be a vector space on E over F. Set

$$\perp\mathcal{V} = \{ f \mid f \notin \mathcal{V} \text{ and } f \text{ is orthogonal to every member of } \mathcal{V} \} .$$

The set $\perp\mathcal{V}$ is a vector space and is called the <u>complementary orthogonal space</u> of $\mathcal{V}$. It is a well known fact from linear algebra that the dimension of the vector space $\perp\mathcal{V}$ is equal to α(E) minus the dimension of the vector space $\mathcal{V}$. If R is a representative matrix for $\mathcal{V}$ and R* is a representative matrix for $\perp\mathcal{V}$, then it is clear from the definition of $\perp\mathcal{V}$ that

$$R^*R^t = O_{a \times b} \tag{1}$$

where a = dimension $\perp\mathcal{V}$ and b = dimension $\mathcal{V}$. Theorem (2.4-6) relates the matroids associated with complementary orthogonal vector spaces.

(2.4-6) $\mathcal{M}^*_{\mathcal{V}} = \mathcal{M}_{\perp\mathcal{V}}$.

In Theorem (2.2-2) we saw that the cobases in $\mathcal{M}_{\mathcal{V}}$ corresponded to the nonzero μ^{th}-order minors of R, where R is $\mu \times n$ and $\mu = \mu(\mathcal{M}_{\mathcal{V}})$ = dimension of $\mathcal{V}$. From Theorems (2.4-2) and (2.4-6) it follows that the bases of $\mathcal{M}_{\mathcal{V}}$ correspond to the nonzero r^{th}-order minors of R^*, where R^* is $r \times n$ and $r = r(\mathcal{M}_{\mathcal{V}})$ = dimension of $\perp\mathcal{V}$.

Another relationship between the structure of $\mathcal{M}_{\mathcal{V}}$ and the matrix R^* follows from (1). By a <u>minimal dependent</u> set of columns of a matrix we mean a set of columns which are linearly dependent and any proper subset of them is linearly independent. It is clear from (1) that the minimal dependent sets of columns of R^* correspond to the representative vectors, in the row space of R, of elementary vectors in $\mathcal{V}$. Consequently the minimal dependent sets of columns of R^* correspond to the members of $\mathcal{C}_{\mathcal{V}}$. A dual statement can be made for R and $\mathcal{C}_{\perp\mathcal{V}}$.

It is useful to be aware of the relationships between the matroid structure and the corresponding matrix properties of R and R^*. These relationships are summarized in Table 2-2.

Let G be an oriented graph and V and I the coboundary space and 1-cycle space, respectively, of G over F. We have previously shown in (2.3-6) that V and I are orthogonal vector spaces and in Section 2.1 pointed out that $\mu(G) + r(G) = \alpha(E(G))$. It is clear that one can construct $\mu(G)$ linearly independent 1-cycles and r(G) linearly independent coboundaries. Consequently V and I are complementary orthogonal vector spaces.

> (2.47-7) Let G be an oriented graph and V and I the coboundary space and 1-cycle space, respectively, of G over F. Then $V = \perp I$.

It follows from (2.3-1), (2.3-3), (2.4-6) and (2.4-7) that $\mathcal{P}(G)$ and $\mathcal{B}(G)$ are dual matroids.

> (2.4-8) Let G be a graph. Then $\mathcal{P}(G)^* = \mathcal{B}(G)$.

Let R_V and R_I be representative matrices of V and I, respectively. According to Table 2-2 and the preceding results, we get:

> (2.4-9) $f(\overline{f})$ is a forest (coforest) of G if and only if $\det[R_V(f)] \neq 0$ $(\det[R_I(\overline{f})] \neq 0)$.

TABLE 2-2

Let $\mathcal{V}$ $(\perp\mathcal{V})$ be a vector space on E over F and R(R*) a representative matrix for $\mathcal{V}$ $(\perp\mathcal{V})$.

(1)	$\mathcal{M}^*_{\mathcal{V}} = \mathcal{M}_{\perp\mathcal{V}}$.		
(2)	b is a base of $\mathcal{M}_{\mathcal{V}}$	<=>	$\det[R^*(b)] \neq 0$.
(3)	b is a base of $\mathcal{M}_{\perp\mathcal{V}}$	<=>	$\det[R(b)] \neq 0$.
(4)	$\bar{b}$ is a cobase of $\mathcal{M}_{\mathcal{V}}$	<=>	$\det[R(\bar{b})] \neq 0$.
(5)	$\bar{b}$ is a cobase of $\mathcal{M}_{\perp\mathcal{V}}$	<=>	$\det[R^*(\bar{b})] \neq 0$.
(6)	f is an elementary vector in $\mathcal{V}$	<=>	$\lVert f \rVert \in \mathcal{C}_{\mathcal{V}}$.
(7)	f is an elementary vector in $\perp\mathcal{V}$	<=>	$\lVert f \rVert \in \mathcal{C}_{\perp\mathcal{V}}$.
(8)	Minimal dependent set of columns of R	<=>	member of $\mathcal{C}_{\perp\mathcal{V}}$.
(9)	Minimal dependent set of columns of R*	<=>	member of $\mathcal{C}_{\mathcal{V}}$.

The reductions and contractions of a matroid and its dual are very simply related. Tutte [Tu 12] has proved the following:

(2.4-10) $(\mathcal{M} \cdot S)^* = \mathcal{M}^* \times S$.

(2.4-11) $(\mathcal{M} \times S)^* = \mathcal{M}^* \cdot S$.

A matroid of the form $(\mathcal{M} \cdot S) \times T$ is called a minor of $\mathcal{M}$. The minors of $\mathcal{M}$ evidently include all the reductions and contractions of $\mathcal{M}$.

Tutte [Tu 12] has also proven the following identities which are useful in dealing with matroids. Let $T \subseteq S \subseteq E$.

(2.4-12) $(\mathcal{M} \times S) \times T = \mathcal{M} \times T$.

(2.4-13) $(\mathcal{M} \cdot S) \cdot T = \mathcal{M} \cdot T$.

(2.4-14) $(\mathcal{M} \cdot S) \times T = (\mathcal{M} \times (E - (S - T))) \cdot T$.

(2.4-15) $(\mathcal{M} \times S) \cdot T = (\mathcal{M} \cdot (E - (S - T))) \times T$.

We list some further properties of the minors of a matroid.

(2.4-16) Every minor of a minor of $\mathcal{M}$ is a minor of $\mathcal{M}$.

(2.4-17) A minor of a regular (binary) matroid is a regular (binary) matroid.

(2.4-18) The minors of $\mathcal{M}^*$ are the duals of the minors of $\mathcal{M}$.

Lastly, we give one additional result concerning the reductions and contractions of $\mathcal{M}$. Let $S \subseteq E$.

(2.4-19) $r(\mathcal{M} \times S) + r(\mathcal{M} \cdot \overline{S}) = r(\mathcal{M})$.

2.5 DUALITY IN GRAPHS

In this section we discuss how a duality theory for graphs can be established through matroid theory. Duality, in general, implies that we have two sets of (dual) quantities and operations such that if a theorem is proved in terms of one set, then the same theorem with dual quantities inserted everywhere yields a true theorem. It is not necessary that the two sets be disjoint. We know, from the previous sections, that if G is any graph, then $\mathcal{P}(G)$ and $\mathcal{B}(G)$, the class of polygons of G and the class of bonds of G, respectively, are matroids. Consequently, any theorem which is true for a general matroid $\mathcal{M}$ is true for both $\mathcal{P}(G)$ and $\mathcal{B}(G)$. Thus, for every matroid-theoretic theorem there exist two graph-theoretic theorems which result from the specialization to the bond and polygon matroids of G. These two derived graph-theoretic theorems are called dual theorems. The duality for graphs, which we have just defined, relies on matroid theory for its rigorous basis. In Table 2-3 we give a list of the dual concepts for graphs and the corresponding matroid-theoretic quantities.

Informally, one sometimes says that if a theorem is true for the polygon concepts, then it is valid in terms of the bond concepts, and vice versa. In general, however, it is necessary for a theorem to be true in terms of matroid concepts or there will be no guarantee that substitution of dual quantities will yield a true theorem. In other words, theorems proved in terms of the polygon (bond) concepts will yield valid theorems in terms of the bond (polygon) concepts if the original theorem is established using only the axioms for matroids and the properties of sets. If, for example, the proof of a theorem relied on some graph-theoretic property which has no matroid-theoretic counterpart, then there is no guarantee that the "dual" theorem is valid. If one chooses to remain within a graph-theoretic framework, then it is necessary to verify that the "dual" theorem is valid; we must prove two theorems.

In the next chapter we will discuss duality in electrical networks in light of the definition of the duality in graphs defined in this section.

TABLE 2-3

DUAL QUANTITIES AND OPERATIONS

Basic Concept	Dual Concepts	
Matroid Concepts	Polygon Concepts	Bond Concepts
$\mathcal{M} = (\mathcal{C}, E)$	$\mathcal{M} = (\mathcal{P}(G), E(G))$	$\mathcal{M} = (\mathcal{B}(G), E(G))$
circuit of $\mathcal{M}$	polygon of G	bond of G
independent set of $\mathcal{M}$	tree of G	cotree of G
base of $\mathcal{M}$	forest of G	coforest of G
$\mathcal{M} \times S$	$(\mathcal{P}(G \cdot S), S)$	$(\mathcal{B}(G \times S), S)$
$\mathcal{M} \cdot S$	$(\mathcal{P}(G \times S), S)$	$(\mathcal{B}(G \cdot S), S)$
$\mathcal{M}^*$	$(\mathcal{B}(G), E(G))$	$(\mathcal{P}(G), E(G))$
$\mathcal{M}^* \cdot S$	$(\mathcal{B}(G \times S), S)$	$(\mathcal{P}(G \cdot S), S)$
base of $\mathcal{M}^*$	coforest of G	forest of G
independent set of $\mathcal{M}^*$	cotree of G	tree of G
circuit of $\mathcal{M}^*$	bond of G	polygon of G

2.6 MATROIDS AND GRAPHS

Let R be a 3×7 matrix over the field of integers modulo 2:

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} .$$

Let $E = \{e_1, \ldots, e_7\}$ and $\mathcal{V}$ be the vector space which has R as its representative matrix. The binary matroid $\mathcal{M}_{\mathcal{V}}$ on E is referred to as being "of Type B I".

The dual of a matroid of Type B I is said to be "of Type B II".

Tutte [Tu 12] has proved the following important results on regular matroids:

(2.6-1) A binary matroid $\mathcal{M}$ is regular if and only if it has no minor of Type B I or B II.

(2.6-2) A binary matroid $\mathcal{M}$ is nonregular if and only if some standard representative matrix R of $\mathcal{M}$ has a submatrix J such that either J or its transpose is of the following form, to within a permutation of columns,

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} .$$

As pointed out by Tutte [Tu 12], this theory is incomplete in the sense that there is no convenient algorithm for determining whether a given binary matroid is regular.

In Fig.2-9 we present the famous Kuratowski graphs.

A matroid is said to be of Type KI or KII if it can be interpreted as the polygon matroid of K_5 or $K_{3,3}$, respectively. A matroid is said to be of Type HI or HII if it can be interpreted as the bond matroid of K_5 or $K_{3,3}$, respectively.

Tutte [Tu 12] has also proved the following fundamental results:

> (2.6-3) A binary matroid is the bond matroid of some graph if and only if it is regular and has no minor of Type KI or KII.
>
> (2.6-4) A binary matroid is the polygon matroid of some graph if and only if it is regular and has no minor in Type HI or HII.

A matroid is called <u>planar</u> if it is both the bond matroid of some graph as well as the polygon matroid of some graph.

Figure 2-10 presents a breakdown of the matroid classes that have been introduced. We also use the terminology that a matroid is said to be <u>graphic</u> if it is the bond matroid of some graph and a matroid is said to be <u>cographic</u> if it is the polygon matroid of some graph.

In Fig.2-10 we also show what we mean by matroids of Type H and K. Specifically, a matroid $\mathfrak{M}$ is of Type H(K) if it is graphic (cographic) and contains as a minor at least one of the minors of Type HI or HII (KI or KII).

In this chapter we have outlined some of the important results in matroid theory as developed by Whitney and Tutte. This material will be referred to in subsequent chapters as we need it.

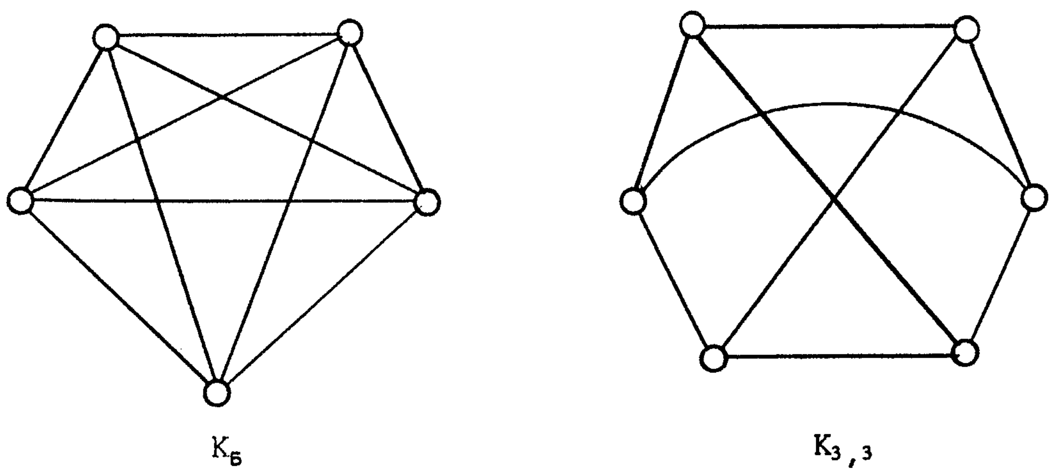


Figure 2-9. Kuratowski Graphs

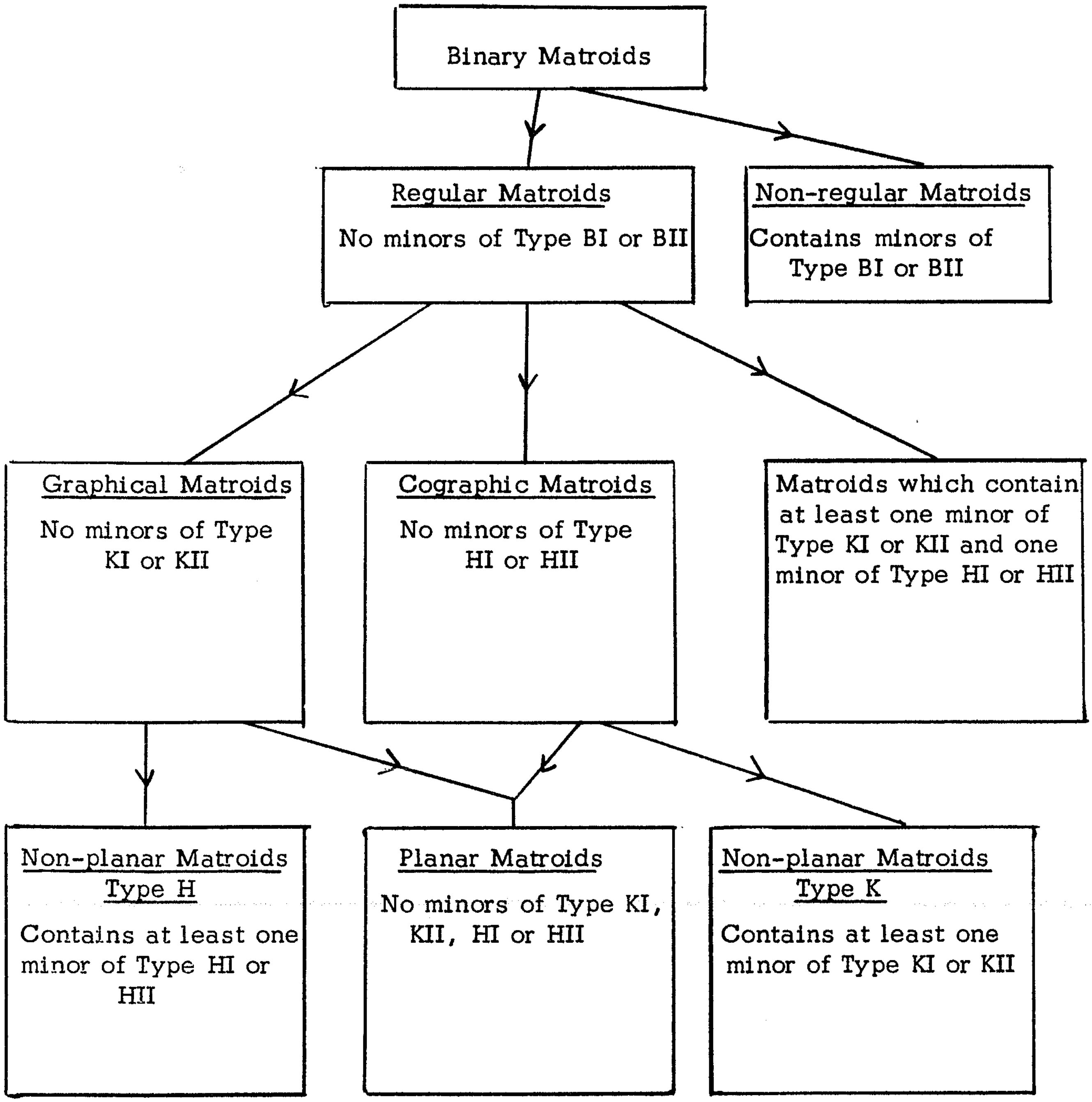


Figure 2-10. Breakown of Matroid Classes

CHAPTER 3. RESISTANCE NETWORKS AND GENERALIZED NETWORKS

3.1 RESISTANCE NETWORKS

In this chapter we introduce the concept of a generalized resistance network and study some of its properties. The generalized network is an extension of the concepts of ordinary p-port resistance networks to matroids. This extension leads to a unified approach to the study of resistance networks and yields new insights as well as results.

In this section we describe a p-port resistance network as a preliminary to the discussion in the next section of the generalized resistance network. A p-port resistance network is an interconnection of two types of elements: port elements and resistance elements. A port element is denoted by a directed edge (Fig.3-1) and the convention used is that the direction of positive current (i) coincides with the direction of the arrow. Positive potential difference (v) means that the arrow points from the vertex of high potential to the vertex of low potential. Note that the product vi represents the instantaneous power delivered to the port element. A resistance element is identical to a port element with the additional requirement that $v = i z$, where $0 < z < \infty$ and z is called the resistance of the element.

The resistance and port elements are interconnected in some manner to form a network and this interconnection is represented by an oriented graph which is called the network graph.

If the network consists of n elements, p of which are port elements and $n-p$ resistance elements, then let

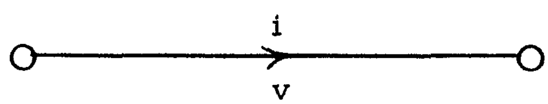


Figure 3-1. Network Element

$$\underline{v}_b^{\,t} = [v_1, \ldots, v_{n-p}]$$

and

$$\underline{i}_b^{\,t} = [i_1, \ldots, i_{n-p}]$$

be the vectors of resistance voltages and currents, respectively.

Similarly

$$\underline{v}_p^{\,t} = [v_{n-p+1}, \ldots, v_n]$$

and

$$\underline{i}_p^{\,t} = [i_{n-p+1}, \ldots, i_n]$$

are the vectors of port voltages and currents, respectively. Also with each resistance element there is an associated resistance z_i, for $i = 1, \ldots, n-p$. The matrix

$$Z_b = \text{diag}\,[z_1, \ldots, z_{n-p}]$$

is called the <u>resistance-element impedance matrix</u>. The vectors $\underline{v}_b$ and $\underline{i}_b$ must satisfy

$$\underline{v}_b = Z_b \underline{i}_b \qquad (1a)$$

or

$$\underline{i}_b = Y_b \underline{v}_b \ , \qquad (1b)$$

where $Y_b = Z_b^{-1}$. Y_b is called the <u>resistance -element admittance matrix</u>.

Let

$$\underline{v} = \begin{bmatrix} \underline{v}_b \\ \hline \underline{v}_p \end{bmatrix} \quad \text{and} \quad \underline{i} = \begin{bmatrix} \underline{i}_b \\ \hline \underline{i}_p \end{bmatrix} \ .$$

The vectors $\underline{v}$ and $\underline{i}$ are called the network voltage and current vectors, respectively.

The vectors $\underline{i}$ and $\underline{v}$ are not only required to satisfy (1a) or (1b) but in addition must satisfy Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL). Thus the algebraic sum of the currents in any bond of the network graph must be zero and the algebraic sum of the voltages around any polygon in the network must be zero. These last two constraints on the network voltage and current vectors are called topological constraints and the relation (1a) is called an Ohm's law constraint.

The topological constraints on the network voltage and current vectors can be stated differently. Let G be the network graph of a p-port resistance network and I and V the 1-cycle space and the coboundary space, respectively, of G over the field F. Then $\underline{i}$ satisfies KCL if and only if $\underline{i}^t$ is the representative vector of some member of I and $\underline{v}$ satisfies KVL if and only if $\underline{v}^t$ is the representative vector of some member of V. Consequently Kirchhoff's laws can be written symbolically as

$$\underline{i} \in I \quad \text{(KCL)} \tag{2}$$

and

$$\underline{v} \in V \quad \text{(KVL)} \quad . \tag{3}$$

The equations (1a) or (1b), (2) and (3) are called the network equations.

In order to retain the familiar properties of Z, the open-circuit (o.c.) impedance matrix, and Y, the short-circuit (s.c.) admittance matrix, of a p-port resistance network we define an auxiliary port-voltage vector $\underline{e}_p = -\underline{v}_p$. Then the o.c. impedance matrix of a network exists if for any prescribed set of port currents $\underline{i}_p$ the network equations uniquely determine the response $\underline{e}_p$. Similarly the s.c. admittance matrix of a network exists if for any prescribed set of port voltages $\underline{e}_p$ the network equations uniquely determine the response $\underline{i}_p$. If Z exists, then the network operation, viewed from the ports, can be expressed as

$$\underline{e}_p = Z\underline{i}_p$$

and if Y exists, then

$$\underline{i}_p = Y\underline{e}_p \quad .$$

Previously in Chapter 1 we indicated that certain matrices were related to p-port resistance networks. At this time, we give the precise definitions for two of these matrices.

A symmetric matrix of real numbers whose main diagonal elements are greater than or equal to the sum of the absolute magnitudes of all the other elements in the same row (column) is called a dominant matrix.

A $p \times p$ symmetric matrix of real numbers is called a paramount matrix if every principal minor of order r is greater than or equal to the absolute value of any r order minor formed from the same rows (columns) for $r = 1, \ldots, p-1$.

The rest of this chapter is devoted to the generalized resistance network and its bearing on p-port resistance networks.

3.2 GENERALIZED NETWORK

In this section we define a resistance network on a regular matroid. As in the case of p-port resistance networks we will consider the generalized network to be an interconnection of two kinds of elements: resistance elements and port elements. In general the generalized network will consist of n elements, p of which are port elements and n - p resistance elements.

Let $\mathcal{M} = (\mathcal{C}, E)$ be a regular matroid on a finite set E. The set E is partitioned into two sets E_p and E_b. The elements in E_p are the port elements and the elements in E_b the resistance elements. Enumerate the elements of E such that

$$E = E_b \cup E_p \quad ,$$

where

$$E_b = \{e_1, e_2, \ldots, e_{n-p}\}$$

and

$$E_p = \{e_{n-p+1}, \ldots, e_n\} \quad .$$

With each element e_i in E we associate two variables u_i and w_i (for i = 1, . . . , n). We define the vectors $\underline{u}$ and $\underline{w}$ as follows:

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}_p \end{bmatrix}$$

and

$$\underline{w} = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}_p \end{bmatrix} \quad ,$$

where

$$\underline{w}_b{}^t = [w_1, \ldots, w_{n-p}]$$

$$\underline{w}_p{}^t = [w_{n-p+1}, \ldots, w_n] \ ,$$

$$\underline{u}_b{}^t = [u_1, \ldots, u_{n-p}]$$

and

$$\underline{u}_p{}^t = [u_{n-p+1}, \ldots, u_n] \ .$$

We have chosen not to use $\underline{v}$ and $\underline{i}$ as variables in the generalized case in order to give a single analysis of the generalized network which later can be specialized to an impedance and/or admittance formulation.

As in Section 3.1, we associate with each member of E_b a positive number d_i ($i = 1, \ldots, n-p$) and require that

$$\underline{w}_b = D\underline{u}_b \ ,$$

where

$$D = \text{diag}\ [d_1, d_2, \ldots, d_{n-p}] \ .$$

D is called the <u>resistance-element immittance matrix.</u>

The next step in defining a generalized network is to write the "topological" constraints for the vectors $\underline{u}$ and $\underline{w}$. Since $\mathcal{M}$ is regular, there exists a regular vector space $\mathfrak{R}$ on E over the field of real numbers such that the supports of the primitive vectors of $\mathfrak{R}$ are in 1-1 correspondence with the circuits of $\mathcal{M}$, that is, $\mathcal{M} = \mathcal{M}_{\mathfrak{R}}$. The vector space $\mathfrak{R}$ is not unique but since $\mathfrak{R}$ is a regular vector space, we think of choosing a particular $\mathfrak{R}$ as fixing the relative orientation of the elements in each of the circuits of $\mathcal{M}$. To see this choose $C \in \mathcal{C}_{\mathfrak{R}}$; then there exists a primitive vector $f \in \mathcal{R}$ such that

$$\|f\| = C \quad .$$

The nonzero values of f are either ± 1 and therefore can be used to determine the relative orientation of the members of C. Thus if $f(e_i) = f(e_j) = \pm 1$, we say that e_i and e_j are <u>similarly oriented in C</u> and if $f(e_k) = -f(e_s) = \pm 1$, we say that e_k and e_s are <u>oppositely oriented in C</u>. By (2.2-4), the choice of $\mathcal{R}$ <u>uniquely</u> determine the relative orientations of the elements in C.

Generalizing KCL and KVL, we require that $\underline{u}^t$ be the representative vector of some member of $\mathcal{R}$ and $\underline{w}^t$ be the representative vector of some member of $\perp\mathcal{R}$, the complementary orthogonal subspace of $\mathcal{R}$. We write the generalized KCL and KVL symbolically as

$$\underline{u} \in \mathcal{R}$$

and

$$\underline{w} \in \perp\mathcal{R} \ .$$

We define a <u>generalized network</u> N as follows:

$$N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E) \quad ,$$

where $\mathcal{M}_{\mathcal{R}}$ is a <u>regular</u> matroid on a finite set E and $\mathcal{R}$ is a corresponding <u>regular</u> vector space on E over the field of real numbers.

The <u>generalized network equations</u> are

$$\underline{u} \in \mathcal{R} \ , \qquad (1)$$

$$\underline{w} \in \perp\mathcal{R} \ , \qquad (2)$$

$$\underline{w}_b = D\underline{u}_b \ , \qquad (3)$$

where $D = \text{diag}\,[d_1, \ldots, d_{n-p}]$.

Equations (1) and (2) are the "topological" constraints on $\underline{u}$ and $\underline{w}$ while equation (3) is an Ohm's law constraint.

At this point we will make the appropriate correspondences between the generalized network and the ordinary impedance and admittance formulations of p-port resistance networks.

Let G be a network graph of a p-port resistance network and partition E(G) according to port and resistance designations. Thus

$$E(G) = E(G)_b \cup E(G)_p \quad ,$$

where

$$E(G)_b = \{e_1, \ldots, e_{n-p}\}$$

and

$$E(G)_p = \{e_{n-p+1}, \ldots, e_n\} \ .$$

The edges in $E(G)_b$ correspond to the resistances and the edges in $E(G)_p$ correspond to the ports. The quantities $\underline{i}^t = (\underline{i}_b{}^t, \underline{i}_p{}^t)$, $\underline{v}^t = (\underline{v}_b{}^t, \underline{v}_p{}^t)$ and $Z_b = \text{diag}\,[z_1, \ldots, z_{n-p}]$, where $0 < z_i < \infty$ for $i = 1, \ldots, n-p$, are defined in 3.1. There are two possible ways to make a correspondence between generalized networks and p-port resistance networks.

Consider the following correspondence. Suppose

$$\underline{u} = \underline{i} \ . \tag{4}$$

Then it follows that

$$\underline{w} = \underline{v} \ , \tag{5}$$

$$\mathfrak{R} = I \ , \tag{6}$$

$$\mathcal{M}_{\mathfrak{R}} = \mathcal{P}(G) \tag{7}$$

and

$$D = Z_b \quad , \tag{8}$$

where I is the 1-cycle space of G over the field of real numbers. Thus the requirement that $\underline{u}$ corresponds to $\underline{i}$ determines the generalized network

$$N_Z = (\rho(G), I, Z_b; E(G)) \quad .$$

If one chooses to have $\underline{v}$ correspond to $\underline{u}$ the generalized network N_Y is obtained:

$$N_Y = (\beta(G), V, Y_b; E(G)) \quad .$$

V is, of course, the coboundary space of G over the field of real numbers.

The subscripts Z and Y reflect the fact that N_Z will lead to an impedance formulation and N_Y yields an admittance formulation. The correspondences between generalized networks and p-port resistance networks are listed, for future reference, in Table 3-1.

Boesch [Bo 1] has shown convenient formulas for Z and Y in terms of topological matrices associated with the network graph and in the next section the generalized network is subjected to a similar analysis and the results are interpreted in terms of matroid structure.

TABLE 3-1 Table of Correspondences

Generalized Networks	Network Equations
$N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E)$	(i) $\underline{u} \in \mathcal{R}$ (ii) $\underline{w} \in \perp \mathcal{R}$ (iii) $\underline{w}_b = D\underline{u}_b$
$N_Z = (\mathcal{P}(G), I, Z_b; E(G))$	(i) $\underline{i} \in I$ (ii) $\underline{v} \in V$ (iii) $\underline{v}_b = Z_b \underline{i}_b$
$N_Y = (\mathcal{B}(G), V, Y_b; E(G))$	(i) $\underline{v} \in V$ (ii) $\underline{i} \in I$ (iii) $\underline{i}_b = Y_b \underline{v}_b$

3.3 ANALYSIS OF GENERALIZED NETWORKS

Having defined a generalized network, the next question to answer is: how does it "work"? In other words, if we specify $\underline{u}_p$, how do the network equations dermine $\underline{u}$ and $\underline{w}$. We first introduce some definitions and notations.

A network $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E)$ is called <u>nondegenerate</u> if one can specify $\underline{u}_p$ arbitrarily and this specification, along with the network equations, uniquely determines $\underline{u}$ and $\underline{w}$. Let $\mathcal{N}$ denote the class of nondegenerate networks.

Suppose $f \in \mathcal{R}$ and $\underline{x}^t$ is a representative vector for f. We define

$$\|\underline{x}\| = \|f\| .$$

Also, as was done in the network equations, we write

$$\underline{x} \in \mathcal{R}$$

to mean that there exists a vector $f \in \mathcal{R}$ such that $\underline{x}^t$ is the representative vector of f. We call $\underline{x}$ elementary (primitive) if there exists an elementary (primitive) vector f in $\mathcal{R}$ such that $\underline{x}^t$ is the representative vector for f.

The next theorem is the main theorem of this section and it characterizes, in terms of matroid structure, those generalized networks which are nondegenerate. Moreover, in the course of proving (3.3-1) we derive explicit expressions for the "response" of a nondegenerate generalized network to an arbitrary port vector $\underline{u}_p$.

(3.3-1) A network $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E)$ is in $\mathcal{N}$ if and only if E_p contains no circuit of $\mathcal{M}_{\mathcal{R}}^*$.

<u>Proof</u>: Let $N \in \mathcal{N}$. Then $\underline{u}_p$ can be specified arbitrarily. Assume that there is a circuit C of $\mathcal{M}^*_{\mathcal{R}}$ such that $C \subseteq E_p$. Then there exists a primitive vector $\underline{x} \in \perp \mathcal{R}$ such that

$$\|\underline{x}\| = C \subseteq E_p \ .$$

Since $\underline{u} \in \mathcal{R}$, it follows that

$$\underline{x}^t \ \underline{u} = 0 \ .$$

Moreover, since $\|\underline{x}\| \subseteq E_p$, there exists a linear relation among the coordinates of $\underline{u}_p$. This contradicts the hypothesis that $N \in \mathcal{N}$ and accordingly no circuit of $\mathcal{M}^*_{\mathcal{R}}$ is contained in E_p.

To show sufficiency, suppose that no circuit of $\mathcal{M}^*_{\mathcal{R}}$ is contained in E_p. Let R^* be a representative matrix for $\perp \mathcal{R}$. Since $\underline{u} \in \mathcal{R}$, it follows that

$$R^* \ \underline{u} = \underline{0} \ . \quad (1)$$

Also since $\underline{w} \in \perp \mathcal{R}$, $\underline{w}$ can be expressed as some linear combination of the rows of R^*, that is,

$$\underline{w} = R^{*t} \ \underline{\varphi} \ , \quad (2)$$

where

$$\underline{\varphi} = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{bmatrix}$$

and

$$r = \text{dimension} \ (\perp \mathcal{R}) \ .$$

If we partition R^* according to $\underline{u}$ and $\underline{w}$, it follows from (1) that

$$R^*_b \ \underline{u}_b + R^*_p \ \underline{u}_p = \underline{0} \ , \quad (3)$$

where

$$R_b^* = [R_b^* \,\vdots\, R_p^*] \ .$$

Using $\underline{u}_b = D^{-1}\,\underline{w}_b$ (D^{-1} exists since $d_i > 0$ for $i = 1, \ldots, n-p$) in (3) we get

$$R_b^* D^{-1}\,\underline{w}_b = -R_p^*\,\underline{u}_p \ . \tag{4}$$

From (2) it follows that

$$\underline{w}_b = R_b^{*t}\,\underline{\varphi} \tag{5}$$

and

$$\underline{w}_p = R_p^{*t}\,\underline{\varphi} \ . \tag{6}$$

Substituting (5) into (4), we get

$$[R_b^* D^{-1} R_b^{*t}]\,\underline{\varphi} = -R_p^*\,\underline{u}_p \ . \tag{7}$$

We show by contradiction that the hypothesis implies $\det[R_b^* D^{-1} R_b^{*t}] \neq 0$.

Assume $\det[R_b^* D^{-1} R_b^{*t}] = 0$. Applying the Binet-Cauchy formula [Ga 1] twice to $\det[R_b^* D^{-1} R_b^{*t}]$, we get

$$\det[R_b^* D^{-1} R_b^{*t}] = \sum_{1 \le i_1 < \ldots < i_r \le n-p} D^{-1}\begin{pmatrix} i_1, \ldots, i_r \\ i_1, \ldots, i_r \end{pmatrix} \left[R_b^* \begin{pmatrix} 1, \ldots, r \\ i_1, \ldots, i_r \end{pmatrix}\right]^2 . \tag{8}$$

If B is a matrix, the notation

$$B\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}$$

represents the determinant of the submatrix of B formed by using rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$.

Since det $[R_b^* D^{-1} R_b^{*t}] = 0$, it is clear from (8) that the rank of R_b^* is less than r. Consequently if $S \subseteq E_b$ and $\alpha(S) = r$, det $[R_b^*(S)] = 0$. Therefore (see Table 2-2,(2)) no $S \subseteq E_b$ is a base of $\mathcal{M}_{\mathcal{R}}$. By (2.4-2), E_p is not contained in any base of $\mathcal{M}_{\mathcal{R}}^*$. Therefore E_p contains a circuit of $\mathcal{M}_{\mathcal{R}}^*$; but this contradicts the hypothesis. Accordingly det $[R_b^* D^{-1} R_b^{*t}] \neq 0$ and from (7) we get

$$\underline{\varphi} = - [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \underline{u}_p \tag{9}$$

Substituting (9) into (5) and (6) and using $\underline{u}_p = D^{-1} \underline{w}_b$, we get the following results.

$$\underline{w} = \begin{bmatrix} \underline{w}_b \\ -- \\ \underline{w}_p \end{bmatrix} = \begin{bmatrix} -R_b^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \text{-------------------} \\ -R_p^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \end{bmatrix} \underline{u}_p \tag{10}$$

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ -- \\ \underline{u}_p \end{bmatrix} = \begin{bmatrix} -D^{-1} R_b^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \text{-----------------------} \\ 1_p \end{bmatrix} \underline{u}_p \,. \tag{11}$$

Equations (10) and (11) show the explicit dependence of $\underline{u}$ and $\underline{w}$ on $\underline{u}_p$. Since (10) and (11) were obtained by applying necessary conditions and are invariant with respect to the choice of R^*, it follows that specification of $\underline{u}_p$ uniquely determines $\underline{u}$ and $\underline{w}$. Therefore $N \in \mathcal{N}$. ∎

An immediate corollary of (3.3-1) is

(3.3-2) Let $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E) \in \mathcal{N}$ and R^* be a representative matrix for $\perp\mathcal{R}$. Partition R^* as $R^* = [R_b^* \,\vdots\, R_p^*]$, where R_b^* and R_p^* correspond to the resistance and port elements, respectively. Then

$$\underline{w} = \begin{bmatrix} \underline{w}_b \\ \hline \underline{w}_p \end{bmatrix} = \begin{bmatrix} -R_b^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \hline -R_p^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \end{bmatrix} \underline{u}_p$$

and

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ \hline \underline{u}_p \end{bmatrix} = \begin{bmatrix} -D^{-1} R_b^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \\ \hline 1_p \end{bmatrix} \underline{u}_p \ .$$

The <u>immittance matrix</u> X_N is defined as

$$X_N = R_p^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^* \ .$$

Therefore X_N characterizes the "operation" of the generalized network in terms of a port description, that is,

$$\underline{w}_p = -X_N \underline{u}_p \ .$$

Figure 3-2 depicts the port description of a generalized network.

An alternate characterization of a nondegenerate network is given by the following theorem which is a consequence of (3.3-1).

(3.3-3) Let $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E)$ and R^* be a representative matrix for $\perp\mathcal{R}$. Partition R^* as $R^* = [R_b^* \,\vdots\, R_p^*]$, where R_b^* and R_p^* correspond to the resistance and port elements respectively. Then N is in $\mathcal{N}$ if and only if $\text{rank}\,(R_b^*) = \text{rank}\,(R^*)$.

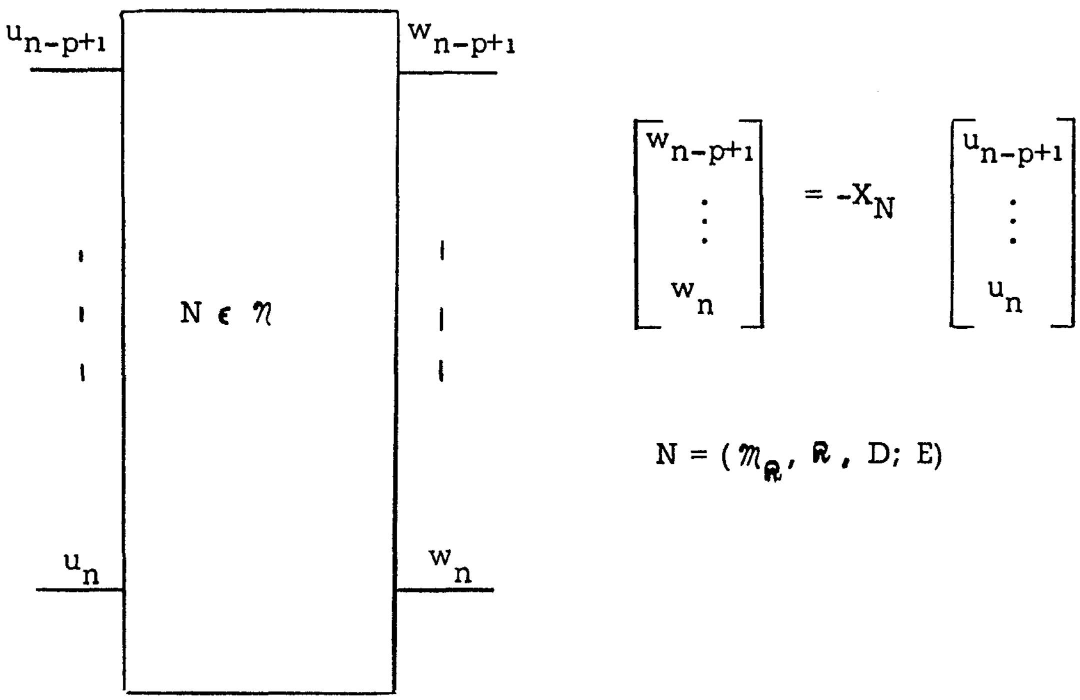


Figure 3-2. Generalized Network

The beauty of matroid theory becomes apparent as one realizes that the matroid structure allows one to visualize the "interconnection" of the elements in E of a generalized network $N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E)$. Theorem (3.3-1) is an excellent example of this since it gives the existence of X_N in terms of the matroid structure. Also matroid theory eliminates the necessity of thinking in terms of admittance or impedance and thus focuses attention on the essential features of the analysis of p-port resistance networks. As we will discuss later, however, the different matroid classes, as depicted in Fig.2-10, allow us to distinguish in a precise way the <u>differences</u> between the admittance and impedance formulations of p-port resistance networks.

Let us return now to Table 3-1 and interpret X_{N_Z} and X_{N_Y}. It is easy to see that $X_{N_Z} = Z$, the o.c. impedance matrix for the resistance network, and $X_{N_Y} = Y$, the s.c. admittance matrix.

TABLE 3-2 Table of Correspondences

$N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E)$	$N_Z = (\mathcal{P}(G), I, Z_b; E(G))$	$N_Y = (\mathcal{B}(G), V, Y_b; E(G))$
$\underline{w}_p = -X_N \underline{u}_p$	$-\underline{v}_p = X_{N_Z} \underline{i}_p$ $X_{N_Z} = Z$	$\underline{i}_p = X_{N_Y} = (-\underline{v}_p)$ $X_{N_Y} = Y$

To conclude this section we show how one obtains the known results on the existence of Z and Y as special cases of (3.3-1).

(3.3-4) Let G be the network graph of a p-port resistance network. Then Z(Y), the o.c. impedance (s.c. admittance) matrix, exists if and only if $G \times E(G)_p$ $(G \cdot E(G)_p)$ contains no bonds (polygons).

Proof: By (3.3-1) and (2.4-3) X_{N_Z} exists if and only if $\mathcal{B}(G)$ has no circuits contained in $E(G)_p$. By the definition of a contraction, $\mathcal{B}(G)$ has no circuits in $E(G)_p$ if and only if $\mathcal{B}(G) \times E(G)_p$ has no circuits. By (2.1-9), $\mathcal{B}(G) \times E(G)_p = \mathcal{B}(G \times E(G)_p)$. Therefore $\mathcal{B}(G) \times E(G)_p$ has no circuits, and Z exists, if and only if $G \times E(G)_p$ contains no bonds.

The proof for Y follows the same pattern as that for Z. ∎

Example 3-1 Let G be the graph in Fig.3-3 and

$$E(G) = E(G)_b \cup E(G)_p \quad ,$$

where

$$E(G)_b = \{e_1, \ldots, e_5\}$$

and

$$E(G)_p = \{e_6, e_7, e_8\} \quad .$$

Z, the o.c. impedance matrix, exists since $G \times E(G)_p$ contains no bonds (see Fig.3-4). Y, the s.c. admittance matrix, does not exist since $G \cdot E(G)_p$ contains a polygon (in this case a loop). $G \cdot E(G)_p$ is shown in Fig.3-5.

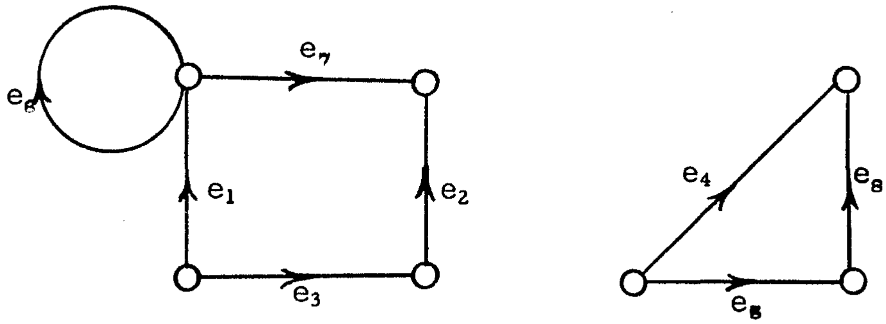


Figure 3-3. Graph of Example 3-1

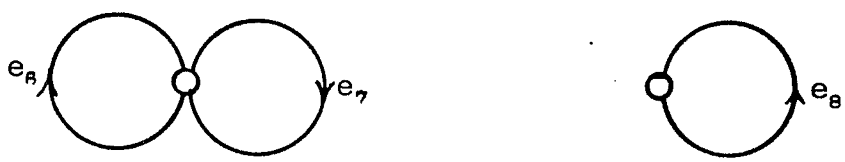


Figure 3-4. $G \times E(G)_P$

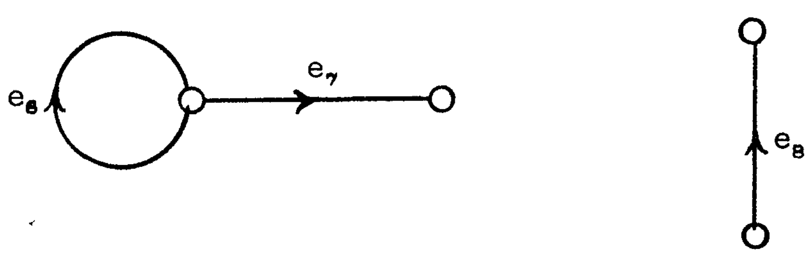


Figure 3-5. $G \cdot E(G)_P$

3.4 PROPERTIES OF X_N

In this section we prove that if $N \in \mathcal{N}$, then X_N is a paramount matrix. The method of proof yields "topological" formulas for the generalized network and consequently extends the concept of a "topological" formula to matroids.

We also treat, in this section, the special case when $r(\mathcal{M}_{\mathcal{R}}) = p$ and derive for this case an additional necessary condition on X_N.

To conclude this section we indicate that the modified topological matrices introduced by Cederbaum [Ce 2] for p-port resistance networks can be extended to generalized networks.

(3.4-1) If $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E)$ belongs to $\mathcal{N}$, then X_N is a paramount matrix.

<u>Proof</u>: $X_N = R_p^{*t} [R_b^* D^{-1} R_b^{*t}]^{-1} R_p^*$, where $R^* = [R_b^* \mid R_p^*]$ is a representative matrix for $\perp \mathcal{R}$, $R_b^* = r \times (n-p)$, $R_p^* = r \times p$ and $r = r(\mathcal{M}_{\mathcal{R}}) = \text{dimension} (\perp \mathcal{R})$. By (3.3-3), $\det [R_b^* D^{-1} R_b^{*t}] \neq 0$ and consequently $n-p \geq r$.

Set

$$A = [R_b^* D^{-1} R_b^{*t}]^{-1} \quad .$$

At this point we introduce a useful notation: let $\bar{i}_s$ represent the index set $i_1 < \ldots < i_s$. Then using the Binet-Cauchy formula [Ga 1] it is not difficult to show that

$$X_N\begin{pmatrix}\bar{i}_s\\ \bar{j}_s\end{pmatrix} = \sum_{\substack{1\le \bar{k}_s \le r\\ 1\le \bar{h}_s \le r}} A\begin{pmatrix}\bar{h}_s\\ \bar{k}_s\end{pmatrix} R_p^*\begin{pmatrix}\bar{h}_s\\ \bar{i}_s\end{pmatrix} R_p^*\begin{pmatrix}\bar{k}_s\\ \bar{j}_s\end{pmatrix} \tag{1}$$

for $s \le r$, and

$$X_N\begin{pmatrix}\bar{I}_s\\ \bar{J}_s\end{pmatrix} = 0 \tag{2}$$

for $s > r$. The summation in (1) is over all index sets $\bar{k}_s$ and $\bar{h}_s$ satisfying

$$1 \le k_1 < \ldots < k_s \le r$$

and

$$1 \le h_1 < \ldots < h_s \le r \quad .$$

Furthermore [Ga 1]

$$A\begin{pmatrix}\bar{h}_s\\ \bar{k}_s\end{pmatrix} = \frac{(-1)^{\beta}\ [R_b^* D^{-1} R_b^{*t}]\begin{pmatrix}\bar{k}'_{r-s}\\ \bar{h}'_{r-s}\end{pmatrix}}{\det\ [R_b^* D^{-1} R_b^{*t}]} \tag{3}$$

for $s < r$, where $\beta = \sum_{i=1}^{s} (h_i + k_i)$. The indices $k'_1 < \ldots < k'_{r-s}$ (i.e. $\bar{k}'_{r-s}$) and $k_1 < \ldots < k_s$ form a complete set of indices on $1, \ldots, r$. Similarly $\bar{h}'_{r-s}$ and $\bar{h}_s$ form a complete set of indices on $1, \ldots, r$.

When $s = r$

$$A\begin{pmatrix}\bar{h}_r\\ \bar{k}_r\end{pmatrix} = \frac{1}{\det\ [R_b^* D^{-1} R_b^{*t}]} \quad . \tag{4}$$

Expand the right hand side of (3) using the Binet-Cauchy formula. The result is

$$[R_b^* D^{-1} R_b^{*t}] \begin{pmatrix} \overline{k'}_{r-s} \\ \overline{h'}_{r-s} \end{pmatrix} = \sum_{1 \le \overline{m}_{r-s} \le n-p} D^{-1} \begin{pmatrix} \overline{m}_{r-s} \\ \overline{m}_{r-s} \end{pmatrix} R_b^* \begin{pmatrix} \overline{k'}_{r-s} \\ \overline{m}_{r-s} \end{pmatrix} R_b^* \begin{pmatrix} \overline{h'}_{r-s} \\ \overline{m}_{r-s} \end{pmatrix} \quad (5)$$

for $s < r$. Substituting (5) and (3) into (1) we get:

$$X_N \begin{pmatrix} \overline{i}_s \\ \overline{j}_s \end{pmatrix} = \sum_{1 \le \overline{m}_{r-s} \le n-p} \frac{D^{-1} \begin{pmatrix} \overline{m}_{r-s} \\ \overline{m}_{r-s} \end{pmatrix}}{\det [R_b^* D^{-1} R_b^{*t}]} H K \quad , \quad (6)$$

where

$$H = \sum_{1 \le \overline{k}_s \le r} (-1)^{\sum_{i=1}^{s} k_i} R_b^* \begin{pmatrix} \overline{k'}_{r-s} \\ \overline{m}_{r-s} \end{pmatrix} R_p^* \begin{pmatrix} \overline{k}_s \\ \overline{j}_s \end{pmatrix}$$

and

$$K = \sum_{1 \le \overline{h}_s \le r} (-1)^{\sum_{i=1}^{s} k_i} R_b^* \begin{pmatrix} \overline{h'}_{r-s} \\ \overline{m}_{r-s} \end{pmatrix} R_p^* \begin{pmatrix} \overline{h}_s \\ \overline{i}_s \end{pmatrix}$$

for $s < r$.

The terms H and K of (6) are easily seen to correspond to the Laplace expansion of minors formed from some r columns of R^* .

We introduce the useful notation:

$$R^* (\overline{m}_{r-s} | \overline{i}_s) = R^* \begin{pmatrix} 1, \ldots\ldots\ldots\ldots\ldots\ldots, r \\ m_1, \ldots, m_{r-s}, n-p+i_1, \ldots, n-p+i_s \end{pmatrix}$$

Then (6) becomes

$$X_N \begin{pmatrix} \overline{i}_s \\ \overline{j}_s \end{pmatrix} = \sum_{1 \le \overline{m}_{r-s} \le n-p} \frac{D^{-1} \begin{pmatrix} \overline{m}_{r-s} \\ \overline{m}_{r-s} \end{pmatrix}}{\det [R_b^* D^{-1} R_b^{*t}]} R^*(\overline{m}_{r-s} | \overline{i}_s) R^*(\overline{m}_{r-s} | \overline{j}_s) \quad (7)$$

for $s < r$.

To obtain the case $s = r$, we combine (4) and (1) to get

$$X_N\begin{pmatrix}\bar{i}_r\\ \bar{j}_r\end{pmatrix} = \frac{R_p^*\begin{pmatrix}1,\ldots,r\\ i_1,\ldots,i_r\end{pmatrix} R_p^*\begin{pmatrix}1,\ldots,r\\ j_1,\ldots,j_r\end{pmatrix}}{\det\,[R_b^* D^{-1} R_b^{*t}]} . \qquad (8)$$

Since $\perp \mathfrak{R}$ is a regular subspace (see 2.2-5) the minors of order r of R^* are restricted to the values $\pm L$ or 0 where L is some positive number. Also the terms $D^{-1}\begin{pmatrix}\bar{m}_{r-s}\\ \bar{m}_{r-s}\end{pmatrix}$ are positive. It should therefore be clear from (7) that a principal minor of order $s < r$ is greater than the absolute magnitude of any s^{th} order minor formed from the same rows. Moreover, consideration of equations (2) and (4) permits one to conclude that X_N is in fact a paramount matrix. ▮

We have the following corollary to (3.4-1)

(3.4-2) Let $N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E) \in \mathcal{N}$ and R^* be a representative matrix for $\perp \mathfrak{R}$. Partition R^* as $R^* = [R_b^* \mid R_p^*]$, where R_b^* and R_p^* correspond to the resistance and port elements, respectively. Then

$$X_N\begin{pmatrix}\bar{i}_s\\ \bar{j}_s\end{pmatrix} = \sum_{1\le\bar{m}_{r-s}<n-p} \frac{D^{-1}\begin{pmatrix}\bar{m}_{r-s}\\ \bar{m}_{r-s}\end{pmatrix}}{\Delta} R^*(\bar{m}_{r-s}\,|\,\bar{I}_s)\, R^*(\bar{m}_{r-s}\,|\,\bar{J}_s)$$

for $s < r$,

$$X_N\begin{pmatrix}\bar{i}_s\\ \bar{j}_s\end{pmatrix} = 0$$

for $s > r$ and

$$X\begin{pmatrix}\bar{i}_r\\ \bar{j}_r\end{pmatrix} = \frac{R_p\begin{pmatrix}1, \ldots, r\\ i_1, \ldots, i_r\end{pmatrix} \quad R_p\begin{pmatrix}1, \ldots, r\\ j_1, \ldots, j_r\end{pmatrix}}{\Delta}$$

where $\Delta = \det [R_b^* D^{-1} R_b^{*t}]$.

It should be apparent that from (3.4-2) we can obtain "topological" formulas for the generalized network. Let us consider the case $s < r$ and $(X_N)_{i,j}$, the i,j element of X_N:

$$(X_N)_{i,j} = \frac{\sum\limits_{1 \le \bar{m}_{r-1} \le n-p} D^{-1}\begin{pmatrix}\bar{m}_{r-1}\\ \bar{m}_{r-1}\end{pmatrix} \frac{R^*(\bar{m}_{r-1}|i)}{L} \quad \frac{R^*(\bar{m}_{r-1}|j)}{L}}{\sum\limits_{1 \le \bar{h}_r \le n-p} D^{-1}\begin{pmatrix}\bar{h}_r\\ \bar{h}_r\end{pmatrix} \frac{R^*\begin{pmatrix}1, \ldots, r\\ h_1, \ldots, h_r\end{pmatrix}}{L} \quad \frac{R^*\begin{pmatrix}1, \ldots, r\\ h_1, \ldots, h_r\end{pmatrix}}{L}} \qquad \text{(A)}$$

where L is a positive number equal to the absolute value of a nonzero r^{th} order minor of R^*. The denominator of (A) was taken from equation (8) in the proof of (3.3-1).

Let $\mathcal{B}_{\mathcal{M}_{\mathcal{R}}}$ denote the class of bases of $\mathcal{M}_{\mathcal{R}}$. Set

$$\mathcal{B}(E_b) = \{ b \mid b \in \mathcal{B}_{\mathcal{M}_{\mathcal{R}}} \text{ and } b \subseteq E_b \} .$$

For any $S = \{ e_{i_1}, \ldots, e_{i_t} \} \subseteq E_b$ we define $[D^{-1} S]$ as

$$[D^{-1} S] = \frac{1}{d_{i_1}} \frac{1}{d_{i_2}} \cdots \frac{1}{d_{i_t}} .$$

Accordingly we can write Δ, the denominator of (A), as

$$\Delta = \sum_{b \in \mathcal{B}(E_b)} [D^{-1} b]$$

The numerator of (A) requires some special attention. A typical term in the numerator of (A) is nonzero if and only if the sets $\{e_{m_1}, \ldots, e_{m_{r-1}}, e_{n-p+i}\}$ and $\{e_{m_1}, \ldots, e_{m_{r-1}}, e_{n-p+j}\}$ are both bases of $\mathcal{M}_{\mathcal{R}}$. Accordingly we define $\mathcal{B}_{i,j}$ as

$$\mathcal{B}_{i,j} = \{S \subseteq E_b \mid S \cup \{e_{n-p+i}\} \in \mathcal{B}_{\mathcal{M}_{\mathcal{R}}} \text{ and } S \cup \{e_{n-p+j}\} \in \mathcal{B}_{\mathcal{M}_{\mathcal{R}}}\}$$

for all $1 \leq i,\ j \leq p$.

Although the members of $\mathcal{B}_{i,j}$ are in 1-1 correspondence with the nonzero terms in the numerator of (A), there is still the matter of the sign of each term when $i \neq j$. Partition the set $\mathcal{B}_{i,j}$ into two sets $\mathcal{B}^+_{i,j}$ and $\mathcal{B}^-_{i,j}$, where for $i \neq j$

$$\mathcal{B}^+_{i,j} = \{S \in \mathcal{B}_{i,j} \mid e_{n-p+i} \text{ and } e_{n-p+j} \text{ are oppositely oriented in the circuit } J(S \cup \{e_{n-p+i}\}, e_{n-p+j}) \text{ of } \mathcal{M}_{\mathcal{R}}\}$$

and

$$\mathcal{B}^-_{i,j} = \{S \in \mathcal{B}_{i,j} \mid e_{n-p+i} \text{ and } e_{n-p+j} \text{ are similarly oriented in the circuit } J(S \cup \{e_{n-p+i}\}, e_{n-p+j}) \text{ of } \mathcal{M}_{\mathcal{R}}\}$$

and for $i = j$

$$\mathcal{B}^+_{i,i} = \mathcal{B}_{i,i}$$

and

$$\mathcal{B}^-_{i,i} = \emptyset\ .$$

(3.4-3) Let $N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E) \in \mathcal{N}$.
Then

$$(X_N)_{i,j} = \frac{\sum_{S \in \mathcal{B}^+_{i,j}} [D^{-1}S] - \sum_{T \in \mathcal{B}^-_{i,j}} [D^{-1}T]}{\sum_{b \in \mathcal{B}(E_b)} [D^{-1}b]}$$

Proof: To prove (3.4-3) we merely have to show that the sets $\mathcal{B}^+_{i,j}$ and $\mathcal{B}^-_{i,j}$ correspond to the negative and positive terms, respectively, of the numerator of $(X_N)_{i,j}$ when $i \neq j$.

Let $S \in \mathcal{B}_{i,j}$ and form the base

$$b = S \cup \{e_{n-p+i}\} .$$

Let R^* be a standard representative matrix for $\underline{|}\,\mathfrak{R}$ with respect to b. Consider the submatrix $(j > i)$

$$R^*(b \cup \{e_{n-p+j}\}) = \begin{array}{c} \overbrace{\hphantom{1 \quad 0 \ldots 0}}^{S} \quad e_{n-p+i} \quad e_{n-p+j} \\ \begin{bmatrix} 1 & 0 \ldots & 0 & 0 & x_1 \\ 0 & 1 \ldots & 0 & 0 & x_2 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 \ldots & 1 & 0 & x_{r-1} \\ 0 & 0 \ldots & 0 & 1 & x_r \end{bmatrix} \end{array}$$

The minimal dependent sets of columns of R^* correspond to circuits in $\mathcal{M}_{\mathfrak{R}}$ (see Table 2-2). By the definition of $\mathcal{B}_{i,j}$ it follows that $x_r = \pm 1$. Moreover, the set $S \cup \{e_{n-p+i}, e_{n-p+j}\}$ is dependent in $\mathcal{M}_{\mathfrak{R}}$ and contains a unique circuit C which has both e_{n-p+i} and e_{n-p+j} as members.

Clearly if $x_r = +1$, e_{n-p+i} and e_{n-p+j} are oppositely oriented in C and if $x_r = -1$, e_{n-p+i} and e_{n-p+j} are similarly oriented in C. Moreover, if $x_r = +1$, the corresponding term in $(X_N)_{i,j}$ will be positive and if $x_r = -1$, the corresponding term will be negative.

The theorem follows. ▌

In (3.4-3) we have extended the notion of a "topological" formula to generalized networks, that is, we can evaluate the entries in X_N by a formula which depends only on the resistance-element immittance matrix and the structure of the matroid $\mathcal{M}_R$.

Example 3-2 Let G be the graph in Fig. 3-6 and $N_Z = (\mathcal{P}(G), I, Z_b; E(G))$, where

$$Z_b = \text{diag}\,[z_1, z_2, z_3]$$

and

$$\mathcal{B}(E_b) = \{\{e_1, e_2, e_3\}\} .$$

Consequently

$$\Delta = \frac{1}{z_1 z_2 z_3} .$$

To calculate $(X_{N_Z})_{1,1}$ we need $\mathcal{B}_{1,1}$:

$$\mathcal{B}_{1,1} = \{\{e_1, e_3\}, \{e_2, e_3\}\} = \mathcal{B}^+_{1,1} .$$

Therefore

$$(X_{N_Z})_{1,1} = \frac{\frac{1}{z_1 z_3} + \frac{1}{z_2 z_3}}{\frac{1}{z_1 z_2 z_3}} = z_2 + z_1 .$$

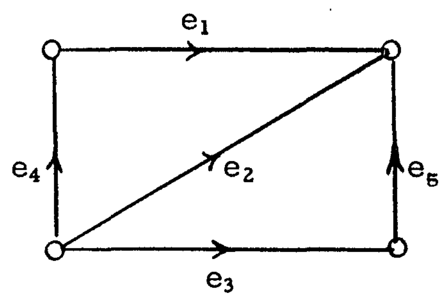


Figure 3-6. Graph of Example 3-2

To calculate $(X_{N_Z})_{1,2}$ we need $\mathcal{B}^{+}_{1,2}$ and $\mathcal{B}^{-}_{1,2}$:

$$\mathcal{B}^{+}_{1,2} = \left\{ \{ e_1, e_3 \} \right\}$$

and

$$\mathcal{B}^{-}_{1,2} = \phi \ .$$

Therefore

$$(X_{N_Z})_{1,2} = \frac{\dfrac{1}{z_1 z_3}}{\dfrac{1}{z_1 z_2 z_3}} = z_2$$

Calculating $(X_{N_Z})_{2,2}$ we find

$$(X_{N_Z})_{2,2} = z_2 + z_3 \ .$$

Thus

$$X_{N_Z} = \begin{bmatrix} z_1+z_2 & z_2 \\ z_2 & z_2+z_3 \end{bmatrix} \ .$$

We now turn to the special case of generalized networks satisfying $\alpha(E_p) = r(\mathcal{M}^{*}_{\mathcal{R}})$. These networks have special significance in the case of N_Z and N_Y. For instance, if $N = N_Z$, then $\alpha(E_p) = r(\mathcal{M}^{*}_{\mathcal{R}})$ becomes $\alpha(E_p) = r(\mathcal{B}(G))$. Thus the number of port elements coincides with the number of elements in a coforest of G. If moreover $N_Z \in \mathcal{N}$, then E_p contains no bond of G, and consequently E_p is a coforest of G. If $N = N_Y \in \mathcal{N}$ and $\alpha(E_p) = r(\mathcal{P}(G))$, then E_p is a forest of G.

The above illustrations are encompased by the following theorem.

(3.4-4) Let $N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E) \in \mathcal{N}$, then $r(\mathcal{M}^*_{\mathfrak{R}}) = \alpha(E_p)$ if and only if E_p is a base of $\mathcal{M}^*_{\mathfrak{R}}$.

Proof: If E_p is a base of $\mathcal{M}^*_{\mathfrak{R}}$, then $r(\mathcal{M}^*_{\mathfrak{R}}) = \alpha(E_p)$.

Conversely, suppose $r(\mathcal{M}^*_{\mathfrak{R}}) = \alpha(E_p)$. Since $N \in \mathcal{N}$, it follows from (3.3-1) that E_p is a base of $\mathcal{M}^*_{\mathfrak{R}}$. ▮

(3.4-5) Let $N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E) \in \mathcal{N}$ and $r(\mathcal{M}^*_{\mathfrak{R}}) = \alpha(E_p)$. Then $X_N = ADA^t$, where A is a totally unimodular matrix.

Proof: By (3.4-5), E_p is a base of $\mathcal{M}^*_{\mathfrak{R}}$. Accordingly $E_b = \overline{E}_p$ is a cobase of $\mathcal{M}^*_{\mathfrak{R}}$. Let R^* be a standard representative matrix of $\mathfrak{R}^*$ with respect to the cobase E_b:

$$R^* = [\, 1_{n-p} \mid R^*_p \,] \ .$$

By (2.2-6), R^*_p is a totally unimodular matrix. Calculating X_N using R^* we find that

$$X_N = R^{*t}_p \, D \, R^*_p \ .$$

Since the transpose of a totally unimodular matrix is totally unimodular, the theorem is proved. ▮

Under the hypothesis of (3.4-5) we know that X_N is a paramount matrix. In the next theorem we give an additional necessary condition on X_N. Unfortunately these two conditions are not sufficient as we will show in Example 3-3.

(3.4-6) Let $Q = [q_{ij}] = ADA^t$, where $A = [a_{ij}]$ is a $p \times b$ totally unimodular matrix and D is a $b \times b$ diagonal matrix with positive diagonal terms. Then

$$Q_{i,r,c} = q_{ii} + |q_{rc}| - |q_{ri}| - |q_{ic}| \geq 0 ,$$

for all $1 \leq i, r, c \leq p$.

<u>Proof:</u> The i,j^{th} element of Q is

$$q_{ij} = \sum_{k=1}^{b} d_k a_{ik} a_{jk} \quad .$$

It is well known [Ce 1] that for fixed i and j all the nonzero products $a_{ik} a_{jk}$, for $k = 1, \ldots, b$, have the same sign. Consequently

$$|q_{ij}| = \sum_{k=1}^{b} d_k |a_{ik} a_{jk}| \quad .$$

Therefore we can write $Q_{i,r,c}$ as

$$Q_{i,r,c} = \sum_{k=1}^{b} d_k [a_{ik}^2 + |a_{rk} a_{ck}| - |a_{rk} a_{ik}| - |a_{ck} a_{ik}|] .$$

We prove $Q_{i,r,c} \geq 0$ by showing that each term in the summation is non-negative.

Case 1: $a_{ik} = 0$. Therefore the only contribution is from $|a_{rk} a_{ck}|$, which is non-negative.

Case 2: $a_{ik} \neq 0$ and $a_{rk} a_{ck} = 0$. Therefore the term

$$d_k [a_{ik}^2 - |a_{rk} a_{ik}| - |a_{ck} a_{ik}|]$$

is non-negative since at least one of the terms $|a_{rk} a_{ik}|$ or $|a_{ck} a_{ik}|$ is zero.

Case 3: $a_{ik} \neq 0$, $a_{rk} a_{ck} \neq 0$. Therefore

$$d_k [a_{ik}^2 + |a_{rk} a_{ck}| - |a_{rk} a_{ik}| - |a_{ck} a_{ik}|] = 0 \quad .$$

We conclude therefore that $\mathbf{Q}_{i,r,c} \geq 0$ for $1 \leq i, r, c \leq p$. ▌

Example 3-3 The matrix

$$Q = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 35 & -30 \\ 0 & -30 & 36 \end{bmatrix}$$

is a paramount matrix [Ce 1]. Q however does not satisfy (3.4-6) since

$$\mathbf{Q}_{2,1,3} = 35 + 0 - 10 - 30 = -5 \quad .$$

Accordingly, the paramountcy condition and the condition of (3.4-6) are independent.

Next let Q' be the paramount matrix

$$Q' = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 12 & 4 & 5 \\ 2 & 9 & 15 & 6 \\ 3 & 5 & 6 & 18 \end{bmatrix} .$$

Cederbaum has shown that **Q'** cannot be displayed as in the hypothesis of (3.4-6). However $Q'_{i,r,c} \geq 0$ for $1 \leq i,r,c \leq 4$. Consequently, paramountcy and the condition that $Q'_{i,r,c} \geq 0$ are not sufficient to guarantee the unimodular decomposition of **Q'**.

To conclude this section we show that the modified topological matrices introduced by Cederbaum [Ce 2] can be extended to generalized networks. In fact, we show that a "modified" matrix exists for a generalized network if $N \in \mathcal{N}$. There seems to be some ambiguity in the literature as to just when a "modified" matrix exists for a p-port resistance network. For instance, in two places [Le 1; Ce 2] networks for which modified matrices exist are called "nonsingular" p-port networks and the meaning of nonsingular is left undefined. We show that a modified topological matrix exists if X_N exists.

Let $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E) \in \mathcal{N}$. By (3.3-1) and Table 2-2, line 2, there exists a representative matrix for $\mathcal{R}$ of the following form

$$R = \left[\begin{array}{c|c} R_{11} & 1_p \\ R_{21} & 0_{p' \times p} \end{array}\right] = \mu \times n \ ,$$

where μ = dimension ($\mathcal{R}$). Moreover, by (2.2-6) we can assume that R is a totally unimodular matrix. We define an augmented matrix R^+:

$$R^+ = \begin{bmatrix} R_{11} & 1_p & 0_{p \times p'} \\ R_{21} & 0_{p' \times p} & 1_{p'} \end{bmatrix} .$$

R^+ is totally unimodular and consequently can be viewed as the representative matrix of a regular vector space $\mathcal{R}^+$. Set

$$N^+ = (\mathcal{M}_{\mathcal{R}^+}, \mathcal{R}^+, D; E \cup E')$$

where $E' = \{ e_{n+1}, \ldots, e_{n+p'} \}$. N^+ is called an <u>augmented network</u> for N and $N^+ \in \mathcal{N}$. Set

$$(R^+)^* = [1_{n-p} \ -R_{11}{}^t \ \vdots \ -R_{21}{}^t] \ .$$

Clearly $(R^+)^*$ is a representative matrix for $\perp \mathcal{R}^+$.

The variables associated with the network N are, as usual,

$$\underline{u} = \begin{bmatrix} \underline{u}_b \\ --- \\ \underline{u}_p \end{bmatrix}$$

and

$$\underline{w} = \begin{bmatrix} \underline{w}_b \\ --- \\ \underline{w}_p \end{bmatrix} .$$

The variables associated with N^+ are

$$\underline{u}^+ = \begin{bmatrix} \underline{u}_b \\ --- \\ \underline{u}_p \\ --- \\ \underline{u}_{p'} \end{bmatrix} = \begin{bmatrix} \underline{u}_b \\ --- \\ \underline{u}^+_{p+p'} \end{bmatrix}$$

and

$$\underline{w}^+ = \begin{bmatrix} \underline{w}_b \\ --- \\ \underline{w}_p \\ --- \\ \underline{w}_{p'} \end{bmatrix} = \begin{bmatrix} \underline{w}_b \\ --- \\ \underline{w}^+_{p+p'} \end{bmatrix} .$$

The augmented network N^+ can be viewed as one which is obtained from N by adding p' additional ports to N.

The following two results are easy consequences of the construction of $\mathcal{R}^+$ from $\mathcal{R}$.

(3.4-7) If $\underline{u}^+ \in \mathcal{R}^+$, then $\underline{u} \in \mathcal{R}$.

(3.4-8) If $\underline{w}^+ \in \mathcal{R}^+$ and $\underline{w}_{p'} = \underline{0}$, then $\underline{w} \in \mathcal{R}$.

Theorem (3.4-9) extends the modified topological matrices of p-port resistance networks to generalized networks.

(3.4-9) Let $N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E) \in \mathcal{N}$ and R be a totally unimodular representative matrix for $\mathfrak{R}$:

$$R = \begin{bmatrix} R_{11} & 1_p \\ R_{21} & O_{p' \times p} \end{bmatrix} = \mu \times n \quad .$$

Then

$$X_N = \hat{R} D \hat{R}^t ,$$

where

$$\hat{R} = R_{11} - R_{11} D R_{21}^t [R_{21} D R_{21}^t]^{-1} R_{21} \quad .$$

Morever, a matrix R in the above form always exists. The matrix $\hat{R}$ is called a <u>modified topological matrix.</u>

<u>Proof</u>: From (3.3-2) it follows that

$$X_{N^+} = \begin{bmatrix} R_{11} D R_{11}^t & R_{11} D R_{21}^t \\ R_{21} D R_{11}^t & R_{21} D R_{21}^t \end{bmatrix} \tag{1}$$

and

$$\underline{w}_p^+ = -(X_{N^+}) \underline{u}_p^+ \quad . \tag{2}$$

Set

$$\underline{u}_{p+p'}^+ = \begin{bmatrix} \underline{u}_p \\ \underline{u}_{p'} \end{bmatrix} = \begin{bmatrix} 1_p \\ (R_{21} D R_{21}^t)^{-1} R_{21} D R_{11}^t \end{bmatrix} \underline{u}_p \quad . \tag{3}$$

Then $\det [R_{21} D R_{21}^t] \neq 0$, since the rows of R_{21} are linearly independent.

If we specify $\underline{u}_p$ arbitrarily in (3) and apply $\underline{u}_{p+p'}^+$ to N^+ we get

$$X_{N^+} \underline{u}^+_{p+p'} = \underline{w}^+_{p+p'} = \begin{bmatrix} \underline{w}_p \\ \underline{0} \end{bmatrix} \quad . \tag{4}$$

In view of (3.4-7) and (3.4-8), the variables $\underline{u}$ and $\underline{w}$, under the conditions imposed by (3), in N^+ satisfy the network equations of N. Since both N and N^+ are in $\mathcal{N}$ the relation between $\underline{w}_p$ and $\underline{u}_p$ in N^+, under the constraint (3), is precisely

$$\underline{w}_p = -X_N \underline{u}_p \quad . \tag{5}$$

Next we obtain an alternative expression for $\underline{w}_p$ in terms of $\underline{u}_p$ operating in N^+ under (3).

Using the constraint (3) and the fact that $\underline{u}^+ \epsilon\, \mathcal{R}^+$ and $\underline{w}^+ \epsilon \perp \mathcal{R}^+$, it is not difficult to see that

$$R_{11}^t \underline{u}_p + R_{21}^t \underline{u}_{p'} = \underline{u}_b \tag{6}$$

$$R_{11} \underline{w}_b + \underline{w}_p = \underline{0} \tag{7}$$

and

$$R_{21} \underline{w}_b = \underline{0} \quad . \tag{8}$$

Combining (6) and (3) we get

$$\underline{u}_b = \hat{R}^t \underline{u}_p \quad . \tag{9}$$

Also from (7) and (8) we get

$$\underline{w}_p = -\hat{R} \underline{w}_b \quad . \tag{10}$$

Using $\underline{w}_b = D \underline{u}_b$, (9) and (10), we arrive at

$$\underline{w}_p = - [\hat{R} D \hat{R}^t] \underline{u}_p \tag{11}$$

Equation (11) describes the relation between the variables $\underline{w}_p$ and $\underline{u}_p$ in N^+ under constraint (3), and (5) describes the identical relation between $\underline{w}_p$ and $\underline{u}_p$. Consequently $X_N = \hat{R} D \hat{R}^t$.

As was previously indicated, a matrix R of the desired form exists if $N \epsilon \mathcal{N}$. $\blacksquare$

3.5 SINGULAR IMMITTANCE MATRICES

As is well known in the case of p-port resistance networks, when an immittance matrix is singular the linear dependence of the columns (or rows) contains information on the port structure of the network. In this section we show how the linear dependence of the columns of X_N is reflected in the structure of the matroid $\mathcal{M}_{\mathfrak{R}}$ associated with N. More precisely, we show that the circuits of $\mathcal{M}_{\mathfrak{R}} \times E_b$ are in 1-1 correspondence with the sets of minimal dependent columns of X_N.

Theorem (3.5-4) deals with a converse problem for paramount matrices. In the previous section we showed that if $N \in \mathcal{N}$, then X_N is a paramount matrix. The converse problem is very difficult, that is, given **Q**, a $p \times p$ paramount matrix, determine a generalized network $N \in \mathcal{N}$ (if one exists) satisfying $X_N = \mathbf{Q}$. This converse problem is called the <u>synthesis problem</u> for generalized networks. In this section we prove an interesting result on singular paramount matrices which has bearing on the synthesis problem. We show that the linear dependence of the columns of a singular paramount matrix cannot be arbitrary and in fact its null space must be regular.

(3.5-1) If $N = (\mathcal{M}_{\mathfrak{R}}, \mathfrak{R}, D; E) \in \mathcal{N}$, then
$\underline{u}_p^t X_N \underline{u}_p = \underline{u}_b^t D \underline{u}_b$.

<u>Proof</u>: Since $\underline{u} \in \mathfrak{R}$ and $\underline{w} \in \perp \mathfrak{R}$, it follows that $\underline{u}_b^t \underline{w}_b + \underline{u}_p^t \underline{w}_p = 0$. Using $\underline{w}_b = D\underline{u}_b$ and $\underline{w}_p = -X_N \underline{u}_p$, the theorem follows. ∎

The next result relates the minimal dependent columns of X_N to the structure of the matroid $\mathcal{M}_{\mathcal{R}}$.

(3.5-2) Let $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E) \in \mathcal{N}$. Then a set of columns of X_N forms a minimal dependent set if and only if the corresponding set of elements in E_p is a circuit of $\mathcal{M}_{\mathcal{R}}$.

Proof: Let $C \subseteq E_p$ be a circuit of $\mathcal{M}_{\mathcal{R}}$. Then there exists an elementary vector $\underline{u}' \in \mathcal{R}$ such that $\|u'\| = C$. Clearly the pair $\underline{u} = \underline{u}'$ and $\underline{w} = \underline{0}$ satisfy the network equations, and since $N \in \mathcal{N}$ it follows that

$$X_N \, \underline{u}'_p = \underline{0} \quad , \tag{1}$$

where $\underline{u}' = \begin{bmatrix} \underline{0} \\ \hline \underline{u}'_p \end{bmatrix}$.

We claim that the columns of (1) which are linearly dependent form a minimal dependent set.

Assume there exists a nonzero vector $\underline{u}''_p$ such that

$$X_N \, \underline{u}''_p = \underline{0}$$

and

$$\|\underline{u}''\| = \|\underline{u}'\| \quad , \tag{2}$$

where $\underline{u}'' = \begin{bmatrix} \underline{0} \\ \hline \underline{u}''_p \end{bmatrix}$.

Since $N \in \mathcal{N}$, $\underline{u}_p$ can be specified arbitrarily and therefore by (3.5-1) $\underline{u}'' \in \mathcal{R}$. But then (2) contradicts the hypothesis and accordingly the dependent columns in (1) form a minimal dependent set.

To show necessity suppose

$$X_N \underline{u}_p = \underline{0} \tag{3}$$

and that the dependent columns of (3) form a minimal dependent set. Again since $N \in \mathcal{N}$, $\underline{u}_p$ can be specified arbitrarily and therefore by (3.5-1) the vector

$$\underline{u} = \begin{bmatrix} \underline{0} \\ --- \\ \underline{u}_p \end{bmatrix}$$

satisfies $\underline{u} \in \mathcal{R}$ and $\|\underline{u}\| \subseteq E_p$.

Assume $\underline{u}$ is not elementary. Then there exists a nonzero vector $\underline{v}^t = [\,\underline{0}^t \quad \underline{v}_p{}^t\,]$ satisfying

$$\|\underline{v}\| \subset \|\underline{u}\|$$

and

$$X_N \underline{v}_p = \underline{0} \ . \tag{4}$$

However, (4) contradicts the hypothesis and accordingly $\underline{u}$ is elementary. Therefore there exists a circuit $C \subseteq E_p$ such that $C = \|u\|$. ∎

Theorem (3.5-2) shows that, in the case of the generalized network, matroid theory allows a geometric interpretation of the singular immittance matrices. For the cases N_Z and N_Y, theorem (3.5-2) specializes to the following well known result.

(3.5-3) Let Z(Y) be the o.c. impedance (s.c. admittance) matrix of a resistance network whose network graph is G. Then the minimal dependent columns of Z(Y) are in a 1-1 correspondence with the polygons (bonds) of G which are contained in E_p.

Previously we defined what is meant by a primitive (elementary) representative vector $\underline{x}^t$ with respect to some vector space $\mathcal{R}$. It should be clear that if U is a collection of n-tuples $\underline{x}$, then we can use the term primitive (elementary) vector in U without reference to a vector space $\mathcal{R}$. Moreover if U is closed under addition of n-tuples and multiplication by a member of F, then we call U a vector space of n-tuples on E over the field F. The reference to a set E is necessary if, for some $\underline{x} \in U$, the notation $\| \underline{x} \|$ is to have meaning. U (a vector space of n-tuples) is called regular if F is the field of real numbers and corresponding to each elementary vector $\underline{x} \in U$ there exists a primitive vector $\underline{x}' \in U$ satisfying

$$\| \underline{x}' \| = \| \underline{x} \| \quad .$$

In the next theorem we characterize the null space of any paramount matrix. If Q is a p $\times$ p matrix, the <u>null space</u> N(Q) of Q is the set of all p-tuples $\underline{x}$ which satisfy $Q\underline{x} = \underline{0}$:

$$N(Q) = \{ \underline{x} \mid Q\underline{x} = \underline{0} \} .$$

(3.5-4) Let Q be a $p \times p$ paramount matrix; then N(Q), the null space of Q, is a regular vector space of p-tuples on $E_p(\alpha(E_p) = p)$.

Proof: Without loss of generality, assume the first r columns of Q form a minimal dependent set.

Assume there exists a principal minor $Q\begin{pmatrix} i_1, \ldots, i_{r-1} \\ i_1, \ldots, i_{r-1} \end{pmatrix} = 0$, where $1 \le i_1 < \ldots < i_{r-1} \le r$. Since Q is paramount, then any $(r-1)^{th}$-order using columns $i_1, \ldots, i_{r-1}$ is zero. Accordingly columns $i_1, \ldots, i_{r-1}$ are linearly dependent; but this contradicts the hypothesis. Therefore every $(r-1)^{th}$- order principal minor formed from the first r columns is nonzero.

Let Q_r be the submatrix formed from the first r rows and columns of Q ; by hypothesis det $[Q_r] \neq 0$. If we let Δ_{ij} be the cofactor obtained from Q_r by crossing out row i and column j, it follows from Jacobi's theorem [Tu 14] that

$$\Delta_{ii} \Delta_{jj} = \Delta_{ij} \Delta_{ji} \quad . \tag{1}$$

However, Q is paramount and consequently

$$\Delta_{kk} \ge | \Delta_{kh} | = | \Delta_{hk} | \quad , \tag{2}$$

for all $1 \le k \le r$ and $1 \le h \le r$. Using (1) and (2) and the fact that $\Delta_{kk} \neq 0$ for $1 \le k \le r$, we conclude that all the first cofactors of Q_r are equal in absolute magnitude.

It follows from the above analysis that the coefficients of the linear relation of the first r columns of Q can be chosen to be ± 1.

Since the first r columns form a minimal dependent set the vector $\underline{x}$, whose coordinates are the coefficients of this linear relation, is elementary in N(Q). Moreover, we have shown that there exists a primitive vector $\underline{x}'$ such that $\|\underline{x}'\| = \|\underline{x}\|$ and $X_N \underline{x}' = \underline{0}$. ▌

Theorem (3.5-4) enables one to exhibit a paramount matrix in a very revealing form.

(3.5-5) Let Q be a $p \times p$ paramount matrix of rank s satisfying $Q\begin{pmatrix}1, \ldots, s\\ 1, \ldots, s\end{pmatrix} \neq 0$. Then Q can be expressed as

$$Q = B^t Q_s B,$$

where B is a $p \times s$ totally unimodular matrix and Q_s is the submatrix formed from the first s rows and columns of Q.

Proof: Partition Q as

$$Q = \left[\begin{array}{c|c} Q_s & Q_{12} \\ \hline Q_{12}^t & Q_{22} \end{array}\right], \qquad (1)$$

where $Q_s = s \times s$,

$Q_{12} = s \times (p-s)$

$Q_{22} = (p-s) \times (p-s)$.

Set

$$T = \left[\begin{array}{c|c} Q_s^{-1} & 0_{s \times (p-s)} \\ \hline -Q_{12}^t Q_s^{-1} & 1_{p-s} \end{array}\right]$$

and form T Q :

$$TQ = \left[\begin{array}{c|c} 1_s & Q_s^{-1} Q_{12} \\ \hline 0_{(p-s)\times s} & Q_{22} - Q_{12}^t Q_s^{-1} Q_{12} \end{array}\right] .$$

Since det $[T] \neq 0$, the rank of TQ is s and accordingly

$$Q_{22} - Q_{12}^t Q_s^{-1} Q_{12} = 0_{(p-s)\times(p-s)} \quad . \tag{2}$$

Setting $B = [1_s \mid Q_s^{-1} Q_{12}]$ and using (1) and (2) we can express **Q** as

$$Q = B^t Q_s B \quad . \tag{3}$$

Let $\underline{x}$ be a p-tuple satisfying $Q\underline{x} = \underline{0}$. Then

$$(B^t Q_s)(B\underline{x}) = \underline{0} \quad . \tag{4}$$

The matrix $B^t Q_s$ is $p \times s$ and of rank s and the matrix $B\underline{x}$ is $s \times 1$. Accordingly (4) implies

$$B\underline{x} = \underline{0} \quad .$$

Conversely, if $\underline{x}$ is a p-tuple satisfying $B\underline{x} = \underline{0}$, then $Q\underline{x} = \underline{0}$.

The above analysis shows that

$$N(Q) = \{\underline{x} \mid B\underline{x} = \underline{0}\} \quad . \tag{5}$$

It should be clear from the construction of B and equation (5) that the row space of B^* is precisely the transpose of the vectors in N(Q), where

$$B^* = [-Q_{12}^t Q_s^{-1} \mid 1_{p-s}] \quad .$$

(Note that $B^* B^t = 0_{(p-s)\times s}$.) By (3.5-4) N(Q) is regular and therefore the theorem follows using (2.4-4) and (2.2-6). ▌

<u>Example 3-4</u> Consider the 6×6 paramount matrix

$$Q = \begin{bmatrix} 4 & -1 & -1 & 3 & -2 & 3 \\ -1 & 4 & -1 & 3 & 3 & -2 \\ -1 & -1 & 4 & -2 & 3 & 3 \\ 3 & 3 & -2 & 6 & 1 & 1 \\ -2 & 3 & 3 & 1 & 6 & 1 \\ 3 & -2 & 3 & 1 & 1 & 6 \end{bmatrix} ,$$

where rank $(\mathbf{Q}) = 3$ and Q_3, the 3×3 submatrix formed from the first 3 rows and columns of $\mathbf{Q}$, satisfies $\det(Q_3) \neq 0$.

$$Q_3^{-1} = \frac{1}{10} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

and therefore

$$B = [\, 1_3 \mid Q_3^{-1} Q_{12} \,] ,$$

where

$$Q_3^{-1} Q_{12} = \frac{1}{10} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 3 \\ 3 & 3 & -2 \\ -2 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

Finally we get

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} .$$

Clearly B is totally unimodular. One can easily verify that

$$\mathbf{Q} = B^t Q_3 B .$$

An open question for generalized networks is to determine whether for every paramount matrix **Q** there exists a generalized network N such that $X_N = \mathbf{Q}$. The case of the singular paramount matrix poses an interesting test in view of (3.5-2). Consequently, if we conjecture that N exists for any **Q**, then Theorem (3.5-4) must also be true. Since we have in fact shown (3.5-4) to be true, the question remains an open one.

3.6 DUALITY

Duality in electrical networks has to do with the relationship between the admittance and impedance formulations of electrical networks. One might erroneously conclude that, with respect to p-port resistance networks, theorems about impedances are immediately true for admittances and vice versa. Cederbaum [Ce 3] has shown the existence of a paramount matrix which is the o.c. impedance matrix of some 4-port resistance network but is <u>not</u> the s.c. admittance matrix of any 4-port resistance network. In this section we will discuss the notion of duality in generalized and p-port resistance networks.

Let $N = (\mathfrak{M}_{\mathfrak{R}}, \mathfrak{R}, D; E)$. We define N^*, the <u>dual network</u> of N as

$$N^* = (\mathfrak{M}_{\perp\mathfrak{R}}, \perp\mathfrak{R}, D^{-1}; E).$$

(3.6-1) Let $N, N^* \in \mathfrak{N}$, then
$(X_N)^{-1} = X_{N^*}$.

<u>Proof</u>: Theorem (3.6-1) is easily seen to be true by comparison of the network equations for N and N^*. ▮

Every abstract network (in $\mathfrak{N}$ or not in $\mathfrak{N}$) has a dual network.

A paramount matrix Q is said to be <u>realized by N</u> if $Q = X_N$. The network N is called a <u>realization</u> of Q.

Two networks $N^{(1)}$ and $N^{(2)}$ are said to be equivalent if $X_{N^{(1)}} = X_{N^{(2)}}$.

We denote by [N], the class of all networks equivalent to N, that is, the set of all networks which realize Q is equal to [N], where $X_N = Q$.

The equivalence class [N] is sometimes denoted by $\mathcal{E}(Q)$, where Q is the paramount matrix satisfying $Q = X_N$. We point out that in this discussion the set E is not fixed and therefore if

$$N^{(1)} = (\mathcal{M}_{\mathcal{R}^{(1)}}, \mathcal{R}^{(1)}, D^{(1)}; E^{(1)})$$

and

$$N^{(2)} = (\mathcal{M}_{\mathcal{R}^{(2)}}, \mathcal{R}^{(2)}, D^{(2)}; E^{(2)})$$

are two different networks in [N], it is not necessary that $E^{(1)} = E^{(2)}$. However, $\alpha(E_p{}^{(1)}) = \alpha(E_p{}^{(2)})$.

A conjecture that so far the author has been unable to prove or disprove is

> (3.6-2) (Conjecture) Let Q be a $p \times p$ paramount matrix; then $\mathcal{E}(Q) \neq \phi$.

This conjecture, in its relation to singular paramount matrices, was discussed in the preceding section following Example 3-4.

Notice that in Fig.2-10 the class of regular matroids partitions into four sets:

(i) H - nonplanar matroids - Type H.

(ii) K - nonplanar matroids - Type K.

(iii) P - planar matroids.

(iv) O - regular matroids not in H, K or P.

Let Q be a $p \times p$ paramount matrix and suppose $\mathcal{E}(Q) \neq \phi$. Then for each $N \in \mathcal{E}(Q)$, the matroid $\mathcal{M}_{\mathcal{R}}$ associated with N is in exactly one of the classes H, K, P or O. Therefore we can partition $\mathcal{E}(Q)$ into four sets: H(Q), K(Q), P(Q) and O(Q), where

$H(Q) = \{N \mid N \in \mathcal{E}(Q)$ and the matroid associated with N is in H.$\}$

$K(Q) = \{N \mid N \in \mathcal{E}(Q)$ and the matroid associated with N is in K.$\}$

$P(Q) = \{N \mid N \in \mathcal{E}(Q)$ and the matroid associated with N is in P.$\}$

$O(Q) = \{N \mid N \in \mathcal{E}(Q)$ and the matroid associated with N is in O.$\}$

We have the following very important results based on Section 2.6.

> (3.6-3) Let Q be a fixed paramount matrix and suppose $\mathcal{E}(Q) \neq \phi$ and $\mathcal{E}(Q)$ is partitioned into the sets H(Q), K(Q), P(Q) and O(Q) defined above. Then Q is the o.c. impedance (s.c. admittance) matrix of some network if and only if at least one of the sets K(Q) or P(Q) (H(Q) or P(Q)) is nonempty.

> (3.6-4) Let Q be a fixed paramount matrix and suppose $\mathcal{E}(Q) \neq \phi$ and that $\mathcal{E}(Q)$ is partitioned into the sets H(Q), K(Q), P(Q) and O(Q). Then Q is both the o.c. impedance matrix of some network and the s.c. admittance matrix of some network if and only if at least one of the sets H(Q) or P(Q) is nonempty and at least one of the sets K(Q) and P(Q) is nonempty.

From (3.6-4) we obtain the well known results that if Q has a planar network realization $N_Z = (\mathcal{P}(G), I, Z_b; E(G))$, then there is a graph G' such that $N_Y = (\mathcal{B}(G'), V', Y'_b\ E(G'))$ and

$$X_{N_Y} = X_{N_Z} .$$

Obviously G' can simply be taken to be G^* the dual graph of G and $Y'_b = Z_b$. We state this as a theorem.

(3.6-5) Let Q be a fixed paramount matrix and suppose $\mathcal{E}(Q) \neq \emptyset$ and $\mathcal{E}(Q)$ is partitioned into the sets H(Q), K(Q), P(Q) and O(Q). Then if $P(Q) \neq \emptyset$, there exists at least two networks N_Z and N_Y in $\mathcal{E}(Q)$ such that $X_{N_Z} = X_{N_Y}$.

The above results are well known but to the best of the author's knowledge have not been previously presented in the context of the theory of regular matroids.

In the remainder of this section we discuss the notion of duality in electrical networks. As we stated in Section 2.5 duality, in general, implies that we have two sets of (dual) quantities and operations such that if a theorem is proved in terms of one set, then the same theorem with dual quantities inserted everywhere yields a true theorem. We have indicated how matroid theory forms a rigorous basis for a duality theory for graphs and in Table 2-3 we have listed the dual concepts for graphs and the corresponding matroid-theoretic quantities.

Since our generalized networks are based on matroid-theoretic ideas, the specialization of theorems on generalized networks to theorems on p-port resistance networks induces a duality theory for p-port resistance networks. Theorems (3.3-4) and (3.5-3) are examples of how specialization of matroid-theoretic results leads to two graph-theoretic results in each case. Notice also that in (3.3-4) it is essential to replace the dual operations as well as the dual quantities.

We can say, in the context of our definition of duality (see Section 2.5), that Z and Y are dual quantities. The extent of this claim is the following. If one proves a theorem for X_N, the immittance matrix of a generalized network in $\mathcal{N}$, then the theorem is immediately true in terms of any X_{N_Z} and X_{N_Y}, where N_Z, $N_Y \in \mathcal{N}$. Thus we have a precise meaning for duality and can accordingly determine whether an appeal to duality is justified in any particular case.

Many books on network theory present "duality" concepts from a much narrower point of view than discussed here and have consequently fostered some incorrect notions. We present a typical approach to "duality". Let $N_Y = (\mathcal{B}(G), V, Y_b; E(G))$ and $N_Z = (\mathcal{P}(G^*), I, Z_b; E(G^*))$ where G^* is a dual graph of G (G is necessarily a planar graph). The network equations for N_Y and N_Z are shown in Table 3-3. In Table 3-3 we

TABLE 3-3 Planar Networks

N_Y (network equations)	N_Z (network equations)
$\underline{i}_b = Y_b \underline{v}_b$	$\underline{v}_b = Z_b \underline{i}_b$
$\underline{v} \in V(G)$	$\underline{i} \in I(G^*)$
$\underline{i} \in I(G)$	$\underline{v} \in V(G^*)$

use the notation V(G) and I(G) for the coboundary and 1-cycle spaces, respectively, of G. Since G^* is a dual graph of G, $I(G^*) = V(G)$ and $V(G^*) = I(G)$ and consequently, if $Z_b = Y_b$, then $X_{N_Y} = X_{N_Z}$. Clearly, if we restrict our attention to networks whose network graphs are planar, then we can make the statement that impedance and admittance are indistinguishable quantities, not dual quantities. The duality of electrical networks as depicted in Table 2-3 is not one in which impedance and admittance blend into one concept; on the contrary, impedances and admittances are different but dual quantities in general.

As we pointed out previously, matroid theory forms the rigorous basis of the duality within the same graph. This concept of duality is so important that we feel some further discussion is justified since all too often appeals to duality are made and no justification is given. We consider the following problem in order to make a point. Suppose we were given Table 2-3 and told that columns 2 and 3 were "dual" quantities but we were unaware of the corresponding matroid-theoretic concepts. Then if we were asked to determine under what conditions we would be justified in appealing to duality for a rigorous proof of a dual theorem, what must we do?

There is one approach to this problem which is by far the most widely used and does not actually use duality to prove the dual theorem. Essentially, the technique is to consciously, or unconsciously, substitute the dual quantities into the proof of the original theorem and make the observation that the proof goes through in the dual case; that is, one proves the dual theorem without an appeal to duality. In effect one must prove two theorems, the original and the "dual" theorem.

A more sophisticated approach to this question would be to prove a theorem concerning the dual quantities, a theorem which justifies an appeal to duality. Such a theorem would necessarily set down the precise conditions under which one could use duality as a rigorous proof of a "dual" theorem. To establish such a rigorous basis for Table 2-3 we would have to invent matroid theory since it is the formal basis of the dual nature of the same graph. Matroids are a generalization of both the bond and polygon concepts and the axioms of matroids are the "rules" under which proofs of theorems must be carried out if duality is justified. Thus we state again that formal duality relies on a single concept within which theorems can be proven and subsequently specialized to two (or more) specific cases.

Accordingly, we should be able to detect whether an appeal to duality in any particular case is justified. For example, suppose it is asserted that, since Z and Y are dual quantities, if a paramount matrix Q can be realized by N_Z, then there exists a network N_Y which also realizes Q. This kind of claim has no basis in our definition of duality. Clearly, there is no appeal to a matroid-theoretic theorem which can be specialized in two ways. There isn't even an appeal to a theorem proven in terms of the polygon (bond) concepts which can be given a dual interpretation. Accordingly, such an assertion cannot be based on duality concepts. Theorem (3.6-3) shows when a paramount matrix can be the o.c. impedance (s.c. admittance) matrix of some network. The conditions are, of course, dual conditions.

Theorem (3.6-5) gives a sufficient condition for two networks N_Y and N_Z to have identical immittance matrices. However, consider the following: suppose $P(Q) = \emptyset$ and $H(Q) \neq \emptyset$, that is, Q is a $p \times p$ paramount matrix which is only realized by networks whose network graphs are nonplanar. An important problem is to determine whether (i) $K(Q) = \emptyset$ in general, (ii) $K(Q) \neq \emptyset$ in general, or (iii) that, depending on Q, either (i) or (ii) is possible. Duality cannot be expected to answer this question or similar ones. A theory of equivalent networks must be developed in order to understand the issues involved. Such a theory is presently nonexistent.

CHAPTER 4. PRINCIPAL MINORS OF A MATROID

4.1 INTRODUCTION: PREVIOUS RESULTS

In this chapter we present and develop the concept of principal minors of a matroid. These results were stimulated by the work of Kishi and Kajitani [Ki 1] in which they introduced the principal partition of a graph and the concept of maximally distant forest pairs. We begin this section with some definitions which will enable us to present their main contributions.

Let G be a finite graph and f_1 and f_2 a pair of forests of G. Let $e_C \in [E(G) - (f_1 \cup f_2)]$. We call a subgraph

$$G_C = G \cdot S \quad ,$$

where $S \subseteq E(G)$, a K-subgraph of G with respect to (f_1, f_2) and e_C if it satisfies the following:

(i) $e_C \in S$.

(ii) $(f_1 \cap S)$ and $(f_2 \cap S)$ are element disjoint forests of $G \cdot S$.

(iii) S is minimal with properties (i) and (ii), that is, if $T \subset S$, then T does not satisfy both (i) and (ii).

A pair of forests, f_1 and f_2, of G is called <u>maximally distant</u> if every pair of forests f'_1 and f'_2 satisfies

$$\alpha(f_1 \cap f_2) \leq \alpha(f'_1 \cap f'_2) \quad .$$

(4.1-1) (Kishi and Kajitani) Let G be a finite graph and f_1 and f_2 a pair of forests of G. Set $E_C = \{e_{C_1}, \ldots, e_{C_m}\} = [E(G) - (f_1 \cup f_2)]$. Then f_1 and f_2 are maximally distant if and only if there exists a K-subgraph of G with respect to (f_1, f_2) and each member of E_C.

In proving (4.1-1), Kishi and Kajitani give a constructive procedure which can be used to generate a maximally distant forest pair from an arbitrary pair of forests.

If (f_1, f_2) is a maximally distant forest pair, then by (4.1-1) there exists a K-subgraph $G \cdot S_i$ for each e_{C_i} in

$$E_C = \{e_{C_1}, \ldots, e_{C_m}\} = [E(G) - (f_1 \cup f_2)] \quad .$$

Let $S = \bigcup_{i=1}^{m} S_i$ and set $G_2 = G \cdot S$. G_2, the reduction of G to S, is called the principal subgraph or principal reduction of G with respect to (f_1, f_2). Kishi and Kajitani have also shown that G_2 is unique in the sense that any maximally distant forest pair generates one and the same principal subgraph G_2.

They also indicate that a dual formulation is possible in terms of coforests of G and that this leads to the unique graph $H_2 = G \times T$, the principal contraction of G to T. Furthermore, they claim, but do not show, that

$$T \cap S = \phi$$

Let $R = [E(G) - (S \cup T)]$. Then the 3-tuple

$$(D_2, H_2, G_2) \quad ,$$

where

$$D_2 = (G \cdot \overline{T}) \times R \ ,$$

$$H_2 = G \times T$$

and
$$G_2 = G \cdot S \ ,$$

is called the principal partition of a graph and it is unique.

In the next section we introduce the notion of the principal minors of a matroid and prove their uniqueness as well as a nesting property for them. In graph-theoretic terms, the principal minors extend the idea of the principal partition of a graph. In Section 4.3 we consider principal minors for the case $k = 1$.

Section 4.4 presents a generalization of maximally distant forests to maximally distant bases of a matroid. Moreover, instead of maximally distant pairs of bases, we introduce r-maximally distant bases for $r = 2,3, \ldots$. The K-subgraphs of Kishi and Kajitani become r-minors in our matroidal theory and (4.1-1) is appropriately generalized to the r-maximally distant bases of a matroid.

Also in Section 4.4 an efficient algorithm is presented for determining a set of r-maximally distant bases. Then the principal minors are related to the concept of r-maximally distant bases and two more algorithms are presented. These algorithms can be used to construct the principal minors of a matroid efficiently. Section 4.4 ends with an example which illustrates the use of Algorithms 2 and 3 in determining principal minors.

In Section 4.5 we treat the 2-principal minors of a matroid and extend to matroids the principal partition of a graph introduced by Kishi and Kajitani.

In the final section of Chapter 4 we give a new matroid-theoretic solution to the two-person switching game and, furthermore, using duality concepts obtain a completely graph-theoretic characterization of this game. Lastly, we consider an alternate form of Edmond's cospanning sets theorem and the notions of hybrid rank and hybrid dimension.

4.2 PRINCIPAL MINORS OF A MATROID

Let $\mathcal{M}$ be a matroid on a finite set E. We define the functions g_k and h_k as

$$g_k(S) = \alpha(S) - k \cdot r(\mathcal{M} \times S)$$

and

$$h_k(T) = \alpha(T) - k \cdot \mu(\mathcal{M} \cdot T)$$

for $k = 1, 2, \ldots$, where $S, T \subseteq E$. The quantities $\overline{g}_k$ and $\overline{h}_k$ are defined as

$$\overline{g}_k = \max_{S \subseteq E} (g_k(S))$$

and

$$\overline{h}_k = \max_{T \subseteq E} (h_k(T))$$

for $k = 1, 2, \ldots$. Since $g_k(\emptyset) = h_k(\emptyset) = 0$, it follows that $\overline{g}_k \geq 0$ and $\overline{h}_k \geq 0$ for $k = 1, 2, \ldots$.

The principal minors of a matroid are defined as follows:

(i) We call $\mathcal{S}_k = \mathcal{M} \times S$ a <u>k^{th} principal minor of the first kind</u> (k-PM1) of $\mathcal{M}$ if it satisfies

(P1) $g_k(S) = \overline{g}_k$.

(P2) S is minimal with property (P1).

(ii) A minor $\mathcal{S}_k^+ = \mathcal{M} \times S^+$ is called a <u>k^{th} augmented principal minor of the first kind</u> (k-APM1) of $\mathcal{M}$ if it satisfies

(AP1) $g_k(S^+) = \overline{g}_k$.

(AP2) S^+ is maximal with property (AP1).

(iii) We call $\mathcal{N}_k = \mathcal{M} \cdot T$ a <u>k^{th} principal minor of the second kind</u> (k-PM2) of $\mathcal{M}$ if it satisfies

(P1)' $h_k(T) = \overline{h}_k$.

(P2)' T is minimal with property (P1)'.

(iv) A minor $\mathcal{N}_k^+ = \mathcal{M} \cdot T^+$ is called a <u>k^{th} augmented principal minor of the second kind</u> (k-APM2) of $\mathcal{M}$ if it satisfies

(AP1)' $h_k(T^+) = \overline{h}_k$.

(AP2)' T^+ is maximal with property (AP1)' .

(4.2-1) If $\mathcal{S}_k$ is a k-PM1 of $\mathcal{M}$ and $\mathcal{N}_k$ is a k-PM2 of $\mathcal{M}$, then $\mathcal{S}_k^*$ is a k-PM2 of $\mathcal{M}^*$ and $\mathcal{N}_k^*$ is a k-PM1 of $\mathcal{M}^*$.

<u>Proof</u>: It follows from (2.4-3) and the definition of nullity that we can write $r(\mathcal{M} \times S) = \mu((\mathcal{M} \times S)^*)$. Consequently, $g_k(S) = \alpha(S) - k\ \mu((\mathcal{M} \times S)^*)$. Using (2.4-11) we obtain $g_k(S) = \alpha(S) - k\ \mu(\mathcal{M}^* \cdot S) = h_k^*(S)$, where h_k^* is defined with respect to $\mathcal{M}^*$. Accordingly, $\mathcal{S}_k^*$ is a k-PM2 of $\mathcal{M}^*$. In a similar manner one can show that $\mathcal{N}_k^*$ is a k-PM1 of $\mathcal{M}^*$. ▌

(4.2-2) If $\mathcal{S}_k^+$ is a k-APM1 and $\mathcal{N}_k^+$ is a k-APM2 of $\mathcal{M}$, then $(\mathcal{S}_k^+)^*$ is a k-APM2 of $\mathcal{M}^*$ and $(\mathcal{N}_k^+)^*$ is a k-APM1 of $\mathcal{M}^*$.

<u>Proof</u>: Same as (4.2-1). ▌

As a consequence of (4.2-1) and (4.2-2) we need only study the principal minors of the first kind.

The following two results are preliminary to Theorems (4.2-5) thru (4.2-8) which establish the existence and uniqueness of the principal minors.

(4.2-3) If $S_1, S_2 \subseteq E$, then $r(\mathcal{M} \times (S_1 \cup S_2)) \leq r(\mathcal{M} \times S_1) + r(\mathcal{M} \times S_2) - r(\mathcal{M} \times (S_1 \cap S_2))$.

Proof: Let b_{12} be a base of $\mathcal{M} \times (S_1 \cap S_2)$. Then there exists a base b of $\mathcal{M} \times (S_1 \cup S_2)$ such that $b_{12} \subseteq b$. Thus we have that

$$\begin{aligned} r(\mathcal{M} \times (S_1 \cup S_2)) &= \alpha(b) \\ &= \alpha(b \cap S_1) + \alpha(b \cap S_2) - \alpha(b_{12}) \\ &\leq (\mathcal{M} \times S_1) + r(\mathcal{M} \times S_2) - r(\mathcal{M} \times (S_1 \cap S_2)) . \end{aligned}$$

The inequality follows since $r(\mathcal{M} \times S_i) \geq \alpha(b \cap S_i)$ for $i = 1,2$ and $r(\mathcal{M} \times (S_1 \cap S_2)) = \alpha(b_{12})$. ▮

(4.2-4) If $S_1, S_2 \subseteq E$, then $g_k(S_1 \cup S_2) \geq g_k(S_1) + g_k(S_2) - g_k(S_1 \cap S_2)$ for $k = 1,2, \ldots$.

Proof: Let k be a positive integer. By (4.2-3) we obtain

$$g_k(S_1 \cup S_2) \geq \alpha(S_1) + \alpha(S_2) - \alpha(S_1 \cap S_2) - k\,(r(\mathcal{M} \times S_1) + r(\mathcal{M} \times S_2) - r(\mathcal{M} \times (S_1 \cap S_2))).$$

Rearranging terms and using the definition of g_k we get

$$g_k(S_1 \cup S_2) \geq g_k(S_1) + g_k(S_2) - g_k(S_1 \cap S_2) \text{ .}$$ ▮

In the following four theorems we establish the existence and uniqueness of the principal minors of a matroid.

(4.2-5) Let $\mathcal{M}$ be a matroid on a finite set E; then $\mathcal{S}_k$, a k-PM1 of $\mathcal{M}$, exists and is unique for $k = 1,2, \ldots$.

Proof: Let k be a positive integer. At least one k-PM1 exists since $\overline{g}_k$ is attained by some set $S \subseteq E$.

Suppose $\mathcal{S}_k^{(1)} = \mathcal{M} \times S_1$ and $\mathcal{S}_k^{(2)} = \mathcal{M} \times S_2$ are two k-PM1 for $\mathcal{M}$, then by (4.2-4) and the fact that $g_k(S_1) = g_k(S_2) = \overline{g}_k$ we get

$$g_k(S_1 \cup S_2) \geq 2\overline{g}_k - g_k(S_1 \cap S_2) \quad . \tag{1}$$

Since $g_k(S_1 \cup S_2)$ and $g_k(S_1 \cap S_2)$ cannot exceed $\overline{g}_k$, it follows from (1) that

$$g_k(S_1 \cap S_2) = \overline{g}_k \quad .$$

Since $S_1 \cap S_2 \subseteq S_i$ and by the minimality of S_i we conclude that $S_i = S_1 \cap S_2$ for $i = 1,2$. Therefore $S_1 = S_2$. ∎

(4.2-6) Let $\mathcal{M}$ be a matroid on a finite set E; then $\mathcal{S}_k^+$, a k-APM1 of $\mathcal{M}$, exists and is unique for $k = 1,2, \ldots$.

<u>Proof</u>: Let k be a positive integer. At least one k-APM1 exists since $\overline{g}_k$ is attained by some set $S \subseteq E$.

Suppose $(\mathcal{S}_k^+)^{(1)} = \mathcal{M} \times S_1$ and $(\mathcal{S}_k^+)^{(2)} = \mathcal{M} \times S_2$ are two k-APM1 of $\mathcal{M}$. From the proof of (4.2-5) we have that

$$g_k(S_1 \cup S_2) \geq 2\overline{g}_k - g_k(S_1 \cap S_2) \quad . \tag{1}$$

Since $g_k(S_1 \cup S_2)$ and $g_k(S_1 \cap S_2)$ cannot exceed $\overline{g}_k$, it follows from (1) that

$$g_k(S_1 \cup S_2) = \overline{g}_k \quad .$$

Since $S_i \subseteq S_1 \cup S_2$ and by the maximality of S_i we conclude that $S_i = S_1 \cup S_2$ for $i = 1,2$. Therefore $S_1 = S_2$. ∎

In view of (4.2-1) and (4.2-2) we can state the following:

(4.2-7) Let $\mathcal{M}$ be a matroid on a finite set E; then $\mathcal{N}_k$, a k-PM2 of $\mathcal{M}$, exists and is unique for $k = 1,2, \ldots$.

(4.2-8) Let $\mathcal{M}$ be a matroid on a finite set E; then $\mathcal{N}_k^+$, a k-APM2 of $\mathcal{M}$, exists and is unique for $k = 1,2, \ldots$.

The four preceding theorems establish the existence and the uniqueness of the various principal minors of a matroid. The next two theorems deal with a nesting propery of the principal minors.

For convenience we introduce the following notation.

Suppose $\mathcal{A} = (\mathcal{C}_A, A)$ and $\mathcal{B} = (\mathcal{C}_B, B)$ are matroids. We write $\mathcal{A} \subseteq \mathcal{B}$ if and only if $A \subseteq B$.

(4.2-9) Let $\mathcal{M}$ be a matroid on a finite set E where $\mathcal{S}_k^+$, $\mathcal{S}_{k+1}^+$ and $\mathcal{S}_k$ are the k-APM1, (k+1)-APM1 and the k-PM1 of $\mathcal{M}$, respectively. Then $\mathcal{S}_k \subseteq \mathcal{S}_k^+$ and $\mathcal{S}_{k+1}^+ \subseteq \mathcal{S}_k$ for $k = 1, 2, \ldots$.

<u>Proof</u>: It follows from the definitions and the uniqueness of $\mathcal{S}_k$ and $\mathcal{S}_k^+$ that $\mathcal{S}_k \subseteq \mathcal{S}_k^+$.

It remains to show that $\mathcal{S}_{k+1}^+ \subseteq \mathcal{S}_k$. Let $\mathcal{S}_{k+1}^+ = \mathcal{M} \times T$ and $\mathcal{S}_k = \mathcal{M} \times S$. By the definitions of g_k and g_{k+1} it follows that

$$g_k(T) = g_{k+1}(T) + r(\mathcal{M} \times T) \qquad (1)$$

and

$$g_k(S \cap T) = g_{k+1}(S \cap T) + r(\mathcal{M} \times (S \cap T)) \; . \qquad (2)$$

According to (4.2-4) we can also write that

$$g_k(S \cup T) \geq g_k(T) - g_k(S \cap T) \qquad (3)$$

Using (1) and (2) in (3) and the fact that $g_{k+1}(T) = \overline{g}_{k+1}$, $g_k(S) = \overline{g}_k$ and $\overline{g}_k \geq g_k(S \cup T)$ we obtain the following

$$\overline{g}_k \geq \overline{g}_k + (\overline{g}_{k+1} - g_{k+1}(S \cap T)) + (r(\mathcal{M} \times T) - r(\mathcal{M} \times (S \cap T))) \; . \qquad (4)$$

Clearly, it follows from (4) that

$$g_{k+1}(S \cap T) = \overline{g}_{k+1} \tag{5}$$

and

$$r(\mathcal{M} \times T) = r(\mathcal{M} \times (S \cap T)) \ . \tag{6}$$

Forming $g_{k+1}(T) - g_{k+1}(S \cap T)$ and using (5) and (6), we get that

$$\alpha(T) = \alpha(S \cap T) \ . \tag{7}$$

Accordingly, $T \subseteq S$ and therefore $\mathcal{S}^{+}_{k+1} \subseteq \mathcal{S}_k$. ▮

In view of (4.2-1) and (4.2-2) we can state:

(4.2-10) Let $\mathcal{M}$ be a matroid on a finite set E where $\mathcal{N}^{+}_{k}$, $\mathcal{N}^{+}_{k+1}$ and $\mathcal{N}_k$ are the k-APM2, (k+1)-APM2 and k-PM2 of $\mathcal{M}$, respectively. Then $\mathcal{N}_k \subseteq \mathcal{N}^{+}_{k}$ and $\mathcal{N}^{+}_{k+1} \subseteq \mathcal{N}_k$.

The preceding results establish the existence, uniqueness and nesting properties of the principal minors of a matroid. Theorems (4.2-5) through (4.2-8) establish the existence and uniqueness while (4.2-9) and (4.2-10) give the nesting properties. In the next section we consider the case $k = 1$.

4.3 THE 1-PRINCIPAL MINORS OF A MATROID

In this section we discuss the special case when $k = 1$, that is, we determine the 1-principal minors of a matroid.

Let $\mathcal{M} = (\mathcal{C}, E)$ be the matroid under discussion in this section. It is not too difficult to show that if $T \subseteq S \subseteq E$, then

$$r(\mathcal{M} \times S) \leq r(\mathcal{M} \times T) + \alpha(S - T) .$$

It follows from the inequality (1) that if $T \subseteq S \subseteq E$, then

$$g_1(S) \geq g_1(T) \quad ,$$

Accordingly, the 1-APM1 of $\mathcal{M}$ is $\mathcal{M}$ itself. Moreover, since $g_1(E) = \alpha(E) - r(\mathcal{M}) = \mu(\mathcal{M})$, it follows that $\overline{g}_1 = \mu(\mathcal{M})$. We also conclude from (4.2-2) that the 1-APM2 of $\mathcal{M}$ is $\mathcal{M}$ and since $h_1(E) = \alpha(E) - \mu(\mathcal{M}) = r(\mathcal{M})$, $\overline{h}_1 = r(\mathcal{M})$. Summarizing the above we have:

> (4.3-1) The 1-APM1 and the 1-APM2 of $\mathcal{M}$ are identical and equal to $\mathcal{M}$, that is, $\mathcal{S}_1^+ = \mathcal{K}_1^+ = \mathcal{M}$. Moreover, $\overline{g}_1 = \mu(\mathcal{M})$ and $\overline{h}_1 = r(\mathcal{M})$.

In the following two theorems we exhibit the 1-PM1 and the 1-PM2 of $\mathcal{M}$.

> (4.3-2) Let $P = \{e \mid e \in E$ and e is a member of at least one circuit of $\mathcal{M}.\}$ Then $\mathcal{S}_1 = \mathcal{M} \times P$.

<u>Proof</u>: Clearly, the members of $\overline{P}$ belong to every base of $\mathcal{M}$, that is, if b is a base of $\mathcal{M}$, then $\overline{P} \subseteq b$. Accordingly, $g_1(P) = \overline{g}_1$.

Next we show that P is minimal with the property $g_1(P) = \overline{g}_1$. Let $T \subset P$ and pick some $e \in (P - T)$. From the definition of P it follows that there exists a base b' of $\mathcal{M}$ such that $e \notin b'$. Therefore

$$g_1(T - \{e\}) = \overline{g}_1 - 1 \ . \tag{1}$$

However, since $T \subseteq (P - \{e\})$,

$$g_1(T) \leq g_1(P - \{e\}) \tag{2}$$

Combining (1) and (2) we get that

$$g_1(T) < \overline{g}_1 \ .$$

Accordingly, $\mathcal{M} \times P$ is the 1-PM1 of $\mathcal{M}$. ▮

(4.3-3) Let $Q = \{e \mid e \in E$ and e is a member of at least one circuit of $\mathcal{M}^*\}$. Then $\mathcal{N}_1 = \mathcal{M} \cdot Q$.

<u>Proof</u>: Theorem (4.3-3) follows from (4.3-2), (4.2-1) and (2.4-10). ▮

In the three preceding theorems we have identified the 1-principal minors of $\mathcal{M}$.

If we let G be a graph and $\mathcal{M} = (\mathcal{P}(G), E(G))$, then (4.3-2) and (4.3-3) have very simple graph-theoretic counterparts.

An edge $e \in E(G)$ is called a <u>separating edge</u> of G if removal of e from G causes the ends of e in G to become disconnected.

(4.3-4) Let G be a graph and $\mathcal{M}$ the polygon matroid of G. Then $\mathcal{J}_1$ is the polygon matroid of the graph obtained from G by removing all separating edges.

(4.3-5) Let G be a graph and $\mathcal{M}$ the polygon matroid of G. Then $\mathcal{N}_1$ is the polygon matroid of the graph obtained from G by removing all loops.

<u>Example 4-1:</u> Let G be the graph in Fig.4-1. It is easy to see that

$$\mathcal{S}_1 = \mathcal{P}(G) \times [E - \{e_5, e_8\}]$$
$$= \mathcal{P}(G) \cdot [E - \{e_5, e_8\}]$$
$$= \mathcal{P}(G_1)$$

and that

$$\mathcal{H}_1 = \mathcal{P}(G) \cdot [E - \{e_1, e_{11}\}]$$
$$= \mathcal{P}(G \times [E - \{e_1, e_{11}\}])$$
$$= \mathcal{P}(H_1) \quad .$$

The graphs G_1 and H_1 are shown in Fig. 4-2 and Fig.4-3.

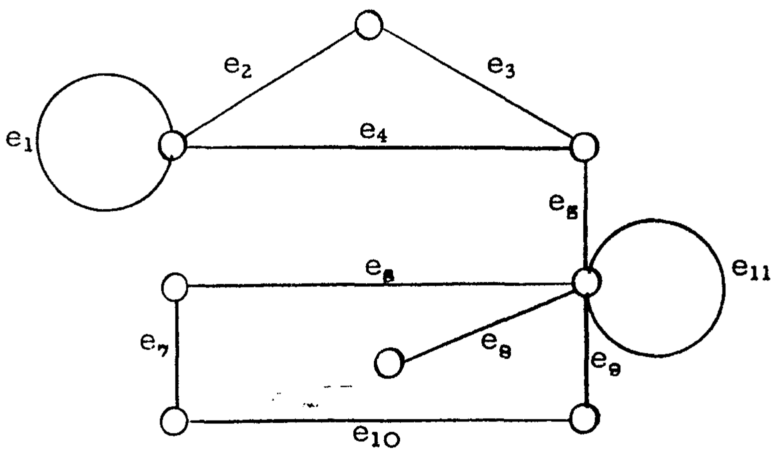


Figure 4-1. Graph G of Example 4-1

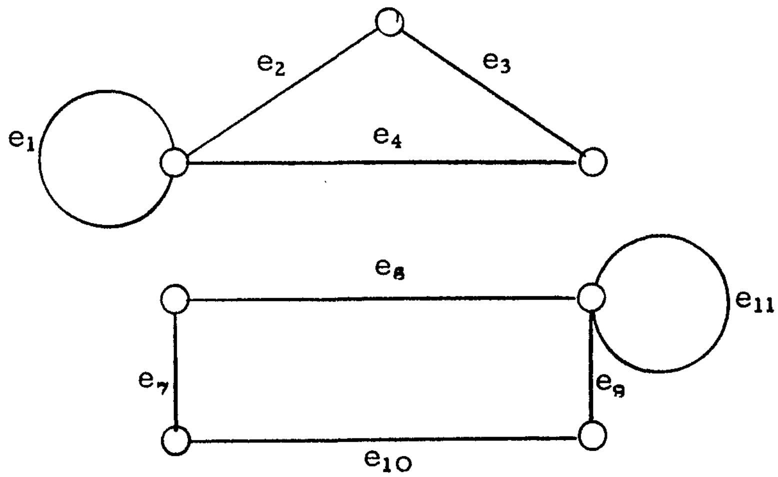


Figure 4-2. Graph G_1 of Example 4-1

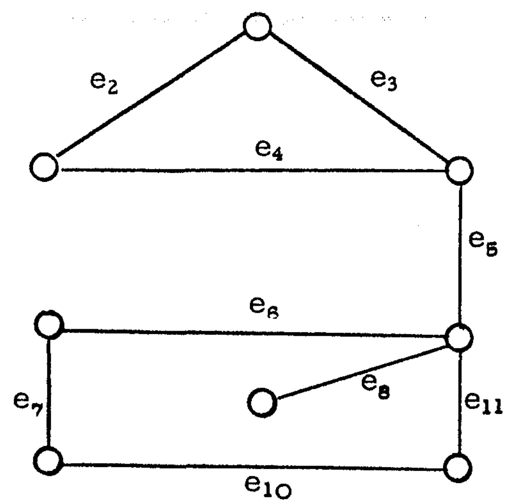


Figure 4-3. Graph H_1 of Example 4-1

4.4 MAXIMALLY DISTANT BASES AND THE PRINCIPAL MINORS

In Section 4.3 we identified $\mathcal{J}_1^+$, $\mathcal{K}_1^+$, $\mathcal{J}_1$ and $\mathcal{K}_1$ for any matroid $\mathcal{M}$ and found that $\overline{g}_1 = \mu(\mathcal{M})$ and $\overline{h}_1 = r(\mathcal{M})$. In this section we determine all the principal minors in a constructive manner using the concept of r-maximally distant bases. Algorithms are presented which can be used to construct any principal minor of a matroid efficiently. We begin with some definitions.

Let $\mathcal{M}$ be a matroid on a finite set E and $\mathcal{B}_{\mathcal{M}}$ the class of bases of $\mathcal{M}$. Let $\mathcal{B}^r_{\mathcal{M}}$ be the class of all r-tuples of bases of $\mathcal{M}$, that is,

$$\mathcal{B}^r_{\mathcal{M}} = \{(b_1, b_2, \ldots, b_r) \mid b_i \in \mathcal{B}_{\mathcal{M}} \text{ for } i = 1, \ldots, r\}$$

Note that $\mathcal{B}^1_{\mathcal{M}} = \mathcal{B}_{\mathcal{M}}$.

For each member $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$ we define a non-negative integer $c(\underline{b})$:

$$c(\underline{b}) = \alpha\,(E - (\bigcup_{i=1}^{r} b_i)) \quad .$$

Define

$$\underline{c}_r = \min_{\underline{b} \in \mathcal{B}^r_{\mathcal{M}}} c(\underline{b}) \quad .$$

If $\underline{b} \in \mathcal{B}^r_{\mathcal{M}}$ and $c(\underline{b}) = \underline{c}_r$, then the coordinates of $\underline{b}$ are called <u>r-maximally distant bases</u> and the r-tuple $\underline{b}$ is called <u>r-maximally distant</u>.

Let $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$. $\mathcal{M} \times S$ is called an r-minor of $\mathcal{M}$ with respect to $\underline{b}$ if S satisfies:

(M1) $(E - (\bigcup_{i=1}^{r} b_i)) \subseteq S$.

(M2) $b_1' = b_1 \cap S, \ldots, b_r' = b_r \cap S$ are element-disjoint bases of $\mathcal{M} \times S$.

(M3) S is minimal with properties (M1) and (M2).

$\mathcal{M} \times S^+$ is called r-augmented minor of $\mathcal{M}$ with respect to $\underline{b} \in \mathcal{B}^r_{\mathcal{M}}$ if S^+ satisfies:

(AM1) $(E - (\bigcup_{i=1}^{r} b_i)) \subseteq S^+$.

(AM2) $b'_1 = b_1 \cap S^+, \ldots, b_r = b_1 \cap S^+$ are element-disjoint bases of $\mathcal{M} \times S^+$

(AM3) S^+ is maximal with properties (AM1) and (AM2).

In order to avoid the use of a disproportionate amount of subscripts (or superscripts) we adopt the following notation (which is quite conventional in computer programming). Let f(S,T,U, ...) be some set-theoretic expression. Then by

$$K \leftarrow f(S,T,U, \ldots)$$

we mean that one calculates the set determined by the expression on the right-hand side of the arrow and then uses K to denote the calculated set. This operation allows one to have the set K on both sides of the arrow. For example, let S, T and K be three finite sets, and write

$$K \leftarrow (S \cap T) \cup K$$

We first calculate the expression on the right-hand side of the arrow, that is, we determine the set $(S \cap T) \cup K$. Then this set is subsequently denoted by the letter K. We can view this operation as one in which the elements in the set K are "changed" or "updated". It is, of course, not necessary for the symbol on the left-hand side of the arrow to appear on the right-hand side also.

The next result is a preliminary one.

(4.4-1) Let b be a base of $\mathcal{M}$ and $e \in E$ and $e \notin b$. Then (e/e')b is a base of $\mathcal{M}$, where e' is any element in J(b,e).

<u>Proof</u>: Assume (e/e') b is not a base for some $e' \in J(b,e)$. Clearly $e' \neq e$ and (e/e')b contains a circuit C satisfying $e' \in C \subseteq (e/e')b$. Furthermore, there exists a circuit C' satisfying

$$e' \in C' \subseteq [(C \cup J(b,e)) - \{e\}] \subseteq b \qquad (1)$$

by the property (C2) of circuits. But (1) contradicts the hypothesis. Therefore (4.4-1) is proved. ▌

The subsequent results are the main results of this section. They relate the concepts of r-maximally distant bases, r-minors, r-augmented minors and the principal minors of a matroid.

(4.4-2) Let $\mathcal{M} = (\mathcal{C}, E)$. Then $\underline{b} \in \mathcal{B}^r_{\mathcal{M}}$ is r-maximally distant if and only if there exists an r-minor of $\mathcal{M}$ with respect to $\underline{b}$.

<u>Proof</u>: Suppose $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$ is r-maximally distant. Our objective is to construct an r-minor of $\mathcal{M}$ with respect to $\underline{b}$. Let

$$D = (E - (\bigcup_{i=1}^{r} b_i))$$

and

$$B = \{e \mid e \in E \text{ and } e \text{ is a member of at least two coordinates of } \underline{b}\} .$$

Each of the members of D forms a unique circuit in every coordinate of $\underline{b}$. Let A_1 denote the union of <u>all</u> such circuits formed by the members of D in each of the coordinates of $\underline{b}$. Each member of A_1 is not a member of certain of the coordinates of b (possibly none) and accordingly forms a unique circuit in each of these coordinates. Let A_2 denote the union of all such circuits formed by each of the members of A_1 in the coordinates of b. In a similar manner A_{n+1} is obtained from A_n for $n = 2, 3, \ldots$. By the construction of the sequence A_n we have that $A_n \subseteq A_{n+1}$.

For the case $r = 1$, $A_1 = A_n$ for $n = 2, 3, \ldots$ and $A_1 \cap B = \emptyset$ since $B = \emptyset$.

Let $r > 1$ and assume that there exists a least positive integer $p \geq 1$ such that $A_p \cap B \neq \emptyset$. We shall show that this assumption leads to a contradiction. Pick some

$$e^* \in (A_p \cap B) \quad .$$

By the construction of the sequence A_n, there exists a sequence

$$e = e_o, e_1 \ldots, e_p = e^* \qquad (1)$$

of distinct elements of E and a sequence

$$b_{i_1}, b_{i_2}, \ldots, b_{i_p} \qquad (2)$$

of bases from b such that

(i) $e \in D$.

(ii) there is a unique circuit C_j satisfying:

(a) $e_{j-1} \in C_j \subseteq b_{i_j} \cup \{e_{j-1}\}$ for $j = 1, \ldots, p$.

(b) $e_j \in C_j$ and $e_j \in b_{i_j}$ for $j = 1, \ldots, p$.

(c) $e_k \notin C_j$ for each k satisfying $j < k \leq p$, where $j = 1, \ldots, p-1$.

We define recursively an updated set of bases b_i as follows: Let

$$b_{i_{p-k+1}} \leftarrow (e_{p-k}/e_{p-k+1})\, b_{i_{p-k+1}} \qquad (3)$$

For $k = 1, \ldots, p$ (the p operations called for in (3) are done in order, that is, first execute (3) with $k = 1$, then $k = 2$, etc.).

We claim that the r-tuple $(b_1, \ldots, b_r)$, after performing the operations defined by (3), satisfies:

(A) $(b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$

(B) $c((b_1, \ldots, b_r)) = \underline{c}_r - 1$.

To show (A) we use mathematical induction. Let $k = 1$ in (3). Thus (3) reads

$$b_{i_p} \leftarrow (e_{p-1}/e_p) b_{i_p} \quad .$$

By (ii) (a) and (ii) (b) and (4.4-1), b_{i_p} (updated) is a base. Consequently, $(b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$ after the first step. Moreover, (ii) (a) and (ii) (b) are satisfied by (1) and (2) for $i = 1, \ldots, p-1$ by (ii) (c).

Assume that at the q^{th} iteration of (3), where $1 \le q < p$, that $(b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$ and (ii) (a) and (ii) (b) are satisfied by (1) and (2) for $i = 1, \ldots, p-q$.

Set $k = q+1$ in (3). Thus (3) reads

$$b_{i_{p-q}} \leftarrow (e_{p-q-1}/e_{p-q}) b_{p-q} \quad .$$

By the induction hypothesis and (4.4-1), $b_{i_{p-q}}$ (updated) is a base. By (ii) (c), it follows that properties (ii) (a) and (ii) (b) are satisfied by (1) and (2) for $j = 1, \ldots, p-(q+1)$. By induction, (A) is proved.

To show (B), note that e_p is the only element in (1) which is not replaced into some coordinate of $(b_1, \ldots, b_r)$. However, $e_p \in B$ and therefore is a member of some coordinate of the final $\underline{b}$. Moreover, $e \in D$ is now a member of b_{i_1} and therefore (B) is proved.

However, (A) and (B) contradict the hypothesis. Accordingly, for every positive integer p, $A_p \cap B = \emptyset$. Therefore, by the finiteness of the set E, there exists a least positive integer s such that

$$A_s = A_{s+1} \tag{4}$$

and

$$A_s \cap B = \emptyset \ . \tag{5}$$

In other words, the procedure originally set forth to construct the sequence A_n must terminate with equations (4) and (5) being satisfied if $\underline{b}$ is r-maximally distant (this being true for $r = 1, 2, \ldots$).

Next we show that $\mathcal{M} \times A_s$ is an r-minor of $\mathcal{M}$ with respect to $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$, where $\underline{b}$ is the original r-maximally distant r-tuple.

Clearly, (M1) is satisfied by A_s. Next since $A_s \cap B = \emptyset$, $b_1 \cap A_s, \ldots, b_r \cap A_s$ are element disjoint independent sets of $\mathcal{M} \times A_s$. By the construction of A_s, $b_i \cap A_s$ is a maximal independent set of $\mathcal{M} \times A_s$ for $i = 1, \ldots, r$. Consequently, (M2) is satisfied. A_s is also minimal with properties (M1) and (M2); to see this suppose $T \subseteq A_s$ satisfies (M1) and (M2). Then by the construction of A_s, it is clear that $A_s \subseteq T$. Consequently, $A_s = T$ and therefore A_s is minimal.

We have shown the existence of an r-minor $\mathcal{M} \times A_s$ of $\mathcal{M}$ with respect to an arbitrary r-maximally distant $\underline{b} \in \mathcal{B}^r_{\mathcal{M}}$.

To show the sufficiency part of the theorem, let $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$ and suppose there exists an r-minor $\mathcal{M} \times S$ of $\mathcal{M}$ with respect to $\underline{b}$. Set

$$D = (E - (\bigcup_{i=1}^{r} b_i)) = \{e_1, \ldots, e_m\} .$$

Choose any $\underline{b}' = (b'_1, \ldots, b'_r) \in \mathcal{B}^r_{\mathcal{M}}$. Then

$$\alpha(E - (\bigcup_{i=1}^{r} b'_i)) \geq \alpha(S - (\bigcup_{i=1}^{r} (b'_i \cap S))) = \alpha(S) - \alpha(\bigcup_{i=1}^{r} (b'_i \cap S)) . \qquad (6)$$

By hypothesis $b_i \cap S$ is a base of $\mathcal{M} \times S$ for $i = 1, \ldots, r$ and these bases are element disjoint. Thus

$$\alpha(\bigcup_{i=1}^{r} (b'_i \cap S)) \leq \alpha(\bigcup_{i=1}^{r} (b_i \cap S)) . \qquad (7)$$

Combining (6) and (7) we get

$$\alpha(E - (\bigcup_{i=1}^{r} b'_i)) \geq \alpha(S) - \alpha(\bigcup_{i=1}^{r} (b_i \cap S) .$$

Since $(b_i \cap S) \subseteq S$ for $i = 1, \ldots, r$, we get

$$\alpha(E - (\bigcup_{i=1}^{r} b'_i)) \geq \alpha(S - (\bigcup_{i=1}^{r} (b_i \cap S))) .$$

Since each of the elements in (E-S) is a member of at least one b_i for $i = 1, \ldots, r$, we get

$$\alpha(E - (\bigcup_{i=1}^{r} b'_i)) \geq \alpha(E - (\bigcup_{i=1}^{r} b_i))$$

or

$$c(\underline{b}') \geq c(\underline{b})$$

for all $\underline{b}' \in \mathcal{B}^r_{\mathcal{M}}$. Therefore, $\underline{b}$ is r-maximally distant. ∎

The proof of (4.4-2) suggests a way of finding an r-maximally distant $\underline{b} \in \mathcal{B}^r_{\mathcal{M}}$ given an arbitrary member of $\mathcal{B}^r_{\mathcal{M}}$.

<u>Algorithm 1</u> - (Determination of r-Maximally Distant Bases)

Let $\mathcal{M}$ be a matroid on a finite set E and let $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$. Let

$$D = [E - \bigcup_{i=1}^{r} b_i]$$

and

$$B = \{e \mid e \in E \text{ and } e \text{ is a member of at least two of the coordinates of } \underline{b}.\}$$

<u>Step 1</u> If $D = \phi$ or $B = \phi$, then $\underline{b}$ is r-maximally distant; otherwise go to Step 2.

<u>Step 2</u> Pick some $e \in D$. Since e is not a member of any coordinate of $\underline{b}$, e forms a unique circuit in each of the bases b_i for $i = 1, \ldots, r$. Let A_1 denote the union of all these circuits. Each member of A_1 is not a member of certain of the coordinates of $\underline{b}$ (possibly none of the coordinates) and thus forms a unique circuit in each of these coordinates. Let A_2 denote the union of <u>all</u> such circuits formed by each of the members of A_1 in the coordinates of $\underline{b}$. In a similar manner A_{n+1} is obtained from A_n for $n = 2, 3, \ldots$.

There are two possible cases:

<u>Case 1</u> There is a least positive integer s such that $A_s = A_{s+1}$ and $A_s \cap B = \phi$. Set

$$D \leftarrow D - \{e\}$$

and return to Step 1.

<u>Case 2</u> There is a least positive integer p such that $A_p \cap B \neq \phi$. Pick some $e^* \in A_p \cap B$. According to the construction of the sequence A_n, there is a sequence

$$e = e_o, e_1, \ldots, e_p = e^*$$

of elements of E and a sequence

$$b_{i_1}, b_{i_2}, \ldots, b_{i_p}$$

of bases from $\underline{b}$ such that e_{j-1} forms a unique circuit in b_{i_j} and this circuit contains e_j for $j = 1, \ldots, p$. Set

$$b_{i_{p-k+1}} \leftarrow (e_{p-k}/e_{p-k+1}) b_{i_{p-k+1}}$$

in sequence starting with $k = 1$ through $k = p$ to obtain the updated version of $\underline{b} = (b_1, \ldots, b_r)$. Set

$$D = [E - \bigcup_{i=1}^{r} b_i]$$

and

$$B = \{ e \mid e \in E \text{ and } e \text{ is a member of at least two of the coordinates of } \underline{b}. \} .$$

Go to Step 1.

The justification of Algorithm 1 is substantially given in the first part of the proof of (4.4-2).

Example 4-2: We illustrate (4.4-2) and Algorithm 1 with an example. Let $\mathcal{M} = (\mathcal{B}(G), E(G))$, where G is the graph in Fig.4-4. The sets $\{e_1, e_4\}$ and $\{e_3, e_8\}$ are coforests of G and consequently bases of $\mathcal{B}(G)$.

Suppose we want to find a 3-maximally distant $\underline{b} = (b_1, b_2, b_3)$. Initially set

$$b_1 = \{e_1, e_4\} ,$$

$$b_2 = b_1$$

and

$$b_3 = \{e_3, e_8\}$$

as depicted in Fig.4-4.

Obviously $(b_1, b_2, b_3) \in \mathcal{B}^3_{\mathcal{B}(G)}$. Initializing Algorithm 1, we calculate

$$D = \{e_2, e_8\}$$

and

$$B = \{e_1, e_4\} .$$

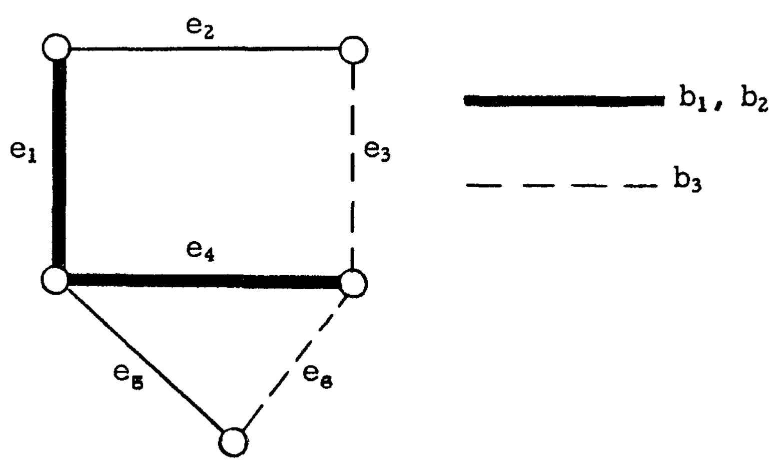


Figure 4-4. Graph G of Example 4-2

Since $D \neq \phi$ and $B \neq \phi$ we go to Step 2 and pick $e = e_2 \in D$. e forms a unique circuit C_1 in b_1 where

$$C_1 = \{e_1, e_2\} .$$

C_1 is also the unique circuit C_2 formed by e in b_2. The unique circuit C_3 formed by e in b_3 is

$$C_3 = \{e_2, e_3\} .$$

Thus

$$A_1 = \{e_1, e_2, e_3\}$$

and

$$A_1 \cap B = \{e_1\} .$$

Therefore we are in Step 2, Case 2. The appropriate sequences are

$$e = e_2, e_1 \qquad (1)$$

and

$$b_1 \quad . \qquad (2)$$

Since $p = 1$, we get

$$b_1 \leftarrow (e_2/e_1)b_1 \quad .$$

Thus

$$\underline{b} = (b_1, b_2, b_3) \quad ,$$

where

$$b_1 = \{e_2, e_4\}$$

$$b_2 = \{e_1, e_4\}$$

$$b_3 = \{e_3, e_8\} \quad .$$

This is illustrated in Fig.4-5.

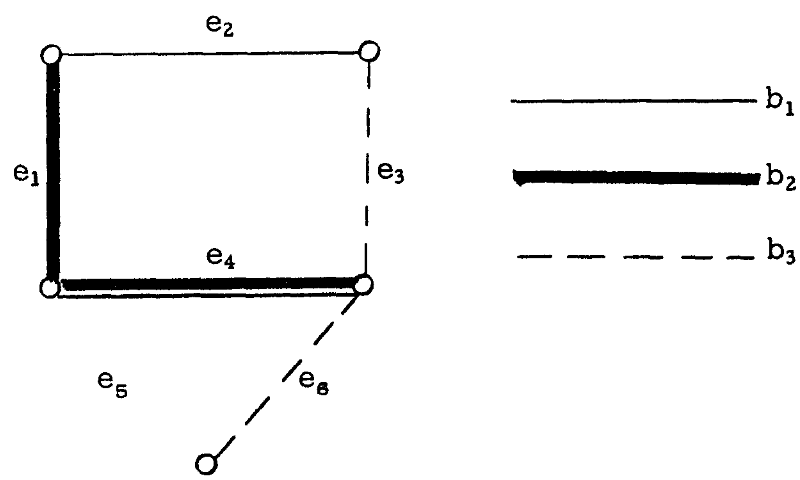


Figure 4-5. Bases b_1, b_2 and b_3

Then D and B become

$$D = \{ e_5 \}$$

and

$$B = \{ e_4 \} .$$

We return to Step 1 and since $D \neq \phi$ and $B \neq \phi$, we go to Step 2 and set $e = e_5 \in D$.

The element e forms a unique circuit C_1 in b_1 where

$$C_1 = \{ e_5 , e_4 , e_2 \} .$$

Since C_1 meets B, we can proceed directly to Step 2, Case 2 and the appropriate sequences are

$$e = e_5 , e_4 \qquad (1)$$

and

$$b_1 . \qquad (2)$$

Since $p = 1$, we get

$$b_1 \leftarrow (e_5 / e_4) b_1 .$$

Thus

$$\underline{b} = (b_1 , b_2 , b_3) ,$$

where

$$b_1 = \{ e_2 , e_5 \} ,$$

$$b_2 = \{ e_1 , e_4 \}$$

and

$$b_3 = \{ e_3 , e_6 \} .$$

This is shown in Fig.4-6.

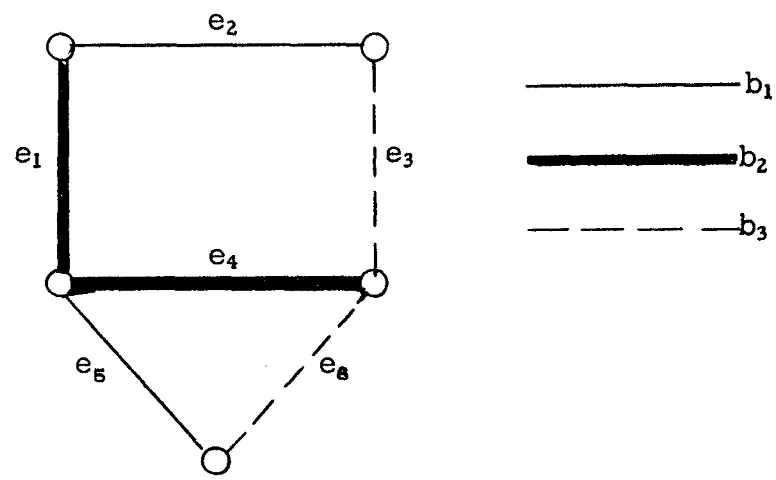


Figure 4-6. Maximally Distant Bases

Calculating D and B, we get

$$D = B = \phi$$

and therefore Algorithm 1 terminates and $\underline{b} \in \mathcal{B}^{3}_{\mathcal{B}(G)}$ is 3-maximally distant.

In the next theorem we show that if $\mathcal{M} \times S$ is any r-minor of $\mathcal{M}$ with respect to an r-maximally distant $\underline{b}$, then $\mathcal{M} \times S$ is the r-PM1 of $\mathcal{M}$.

(4.4-3) Let $\mathcal{M} = (\mathcal{C}, E)$ and $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^{r}_{\mathcal{M}}$ be r-maximally distant. Then $\mathcal{M} \times S$, an r-minor of $\mathcal{M}$ with respect to $\underline{b}$ is equal to $\mathcal{S}_r$, the r-PM1 of $\mathcal{M}$; that is, $\mathcal{S}_r = \mathcal{M} \times S$.

<u>Proof</u>: We must show that (i) $g_r(S) = \overline{g}_r$ and (ii) S is minimal with property (i).

To prove (i) suppose $T \subseteq E$. Then

$$\alpha(T) \leq \bigcup_{i=1}^{r} \alpha(b_i \cap T) + \alpha(D \cap T) \tag{1}$$

and

$$r \cdot r\,(\mathcal{M} \times T) \geq \sum_{i=1}^{r} \alpha(b_i \cap T) \quad , \tag{2}$$

where

$$D = (E - (\bigcup_{i=1}^{r} b_i) \quad .$$

Using (1) and (2) in the expression for $g_r(T)$ we get

$$g_r(T) \leq \alpha(D \cap T) = \alpha(D').$$

By hypothesis $b_1 \cap S, \ldots, b_r \cap S$ are element disjoint bases of $\mathcal{M} \times S$. Thus

$$\alpha(S) = \sum_{i=1}^{r} \alpha(b_i \cap S) + \alpha(D)$$

and

$$r \cdot r\,(\mathcal{M} \times S) = \sum_{i=1}^{r} \alpha(b_i \cap S) \quad .$$

Consequently, $g_r(S) = \alpha(D)$. Therefore

$$g_r(T) \leq g_r(S) \ .$$

Since T was arbitrary, $g_r(S) = \overline{g}_r$.

We prove (ii) by showing that if $T \subset S$, then $g_r(T) < g_r(S)$. Assume $T \subset S$.

<u>Case 1</u> The $\alpha(b_i \cap T)$'s are not all equal. Let $\overline{\alpha} = \max(\alpha(b_i \cap T))$. Then $r(\mathcal{M} \times T) \geq \overline{\alpha}$. Using this and (1) in the expression for $g_r(T)$ we get

$$g_r(T) \leq \alpha(D') - \sum_{i=1}^{r} (\overline{\alpha} - \alpha(b_i \cap T)) < \alpha(D')$$

Therefore $g_r(T) < g_r(S)$.

<u>Case 2</u> The $\alpha(b_i \cap T)$'s are all equal. We treat this case by contradiction. Assume $g_r(T) = \overline{g}_r$. Therefore, using (1) and (2) and the fact that $T \subset S$, it follows that

$$\alpha(T) = \sum_{i=1}^{r} \alpha(b_i \cap T) + \alpha(D') \tag{3}$$

$$r \cdot r(\mathcal{M} \times T) = \sum_{i=1}^{r} \alpha(b_i \cap T) \tag{4}$$

and

$$\alpha(D') = \alpha(D) \ . \tag{5}$$

It follows from (4) and the hypothesis, that $(b_1 \cap T), \ldots, (b_r \cap T)$ are element disjoint bases of $\mathcal{M} \times T$. Also, (5) and (3) imply that $D \subseteq T$. Therefore the set T satisfies conditions (M1) and (M2) for an r-minor of $\mathcal{M}$ with respect to $\underline{b}$. This, however, is a contradiction since by hypothesis $\mathcal{M} \times S$ is an r-minor of $\mathcal{M}$ with respect to $\underline{b}$. Therefore, $g_r(T) < g_r(S)$.

Consequently S is minimal with the property $g_r(S) = \overline{g}_r$ and we conclude that $\mathcal{J}_r = \mathfrak{M} \times S$. ∎

The uniqueness of the r-principal minors of $\mathfrak{M}$ guarantees that there is one and only one r-minor with respect to <u>any</u> r-maximally distant $\underline{b} \in \mathfrak{B}^r_{\mathfrak{M}}$. The following is an immediate corollary to (4.4-3).

> (4.4-4) Let $\mathfrak{M} = (\mathcal{C}, E)$ and $\underline{b} = (b_1, \ldots, b_r) \in \mathfrak{B}^r_{\mathfrak{M}}$ be r-maximally distant. Then $\overline{g}_r = \underline{c}_r = \alpha(E - (\bigcup_{i=1}^{r} b_i))$.

An alternative characterization of $\mathcal{J}_r$ is given in the following theorem.

> (4.4-5) Let $\mathfrak{M} = (\mathcal{C}, E)$ and $\underline{b} = (b_1, \ldots, b_r) \in \mathfrak{B}^r_{\mathfrak{M}}$ be r-maximally distant. Set $D = (E - (\bigcup_{i=1}^{r} b_i))$. Let S be the union of all sets D formed by spanning over all r-maximally distant r-tuples $\underline{b}$. Then $\mathcal{J}_r = \mathfrak{M} \times S$.

We omit the proof of (4.4-5) since the reader should have no difficulty supplying it. At this point we state the algorithm for constructing $\mathcal{J}_r$, the r-PM1 of $\mathfrak{M}$.

<u>Algorithm 2</u> (Determination of $\mathcal{J}_r$, the r-PM1 of $\mathfrak{M}$)

Let $\mathfrak{M} = (\mathcal{C}, E)$ and $\underline{b} = (b_1, \ldots, b_r) \in \mathfrak{B}^r_{\mathfrak{M}}$ be r-maximally distant. Set

$$D = (E - (\bigcup_{i=1}^{r} b_i)) \quad .$$

The members of D form unique circuits in each of the coordinates of $\underline{b}$. Let A_1 denote the union of <u>all</u> such circuits formed by the members of D in each of the coordinates of $\underline{b}$. Each member of A_1 is not a member of certain of the coordinates of $\underline{b}$ and accordingly forms a unique circuit in each of these coordinates. Let A_2 denote the union of <u>all</u> such circuits formed by each of the members of A_1 in the coordinates of $\underline{b}$. In a similar manner A_{n+1} is obtained from A_n for $n = 2, 3, \ldots$.

There exists a least positive integer s such that

$$A_s = A_{s+1} \quad .$$

Let $S = A_s$. Then $\mathcal{S}_r = \mathcal{M} \times S$.

Theorem (4.4-2) is also true for r-augmented minors as we show in the following theorem.

(4.4-5) Let $\mathcal{M} = (\mathcal{C}, E)$. Then $\underline{b} \in \mathcal{B}^r_{\mathcal{M}}$ is r-maximally distant if and only if there exists an r-augmented minor of $\mathcal{M}$ with respect to $\underline{b}$.

Proof: Suppose $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$ is r-maximally distant. As in the proof of (4.4-2), our objective is to construct an r-augmented minor with respect to $\underline{b}$. Set

$$D = (e - (\bigcup_{i=1}^{r} b_i))$$

and

$$B = \{e \mid e \in E \text{ and } e \text{ is a member of at least two coordinates of } \underline{b}\} .$$

Using Algorithm 2 we construct $\mathcal{S}_r = \mathcal{M} \times S$, the r-minor with respect to $\underline{b}$. Set

$$F = [\, E - (B \cup S) \,]$$

and

$$S^+ = S \quad .$$

We claim that the following algorithm gives an r-augmented minor of $\mathcal{M}$ with respect to $\underline{b}$.

Step 1 If $F = \emptyset$, $\mathcal{M} \times S^+$ is an r-augmented minor of $\mathcal{M}$ with respect to $\underline{b}$, otherwise, go to Step 2.

Step 2 Pick some $e \in F$. The element e forms a unique circuit in all but one of the coordinates of $\underline{b}$. Let A_1 be the union of all such circuits formed by e in the coordinates of $\underline{b}$. Each member of A_1 is not a member of certain coordinates of $\underline{b}$ and accordingly forms a unique circuit in each of these coordinates. Let A_2 denote the union of all such circuits formed by each of the members of A_1 in the coordinates of $\underline{b}$. In a similar manner A_{n+1} is obtained from A_n for $n = 2, 3, \ldots$. There are two possibilities:

(1) There exists a least positive integer p such that $A_p \cap B \neq \emptyset$.

$$F \leftarrow F - \{e\}$$

and return to Step 1.

(2) There exists a least positive integer s such that $A_s = A_{s+1}$ and $A_s \cap B = \emptyset$. In this case we set

$$S^+ \leftarrow S^+ \cup A_s ,$$

$$F \leftarrow F - \{e\}$$

and return to Step 1.

To show that $\mathcal{M} \times S^+$ is an r-augmented minor of $\mathcal{M}$ with respect to $\underline{b}$, it suffices to show that S^+ is maximal with properties (AM1) and (AM2). Assume that there is a set T which also satisfies (AM1) and (AM2) with respect to $\underline{b}$ and $S \subset T$. Then there is an element e in $F \cap (T - S^+)$. Clearly, the preceding algorithm when applied to e must terminate in Step 2, case (2) and accordingly $e \in S^+$. But this is a contradiction and therefore S^+ is maximal.

The sufficiency is proved in the same manner as the sufficiency part of (4.4-2). ▮

The next theorem is the counterpart of (4.4-3) for the augmented principal minors.

(4.4-7) Let $\mathcal{M} = (\mathcal{C}, E)$ and $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}^r_{\mathcal{M}}$ be r-maximally distant. Then $\mathcal{M} \times S^+$, an r-augmented minor of $\mathcal{M}$ with respect to $\underline{b}$ is equal to $\mathcal{S}^+_r$, the r-APM1 of $\mathcal{M}$; that is, $\mathcal{S}^+_r = \mathcal{M} \times S^+$.

Proof: We must show that (i) $g_r(S^+) = \overline{g}_r$ and (ii) that S^+ is maximal with property (i).

The proof of (i) is the same as the proof of (i) in (4.4-3).

We must show that S^+ is maximal with property (i). Since $\mathcal{M} \times S^+$ is an r-augmented minor of $\mathcal{M}$ with respect to $\underline{b}$, it follows that if $S^+ \subset T$, then T does not satisfy property (AM2).

Case 1: $(T \cap b_i) \cap B = \phi$ for $i = 1, \ldots, r$.

$$g_r(T) = \alpha(T) - r \cdot r(\mathcal{M} \times T)$$
$$= \alpha(D) + \sum_{i=1}^{r} (b_i \cap T) - r \cdot [r(\mathcal{M} \times T)].$$

By hypothesis there exists an i_1 such that $(b_{i_1} \cap T) < r(\mathcal{M} \times T)$. Therefore

$$g_r(T) < \alpha(D) = \overline{g}_r .$$

Case 2: There exists an i_2 such that $(T \cap b_{i_2}) \cap B \neq \phi$. Consequently

$$\alpha(T) < \alpha(D) + \sum_{i=1}^{r} \alpha(T \cap b_i)$$

and therefore

$$g_r(T) < \alpha(D) = \overline{g}_r .$$

Thus we have shown that if $S^+ \subset T$, $g_r(T) < \overline{g}_r$ and accordingly S^+ is maximal with property (i). ∎

We conclude this section with Algorithm 3 which is used for the determination of $\mathcal{S}^+_r$, the k-APM1 of $\mathcal{M}$. The justification of this algorithm is given in the proof of (4.4-6).

Algorithm 3 (Determination of $\mathcal{S}_r^+$, the r-APM1 of $\mathcal{M}$)

Let $\mathcal{M} = (\mathcal{C}, E)$ and $\underline{b} = (b_1, \ldots, b_r) \in \mathcal{B}_{\mathcal{M}}^r$ be r-maximally distant $(r > 1)$. Form $\mathcal{S}_r = \mathcal{M} \times S$, the k-PM1 of $\mathcal{M}$ (Algorithm 2).

Set

$$B = \{e \mid e \in E \text{ and } e \text{ is a member of at least two coordinates of } \underline{b}\},$$

$$F = [E - (S \cup B)]$$

and $S^+ = S$.

Step 1 If $F = \phi$, then $\mathcal{S}_r^+ = \mathcal{M} \times S^+$, otherwise go to Step 2.

Step 2 Pick some $e \in F$. The element e forms a unique circuit in all but one of the coordinates of $\underline{b}$. Let A_1 be the union of all such circuits formed by e in the coordinates of $\underline{b}$. Each member of A_1 is not a member of certain coordinates of $\underline{b}$ and accordingly forms a unique circuit in each of these coordinates. Let A_2 denote the union of all such circuits formed by each of the members of A_1 in the coordinates of $\underline{b}$. In a similar manner A_{n+1} is obtained from A_n for $n = 2, 3, \ldots$.

There are two cases:

Case 1: There is a least positive integer p such that $A_p \cap B \neq \phi$. Set

$$F \leftarrow F - \{e\}$$

and return to Step 1.

Case 2: There is a least positive integer s such that $A_s = A_{s+1}$ and $A_s \cap B = \phi$. Set

$$S^+ \leftarrow S^+ \cup A_s,$$

$$F \leftarrow F - \{e\}$$

and return to Step 1.

The preceding algorithms provide us with efficient means for constructing the principal minors of a matroid. To conclude this section we work out an example illustrating the use of Algorithms 2 and 3.

Example 4-3. Consider the graph G in Fig.4-7. $E(G) = \{1, \ldots, 55\}$. Let $\mathcal{M} = (\mathcal{P}(G), E(G))$. Applying Algorithm 1 we obtain a 2-maximally distant $\underline{b} = (b_1, b_2) \in \mathcal{B}^2_{\mathcal{P}(G)}$. The bases (forests of G) b_1 and b_2 are shown in Fig.4-8.

Let us apply Algorithm 2 using the r-maximally distant ordered pair $\underline{b} = (b_1, b_2)$. Initializing the algorithm we find that

$$D = \{13, 14, 15\} .$$

We find that

$$A_1 = \{13, 14, 15, 12, 19, 9, 11, 16, 17, 18, 8, 21\}$$

$$A_2 = A_1 \cup \{10\} .$$

$$A_3 = A_2 .$$

Therefore $s = 2$ and

$$S = \{8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21\} .$$

Since $\mathcal{M} = (\mathcal{P}(G), E(G))$ we get that

$$\mathcal{J}_2 = \mathcal{P}(G \cdot S) .$$

The graph $G_2 = G \cdot S$ is shown in Fig.4-9.

Our next task is to find $\mathcal{J}_2^+$ by applying Algorithm 3. We will give the results of the application of this algorithm in preference to presenting the details which are easy but tedious to carry out. We find that

$$S^+ = \{1, \ldots, 31\}$$

and therefore $\mathcal{J}_2^+ = \mathcal{P}(G \cdot S^+)$. The graph $G_2^+ = G \cdot S^+$ is shown in Fig.4-10. It turns out that $\mathcal{J}_3^+$ is the null matroid and so in summary we have that $\mathcal{J}_1^+ = \mathcal{J}_1 = \mathcal{P}(G)$, $\mathcal{J}_2^+ = \mathcal{P}(G_2)$ and $\mathcal{J}_3^+ = \Omega$. Also we note that

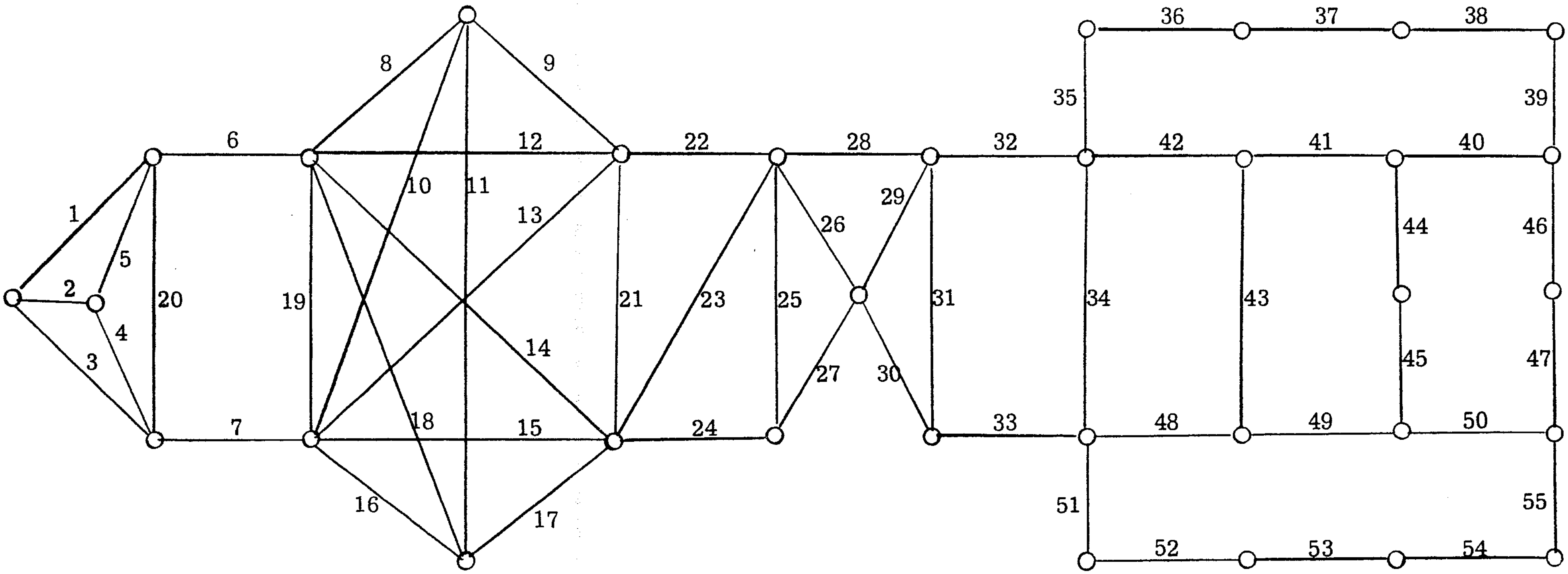


Figure 4-7. Graph G of Example 4-3

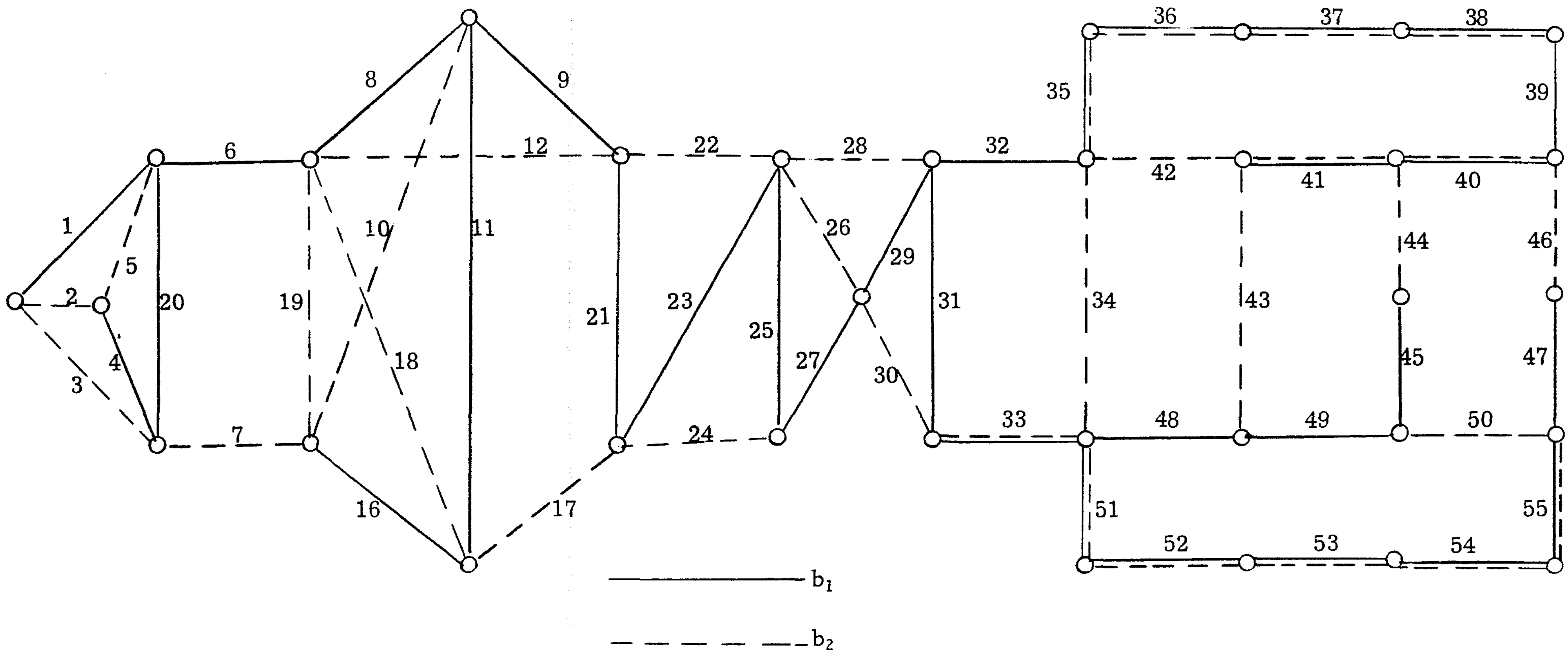


Figure 4-8. Forest Pair of G

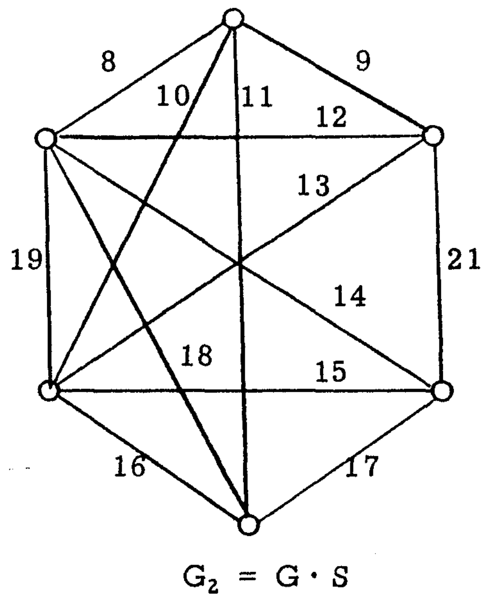
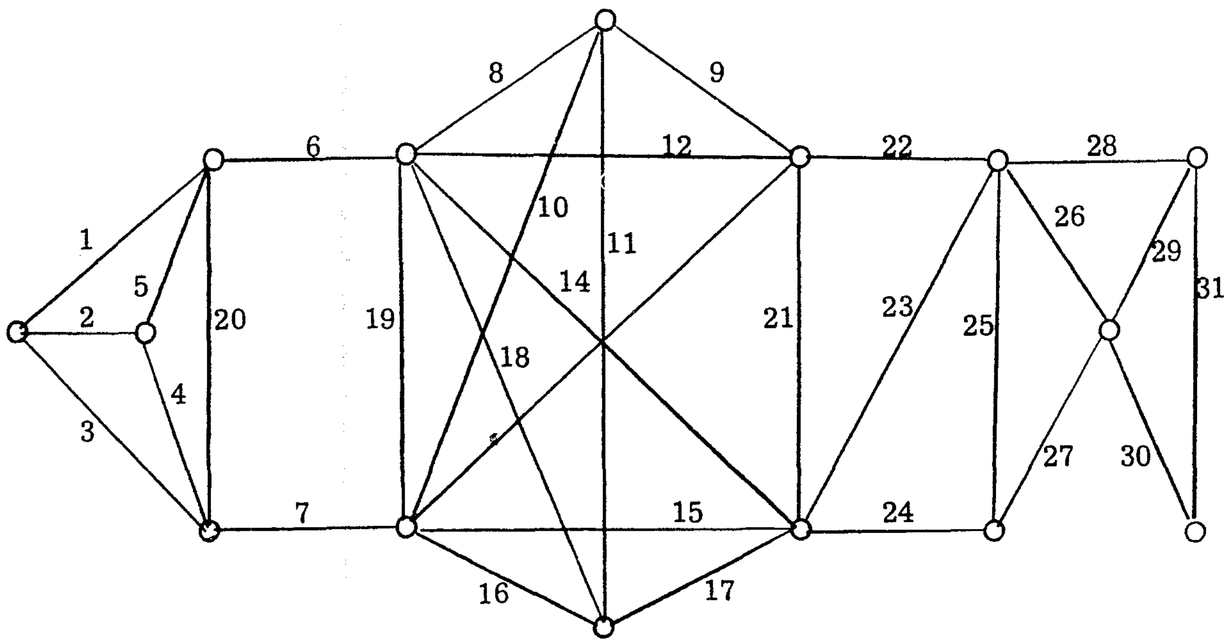


Figure 4-9. Graph G_2 of Example 4-3



$G_2^+ = G \cdot S^+$

Figure 4-10. Graph G_2^+ of Example 4-3

$$\mathcal{M} = \mathcal{J}_1^{+} = \mathcal{J}_1 \supset \mathcal{J}_2^{+} \supset \mathcal{J}_2 \supset \mathcal{J}_3^{+} = \Omega \quad .$$

Since $\mathcal{J}_3^{+} = \Omega$, we have determined all of the non-null principal minors of the first kind for $\mathcal{P}(G)$. To find the principal minors of the second kind we must consider the matroid $\mathcal{P}(G)^{*} = \mathcal{B}(G)$ (this follows from (4.21) and (4.2-2).

Let $\underline{b} = (b_1, b_2)$ be a 2-maximally distant ordered pair with respect to the matroid $\mathcal{B}(G)$. The coordinates of $\underline{b}$ are shown in Fig.4-11. Let us apply Algorithm 3 with respect to $\mathcal{B}(G)$ and $\underline{b}$. The first task is to apply Algorithm 2. Initializing Algorithm 2 we get

$$D = \{33,35, \ldots, 38,40,41,51, \ldots, 55\} \quad .$$

We find that

$$A_1 = \{33,35, \ldots, 38,40,41,51, \ldots, 55,32,34,43,44,46,39,42,$$
$$49,45,47,50\}$$

$$A_2 = A_1 \cup \{48\}$$

$$A_3 = A_2 \quad .$$

Therefore $s = 2$ and

$$S = \{32, \ldots, 55\} \quad .$$

Therefore by (4.2-1) $\mathcal{K}_2^{*} = \mathcal{B}(G) \times S$.

$$\mathcal{K}_2^{*} = \mathcal{B}(G) \times S = \mathcal{B}(G \times S) \quad . \qquad \text{(by 2.1-9)}$$

$$\mathcal{K}_2 = \mathcal{B}(G \times S)^{*} = \mathcal{P}(G \times S) \quad . \qquad \text{(by 2.4-8)}$$

The graph $H_2 = G \times S$ is shown in Fig.4-12.

At this point we initialize Algorithm 3 and find

$$B = \{13, 14, 15\} \ ,$$

$$F = \{1, \ldots, 12, 16, \ldots, 31\}$$

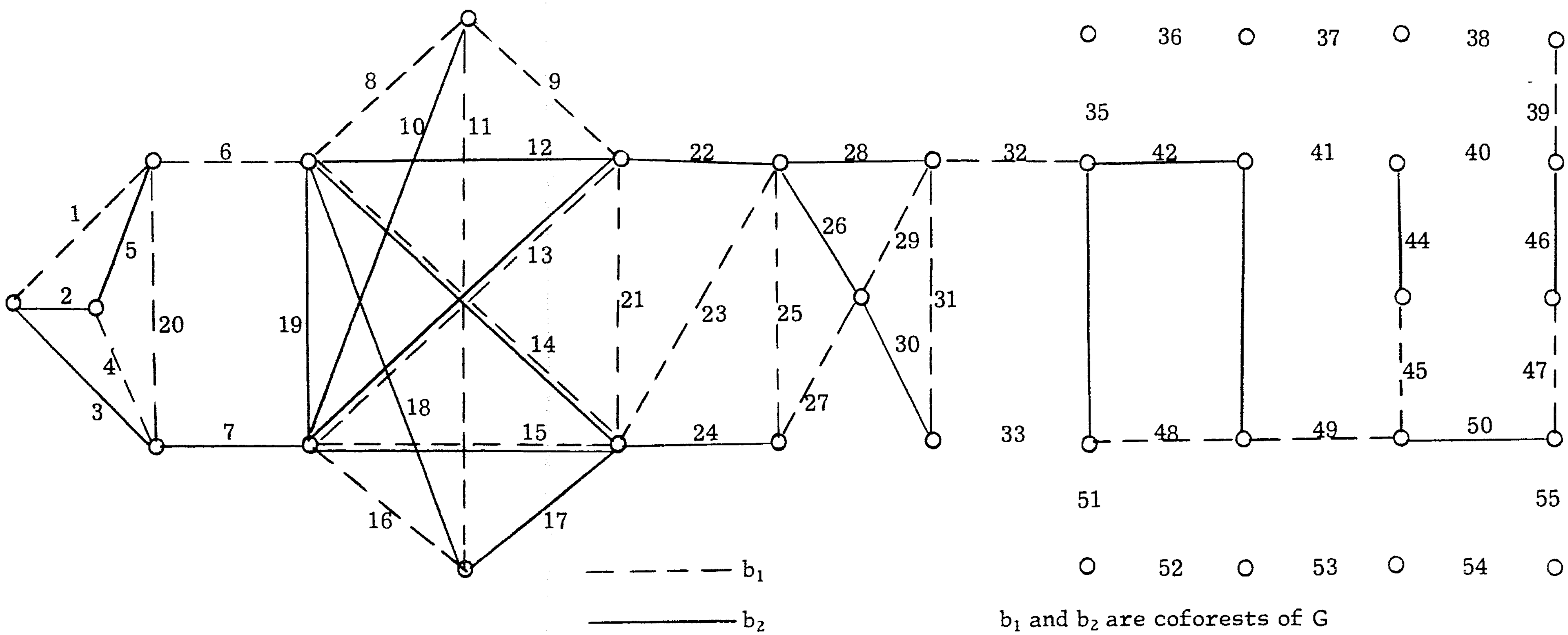
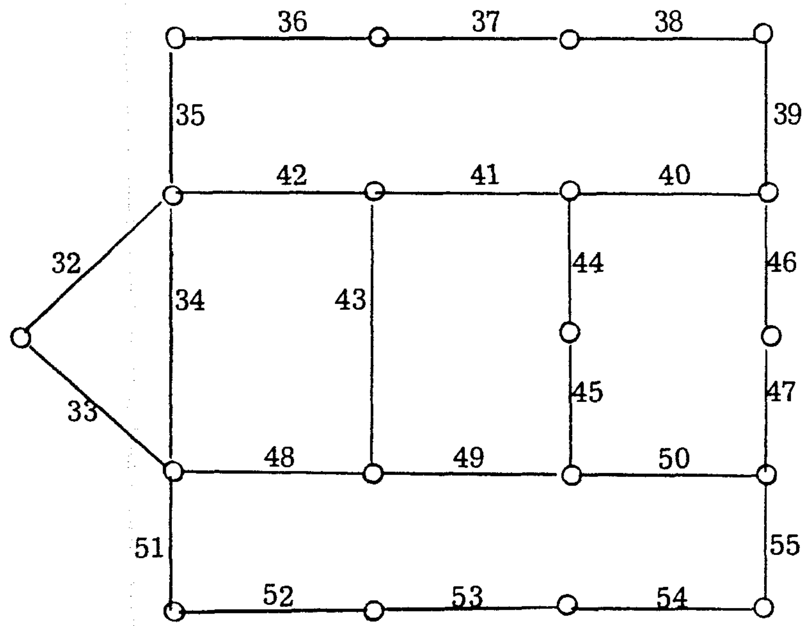


Figure 4-11. Coforest Pair of G



H_2

Figure 4-12. Graph H_2 of Example 4-3

and

$$S^{+} = \{32, \ldots, 55\}$$

<u>Step 1</u> Since $F \neq \emptyset$, we go to Step 2.

<u>Step 2</u> Pick $e = 1 \in F$. The results of Step 2 are

$$F = \{8, \ldots, 12, 16, \ldots, 19, 21, \ldots, 31\}$$

and

$$S^{+} = \{1, \ldots, 7, 20, 32, \ldots, 55\}$$

Return to Step 1.

<u>Step 1</u> Since $F \neq \emptyset$, go to Step 2.

<u>Step 2</u> Pick $e = 8 \in F$.

The final result of applying Algorithm 4-3 is

$$S^{+} = \{1, \ldots, 7, 20, 22, \ldots, 55\} .$$

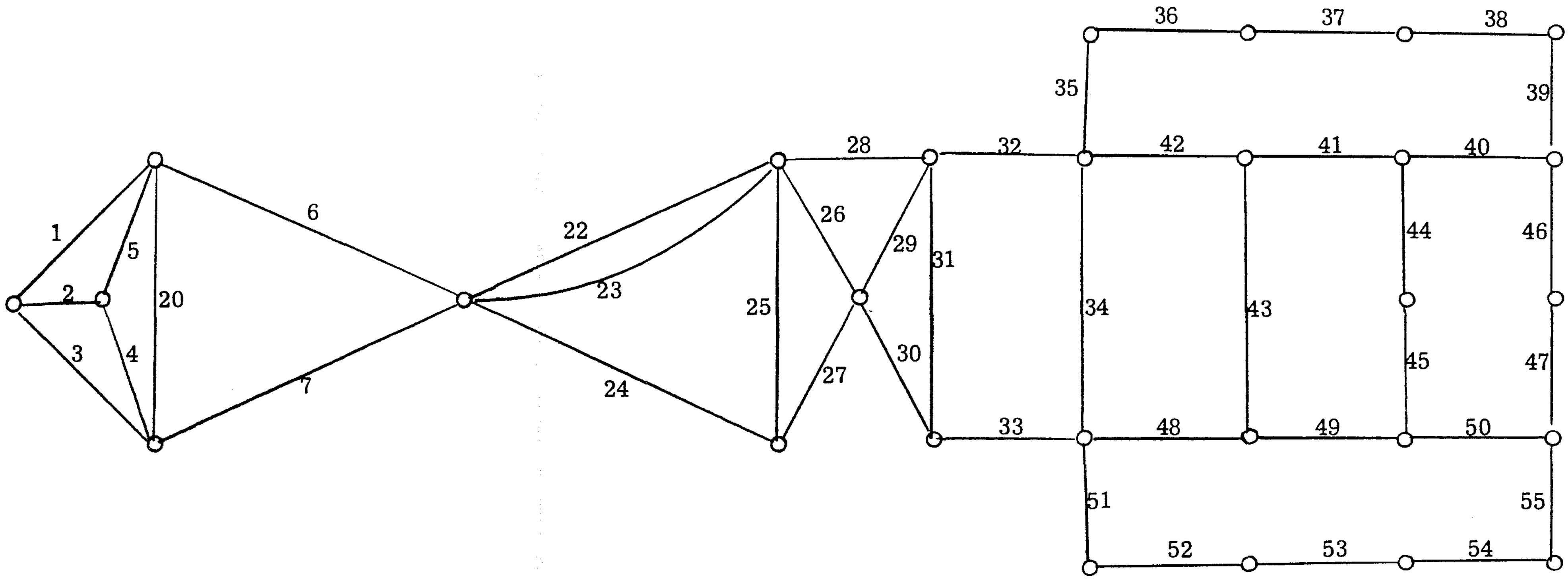
Therefore we have that

$$\mathcal{K}_2^{+} = \mathcal{P}(H_2^{+})$$

where

$$H_2^{+} = G \times S^{+} \qquad \text{(See Fig.4-13)} .$$

Since $\mathcal{K}_3 \subseteq \mathcal{K}_3^{+} \subset \mathcal{K}_2$ (by 4.2-10) and since $(\mathcal{M} \cdot S) \cdot T = \mathcal{M} \cdot T$ if $T \subseteq S \subseteq E$ (by (2.4-13), we can restrict our attention to the graph H_2 in Fig.4-11. To find $\mathcal{K}_3^{+}$ we must locate a 3-maximally distant 3-tuple $\underline{b} = (b_1, b_2, b_3) \in \mathcal{B}^3_{\mathcal{B}(H_2)}$. Consider the choice of $\underline{b}$ (not necessarily 3-maximally distant) in Fig.4-14. Applying Algorithm 1 to $\underline{b}$ yields an updated $\underline{b}$ which is 3-maximally distant. The new coordinates of $\underline{b}$ are shown in Fig.4-15.



$H_2^+ = G \times S^+$

Figure 4-13. Graph H_2^+ of Example 4-3

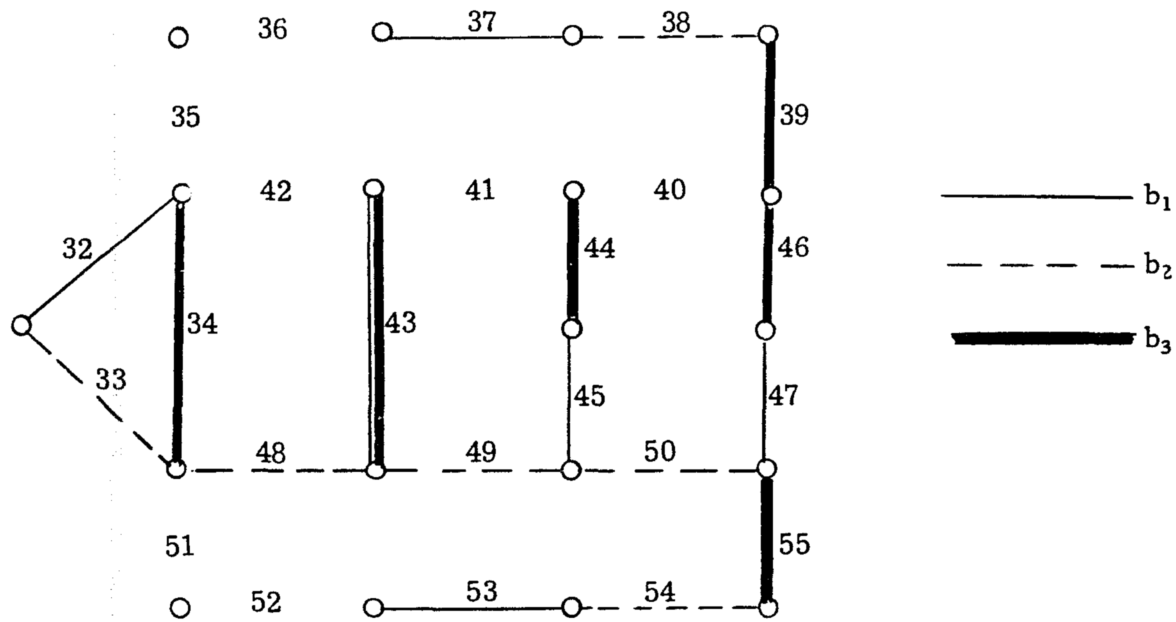


Figure 4-14. Three Coforests of H_2

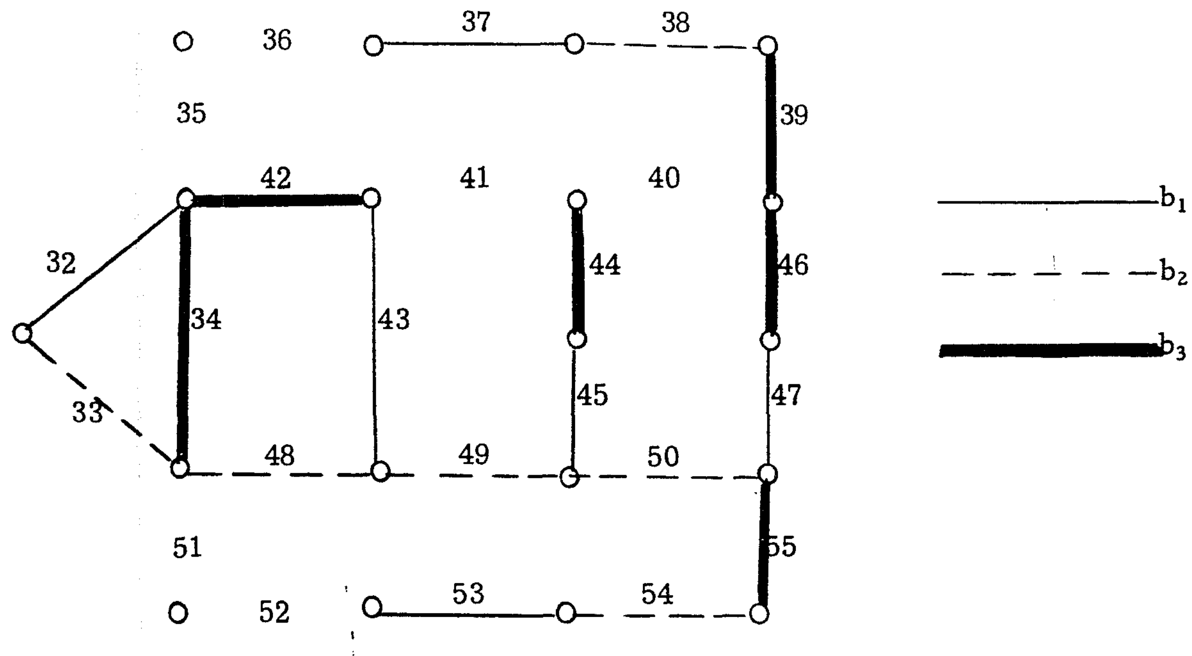


Figure 4-15. Maximally Distant Coforests of H_2

Using b we apply Algorithm 3 with the result that

$$\mathcal{N}_3^{+} = \mathcal{P}(H_2) = \mathcal{N}_2$$

and

$$\mathcal{N}_3 = \mathcal{P}(G \times S) \quad ,$$

where

$$S = \{35, \ldots, 41, 44, \ldots, 47, 49, \ldots, 55\} .$$

Let $H_3 = G \times S$. H_3 is shown in Fig. 4-16.

Finally we get that

$$\mathcal{N}_4^{+} = \mathcal{N}_3$$

and

$$\mathcal{N}_4 = \mathcal{P}(G \times S) \quad ,$$

where

$$S = \{35, \ldots, 39, 51, \ldots, 55\} .$$

$H_4 = G \times S$ and is shown in Fig. 4-17.

It turns out that $\mathcal{N}_5^{+} = \mathcal{N}_4$ and

$$\mathcal{N}_5 = \Omega \text{ , the null matroid.}$$

Thus we see that Algorithm 3 applied to $\mathcal{P}(G)$ and/or $\mathcal{R}(G)$ uniquely decomposes a graph into certain unique reduced contractions and reductions. Another point worth noticing in this example is that after $\mathcal{J}_2^{+}$ and $\mathcal{N}_2^{+}$, the higher order principal minors of the first and second kind do not have any elements in common. In (4.5-2) we show that, in general, the principal minors of the first and second kind are, disjoint after $\mathcal{N}_2^{+}$ and $\mathcal{J}_2^{+}$.

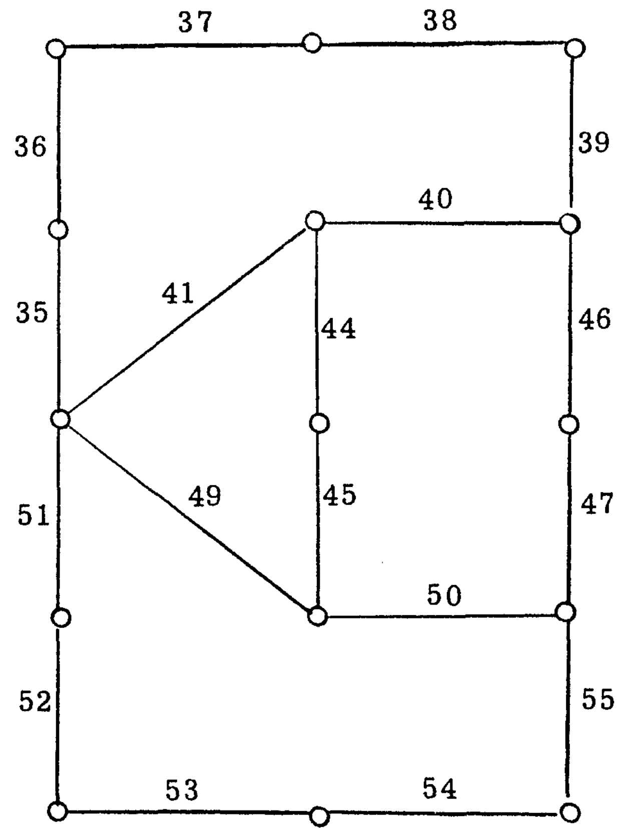


Figure 4-16. Graph H_3 of Example 4-3

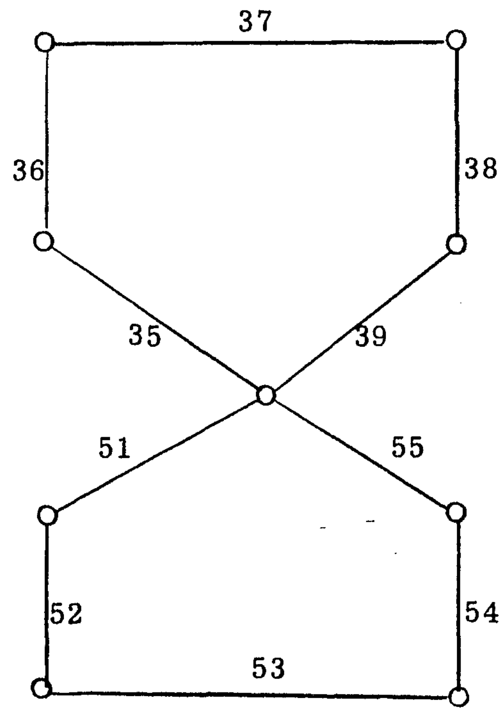


Figure 4-17. Graph H_4 of Example 4-3

4.5 THE PRINCIPAL PARTITION OF A MATROID

In this section we present an extension to matroids of a graph-theoretic result of Kishi and Kajitani wherein they introduced the notion of the principal partition of a graph.

(4.5-1) Let $\mathcal{M} = (\mathcal{C}, E)$ and suppose $\underline{b} = (b_1, b_2) \in \mathcal{B}^2_{\mathcal{M}}$ is 2-maximally distant. Then $(\overline{b}_1, \overline{b}_2) \in \mathcal{B}^2_{\mathcal{M}^*}$ is 2-maximally distant in $\mathcal{M}^*$.

Proof: Let c^* be defined with respect to $\mathcal{M}^*$. Using (2.4-3) it is not difficult to show that for every $\underline{b} = (b_1, b_2) \in \mathcal{B}^2_{\mathcal{M}}$,

$$2 \cdot r(\mathcal{M}) - c^*((\overline{b}_1, \overline{b}_2)) + c(\underline{b}) = \alpha(E)$$

or equivalently

$$c^*((\overline{b}_1, \overline{b}_2)) = c(\underline{b}) + [\, 2r(\mathcal{M}) - \alpha(E)\,]$$

Therefore $\underline{c}_2^*$ the minimum of $c^*((b_1, b_2))$ is attained for precisely those $\underline{b} \in \mathcal{B}^2_{\mathcal{M}}$ which satisfy $c(\underline{b}) = \underline{c}_2$. ∎

(4.5-2) Let $\mathcal{M} = (\mathcal{C}, E)$ and $\mathcal{S}_2 = \mathcal{M} \times S$ and $\mathcal{K}_2 = \mathcal{M} \cdot T$ be the 2-principal minors of the first and second kind, respectively. Then $S \cap T = \emptyset$.

Proof: The theorem is trivially true if either S or T is the null set. Therefore we take $S \neq \emptyset$ and $T \neq \emptyset$. Let $\underline{b} = (b_1, b_2) \in \mathcal{B}^2_{\mathcal{M}}$ be 2-maximally distant. It follows from (4.5-1), (4.4-3) and (4.2-1) that

$$[E - (b_1 \cup b_2)] \subseteq S \quad , \tag{1}$$

$$(b_1 \cap b_2) \subseteq T \quad , \tag{2}$$

$$(E - (b_1 \cup b_2)) \cap T = \phi \quad (3)$$

and

$$(b_1 \cap b_2) \cap S = \phi \quad (4)$$

Assume $e \in S \cap T$. Using (1) through (4) we conclude

$$e \in [(b_2 - b_1) \cup (b_1 - b_2)]$$

Since $S \neq \phi$, it follows that $(E - (b_1 \cup b_2)) \neq 0$. By Algorithm 2, there is a sequence $e_o, e_1, \ldots, e_u$ such that $e_o \in [E - (b_1 \cup b_2)]$, $e_u = e$ and e_{i-1} forms a unique circuit in b_1 or b_2 and this circuit contains e_i for $i = 1, \ldots, u$. By (2.4-5), consideration of the construction of $\mathcal{K}_2^*$ in $\mathcal{M}^*$ using Algorithm 2 and the fact that $e \in T$, we conclude that $e_{u-1} \in T$. Moreover, if we apply (2.4-5) repeatedly, we conclude that

$$e_o \in T \quad . \quad (5)$$

However, (5) contradicts (2). The theorem follows. ▮

(4.5-3) Let $\mathcal{M} = (\mathcal{C}, E)$ and $\mathcal{J}_2 = \mathcal{M} \times S$ and $\mathcal{K}_2 = \mathcal{M} \cdot T$ be the 2-PM1 and 2-PM2 of $\mathcal{M}$, respectively. Then $\mathcal{J}_2^+ = \mathcal{M} \times \overline{T}$ and $\mathcal{K}_2^+ = \mathcal{M} \cdot \overline{S}$.

<u>Proof</u>: Let $\Delta = (E - (S \cup T))$. Then $\overline{T} = S \cup \Delta$. Let $\underline{b} = (b_1, b_2) \in \mathcal{B}^2_{\mathcal{M}}$ be 2-maximally distant and set $b_i' = b_i \cap \overline{T}$ for $i = 1, 2$. Clearly, b_i' is an independent set in $\mathcal{M} \times \overline{T}$. Pick any $e \in \overline{T}$ satisfying $e \notin b_i'$. We show by contradiction that $J(b_1, e) \subseteq \overline{T}$.

Assume $J(b_1, e) \cap T \neq \phi$, that is, there exists an $e' \in J(b_1, e) \cap T$. By the construction of T in $\mathcal{M}^*$ using $(\overline{b_1}, \overline{b_2})$ and by (2.4-5), it follows that $e \in T$, contrary to the choice of e. Accordingly, $J(b_1, e) \subseteq \overline{T}$ and it follows that b'_1 is a base of $\mathcal{M} \times \overline{T}$. Similarly, b'_2 is a base of $\mathcal{M} \times \overline{T}$.

From the above considerations it follows that $g_2(T) = g_2(S) = \overline{g}_2$.

It remains to show that $\overline{T}$ is maximal with the property $g_2(T) = \overline{g}_2$.

Assume there exists a set R satisfying $\overline{T} \subset R$ and $g_2(R) = \overline{g}_2$. Using the characterization of $\mathcal{K}_2^*$ given in (4.4-5) we can assume, without loss of generality, that

$$(b_1 \cap b_2) \cap R \neq \phi \ .$$

But then

$$\alpha(R) < \alpha(E-(b_1 \cup b_2)) + \alpha(b_1 \cap R) + \alpha(b_2 \cap R)$$

and accordingly

$$g_2(R) < \alpha\,(E-(b_1 \cup b_2)) = \overline{g}_2 \ .$$

The theorem follows. ∎

<u>Example 4-4</u>. Let $\mathcal{M}$ be the polygon matroid of the graph in Fig.4-7. In Example 4-3 we found that $\mathcal{S}_2 = \mathcal{P}(G_2)$ and $\mathcal{K}_2 = \mathcal{P}(H_2)$ where $G_2 = G \cdot S$ and $H_2 = G \times T$ are shown in Figs. 4-9 and 4-12, respectively:

$$S = \{8, \ldots, 19, 21\}$$

and

$$T = \{32, \ldots, 35\} .$$

Set $\Delta = (E-(S \cup T))$ and consequently

$$\overline{T} = S \cup \Delta = \{1, \ldots, 31\} .$$

According to (4.5-3)

$$\mathcal{S}_2^{+} = \mathcal{P}(G) \times \overline{T}$$

$$= \mathcal{P}(G_2^{+})$$

where $G_2^+ = G \cdot \overline{T}$. G_2^+ is shown in Fig.4-10. Similarly

$$\overline{S} = T \cup \Delta = \{1, \ldots, 7, 20, 22, \ldots, 55\}$$

and according to (4.5-3)

$$\mathcal{N}_2^+ = \mathcal{P}(G) \cdot \overline{S} = \mathcal{P}(H_2^+) \ ,$$

where $H_2^+ = G \times \overline{S}$. H_2^+ is shown in Fig.4-13. H_2^+ and G_2^+ are the same graphs which were obtained by applying Algorithm 3 directly.

We are now in a position to define the principal partition of a matroid. Let $\mathcal{J}_2 = \mathcal{M} \times S$ and $\mathcal{N}_2 = \mathcal{M} \cdot T$ be the 2-PM1 and the 2-PM2, respectively, of $\mathcal{M}$. Let $\Delta = (E - (S \cup T))$ and

$$\mathcal{D}_2 = (\mathcal{M} \times \overline{T}) \cdot \Delta = (\mathcal{M} \cdot \overline{S}) \times \Delta \ .$$

The alternative expressions for $\mathcal{D}_2$ can be shown to be equal by using (2.4-14) or (2.4-15). By (4.5-3) we can write that

$$\mathcal{D}_2 = \mathcal{J}_2^+ \cdot \Delta = \mathcal{N}_2^+ \times \Delta \ ,$$

where $\mathcal{J}_2^+$ and $\mathcal{N}_2^+$ are the 2-APM1 and the 2-APM2, respectively, of $\mathcal{M}$. The unique three-tuple

$$(\mathcal{D}_2, \mathcal{N}_2, \mathcal{J}_2)$$

is called the principal partition of the matroid $\mathcal{M}$. By (4.2-5), (4.2-7) and (4.5-2) the sets S, T and Δ uniquely partition the set E.

> (4.5-4) Let $\mathcal{M} = (\mathcal{C}, E)$ and $(\mathcal{D}_2, \mathcal{N}_2, \mathcal{J}_2)$ be the principal partition of $\mathcal{M}$. Then $(\mathcal{D}_2^*, \mathcal{J}_2^*, \mathcal{N}_2^*)$ is the principal partition of $\mathcal{M}^*$.

Proof: By (4.2-1), $\mathcal{S}_2^*$ is the 2-PM2 and $\mathcal{K}_2^*$ is the 2-PM1 of $\mathcal{M}^*$, respectively.

Let $\mathcal{S}_2 = \mathcal{M} \times S$, $\mathcal{K}_2 = \mathcal{M} \cdot T$ and $\Delta = (E - S \cup T))$. Then $\mathcal{D}_2 = (\mathcal{M} \times T) \cdot \Delta$.

$$\mathcal{D}_2^* = ((\mathcal{M} \times \overline{T}) \cdot \Delta)^*$$

$$= (\mathcal{M} \times \overline{T})^* \times \Delta \qquad \text{(by 2.4-10)},$$

$$= (\mathcal{S}_2^+)^* \times \Delta \quad .$$

However, $(\mathcal{S}_2^+)^* \times \Delta$ is, by definition, the first coordinate of the principal partition of $\mathcal{M}^*$. ∎

(4.5-5) Let $\mathcal{M} = (\mathcal{C}, E)$ and $\mathcal{S}_k^+ = \mathcal{M} \times S^+$ and $\mathcal{S}_k = \mathcal{M} \times S$ be the k-APM1 and the k-PM1 of $\mathcal{M}$, respectively. Let $\Delta = (S^+ - S)$. Then Δ can be partitioned into k sets $\Delta_1, \ldots, \Delta_k$ such that Δ_i is a base of $\mathcal{D}_k$ for i = 1, . . . , k, where $\mathcal{D}_k = \mathcal{S}_k^+ \cdot \Delta$.

Proof: The theorem is trivial if $\Delta = \phi$. Therefore we take $\Delta \neq \phi$. Let $\underline{b} = (b_1, \ldots, b_k) \in \mathcal{B}_{\mathcal{M}}^k$ be k-maximally distant. Let

$$\Delta_i = \Delta \cap b_i$$

for i = 1, . . . , k. Clearly the Δ_i's partition Δ.

We show that the Δ_i's are independent sets of $\mathcal{D}_k$ by contradiction.

Assume, without loss of generality, that Δ_1 is not an independent set in $\mathcal{D}_k$. Therefore, there is a circuit $C \subseteq S^+$ which satisfies

$$C \cap \Delta \subseteq \Delta_1$$

and

$$C \cap \Delta \neq \phi \ .$$

Clearly C must contain members of S^+ which are also members of either

$$\bigcup_{i=2}^{k} b_i \cap S^+$$

or

$$(E - (\bigcup_{i=1}^{k} b_i)) .$$

Set

$$R = C \cap [(\bigcup_{i=2}^{k} (b_i \cap S^+)) \cup (E - (\bigcup_{i=1}^{k} b_i))] .$$

Each of the members of R forms a unique circuit in $b_1 \cap S$ since $b_1 \cap S$ is a base of $\mathcal{M} \times S$. Enumerate the elements of E such that

$$R = \{ e_1, \ldots, e_p \}$$

and let C_i be the unique circuit formed by e_i in $b_1 \cap S$ for $i = 1, \ldots, p$. We define the circuit $C^{(k)}$ recursively as

$$C^{(o)} = C$$

and $C^{(k)}$ is a circuit which satisfies

$$e \in C^{(k)} \subseteq ((C_k \cup C^{(k-1)}) - \{e_k\}) ,$$

where e is a specific element in $C \cap \Delta$. It is not difficult to verify that the above scheme determines a sequence (not necessarily unique) of circuits $C^{(k)}$ for $i = 1, \ldots, p$ and that

$$C^{(p)} \subseteq b_1 \cap S^+ .$$

This is a contradiction, however, since $b_1 \cap S^+$ is an independent set in $\mathcal{S}_k^+$. Therefore the Δ_i's are maximal independent sets in $\mathcal{S}_k$. We apply (2.4-19) to $\mathcal{M} \times S^+$ and find that

$$r((\mathcal{M} \times S^+) \times S) + r((\mathcal{M} \times S^+) \cdot \Delta) = r(\mathcal{M} \times S^+) .$$

Therefore

$$r(\mathcal{B}_k) = r(\mathcal{M} \times S^+) - r(\mathcal{M} \times S)$$

$$= \alpha(b_i \cap S^+) - \alpha(b_i \cap S)$$

$$= \alpha(\Delta_i)$$

for $i = 1, \ldots, k$.

Accordingly, Δ_i is a base of $\mathcal{B}_2$ for $i = 1, \ldots, k$. ▮

Example 4-5: Let G be the graph in Fig.4-18 and $\underline{b} = (b_1, b_2)$ 2-maximally distant in $\mathcal{P}(G)$.

It turns out that

$$\mathcal{S}_2^+ = \mathcal{P}(G) \quad ,$$

$$\mathcal{S}_2 = \mathcal{P}(G_2)$$

and

$$\mathcal{X}_2 = \Omega \quad ,$$

where G_2 is shown in Fig.4-19.

Calculating Δ we find that

$$\Delta = \{8,9\}.$$

As in the proof of (4.5-4), set

$$\Delta_1 = \Delta \cap b_1 = \{9\}$$

and

$$\Delta_2 = \Delta \cap b_2 = \{8\} .$$

$\mathcal{B}_2 = \mathcal{P}(G) \cdot \Delta = \mathcal{P}(G \times \Delta)$, where $G \times \Delta$ is shown in Fig.4-20. Clearly Δ_1 and Δ_2 are element disjoint bases of $\mathcal{P}(G \times \Delta)$.

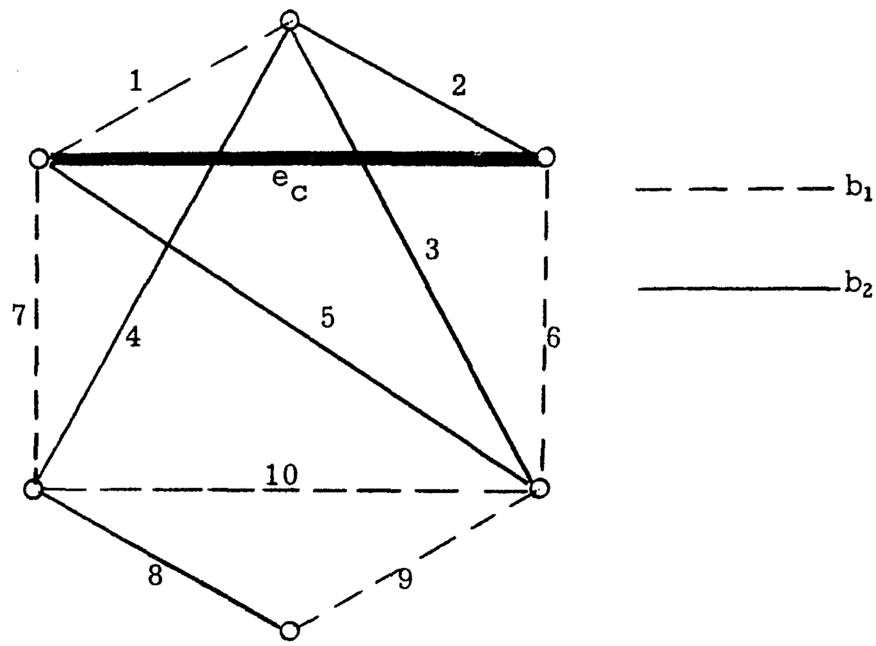


Figure 4-18. Graph G of Example 4-5

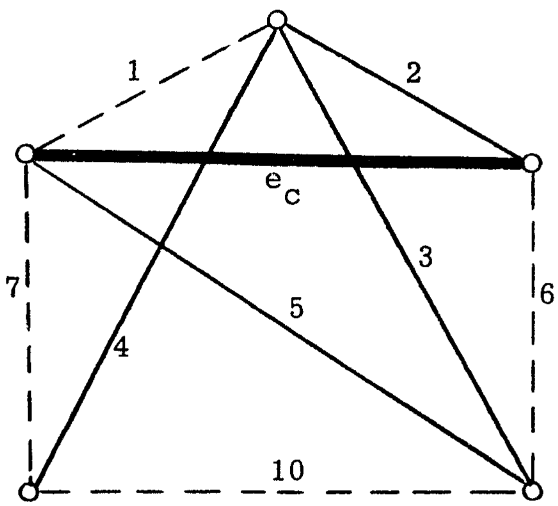


Figure 4-19. Graph G_2 of Example 4-5

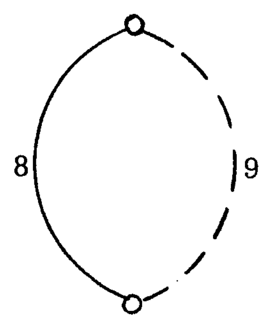


Figure 4-20. Graph G ✗ Δ of Example 4-5

<u>Example 4-6:</u> Let G be the graph of Fig.4-7 and $\mathcal{M}$ the <u>bond</u> matroid of G. From Figs. 4-13 and 4-16 we get

$$G_3^+ = \{32, \ldots, 55\}$$

and

$$G_3 = \{35, \ldots, 41, 44, \ldots, 47, 49, \ldots, 55\}.$$

(Note that we are working with the dual case of Example 4-3). Calculating $\boldsymbol{\Delta}$ we get

$$\boldsymbol{\Delta} = \{32, 33, 34, 42, 43, 48\}$$

and therefore

$$\mathcal{B}_3 = (\mathcal{B}(G) \times G_3^+) \cdot \boldsymbol{\Delta}$$
$$= \mathcal{B}(D_3) \quad ,$$

where $D_3 = (G \times G_3^+) \cdot \boldsymbol{\Delta}$. D_3 is shown in Fig.4-21. The partition of $\boldsymbol{\Delta}$ into $\boldsymbol{\Delta}_1$, $\boldsymbol{\Delta}_2$ and $\boldsymbol{\Delta}_3$ is indicated in Fig.4-21. The $\boldsymbol{\Delta}_i$'s correspond to coforests of D_3.

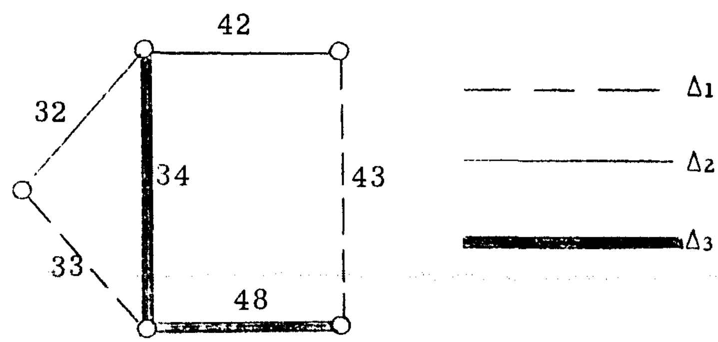


Figure 4-21. Graph D_3 of Example 4-6

4.6 APPLICATIONS OF THE PRINCIPAL MINORS OF A MATROID

In this section we cover three topics:

(i) the Shannon switching game

(ii) Edmonds' cospanning sets theorem and

(iii) the concept of hybrid rank and hybrid dimension.

(i) The Shannon Switching Game

The Shannon switching game [Le 3] is played by two players, one called "cut" and the other called "short". The game is played on a graph with respect to two distinguished vertices. The players play alternately, each one tagging edges of the graph. It is the objective of the short player to tag a set of edges which forms a path between the two distinguished vertices (i.e. to tage a set of edges which shorts the two distinguished vertices). The objective of the cut player is to tag a set of edges which contains at least one edge from every path between the distinguished vertices. Each edge of the graph can be tagged at most once.

Example 4-7 Consider the graph in Fig.4-22 in which x and y are the distinguished vertices and suppose the short player goes first. If s_i is the tag of the short player on the i^{th} play and c_j of the tag of the cut player on the j^{th} play, then the labeled graph in Fig.4-22 represents a game in which the cut player has won with the short player going first.

We can distinguish between three kinds of games:

(i) If the short player, going second, can win against every strategy of the cut player, then the game is called a short game.

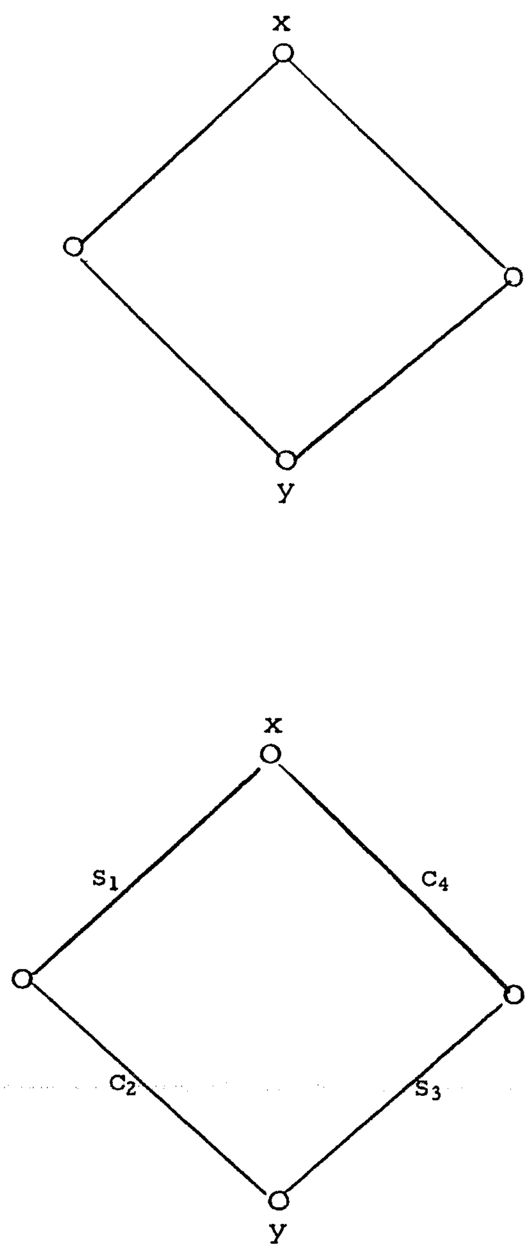


Figure 4-22. Graphs of Example 4-7

(ii) If the cut player, going second, can win against every strategy of the short player, then the game is called a cut game.

(iii) If the player going first (but not second) can win against every strategy of the other player, then the game is called a neutral game.

The classification into short, cut and neutral games logically includes all possible games. Moreover, a switching game can be in only one category. Therefore given a graph G and two distinguished vertices v_1 and v_2, the switching game played on G with respect to v_1 and v_2 is uniquely characterized as a short, cut or neutral game.

Lehman has given a winning strategy for the two-person (Shannon) switching game; his main results are given in a matroid-theoretic context. Following Lehman we will define the two-person switching game on a matroid. To do this, however, we need some definitions.

Let $\mathcal{M} = (\mathcal{C}, E)$ and T, $S \subseteq E$. We say that S spans T with respect to $\mathcal{M}$ if

$$T \subseteq \mathcal{S}(S)_{\mathcal{M}} ,$$

where

$$\mathcal{S}(S)_{\mathcal{M}} = \{e \mid e \in S \text{ or there exists a circuit } C \text{ of } \mathcal{M} \text{ such that } \{e\} = C \cap \overline{S}\} .$$

The set $\mathcal{S}(S)_{\mathcal{M}}$ is called the span of S with respect to $\mathcal{M}$. The sets S and T are called cospanning sets with respect to $\mathcal{M}$ if

$$\mathcal{S}(S)_{\mathcal{M}} = \mathcal{S}(T)_{\mathcal{M}} .$$

Let e be a distinguished element in E. The switching game on $\mathcal{M}$

with respect to e is played by two players, "short" and "cut". The players play alternately tagging the members of $(E - \{e\})$ and it is the objective of the short player to tag a set of edges in $(E - \{e\})$ which spans $\{e\}$ with respect to $\mathfrak{M}$. The cut player attempts to tag a subset of $(E - \{e\})$ which makes the objective of the short player unattainable. The elements in $(E - \{e\})$ can be tagged at most once. The players have complete information.

We call a switching game on $\mathfrak{M}$ with respect to e a

(i) short game if the short player, playing second, can win against every strategy of the cut player.

(ii) cut game if the cut player, playing second, can win against every strategy of the short player.

(iii) neutral game if the player going first (but not second) can win against every strategy of the other player.

A switching game on $\mathfrak{M}$ with respect to $e \in E$ is denoted by $\mathfrak{J} = (\mathfrak{M}, e)$. The class of all games can be partitioned into three classes depending on whether the game is a short, cut or neutral game. The classification of any game will be shown to be intimately related to the principal partition of a matroid.

Lehman has proved the following:

> (4.6-1) (Lehman) Let $\mathfrak{J} = (\mathfrak{M}, e)$ be a two-person switching game. Then exactly one of the following statements holds:

(i) $\mathfrak{M}$ contains two disjoint cospanning independent sets spanning but not containing e: $\mathfrak{J}$ is a short game.

(ii) $\mathcal{M}^*$ contains two disjoint cospanning independent sets spanning but not containing e: $\mathfrak{J}$ is a cut game.

(iii) $\mathcal{M}$ and $\mathcal{M}^*$ each contain two disjoint cospanning independent sets, in each case e being a member of one independent set: $\mathfrak{J}$ is a neutral game.

We need a preliminary result concerning the span operator.

(4.6-2) Let $\mathcal{M} = (\mathcal{C}, E)$ and $N \subseteq E$. If $A_1, A_2 \subseteq N$ and $\mathcal{S}(A_1)_{\mathcal{M}\times N} = \mathcal{S}(A_2)_{\mathcal{M}\times N}$, then $\mathcal{S}(A_1)_{\mathcal{M}} = \mathcal{S}(A_2)_{\mathcal{M}}$.

Proof: Clearly, $\mathcal{S}(A_i)_{\mathcal{M}\times N} \subseteq \mathcal{S}(A_i)_{\mathcal{M}}$ for i = 1,2. We first show that $\mathcal{S}(A_1)_{\mathcal{M}} \subseteq \mathcal{S}(A_2)_{\mathcal{M}}$.

Case 1: Suppose $\mathcal{S}(A_1)_{\mathcal{M}\times N} = \mathcal{S}(A_1)_{\mathcal{M}}$. Since $\mathcal{S}(A_1)_{\mathcal{M}\times N} = \mathcal{S}(A_2)_{\mathcal{M}\times N} \subseteq \mathcal{S}(A_2)_{\mathcal{M}}$, it follows that $\mathcal{S}(A_1)_{\mathcal{M}} \subseteq \mathcal{S}(A_2)_{\mathcal{M}}$.

Case 2: Suppose next that $\mathcal{S}(A_1)_{\mathcal{M}\times N} \subset \mathcal{S}(A_1)_{\mathcal{M}}$. Pick some e in $(\mathcal{S}(A_1)_{\mathcal{M}} - \mathcal{S}(A_1)_{\mathcal{M}\times N})$.

There exists a $C \subseteq \mathcal{C}$ satisfying:

(i) $e \in C$

and

(ii) $(C - \{e\}) \subseteq A_1 \subseteq \mathcal{S}(A_2)_{\mathcal{M}\times N}$.

(Furthermore, $e \in E-N$.

If $(C - \{e\}) \subseteq A_2$, then $e \in \mathcal{S}(A_2)_{\mathcal{M}}$. Accordingly, suppose

$$\Delta = (C - \{e\}) \cap (\mathcal{S}(A_2)_{\mathcal{M}\times N} - A_2) \neq \phi .$$

Enumerate the elements of E such that

$$\Delta = \{e_1, \ldots, e_p\} .$$

There is a circuit $C_i \in \mathcal{C}$ satisfying $e_i \in C_i$ and $(C_i - \{e\}) \subseteq A_2$ for $i = 1, \ldots, p$. We define a sequence of circuits $C^{(k)}$ by

$$C^{(o)} = C \quad (1)$$

and

$$e \in C^{(k)} \subseteq (C^{(k-1)} \cup C_k - \{e_k\}) \quad (2)$$

for $k = 1, \ldots, p$. One can easily verify that there is a sequence

$$C^{(o)}, C^{(1)}, \ldots, C^{(p)}$$

of circuits (not necessarily unique) of $\mathcal{M}$ which satisfies (1) and (2). Clearly, $(C^{(p)} - \{e\}) \subseteq A_2$. Therefore $e \in \mathscr{S}(A_2)_{\mathcal{M}}$.

Since e is arbitrary it follows that

$$(\mathscr{S}(A_1)_{\mathcal{M}} - \mathscr{S}(A_1)_{\mathcal{M} \times N}) \subseteq \mathscr{S}(A_2)_{\mathcal{M}} .$$

However, since $\mathscr{S}(A_1)_{\mathcal{M} \times N} = \mathscr{S}(A_2)_{\mathcal{M} \times N} \subseteq \mathscr{S}(A_2)_{\mathcal{M}}$ it follows that $\mathscr{S}(A_1)_{\mathcal{M}} \subseteq \mathscr{S}(A_2)_{\mathcal{M}}$.

The above argument can be repeated with A_1 and A_2 interchanged. Consequently, $\mathscr{S}(A_1)_{\mathcal{M}} = \mathscr{S}(A_2)_{\mathcal{M}}$. ▮

In the following theorem we show how the principal partition of a matroid and the classifications of a two-person switching game are related.

For convenience we introduce the following notation. If $\mathfrak{R} = (\mathcal{M} \cdot S) \times T$ is a minor of $\mathcal{M} = (\mathcal{C}, E)$, where $T \subseteq S \subseteq E$, then we write $e \in \mathfrak{R}$ if and only if $e \in T$.

(4.6-3) Let $\mathfrak{T} = (\mathcal{M}, e)$ be a two-person switching game and $(\mathcal{D}_2, \mathcal{N}_2, \mathcal{S}_2)$ the principal partition of $\mathcal{M}$. Then $\mathfrak{T}$ is a short (cut) [neutral] game if and only if $e \in \mathcal{S}_2$ ($\in$ [illegible]) [$e \in \mathcal{D}_2$].

Proof: Suppose $e \in \mathcal{S}_2 = \mathcal{M} \times S$. By (4.4-3), (4.4-5) and the definition of a 2-minor of $\mathcal{M}$, there exists a 2-maximally distant $\underline{b} = (b_1, b_2)$ such that

(i) $(b_1 \cap S)$ and $(b_2 \cap S)$ are disjoint bases of $\mathcal{S}_2$ and

(ii) $e \in (S - (b_1 \cup b_2))$.

Accordingly, $b_1 \cap S$ and $b_2 \cap S$ are disjoint cospanning independent sets with respect to $\mathcal{S}_2$. By (4.6-2), $b_1 \cap S$ and $b_2 \cap S$ are disjoint cospanning independent sets with respect to $\mathcal{M}$. Moreover, since $b_1 \cap S$ and $b_2 \cap S$ are bases of $\mathcal{S}_2$, they span $\{e\}$ with respect to $\mathcal{S}_2$ and consequently they span $\{e\}$ with respect to $\mathcal{M}$. Therefore by (4.6-1), (i), $\mathfrak{T}$ is a short game.

Next suppose $e \in \mathcal{N}_2 = \mathcal{M} \cdot T$. In view of (4.2-1), the above reasoning applied to $\mathcal{M}^*$ leads to the conclusion that $\mathfrak{T}$ is a cut game.

Finally, suppose $e \in \mathcal{D}_2 = \mathcal{S}_2^+ \cdot \Delta = \mathcal{N}_2^+ \times \Delta$, where $\Delta = (E - (S \cup T))$ and $\mathcal{S}_2^+$ and $\mathcal{N}_2^+$ are the 2-APM1 and the 2-APM2, respectively, of $\mathcal{M}$. Accordingly, $e \in \mathcal{S}_2^+$ and $e \in \mathcal{N}_2^+$. By (4.5-3), (4.4-7) and the definition of a 2-augmented minor of $\mathcal{M}$, if $\underline{b} = (b_1, b_2) \in \mathcal{B}^2_{\mathcal{M}}$ is 2-maximally distant, then

(i) $b_1 \cap \overline{T}$ and $b_2 \cap \overline{T}$ are disjoint bases of $\mathcal{S}_2^+$ and

(ii) e is a member of either $b_1 \cap \overline{T}$ or $b_2 \cap \overline{T}$.

By (4.6-2), $b_1 \cap \overline{T}$ and $b_2 \cap \overline{T}$ are disjoint cospanning independent sets with respect to $\mathcal{M}$.

A similar argument with respect to $(\mathcal{K}_2^+)^*$ in $\mathcal{M}^*$ shows that $\overline{b_1} \cap \overline{S}$ and $\overline{b_2} \cap \overline{S}$ are disjoint cospanning independent sets with respect to $\mathcal{M}^*$ and e is a member of either $\overline{b_1} \cap \overline{S}$ or $\overline{b_2} \cap \overline{S}$.

Clearly, the conditions of (4.6-1), (iii) are fulfilled and therefore $\mathfrak{F}$ is a neutral game.

The necessity part of the theorem follows since the sets S, T and Δ are unique and partition the set E. ∎

Algorithms 1 and 2 enable one to construct the principal partition of $\mathcal{M}$ as well as the disjoint cospanning independent sets referred to in (4.6-1). The strategy one uses is based on these two disjoint cospanning independent sets.

1. <u>Strategy for short games</u>. If $\mathfrak{F} = (\mathcal{M}, \mathcal{C})$ is a short game, then by (4.6-3) $e \in \mathcal{S}_2 = \mathcal{M} \times S$, the 2-PM1 of $\mathcal{M}$. By Theorems (4.4-3) and (4.4-5) there exists a 2-maximally distant $\underline{b} = (b_1, b_2) \in \mathfrak{B}^2_{\mathcal{M}}$ satisfying (i) $b_1 \cap S$ and $b_2 \cap S$ are element disjoint bases of $\mathcal{M} \times S$, (ii) $e \in (S - b_1 \cup b_2)$. Set

$$M = b_1 \cap S \ ,$$

$$N = b_2 \cap S$$

and

$$\mathcal{M}' = \mathcal{M} \times (M \cup N \cup \{e\}) \ .$$

The objective of the short player is to win the game in $\mathcal{M}'$ since by (4.6-2) this is equivalent to winning the game in $\mathcal{M}$. Since we intend to exhibit a strategy for the short player which only uses elements in M ∪ N, it is not to the cut player's advantage to tag elements outside of M ∪ N.

Suppose the cut player, playing first, tags $e_{i_1} \in M$. Since M is a base of $\mathcal{M}'$, $N \cup \{e\}$ is a base of $(\mathcal{M}')^*$ and consequently e_{i_1} forms a unique circuit C_1 of $(\mathcal{M}')^*$ in $N \cup \{e\}$.

We show by contradiction that $C_1 \cap N \neq \emptyset$. Assume $C_1 \cap N = \emptyset$. Then $C \subseteq M \cup \{e\}$, but since N is a base of $\mathcal{M}$, $M \cup \{e\}$ is a base of $(\mathcal{M}')^*$. Thus we have a contradiction and accordingly $C \cap N \neq \emptyset$. Pick some $e_{i_2} \in C \cap N$. The short player tags e_{i_2}. Then set

$$M \leftarrow (e_{i_2}/e_{i_1})M \quad ,$$

$$N \leftarrow N$$

and

$$\mathcal{M}' = \mathcal{M} \times (M \cup N \cup \{e\}) \quad .$$

Using (2.4-5) it follows that M is a base of $\mathcal{M}'$. The set $M \cap N = \{e_{i_2}\}$ is the set of elements tagged by the short player. Now suppose the cut player plays $e_{i_3} \in (N - (M \cap N))$; since N is a base of $\mathcal{M}'$, e_{i_3} forms a unique circuit C_2 of $(\mathcal{M}')^*$ in $(M - (M \cap N)) \cup \{e\}$.

We again show by contradiction that $C_2 \cap (M - (M \cap N)) \neq \emptyset$. Assume $C_2 \cap (M - (M \cap N)) = \emptyset$. Then $C_2 \subseteq (N - (M \cap N)) \cup \{e\}$ is a base of $(\mathcal{M}')^*$. This is a contradiction and accordingly $C_2 \cap (M - (M \cap N)) \neq \emptyset$. Pick any $e_{i_4} \in C_2 \cap (M - (M \cap N))$. The short player tags e_{i_4}. Then set

$$M \leftarrow M$$

$$N \leftarrow (e_{i_4}/e_{i_3})N$$

and

$$\mathcal{M}' \leftarrow \mathcal{M} \times (M \cup N \cup \{e\})$$

Again by (2.4-5), N is a base of $\mathcal{M}'$. Also $M \cap N = \{e_{i_2}, e_{i_4}\}$ is equal to the set of elements tagged by the short player.

This strategy, if repeated $\alpha(M) - 2$ more times, yields a victory for the short player. We summarize the short player's strategy as follows.

<u>Strategy: Short Game</u>

Find a 2-maximally distant $\underline{b} = (b_1, b_2) \in \mathcal{B}^2_{\mathcal{M}}$ satisfying

(i) $b_1 \cap S$ and $b_2 \cap S$ are element-disjoint bases of $\mathcal{S}_2 = \mathcal{M} \times S$, the 2-PM1 of $\mathcal{M}$.

(ii) $e \in (S - b_1 \cup b_2)$

Set $M = b_1 \cap S$, $N = b_2 \cap S$ and $\mathcal{M}' = \mathcal{M} \times (N \cup M \cup \{e\})$

The recursive part of the strategy is as follows.

If on the k^{th} play, where k is odd, the cut player tags $e_{i_k} \in (M - (M \cap N))$ $(e_{i_k} \in (N - (M \cap N)))$, then e_{i_k} forms a unique circuit C of $(\mathcal{M}')$ in $(N - (M \cap N)) \cup \{e\}((M-(M \cap N)) \cup \{e\})$. Pick some $e_{i_{k+1}} \in C \cap (N - (M \cap N))$ $(e_{i_{k+1}} \in C \cap (M - (M \cap N)))$. The short player tags $e_{i_{k+1}}$. Then set

$$M \leftarrow (e_{i_{k+1}}/e_{i_k})M \ (M) \quad ,$$

and
$$N \leftarrow N\,((e_{i_{k+1}}/e_{i_k})N)$$

2. <u>Strategy for the cut games</u>. Let $\mathcal{J} = (\mathcal{M}, \mathcal{C})$ be a cut game. Then by (4.6-3) $e \in \mathcal{K}_2$, the 2-PM2 of $\mathcal{M}$. By (4.5-4) however, it is easy to see that $(\mathcal{M}^*, e)$ is a short game. Consequently, a winning strategy is available to the cut player playing in $\mathcal{M}^*$ with respect to e. To see this, suppose the cut player plays the previous strategy of the short player in $\mathcal{M}^*$ with respect to e. Then by (4.6-3) the cut player will tag a set of elements T such that $T \cup \{e\}$ is a circuit of $\mathcal{M}^*$. Assume that the short player simultaneously tagged a set R such that $R \cup \{e\}$ is a circuit of $\mathcal{M}$. This is impossible since $R \cap T = \phi$ and thus $\alpha((R \cup \{e\}) \cap (T \cup \{e\})) = 1$.

Consequently it is impossible for the short player to have tagged a set of elements which span e in $\mathcal{M}$. Therefore the strategy for the cut player in a cut game is the same strategy that the short player uses in a short game, except that the cut player plays with respect to the dual matroid $\mathcal{M}^*$.

3. Strategy for the neutral games. Let $(\mathcal{M}, \mathcal{C})$ be a neutral game. Then the short(cut) player playing first uses the same strategy as is used for the short(cut) game with the condition that the other player has tagged the edge e on a fictitious first play. If the short(cut) player goes first the matroid $\mathcal{S}_2^+$, the 2-APM1 ($\mathcal{N}_2^+$, the 2-APM2) of $\mathcal{M}$ is used to play the neutral game.

Although Lehman obtained these matroid-theoretic strategies, he failed to see how they specialized to graphs to give a complete graph-theoretic solution to the two-person switching game. In what follows we state a complete graph-theoretic solution to the two-person switching game. These results constitute an instance of the graph-theoretic duality presented in Section 2.5.

(4.6-3A) Let $\mathfrak{G} = (G, e)$ be a switching game and (D_2, H_2, G_2) the principal partition of G. Then $\mathfrak{G}$ is a short(cut) [neutral] game if and only if $e \in E(G_2)$ $(e \in E(H_2))$ $[e \in E(D_2)]$.

STRATEGY: $\mathfrak{J} = (G, e)$ IS A SHORT GAME

Determine $G_2 = G \cdot S$, the principal reduction of G, and a 2-maximally distant forest pair $\underline{f} = (f_1, f_2)$ of G which satisfies $e \notin f_1 \cup f_2$. Set

$$M = f_1 \cap S \ ,$$

$$N = f_2 \cap S$$

and
$$G' = G \cdot (M \cup N \cup \{e\}) \ .$$

The basic strategy of the short player is given recursively as follows. On the k^{th} play, where k is odd, the cut player tags edge $e_{i_k} \in [(M \cup N) - (M \cap N)]$. If $e_{i_k} \in M(N)$, then e_{i_k} forms a unique bond in $(N-(M \cap N)) \cup \{e\}(M-(M \cap N)) \cup \{e\})$ and this bond contains an edge in $N - (M \cap N)\ (M-(M \cap N))$. Set

$$M \leftarrow (e_{i_{k+1}}/e_{i_k})M(M)$$

$$N \leftarrow N((e_{i_{k+1}}/e_{i_k})\,N)$$

and
$$G' \leftarrow G \cdot (M \cup N \cup \{e\}) \ .$$

STRATEGY: $\mathfrak{J} = (G, e)$ IS A CUT GAME

Determine $H_2 = G \times T$, the principal contraction of G, and a 2-maximally distant coforest pair $\underline{\tilde{f}} = (\overline{f_1}, \overline{f_2})$ of G which satisfies $e \notin \overline{f_1} \cup \overline{f_2}$. Set

$$M = \overline{f_1} \cap T \ ,$$

$$N = \overline{f_2} \cap T$$

and
$$H' = G \times (M \cup N \cup \{e\}) \ .$$

The basic strategy of the cut player is given recursively as follows. On the k^{th} play, where k is odd, the short player tags edge $e_{i_k} \in [M \cup N - (M \cap N)]$.

If $e_{i_k} \in M(N)$, then e_{i_k} forms a unique polygon in $(N-(M \cap N)) \cup \{e\}$ $((M-(M \cap N)) \cup \{e\})$ and this polygon contains an edge $e_{i_{k+1}}$ in $N-(M \cap N)$ $(M-(M \cap N))$. Set

$$M \leftarrow (e_{i_{k+1}}/e_{i_k})M(M) \quad ,$$

$$N \leftarrow N((e_{i_{k+1}}/e_{i_k})N)$$

and

$$H' \leftarrow G \times (M \cup N \cup \{e\}) \ .$$

STRATEGY: $\mathfrak{F} = (G, e)$ IS A NEUTRAL GAME

The short(cut) player playing first uses the same strategy as is used for a short(cut) game with the condition that the other player has tagged the edge e on a fictitious first play.

Thus if the short(cut) player goes first the graph G_2^+ (H_2^+) is used to play the neutral game.

Example 4-8. Let G be the graph of Fig.4-18 and $\mathfrak{F} = (\mathcal{P}(G), e_C)$. $\mathcal{S}_2 = \mathcal{P}(G_2)$ where $G_2 = G \cdot S$ and $S = \{1,2,3,4,5,6,7,10,e_C\}$. Since $e_C \in \mathcal{S}_2$, $\mathfrak{F}$ is a short game. The sets M and N of the previous discussion are

$$M = \{1,6,7,10\}$$

and

$$N = \{2,3,4,5\} \ .$$

(See Fig.4-18.) The cut player goes first and tags $2 \in N$. The second play (which is not unique) is made by the short player who tags 6 in M. The subsequent plays in $\mathcal{S}_2$ are indicated in Fig.4-23. Clearly,

$e_C \in \mathcal{S}(\{6,3,4,7\})_{\mathcal{P}(G)}$.

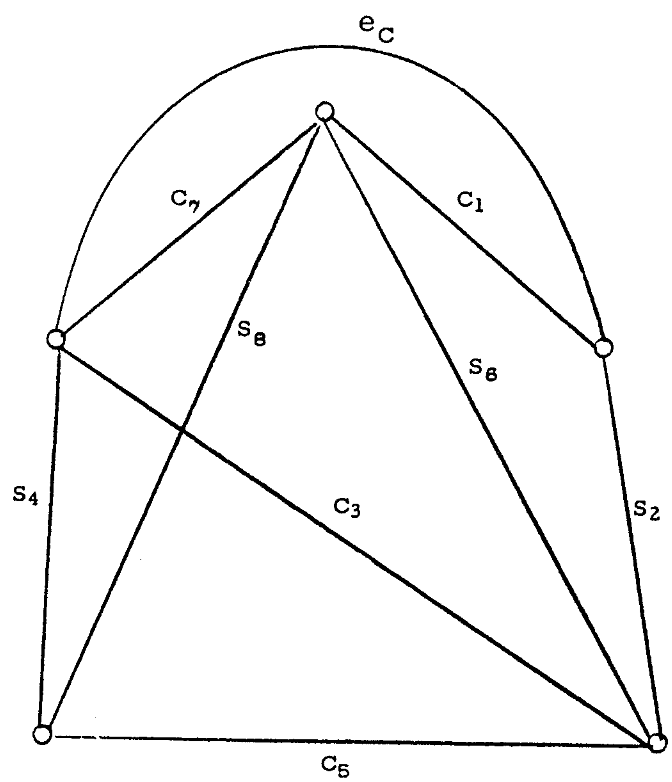


Figure 4-23. Graph of Example 4-8

(B) Edmonds' Cospanning-Sets Theorem

Let $\mathcal{M} = (\mathcal{C}, E)$ be a matroid. We previously introduced the notion of two cospanning sets. This notion can be generalized to k cospanning sets, where k is any integer greater than 1. If $A_1, \ldots, A_k \subseteq E$, we say that the sets $A_1, \ldots, A_k$ are cospanning sets with respect to $\mathcal{M}$ if

$$\mathscr{S}(A_1)_{\mathcal{M}} = \mathscr{S}(A_2)_{\mathcal{M}} = \ldots = \mathscr{S}(A_k)_{\mathcal{M}} .$$

Edmonds stated and proved the following theorem:

> (4.6-4) (Edmonds' Cospanning-Sets Theorem). For any matroid $\mathcal{M} = (\mathcal{C}, E)$ and any subsets N and K of E, there exists as many as k disjoint subsets of N which span each other and which span K, if and only if there is no reduction $\mathcal{M} \cdot M'$ of $\mathcal{M}$ where $N \cap M'$ partitions into as few as k sets such that each is independent in $\mathcal{M} \cdot M'$ and such that at least one of them does not span $K \cap M'$.

In this section we give an alternative form of (4.6-4) which makes use of the principal minors of a matroid. Our Theorem (4.6-6) emphasizes a constructive approach. It is convenient to prove the following preliminary result.

> (4.6-5) Let $\mathcal{M} = (\mathcal{C}, E)$ and $A \subseteq T \subseteq S \subseteq E$.
> Then $\mathscr{S}(\mathscr{S}(A)_{\mathcal{M} \times T})_{\mathcal{M} \times S} = \mathscr{S}(A)_{\mathcal{M} \times S}$.

<u>Proof:</u> Clearly, $\mathscr{S}(A)_{\mathcal{M} \times T} \subseteq \mathscr{S}(A)_{\mathcal{M} \times S}$. Suppose $e \in (\mathscr{S}(\mathscr{S}(A)_{\mathcal{M} \times T})_{\mathcal{M} \times S} - \mathscr{S}(A)_{\mathcal{M} \times T})$. Then there is a circuit $C \in \mathcal{C}$ such that $e \in C$ and $(C - \{e\}) \subseteq \mathscr{S}(A)_{\mathcal{M} \times T}$. By a construction similar to that used in the proof of (4.6-2), we can find a $C' \in \mathcal{C}$ such that $e \in C'$ and $(C' - \{e\}) \subseteq A$.

Therefore $e \in \mathscr{S}(A)_{\mathcal{M} \times S}$. Consequently,

$$\mathscr{S}(\mathscr{S}(A)_{\mathcal{M} \times T})_{\mathcal{M} \times S} \subseteq \mathscr{S}(A)_{\mathcal{M} \times S}$$

Also since $A \subseteq \mathscr{S}(A)_{\mathcal{M} \times T}$, it follows that

$$\mathscr{S}(A)_{\mathcal{M} \times S} \subseteq \mathscr{S}(\mathscr{S}(A)_{\mathcal{M} \times T})_{\mathcal{M} \times S} .$$

The theorem follows. ▮

(4.6-6) Let $\mathcal{M} = (\mathcal{C}, E)$ be a matroid and K and N any two subsets of E. Let $\mathcal{M} \times S^+$ be the k-APM1 of $\mathcal{M} \times N$. Then there exists k disjoint subsets N_i of N which span each other and span K if and only if $K \subseteq \mathscr{S}(S^+)_{\mathcal{M}}$.

Proof: Suppose $K \subseteq \mathscr{S}(S^+)_{\mathcal{M}}$. Clearly there exists k disjoint subsets N_i of $S^+ \subseteq N$ which span each other with respect to $\mathcal{M} \times S^+$, that is,

$$\mathscr{S}(N_i)_{\mathcal{M} \times S^+} = S^+$$

for $i = 1, \ldots, k$. By (4.6-2)

$$\mathscr{S}(N_1)_{\mathcal{M}} = \mathscr{S}(N_2)_{\mathcal{M}} = \ldots = \mathscr{S}(N_k)_{\mathcal{M}} .$$

By (4.6-5) $\mathscr{S}(\mathscr{S}(N_i)_{\mathcal{M} \times S^+})_{\mathcal{M}} = \mathscr{S}(N_i)_{\mathcal{M}}$.

By hypothesis $K \subseteq \mathscr{S}(N_i)_{\mathcal{M}}$ for $i = 1, \ldots, k$.

Next suppose there exists k disjoint subsets N_i of N which are cospanning and span K with respect to $\mathcal{M}$. We must show that $K \subseteq \mathscr{S}(S^+)_{\mathcal{M}}$, where $\mathcal{M} \times S^+$ is the k-APM1 of $\mathcal{M} \times N$.

Let $b_i \subseteq N_i$ be a minimal set satisfying $\mathscr{S}(b_i)_{\mathcal{M}} = \mathscr{S}(N_i)_{\mathcal{M}}$ for $i = 1, \ldots, k$. The b_i's are independent sets of $\mathcal{M}$. Set $B = \bigcup_{i=1}^{k} b_i$. It is not difficult to see that the b_i's are disjoint bases of $\mathcal{M} \times B$. By (4.6-2) the b_i's are cospanning sets with respect to $\mathcal{M} \times N$.

Let $\mathcal{M} \times S^+$ be the k-APM1 of $\mathcal{M} \times N$. Then by (4.4-7) and the definition of a k-augmented minor, there exists k disjoint bases b'_i of $\mathcal{M} \times S^+$. By (4.6-2) the sets b'_i's are cospanning independent sets of $\mathcal{M} \times N$.

Set $d_i = b_i \cup b'_i$ for $i = 1, \ldots, k$. The d_i's are obviously disjoint. It is not very difficult to show that

$$\mathscr{S}(d_1)_{\mathcal{M} \times N} = \mathscr{S}(d_2)_{\mathcal{M} \times N} = \ldots = \mathscr{S}(d_k)_{\mathcal{M} \times N} .$$

Let $d'_i \subseteq d_i$ be a minimal set satisfying

$$\mathscr{S}(d_i')_{\mathcal{M} \times N} = \mathscr{S}(d_i)_{\mathcal{M} \times N}$$

for $i = 1, \ldots, k$. To form d_i' one can add to the set b_i' only those members of b_i which are not spanned by b_i' with respect to $\mathcal{M} \times N$. With this construction of d_i' in mind it should be clear that

$$g_k(\mathscr{S}(d_i')_{\mathcal{M} \times N}) = \overline{g}_k \quad ,$$

where $\overline{g}_k$ is defined with respect to $\mathcal{M} \times N$. Accordingly,

$$S^+ = \mathscr{S}(d_i')_{\mathcal{M} \times N}$$

and therefore

$$B \subseteq S^+ \quad .$$

Since $\mathscr{S}(B)_{\mathcal{M}} \supseteq K$ and $S^+ \supseteq B$, it follows that $\mathscr{S}(S^+)_{\mathcal{M}} \supseteq K$. ▌

(C) Hybrid Rank and Hybrid Dimension

Let $\mathcal{M} = (\mathcal{C}, E)$. We define the function r_H and $r_H(S) = r(\mathcal{M} \times S) + \mu(\mathcal{M} \cdot \overline{S})$, where $S \subseteq E$. The number $r_H(S)$ is called the

<u>hybrid rank of $\mathcal{M}$ with respect to S.</u>

We define

$$\underline{r}_H = \min_{S \subseteq E} r_H(S) .$$

$\underline{r}_H$ is called the <u>hybrid rank</u> of $\mathcal{M}$. Note that $r_H(E) = r(\mathcal{M})$ and $r_H(\phi) = \mu(\mathcal{M})$. Consequently,

$$\underline{r}_H \le \min (r(\mathcal{M}), \mu(\mathcal{M})) .$$

The notion of the hybrid rank of a graph was introduced by Tsuchiya et al. [Ts 1] . It was shown that the hybrid rank of a graph has important implications in electrical network theory. In this section we give the extension of their results to matroids and vector spaces.

(4.6-7) Let $\mathcal{M} = (\mathcal{C}, E)$. Then $\underline{r}_H = \mu(\mathcal{M}) - \overline{g}_2 = r(\mathcal{M}) - \overline{h}_2$ where $\overline{g}_2$ and $\overline{h}_2$ are defined with respect to $\mathcal{M}$.

<u>Proof</u>: From (2.4-19) we get that

$$\mu(\mathcal{M} \times S) + \mu(\mathcal{M} \cdot \overline{S}) = \mu(\mathcal{M}) .$$

Using (1) in the expression for $r_H(S)$ we find:

$$\begin{aligned} r_H(S) &= r(\mathcal{M} \times S) + \mu(\mathcal{M}) - \mu(\mathcal{M} \times S) \\ &= 2r(\mathcal{M} \times S) + \mu(\mathcal{M}) - \mu(\mathcal{M} \times S) - r(\mathcal{M} \times S) \\ &= 2r(\mathcal{M} \times S) + \mu(\mathcal{M}) - \alpha(S) \\ &= \mu(\mathcal{M}) - g_2(S) . \end{aligned} \tag{2}$$

It follows from (2) that

$$\underline{r}_H = \mu(\mathcal{M}) - \overline{g}_2 .$$

Similarly, using (2.4-19) directly in the expression for $r_H(S)$ we obtain

$$\underline{r}_H = r(\mathcal{M}) - \overline{h}_2 \quad . \quad \blacksquare$$

The above ideas can be extended to finite dimensional vector spaces. Let $\mathcal{V}$ be a vector space on E over the field F. If f is any vector of $\mathcal{V}$ and $S \subseteq E$, then we say that g is the projection of f on S if $g(s) = f(s)$ for all $s \in S$.

The class of projections on S of the vectors of $\mathcal{V}$ form a vector space on S over F. We call this vector space the projection of $\mathcal{V}$ on S and denote it by $\mathcal{V}(S)$.

Let dim ($\mathcal{V}$) be equal to the dimension of the vector space $\mathcal{V}$.

The following results, which are proved by Tutte [Tu 12], are preliminary ones.

(4.6-8) (Tutte) $\dim(\mathcal{V}) = \mu(\mathcal{M}_{\mathcal{V}})$.

(4.6-9) (Tutte) $\mathcal{M}_{\mathcal{V}(S)} = \mathcal{M}_{\mathcal{V}} \cdot S$.

(4.6-10) (Tutte) $\mathcal{M}_{\perp\mathcal{V}} = (\mathcal{M}_{\mathcal{V}})^*$.

We define the hybrid dimension of $\mathcal{V}$ with respect to $S \subseteq E$ as

$$d_H(S) = \dim(\mathcal{V}(S)) + \dim((\perp\mathcal{V})(\overline{S})) \ .$$

The hybrid dimension of $\mathcal{V}$ is defined as

$$\underline{d}_H = \min_{S \subseteq E} d_H(S) \quad .$$

Clearly $\underline{d}_H \leq \min(\dim(\mathcal{V}), \dim(\perp\mathcal{V}))$.

(4.6-11) Let $\mathcal{V}$ be a vector space on E over F. Then $\underline{d}_H = \dim(\mathcal{V}) - \overline{g}_2 = \dim(\perp\mathcal{V}) - \overline{h}_2$, where $\overline{g}_2$ and $\overline{h}_2$ are defined with respect to $\mathcal{M}_{\mathcal{V}}$.

Proof: Using (4.6-8), (4.6-9) and (4.6-10) we can write

$$d_H = \mu(\mathcal{M}_{\mathcal{V}(S)}) + \mu(\mathcal{M}_{(\perp\mathcal{V})(\overline{S})})$$

$$= \mu(\mathcal{M}_{\mathcal{V}} \cdot S) + \mu(\mathcal{M}_{\perp\mathcal{V}} \cdot \overline{S})$$

$$= \mu(\mathcal{M}_{\mathcal{V}} \cdot S) + \mu(\mathcal{M}^*_{\mathcal{V}} \cdot \overline{S}).$$

By (2.4-11) and (2.4-2) we have that

$$d_H(S) = \mu(\mathcal{M}_{\mathcal{V}} \cdot S) + \mu((\mathcal{M}_{\mathcal{V}} \times \overline{S})^*)$$

$$= \mu(\mathcal{M}_{\mathcal{V}} \cdot S) + r(\mathcal{M}_{\mathcal{V}} \times \overline{S}) \quad .$$

However, comparison of our last expression for $d_H(S)$ with the expression for $r_H(\overline{S})$ with respect to $\mathcal{M}_{\mathcal{V}}$ gives

$$d_H(S) = r_H(\overline{S}) \quad .$$

Consequently,

$$\underline{d}_H = \underline{r}_H \quad .$$

The theorem follows upon using (4.6-7), (4.6-8) and (4.6-10). ▮

CHAPTER 5. CONCLUSIONS AND SUMMARY OF CONTRIBUTIONS

In this chapter we summarize the contributions made in this dissertation and, where possible, indicate areas and specific problems for future research.

The concept of a generalized network has been introduced in Chapter 3. A generalized network $N = (\mathcal{M}_{\mathfrak{R}}; \mathfrak{R}; D; E)$ can be thought of as a p-port resistance network constructed on a regular matroid $\mathcal{M}_{\mathfrak{R}}$. We view the matroid $\mathcal{M}_{\mathfrak{R}}$ as the "unoriented" description of the interconnection of the members of E. The regular vector space R, as pointed out in section 3.2, admits a definition of the relative orientation of the elements in any circuit of $\mathcal{M}_{\mathfrak{R}}$ which is consistent. We point out that, in view of Theorem (2.2-4), the choice of $\mathfrak{R}$ being regular is not essential to the concept of relative orientation and, in fact, many of our results are valid for non-regular spaces. The result in which regularity is crucial is Theorem (3.4-1), where it is shown that if $N \in \mathcal{N}$, then $\mathbf{X}_N$ is a paramount matrix. Thus the choice of regular matroids for generalized networks can be motivated by the fact that the o.c. impedance and s.c. admittance matrices of p-port resistance networks are also paramount matrices.

The choice of regular matroids can be motivated from another point of view. As is shown in Fig.2-10, the class of regular matroids in the immediate generalization of the graphic and cographic matroids and consequently shares many of their properties. It is not unreasonable to expect, therefore, that the study of generalized networks (defined in terms of regular matroids) will increase our understanding of the p-port resistance network.

A general proof of the paramountcy condition is given for generalized networks. The method of proof for Theorem (3.4-1) gives an interesting alternative way of calculating any minor of $\mathbf{X}_N$. In (3.4-2) the alternative expression for $\mathbf{X}_N \left(\frac{\overline{i}}{j}\right)$ is given and in view of Table 2-2, line (5) this expression suggests topological formulas for $(X_N)_{i,j}$.

Accordingly, the necessary notation is developed and in (3.4-3) the topological formula for $(X_N)_{i,j}$ is set down explicitly. This is an example of how the matroid structure enters into the description of the properties of generalized networks.

An additional necessary condition is derived for the matrix triple product $Q = ADA^t$, where A is a $p \times b$ totally unimodular matrix and D is a $b \times b$ diagonal matrix with positive diagonal terms. It is shown in (3.4-6) that $Q_{i,r,c} \geq 0$ for all choices of i,r,c satisfying $1 \leq i,r,c \leq p$. In Example 3-3 we show that this new condition is not implied by the paramountcy condition. However, even if a matrix Q is paramount and satisfies $Q_{i,r,c} \geq 0$ for all $1 \leq i,r,c \leq p$, this is not enough to guarantee that Q can be expressed as ADA^t, where ADA^t satisfies the condition of (3.4-6).

The notion of modified topological matrices has been extended to generalized networks. It was shown that if $N = (\mathcal{M}_{\mathcal{R}}, \mathcal{R}, D; E) \in \mathcal{N}$, then X_N can always be expressed as

$$X_N = \hat{R} D \hat{R}^t$$

where $\hat{R} = R_{11} - R_{11} D R_{21}^t [R_{21} D R_{21}^t]^{-1} R_{21}$ and $R = \left[\begin{array}{c|c} R_{11} & 1_p \\ \hline R_{21} & O_{p' \times p} \end{array}\right] = \mu \times n$ is a representative matrix for $\mathcal{R}$.

A new result on paramount matrices is given in section 3.5. We show in (3.5-4) that the null space of a paramount matrix (considered as an operator) is a regular vector space. Accordingly, the linear dependence of the columns of a paramount matrix must be such that the coefficients of a linear combination of a minimal dependent set of columns can always be chosen as ±1. This result is translated into a representation theorem for any paramount matrix. Theorem (3.5-5) says that if Q is a $p \times p$ paramount matrix of rank s satisfying $Q\begin{pmatrix} 1\,2 \ldots s \\ 1\,2 \ldots s \end{pmatrix} \neq 0$, then $Q = B^t Q_s B$, where B is a $p \times s$ totally unimodular matrix and Q_s is the submatrix formed from the first s rows and columns of Q.

A conjecture that so far the author has been unable to prove or disprove is: let Q be a $p \times p$ paramount matrix. Then $\mathcal{E}(Q) \neq \phi$. It is believed that given any $p \times p$ paramount matrix Q there exists at least one generalized network N which realizes Q, that is, $X_N = Q$. The singular paramount matrices posed an interesting test for this conjecture. According to (3.5-2) the minimal dependent columns of X_N are in 1-1 correspondence with the circuits of $\mathcal{M}_{\mathcal{R}} \times E_p$. If the conjecture is true, then (3.5-4) follows immediately. On the other hand, if (3.5-4) were not true, then the conjecture would be false. Since we were able to prove (3.5-4), the conjecture still stands.

In Theorems (3.6-3), (3.6-4) and (3.6-5) we use the results of section 2.6 to restate some known results in a matroid-theoretic context.

Finally we discuss the use of matroid theory as a rigorous basis for a graph-theoretic duality concept. "Duality" used on an "intuitive" level can lead to erroneous conclusions, therefore, we set down the precise circumstances under which an appeal to duality is justified.

The general synthesis problem for generalized as well as p-port resistance remains unsolved. Also there is no adequate theory of equivalent networks. Such a theory would be useful for the optimization of certain properties of networks such as sensitivity. These important problems constitute areas for future research.

In Chapter 4 we studied a basic structural property of matroids (and graphs), namely, the principal minors of a matroid. We introduced these new concepts through the functions g_k and h_k $(k=1,2, \ldots)$. Although we defined two kinds of principal minors, it was shown in (4.2-1) and (4.2-2) that by using matroid duality we could restrict our attention to the k-PM1 and the k-APM1. Using the properties of matroids and the definitions of k-PM1 and k-APM1, we establish the existence and the uniqueness of $\mathcal{S}_k$, the k-PM1 of $\mathcal{M}$, and $\mathcal{S}_k^+$, the k-APM1 of $\mathcal{M}$ for $k=1,2, \ldots$. Another result of section 4.2 is the nesting property of principal minors. It was shown that

$$\mathcal{S}_k^+ \supseteq \mathcal{S}_k$$

and

$$\mathcal{S}_k \supseteq \mathcal{S}_{k+1}^+$$

for $k = 1,2, \ldots$.

Section 4.3 treats the 1-principal minors. We found that

$$\mathcal{S}_1^{+} = \mathcal{N}_1^{+} = \mathcal{M}$$

and $\overline{g}_1 = \mu(\mathcal{M})$, that is, the 1-APM1 and the 1-APM2 of $\mathcal{M}$ are the same and are equal to the matroid $\mathcal{M}$. The 1-PM1 of $\mathcal{M}$ was shown to consist essentially of the "cyclically connected" part of $\mathcal{M}$. Specifically, it was shown that if

$$P = \{e \mid e \in E \text{ and } e \text{ is a member of at least one circuit of } \mathcal{M}\},$$

then $\mathcal{S}_1 = \mathcal{M} \times P$. Dually we found that $\mathcal{N}_1 = \mathcal{M} \cdot Q$, where

$$Q = \{e \mid e \in E \text{ and } e \text{ is a member of at least one circuit of } \mathcal{M}^*\}.$$

The specialization of these results to graphs is illustrated in Example 4-1.

To facilitate the study of the principal minors of a matroid, two new concepts were introduced in section 4.4:

(1) r-maximally distant bases and

(2) r-minors.

These concepts are extensions of two corresponding graph-theoretic notions first introduced by Kishi and Kajitani [Ki 1]. In Theorem (4.4-2) we relate these two concepts. The main theorems of section 4 are (4.4-3) and (4.4-7) wherein we expose the structure of the principal minors of a matroid. In this section we also provide algorithms which can be used to construct the principal minors of a matroid efficiently.

In section 5 we define the principal partition of a matroid. This is a straightforward generalization of the principal partition of a graph.

In the final section of Chapter 4 we treat three applications of the matroid-theoretic results of the previous sections. Theorem (4.6-3) gives a new characterization of the two-person switching game. Furthermore, we have shown how duality in graphs allows one to obtain a complete graph-theoretic solution to the two-person switching game.

We have also treated a theorem of Edmonds' called the cospanning sets theorem. We have applied some of our results to obtain a more direct version of this theorem in (4.6-6).

Finally, we have included extensions of the hybrid rank of a graph to the hybrid rank of a matroid and the hybrid dimension of a vector space.

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