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TIME SUB-OPTIMAL CONTROL OF A CLASS OF LINEAR SYSTEMS SUBJECT TO SEVERAL SIMULTANEOUS INPUT CONSTRAINTS

by

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Simultaneous Input Constraints by Arthur Van Gelder

Adviser: George M. Kranc

ABSTRACT

The research described herein had as a primary objective the development of a simple and versatile controller which would result in "near optimum" performance for a certain class of systems and input constraints. First the general forn of the time-optimal input to a linear system is established when the input is subject to several simultaneous norm constraints of the form $\|u\|_{pi} = \left[\int_{t_0}^{T} |u|^{Pi} dt \right]^{1/Pi} \leq L_i$. This is done using the calculus of variations in conjunction with a "multi-norm." The results permit the investigation of combinations of constraints which cannot be easily treated by "conventional" methods such as the maximum principle and dynamic programming. A general computational procedure which can be used for any combination of input norm constraints is given and illustrated by an example.

The optimal solutions for several physically meaningful constraint combinations are then considered in detail for some time-invariant second order systems. Based upon the optimal solutions, several suboptimal solutions are proposed and investigated. It is shown that a simple coast-bang-off-bang solution results in near optimum performance for an underdamped second order system subject to simultaneous input area and magnitude constraints. It is then argued that this solution can also be often effectively used with other constraint combinations and with higher order systems whose behavior is governed by a predominant pair of complex conjugate poles.

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CHAPTER I

INTRODUCTION

The methods of analysis and design and synthesis of automatic control systems have advanced rapidly during the past two decades. During and immediately following World War II significant improvements in system performance were obtained. At that time, the performance criteria were usually expressed in terms of gain and phase margin, damping ratio, bandwidth, frequency response, steady-state error, etc. More recently, the emphasis has shifted to optimization of the system design from the point of view of minimum expenditure of control effort, minimum terminal time, maximum payload, or other similar requirements. Of course optimal use of available resources has always been an implicit goal of every engineering design. However, the demands of increased performance in the defense and aerospace industries has required a significant departure from the "classical" techniques of control system engineers. This has revitalized interest in certain areas in classical mathematics which provide the foundations for optimality as well as stimulating further research, both by mathematicians and engineers, to improve presently available methods and techniques.

Although extensive interest in optimal control is of fairly recent origin, there has been an almost unbelievable amount of literature generated in the area. An excellent survey of a portion of this literature oriented primarily towards aeronautical and astronautical applications by Paiewonsky[1] contains 362 references. The recent book by Athans and Falb[2] has over 200 references and that by Tou[3] has over 150 references. While it would be both redundant and presumptuous to attempt any detailed historical survey here, some general background material is necessary to place the work presented here in its proper perspective.

1.1 The Optimal Control Problem

Any intelligent discussion of a control system depends upon defining an explicit measure of performance quality. Sometimes a system may only be required to exhibit "good" dynamic response characteristics and low sensitivity to external disturbances. This would be typical of a "classical" or conventional control problem. If the system must satisfy further demands--such as minimum expenditure of fuel, or performing a stipulated mission in minimum time--then a more sophisticated approach is necessary.

The optimal control problem can be divided into two major parts. First is the proper formulation of the problem, which may be far from trivial, and second is the solution of the problem by some appropriate procedure. Since the formulation of a mathematical model for a real system involves a compromise between the needs of analytic tractability and of including all significant dynamic features, the results of any analysis must be interpreted with due regard for these factors.

Broadly speaking, the system is often described by a set of differential equations with various initial and terminal conditions specified. The problem is then to find the control function which will minimize (or maximize) some specified performance index or functional of the control and state variables. This is the classical problem of Lagrange or

Mayer[3], depending upon the formulation, and has to mathematicians a well established solution. However, this "solution" is often in the form of a system of non-linear differential equations with two-point boundary conditions. This can lead to considerable computational difficulties. Further, with this classical formulation, there is difficulty in handling inequality constraints on the control function or the state variables. Since any real system is subject to such physical constraints, the direct classical approach had limited usefulness in the solution of optimal control problems. As a result of these limitations, several new approaches and improvements in the classical techniques have recently been developed. Among these are the maximum principle, dynamic programming, gradient methods, and functional analysis.

Pontryagin's maximum principle[2,3,4] may be viewed as an outgrowth of the Hamiltonian approach to variational problems. The maximum principle leads to a solution which is (in general) a set of non-linear differential equations with two point boundary conditions so that the same computational difficulties exist as with the calculus of variations. It should also be noted that the maximum principle provides only a set of necessary conditions for the existence of a solution. However, in most practical problems, physical considerations will often confirm sufficiency and uniqueness. It appears that the maximum principle is completely identical to the classical methods when the inequality constraints are included via the Valentine condition[5]. The popularity of the maximum principle is due primarily to its elegant format and the systematic "ease" with which it can be applied.

Dynamic programming was developed by Bellman[6] in the mid-1950's and may be viewed as an outgrowth of the Hamilton-Jacobi approach to variational problems. It is a genuinely original contribution to a long outstanding problem. It has also helped development in such diverse areas as management science, information theory, learning theory, optimal control, optimal filtering, adaptive control, and variational calculus. Dynamic programming is a multistage decision process and the fundamental premise is embodied in the Principle of Optimality[6]:

> An optimal policy has the property that whatever the initial state and the initial decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

As in all other techniques, there are serious computational problems involved in the application of dynamic programming: It is limited in this case by computer storage capacity. A significant point in its favor, however, is its almost complete indifference to the presence of "pathological" functions.

Gradient methods generally yield only local extrema since small deviations from a reference condition are analyzed. Although the basic procedure of the method of steepest descent in function space is easily established, a variety of practical complications has precluded the formulation of a simple straightforward prodedure which will always work. Nonetheless, the gradient method is a very powerful computational technique.

Although all the procedures discussed above can properly be considered "functional analysis," this term has recently been used to describe a method based upon results obtained by Krassovskii[7] and Kulikowski[8,9] and developed further by Kranc and Sarachik[10,11] and Kreindler[12,13]. This technique makes use of certain properties of functionals in normed spaces. Computational techniques based upon this method have been discussed by Klafter[14] and Vachtsevanos[15].

For any given problem, therefore, one may select one of several superficially unrelated methods to achieve optimization. It is not always obvious which method is best suited to a given problem although usually one will be better than the others.

1.2 Time Optimal Control

Time optimal control is one of the more interesting control problems because, in addition to the existence of relatively simple solutions for some cases, the solutions provide insight into some of the more general aspects of optimization. A review of a portion of the literature on time-optimal control was made by Kreindler[12]. Originally, most of the literature was concerned with the situation when the magnitude of the input u is bounded, $|u| \leq 1$, with no additional constraints. It is now well known and rigorously established (see reference 2 for example) that the optimal input for such a situation is the bang-bang input, i.e., $u = \pm 1$ only. Most of the investigations of time-optimal control of linear systems have only dealt with the case where the poles or characteristic roots of the system are real, distinct, and negative. For this case, the input to an n-th order system changes sign at most $n - 1$ times[16]. Bushaw[17] considered the general second order regulator problem including those cases where the system has oscillatory characteristics. Under these circumstances, a large number of input switchings may be required.

If the control variable is further constrained by limitations on the total available control effort, the complexity of the solution naturally increases. The form of the solution to this problem is often identical to that which is obtained for a minimum effort problem with bounded control variables. Of the investigations of minimum effort problems which have been reported, the work of Flugge-Lotz and Craig[18] and Kreindler[13] is most pertinent to the work presented here--although the work here is of a more general nature.

1.3 Scope of This Work and Chapter Description

The previous sections briefly discussed the optimal control problem. However, after the optimal solution is obtained for a specific problem, it is usually economically unfeasible to implement the optimal solution in an engineering design. Although it may be expected that future technological advances will alleviate some of the difficulties, large scale implementation of optimal designs does not appear likely. However, one may safely predict a rapid increase in the use of sub-optimal control systems, i.e., a system whose controller is relatively simple in form (compared to that of the optimal controller) but which results in near optimum performance.

One of the major pruposes of the work reported here is to develop such a sub-optimal controller. However, discussion of this topic cannot proceed in a sensible manner without first determining the optimal solution for various cases of interest. These not only provide a basis for comparison, but also often suggest an acceptable form for the sub-optimal input. Chapter II contains, in part, the derivation and interpretation of an equation (2-18) which yields the general form of the time-optimal input to an n-th order linear system subject to several simultaneous input norm constraints. This single equation contains a great deal of information which previously had to be derived piece by piece, i.e., each combination of constraints required a new of different derivation. In addition, the formulation is such that the form of the time-optimal input can be readily established for numerous constraint combinations which are difficult to treat by other methods. An outline of a general computational scheme is presented and illustrated by an example.

Chapter III treats the time-optimal solution for a linear, time-invariant, underdamped, second-order system where the input is subject to an area constraint but is unbounded in magnitude. This leads to an impulsive input which, of course, can be at best only crudely approximated in most real systems. Several interesting conclusions can be drawn from the results, however. In particular, it is shown that an extremely simple sub-optimal impulsive input requiring virtually no computation is possible. This leads to a somewhat analogous sub-optimal solution for the case where the input is bounded. This matter is discussed in Chapter V.

Chapter IV treats the harmonic oscillator. This is the simplest system where the maximum number of input switchings is not known a priori. It is expected that the results obtained for the pure oscillatory case will provide insight into the solution for the lightly damped case. This is indeed so.

Chapter V includes a detailed investigation of the time-optimal solution of a single input, linear, time-invariant, underdamped, second-order system when the input magnitude is bounded and the total allowable input area is constrained, i.e., $|u| \leq 1$ and $\int_0^T |u| \, dt \leq L_1$. Based upon the optimal solution, several sub-optimal solutions are proposed and investigated. It is shown that a simple coast-bang-bang and a coast-bang-off-bang solution both result in near optimum performance.

In Chapter VI it is shown that the sub-optimal inputs developed in the previous chapter can be used effectively in some situations when the input energy and magnitude rather than the area and magnitude are constrained. It is also suggested that the sub-optimal solution can be used for higher order systems whose behavior is governed by a predominant pair of complex conjugate poles.

Except in Chapter II, where a general linear system is considered, the discussion is limited to fixed or time-invariant systems and to the regulator problem, i.e., the terminal state is the origin.

CHAPTER II

GENERAL FORM OF THE TIME OPTIMAL INPUT

This chapter contains the derivation and interpretation of an equation which yields the general form of the time optimal input to an n-th order linear system subject to several simultaneous constraints on the input. It is assumed that there are k constraints and that they can all be expressed as a norm in L_p - space of the input u(t) being less than or equal to some given number L:

$$\|u\|_{p_i} \triangleq \left[\int_{t_o}^{T} |u(t)|^{p_i} \, dt\right]^{1/p_i} \leq L_i \qquad (2.1)$$

where $1 \leq p_i \leq \infty$ and $i = 1, 2, \ldots, k$. While only a single-input system is considered here, it appears that the extension to multi-input systems would be straightforward by applying and extending some of the procedures developed by Sarachik and Kranc[11]. The expected form of the time optimal input for the multi-input case is suggested based upon the results established here.

A necessary and sufficient condition for the existence of a time optimal solution is developed. However, this condition is useful only if the system is simple enough so that an analytic expression for the input can be determined. Since this is generally not possible, this condition is little more than a formal statement which is almost self-evident from physical considerations. Several necessary conditions for the existence of the solution are given. These are rather weak conditions but would nevertheless be of considerable value in any computational scheme used to find the optimal input. In addition, a computational procedure for a general combination of constraints is outlined in some detail. The procedure is illustrated by an example.

Some results similar to those obtained here were obtained previously by Kreindler[13]. The procedure here is simpler, however, and the results here are of a more general nature. In addition, Kreindler did not consider the computational problem.

2.1 Mathematical Formulation

The linear system to be controlled is described by the state equations

$$\underline{\dot{x}}(t) = A(t)\,\underline{x}(t) + \underline{b}(t)\,u(t) \qquad (2.2)$$

where $\underline{x}(t)$ and $\underline{b}(t)$ are n-vectors (column) and $A(t)$ is an $n \times n$ matrix. The solution of this vector differential equation is

$$\underline{x}(t) = \phi(t, t_o)\,\underline{x}(t_o) + \int_{t_o}^{t} \phi(t, \tau)\,\underline{b}(\tau)\,u(\tau)\;d\tau \qquad (2.3)$$

where $\phi(t, \tau)$ is the system transition matrix $(n \times n)$. This superposition integral leads to an alternate description of the system which is of greater practical utility. In many physical problems, one is often interested in only a relatively small number of outputs although the system may be of high order. Denoting the system outputs by the m-vector $\underline{y}(t)$ (where $m \leq n$) and with $\underline{y}(t)$ related to $\underline{x}(t)$ by the $m \times n$ matrix $M(t)$

$$\underline{y}(t) = M(t)\,\underline{x}(t) \qquad (2.4)$$

we have

$$\underline{y}(t) = M(t)\phi(t, t_o)\,\underline{x}(t_o) + \int_{t_o}^{t} M(t)\phi(t, \tau)\,\underline{b}(\tau)\,u(\tau)\quad d\tau \qquad (2.5)$$

This equation can be considerably simplified by letting

$$\underline{e}(t) = \underline{y}(t) - M(t)\phi(t, t_o)\,\underline{x}(t_o) \qquad (2.6)$$

and

$$\underline{h}(t, \tau) = M(t)\phi(t, \tau)\,\underline{b}(\tau) \qquad (2.7)$$

This yields the relatively simple description

$$\underline{e}(t) = \int_{t_o}^{t} \underline{h}(t, \tau)\, u(\tau) \; d\tau \tag{2.8}$$

It is to be noted that $\underline{e}(t)$ represents the difference between the actual system output and the effect of the initial state upon the output. It will be assume that this effect is known and that $\underline{e}(t)$ is continuous. Furthermore, $\underline{h}(t, \tau)$ can be interpreted as the response $\underline{y}(t)$ to an impulsive input $u(t) = \delta(t-\tau)$ with $x(t_o) = 0$. Hence (2.8) does not require explicit knowledge of $M(t)$, $\phi(t, \tau)$, or $\underline{b}(t)$ individually. It will be possible in many cases to determine $\underline{h}(t, \tau)$ and $e(t)$ from experimental data.

It will be assumed that the system is output controllable, i.e., there exists an input $u(t)$ on the finite intermal $t_o \leq t \leq T$ which will transfer the system output from any finite initial point at $t = t_o$ to any finite output $\underline{y}(T)$. It is further assumed that $h_i(t, \tau)$, the m components of $\underline{h}(t, \tau)$, are bounded on finite intervals of t and τ.

It should be noted at the outset that there are several major practical difficulties involved in the application of the results of the following derivations. First, one does not usually know a priori that a solution exists which satisfies the terminal conditions with the input constrained in some given manner. Moreover, if a solution does exist, it will not usually be known on the boundary of which constraint(s) the input operates. These problems -- while certainly of great importance -- have nothing to do with the derivations which follow. They are considered separately in Section 2.3. The problem to be considered here is to find the input $u(t)$ which satisfies all the constraints of (2.1) and drives the system to the desired terminal state in minimum time.

Two methods of attacking this problem will now be considered. The first is a straightforward calculus of variations procedure which leads simply and directly to the desired result. However, using this approach it is necessary to assume that the input operates on the boundary of at least one of the constraints although one could argue that this must be so on physical grounds. The second approach uses a "multi-norm" which is minimized using the calculus of variations. This approach has several advantages. First, it establishes the positiveness of some of the "Lagrangian multipliers" which are introduced into the mathematical formulation of the problem. Second, the input for a class of minimum norm problems is determined as a result of this procedure. Finally, it is formally established that the input must operate on the boundary of at least one of the constraints. However, the second procedure has the disadvantage of introducing parameters into the solution which, while apparently analogous to some of the Lagrangian multipliers of the first procedure, leave something to be desired in the way of formal mathematical rigor. The results of the two procedures, however, are identical and they reduce to previously established results for several special cases which are investigated.

First Approach

One possible approach to the solution of this problem is as follows: For a fixed terminal time T, use a direct calculus of variations approach to minimize one of the norms of (2.1), say $p_i = j$, subject to the terminal conditions and the constraints on the other norms introduced by means of Lagrangian multipliers. This is repeated until the smallest T is found which makes $\|u\|_j = L_j$. For the purpose of illustration, consider the case of three constraints with $p_1 = 1$, $p_2 = 2$, and $p_3 = \infty$. These correspond to the physically meaningful area, "energy", and magnitude constraints, respectively. It will be convenient to let $p_3 = n$ and then later let $n \to \infty$. Let us minimize $\|u\|_1$ for a fixed terminal time T. We must therefore minimize the functional

$$J = \|u\|_1 + \underline{\lambda}' [\underline{e}(T) - \int_{t_0}^{T} \underline{h}(T,t)\, u(t)\, dt] + w_2(\|u\|_2 - L_2) + w_\infty(\|u\|_n - L_\infty) \tag{2.9}$$

where $\underline{\lambda}$ is a column vector of m Lagrangian multipliers, the prime ' denotes transpose, and w_2 and w_∞ are also Lagrangian multipliers. A necessary condition for J to be a minimum is that the first variation in J for a variation in u should be zero. This yields

$$\left(1 + w_2 \frac{|u|}{\|u\|_2} + w_\infty \frac{|u|^{n-1}}{\|u\|_n^{n-1}}\right) \operatorname{sgn} u - \underline{\lambda}' \underline{h}(T,t) = 0 \tag{2.10}$$

where

$$\operatorname{sgn} u = \begin{cases} +1 & u > 0 \\ -1 & u < 0 \end{cases} \tag{2.11}$$

and is undefined for $u = 0$. Since we desire a solution for u such that $\|u\|_{p_i} = L_i$ (to satisfy the constraints), (2.10) can be written as

$$\left(1 + w_2 \frac{|u|}{L_2} + w_\infty \left|\frac{u}{L_\infty}\right|^{n-1}\right) \operatorname{sgn} u = \underline{\lambda}' \underline{h}(T,t) \tag{2.12}$$

This equation must now be solved for u. It is clear that the u which minimizes $\|u\|_1$ for a given T will be known to within $m + 2$ constants, the m λ's and the two w's. There are also $m + 2$ equations available to evaluate these parameters: The m terminal conditions and the two constraint integrals. It is important to note that although we have set $\|u\|_2 = L_2$ and $\|u\|_n = L_\infty$ in arriving at (2.12) this does not automatically force the resulting u to satisfy these conditions. The u which satisfies (2.12) must be substituted into the constraint integrals and these, along with the terminal conditions, provide a sufficient number of equations to permit the evaluation of the Lagrangian multipliers.

A plot of $\underline{\lambda}' \underline{h}(T, t)$ vs. u is given in Figure 2-1 for (1) $n = \infty$ and (2) n large but finite. Here both w_2 and w_∞ have been taken as positive. Although there is no mathematical justification for this, such must be the case if u is to be uniquely related to $\underline{\lambda}' \underline{h}(T, t)$. This will be established below using a somewhat different approach.

It is to be noted that the input automatically satisfies the magnitude constraint, independent of w_∞, i.e., the form of the input guarantees that the magnitude constraint will be satisfied. The reason for this is evident when (2.12) and Figure 2-1 are considered carefully. For $n = \infty$, the $\left|\frac{u}{L_\infty}\right|^{n-1}$ term in (2.12) is zero for $|u| < L_\infty$. Since $\underline{\lambda}' \underline{h}(T, t)$ must remain finite, $|u|$ cannot exceed L_∞. Hence any positive, finite w_∞ does not influence the solution. However, if n is large, but finite, the value of w_∞ does influence the solution and must be determined along with the other Lagrangian multipliers. The magnitude constraint has been "forced" into the form of an integral constraint by letting $p_3 \rightarrow \infty$. This limit introduces a degeneracy in the relationship between u and $\underline{\lambda}' \underline{h}(t, t)$. In general, one Lagrangian multiplier is needed for each integral constraint (except the magnitude constraint) in the minimum norm problem for a fixed terminal time T.

Some additional comments are in order regarding this procedure. First, the same result will be obtained regardless which norm is minimized. (2.12) would differ only in that the first term would be the Lagrangian multiplier w_1 rather than unity, and one of the other w's would not appear -- the one corresponding to whichever norm is being minimized. However, the equation could be divided through by w_1 leading identically to an equation in the form of (2.12). Second, it has been assumed that the input operates on the boundary of all the constraints. This, of course, if generally not known a priori so that all possible constraint combinations would have to be investigated. Further, it is possible that the terminal conditions could not be met for any T if the

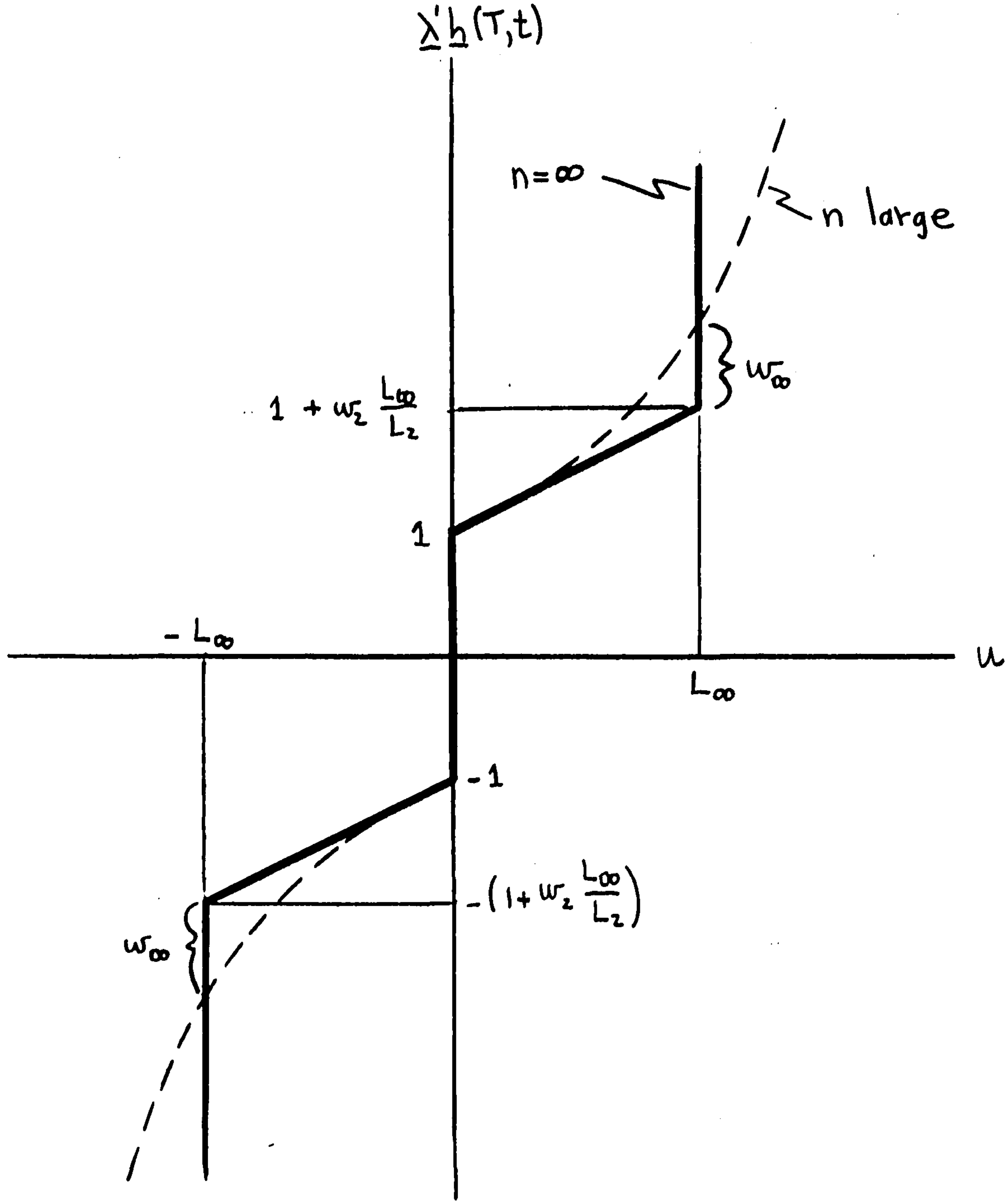


Figure 2-1

$\underline{\lambda}'\underline{h}(T, t)$ as a function of u.

constraints are too severe. The necessary and sufficient conditions for the existence of a solution will be discussed below following a somewhat different approach to the same problem. This approach will establish the positiveness of the w's.

Second Approach

First let (2.1) be rewritten as

$$\left\| \frac{u}{L_i} \right\|_{p_i} \triangleq \left[\int_{t_o}^{T} \left| \frac{u}{L_i} \right|^{p_i} dt \right]^{1/p_i} \leq 1 \tag{2.13}$$

The approach used in the derivation which follows will be to first attempt to minimize the largest of the norms of (2.13) for a given time T. This is repeated for different values of T until the largest of the norms is unity. If the largest is unity, then certainly all the norms are less than or equal to unity. This procedure makes use of the multinorm concept introduced by Sarachik and Kranc.[11]

Consider the following cost functional:

$$J_p = \left[\sum_{i=1}^{k} w_i \left\| \frac{u}{L_i} \right\|_{p_i}^{p} \right]^{1/p} \tag{2.14}$$

where there are k constraints and the w_i are positive, finite weighting factors. In the limit as $p \to \infty$, we have

$$K = \lim_{p\to\infty} = \max_i \left\{ \left\| \frac{u}{L_i} \right\|_{p_i} \right\} \tag{2.15}$$

Note that when the limit is taken, the resulting expression for J_p is independent of the w's. However, the u which results from the minimization of J_p for a given T, followed by the limiting process, does depend upon the w's as will be shown. The result obtained in this manner does agree with that obtained by the first approach. However, there appears to be no satisfactory explanation based on the second approach only as to why the parameters w have no effect upon the limit of J_p and yet still appear in the final result for u.

The minimization of J_p followed by taking the limit as $p \to \infty$ is the minimization of the largest of the norms $\left\| \frac{u}{L_i} \right\|_{p_i}$ for a given T. The minimum value of the largest of the norms could be obtained in a number of ways. First, one of the norms could be larger than all of the others; or two of the norms could be equal and larger than the rest; or three could be equal and larger than the rest; etc. Since the system has been assumed to be output controllable, for every T there is a K, the value of the minimum norm. For the time optimal problem, we seek the smallest T for which K = 1 so that $\|u\|_{p_i} \leq L_i$ for all i and where the input must of necessity operate on the boundary of at least one constraint.

The terminal conditions can be introduced by means of Lagrangian multipliers $\underline{\lambda}$ as before leading to the following functional to be minimized:

$$I = J_p + \underline{\lambda}' \left[\underline{e}(T) - \int_{t_o}^{T} \underline{h}(T, t)\, u(t) \quad dt \right]. \tag{2.16}$$

Taking the first variation in I for a variation in u and setting it equal to zero, we obtain a necessary condition for I to be a minimum:

$$\sum_{i=1}^{k} \frac{w_i}{L_i} \frac{\left\| \frac{u}{L_i} \right\|_{p_i}^{p-p_i}}{J_p^{p-1}} \left(\frac{|u|}{L_i} \right)^{p_i - 1} \operatorname{sgn} u = \underline{\lambda}' \, \underline{h}(T, t) \tag{2.17}$$

In the limit for $p \to \infty$, this reduces to

$$\sum_{i=1}^{k'} \frac{w_i}{L_i} \left(\frac{|u|}{K L_i} \right)^{p_i - 1} \operatorname{sgn} u = \underline{\beta}' \, \underline{h}(T, t) \tag{2.18}$$

where

$$\underline{\beta} = \left[\sum_{i=1}^{k'} w_i \right] \underline{\lambda} \tag{2.19}$$

and the constraints have been ordered so that the first k' norms are equal to each other and larger than the rest, i.e.,

$$\left\| \frac{u}{L_i} \right\|_{p_i} = K \qquad i = 1, 2, \ldots, k'$$

$$\left\| \frac{u}{L_i} \right\|_{p_i} < K \qquad i = k'+1, k'+2, \ldots, k \tag{2.20}$$

In particular, if there are three norms of interest, say $p_1 = 1$, $p_2 = 2$, and $p_3 = n \to \infty$, and the minimum value of the maximum norm is obtained when all three are equal, we ha[ve]

$$\left[\frac{w_1}{L_1} + \frac{w_2}{L_2} \frac{|u|}{KL_2} + \frac{w_\infty}{L_\infty} \left(\frac{|u|}{KL_\infty} \right)^{n-1} \right] \operatorname{sgn} u = (w_1 + w_2 + w_\infty) \underline{\lambda}' \underline{h}(T, t) \tag{2.21}$$

A necessary and sufficient condition for the existence of a time optimal solution is that for some T, the value of the minimum norm K is less than or equal to unity. For the time optimal problem, we seek the smallest T for which $K = 1$. Hence the time optimal input must satisfy the following equation when the input operates on the boundary of all three constraints:

$$\left[\frac{w_1}{L_1} + \frac{w_2}{L_2} \frac{|u|}{L_2} + \frac{w_\infty}{L_\infty} \left| \frac{u}{L_\infty} \right|^{n-1} \right] \operatorname{sgn} u = (w_1 + w_2 + w_\infty) \underline{\lambda}' \underline{h}(T, t) \tag{2.22}$$

Obviously both sides of this equation can be multiplied by $\frac{L_1}{w_1}$ so that

$$\left[1 + \frac{L_1}{L_2} \frac{w_2}{w_1} \frac{|u|}{L_2} + \frac{L_1}{L_\infty} \frac{w_\infty}{w_1} \left| \frac{u}{L_\infty} \right|^{n-1} \right] \operatorname{sgn} u = L_1 \left[1 + \frac{w_2}{w_1} + \frac{w_\infty}{w_1} \right] \underline{\lambda}' \underline{h}(T, t) \tag{2.23}$$

This is in the same form as (2.12). As discussed previously, the resulting u is independent of w_∞/w_1, as $n \to \infty$ so that only m λ's and the value of w_2/w_1 need be determined.

2.2 Interpretation

In the interest of clarity, it is useful at this time to consider only those values of p_i which have direct physical significance. It is readily apparent from (2.1) that $p_i = 1$, 2, and ∞ correspond to the norm of u(t) being equal to the total input area, the square root of the input "energy," and the maximum magnitude of the input, respectively. The results obtained below can be easily extended to include any other values of p_i, $1 \le p_i \le \infty$.

For the particular case of interest, $k = 3$ and $p_1 = 1$, $p_2 = 2$, and $p_3 = \infty$. As above we will let $p_3 = n$ and later let $n \to \infty$. There are seven possible situations which must be investigated: operation on the boundary of only one input constraint (3 cases), operation on the boundary of two of the constraints (3 cases), and operation on the boundary of all three constraints. These will now be treated in turn. In the following, it is convenient to drop the explicit expression of the argument of $\underline{h}(T,t)$, i.e., $\underline{h} = \underline{h}(T,t)$.

Case A: Area Constraint Alone

It is easy to show that, in general, with one exception, when the input operates on the boundary of say constraints numbers 1 and 2, the input is identically that which would be obtained if only constraints numbers 1 and 2 were originally specified. This is evident from (2.18). The exception is that it is impossible to operate on the boundary only the area constraint if other constraints are specified. The reason for this is that the area constraint, by itself, leads to an impulsive input[12]. This will violate any other specified constraint. In the interest of completeness, however, it will now be shown that an impulsive input will satisfy (2.18) if only an area constraint, $p_1 = 1$, is specified. In this case (2.18) becomes (with $K = 1$)

$$\frac{1}{L_1} \text{ sgn } u = \underline{\lambda}'\underline{h} \tag{2.24}$$

This equation can be satisfied when

$$u = 0 \tag{2.25}$$

$$\text{sgn } u = L_1 \underline{\lambda}'\underline{h} \tag{2.26}$$

Since sgn u is undefined for $u = 0$ [see (2.11)] this is not inconsistent. Obviously u cannot be identically zero otherwise $\|u\|_1 = 0$. The only instants of time when u can be non zero are when

$$|\underline{\lambda}'\underline{h}| = 1/L_1 \tag{2.27}$$

Furthermore, $|\underline{\lambda}'\underline{h}|$ cannot exceed $1/L_1$ since this requires $|\text{sgn}\, u| > 1$. Therefore we must have

$$|\underline{\lambda}'\underline{h}| \le 1/L_1 \tag{2.28}$$

with equality at a finite number of points in time. At these times, the input is impulsive. This condition results in an input which has at most n impulses in an n-th order time-invariant system.

Case B: Magnitude Constraint Alone

Here (2.18) becomes

$$\frac{1}{L_\infty}\left(\frac{|u|}{L_\infty}\right)^{n-1} \text{sgn}\, u = \underline{\lambda}'\underline{h} \tag{2.29}$$

From this we have

$$\text{sgn}\, u = \text{sgn}\, \underline{\lambda}'\underline{h} \tag{2.30}$$

and

$$|u|^{n-1} = L_\infty^{n} |\underline{\lambda}'\underline{h}| \tag{2.31}$$

so that

$$|u| = L_\infty^{\frac{n}{n-1}} |\underline{\lambda}'\underline{h}|^{\frac{1}{n-1}} = L_\infty \text{ as } n \to \infty \tag{2.32}$$

Therefore

$$u = L_\infty \,\text{sgn}\, \underline{\lambda}'\underline{h} \tag{2.33}$$

This is the well known bang-bang solution which is well established for this case[2,3,4]. It is evident that since only the sign of $\underline{\lambda}'\underline{h}$ is important, $\underline{\lambda}$ can only be determined to within a multiplicative constant.

Case C: Energy Constraint Alone

Here (2.18) yields

$$\frac{1}{L_2^2} |u| \,\text{sgn}\, u = \underline{\lambda}'\underline{h} \tag{2.34}$$

so that

$$u = L_2^2 \underline{\lambda}'\underline{h} \tag{2.35}$$

As a point of interest, it can be shown[2] that this input is linearly related to the states.

Case D: Area and Magnitude Constraints

Proceeding as above

$$\left[\frac{w_1}{L_1} + \frac{w_\infty}{L_\infty}\left(\frac{|u|}{L_\infty}\right)^{n-1}\right] \operatorname{sgn} u = (w_1 + w_\infty)\underline{\lambda}'\underline{h} \tag{2.36}$$

The u(t) which satisfies (2.36) is easily found. If $u \neq 0$, $\operatorname{sgn} u = \operatorname{sgn} \underline{\lambda}'\underline{h}$ and

$$|u|^{n-1} = w_1 L_\infty^{\ n}\left[\,|(1 + w_\infty/w_1)\underline{\lambda}'\underline{h}| - 1/L_1\right]/w_\infty \tag{2.37}$$

When $|(1 + w_\infty/w_1)\underline{\lambda}'\underline{h}| > 1/L_1$, this results in $|u| = L_\infty$ as $n \to \infty$ as in Case B. Therefore when $|(1 + w_\infty/w_1)\underline{\lambda}'\underline{h}| > 1/L_1$

$$u = L_\infty \operatorname{sgn} \underline{\lambda}'\underline{h} \tag{2.38}$$

When $|(1 + w_\infty/w_1)\underline{\lambda}'\underline{h}| < 1/L_1$, (2.36) can only be satisfied if $u = 0$ and $\operatorname{sgn} u = L_1(1 + w_\infty/w_1)\underline{\lambda}'\underline{h}$. Since sgn u is undefined at $u = 0$ except that $|\operatorname{sgn}(0)| < 1$, this is not an unreasonable condition. The resulting control action is the well known bang-off-bang input which is readily derivable by other means:

$$u = \begin{cases} 0 & |(1 + w_\infty/w_1)\underline{\lambda}'\underline{h}| < 1/L_1 \\ L_\infty \operatorname{sgn} \underline{\lambda}'\underline{h} & |(1 + w_\infty/w_1)\underline{\lambda}'\underline{h}| > 1/L_1 \end{cases} \tag{2.39}$$

It is to be noted that only the ratio of the two weighting factors appears in the input so that one may be arbitrarily specified. Moreover, since the w_∞/w_1 and $\underline{\lambda}'$ terms always appear as a product, both weighting factors are completely arbitrary in this case. This will be discussed in detail in the last part of this section after all of the cases are treated.

Case E: Energy and Magnitude Constraints

Proceeding as before, (2.18) yields for this case

$$\left[\frac{w_2}{L_2^2}\,|u| + \frac{w_\infty}{L_\infty}\left|\frac{u}{L_\infty}\right|^{n-1}\right] \operatorname{sgn} u = (w_2 + w_\infty)\underline{\lambda}'\underline{h} \tag{2.40}$$

For this case we have $\operatorname{sgn} u = \operatorname{sgn} \underline{\lambda}'\underline{h}$ for all t. It is evident that $u = 0$ only when

$\underline{\lambda}'\,\underline{h} = 0$. At those times when $|u| < L_\infty$, the second term on the left hand side of (2.40) becomes zero as $n \to \infty$ resulting in

$$u = L_2^2 (1 + w_\infty/w_2) \underline{\lambda}'\,\underline{h} \tag{2.41}$$

We can also write

$$|u|^{n-1} = w_2 L_\infty^n \left[|(1 + w_\infty/w_2) \underline{\lambda}'\,\underline{h}| - |u|/L_2^2 \right]/w_\infty \tag{2.42}$$

When the quantity in brackets is positive, we have as in Cases B and D, $|u| = L_\infty$ as $n \to \infty$. The final result is then

$$u = \begin{cases} L_2^2 (1 + w_\infty/w_2) \underline{\lambda}'\,\underline{h} & |(1 + w_\infty/w_2) \underline{\lambda}'\,\underline{h}| < L_\infty/L_2^2 \\ L_\infty \operatorname{sgn} \underline{\lambda}'\,\underline{h} & |(1 + w_\infty/w_2) \underline{\lambda}'\,\underline{h}| > L_\infty/L_2^2 \end{cases} \tag{2.43}$$

The form of the input in this case is again in agreement with that which is well established by other means. As in Case D, only the ratio of the weighting factors appears in the expression for the input and this ratio always multiplies $\underline{\lambda}'$ so that again the weighting factors can be arbitrarily chosen.

Case F: Area and Energy Constraints

Proceeding as above, we have

$$\left[\frac{w_1}{L_1} + \frac{w_2 |u|}{L_2^2} \right] \operatorname{sgn} u = (w_1 + w_2) \underline{\lambda}'\,\underline{h} \tag{2.44}$$

The resulting u has an off interval where $|(1 + w_2/w_1) \underline{\lambda}'\,\underline{h}| < 1/L_1$ (which requires $\operatorname{sgn} u = L_1 (1 + w_2/w_1) \underline{\lambda}'\,\underline{h}$ and an "on" interval which will now be determined. For $u \neq 0$, we have $\operatorname{sgn} u = \operatorname{sgn} \underline{\lambda}'\,\underline{h}$ so that solving (2.44) for u we obtain

$$u = \begin{cases} 0 & |(1 + w_2/w_1) \underline{\lambda}'\,\underline{h}| < 1/L_1 \\ \frac{w_1}{w_2} L_2^2 \left[(1 + w_2/w_1) \underline{\lambda}'\,\underline{h} - \frac{1}{L_1} \operatorname{sgn} \underline{\lambda}'\,\underline{h} \right] & |(1 + w_2/w_1) \underline{\lambda}'\,\underline{h}| > 1/L_1 \end{cases} \tag{2.45}$$

The form of the input for this case has not appeared previously. It is to be noted that although the weighting factors again only appear as a ratio, this ratio appears in the equation for u other than just multiplying $\underline{\lambda}'$. Hence, one weighting factor can still be selected arbitrarily but the other can not.

Case G: Area, Energy and Magnitude Constraints

For this case we have

$$\left[\frac{w_1}{L_1} + \frac{w_2}{L_2^2} |u| + \frac{w_\infty}{L_\infty^n} |u|^{n-1}\right] \operatorname{sgn} u = (w_1 + w_2 + w_\infty)\underline{\lambda}' \underline{h} \tag{2.46}$$

It is easily shown that the form of the input for this case is

$$u = \begin{cases} 0 & |(1 + w_2/w_1 + w_\infty/w_1)\,\underline{\lambda}'\underline{h}| < 1/L_1 \\ \dfrac{w_1}{w_2} L_2^2 \left[(1 + \dfrac{w_2}{w_1} + \dfrac{w_\infty}{w_1})\,\underline{\lambda}'\underline{h} - \dfrac{1}{L_1} \operatorname{sgn} \underline{\lambda}'\underline{h}\right] & \dfrac{1}{L_1} < |(1 + \dfrac{w_2}{w_1} + \dfrac{w_\infty}{w_1})\underline{\lambda}'\underline{h}| < \dfrac{1}{L_1} + \dfrac{w_2}{w_1}\dfrac{L_\infty}{L_2^2} \\ L_\infty \operatorname{sgn} \underline{\lambda}'\underline{h} & |(1 + w_2/w_1 + w_\infty/w_1)\underline{\lambda}'\underline{h}| > 1/L_1 + \dfrac{w_2}{w_1}\dfrac{L_\infty}{L_2^2} \end{cases} \tag{2.47}$$

This input is composed of three types of regions: (1) There are intervals where $|u| = L_\infty$, i.e., saturation intervals due to the magnitude constraint. (2) There are intervals where $u = 0$. The off intervals result from the area constraint. (3) There are regions where the input is linearly related to the system inpulsive responses (running backwards in time). This is the result of the energy constraint. It is easy to see that (2.47) reduces to the cases considered above. As expected, one of the weighting factors can be arbitrarily specified. Furthermore, since the term w_∞/w_1 always appears as a multiplier of $\underline{\lambda}'$, a second of the weighting factors can be arbitrarily chosen.

General Comments

Although it is to be expected that some of the weighting factors can be chosen arbitrarily from the original formulation of the minimization problem, some further discussion is useful concerning this topic. The simplest way of looking at this is to consider the information available to evaluate parameters in the solution. There are m terminal conditions from the m controlled outputs and, in addition, k integral conditions for the constraints which must be satisfied at the terminal time T. An exception is the magnitude constraint condition which is automatically satisfied by the form of the input, i.e., the form of the input guarantees that $\|u\|_{\infty} = |u|_{max} \leq L_{\infty}$. This is not true for any of the other norm constraints. Hence for the constraint combinations considered, only when the input operates on the boundary of both the area and energy constraints is a weighting factor required. In general, we must find m components of $\underline{\lambda}$, the terminal time T, and k_1' weighting factors, where k_1' is the number of constraints -- other than magnitude -- upon whose boundary the input operates. Hence, the number of non-arbitrary parameters in the form of the input is given by

$$N = m + 1 + (k_1' - 1) = m + k_1' \tag{2.48}$$

The $(k_1' - 1)$-term results from the fact that one of the weighting factors can always be arbitrarily specified.

The interpretation of (2.18), which is summarized in (2.47), for a specific constraint combination contains a great deal of information which previously had to be derived piece by piece, i.e., each combination of constraints required a new derivation. Furthermore, it can treat constraint combinations which cannot be easily attacked by other methods. It should be pointed out that the solution for u for the various cases considered could have been accomplished by plotting $\underline{\lambda}' \underline{h}$ vs u as was done in Figure 2-1 for (2.12).

In the derivation of the general form of the input, it has been assumed that a solution does exist, i.e., the specified constraints are such that there is at least one input satisfying all the constraints which will drive the system to the desired terminal state in some finite time T. If there is more than one such input, the one resulting in the smallest value of T is the time optimal input. It is shown that the input operates on the boundary of at least one constraint. However, if more than one constraint is specified, it is not known a priori upon the boundary of which constraint(s) the input operates.

As previously indicated, the results obtained above can be easily extended to include any input norm in L_p - space. The extension to the multi-input case would seem to be straightforward in principle, although rather involved in detail, and was not considered. However, based upon the results obtained above and those in Sarachik and Kranc[11], the following would seem to be a reasonable conjecture. The form of u_j, the j-th input, is govered only by the constraints placed upon u_j. As a specific example, if the system has two inputs and the constraints are

$$\int_{t_o}^{T} [u_1^2 + u_2^2]\, dt \leq E \tag{2.49}$$

$$\|u_2\|_\infty = \max |u_2(t)| \leq M \tag{2.50}$$

and the impulsive response vectors are $\underline{h}_1(T, t)$ and $\underline{h}_2(T, t)$ from inputs 1 and 2, respectively, then

$$u_1 = \underline{\lambda}_1' \underline{h}_1(T, t) \tag{2.51}$$

$$u_2 = \begin{cases} \underline{\lambda}_2' \underline{h}_2(T, t) & |\underline{\lambda}_2' \underline{h}_2(T, t)| < M \\ M \operatorname{sgn} \underline{\lambda}_2' \underline{h}_2(T, t) & |\underline{\lambda}_2' \underline{h}_2(T, t)| > M \end{cases} \tag{2.52}$$

where $\underline{\lambda}_1$ and $\underline{\lambda}_2$ are each m-vectors, m being the number of outputs being controlled.

2.3 Some Necessary Conditions

Numerous conditions necessary for the existence of a solution were found. These would be of value in any computational scheme since they serve to reduce the space which must be searched for the optimal solution. Unfortunately, most of the conditions developed are rather "weak" so that it would not pay to apply them blindly in hopes of saving considerable computation time.

If R_i is the reachable region (set of reachable states) from a given initial state with $\|u\|_i \leq L_i$ and no additional constraints, then the reachable region for k simultaneous constraints must be a subset of the intersection of all the R_i's, $i = 1, 2, \ldots, k$. A simple illustration is shown in Figure 2-2 for simultaneous area, energy, and magnitude constraints in 2-space with $\underline{x}(t_o) = 0$. The singly shaded area is the intersection of the three individually reachable regions. Since the time required to reach a point depends not only on the location of the point but also upon the form of the input, it is to be expected that the jointly reachable region will be subset of the singly shaded area as shown by the doubly shaded area. With $\underline{x}(t_o) = 0$, all the reachable regions will be symmetrical about the origin in the state space.

A somewhat tighter or stronger condition is available by considering R_{it}, an individually reachable region for a given time $t = T - t_o$. Since each R_{it} must be a subset of the corresponding R_i, the intersection of the R_{it} regions must be a subset of the intersection of the R_i regions.

It is of course possible that some R_i is itself a subset of some R_j (or similarly $R_{it} \in R_{jt}$). This does not imply that the j-th constraint does not influence the optimal input. For example, consider the case of simultaneous area and magnitude constraints with individually reachable regions as shown in Figure 2-3. Here $R_1 \in R_\infty$, i.e., the jointly reachable region is a subset of R_1 (due to the area constraint

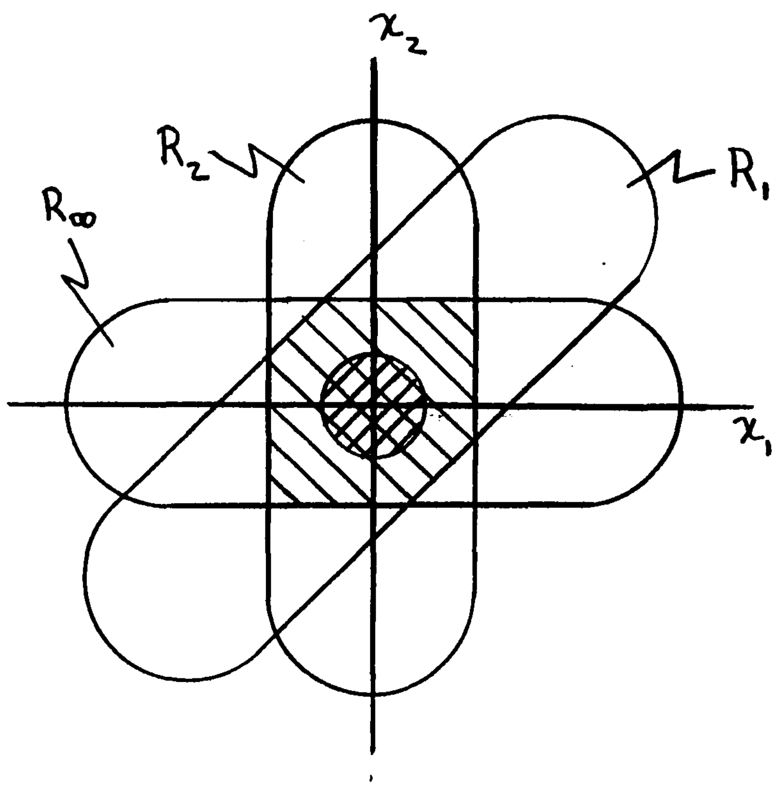


Figure 2-2 Reachable Regions with Three Constraints

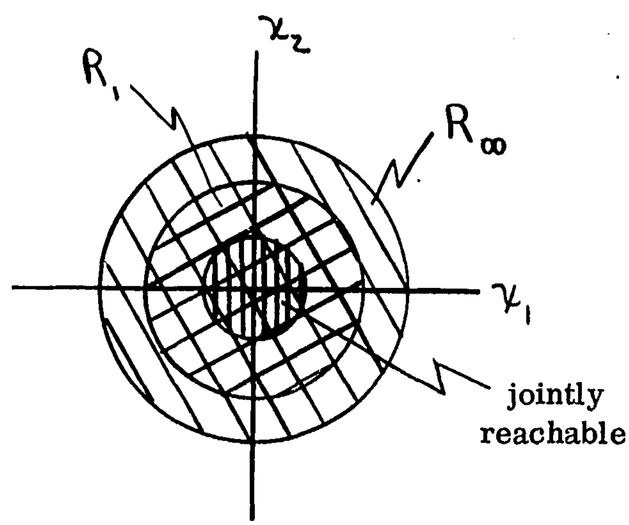


Figure 2-3 Reachable Regions when $R_1 \in R_\infty$

acting alone) which is itself a subset of R_∞ (due to the magnitude constraint acting alone). However, the optimal input will certainly not operate on the boundary of the area constraint alone since this results in an impulsive input which obviously violates the magnitude constraint. Since the individually reachable regions are obtained for inputs of different forms, and since the jointly reachable region is obtained by an input of still another form, no strong, general sufficiency conditions could be developed from the intersection of the individually reachable regions.

There are several relationships which can be used to eliminate some of the space which must be searched to determine the optimal input. First, a lower bound for T can be found. One procedure would be to find the largest values of time required for the terminal state to be in the individually reachable regions R_{it}. The value obtained for any R_{it} is obviously a lower bound, but the largest of these will certainly be the closest to the optimal time.

For terminal times T_1 such that $L_\infty T_1 \leq L_1$, the area constraint will automatically be satisfied by the magnitude constraint. Hence for value of $T \leq T_1$, the area constraint does not apply if there is a magnitude constraint. For example, the $L_1 - L_\infty$ problem reduces to an L_∞ or bang-bang problem for $T \leq L_1/L_\infty$. In a similar manner, the $L_2 - L_\infty$ problem reduces to an L_∞ problem for $T \leq (L_2/L_\infty)^2$. Another useful result occurs if $L_\infty = 1$. In this case $\|u\|_2^2 \leq \|u\|_1$ for all T and any u. If $L_1 \leq L_2^2$, then the energy constraint does not apply.

Certain lower limits on the magnitude of the components of the m-vector $\underline{\lambda}$ can be established. For example, for operation on the boundary of both the magnitude and area constraints, we have from (2.39) that $\underline{\lambda}$ must be such that

$|2\underline{\lambda}'\,\underline{h}(T,t)| > 1/L_1$ for some finite interval of time. (For definiteness, we have set $w_\infty = w_1$.) Since the impulsive response has been assumed to be known, some information concerning the components of $\underline{\lambda}$ is contained in this relationship. Certainly they all can not be infinitesimally small. This means that the $\underline{\lambda}$-vector must be large enough so that its tip lies outside of some region centered a the origin. The region itself, of course, depends upon $\underline{h}(T,t)$, t, and L_1. As a specific example, consider the simple system shown in Figure 2-4 with $L_1 = \frac{1}{2}$.

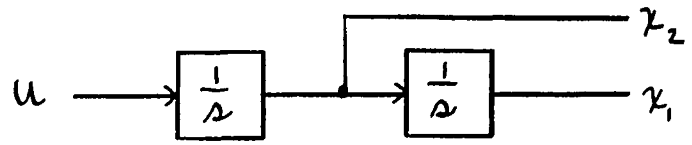


Figure 2-4

Block Diagram of a Double Integrator

We have $h_2(t) = 1$ and $h_1(t) = t$ so that $|\lambda_1(T-t) + \lambda_2| > 1$ for some range of t for the input to be non-zero. While this procedure may have some value as an aid in achieving some physical insight into the optimal solution, it appears that it is difficult to apply -- even in the simple example.

2.4 General Computational Procedure

An outline of one possible procedure for computing the optimal input will now be given. Once again, for definiteness, only area, energy, and magnitude constraints are considered. In essence, the procedure is a repeated search in m + 1 space. Since m outputs are to be controlled, one must determine the m-vector $\underline{\lambda}$. In addition, the optimal terminal time T must be found. (In the case of a magnitude constraint alone, the space is reduced by 1; if both area and energy constraints apply, the space must be increased by 1 -- a non-arbitrary weighting factor.) Hence, for any assumed constraint combination, the input is known to within about m constants independent of the order of the system. While the procedure described here may be somewhat inefficient, it is basically simple and straightforward.

The major difficulty is that one does not know a priori upon the boundary of which constraint(s) the optimal input operates (or even if a solution exists). Lacking any useful existence conditions, it is necessary to incorporate in the scheme a means to halt the computation if it becomes evident that no solution exists.

The form of the input becomes more complicated as the number of actual input constraints increases. Hence more calculation is required to evaluate the input norms to see if all the constraints are satisfied by a given input. Furthermore, if the system is oscillatory, it is usually not possible to determine the number of sign changes in the input before the outset of the calculation. However, if one considers first the magnitude constraint alone, or even simultaneous area and magnitude constraints, the input is piecewise constant -- either bang-bang or bang-off-bang, respectively. In either case, the magnitude constraint is automatically satisfied while the input area and energy can be easily calculated. In this way, a lower-bound for T and approximate values for the λ_i are readily established. Furthermore, one could use a sub-optimal solution (of the form, for example, discussed in Chapters V and VI) to determine an upper bound for T -- although this will not always work since the chosen sub-optimal solution may not exist, even if the optimal solution does. Hence by solving a relatively simple problem, one has a reasonable starting point for the solution of the problem at hand.

The procedure is outlined in Figure 2-5. This diagram is deceptively simple

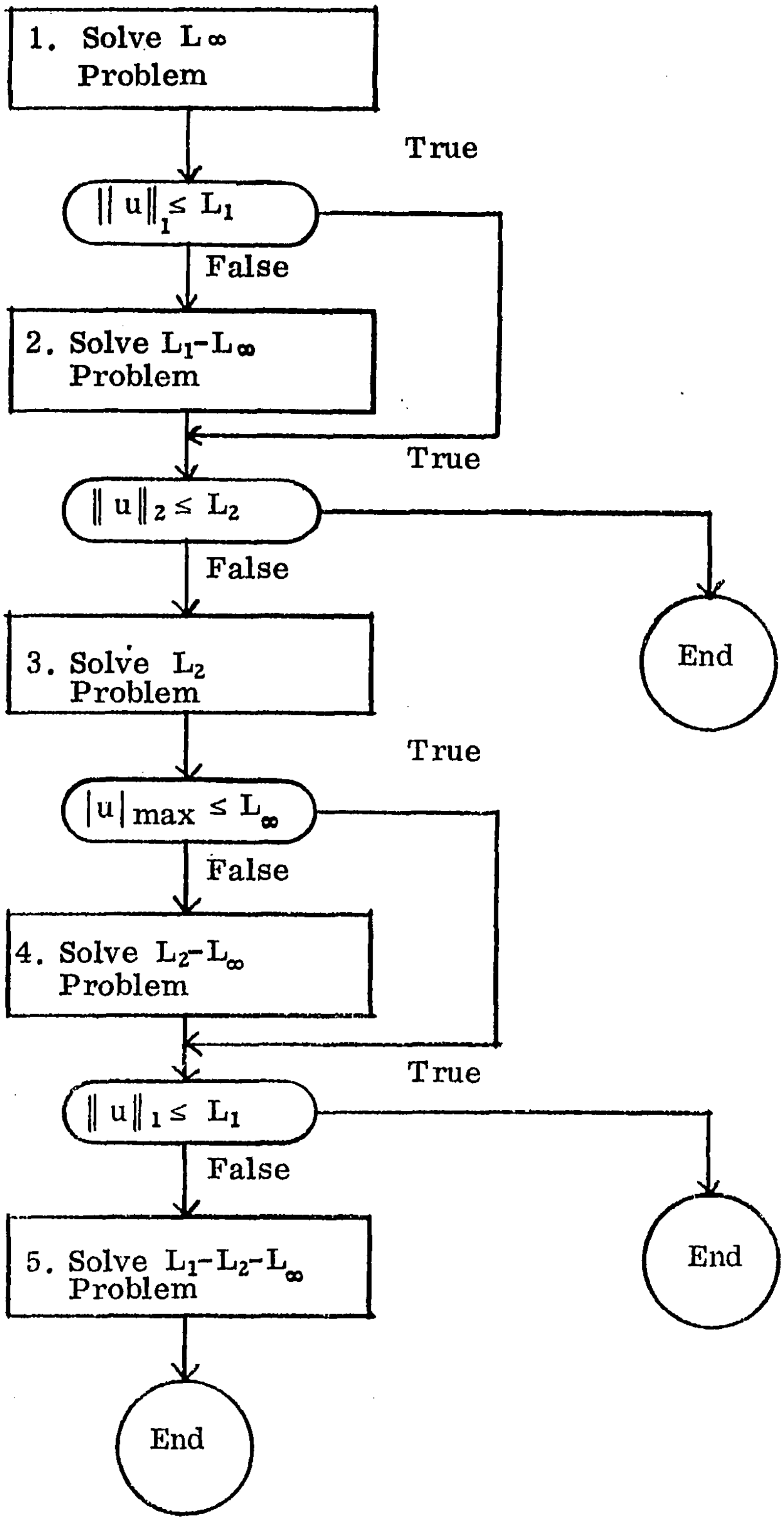


Figure 2-5 Outline of General Computational Procedure

since the rectangular "Solve . . . Problem" boxes involve considerable computation. In addition, they contain whatever logic is necessary to prevent a fruitless search for a non-existent solution. The general approach is to attempt to find the optimal solution by first considering the magnitude and energy constraints individually. (The area constraint alone leads to an impulsive input so it need not be considered.) Numerous techniques for solving each of these cases have been discussed in the literature (reference 1, Sections V and VI lists over 50 articles) and will not be considered here. The procedure outlined here is a means of incorporating previously developed computational techniques in a general scheme which should be useful when several input constraints are specified.

The procedure starts with the magnitude constraint considered by itself in Block No. 1 of Figure 2-5. Although the computation involved here would be no less than that with the energy constraint alone, the evaluation of the norms is simpler here since u is piecewise constant. Further, it may be reasonably expected that for time-optimal control, many physical systems will operate with the input at the saturation value for some -- if not most of the time. Hence while the actual optimal input with additional constraints will generally not be of the bang-bang type, this input may provide a good first approximation for T and $\underline{\lambda}$ which is a virtual necessity in many iteration procedures.

After determining the bang-bang input, $\|u\|_1$ and $\|u\|_2$ are evaluated and compared to their constraints, i.e., is $\|u\|_1 = L_\infty T \leq L_1$ and if so, is $\|u\|_2^2 = L_\infty^2 T \leq L_2^2$. If both of these conditions are satisfied, then the bang-bang solution is indeed optimal and the problem is solved. If $\|u\|_1 > L_1$, then the $L_1 - L_\infty$ problem is solved in Block No. 2 and the resulting $\|u\|_2$ is compared with L_2. If $\|u\|_2 > L_2$, either from the L_∞ -

problem or the $L_1 - L_\infty$ problem, the next attempt is to try the energy constraint by itself, the L_2 problem, as indicated in Block No. 3. The other norms are then evaluated and compared to their constraints. If $\|u\|_\infty > L_\infty$ problem must be solved as shown in Block No. 4. The area norm is then evaluated and compared with L_1. If this constraint is satisfied, the problem is solved, if not, we go to Block No. 5 where all three constraints are applied to the input.

This procedure can be simplified somewhat by the elimination of Blocks 1 and 3 and the norm comparison which follows each. These are special cases of Blocks 2 and 4, respectively, and actually involve no less computation. (It is for this reason that the $L_1 - L_2$ problem is not treated separately -- it is a special case of the three constraint problem and involves as much computation.) Although as one moves from block to block the form of the input changes, the details of the computation remain the same. The only difference is that an m-space must be searched in Block No. 1, an (m+ 1) - space in Blocks 2, 3, and 4, and an (m+ 2) - space in Block No. 5.

Because the computation time increases considerably as the dimension of the space to be searched is increased, it is desirable to solve first the simplest problem which leads to good approximations to T and $\underline{\lambda}$. For this reason, it may be useful in some cases to consider the area constraint alone as a first trial since this reduces the evaluation of the terminal conditions of (2.8) to summations. This topic is considered in more detail in Chapter III for an underdamped second order system. A sub-optimal solution such as the coast-bang-bang solution discussed in Chapters V and VI may also be useful for this purpose.

2.5 An Illustrative Example

In order to illustrate the time optimal input to a system with various input constraint combinations, as well as demonstrating the usefulness of the approach taken in the general procedure outlined in the last section, consider the following simple but non-trivial example. The single input system consisting of two integrators with no feedback as shown in Figure 2-4 is to be driven from the initial state $\underline{x}(0) = \begin{bmatrix} -17.5 \\ -1 \end{bmatrix}$ to the origin in minimum time T subject to the following constraints on the input: $\|u\|_1 \leq 6$, $\|u\|_2 \leq \sqrt{5}$, and $\|u\|_\infty \leq 1$. The numbers have been chosen so that a solution does exist and so that the input operates on the boundary of all three constraints. The system can be described by the state equations

$$\dot{x}_1 = x_2 \tag{2.53}$$

$$\dot{x}_2 = u \tag{2.54}$$

The impulse response and transition matrices are easily shown to be

$$\underline{h}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} \tag{2.55}$$

$$\underline{\phi}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \tag{2.56}$$

respectively. This leads to

$$\underline{e}(T) = -\phi(T)\underline{x}(0) = -\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -17.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 17.5 + T \\ 1 \end{bmatrix} \tag{2.57}$$

Hence

$$\begin{bmatrix} 17.5 + T \\ 1 \end{bmatrix} = \int_0^T \begin{bmatrix} T - t \\ 1 \end{bmatrix} u(t) \; dt \tag{2.58}$$

which results in

$$\int_0^T u(t) \; dt = 1 \tag{2.59}$$

$$\int_0^T t \; u(t) \; dt = -17.5 \tag{2.60}$$

These are the terminal conditions which must be satisfied by the input. Further, the constraints are such that the input must also satisfy the following:

$$\|u\|_1 = \int_0^T |u| \ dt \leq L_1 = 6 \qquad (2.61)$$

$$\|u\|_2^2 = \int_0^T u^2 \ dt \leq L_2^2 = 5 \qquad (2.62)$$

$$\|u\|_\infty = |u|_{max} \leq L_\infty = 1 \qquad (2.63)$$

In order to make maximum use of this example, all possible constraint combinations will be considered. Further, the effect of different values of L_2 will be investigated. In this way we can establish a clear picture of some of the interrelationships which exist. The results are tabulated and discussed at the of this section. Finally a simple sub-optimal solution is given and briefly discussed. This serves as an introduction to the material which follows in later chapters.

We have in general for this example

$$\underline{\lambda}'\underline{h} = \underline{\lambda}'\underline{h}(T-t) = \lambda_1(T-t) + \lambda_2 \qquad (2.64)$$

Before proceeding with this example, it should be pointed out that only if both energy and area constraints apply is it necessary to have a weighting factor other than unity. Hence in several of the cases below, $\underline{\lambda}$ can only be found to within a multiplicative constant. It is useful to include the weighting factors, however, to see if there is any relationship between them which would be useful in a general computational scheme.

<u>Area Constraint Alone</u> $\|u\|_1 \leq 6$

For this case

$$u = A\,\delta(t) - B\,\delta(t-T) \qquad (2.65)$$

It is easy to show that both A and B must be positive. This corresponds to $\underline{\lambda}'\underline{h}$ as shown in Figure 2-6. This form of $\underline{\lambda}'\,\underline{h}$ is applicable to all the cases considered here. In all the cases treated in this example it is easier to solve for the "switching times" rather than solve for $\underline{\lambda}$ directly. The constraint and terminal conditions yield

$$\begin{aligned} 6 &= A + B \\ 1 &= A - B \\ -17.5 &= -BT \end{aligned} \tag{2.66}$$

so that

$$\begin{aligned} A &= 3.5 \\ B &= 2.5 \\ T &= 7 \end{aligned} \tag{2.67}$$

From (2.28) we have that $|\underline{\lambda}'\,\underline{h}| = 1/L_1 = 1/6$ at $t = 0$ and at $t = T$, so that

$$\underline{\lambda}' = \frac{1}{21}\,[1 \;\; -3.5] \tag{2.68}$$

It is obvious that because of the impulsive nature of the input, both $\|u\|_2$ and $\|u\|_\infty \to \infty$.

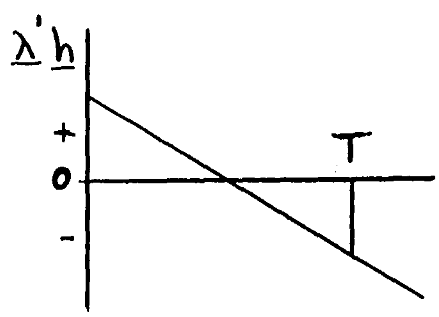


Figure 2-6

$\underline{\lambda}'\underline{h}(T-t)$ For Double Integrator Example

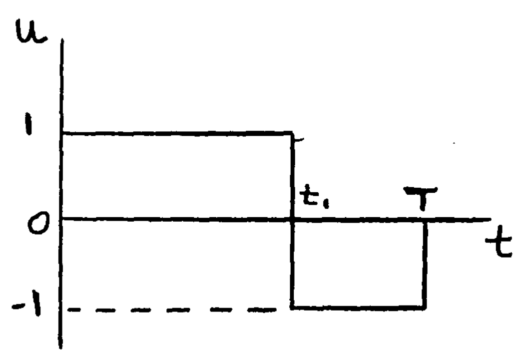


Figure 2-7

Input with Magnitude Constraint

Magnitude Constraint Alone $\|u\|_\infty = |u|_{max} \leq 1$

Here

$$u = \text{sgn}\ \underline{\lambda}'\ \underline{h} \tag{2.69}$$

and is shown in Figure 2-7. The constraint is automatically satisfied by the form of the input so that $\underline{\lambda}$ can be found only to within a constant. From the terminal conditions, we have

$$\begin{aligned} 1 &= t_1 - (T - t_1) \\ -17.5 &= t_1^2/2 - (T^2 - t_1^2)/2 \end{aligned} \tag{2.70}$$

This results in

$$\begin{aligned} t_1 &= 1 + \sqrt{18} = 5.243 \\ T &= 1 + 2\sqrt{18} = 9.485 \end{aligned} \tag{2.71}$$

and

$$\underline{\lambda}' = k\ [1 - 4.243] \tag{2.72}$$

It is easy to see that $\|u\|_1 = \|u\|_2^2 = T = 9.485$ so that both norms exceed their respective constraints.

Energy Constraint Alone $\|u\|_2 \leq \sqrt{5}$

In order to determine the relationship between L_2 and T as well as solving this specific example, this case will be solved in terms of a general L_2. We have

$$u = L_2^2\ [\lambda_1(T - t) + \lambda_2] = a - bt \tag{2.73}$$

The input is shown in Figure 2-8. The constraint and terminal conditions yield

$$\begin{aligned} L_2^2 &= a^2 T - abT^2 + b^2 T^3/3 \\ 1 &= aT - bT^2/2 \\ -17.5 &= aT^2/2 - bT^3/3 \end{aligned} \tag{2.74}$$

From the terminal conditions we have

$$a = (105 + 4T)/T^2$$
$$b = (210 + 6T)/T^3 \quad (2.75)$$

which when substituted into the constraint equation yields

$$L_2^2 = (4T^2 + 210\,T + 3675)/T^3 \quad (2.76)$$

For this particular example, $L_2^2 = 5$ which gives

$$T = 10.875 \quad (2.77)$$

and

$$\underline{\lambda}' = 0.0428\,[1 - 5.23] \quad (2.78)$$

The other norms are readily evaluated to be

$$\|u\|_\infty = 1.255$$
$$\|u\|_1 = 6.36 \quad (2.79)$$

Again, both exceed their respective constraints.

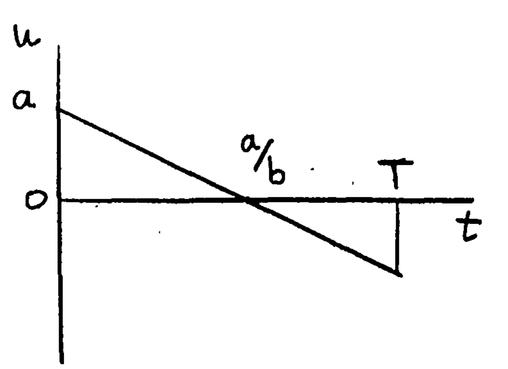


Figure 2-8

Input with Energy Constraint

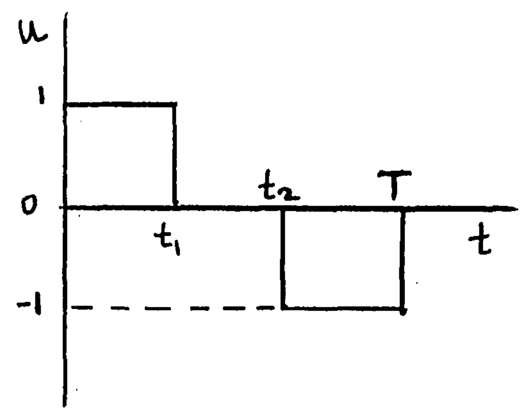


Figure 2-9

Input with Magnitude and Area Constraint

<u>Magnitude and Area Constraints</u> $\|u\|_\infty \le 1, \ \|u\|_1 \le 6$

The input is given by

$$u = \begin{cases} 0 & |\underline{\gamma}'\,\underline{h}| < 1/L_1 = 1/6 \\ \operatorname{sgn} \underline{\gamma}'\,\underline{h} & |\underline{\gamma}'\,\underline{h}| > 1/L_1 = 1/6 \end{cases} \quad (2.80)$$

where

$$\underline{\gamma} = (1 + w_\infty/w_1)\,\underline{\lambda} \quad (2.81)$$

The input is shown in Figure 2-9. The constraint and terminal conditions yield

$$
\begin{aligned}
6 &= t_1 + (T - t_2) \\
1 &= t_1 - (T - t_2) \\
-17.5 &= t_1^2/2 - (T^2 - t_2^2)/2
\end{aligned}
\tag{2.82}
$$

which when solved result in

$$
\begin{aligned}
t_1 &= 3.5 \\
t_2 &= 8.2 \\
T &= 10.7
\end{aligned}
\tag{2.83}
$$

and $\|u\|_2^2 = 6$ which exceeds the constraint. We also obtain

$$
\underline{\lambda}' = \frac{w_1}{w_1 + w_\infty} \; 0.071 \; [1 - 4.84]
\tag{2.84}
$$

Magnitude and Energy Constraints $\|u\|_\infty \leq 1,\ \|u\|_2 \leq \sqrt{5}$

In this case it is not known a priori whether or not the input saturates only on the positive side. For the particular constraint values given, however, this is the case and the optimal input is as shown in Figure 2-10. As in the case with the energy constraint alone, we will consider a general L_2 in order to determine the influence of this particular constraint in addition to solving this specific example. We have

$$
u = \begin{cases} L_2^2 \, \underline{\gamma}' \underline{h} & |\underline{\gamma}' \underline{h}| < 1/L_2^2 \\ \operatorname{sgn} \underline{\gamma}' \underline{h} & |\underline{\gamma}' \underline{h}| > 1/L_2^2 \end{cases}
\tag{2.85}
$$

where

$$
\underline{\gamma} = (1 + w_\infty / w_2)\, \underline{\lambda}
\tag{2.86}
$$

Application of the energy constraint and the terminal conditions yields

$$
T = 4 - 3\, L_2^2 + 2\sqrt{3(L_2^4 - 2\, L_2^2 + 25)}
\tag{2.87}
$$

which is valid for $4.464 < L_2^2 \leq 9.485$. For the particular example with $L_2^2 = 5$, we have

$$T = 10.908 \tag{2.88}$$

with $t_1 = 1.523$, $t_2 = 10.385$, $\|u\|_1 = 6.478$ and

$$\underline{\lambda}' = \frac{w_2}{w_2 + w_\infty} \; 0.0451 \; [1 \; - \; 4.94] \tag{2.89}$$

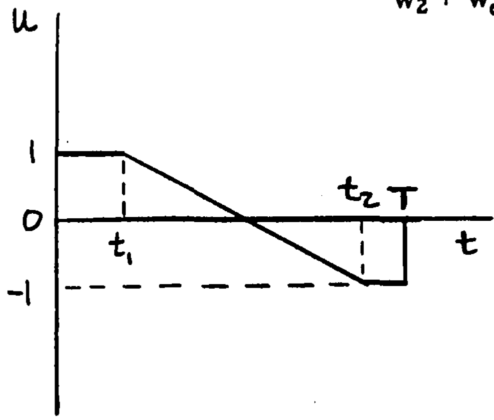


Figure 2-10

Input with Magnitude and Energy Constraints

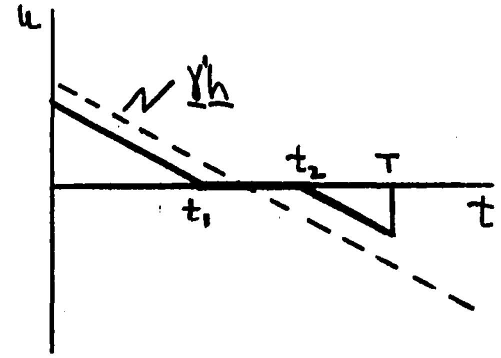


Figure 2-11

Input with Energy and Area Constra

Energy and Area Constraints $\|u\|_1 \leq 6$, $\|u\|_2^2 \leq 5$

For this case the input is

$$u = \begin{cases} 0 & |\underline{\gamma}'\,\underline{h}| \leq 1/6 \\ 5\, w_1/w_2\, [\underline{\gamma}'\,\underline{h} - \frac{1}{6}\,\text{sgn}\,\underline{\gamma}'\,\underline{h}] & |\underline{\gamma}'\,\underline{h}| \geq 1/6 \end{cases} \tag{2.90}$$

where

$$\underline{\gamma} = (1 + w_2/w_1)\,\underline{\lambda} \tag{2.91}$$

The input is shown in Figure 2-11. One obtains directly

$$T = 10.918 \tag{2.92}$$

with $t_1 = 5.235$, $t_2 = 6.493$, and $\|u\|_\infty = 1.333$. In addition it is easily shown that

$$w_1/w_2 = 0.222 \tag{2.93}$$

and

$$\underline{\lambda}' = 0.0422\,[1 \; - \; 5.15] \tag{2.94}$$

In this case the ratio of the weighting factors is not arbitrary as opposed to the previous two cases.

<u>All Three Constraints</u>

The form of the time-optimal input when it operates on the boundary of all three constraints can be expressed by

$$u = \begin{cases} 0 & |\underline{\gamma}'\,\underline{h}| < 1/6 \\ 5\,\dfrac{w_1}{w_2}\left[\underline{\gamma}'\,\underline{h} - \dfrac{1}{6}\,\mathrm{sgn}\,\underline{\gamma}'\,\underline{h}\right] & 1/6 < |\underline{\gamma}'\,\underline{h}| < \dfrac{1}{6} + \dfrac{w_2}{5\,w_1} \\ \mathrm{sgn}\,\underline{\gamma}'\,\underline{h} & |\underline{\gamma}'\,\underline{h}| > \dfrac{1}{6} + \dfrac{w_2}{5\,w_1} \end{cases} \tag{2.95}$$

where

$$\underline{\gamma} = (1 + w_2/w_1 + w_\infty/w_1)\,\underline{\lambda} \tag{2.96}$$

As in all the cases considered, $\underline{\lambda}'\,\underline{h}$ must be as shown in Figure 2-6 if the terminal conditions are to be satisfied. The resulting input is shown in Figure 2-12. It should be noted that although the input is known to saturate during the positive interval $0 < t < t_1$, it is not known a priori that the input saturates during the negative portion, i.e., another possible input form exists with the terminal time T such that $T < t_4$. This latter form however, can be shown to be valid only when $L_2^2 < 4\frac{1}{3}$.

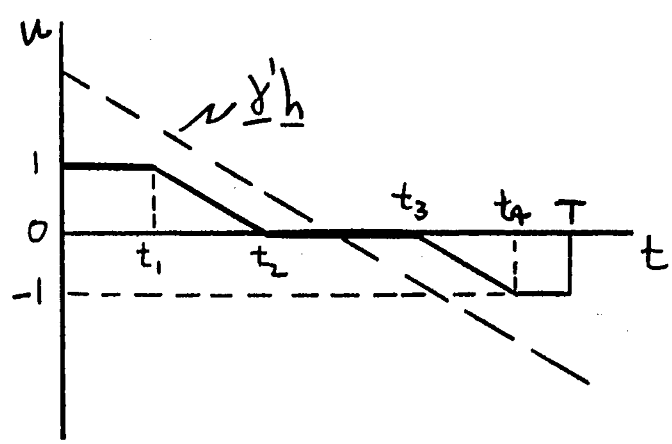


Figure 2-12

Input with All Three Constraints

From the form of the input we have

$$t_4 - t_3 = t_2 - t_1 \tag{2.97}$$

From the constraints we have

$$6 = t_1 + (T - t_4) + (t_2 - t_1) \tag{2.98}$$

$$5 = t_1 + (T - t_4) + \frac{2}{3}(t_2 - t_1) \tag{2.99}$$

The terminal condition on x_2 yields

$$1 = t_1 - (T - t_4) \tag{2.100}$$

Combining these with the terminal condition on x_1 we obtain

$$T = 11 \tag{2.101}$$

with $t_1 = 2$, $t_2 = 5$, $t_3 = 7$, and $t_4 = 10$. $\underline{\gamma}$ is easily found by using $u(t_2) = u(t_3) = 0$. This yields

$$\underline{\gamma}' = \frac{1}{6}[1 - 5] \tag{2.102}$$

From $u(t_1) = 1$ it is easily shown that $w_2/w_1 = 2.5$. Hence for $\underline{\lambda}$ we have

$$\underline{\lambda}' = \frac{w_1}{w_\infty + 3.5\, w_1}\,\underline{\gamma}' = \frac{w_1/6}{w_\infty + 3.5\, w_1}\,[1 - 5] \tag{2.103}$$

<u>Summary</u>

The results of the specific numerical example are tabulated in Table 2-1. As anticipated, the cases with a magnitude constraint result in unspecified weighting factors. In the case of the bang-bang input (magnitude constraint alone), this cannot be resolved since the amplitude of $\underline{\lambda}'\,\underline{h}$ does not influence the input. In the other cases, however, if one takes say $w_\infty = 1$, and makes the remaining weighting factor much greater than 1, it is obvious that excellent consistency is obtained for the value of $\underline{\lambda}$ in all the various cases. Actually, even making w_1/w_∞ greater than about 1 or 2 leads to good agreement. More importantly, it is obvious that

TABLE 2-1 Results of Double Integrator Example

Constraint(s) Applied: Area $L_1 = 6$	Constraint(s) Applied: Energy $L_2^2 = 5$	Constraint(s) Applied: Magnitude $L_\infty = 1$	Terminal Time T	Values of Input Norms: $\|u\|_1$	Values of Input Norms: $\|u\|_2^2$	Values of Input Norms: $\|u\|_\infty$	$\underline{\lambda'}$	$\left(\frac{w_i}{w_\infty} >> 1\right)$ $\underline{\lambda'}$
x	x	x	11.000	6	5	1	$\frac{w_1/6}{w_\infty + 3.5\, w_1}$ [1 - 5]	[0.0476 - 0.238]
		x	9.485	9.485	9.485	1	k [1 - 4.243]	k [1 - 4.243]
	x		10.875	6.36	5	1.255	0.0428 [1 - 5.23]	[0.0428 - 0.224]
x			7.000	6	∞	∞	0.0476 [1 - 3.5]	[0.0476 - 0.167]
	x	x	10.908	6.478	5	1	$\frac{0.0451\, w_2}{w_\infty + w_2}$ [1 - 4.94]	[0.0451 - 0.223]
x		x	10.700	6	6	1	$\frac{0.071\, w_1}{w_\infty + w_1}$ [1 - 4.84]	[0.071 - 0.343]
x	x		10.918	6	5	1.333	0.0422 [1 - 5.15]	[0.0422 - 0.217]

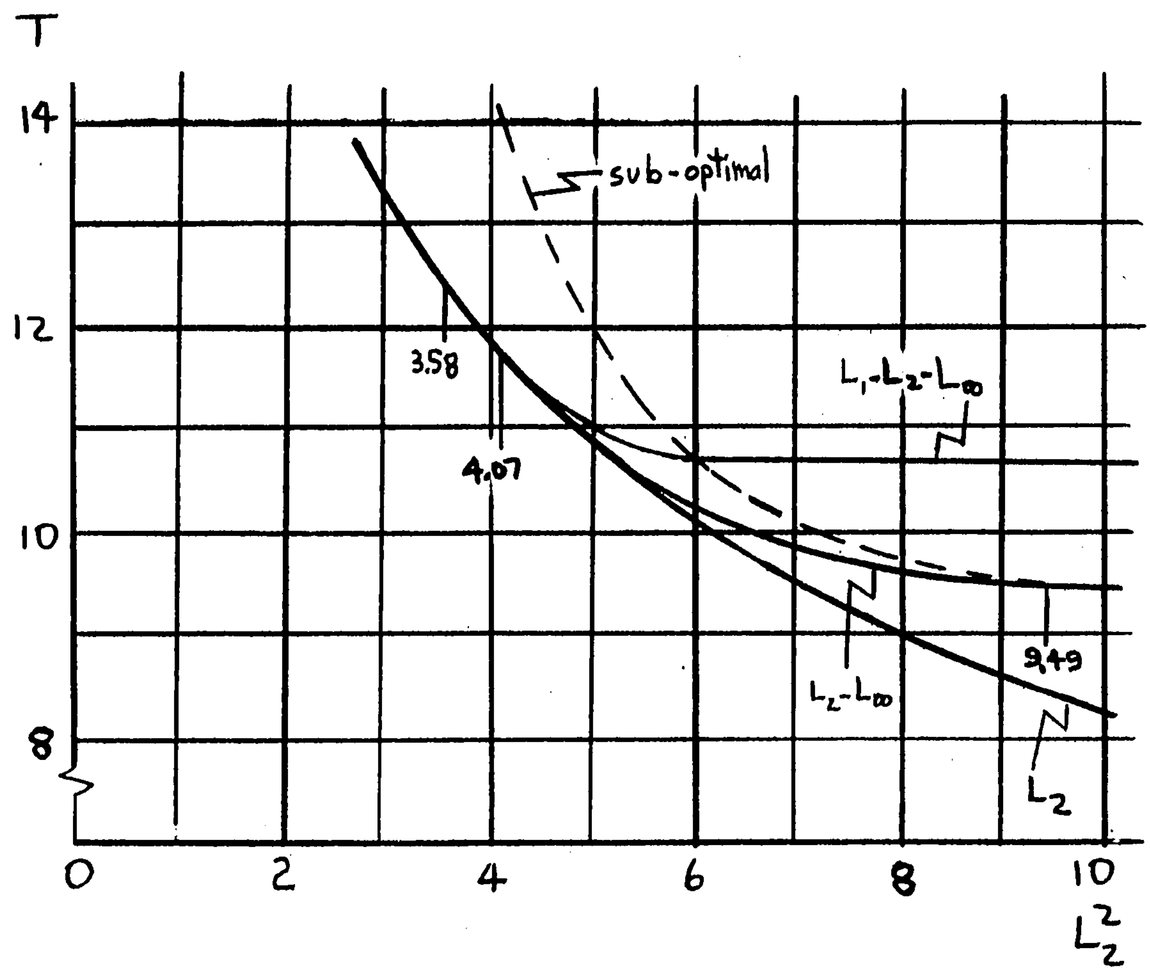


Figure 2-13 Optimal Time Versus L_2^2 for Various Constraints Combinations

the ratio of λ_2 to λ_1 is near the optimal solution value of - 5 for the cases treated, being within 5% except for the bang-bang solution (15%) and the impulsive input solution (30%). This indicates that even a crude first trial which exceeds almost all the constraints by a considerable amount can still provide very useful information for use in any general computational procedure such as the one discussed in the previous section. The weighting factors, where arbitrary, might be of some value in obtaining convergence in an iteration procedure. This topic would appear to warrant future study.

Variation of T with L_2

It is instructive to consider the minimum time solution as a function of one of the constraints. We have used a general L_2 in several places in the example treated above in order to illustrate this. If the system treated is subject to an energy constraint alone, then the relationship between T and L_2^2 is obtained from (2.76). The results are plotted in Figure 2-13 and labeled as L_2. It can be shown that for $L_2^2 > 3.58$, this form of the input violates the magnitude constraint although the area constraint remains satisfied until a somewhat larger value of L_2^2.

If the system is subject to energy and magnitude constraints then the relationship between T and L_2^2 is given by (2.87) for $4.464 \leq L_2^2 \leq 9.485$. For $L_2^2 < 4.464$, the input saturates only during the positive portion of the input. Since this is a small range, it was not investigated in detail. One point is interesting and will be discussed shortly. For $L_2^2 > 9.485$, the input reduces to the bang-bang form and the input operates only on the boundary of the magnitude constraint so that an increase in L_2 will not reduce T. The results obtained from

(2.87) are plotted in Figure 2-13 labeled as $L_2 - L_\infty$. It is easy to show that all of the points obtained using (2.87) violate the area constraint. Hence the transition between the $L_2 - L_\infty$ problem and the problem where all three constraints apply occurs for some L_2^2, $3.58 < L_2^2 < 4.464$, which corresponds to an input in the form shown in Figure 2-10 with $T < t_2$. It is possible to show that an input of this form results in $\|u\|_1 = 6$ if $L_2^2 = 4.07$ which results in $T = 11.80$.

It is convenient at this time to consider a sub-optimal input whose form is bang-off-bang as shown in Figure 2-9. With $L_\infty = 1$, $\|u\|_1 = \|u\|_2^2$ so that the relationship between L_2^2 and T is the same as that between L_1 and T in an $L_1 - L_\infty$ problem. The results obtained are shown in Figure 2-13. For the specific numerical example treated with $L_2^2 = 5$, the sub-optimal solution results in an increase in terminal time of at 8% . For values of L_2^2 greater than 6, the sub-optimal solution becomes the optimal solution since the input will not operate on the boundary of the energy constraint under this condition.

If one considers the same problem with no area constraint specified, then the sub-optimal solution is not optimal unless L_2^2 greater than 9.485. The increase in terminal time over the optimal is again small over a wide range of L_2^2.

It is also interesting to note that there is a range of L_2^2 where this sub-optimal solution might result in satisfactory performance when only an energy constraint is applicable. This range could certainly be increased if one "optimized" the value of L_∞. These matters will be pursued in greater detail in Chapter VI.

CHAPTER III

UNDERDAMPED SECOND-ORDER SYSTEM -- AREA CONSTRAINT

It is well known that an input area constraint -- with no other constraints -- leads to an impulsive input[12]. While this type of input may have little practical value in a physical system, certain useful and interesting information can be established by investigation of such an input. One of the most useful results is the establishment of a lower bound on the minimum time needed for the time-optimal control of a system subject to an input area constraint in addition to a second constraint. This is useful in any search procedure used to compute the time-optimal input for such a situation.

It is shown in this chapter that the switching surfaces for the second-order underdamped system subject to an input area constraint are straight lines in state-space. It is also shown that for most initial conditions sub-optimal impulsive inputs exist which require little or no computation and result in a small increase in the terminal or solution time. This suggests that the terminal time may not be critically dependent upon the value of the input area constraint, i.e., a substantial reduction in terminal time can only be accomplished by a considerable increase in the input area. This will be discussed in detail in Chapter V.

The discussion and analysis in this chapter are essentially geometric in nature. This results in a considerable simplification of the details of the proofs and a simple interpretation of the final results.

3.1 Mathematical Formulation

The system is described by the normalized differential equation

$$\ddot{x} + 2\zeta\dot{x} + x = u \tag{3.1}$$

where $0 < \zeta < 1$. By choosing $x_1 = x$ and $x_2 = \dot{x}_1$ as the state variables, the system can be described by the following pair of state equations

$$\dot{x}_1 = x_2 \tag{3.2}$$

$$\dot{x}_2 = -x_1 - 2\zeta x_2 + u \tag{3.3}$$

The system described by these equations is shown in Fig. 3-1. It is desired to transfer the system from the initial state $x_1(0) = x_{1o}$, $x_2(0) = x_{2o}$ to the terminal state $x_1(T) = x_2(T) = 0$ in the minimum time T subject to the input area constraint

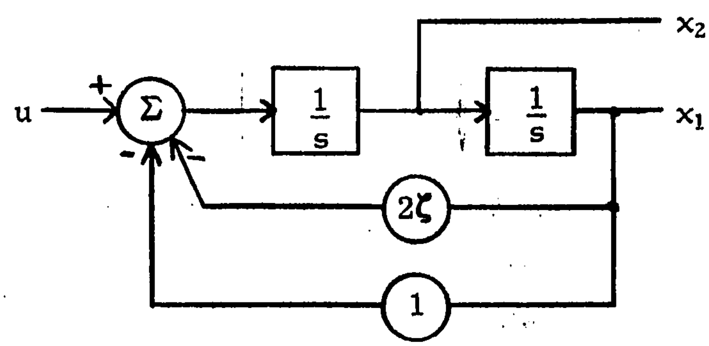


Fig. 3-1 Block Diagram of Underdamped Second Order System

$$\int_0^T |u|\, dt \leq L \tag{3.4}$$

As will be shown, the input required consists of, in general, two impulses, one positive and one negative, one occurring at time $t = T$ and one occurring at some earlier time $t = t_1$ (where t_1 may be 0).

$$u(t) = K_1\,\delta(t - t_1) - K_2\,\delta(t - T) \tag{3.5}$$

where $|K_1+K_2| = L$ and with both K_1 and K_2 positive or both negative. $\delta(x)$ is a unit impulse occurring at $x = 0$.

The impulsive response of the system is

$$h_1(t) = \frac{1}{\omega} e^{-\zeta t} \sin\omega t \tag{3.6}$$

$$h_2(t) = \frac{1}{\omega} e^{-\zeta t} (\omega \cos\omega t - \zeta \sin \omega t) \tag{3.7}$$

where

$$\omega = \sqrt{1 - \zeta^2} \tag{3.8}$$

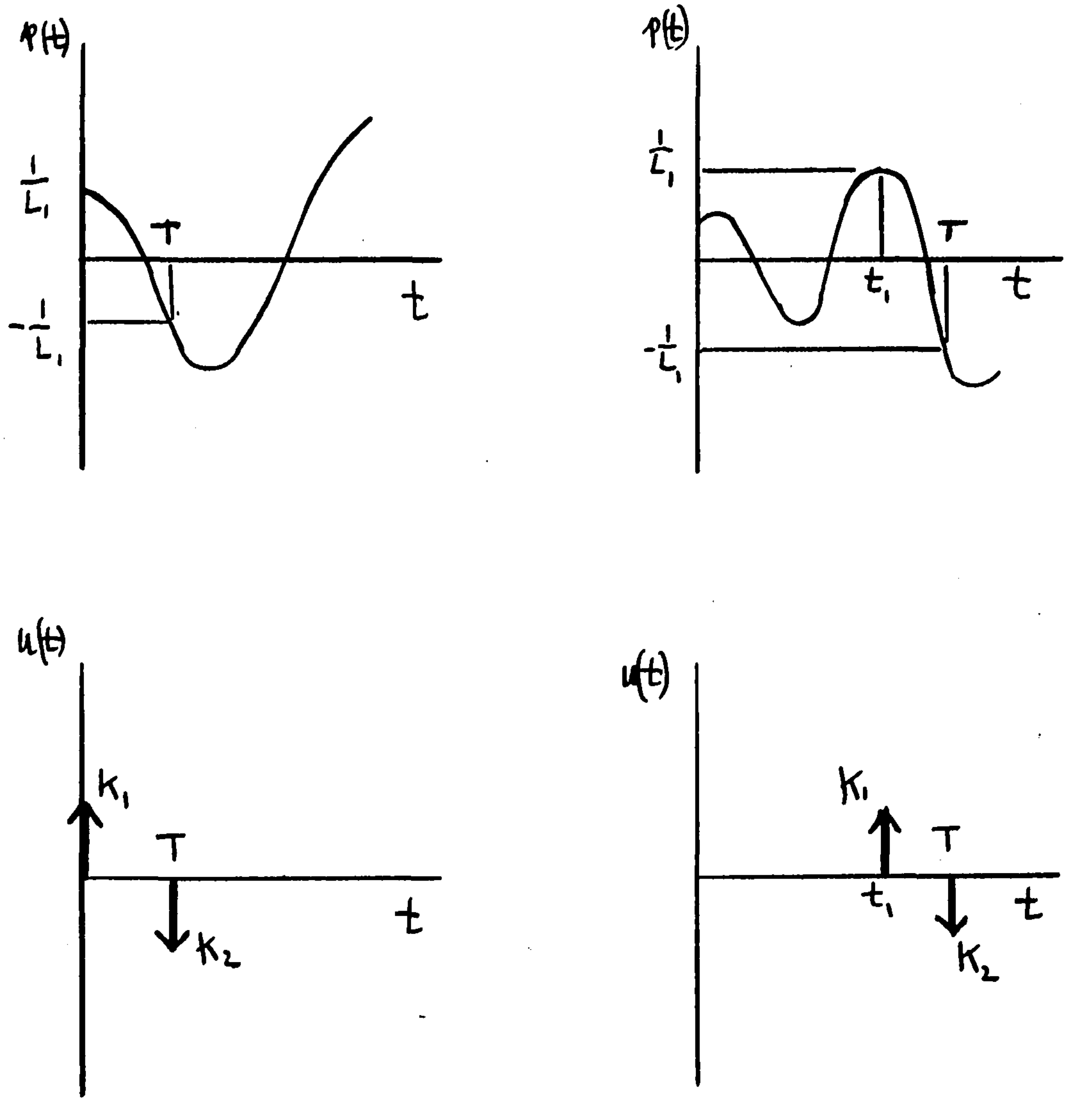
Therefore the input is determined using (2.28) where

$$\underline{\lambda}'\underline{h}\,(T-t) = p(t) = \lambda_1 h_1(T-t) + \lambda_2 h_2(T-t) = K e^{\zeta t} \cos(\omega t - \phi) \tag{3.9}$$

where λ_1 and λ_2, and K and ϕ, are constants to be determined. Figure 3-2 illustrates the only two possible inputs which satisfy the conditions imposed by the $\underline{\lambda}'\underline{h}$ function. (In this figure K_1 and K_2 have been taken as positive for the purpose of illustration). Figure 3-2a shows that one possible input consists of an impulse at $t = 0$ and a second impulse occurring at $t = T$. It is obvious from the figure that $\omega T < \pi$.

Figure 3-2b illustrates the only possible situation if there is no input impulse at $t = 0$. The time between pulses is again such that $\omega(T - t_1) < \pi$. The results of the second situation can be used to develop the input required in the first case. Hence the general case requiring two impulses, with neither occurring at $t = 0$, will be considered first. The only computational problem involved is finding $\theta = \omega(T - t_1)$ by solving a transendental equation of the form

$$e^{b\theta} \cos(\theta + \delta) = \omega \tag{3.10}$$



(a) (b)

Fig. 3-2 p(t) and Possible Inputs

where b, δ, and ω are known constants in terms of ζ. This equation can be solved for θ to any desired degree of accuracy. Within this limitation, the solution which is obtained in the next section is exact.

3.2 Determination of Optimal Input

Consider first initial conditions and an input constraint that lead to p(t) corresponding to Fig. 3-2b, i.e., no input impulse at t = 0. The local maxima and minima of p(t) can be easily found by differentiating (3.9):

$$\frac{dp}{dt} = 0 = K e^{\zeta t_m} [-\omega \sin(\omega t_m - \phi) + \zeta \cos(\omega t_m - \phi)] \tag{3.11}$$

or

$$\omega t_m - \phi = \tan^{-1} \frac{\zeta}{\omega} \triangleq \delta \tag{3.12}$$

where the angle is in the first quadrant (see Fig. 3-3). Substitution of (3.12) into (3.9) yields

$$p(t) = K e^{\zeta t} \cos(\omega t - \omega t_m + \delta) \tag{3.13}$$

and

$$p(t_m) = K e^{\zeta t_m} \cos\delta = K e^{\zeta t_m} \omega \tag{3.14}$$

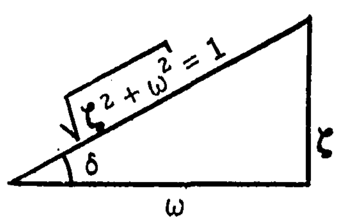


Fig. 3-3 Determination of δ

Since $p(T) = -p(t_1)$, by letting $t_m = t_1$ we obtain

$$p(T) = K e^{\zeta T} \cos(\omega(T-t_1) + \delta) = -K e^{\zeta t_1} \qquad (3.15)$$

or

$$e^{\zeta(T-t_1)} \cos(\omega(T-t_1)+\delta) = -\omega \qquad (3.16)$$

Letting $\theta = \omega(T - t_1)$ we obtain the desired result

$$e^{\zeta(\theta/\omega)} \cos(\theta + \delta) = -\omega \qquad (3.17)$$

The value of θ lies in the range

$$\frac{\pi}{2} - \delta < \theta < \pi \qquad (3.18)$$

Since ω and hence δ are known in terms of ζ, θ can be found in terms of ζ. A short table of results is given in Table 3-1. It is to be noted that for those

TABLE 3-1 Angle Between Impulses as a Function of the Damping Ratio

Damping Ratio ζ	Angle Between Impulses (degrees)
0.1	135.89
0.2	118.44
0.3	104.97
0.4	93.20
0.5	82.22
0.6	71.43
0.7071	59.50
0.8	48.15
0.9	33.37
0.95	23.37
0.99	10.38

initial conditions and input constraints for which the input is of this form, the interval between impulses depends only upon the parameter ζ, the damping ratio. It is completely independent of the constraint and the initial conditions. This fact leads to very simple switching surfaces (or switching lines since this system is completely described in two-space).

Figure 3-4 shows a state-space diagram for some given value of damping ζ with the corresponding angle θ between pulses as calculated from Eq. 3.17. Non-orthogonal axes are used as shown since this results in a trajectory which is a logarithmic spiral[20]. There is no loss in generality by assuming L = 1 since the magnitude of the states can easily be appropriately scaled.

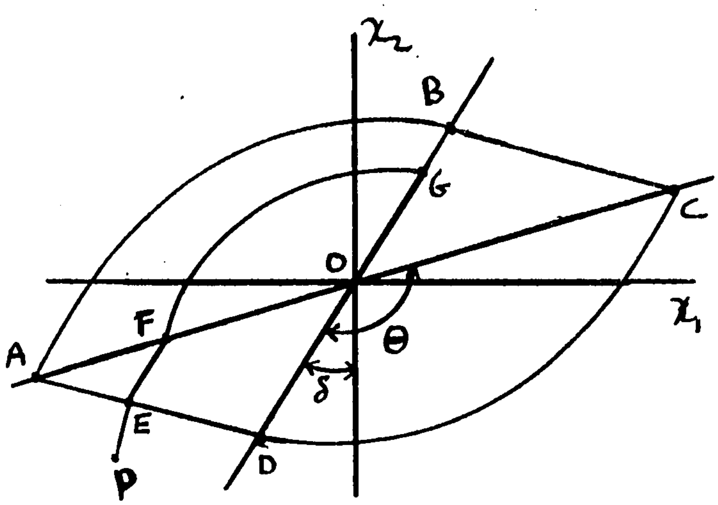


Fig. 3-4 Typical Trajectory with Two Input Impulses

The point A is the initial condition which results in zero input at $t = t_1 = 0$ and the full allowed input area at t = T by means of a negative impulse. The trajectory is ABO where AB is a logarithmic spiral and distance BO is unity (L). The symmetrica point which results in a positive impulse at t = T is point C with trajectory CDO where again distance DO is 1.

It will now be shown that an input (whose value will be determined in the next section) should be applied at t = 0 only if the initial state is in the region whose boundary consists of the spiral segments AB and CD and the straight line segments BC and DA. Furthermore it will be shown that for an initial state outside this region, say for example point P, the system should be allowed to coast until the state is on one of the straight line segments, say point E. At this time an impulse of strength of distance EF should be applied, i.e., the system is driven to line AOC. The system then again coasts for an angle θ along the spiral arc FG. At this time an impulse of strength of distance GO is applied. To prove that this is the optimal input, it is sufficient to show that only points on line AD (or BC) can be driven to the origin by two impulses of opposite sign whose total strength is 1 and which are applied at an angle of θ apart.

Let us concern ourselves only with segment AD since the other case is symmetrical. Obviously, the state of the system must be on line AO after application of the first impulse at $t = t_1$. Equally clear is the fact that the initial states which can be driven to line AO by the first (positive) impulse are such that $x_{10} < 0$ and x_{20} lies below line AO. Consider the general trajectory EFGO. Let distance GO = k where $0 < k < 1$. Distance OF is then

$$\widehat{OF} = k e^{\zeta\theta/\omega} \tag{3.19}$$

Referring to Fig. 3-5 for the rectangular coordinates of point F in space y_1-y_2

$$y_{1F} = -\widehat{OF}\cos(\theta - \delta - 90^{\circ}) = -\widehat{OF}\sin(\theta - \delta) \tag{3.20a}$$

$$y_{2F} = -\widehat{OF}\sin(\theta - \delta - 90^{\circ}) = \widehat{OF}\cos(\theta - \delta) \tag{3.20b}$$

Point E' is found by going a distance $1 - k$ in the $-x_2$ direction. We seek to show that point E' is the same as point E, i.e., it lies on line AD.

$$y_{1E'} = y_{1F} - (1 - k)\sin\delta \tag{3.21a}$$

$$y_{2E'} = y_{2F} - (1 - k)\cos\delta \tag{3.21b}$$

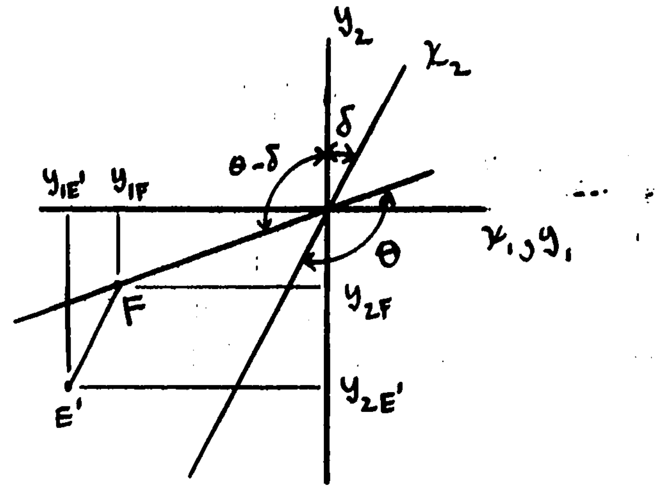


Fig. 3-5 Determination of Switching Line

Using (3.19) with (3.20) in (3.21) we obtain

$$y_{1E'} = -\sin\delta - k(e^{\zeta\theta/\omega}\sin(\theta-\delta) - \sin\delta) \tag{3.22a}$$

$$y_{2E'} = -\cos\delta + k(e^{\zeta\theta/\omega}\cos(\theta-\delta) + \cos\delta) \tag{3.22b}$$

For a given value of ζ, $y_{1E'}$ and $y_{2E'}$ are linearly related to k. Hence, by eliminating k from (3.22), we see that $y_{1E'}$ is linearly related to $y_{2E'}$. The "end points" of this straight line are easily established by taking k = 0 and k = 1. First, for k = 0

$$y_{1E'}(0) = -\sin\delta \tag{3.23a}$$

$$y_{2E'}(0) = -\cos\delta \tag{3.23b}$$

These are the coordinates of point D. Now, with k = 1 we have

$$y_{1E'}(1) = -e^{\zeta\theta/\omega}\sin(\theta-\delta) \tag{3.24a}$$

$$y_{2E'}(1) = e^{\zeta\theta/\omega}\cos(\theta-\delta) \tag{3.24b}$$

These are the coordinates of point A. Therefore point E' lies on line AD and hence is the same as point E. Thus it has been established that only those states on lines AD and BC can be driven to the origin by two impulses of opposite sign, with total strength 1, and applied at an angle θ apart.

It has further been established that the "switching surfaces" for initial states outside of region ABCDA are the straight line segments AD and BC along with the x_2-axis BOD. Furthermore, the strength of the impulse applied at $t=t_1$ should be such so as to drive the system from line AD (or BC) to line AOC. The system then coasts until $x_1=0$ (line BOD) and then is driven to the origin by an impulse using the remaining input area.

If the initial state lies in the region ABCDA, an input is required at $t=0$ and a second of opposite sign at $t=T$ where $\omega T = \psi < \theta$. Since the region is symmetrical about the origin, only the region ABODA need be discussed. The input in this case will be a positive impulse followed by a negative impulse. Letting the second impulse have a strength k, the required input is of the form

$$u(t) = (1-k)\,\delta(t) - k\delta\,(t-T) \qquad (3.25)$$

The trajectory from some arbitrary initial state E' in Fig. 3-6 would then be straight line E'F' at $t = 0$ due to the impulse $1-k$, the spiral arc F'G' where the system coasts until $x_1=0$ at point G'. Distance OG' is of course equal to k.

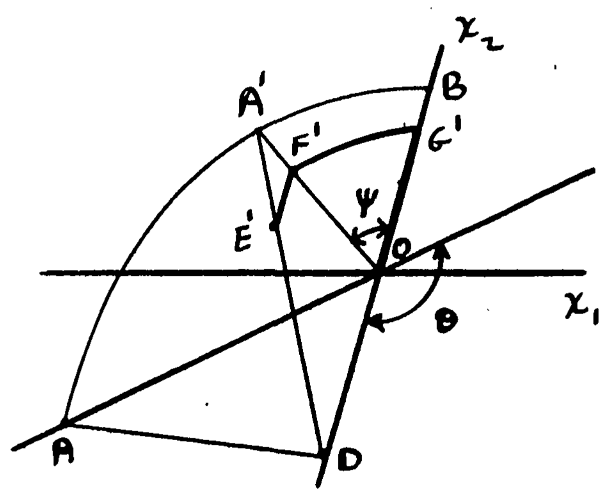


Fig. 3-6 Determination of Switching Line for Initial Conditions Near Origin

Consider for the moment the locus of all possible initial states which require a given value of $\psi = \omega T$. This is precisely the same problem which is solved above where the angle between impulses was θ. By using exactly the same argument, the locus is easily shown to be the straight line A'D. The required values for k and $T = \psi/\omega$ are again easily found geometrically:

Through point D and the initial state draw a straight line, thereby establishing point A'. (Arc AB has of course been previously determined in order to establish that an input is required at $t = 0$.) Construction of straight line A'O now allows the determination of the angle ψ and the distance E'F' which is the strength of the impulse to be applied at $t = 0$. The system then coasts until $x_1 = 0$ at which time it is driven to the origin with the remaining area.

3.3 Sub-Optimal Inputs

A simple sub-optimal impulsive input requiring no calculations and resulting in only a small increase in terminal time for most initial states is the following: Apply no input until $x_1 = 0$ and $-1 \leq x_2 \leq 1$. At this time the system is driven to the origin by an impulse of appropriate strength. For initial states outside of region ABCDA, this would result in a trajectory, for example, of PEHO in Fig. 3-7 instead of PEFGO. This would result in an increase in terminal time corresponding to angle AOE $= \rho$ and a decrease in the required area from distance BO = 1 to distance HO < 1. The maximum value of ρ would be angle AOD which is $180^\circ - \theta$. The corresponding maximum decrease in required area would be $1 - e^{\zeta \pi/\omega}$. The percentage increase in time will in general be smaller for initial states which are relatively far from the origin. However it would be meaningless to investigate this in detail since there is no such thing as a "typical" initial state.

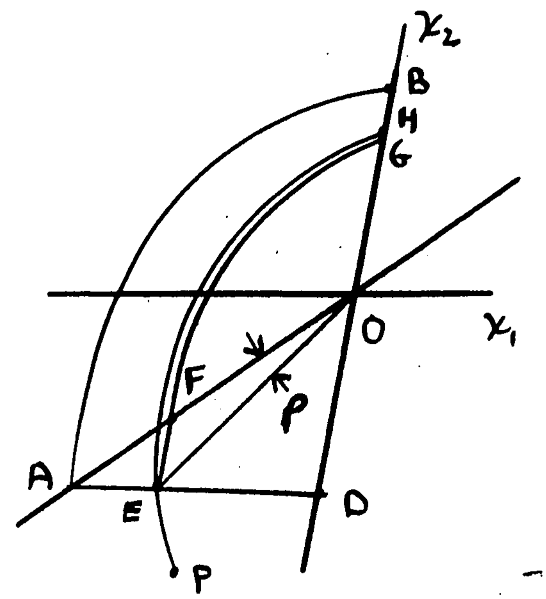


Fig. 3-7 Sub-Optimal Switching Lines

The same sub-optimal input could also be used for initial states inside region ABCDA. However, by letting the system coast until $x_1 = 0$, the terminal time could be increased by as much as $\Delta t = \pi/\omega$. The percentage increase could be very large for initial states with $x_{1o} \approx 0$ and $x_{2o} \approx -1$ in region ABODA, for example. This still is the simplest input however since no other input consisting of a single impulse could drive the system to the origin. Furthermore, as stated above, no calculation is involved in the determination of this input (other than determining when $|x_2| < 1$ when $x_1 \equiv 0$) and the value of the initial state does not affect the switching lines.

If one allows for two impulses, then it would seem reasonable to use the optimal solution for initial states outside of region ABCDA since there doesn't appear to be any two impulse input which is simpler than the optimal input. However, for initial states within region ABCDA the switching surfaces depend upon the initial states. There appears to be no simple two impulse input for this region which does not involve considerable computation -- more than the optimal solution. Therefore it would seem fruitless to use any sub-optimal input for this region other than the single impulse input discussed previously. Hence a single input impulse is the most satisfactory sub-optimal solution since it involves no computation and leads to only a small increase in the terminal time except for some initial states near the origin. If these states are of primary concern, then the optimal solution is desirable.

Conclusion

It has been shown that the optimal input to an underdamped second order system required to drive the system to the origin from an arbitrary initial state subject to an input area constraint is easily established geometrically in state space. It has also been shown that the required switching surfaces for all but a small region of initial states are independent of the initial state. Furthermore, a simple sub-optimal input consisting of a single impulse exists and results in only a small increase in the terminal time (except again for the same small region of initial states.) While this chapter yields useful information as to the lower bound on terminal time if the input is subject to an area constraint plus an additional constraint, it has in addition some direct application. In particular, if the system time constants are large, it may be possible to consider an input with area and magnitude (or energy) constraints as effectively an impulse, i.e., when the system response time is long compared to the duration of the input, the exact form of the input is unimportant and it may be considered to be impulsive with little loss in accuracy. Furthermore, the results imply that when the area is constrained in a regulator problem, it is advisable to let the system coast "as long as possible" and then drive it to the origin using the full input available. This will form the basis for one of the sub-optimal solutions considered in Chapter V.

CHAPTER IV

SECOND ORDER SYSTEM WITH NO DAMPING (OSCILLATOR)

This chapter is concerned with the simplest system in which the maximum number of zero-crossings of the $\underline{\lambda}'\underline{h}$ function is not known a priori: a simple harmonic oscillator. Because of this fact, it is not always possible to find explicit relationships for the minimum solution time in terms of the initial state and the constraints. In the cases where only a single constraint is applied to the input, it is possible to develop relationships between the optimal terminal time T and the constraint L_i. For the area constraint alone, analytic expressions are readily found giving T in terms of L_1 and vice versa. For the energy constraint alone, an analytical relationship for L_2 in terms of T is derived. For the magnitude constraint alone, a graphical representation of the switching lines in the state space enables the calculation of T for a given L_∞. This particular case has recieved considerable attention in the literature[2,4] and is discussed only briefly here. When more than one constraint is involved, the problem becomes quite complex. In the case of area and magnitude constraints, it is possible to find T and L_1 for a given initial state and for a given L_∞ if the portion of a cycle that the input is in saturation is specified. This is most easily accomplished by the use of appropriate switching lines in the state space. A reduction of the portion of a cycle in saturation reduces L_1 and increases T. A similar procedure in the case of energy and magnitude constraints is not generally possible unless the system reaches the terminal state during a saturation portion of the input. The other combinations of these constraints again lead to no general relationships. Outside of the magnitude constrained case noted above, the only case to receive any detailed attention in the literature is that of the combination of area and magnitude constraints[12,13,18]. Since the resulting bang-off-bang input is the basis for the sub-optimal scheme presented in Chapters V and VI, this case is analyzed here in some detail.

4.1 Mathematical Formulation

The system considered here is illustrated in Figure 4-1.

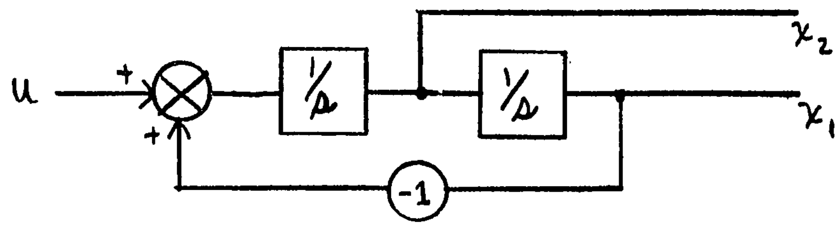


Figure 4-1. Block Diagram of Oscillator

It is described by the state equations

$$\dot{x}_1 = x_2 \qquad (4.1)$$

$$\dot{x}_2 = - x_1 + u \qquad (4.2)$$

The impulse response and transition matrix are easily shown to be, respectively,

$$\underline{h}(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \qquad (4.3)$$

$$\phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \qquad (4.4)$$

Let it be desired to drive the system from some arbitrary initial state $\underline{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ to the origin in minimum time T subject to various constraint combinations on the input. Making use of (2.6) thru (2.8) we obtain

$$\underline{e}(T) = - \phi(T)\underline{x}(0) = - \begin{bmatrix} x_{10}\cos T + x_{20}\sin T \\ -x_{10}\sin T + x_{20}\cos T \end{bmatrix} = \int_0^T \begin{bmatrix} \sin(T-T) \\ \cos(T-t) \end{bmatrix} u(t)\, dt \qquad (4.5)$$

The resulting pair of equations can be solved to yield

$$x_{10} = \int_0^T \sin t \; u(t) \; dt \qquad (4.6)$$

$$x_{20} = -\int_0^T \cos t \; u(t) \; dt \qquad (4.7)$$

In addition, we have

$$\underline{\lambda}'\underline{h} = \underline{\lambda}'\underline{h}(T-t) = \lambda_1 \sin(T-t) + \lambda_2 \cos(T-t) = K \cos(t-\psi) \qquad (4.8)$$

The constraints which are to be considered, individually and in various combinations, are

$$\|u\|_1 = \int_0^T |u| \; dt \le L_1 \qquad (4.9)$$

$$\|u\|_2^{\,2} = \int_0^T u^2 \; dt \le L_2^{\,2} \qquad (4.10)$$

$$\|u\|_\infty = \max |u| \le L_\infty \qquad (4.11)$$

4.2 Area Constraint

Since from (2.28) it is necessary that $|\underline{\lambda}'\underline{h}| \le 1/L_1$ with equality at at least one instant of time, we have as a possibility $K = 1/L_1$ with impulses occurring at $t = \psi + n\pi$ where n is an integer as shown in Figure 4-2.

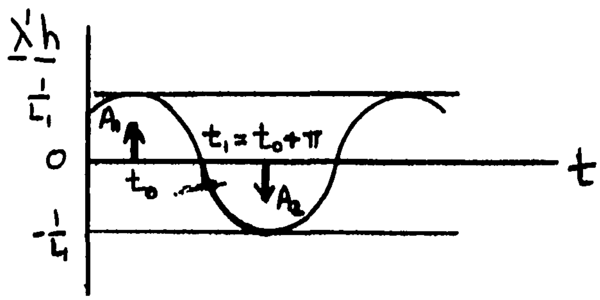


Figure 4-2. Possible Input to Oscillator with Area Constraint

Since the trajectory will be a circular arc between pulses, it is clear that there will be no solution unless

$$L_1 \ge R_o = \sqrt{x_{10}^{\,2} + x_{20}^{\,2}} \qquad (4.12)$$

Further, an input as shown in Figure 4-2 will not be optimal since the same terminal state will be reached at time t_o if an impulse of strength $A_1 + A2$ is applied at $t = t_o$. It is equally clear that, at most, two impulses are required and that these are to be applied at $t = 0$ and $t = T < \pi$ as shown in Figure 4-3. Only, if $L_1 = R_o$ will there be no impulse at t=0 (unless of course $x_{10} = 0$ in which case $T = 0$).

It is readily established that the impulse applied at $t = 0$ should be positive if $x_{10} \leq 0$. This is most easily shown by working backwards from the origin when a negative pulse is applied at $t = T$ resulting in a "trajectory" at $t = T$ of line AO in Figure 4-4, for example.

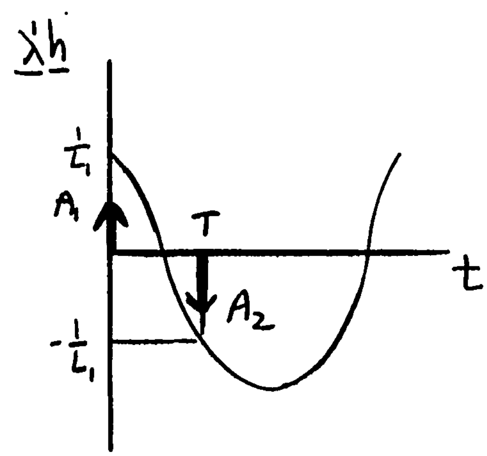


Figure 4-3. Optimal Input to Oscillator with Area Constraint

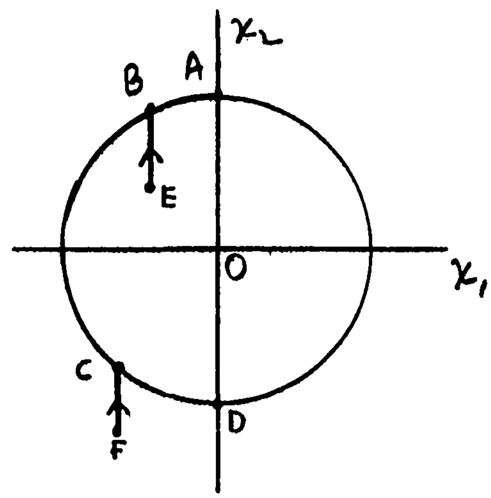


Figure 4-4. Typical Trajectory

The circular arc ABCD is the locus of points which can reach point A in a time $t < \pi$. Since the impulse at $t = 0$ must be of opposite sign as that at $t = T$, we must find those points which can reach this arc by means of a positive impulse. Two such points are E and F. This clearly shows that for $x_{10} < 0$, the input must be a positive impulse at $t = 0$ and a negative impulse at $t = T$. The converse is obviously true for $x_{10} > 0$. Hence we can write

$$u = \left[-A_1 \, \delta(t) + A_2 \, \delta(t-T) \right] \text{sgn } x_{10} \qquad (4.13)$$

where A_1 and A_2 are positive and

$$A_1 + A_2 = L_1 \qquad (4.14)$$

The determination of A_1, A_2, and T can be accomplished by substituting (4.13) in (4.6) and (4.7) which yields

$$x_{10} = A_2 \sin T \operatorname{sgn} x_{10} \qquad (4.15\ a)$$

$$x_{20} = \left[A_1 - A_2 \cos T\right] \operatorname{sgn} x_{10} \qquad (4.15\ b)$$

These can be written as

$$|x_{10}| = A_2 \sin T \qquad (4.16\ a)$$

$$x_{20} \operatorname{sgn} x_{10} = A_1 - A_2 \cos T \qquad (4.16\ b)$$

Using (4.14) we have

$$x_{20} \operatorname{sgn} x_{10} = L_1 - A_2(1 + \cos T) = L_1 - \frac{|x_{10}|}{\sin T}(1 + \cos T)$$

$$= L_1 - \frac{|x_{10}|}{\tan T/2} \qquad (4.17)$$

which can be written as

$$\tan \frac{T}{2} = \frac{|x_{10}|}{L_1 - x_{20} \operatorname{sgn} x_{10}} \qquad (4.18)$$

where of course, $0 < T/2 < \pi/2$.

For a given L_1, T is found from (4.18). Then, from (4.16 a) and (4.14) we have

$$A_2 = |x_{10}| / \sin T \qquad (4.19)$$

$$A_1 = L_1 - A_2 \qquad (4.20)$$

A_1 can be expressed directly in terms of L_1 if T is eliminated from (4.16):

$$A_1 = \frac{L_1^2 - R_o^2}{2\,(L_1 - x_{20} \operatorname{sgn} x_{10})} \qquad (4.21)$$

4.3 Magnitude Constraint

This case has received considerable attention by various investigators (see, for example, reference 1 for a detailed list). The time optimal input is the well-known "bang-bang" input

$$u = L_\infty \operatorname{sgn} \lambda' \underline{h}(T - t) \qquad (4.22)$$

It is clear from (4.22) and (4.8) that the input is alternately positive and negative with the duration of each full piecewise constant interval equal to π. The first and last intervals may be less than π. It can be shown that the trajectories for $u = + L_\infty$ are circular arcs with a common center at $(x_1, x_2) = (L_\infty, 0)$ while those for $u = -L_\infty$ are circular arcs with a common center at $(-L_\infty, 0)$. The only two arcs which pass through the origin, i.e., reach the terminal state, are those whose radius is L_∞. Since the terminal interval must be less or equal to than π, these arcs are semi-circles. The trajectory during the final interval must correspond to a portion of one of these arcs. Working backwards from the origin with the knowledge that each full interval lasts for π seconds, the switching lines shown in Figure 4-5 are easily constructed.

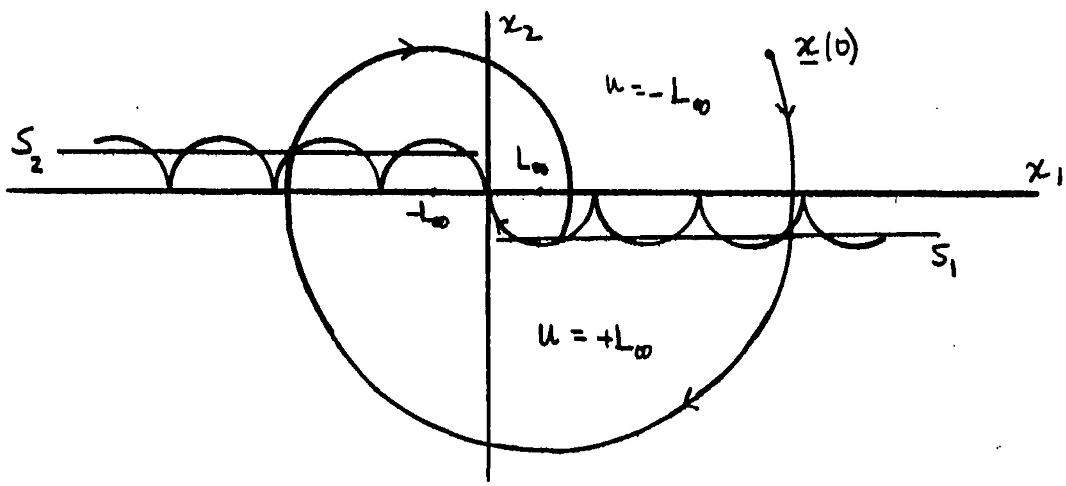


Figure 4-5. Optimal Bang-Bang Switching Lines

The switching lines are semicircles which are uniformly spaced along the x_1-axis. Above the switching lines, the input should be $u = - L_\infty$ and below the lines, $u = + L_\infty$. A trajectory for some arbitrary initial state in the first quadrant is also shown in Figure 4-5. It is clear that for a given initial state, the actual switching points lie on the two straight lines S_1 and S_2. Different initial states will of course in general lead to different pairs of straight lines. It should be noted that if these lines are extended to the x_2-axis, the last portion of the trajectory must violate these simple switching lines.

Although one can develop equations which could be used to determine

T in terms of L_∞ , it is clear that this yields no more information than does the graphical construction. It is easily shown that an increase in L_∞ will result in a decrease in T. It is also apparent that the "inverse" problem is most easily solved by successive approximations, i.e., to find the value of L_∞ which drives the system from some given initial state to the origin in a given terminal time T, the easiest procedure is to assume an L_∞ and solve for T. If T is too large, decrease L_∞ ; if T is too small, increase L_∞ .

The straight line switching curves involve knowledge of the initial state and some computation. The ease of implementation over that of the semi-circular switching curves offsets this to some extent. The semi-circular arcs would be more difficult to implement in an engineering design but do not depend upon the initial condition. Perhaps a reasonable compromise would be a sub-optimal switching line such as shown in Figure 4-6.

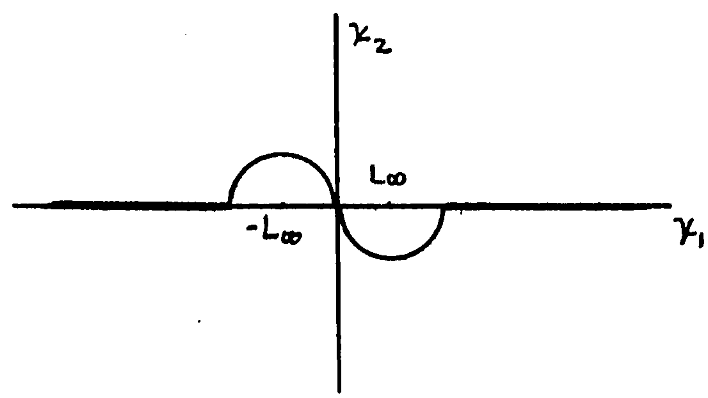


Figure 4-6. Sub-Optimal Bang-Bang Switching Lines

It can be shown[2] that this switching line results in a maximum degradation in time of about 4.3% and this, in general, decreases toward zero for initial states far from the origin.

4.4 Energy Constraint

For this case we have

$$u = L_2^{2} \lambda' h = A \sin t + B \cos t \qquad (4.23)$$

By direct substitution of this input in (4.6), (4.7), and (4.10) we obtain

$$x_{10} = A \left[\frac{T}{2} - \frac{\sin 2T}{4} \right] + B \, \frac{1-\cos 2T}{4} \qquad (4.24)$$

$$x_{20} = A \, \frac{\cos 2T - 1}{4} - B \left[\frac{T}{2} + \frac{\sin 2T}{4} \right] \qquad (4.25)$$

$$\|u\|_2^{2} = \frac{A^2 + B^2}{2} T + \frac{B^2 - A^2}{4} \sin 2T - \frac{AB}{2} (\cos 2T - 1) \qquad (4.26)$$

One can find A and B in terms of T and the initial state (the solution of the "inverse" problem) from (4.24) and 4.25). This yields

$$A = \frac{x_{10}(2T + \sin 2T) + x_{20}(1 - \cos 2T)}{T^2 - \sin^2 T} \qquad (4.27)$$

$$B = -\frac{x_{10}(1 - \cos 2T) + x_{20}(2T - \sin 2T)}{T^2 - \sin^2 T} \qquad (4.28)$$

Substitution of these into (4.26) yields $\|u\|_2^{2}$ in terms of T. This is the solution of the minimum energy problem for a given T. Since $\|u\|_2^{2}$ decreases as T increases, the minimum time problem can be solved by successive approximations. This would appear to be far simpler than the "direct" solution of the non-linear equation which results from the substitution of (4.27) and (4.28) into (4.26) with $\|u\|_2 = L_2$.

In order to establish a reasonable initial guess or trial for T, consider those conditions for which $T + n\pi$ leads to an optimal solution (where n is an integer). The equations reduce to

$$\|u\|_2^{2} = \frac{A^2 + B^2}{2} n\pi \qquad (4.29)$$

$$A = \frac{2\, x_{10}}{n\pi} \qquad (4.30)$$

$$B = -\frac{2\,x_{20}}{n\pi} \tag{4.31}$$

which results in

$$\|u\|_2^{\,2} = \frac{2}{n\pi}\,(x_{10}^{\,2} + x_{20}^{\,2}) = \frac{2}{n\pi}\,R_o^{\,2} \tag{4.32}$$

so that

$$T = n\pi = 2\,\frac{R_o^{\,2}}{\|u\|_2^{\,2}} \tag{4.33}$$

This equation establishes that in general

$$T_{min} < T < T_{min} + \pi \tag{4.34}$$

where

$$T_{min} = \left[\frac{2R_o^{\,2}}{\pi L_2^{\,2}}\right]\pi \tag{4.35}$$

where $[y]$ represents the largest integer in y. For example, if $\frac{2R_o^{\,2}}{\pi L_2^{\,2}} =$ 6.5, it is evident that a somewhat larger value of L_2 will result in $T_{opt} = 6\pi$ while a somewhat smaller value will result in $T_{opt} = 7\pi$. Hence we have not only found T_{opt} within a range of π seconds, a good first approximation should be

$$T_{opt} \doteq \frac{2\,R_o^{\,2}}{L_2^{\,2}} \tag{4.36}$$

4.5 Magnitude and Area Constraints

This case has received some consideration as a minimum fuel (area) problem subject to an input magnitude constraint [18]. The input is bang-off-bang as discussed in Chapter II. With L_∞ taken as 1 for convenience, the trajectories corresponding to each piecewise constant input interval are circular arcs centered on the x_1-axis at -1, 0, +1 for u = -1, 0, +1, respectively. Switching curves can easily be constructed for any given value of Θ, the duration of a full saturation interval. Once again, consider working backwards from the origin. During the final interval, the input must be other than zero, i.e., the system must be powered, in order to reach the origin. Let us assume that u = +1 and that the last pulse is a full pulse, i.e., it is applied for a duration of θ seconds. Hence at time T $-\theta$ the state of the system would be at point A in Figure 4.7a. We have

$$r_o^2 = 1^2 + 1^2 - 2(1)(1)\cos\theta = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2} \quad (4.37)$$

so that

$$r_o = 2\sin\frac{\theta}{2} \quad (4.38)$$

The system must have coasted for an interval of $\pi - \theta$ in reaching point A. Hence at time T $-\pi$, the system was at point B. The system reaches point B from point C after θ seconds with u = - 1. It will now be shown that angle COB = θ. From Figure 4-7b we have

$$r_1 \sin\gamma = r_o \cos\theta/2$$

$$r_1 \cos\gamma = 1 + r_o \sin\theta/2$$

From 4-7C we have

$$y_1 = r_1 \cos(\pi - (\theta + \gamma)) = -r_1 \cos(\theta + \gamma)$$

$$y_2 = r_1 \sin(\pi - (\theta + \gamma)) = r_1 \sin(\theta + \gamma)$$

After some trigonemetric manipulation we obtain

$$y_1 = -\cos\theta + r_o \sin\theta/2$$

$$y_2 = \cos\theta/2\,(2\sin\theta/2 + r_o)$$

$$1 + y_1 = \sin\theta/2\,(2\sin\theta/2 + r_o)$$

Therefore

$$\phi = \tan^{-1}\frac{y_2}{1 + y_1} = \tan^{-1}\frac{\cos\theta/2}{\sin\theta/2} = 90^\circ - \theta/2 \quad (4.39)$$

Hence COA is a straight line and angle COB = θ. Further, the radius of

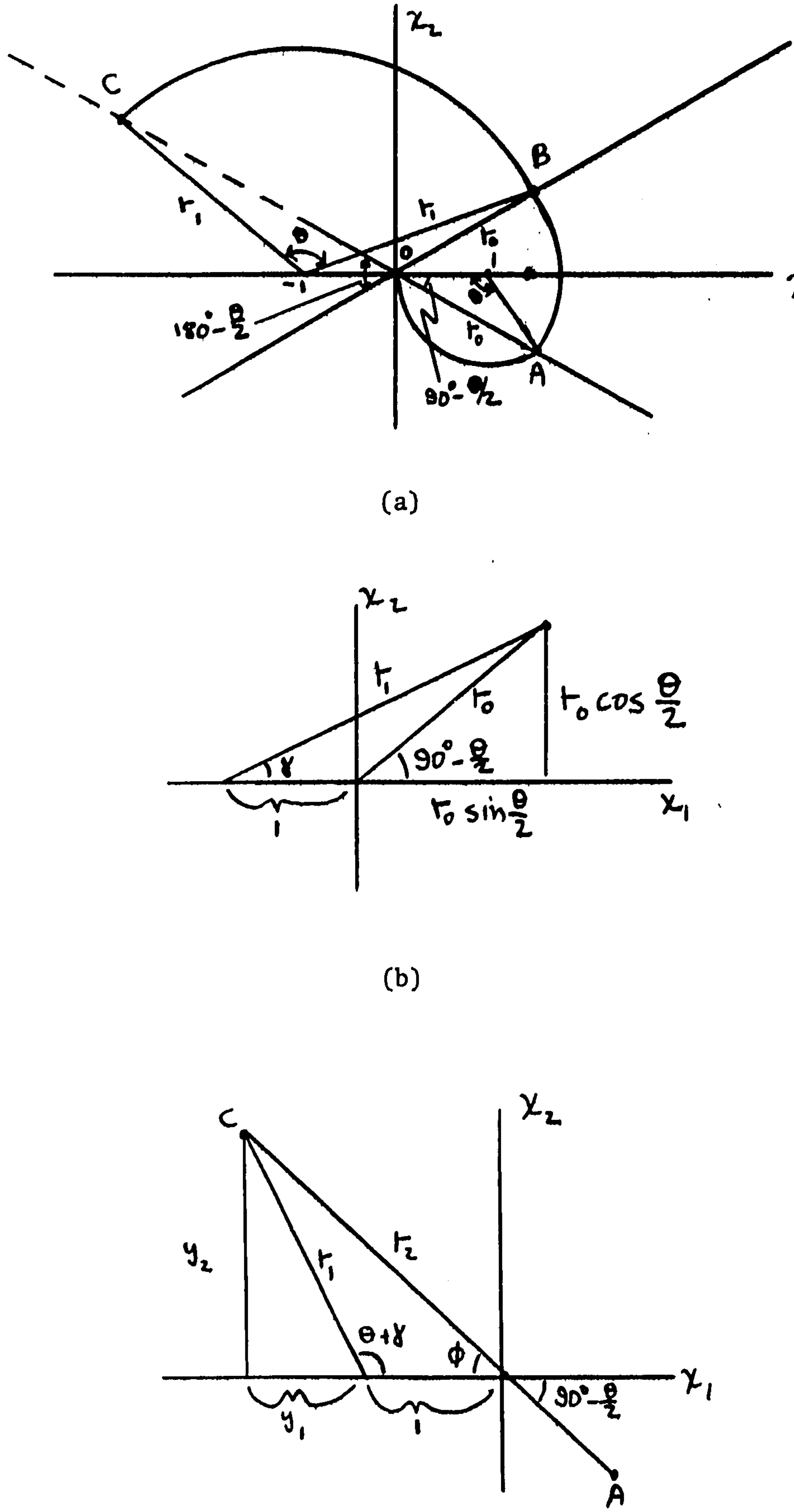


Figure 4-7. Development of Bang-Off-Bang Switching Lines

the trajectory of the next (previous) off interval is

$$r_2 = \sqrt{y_2^{\,2} + (1 + y_1)^{\,2}} = r_o + 2 \sin \theta/2 \qquad (4.40)$$

It will now be shown that as this procedure is continued, all switching points will lie on the same straight lines which have just been established and that each "on" pulse reduces the distance to the origin by $2 \sin \theta/2$. Consider the general situation illustrated in Figure 4-8 where point K corresponds to the beginning of an off interval.

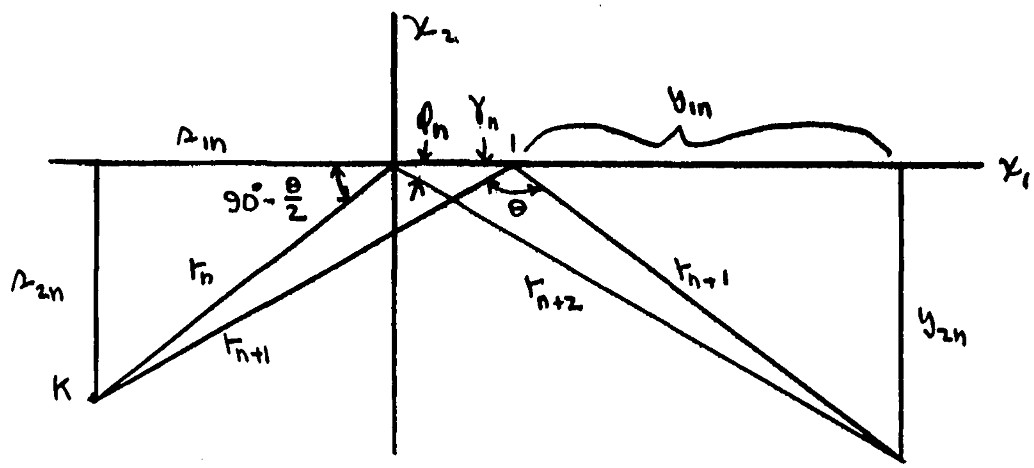


Figure 4-8. Effect of Full "On" Pulse

We have

$$s_{1n} = r_n \sin \theta/2$$

$$s_{2n} = r_n \cos \theta/2$$

$$r_{n+1} \sin \gamma_n = s_{2n}$$

$$r_{n+1} \cos \gamma_n = 1 + s_{1n}$$

Now

$$y_{1n} = r_{n+1} \cos(180^\circ - (\theta + \gamma_n)) = - \cos\theta + r_n \sin\theta/2$$

$$y_{2n} = r_{n+1} \sin(180^\circ - (\theta + \gamma_n)) = \sin\theta + r_n \cos\theta/2$$

And, as above,

$$\phi_n = \tan^{-1} \frac{y_{2n}}{1 + y_{1n}} = 90^\circ - \theta/2 \qquad (4.41)$$

$$r_{n+2} = r_n + 2 \sin\theta/2 \qquad (4.42)$$

Hence, when the last pulse is a full pulse of duration θ, the switching curves are straight lines through the origin at an angle of $\theta/2$ with respect to the x_2-axis. Each full pulse reduces the distance to the origin by 2 sin $\theta/2$. Obviously identical results are obtained if the terminal pulse is a full negative pulse.

Now consider a partial last pulse of duration $\beta<\theta$ as shown in Figure 4-9. A little thought clearly indicates that the locus of points which will reach curve C_1 after an off or coast interval of $\pi-\theta$ is curve C_2. To reach C_2 after an on interval with u = - 1, a point must be on curve C_3. All of these curves are identical portions of circles whose centers lie on lines through points $(x_1, X_2) = (\pm 1, 0)$ at an angle of $90^{\circ} - \theta/2$ with respect to the x_1-axis, i.e., parallel to the straight lines established above. The distance between centers is 2 sin $\theta/2$. The switching lines are shown in Figure 4-10.

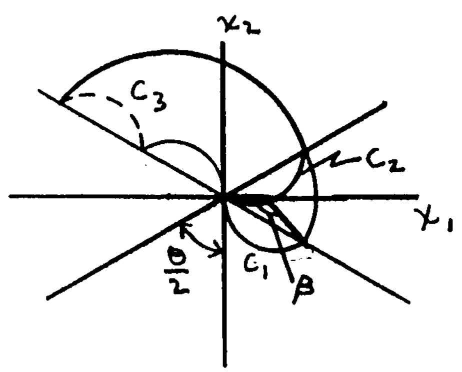


Figure 4-9. Effect of Partial Last Pulse

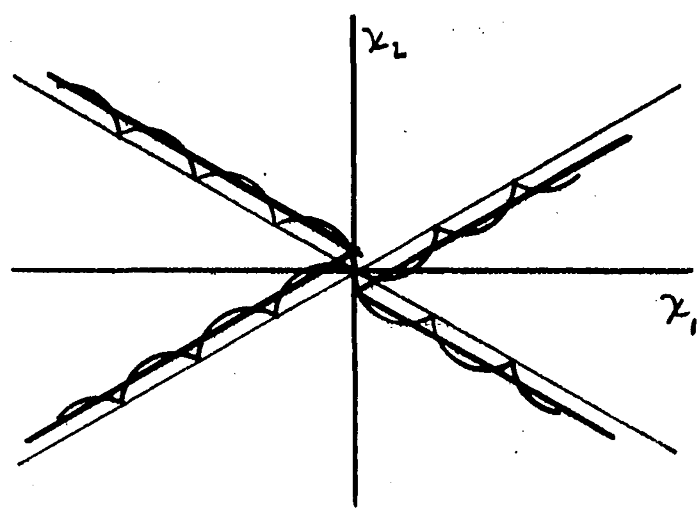


Figure 4-10. Optimal Switching Lines For Area and Magnitude Constraints

As in the case with a magnitude constraint alone, the switching points in any given situation will lie on straight lines which do not pass through the origin (except when the last pulse if a full pulse) as shown in Figure 4-10. These straight line switching curves are of course a function of the initial state as well as the value of the area constraint since this influences Θ. (In the bang-bang problem, $\Theta = 180°$.) The last pulse again must violate these simple switching lines if they are extended to the x_2-axis. It is to be noted that each full pulse reduces the distance along equivalent switching lines by 2 sin $\Theta/2$. This results in a reduction of $r = \sqrt{x_1^2 + x_2^2}$ by a little less than 2 sin $\Theta/2$.

Equations can be developed to enable an analytic solution with simultaneous transendental equations but once again it would appear to be more convenient to consider the solution graphically by successive approximations--both for the minimum time problem and the inverse or minimum norm problem. In particular, to solve the minimum time problem for a given L_1, choose a Θ, construct the switching curves, and from the resulting trajectory compute $\|u\|_1$. If this is less than L_1, increase Θ; if not, decrease Θ. This process is continued until $\|u\|_1 = L_1$. The identical procedure is used in a minimum fuel problem with a specified T.

It is not necessary to compute the entire trajectory to find out if the constraint on $\|u\|_1$ is satisfied. All that is necessary is to find the first switching point, for example point P in Figure 4-11.

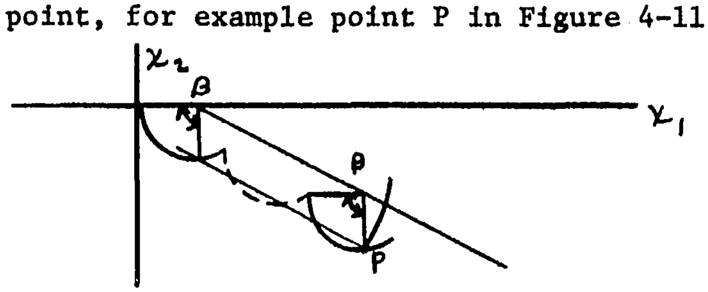


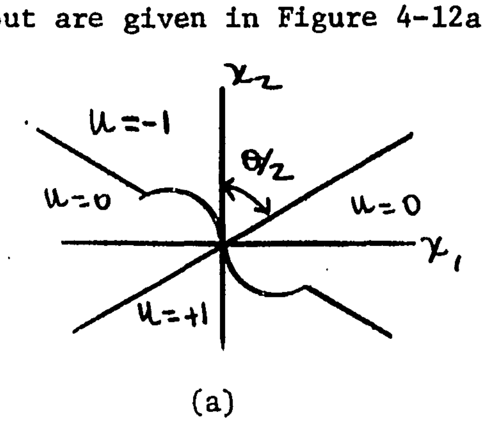
Figure 4-11. Determination of Duration of Partial Last Pulse

This immediately establishes β, the duration of the final pulse. If the system is powered at t = 0 and the duration of the first partial pulse is γ, then

$$\|u\|_1 = \gamma + n\theta + \beta \tag{4.43}$$

where n is the number of full pulses which would be obvious by inspection in any given case. If the system is in a coast region at t = 0, then $\gamma = 0$.

There are numerous possible sub-optimal inputs which can be expected to result in satisfactory performance or provide a reasonable estimate or first approximation to the optimal θ. The switching curves for one such input are given in Figure 4-12a.



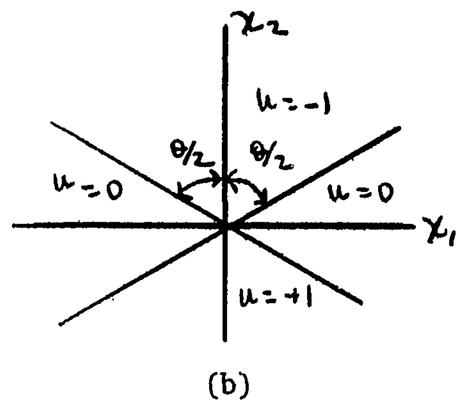


Figure 4-12. Sub-Optimal Switching Lines

They are symmetrical straight lines through the origin except for the circular arcs which are required for the final powered portion in order to reach the origin.

A somewhat simpler set of switching lines can be obtained by forcing the final pulse to be a full pulse. In this case the switching lines are straight through the origin as in Figure 4-12b. Of course it is to be expected that this input will result in $\|u\|_1 < L_1$, i.e., the resulting input will not use all of the allowable fuel. This form of input leads to a particularly simple solution if the initial state is in an off region. In this case we have

$$n\theta = \|u\|_1 \tag{4.44}$$

$$2n \sin \theta/2 = r_o \tag{4.45}$$

where n is an integer. If θ_f is the duration of a full pulse which would result in the use of all the fuel, i.e., $\|u\|_1 = L_1$, when there is no restriction on n in (4.44) and (4.45) we have

$$\frac{\sin \theta_f/2}{\theta_f/2} = \frac{r_o}{L_1} \tag{4.46}$$

The desired value of θ is the largest θ, $\theta < \theta_f$ which satisfies (4.44) and (4.45) subject to n being an integer. The value of n which accomplishes this is

$$n = \left[\frac{L_1}{\theta_f}\right] + 1 \tag{4.47}$$

If L_1/θ is an integer, the 1 should not be added in (4.47). Substitution of this value for n in (4.45) yields the desired θ.

Example: As an illustration of the above, consider $x_{20} = 0$, $x_{10} = 6.57 = r_o$, and $L_1 = 7.85$. From (4.46)

$$\frac{\sin \theta_f/2}{\theta_f/2} = 6.57/7.85 = 0.837$$

so that

$$\theta_f = 2.03 \text{ rad.} = 116^o$$

Using (4.47)

$$n = \left[\frac{7.85}{2.03}\right] + 1 = [3.87] + 1 = 4$$

From (4.45) we have

$$\sin \theta/2 = r_o/2n = 6.57/8 = 0.821$$

$$\theta = 1.93 \text{ rad} = 110.4^\circ$$

and

$$\|u\|_1 = n\theta = 7.72 < L_1 = 7.85$$

This sub-optimal solution is composed of 4 full on pulses and 3½ off intervals so that

$$T_s = 3\tfrac{1}{2}\pi + 1.93/2 = 11.96 \text{ sec}$$

The optimal solution for this example consists of three full on pulses of 120^o and a partial last pulse of 90^o which results in $T_{opt} = 11.57$ seconds. The increase in time resulting from the sub-optimal solution is about 3.4% over the optimal value. On the other hand, it uses about 1.65% less fuel.

Investigation of a number of examples indicates that

$$\Delta T\ \% \cong -2\ \Delta \|u\|_1 \quad \%$$

Some modification of this procedure is necessary if the initial state is located in an on region since the partial first pulse must be accounted for. Consider Figure 4-13 where the initial state, point P, is in the u = -1 region.

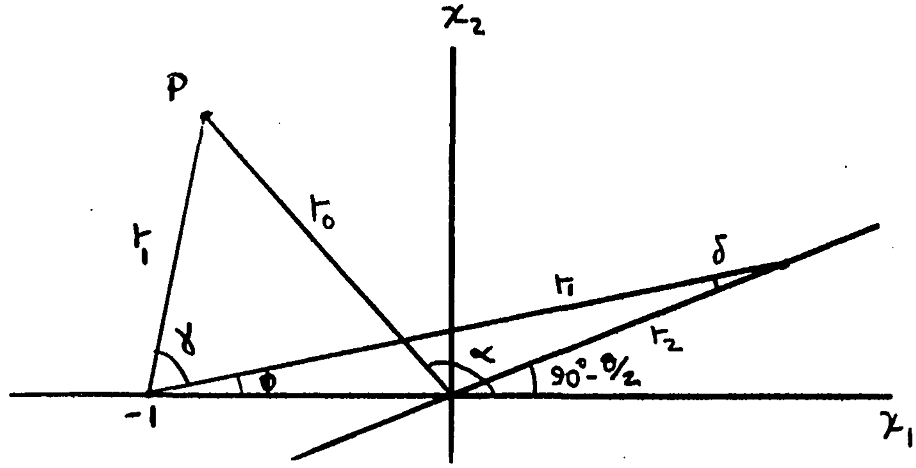


Figure 4-13. Effect of Partial First Pulse

We have

$$r_1^2 = 1 + r_o^2 - 2\ r_o \cos\ (180° - \alpha) = 1 + r_o^2 + 2x_{10} \tag{4.49}$$

$$r_1^2 = 1 + r_2^2 - 2\ r_2 \cos\ (90° + \theta/2) = 1 + r_2^2 + 2\ r_2 \sin\theta/2$$

Hence (4.50)

$$r_2^2 + 2\ r_2 \sin\theta/2 = r_o^2 + 2\ x_{10} \tag{4.51}$$

But

$$r_2 = 2n \sin\theta/2 \tag{4.52}$$

where n is the number of full pulses. Substituting (4.52) in (4.51)

$$\sin^2\frac{\theta}{2} = \frac{r_o^2 + 2\ x_{10}}{4n(n+1)} \tag{4.53}$$

Also

$$\gamma + \phi = \tan^{-1} \frac{x_{20}}{1 + x_{10}} \tag{4.54}$$

and

$$\sin\delta = \frac{1}{r_1} \sin\ (90° + \theta/2) = \frac{1}{r_1} \cos\theta/2 \tag{4.55}$$

where

$$\phi = 90^o - \theta/2 - \delta \qquad (4.56)$$

For a given value of n, one can find θ from (4.53). Since r_1 is determined by the initial state, δ, ϕ, and finally γ can be found from (4.55), (4.56), and (4.54). The area constraint requires

$$n\theta + \gamma \leq L_1 \qquad (4.57)$$

If this is not satisfied, one must choose a larger n which will result in a smaller θ. The process is continued until the smallest n which leads to (4.57) being satisfied is found. A good approximation for n can be obtained by assuming that the relative effect of the initial partial pulse is the same as that of a full pulse. This means that (4.46) and (4.47) can be used to obtain n. This will often yield the correct value although most combinations of initial states and constraints require n to be 1 less than that given by (4.47). No conditions were found which required a value of n greater than that given by (4.47).

<u>Example</u>: $x_{10} = 0$, $x_{20} = 5.67 = r_o$, $L_1 = 6.76$. Here the optimal solution consists of a partial first pulse of 1.01 rad. (58°), 2 full pulses of 2.09 rad. (120°), and a final pulse of 1.57 rad. (90°) leading to T_{opt} = 9.90 seconds. For the sub-optimal solution we have as a trial

$$\frac{\sin \theta_f/2}{\theta_f/2} = \frac{r_o}{L_1} = 0.838$$

$$\theta_f = 2.02 \text{ rad.}$$

$$n \doteq \left[\frac{6.76}{2.02}\right] = [3.55] = 3$$

From (4.53), $\theta = 1.93$ rad. $= 110.4^o$. Since $r_1 = \sqrt{1^2 + 5.67^2} = 5.76$ we have from (3.55) and (3.56)

$$\delta = 0.099 \text{ rad}$$

$$\phi = 0.51 \text{ rad.}$$

From (4.54) we obtain $\gamma = 0.89$ rad. $= 51^o$. Checking the constraint condidition (3.57)

$$\|u\|_1 = n\theta + \gamma = 3(1.93) + 0.89 = 6.68 < L_1 = 6.76$$

We also have that T is composed of γ plus 3 full pulses and 3 coast intervals so that

$$T_s = 3\pi + \gamma = 10.31 \text{ sec.}$$

The increase in time is about 4.1% while the decrease in fuel is about 1.2%. There appears to be no simple approximate relationship such as (4.48) relating these two quantities.

Certain general conclusions can be drawn regarding the value of the input for various system states. The input has little or no effect when $x_2 \cong 0$ so it is not surprising that--except near the origin --u = 0 along the x_1 - axis. The input has maximum effect when $x_1 \equiv 0$ so that the full input should be applied when the state of the system is on the x_2-axis. These two facts are easily established by consideration of Figure 4-14 where r_o is the radius of the circular arc of a u =-1 powered trajectory and r is the distance to the origin.

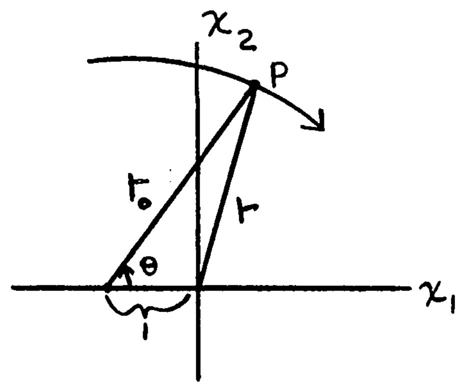


Figure 4-14. Determination of Points where Input has Maximum Effect

We seek to establish when dr/dθ is maximum and minimum. (Note: the angle θ in this Figure is not the duration of a full on interval). We have

$$r^2 = 1 + r_o^2 - 2 r_o \cos\theta$$

$$2 r\, dr = 2 r_o \sin\theta\, d\theta$$

$$\frac{dr}{d\theta} = \frac{r_o}{r}\sin\theta = \frac{r_o \sin\theta}{\sqrt{1 + r_o^2 - 2 r_o \cos\theta}} \qquad (4.58)$$

$$\frac{d^2r}{d\theta^2} = r_0 \quad \frac{r\cos\theta - r_0\sin^2\theta/r}{r^2} = 0$$

so that

$$\cos\theta(1 + r_o^2 - 2\, r_o \cos\theta) = r_o \sin^2\theta = r_o(1 - \cos^2\theta)$$

$$\cos\theta(1 + r_o^2) - r_o(1 + \cos^2\theta) = 0$$

and
$$\frac{\cos\theta}{1 + \cos^2\theta} = \frac{r_o}{1 + r_o^2} = \frac{1/r_o}{1 + 1/r_o^2} \tag{4.59}$$

Although (4.59) can be solved for cos θ by means of the quadratic formula, it is evident that the two solutions are

$$\cos\theta = r_o \qquad r_o < 1 \tag{4.60}$$

$$\cos\theta = 1/r_o \qquad r_o > 1 \tag{4.61}$$

Since r_o must be greater than or equal to 1 (the last powered trajectory has $r_o = 1$, all others have $r_o > 1$), the solution of (4.61) is the desired result. Inspection of Figure 4-14 indicates that point P must lie on the positive x_2-axis in order to satisfy (4.61). For a $u = +1$ powered trajectory, point P will lie on the negative x_2- axis due to the symmetry involved.

The points where the input is ineffective are easily found by inspection of (4.58). Clearly $dr/d\theta = 0$ when $\sin\theta = 0$ or $\theta = 0^o$, 180^o. Furthermore, if $-\pi < \theta < 0$, i.e., if P is in the third or fourth quadrant for $u = -1$, (4.58) indicates that $dr/d\theta < 0$ so that a negative input would actually be detrimental--at least as far as efficient use of input fuel is concerned. In the magnitude constrained bang-bang time-optimal case, some reduction of time is obtained by allowing the input to be negative in certain regions below the x_1-axis (see Figure 4-5).

At this point it should be noted that although the switching surfaces discussed in this section--both the optimal and sub-optimal--give the illusion of closed loop control, some care must be exercised as to praising their virtues to highly. It is indeed true that the switching lines do provide closed loop control in that the input at any time is determined by the state of the system at that time. However the resulting trajectory is optimal (or nearly so in the sub-optimal case) only for the conditions for which it was derived. If there is some disturbance introduced into the system, the remaining trajectory will generally not be optimal. Moreover, with disturbances the desired terminal state will generally not be reached. On the other hand, it is true that the implementation of the switching lines would in general result in a smaller terminal error than an open loop control, for example, specification of the switching times.

The simplest means of overcoming the problem of disturbances occurring after t = 0 would be to recompute θ for the switching lines during each off interval. This would enable the controller to account for any disturbances introduced until $r(t) = \sqrt{x_1^2 + x_2^2} > L_1' = L_1 - \int_0^t |u|\, dt$.

Under this condition the origin cannot be reached. It would also be possible to compute the switching lines as a continuous function of time. This of course involves considerably more computation.

4.6 Energy and Magnitude Constraints

From the results obtained in Chapter II, the optimal input is of the form shown in Figure 4-15

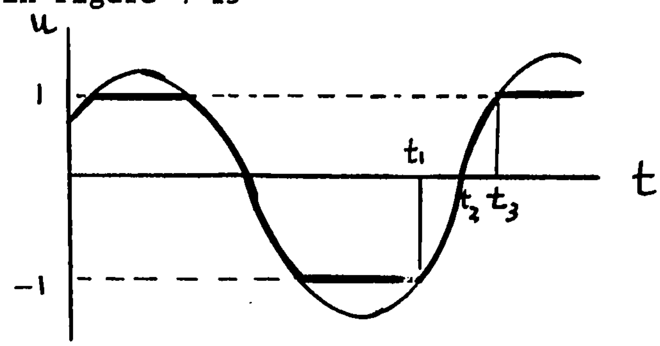


Figure 4-15. Optimal Input with Energy and Magnitude Constraints

$$u = \begin{cases} \underline{\lambda}'\,\underline{h}(T-t) & |\underline{\lambda}'\underline{h}| < 1 \\ \operatorname{sgn} \underline{\lambda}'\,\underline{h} & |\underline{\lambda}'\,\underline{h}| > 1 \end{cases}$$

where L_∞ has been taken as unity for convenience. When one attempts to develop switching surfaces in the manner of the previous section, an inconsistency appears. The difficulty arises because of the fact that the value of u can be anything between and including - 1 and 1. Consider first what happens when the origin is reached while the input is in the saturation region. As in the previous section, one can work backwards from the origin and develop the optimal switching curves for a given value of the full saturation interval. Now, however, if one attempts to "complete" the switching curves by working backwards from the origin when the terminal time is on a sinusoidal portion of the input, it soon becomes apparent that the results will be difficult to interpret. The reason for this is that the final portion of the trajectory depends upon the duration of the trajectory! Such is not the case when the input is at the saturation value at $t = T$. In particular, consider the case where the full saturation interval is 90°. Let the origin be reached after a full sinusoidal interval of 90 corresponding to t, $t_1 \leq t \leq t_3$ in Figure 4-15. The state of the system at $t = t_1$ is point A of Figure 4-16 and the trajectory is as shown.

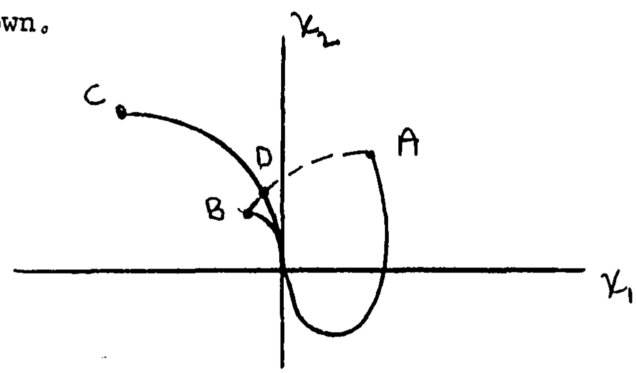


Figure 4-16. Demonstration of Difficulty in Interpreting Switching Lines for Energy and Magnitude Constraints

Now consider the state at t_1 which reaches the origin at $t = t_2$. This will be point B of Figure 4-16. Not only is trajectory BO not a part of trajectory AO, it lies <u>below</u> trajectory CO--the trajectory for the full negative saturation interval which causes the origin to be reached. This means that the locus of states at t_1 which can be driven to the origin by some portion of the sinusoidal interval will cross trajectory CO, say at point D. This appears to imply that from point D, two optimal trajectories are possible. Although there are systems and situations where the optimal input may not be unique, this is not one of them. This apparent inconsistency is easily resolved when one remembers that the trajectories drawn in Figure 4-16 are for a particular value for the duration of the full saturation interval. Hence there may be several inputs of the desired form which will drive the system from a given initial state to the origin. However, only one of these will result in the minimum time solution, $T = T_o$, with the specified L_2. For those systems where more than one input satisfies all the constraints, the one resulting in the smallest T is optimal.

It should be pointed out that whatever the input is in this case, i.e., whether it terminates on a saturation interval or on a sinusoidal interval, when $x_1 \cong 0$, $u = -\operatorname{sgn} x_2$, and that u passes through zero when $x_2 \cong 0$. This agrees with the results of the previous section that the input is most effective when $x_1 \cong 0$ and least effective when $x_2 \cong 0$.

No sub-optimal inputs of the form shown in Figure 4-15 were investigated. It would seem that one which reached the origin at the end of--or at least on the saturation interval would be simplest. The application of a bang off-bang sub-optimal controller to this constraint combination is discussed by Chapter VI.

4.7 Other Constraint Combinations

Since simultaneous energy and area constrainst lead to a "weighting factor" in the solution, it appears that no simple results can be obtained for this constraint combination--especially in view of the results of the previous section. Hence no attempt was made to find a explicit form of the solution.

The addition of a magnitude constraint to the other two would certainly, in general, increase the complexity of the solution. Again, no explicit solution was attempted.

CHAPTER V

UNDERDAMPED SECOND ORDER SYSTEM - AREA AND MAGNITUDE CONSTRAINTS

When damping is introduced into the system discussed in Chapter IV, the complexity of the optimal solution for given constraints increases considerably. Not only is the number of input zero crossings unknown a priori, but the duration of each different portion of the input changes every "cycle." For the case of a magnitude constraint alone, switching lines can be easily constructed--although they would be somewhat difficult to implement in an engineering design. When both area and magnitude constraints are imposed upon the input, it is not possible to define optimal switching lines (except by connecting the end points of segments of the trajectory after the optimum trajectory is found). However, based upon the results of Chapter IV and the form of the input established for this case, several sets of sub-optimal switching lines are suggested and investigated. No clear-cut decision is reached as to the "best" sub-optimal solution to use in a specific engineering design because of a number of undetermined factors. First, the original statement of the problem--minimum time solution subject to input constraints--may not be the most useful or most meaningful from a practical standpoint. In general, one expects a trade-off between the fuel consumption and the terminal time for a given form of solution. Since it may be possible to reduce the fuel consumption considerably by allowing a relatively small increase in time, or vice versa, some compromise is normally desirable--based upon the specific requirements of a specific situation. Second, the trade-off between solution time on the one hand and relative simplicity of the engineering implementation on the other cannot be discussed in general terms. Third, it will not be necessary for the input to force the system to the terminal state (the origin) in many designs, but rather to drive the system from the initial state relatively quickly to some small region near the origin where the system's inherent damping will cause the state to approach the origin asymptotically. The size and shape of such a region depend again upon the specific problem and of course upon the sophistication of the implementation desired. One would expect considerable freedom to be available in the choice of this region in many designs.

The results of this chapter provide a sound basis for the establishment of an extremely simple sub-optimal control scheme which should be satisfactory for a wide variety of situations. In addition, a second more sophisticated control scheme whose performance sometimes hardly differs from the optimal is presented. Application to a minimum fuel problem subject to a constraint on the terminal time is also considered.

5.1 Optimal Solution

Consider the system of Figure 5-1 which is described by the state equations:

$$\dot{x}_1 = x_2 \qquad (5.1)$$

$$\dot{x}_2 = - x_1 - 2\zeta x_2 + u \qquad (5.2)$$

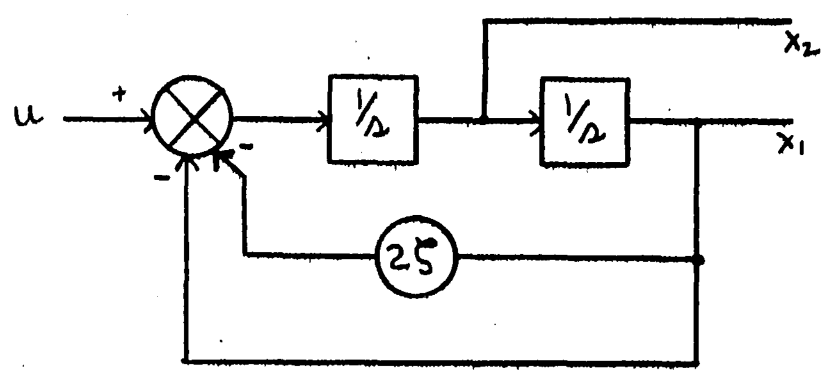


Figure 5-1. Block Diagram of Underdamped Second Order System

The impulse response and transition matrix are, respectively,

$$\underline{h}\ (t) = e^{-\zeta t} \begin{bmatrix} \frac{1}{\omega} \sin \omega t \\ \cos \omega t - \frac{\zeta}{\omega} \sin \omega t \end{bmatrix} \qquad (5.3)$$

$$\phi\ (t) = e^{-\zeta t} \begin{bmatrix} \cos \omega t + \frac{\zeta}{\omega} \sin \omega t & \sin \omega t \\ -\frac{1}{\omega} \sin \omega t & \cos \omega t - \frac{\zeta}{\omega} \sin \omega t \end{bmatrix} \qquad (5.4)$$

$$\text{where } \omega = \sqrt{1 - \zeta^2} \qquad (5.5)$$

It is desired to drive the system from some given initial state

$\underline{x}\ (0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ to the origin in minimum time T subject to the input constraints

$$\|u\|_\infty = \max |u| \leq 1 \qquad (5.6)$$

$$\|u\|_1 = \int_0^T |u|\, dt \leq L_1 \qquad (5.7)$$

where L_1 is some given number. It should be noted that a solution exists for any finite but arbritarily small L_1 since, because of the damping, the system can be allowed to coast until the state is only an infinitesimal distance away from the origin. From that point the system can be driven to the origin using only an infinitesimal amount of input fuel. For this reason, a minimum fuel problem subject only to an input magnitude constraint is meaningless for this system unless the terminal time is limited to less than some value.

When (5.3) and (5.4) are substituted in (2.8), one obtains after some manipulation

$$\omega x_{10} = \int_0^T e^{\zeta t} \sin \omega t\ u(t)\ dt \qquad (5.8)$$

$$\zeta x_{10} + x_{20} = \int_0^T e^{\zeta t} \cos \omega t\ u(t)\ dt \qquad (5.9)$$

These are the terminal conditions which must be satisfied by u(t). As expected these reduce to (4.6) and (4.7), the corresponding equations for the harmonic oscillator, when the damping ratio $\zeta = 0$.

We have, in addition

$$\underline{\lambda}'\underline{h} = \underline{\lambda}'\underline{h}(T - t) = K e^{\zeta t} \cos(\omega t - \phi) \qquad (5.10)$$

which is shown in Figure 5-2 with the corresponding u(t) also illustrated.

$$u(t) = \begin{cases} \operatorname{sgn} \underline{\lambda}'\underline{h} & |\underline{\lambda}'\underline{h}| > 1 \\ 0 & |\underline{\lambda}'\underline{h}| < 1 \end{cases} \qquad (5.11)$$

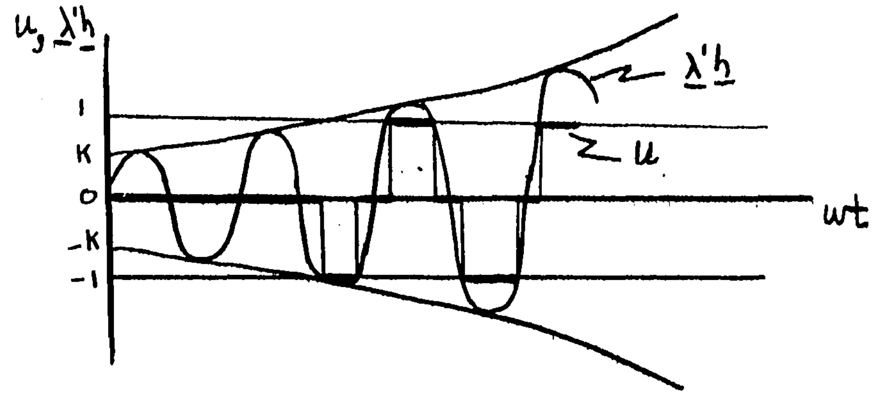


Figure 5-2. $\underline{\lambda}'\underline{h}$ and u With Area and Magnitude Constraints

It is to be noted that each full "on" pulse is longer in duration than the previous on pulse, and correspondingly, the off intervals decrease in duration. The final on pulse will naturally not generally be a "full" pulse. A number of different situations are possible at t = 0 for the input depending upon the values of K and ϕ. First, the system may be powered at $t = 0^+$ so that there is a partial first pulse. Second, the system may be in a "short" off region. Both of these correspond to K greated than about 1. On the other hand, if K is substantially less than unity, the system will coast for a number of "cycles" before the input is applied. This is the case shown in Figure 5-2. A "typical" trajectory can be constructed by working backwards in time from the origin. Due to the time varying nature of the duration of the on interval, it is not possible to construct switching surfaces corresponding to those in Chapter IV. For example, consider Figure 5-3 where, working back from the origin, trajectory OABC is obtained for a "full" final pulse, the final + 1 powered portion being AO. (A'O is the trajectory for a final - 1 powered portion.)

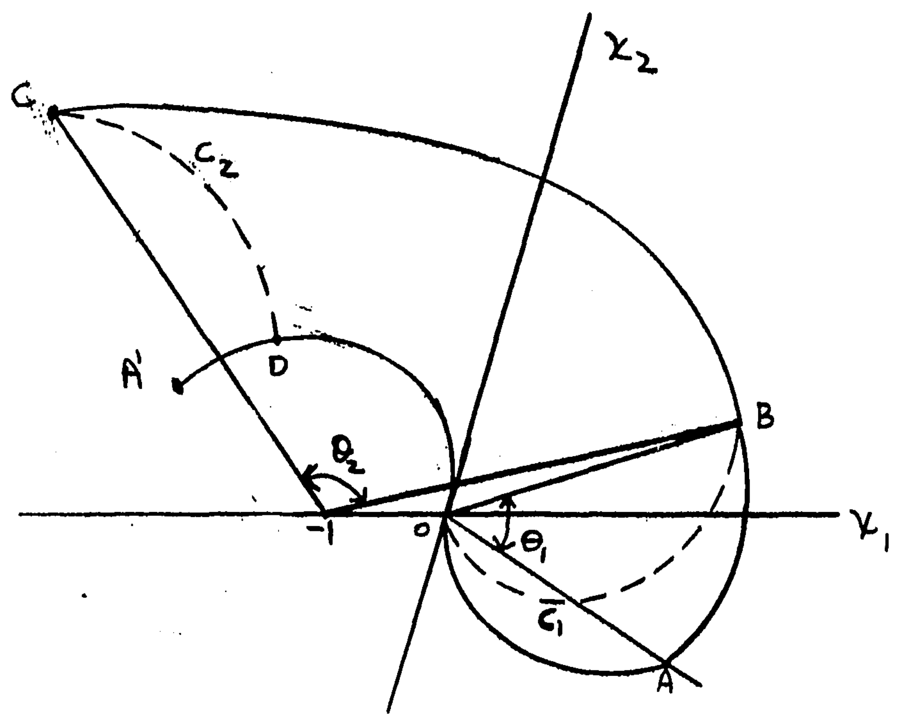


Figure 5-3. Attempted Construction of Optimal Switching Lines.

The locus of points which can reach curve AO by means of the appropriate off interval θ_1 is curve C_1. The locus of points which can reach C_1 by means of a -1 powered interval of the appropriate duration $\omega t = \theta_2$ is curve C_2. Since θ_2 is of necessity less than the terminal full powered interval, the intersection of C_2 with A'O is at point D. (With no damp- the intersection is at point A'.) If one now attempts to continue to construct switching lines based upon a final full pulse of some specific value, a meaningless garble of curves is obtained. Only in the case where the input is of the bang-bang form, i.e., only the input magnitude is constrained, can switching lines be constructed. These lines were first established by Bushaw[17] and are as shown in Figure 5-4.

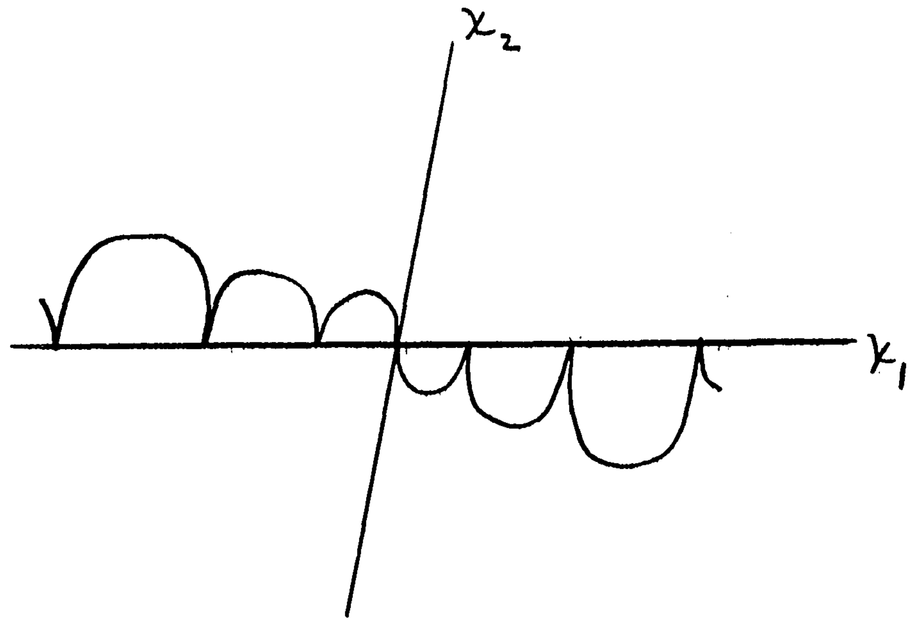


Figure 5-4. Optimal Switching Lines

The switching lines in this case are segments of spirals. With both constraints applicable, one cannot describe the trajectory from an initial state to the origin in terms of a final full pulse as could be done in the oscillator case. The reason is obvious: all full pulses in the oscillator case had the same duration and the same effect in driving the state toward the origin. With damping, the term "full pulse" loses its significance since the duration of each on interval is different from that of all others as is its effect.

5.2 Sub-Optimal Solutions

A variety of sub-optimal solutions using only the discrete values of +1, 0, and - 1 for the input are possible. Several relatively simple ones were investigated in detail using as a "design criterion" that the primary concern should be with those combinations of initial states and allowable input areas that result in several input switchings in the optimal solution. This is the situation which has received no attention previously although it is probably the most interesting situation both from the theoretical and practical points of view. In the following, "coast" is used to describe the interval when u = 0 before any non-zero input is applied while "off" is used to describe the interval when u = 0 between on pulses of $u = \pm 1$.

In order to reduce the amount of computation required, it was decided to use certain specific combinations of initial states and constraints which enable a reasonable estimate to be made for the performance resulting from any initial state and constraints combination. This was accomplished by working backwards from the origin with a specified optimal input. In this way, the optimal terminal time for the resulting initial state and constraint is easily found. For initial states far from the origin, the optimal and all sub-optimal solutions considered involve an initial coast interval. Hence it is only necessary to consider initial states with $X_{20} = 0$ with constraints such that the optimal input becomes non-zero during the first "half-cycle." The results obtained thereby are "worse case" results since for initial states father from the origin, the sub-optimal and optimal solutions are identical to this point (they both coast).

For initial states close to the origin, the optimal input has very few switchings so that the system will reach the region near the origin quickly even if no input is applied. Because of this, it was again felt that initial states with $X_{20} = 0$ would prove satisfactory in predicting the overall performance capability of a sub-optimal scheme.

Several sub-optimal solutions are discussed in detail below. Typical results are given in Tables 5-1, 5-2, and 5-3 for $\zeta = 0.1$, 0.3, and 0.5, respectively. The results for values of the damping ratio greater than 0.5 are similar. The numerical details were carried out using an IBM 7040 digital computer.

Coast-Bang-Bang

When the optimal input is similar to that shown in Figure 5-2, it is to be expected that good results could be obtained by allowing the system to coast for some time and then applying a bang-bang input to drive the system to the origin. Since the optimal bang-bang switching surfaces are too complicated to implement in a simple engineering design, it is first necessary to consider a sub-optimal bang-bang solution. Based upon the sub-optimal bang-bang solution suggested for the harmonic oscillator, one is immediately led to the sub-optimal solution using the switching lines shown in Figure 5-5.

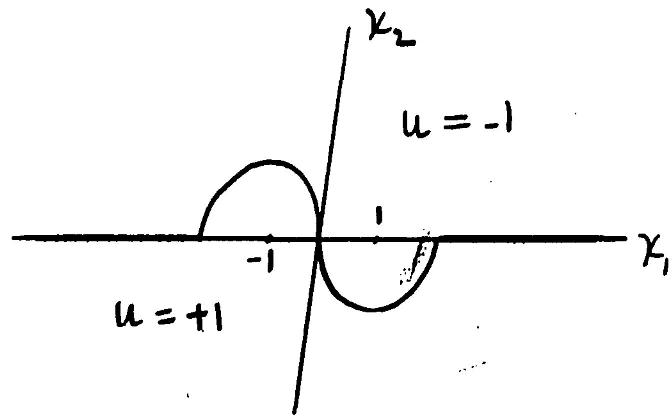


Figure 5-5. Sub-Optimal Bang Bang Lines

These consist of the x_1-axis except near the origin where the trajectory of a final powered portion is used. These sub-optimal switching lines were investigated in detail. The percent increase in solution time over the optimal for initial conditions along the x_1-axis (worse case) is shown in Figure 5-6 for $\zeta = 0.1$ which shows a maximum increase of about 3.35%.

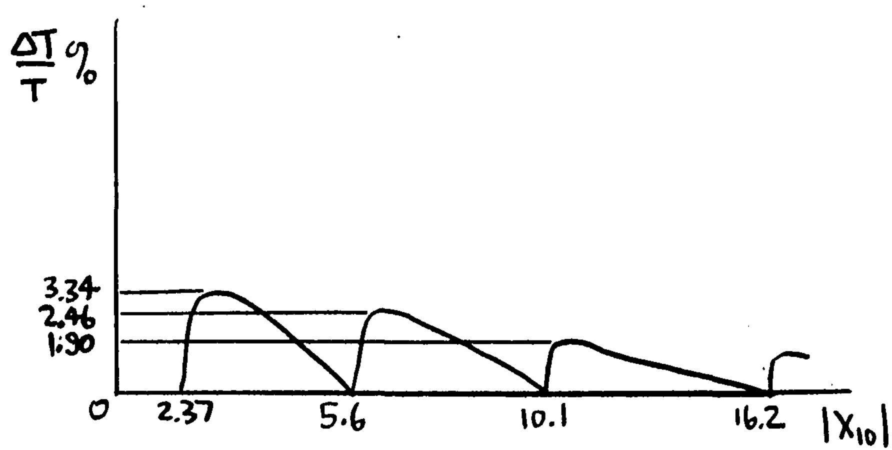


Figure 5-6. Increase in Terminal Time Using Sub-Optimal Switching Lines

The results are typical of what occurs for any value of the damping ratio.

In general, the maximum increase decreases as the damping ratio increases. For ζ=0, the maximum increase is about 4.3%. The increase in time tends to be smaller for initial states which are more distant from the origin.

This set of switching lines can now be incorporated in a controller which allows the system to coast until it is possible to drive the system to the origin with these switching lines without exceeding the amount of input fuel available. Such a scheme is illustrated in Figure 5-7.

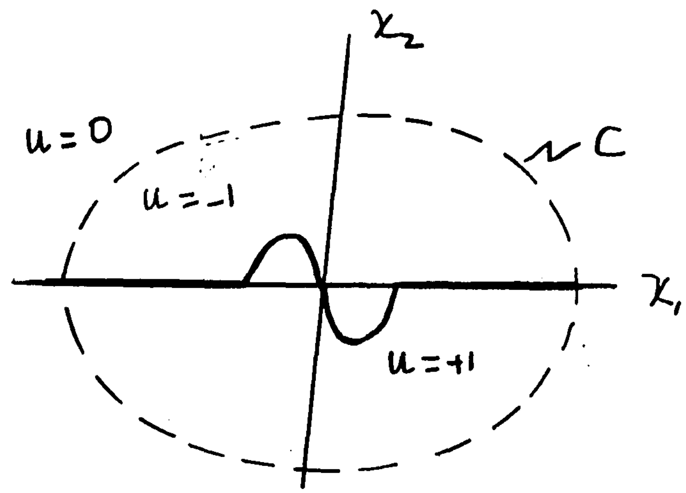


Figure 5-7. Coast Bang-Bang Switching Lines

Curve C is the locus of points which can be driven to the origin using the sub-optimal bang-bang input for some given value of allowable input area. The curve C is approximately a circle for small values of the damping ratio zeta but this is a very crude approximation for zeta greater than about 0.2. A detailed study of the performance of such a scheme was not attempted. Rather, an analysis using a degenerate form of curve C was made and will now be discussed.

The area required for a bang-bang solution from initial states on the x_1-axis at the cusps of the Bushaw curves (see Figure 5-4) can be easily found since these points correspond to an integral number of full 180^o on intervals. The distance from the n-th cusp to the origin is simply related to the required area:

$$r_\pi = (1 + e^{s\pi})(1 + e^{s\pi} + e^{2s\pi} + \dots + e^{(n-1)s\pi})$$

$$= \coth \frac{s\pi}{2} (e^{L_1} - 1) \qquad (5.12)$$

where $s = \zeta/\omega$ and L_1 is the required area. For a given L_1, (5.12) yields (approximately) the distance from the origin of the point on the x_1-axis which can be driven to the origin by the sub-optimal (or optimal) bang-bang input. A simple way to implement a sub-optimal coast-bang-bang solution is to allow the system to coast until the trajectory crosses the x_1-axis at a distance $|x_1| < r_\pi$. (About 40% of the initial state-constraint combinations require the use of a somewhat smaller value for r_π to insure that the origin is reached.) This procedure was investigated using a digital computer for a sufficient number of conditions to validate the results. Typical results are given in Tables 5-1, 5-2, and 5-3. It is to be noted that the increase in terminal time over the optimal is quite small—usually less than 10%—considering the comparitive simplicity of the sub-optimal switching lines. Furthermore, the input area of the sub-optimal solution is always less than L_1 (except possibly for initial states at the cusps of the Bushaw curves). Obviously some improvement in performance can be obtained with a more sophisticated controller which uses more of the allowable area L_1 by implementing curve C (or an approximation to C) of Figure 5-7.

Straight Line Switching Lines

A second sub-optimal solution (of which the previous is a special case) allows the system to coast until the state is within some distance from the origin and then uses straight lines in the state space to control a bang-off-bang input as shown in Figure 5-8.

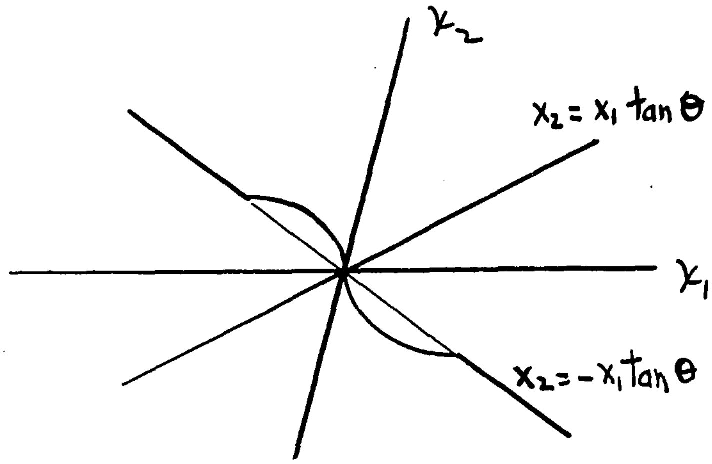


Figure 5.8. Straight Line Switching Lines

The switching lines are $x_2 = \pm x_1 \tan \theta$, where θ is an angle which must be calculated. Near the origin the appropriate portion of the final powered trajectory is used in lieu of the $x_2 = - x_1 \tan \theta$ line to insure that the origin is reached. A detailed investigation of this solution was conducted using a digital computer and some typical results are given in Tables 5-1, 5-2, and 5-3. The results show conclusively that such switching lines rarely offer any improvement in performance over the previously discussed coast-bang-bang system. Furthermore, the best initial coast interval and value of θ to be used in any particular case is a very complicated function of the initial state and allowable input area and hence considerable computation is required to find theses quantities-- probably comparable to that required to find the optimal solution. In addition, even for those cases where straight line switching lines result in better performance than the coast-bang-bang solution, another solution whose sophistication lies between the two already discussed leads to still better performance. Therefore, straight line switching lines--other than the coast-bang-bang solution--were deemed to be inferior to the others and of no practical utility.

Variable θ Coast-Bang-Off-Bang

A sub-optimal solution which requires a somewhat more sophisticated implementation than the coast-bang-bang-bang solution, but less than the straig line case, and sometimes leads to a considerable improvment in performance--while rarely being significantly inferior will now be discussed. This solution uses switching lines of the form just discussed and shown in Figure 5-8 with one important difference: The value of θ used is changed each "half-cycle." An investigation of the duration of the optimal on intervals indicated that a very good approximation for the duration of an interval is

$$\theta_{on} = \frac{r_{\pi}}{R_c} 180^0 \qquad (5.13)$$

where R_c is the distance of the state from the origin when the trajectory crosses the x_1-axis during the off interval preceeding θ_{on}. r_{π} is as defined by (5.12) when L_1 is the total area remaining. It is of course also necessary to determine when the first on pulse should be applied. In the interest of simplicity, it was decided to use a predetermined value of x_1-axis crossing, R_{st}, for a given L_1 so that the first input pulse is applied during the "half-cycle" after $R_c < R_{st}$. R_{st} was chosen as approximately the value of axis crossing just after which the optimal input first on pulse was required. The expression for R_{st} which was developed is

$$R_{st} = L_1\left((22.5\,\zeta^2 - 3.75\,\zeta + 0.35)L_1 + 0.5\right) \qquad (5.14)$$

A large number of examples were investigated and some typical results are given in Tables 5-1, 5-2, and 5-3.

Two important conclusions were reached concerning this sub-optimal solution. First, when the terminal on interval of the optimal solution is almost a "full" pulse, this sub-optimal solution is virtually identical to the optimal solution. For cases where the optimal input results in a partial last pulse significantly less than a full pulse, the performance deteriorates somewhat and there is little difference in performance between this solution and the coast-bang-bang solution. Second, the approximation for r_π is poor near the origin so that the variable θ solution does not reach the origin for initial states near the origin. However, if one is concerned with driving the system to a region near the origin in minimum time, subject to input area and magnitude constraints, this solution will usually reduce the distance of the state from the origin by over 99% in <u>about the same time</u> that the optimal solution requires to reach the origin. This is true for both initial states near to and far from the origin. This may be entirely satisfactory and even highly desirable in many practical situations.

<u>General Comments</u>

The results obtained above are generally "worse case" examples inasmuch as they were chosen so that an input is required during the first "half-cycle." Initial states which are further from the origin, with the same allowable input area, will in general lead to even better results since the duration of the initial coast interval will be virtually identical in both the optimal and sub-optimal solutions.

Hence, longer initial coast intervals in the optimal solution result in correspondingly less percentage increase in terminal time for the sub-optimal solutions.

The actual implementation of either the coast-bang-bang or variable θ sub-optimal solutions in a practical engineering design might include one significant modification over what has already been discussed in detail above. As previously suggested, it is possible that the terminal state need only be approached and not necessarily reached exactly. In this case a dual-mode controller might prove desirable. One of the solutions discussed above could be used to drive the system to some region near the origin, after which the system would be allowed to coast so that the state would approach the origin asymptotically. This would allow large initial "errors" to be rapidly--and near optimally-- reduced while avoiding any possibility of "chattering" near the origin. This would also eliminate the operation of the controller for small initial errors which may be harmless. By using the system's inherent damping near the origin, a considerable saving in input fuel (area) may be effected. The usefulness of such a system and the appropriate size and shape of the final coast or off region near the origin depend entirely upon the specific situation and cannot be discussed any further in general terms. One point is worth noting, in general, however: The use of an off region near the origin would eliminate the need for implementing the final powered portion of the trajectory. This would result in a considerable simplification of the controller.

It should be pointed out that the expression for r_{π}, while involving both an exponential and a hyperbolic function, reduces to just an exponential for a given value of the damping ratio, zeta. Hence for a specified value of zeta, the controllers discussed above could be implemented by analog methods and certainly can be used for real-time control.

It is clear that the variable θ sub-optimal solution has an additional feature not previously mentioned. It is capable of compensating for disturbances introduced into the system while the controller is operating. It is this self-correcting feature that makes this solution very attractive since <u>the solution remains near optimal even in the presense of external disturbances</u>.

One final comment concerning the straight line coast-bang-off-bang solution should be made. Some consideration was given to using two (or perhaps even four) different values of θ for the switching lines in different quadrants of the state space, e.g., $x_2 = x_1 \tan \theta_1$ and $x_2 = -x_1 \tan \theta_2$. However, this would require even more computation than the single value for θ. Furthermore, there is no reason to expect that this could significantly improve the system performance.

5.3 Minimum Fuel Problem

The coast-bang-bang sub-optimal solution is readily adaptable to the minimum fuel, time constrained problem. Consider Figure 5-7 where curve C is the locus of points which can be driven to the origin with the sub-optimal bang-bang input in $T = L_1$ seconds where T is the maximum allowable solution time and L_1 is the resulting area. (They are numerically equivalent since $|u| \equiv 1$ so that $\int_0^T |u|\, dt \equiv T$.) If the initial state is outside of the region enclosed by C, then the origin cannot be reached in T seconds with the sub-optimal bang-bang input. If the initial state is on C then a solution exists, $\|u\|_1 = L_1$, and the input should be applied immediately: $u = -\operatorname{sgn} x_2$ until the state of the system nears the origin. If the initial state is within C then a solution exists, $\|u\|_1 < L_1$, and the system should be allowed to coast until the state is on curve C_τ where $\tau = T - t$, the "time-to-go." That is, the curve C continuously contracts so that it corresponds to the allowable time-to-go. This involves the same practical implementation difficulties as discussed previously in connection with curve C since the curve is not a simple function of $\tau(L_1)$. However, once might again use the x_1-axis crossing as an indication as to when the system should become powered. It is obvious that since states outside of C_τ cannot be driven to the origin in τ seconds, the last x_1-axis crossing within C_τ must be used.

TABLE 5-1 Comparison of Optimal and Sub-Optimal Solutions, $\zeta = 0.1$

x_{10}	Terminal Time			
	Optimal	Straight Line	Coast-Bang-Bang	Coast-Bang-Off-Bang
141.3	31.54	33.33	32.29	31.56
136.2	30.94	31.18	31.25	31.16
124.2	30.29	30.74	30.61	30.73
110.4	29.57	30.38	29.98	30.36
101.6	28.78	30.19	29.36	30.15
21.2	15.56	17.59	17.85	15.29[1]
19.6	14.99	15.25	17.61	15.19[2]
18.4	14.68	15.03	17.39	15.02
15.5	13.98	14.66	16.66	14.61[3]
13.3	13.15	14.44	15.22	11.95[4]

Notes: 1 $r_T = 0.19$; 2 $r_T = 0.02$; 3 $r_T = 0.03$; 4 $r_T = 0.24$

TABLE 5-2 Comparison of Optimal and Sub-Optimal Solutions, $\zeta = 0.3$

x_{1O}	Terminal Time			
	Optimal	Straight Line	Coast-Bang-Bang	Coast-Bang-Off-Bang
298.5	16.45	18.56	16.96	16.48
278.0	16.02	16.17	16.17	16.16
224.8	15.48	15.80	15.64	15.80
157.6	14.74	15.45	14.99	15.43
111.8	13.70	15.28	14.15	15.19
39.2	9.83	12.04	10.61	9.90 [1]
36.0	9.40	9.57	9.72	9.61
28.4	8.87	9.21	9.12	9.24
19.1	8.12	8.88	8.49	8.88
13.1	7.02	8.75	7.82	8.69

Note: [1] $r_T = 0.01$

TABLE 5-3 Comparison of Optimal and Sub-Optimal Solutions, $\zeta = 0.5$

x_{1O}	Terminal Time			
	Optimal	Straight Line	Coast-Bang-Bang	Coast-Bang-Off-Bang
311.2	10.87	13.56	11.72	10.91 [1]
286.9	10.59	10.67	11.44	10.73
218.5	10.16	10.25	10.43	10.42
120.8	9.45	9.97	9.83	10.05
48.7	7.99	9.86	9.14	9.87
48.2	7.19	9.95	7.82	7.26 [2]
37.7	6.73	6.83	6.88	6.90
43.3	5.21	6.50	6.37	6.54
15.1	5.79	6.34	5.99	6.35
8.0	5.08	6.24	5.52	6.19

Notes: [1] $r_T = 0.01$; [2] $r_T = 0.03$

CHAPTER VI

ADDITIONAL REMARKS

This chapter briefly considers the extension of the sub-optimal solutions of the previous chapter to the case where the input energy and magnitude rather than the input area and magnitude are constrained. It is also shown that the solution can be used when the input area and energy are constrained and when all three constraints are applicable. The extension of the results to a certain class of higher order systems is also briefly considered. No detailed numerical examples are given in this chapter since the concepts involved are very simple and the situations where "reasonable results" are obtainable are easily predicted.

6.1 Energy and Magnitude Constraints

Consider the same underdamped second order system treated in the previous chapter and shown in Figure 5-1. The constraints are now

$$\|u\|_2^{\,2} = \int_0^T u^2 \; dt \leq L_2^{\,2} \tag{6.1}$$

$$\|u\|_\infty = \max \; |u| \leq 1 \tag{6.2}$$

The optimal input is

$$u = \begin{cases} \underline{\lambda}'\,\underline{h} & |\underline{\lambda}'\,\underline{h}| < 1 \\ \operatorname{sgn}\; \underline{\lambda}'\,\underline{h} & |\underline{\lambda}'\,\underline{h}| > 1 \end{cases} \tag{6.3}$$

where the $\underline{\lambda}'\,\underline{h}$ function is the same as (5.10) which is repeated here for convenience.

$$\underline{\lambda}'\,\underline{h} = K\, e^{\zeta t} \;\cos(\omega t - \phi) \tag{6.4}$$

It is evident that there are two important differences from the Chapter V case. First, the optimal terminal time no longer must occur during a saturation interval since the system is now always powered (except instantaneously at a finite number of zero crossings). Second, the initial coast interval and the off intervals of the area constrained case have been replaced by a "linear" input.

It is to be noted that during the intervals when $|\lambda' \underline{h}| < 1$, $u^2 = |\lambda' \underline{h}|^2 < |\lambda' \underline{h}| < 1$. Hence one expects a relatively small amount of energy to be consumed during these intervals if the optimal input saturates during several "cycles." The case where the input does not saturate--only the energy constraint applies--has been considered by many others (see, for example, references 1 and 2). Flugge-Lotz and Marbach[19] have shown that the same controller can sometimes be used effectively even if saturation occurs. However, if the optimal input is saturated for an appreciable portion of the time, the input approaches a coast-bang-off-bang input. Since $|u| \leq 1$, it is evident that $\|u\|_2^2 \leq \|u\|_1$. This inequality approaches an equality as the portion of the input in saturation increases. Therefore it is possible to use the sub-optimal solutions of the previous chapter for this case if L_1 in the previous chapter is now replaced by L_2^2. "Good" results are to be expected when the input is at the saturation level for an appreciable portion of the total time.

The validity of this solution has already been illustrated in Chapter II using a double integrator with no feedback. The results were given in Figure 2-13. The bang-off-bang sub-optimal input considered there resulted in an increase in terminal time over the optimal L_2-L_∞ solution of less than 10% over a wide range of values of L_2. In a system with feedback, the system's inherent damping will tend to "swamp out" the effect of the input (see below) so that satisfactory performance can be safely predicted for a sizable range of initial state and constraint combinations. Even in the case where the optimal input is not saturated for an appreciable portion of the time, this sub-optimal input may be useful because of its simplicity. Furthermore, for damping ratios greater than about 0.3, the form and value of the input will have little effect upon the time required to reduce the distance of the state from the origin from large initial values to some reasonable value--as is borne out by the results of the previous chapter. Hence it may be concluded that the results of the previous chapter are directly applicable to the case where the input energy and magnitude are constrained. It should be noted, in passing, that the sub-optimal solutions of the previous chapter can also be applied to the case of an input energy constraint alone by arbitrarily assigning a value for $L_\infty^- = |u|$ max. The result obtained for the double integrator case is again shown in Figure 2-13 which indicates that reasonable performance may be obtained over a wide range of conditions.

The circumstances involved in any specific situation must be carefully considered, however, if near optimal performance is desired since this type of sub-optimal solution might best be described as a sub-optimal solution, once removed, i.e., it is a sub-optimal to a sub-optimal. Once again one is faced with a trade-off between simplicity and performance. As previously stated, little can be said in general concerning this trade-off.

6.2 Energy, Area, and Magnitude Constraints

As far as the writer knows, this case is only of academic interest and is included only for completeness. The constraints are now

$$\|u\|_1 = \int_0^T |u| \; dt \leq L_1 \tag{6.5}$$

$$\|u\|_2^{\,2} = \int_0^T u^2 \; dt \leq L_2^2 \tag{6.6}$$

$$\|u\|_\infty = \max \; |u| \leq 1 \tag{6.7}$$

and the optimal input is given by

$$u = \begin{cases} 0 & |\underline{\lambda}'\underline{h}| < w_1 \\ \underline{\lambda}'\underline{h} - w_1 \operatorname{sgn} \underline{\lambda}'\underline{h} & w_1 < |\underline{\lambda}'\underline{h}| < w_1 + 1 \\ \operatorname{sgn} \underline{\lambda}'\underline{h} & |\underline{\lambda}'\underline{h}| > w_1 + 1 \end{cases} \tag{6.8}$$

where $\underline{\lambda}'\underline{h}$ is as given in (6.4). If $L_2^2 > L_1$, then as discussed in Chapter II, the energy constraint does not apply since $\|u\|_2^2 < \|u\|_1$. With $L_2^2 < L_1$, the sub-optimal solution of the previous section can be used. Once again the example of Chapter II can be used as an indication of the performance which might be expected. Figure 2-13 shows a rather rapid deterioration in performance as L_2^2 is reduced substantially below L_1. Hence unless $L_2^2 \approx L_1$ (or $L_2^2 > L_1$), the sub-optimal solution may be expected to result in a considerable percentage increase in the solution time over the optimal time. However, with damping in the system, the increase whould not be as great as indicated by Figure 2-13.

Special Case: Energy and Area Constraints

If there is no magnitude constraint, it is necessary to artificially select one in order to use the sub-optimal solutions which have been developed here. With an area constraint alone the selection is completely arbitrary, but in this case there is a sound basis for selecting a particular value. From (6.5) and (6.6) we have

$$L_\infty T_{on} = L_1 \qquad (6.9)$$

$$L_\infty^2 T_{on} = L_2^2 \qquad (6.10)$$

where T_{on} is the total on time, i.e., the total time when $|u| = L_\infty$, the saturation level. From these we have

$$L_\infty = L_2^2/L_1 \qquad (6.11)$$

It is now necessary to normalize the system equations so that $L_\infty = 1$. In this way this case reduces to the one just discussed. (6.11) also provides a means for improving the performance of the case where all three constraints apply. It was noted above that the performance in such a case is poorest when L_2^2 is less than L_1. Rather than operate on the boundary of the magnitude constraint in this situation, it would seem desirable to operate on the boundary of the other two constraints with the value of u_{max} being determined by (6.11).

6.3 Extension to Higher Order Systems

This section briefly outlines a procedure which can be used to apply the results of Chapter V to a limited class of higher order systems whose input is constrained in both area and magnitude. (Of course the results developed in the previous sections can be used in combination with those developed here if other constraint combinations are present.)

All that is necessary is to find an equivalent zeta (the damping ratio) and omega (the damped natural frequency) for a second order system whose behavior is essestially the same as the system under consideration. This, naturally, is only possible if the system behaves essentially like an underdamped second order system, i.e., it has a predominant pair of complex conjugate poles with all other poles well removed from the origin of the s-plane. The design of control systems using classical procedures based upon the step input response of a second order system is well established. Since the sub-optimal inputs under consideration may be considered to be a series of step inputs, it is reasonable to expect that they can be satisfactorily used to control two of the outputs of the class of higher order systems whose behavior is governed by a pair of complex conjugate poles.

The required values for zeta and omega can easily be determined if the system is initially allowed to coast until at least two successive x_1-axis (or x_2-axis) crossings are accomplished. Let the distance from the origin to these two crossings be denoted by R_{c1} and R_{c2} and the elapsed time between the crossings be denoted by t_1. For a second order system, we would have

$$\omega t_1 = \pi \tag{6.12}$$

$$R_{c2} = R_{c1} e^{-\zeta t_1} \tag{6.13}$$

Hence, if the system under consideration behaves essentially as a second order system, we have

$$\zeta = \frac{1}{t_1} \ln R_{c1}/R_{c2} \tag{6.14}$$

$$\omega = \pi / t_1 \tag{6.15}$$

In order to use the results of Chapter V, all that remains is to do is a normalization of the time scale since in Chapter V

$$\zeta^2 + \omega^2 = 1 \tag{6.16}$$

With this accomplished, the results of Chapter V can be applied directly. As stated above, reasonable performance can be expected if the system has a predominant pair of complex conjugate poles.

6.4 Suggestions for Future Work

The research reported here points out that substantial additional work is required in this area if sub-optimal controllers are to be usefully and effectively implemented in practical engineering designs. Some work is needed to develop a simple implementation for the curve C in Chapter V, for example, so that a coast-bang-bang input would use all--or almost all--of the available fuel. It is also necessary to develop a controller for higher order systems whose behavior is not governed solely by a predominant pair of complex conjugate poles since many physical systems are truly third or fouth--or even higher--order. The extension to multi-input systems is also important since an increasingly larger body of systems falls into this category--and are generally the ones where "optimal control" can be used to advantage. Although an educated guess was made as to the form of the optimal inputs for a multi-input system with various input constraint combinations, it remains to establish that this is indeed the case. This would then form the basis for the investigation of practical sub-optimal solutions.

Probably the most important single item which needs considerable future study is the development of a procedure by which it can be readily established, in any given case, upon the boundary of which constraints the input operates. The procedure suggested in Chapter II is a trial and error method which involves assuming various constraint combinations and checking the results obtained to see if all other constraints are satisfied. While this might be satisfactory for a single input system, a large number of trials may be required in a multi-input system.

A final item which must certainly be considered is the effect of parameter variations upon the solution. It is obvious from even a cursory inspection of the results obtained in Chapter V that the value of zeta has a considerable influence upon the solution. The sensitivity of the control schemes investigated to an error in the determination of zeta or one of the other system parameters merits a detailed study.

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