Poincaré Duality Induces A BV-Structure On Hochschild Cohomology

by

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Abstract

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Adviser: Professor Dennis Sullivan

In this thesis we develop the notion of Poincaré-duality for A_∞-algebras. Algebraically this allows a synthesis of Gerstenhaber's G-algebra structure on one Hochschild complex with Connes' cyclic structure on the other Hochschild complex. We obtain a BV-algebra by combining both structures. Geometrically, the notion of Poincaré-duality for A_∞-algebras can be constructed in the intersection theory of a closed manifold. We obtain an algebraic model for the BV-algebra of string topology verified modulo one point.

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Für Barbara

Contents

1 Introduction

The original problem for this thesis was to construct an algebraic model for the string topology [6] on the homology of the free loop space of a simply connected manifold. More explicitly, in the paper [6] by M. Chas and D. Sullivan it was shown that the homology of the free loop space is a BV-algebra. This BV-algebra consists of a Δ-operator with $\Delta^2 = 0$, coming from the circle action on the free loop space, and a product, which combines the intersection-product on chains in a manifold with composition of loops. These two operations fit together in the sense that the deviation of Δ from being a derivation of the product is the bracket of a Gerstenhaber-algebra.

Algebraically, in M. Gerstenhaber's paper [8] it was shown that for any associative algebra A, Hochschild-cohomology $H^\bullet(A, A)$ has a Gerstenhaber-structure. More generally, there is a Gerstenhaber-structure on Hochschild-cohomology $H^\bullet(A, A)$ of every A_∞-algebra A, see [10]. On the other hand, the Hochschild-cochain-complex $C^\bullet(A, A^\bullet)$ of A with values in its dual bimodule $A^\bullet$ carries a B-operator with $B^2 = 0$, which was used by A. Connes [4] to define cyclic homology. In this thesis, the notion of an ∞-Poincaré-duality-structure for A_∞-algebras is developed, which is a special kind of quasi-isomorphism of the Hochschild-spaces $C^\bullet(A, A)$ and $C^\bullet(A, A^\bullet)$, determined by certain tree operations. This allows one to combine the B-operator on $C^\bullet(A, A^\bullet)$ with the multiplication of the Gerstenhaber-algebra on $C^\bullet(A, A)$ to give a BV-algebra on homology. Furthermore, the Gerstenhaber-bracket induced by this BV-algebra is the one described by Gerstenhaber in [8].

The question now is to compare this abstract BV-algebra with the one from string topology [6]. It is known that for simply connected spaces X, the chains of the free loop space is quasi-isomorphic to the Hochschild-cochain-complex $C^\bullet(A, A^\bullet)$ of the cochain-algebra $A = C^\bullet(X)$ with values in its dual bimodule $A^\bullet$; compare J.D.S. Jones [12]. Jones' paper also identifies the Δ-operator on the free loop space homology with the B-operator. In order to complete the identification of the BV-structures, one coming from our ∞-Poincaré-duality-algebra, the other from string topology, two steps would have to be considered. First, one would have to verify that the cochains $C^\bullet(X)$ of a manifold indeed possess an ∞-Poincaré-duality-structure which can be used for the above proof. M. Zeinalian and the author have shown (see chapter 4) that using Poincaré-duality on a compact oriented manifold allows one to construct a suitable chain-map of the Hochschild-spaces. Showing that this map is a quasi-isomorphism is work in progress. A more stringent version of Poincaré Duality was already given before. But it seems almost impossible to realize this notion on the chain level of a given closed manifold. Secondly, one would have to show that the multiplication on the free loop space homology coincides with the one on $H^\bullet(C^\bullet(X), C^\bullet(X))$ when using the Jones' identification and the constructed Poincaré duality quasi-isomorphism between $C^\bullet(C^\bullet(X), C^\bullet(X))$ and $C^\bullet(C^\bullet(X), C_\bullet(X))$ from chapter 4. A closely related result was announced in a paper by R. Cohen and J.D.S. Jones [5].

2 ∞ Poincaré Duality on A_∞ Algebras

Convention: In all of this thesis, the convention will be used that the differential in a complex points downward, so that for example chains are concentrated in positive, and cochains are concentrated in negative degree.

2.1 A_∞ Algebras

Let us review the usual definitions about A_∞-algebras that are important to the discussion of this paper, and that can be found in many sources (compare [9] section 1, and [17]).

Definition 2.1. *A **coalgebra** (C, Δ) over a ring R consists of an R-module C and a comultiplication $\Delta : C \to C \otimes C$ of degree 0 satisfying coassociativity:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes id} \\ C \otimes C & \xrightarrow[id \otimes \Delta]{} & C \otimes C \otimes C \end{array}$$

*Then a **coderivation** on C is a map $f : C \to C$ such that*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes id + id \otimes f} \\ C & \xrightarrow[\Delta]{} & C \otimes C \end{array}$$

Definition 2.2. *Let $V = \bigoplus_{j \in \mathbf{Z}} V_j$ be a graded module over a given ground ring R. The **tensor-coalgebra** of V over the ring R is given by*

$$TV := \bigoplus_{i \geq 0} V^{\otimes i}.$$

$$\Delta : TV \to TV \otimes TV, \qquad \Delta(v_1, ..., v_n) := \sum_{i=0}^{n} (v_1, ..., v_i) \otimes (v_{i+1}, ..., v_n).$$

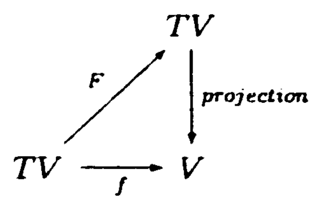
Let $A = \bigoplus_{j \in \mathbf{Z}} A_j$ be a graded module over a given ground ring R. Define its ***suspension*** *sA to be the graded module $sA = \bigoplus_{j \in \mathbf{Z}} (sA)_j$ with $(sA)_j := A_{j-1}$. The suspension map $s : A \to sA$, $s : a \mapsto sa := a$ is an isomorphism of degree +1.*

Now the ***bar complex*** *of A is given by $BA := T(sA)$.*

An ***$\mathbf{A}_\infty$-algebra*** *on A is given by a coderivation D on BA of degree -1 such that $D^2 = 0$.*

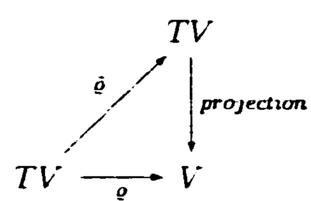
Let's try to understand this definition.

The tensor-coalgebra has the property to lift every module map $f : TV \to V$ to a coalgebra-map $F : TV \to TV$:



A similar property for coderivations on TV will make it possible to understand the definition of A_∞-algebras in a different way.

Lemma 2.3. *(a) Given a map $\varrho : V^{\otimes n} \to V$ of degree $|\varrho|$, which can be viewed as a map $\varrho : TV \to V$ by letting its only non-zero component being given by the original ϱ on $V^{\otimes n}$. Then ϱ lifts uniquely to a coderivation $\bar{\varrho} : TV \to TV$ with*



by taking

$$\tilde{\varrho}(v_1, ..., v_k) := 0, \quad for \;\; k < n,$$

$$\tilde{\varrho}(v_1, ..., v_k) := \sum_{i=0}^{k-n} (-1)^{|\varrho|\cdot(|v_1|+...+|v_i|)}(v_1, ..., \varrho(v_{i+1}, ..., v_{i+n}), ..., v_k), \quad for \;\; k \geq n.$$

Thus $\tilde{\varrho}\,|_{V^{\otimes k}}: V^{\otimes k} \to V^{\otimes k-n+1}$.

(b) There is a one-to-one correspondence between coderivations $\sigma : TV \to TV$ *and systems of maps* $\{\varrho_i : V^{\otimes i} \to V\}_{i\geq 0}$, *given by* $\sigma = \sum_{i\geq 0} \tilde{\varrho}_i$.

Proof. (a) The argument here is dual to the way one lifts derivations on TA. To be precise one should use induction on the output-component of $\tilde{\varrho}$. Denote by $\tilde{\varrho}^j$ the component of $\tilde{\varrho}$ mapping $TV \to V^{\otimes j}$. Then $\tilde{\varrho}^1, ..., \tilde{\varrho}^{m-1}$ determine uniquely the component $\tilde{\varrho}^m$, because of the coderivation property of $\tilde{\varrho}$. Let's derive an equation with which this can be seen easily.

$$\begin{aligned}
\Delta(\tilde{\varrho}(v_1, ..., v_k)) &= (\tilde{\varrho} \otimes id + id \otimes \tilde{\varrho})(\Delta(v_1, ..., v_k)) = \\
&= (\tilde{\varrho} \otimes id + id \otimes \tilde{\varrho})(\sum_{i=0}^{k}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k)) = \\
&= \sum_{i=0}^{k} \tilde{\varrho}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k) + \\
&\quad +(-1)^{|\tilde{\varrho}|\cdot(|v_1|+...+|v_i|)}(v_1, ..., v_i) \otimes \tilde{\varrho}(v_{i+1}, ..., v_k).
\end{aligned}$$

Now, projecting both sides to $\bigoplus_{i+j=m} V^{\otimes i} \otimes V^{\otimes j} \subset TV \otimes TV$ yields

$$\begin{aligned}
\Delta(\tilde{\varrho}^m(v_1, ..., v_k)) &= \sum_{i=0}^{k} \tilde{\varrho}^{m+i-k}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k) + \\
&\quad +(-1)^{|\tilde{\varrho}|\cdot(|v_1|+...+|v_i|)}(v_1, ..., v_i) \otimes \tilde{\varrho}^{m-i}(v_{i+1}, ..., v_k).
\end{aligned}$$

So the righthand side depends only on $\tilde{\varrho}^j$ with $j < m$, except for the uninteresting terms $\tilde{\varrho}^m(v_1, ..., v_k) \otimes 1$ and $1 \otimes \tilde{\varrho}^m(v_1, ..., v_k)$, where $1 \in TV$. With this, an induction argument shows that $\tilde{\varrho}^m$ is only nonzero on $V^{\otimes k}$ for $k = m + n - 1$, where it is

$$\tilde{\varrho}^m(v_1, ..., v_{m+n-1}) = \sum_{i=0}^{m-1} (-1)^{|\varrho| \cdot \epsilon_i}(v_1, ..., \varrho(v_{i+1}, ..., v_{i+n}), ..., v_{m+n-1}),$$

with $\epsilon_i = |v_1| + ... + |v_i|$.

(b) Observe that being a coderivation is a linear condition, and so the sum of coderivations is again a coderivation. Thus the map

$$\alpha : \{\{\varrho_i : V^{\otimes i} \to V\}_{i \geq 0}\} \to Coder(TV), \quad \{\varrho_i : V^{\otimes i} \to V\}_{i \geq 0} \mapsto \sum_{i \geq 0} \tilde{\varrho}_i$$

is well defined. Its inverse β is given by $\beta : \sigma \mapsto \{pr_V \circ \sigma|_{V^{\otimes i}}\}_{i \geq 0}$, because the explicit lifting property of (a) shows that $\beta \circ \alpha = id$, and the uniqueness part of (a) shows that $\alpha \circ \beta = id$.

□

Let's apply this to Definition 2.2.

Proposition 2.4. *Let (A, D) be an A_∞-algebra. Now let D be given by a system of maps $\{D_i : sA^{\otimes i} \to sA\}_{i \geq 1}$, (where $D_0 = 0$ is assumed.) just like in Lemma 2.3.(b), and rewrite them as $m_i : A^{\otimes i} \to A$ given by $D_i = s \circ m_i \circ (s^{-1})^{\otimes i}$.*

Then the condition $D^2 = 0$ is equivalent to the following system of equations:

$$
\begin{aligned}
m_1(m_1(a_1)) &= 0, \\
m_1(m_2(a_1, a_2)) - m_2(m_1(a_1), a_2) - (-1)^{|a_1|} m_2(a_1, m_1(a_2)) &= 0, \\
m_1(m_3(a_1, a_2, a_3)) - m_2(m_2(a_1, a_2), a_3) + m_2(a_1, m_2(a_2, a_3)) + & \\
+ m_3(m_1(a_1), a_2, a_3) + (-1)^{|a_1|} m_3(a_1, m_1(a_2), a_3) + & \\
+ (-1)^{|a_1|+|a_2|} m_3(a_1, a_2, m_1(a_3)) &= 0, \\
\ldots & \\
\sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\varepsilon} \cdot m_{k-i+1}(a_1, \ldots, m_i(a_j, \ldots, a_{j+i-1}), \ldots, a_k) &= 0, \\
\textit{where } \varepsilon = i \cdot \sum_{l=1}^{j-1} |a_l| + (j-1) \cdot (i+1) + k - i & \\
\ldots &
\end{aligned}
$$

Proof. The only difficulty is to determine the signs when replacing the D_r's by the m_r's. First notice that

$$
\begin{aligned}
& D_k(sa_1, \ldots, sa_k) = \\
= \; & s \circ m_k \circ (s^{-1})^{\otimes k}(sa_1, \ldots, sa_k) = \\
= \; & (-1)^{\sum_{j=1}^{k-1}(|a_j|+1)} s \circ m_k \circ ((s^{-1})^{\otimes k-1} \otimes id)(sa_1, \ldots, sa_{k-1}, s^{-1}sa_k) = \\
= \; & (-1)^{\sum_{j=1}^{k-2} 2(|a_j|+1) + |a_{k-1}|+1} s \circ m_k \circ ((s^{-1})^{\otimes k-2} \otimes (id)^{\otimes 2}) \\
& \qquad (sa_1, \ldots, sa_{k-2}, s^{-1}sa_{k-1}, s^{-1}sa_k) = \\
& \ldots \\
= \; & (-1)^{\sum_{j=1}^{k}(k-j)(|a_j|+1)} s \circ m_k(a_1, \ldots, a_k).
\end{aligned}
$$

Therefore it follows from Lemma 2.3.(a) applied to the degree -1 coderivation D, that

$$pr_{sA} \circ D^2(sa_1, ..., sa_k) = \tag{2.5}$$

$$= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\sum_{l=1}^{j-1}(|a_l|+1)} D_{k-i+1}(sa_1, ..., D_i(sa_j, ..., sa_{j+i-1}), ..., sa_k) =$$

$$= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\sum_{l=1}^{j-1}(|a_l|+1)+\sum_{l=j}^{j+i-1}(j+i-l-1)\cdot(|a_l|+1)} \cdot$$

$$\cdot D_{k-i+1}(sa_1, ..., s \circ m_i(a_j, ..., a_{j+i-1}), ..., sa_k) =$$

$$= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\varepsilon} m_{k-i+1}(a_1, ..., m_i(a_j, ..., a_{j+i-1}), ..., a_k),$$

where ε can be determined by using the fact that $m_r = \pm s^{-1} \circ D_r \circ s^{\otimes r}$ is of degree $-1 - 1 + r = r - 2$. Instead of doing the general case it is more instructive to look at four special cases where i and j are either even or odd. Let's take $k = 8$. As seen above, signs occur from applying D_i, transforming D_i into m_i, and transforming D_{k-i+1} into m_{k-i+1}.

$i = 3,\ j = 3:$

	a_1	a_2	$(a_3$	a_4	$a_5)$	a_6	a_7	a_8
$D_i:$	$\lvert a_1\rvert + 1$	$\lvert a_2\rvert + 1$						
$m_i:$				$\lvert a_4\rvert + 1$				
$m_{k-i+1}:$	$\lvert a_1\rvert + 1$		$(\lvert a_3\rvert +$	$\lvert a_4\rvert + \lvert a_5\rvert$	$+3 - 1)$		$\lvert a_7\rvert + 1$	

$i = 3,\ j = 4:$

	a_1	a_2	a_3	$(a_4$	a_5	$a_6)$	a_7	a_8
$D_i:$	$\lvert a_1\rvert + 1$	$\lvert a_2\rvert + 1$	$\lvert a_3\rvert + 1$					
$m_i:$					$\lvert a_5\rvert + 1$			
$m_{k-i+1}:$	$\lvert a_1\rvert + 1$		$\lvert a_3\rvert + 1$				$\lvert a_7\rvert + 1$	

So, if i is odd, then the lower two rows (m_i and m_{k-i+1}) show that here the "a_r"-terms are exactly $\sum_{l=1}^{k}(k-l)\cdot(|a_l|+1)$, and for $j=3$ there is an additional -1. The top (D_i-)row has the terms $\sum_{l=1}^{j-1}(|a_l|+1) = (\sum_{l=1}^{j-1}|a_l|)+j-1$. The additional -1 can be put together with the $j-1$ to give a constant depending only on k. Thus the term for ε is given by

$$\varepsilon = \sum_{l=1}^{k}(k-l)\cdot(|a_l|+1) + \sum_{l=1}^{j-1}|a_l| + k - 1$$

$i=4,\ j=3:$

	a_1	a_2	$(a_3$	a_4	a_5	$a_6)$	a_7	a_8
D_i :	$\lvert a_1\rvert+1$	$\lvert a_2\rvert+1$						
m_i :			$\lvert a_3\rvert+1$		$\lvert a_5\rvert+1$			
m_{k-i+1} :		$\lvert a_2\rvert+1$					$\lvert a_7\rvert+1$	

$i=4,\ j=4:$

	a_1	a_2	a_3	$(a_4$	a_5	a_6	$a_7)$	a_8
D_i :	$\lvert a_1\rvert+1$	$\lvert a_2\rvert+1$	$\lvert a_3\rvert+1$					
m_i:				$\lvert a_4\rvert+1$		$\lvert a_6\rvert+1$		
m_{k-i+1}:		$\lvert a_2\rvert+1$		$(\lvert a_4\rvert+$	$\lvert a_5\rvert+\lvert a_6\rvert$	$+\lvert a_7\rvert+$	$4-1)$	

So, if i is even then the "a_r"-terms (from all three rows) are exactly $\sum_{l=1}^{k}(k-l)\cdot(|a_l|+1)$. The only j-dependence is the additional -1 in the $i=4,\ j=4$ case, which will induce an alternating sign (, which starts depending on k). So, this case gives:

$$\varepsilon = \sum_{l=1}^{k}(k-l)\cdot(|a_l|+1) + k + j - 1.$$

Putting both cases together, and bringing the common term $\sum_{l=1}^{k}(k-l)\cdot(|a_l|+1)$ (for both i even and i odd) to the left, one gets the sign

$$\begin{aligned}\varepsilon-(\sum_{l=1}^{k}(k-l)\cdot(|a_l|+1)) &= i\cdot(\sum_{l=1}^{j-1}|a_l|+k-1)+(i+1)\cdot(k+j-1)= \\ &= i\cdot\sum_{l=1}^{j-1}|a_l|+(j-1)\cdot(i+1)+k-i.\end{aligned}$$

Thus, dividing the equation $D^2=0$ by the sign $(-1)^{\sum_{l=1}^{k}(k-l)\cdot(|a_l|+1)}$ yields the result. □

Example 2.6. *Any differential graded algebra (A,∂,μ) gives an A_∞-algebra-structure on A by taking $m_1:=\partial$, $m_2:=\mu$ and $m_k:=0$ for $k\geq 3$. Then the equations from Proposition 2.4. are the defining conditions of a differential graded algebra:*

$$\begin{aligned}\partial^2(a) &= 0, \\ \partial(a\cdot b) &= \partial(a)\cdot b+(-1)^{|a|}a\cdot\partial(b), \\ (a\cdot b)\cdot c &= a\cdot(b\cdot c).\end{aligned}$$

There are no higher equations.

Definition 2.7. *Given an A_∞-algebra (A,D). Then the **Hochschild-cochain-complex of** A is defined to be the space $C^\bullet(A):=CoDer(BA,BA)$ of coderivations on BA with the differential $\delta: C^\bullet(A)\to C^\bullet(A)$ given by $\delta(f):=[D,f]=D\circ f-(-1)^{|f|}f\circ D$. It is $\delta^2=0$, because as D is of degree -1 and $D^2=0$ one follows that $\delta^2(f)=[D,D\circ f-(-1)^{|f|}f\circ D]=D\circ D\circ f-(-1)^{|f|}D\circ f\circ D-(-1)^{|f|+1}D\circ f\circ D+(-1)^{|f|+|f|+1}f\circ D\circ D=0$.*

2.2 A_∞ Bimodules

Given an A_∞-algebra (A, D). The goal is now to define the concept of an A_∞-bimodule over A, which was already given in [9] and [14].

This should be a generalization of two facts. First, one can define the Hochschild-cochain-complex for any algebra with values in a bimodule, and so one should still be able to make that definition in the infinite case. Second, any algebra is a bimodule over itself by left- and right-multiplication, which should again still hold (see section 2.1.4.).

The following space and map are important ingredients.

Definition 2.8. *For modules V and W over R, one writes*

$$T^W V := R \oplus \bigoplus_{k\geq 0, l\geq 0} V^{\otimes k} \otimes W \otimes V^{\otimes l}.$$

Furthermore, let

$$\Delta^W : T^W V \to (TV \otimes T^W V) \oplus (T^W V \otimes TV),$$

be given by

$$\begin{aligned}\Delta^W(v_1, \ldots, v_k, w, v_{k+1}, \ldots, v_{k+l}) \; &:= \; \sum_{i=0}^{k} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, w, \ldots, v_n) + \\ &+ \sum_{i=k}^{k+l} (v_1, \ldots, w, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{k+l}).\end{aligned}$$

Again for modules A and M let $B^M A$ be given by $T^{sM} sA$, where s is the suspension from Definition 2.2.

Observe that $T^W V$ is not a coalgebra, but rather a bi-comodule over TV. Here is a definition for a coderivation from TV to $T^W V$.

Definition 2.9. *A* ***coderivation*** *from TV to $T^W V$ is defined to be a map $f : TV \to T^W V$ that makes the following diagram commute:*

$$\begin{array}{ccc} TV & \xrightarrow{\Delta} & TV \otimes TV \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id\otimes f + f\otimes id} \\ T^W V & \xrightarrow[\Delta^W]{} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \end{array}$$

For modules A and M let $C^\bullet(A, M) := CoDer(BA, B^M A)$ be the space of coderivations in the above sense. This space is called the ***Hochschild-cochain-complex of A with values in M.***

Lemma 2.10. *(a) Given a map $\varrho : V^{\otimes n} \to W$ of degree $|\varrho|$, which can be viewed as a map $\varrho : TV \to W$ by letting its only non-zero component being given by the original ϱ on $V^{\otimes n}$. Then ϱ lifts uniquely to a coderivation $\tilde{\varrho} : TV \to T^W V$ with*

$$\begin{array}{ccc} & & T^W V \\ & {\scriptstyle \tilde{\varrho}}\nearrow & \downarrow{\scriptstyle projection} \\ TV & \xrightarrow[\varrho]{} & W \end{array}$$

by taking

$$\tilde{\varrho}(v_1, \ldots, v_k) := 0, \qquad for\ \ k < n,$$

$$\tilde{\varrho}(v_1, \ldots, v_k) := \sum_{i=0}^{k-n} (-1)^{|\varrho|\cdot(|v_1|+\ldots+|v_i|)} (v_1, \ldots, \varrho(v_{i+1}, \ldots, v_{i+n}), \ldots, v_k),$$

$$for\ \ k \geq n.$$

Thus $\tilde{\varrho}\,|_{V^{\otimes k}} : V^{\otimes k} \to \bigoplus_{i+j=k-n} V^{\otimes i} \otimes W \otimes V^{\otimes j}$.

(b) There is a one-to-one correspondence between coderivations $\sigma : TV \to T^W V$ and systems of maps $\{\varrho_i : V^{\otimes i} \to W\}_{i\geq 0}$, given by $\sigma = \sum_{i\geq 0} \tilde{\varrho}_i$.

Proof. (a) The proof is similar to the one of Lemma 2.3.(a). Now $\tilde{\varrho}^j$ is meant to be the component of $\tilde{\varrho}$ mapping $TV \to \bigoplus_{r+s=j} V^{\otimes r} \otimes W \otimes V^{\otimes s}$. Let's do again induction on m for $\tilde{\varrho}^m$. The equation

$$\begin{aligned}
\Delta^W(\tilde{\varrho}(v_1, ..., v_k)) &= (\tilde{\varrho} \otimes id + id \otimes \tilde{\varrho})(\Delta(v_1, ..., v_k)) = \\
&= (\tilde{\varrho} \otimes id + id \otimes \tilde{\varrho})(\sum_{i=0}^{k} (v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k)) = \\
&= \sum_{i=0}^{k} \tilde{\varrho}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k) + \\
&\quad +(-1)^{|\tilde{\varrho}|\cdot(|v_1|+...+|v_i|)}(v_1, ..., v_i) \otimes \tilde{\varrho}(v_{i+1}, ..., v_k)
\end{aligned}$$

is being projected to the component

$$\bigoplus_{r+s+t=m} (V^r \otimes W \otimes V^s) \otimes V^t + V^r \otimes (V^s \otimes W \otimes V^t) \subset T^W V \otimes TV + TV \otimes T^W V.$$

This gives

$$\begin{aligned}
\Delta^W(\tilde{\varrho}^m(v_1, ..., v_k)) &= \sum_{i=0}^{k} \tilde{\varrho}^{m+i-k}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k) + \\
&\quad +(-1)^{|\tilde{\varrho}|\cdot(|v_1|+...+|v_i|)}(v_1, ..., v_i) \otimes \tilde{\varrho}^{m-i}(v_{i+1}, ..., v_k).
\end{aligned}$$

The righthand side depends only on $\tilde{\varrho}^j$ with $j < m$, except for the uninteresting terms $\tilde{\varrho}^m(v_1, ..., v_k) \otimes 1$ and $1 \otimes \tilde{\varrho}^m(v_1, ..., v_k)$. An induction shows that $\tilde{\varrho}^m$ is only nonzero on $V^{\otimes k}$ for $k = m + n - 1$, where it is

$$\tilde{\varrho}^m(v_1, ..., v_{m+n-1}) = \sum_{i=0}^{m-1} (-1)^{|\varrho|\cdot\epsilon_i}(v_1, ..., \varrho(v_{i+1}, ..., v_{i+n}), ..., v_{m+n-1}),$$

where $\epsilon_i = |v_1| + ... + |v_i|$.

(b) Then maps

$$\alpha : \{\{\varrho_i : V^{\otimes i} \to W\}_{i\geq 0}\} \to Coder(TV, T^W V),$$

$$\{\varrho_i : V^{\otimes i} \to W\}_{i\geq 0} \mapsto \sum_{i\geq 0} \tilde{\varrho}_i$$

$$\beta : Coder(TV, T^W V) \to \{\{\varrho_i : V^{\otimes i} \to W\}_{i\geq 0}\},$$

$$\sigma \mapsto \{pr_W \circ \sigma|_{V^{\otimes i}}\}_{i\geq 0}$$

are inverse to each other by (a).

□

Of course one wants to put a differential on $C^\bullet(A, M)$ just like in section 2.1.1.

Proposition 2.11. *Given an A_∞-algebra (A, D) and a module M. Let $D^M : B^M A \to B^M A$ be a map of degree -1. Then the induced map $\delta^M : CoDer(BA, B^M A) \to CoDer(BA, B^M A)$, given by $\delta^M(f) := D^M \circ f - (-1)^{|f|} f \circ D$, is well-defined, (i.e. it maps coderivations to coderivations.) if and only if the following diagram commutes:*

$$\begin{array}{ccc} B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\ \downarrow{\scriptstyle D^M} & & \downarrow{\scriptstyle \kappa} \\ B^M A & \xrightarrow[\Delta^M]{} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \end{array} \tag{2.12}$$

where the vertical map on the right is given by $\kappa = (id \otimes D^M + D \otimes id) \oplus (D^M \otimes id + id \otimes D)$.

Proof. Let $f : BA \to B^M A$ be a coderivation. One needs to investigate under which conditions $\delta^M(f)$ is also a coderivation. This means, that

$$(id \otimes \delta^M(f) + \delta^M(f) \otimes id) \circ \Delta = \Delta^M \circ \delta^M(f),$$

or

$$(id \otimes (D^M \circ f) - (-1)^{|f|} id \otimes (f \circ D) + (D^M \circ f) \otimes id - (-1)^{|f|} (f \circ D) \otimes id) \circ \Delta =$$

$$= \Delta^M \circ D^M \circ f - (-1)^{|f|} \Delta^M \circ f \circ D.$$

Now, using the coderivation property for f and D, one gets the following identity

$$\begin{aligned} \Delta^M \circ f \circ D &= (id \otimes f) \circ \Delta \circ D + (f \otimes id) \circ \Delta \circ D = \\ &= (id \otimes f) \circ (id \otimes D) \circ \Delta + (id \otimes f) \circ (D \otimes id) \circ \Delta + \\ &\quad + (f \otimes id) \circ (id \otimes D) \circ \Delta + (f \otimes id) \circ (D \otimes id) \circ \Delta = \\ &= (id \otimes (f \circ D) + (-1)^{|f|} D \otimes f + f \otimes D + (f \circ D) \otimes id) \circ \Delta, \end{aligned}$$

and so the requirement for $\delta^M(f)$ above reduces to

$$\Delta^M \circ D^M \circ f =$$

$$= (id \otimes (D^M \circ f) + (D^M \circ f) \otimes id + D \otimes f + (-1)^{|f|} f \otimes D) \circ \Delta =$$

$$= (id \otimes D^M + D \otimes id) \circ (id \otimes f) \circ \Delta + (D^M \otimes id + id \otimes D) \circ (f \otimes id) \circ \Delta =$$

$$= (id \otimes D^M + D \otimes id) \circ \Delta^M \circ f + (D^M \otimes id + id \otimes D) \circ \Delta^M \circ f.$$

The last step looks strange, because $\Delta^M \circ f = (id \otimes f) \circ \Delta + (f \otimes id) \circ \Delta$. But as $id \otimes D^M + D \otimes id : BA \otimes B^M A \to BA \otimes B^M A$, this map doesn't pick up any part from $(f \otimes id) \circ \Delta : BA \to B^M A \otimes BA$. A similar argument applies to $D^M \otimes id + id \otimes D$.

So, D^M has to satisfy

$$\Delta^M \circ D^M \circ f = (id \otimes D^M + D \otimes id + D^M \otimes id + id \otimes D) \circ \Delta^M \circ f,$$

for all coderivations $f : TA \to T^M A$. By Lemma 2.10., one sees that there are enough coderivations, to make this condition equivalent to

$$\Delta^M \circ D^M = (id \otimes D^M + D \otimes id + D^M \otimes id + id \otimes D) \circ \Delta^M,$$

which is the claim. □

Again one wants to describe D^M by a system of maps.

Lemma 2.13. *(a) Given a module V and a coderivation ψ on TV with an associated system of maps $\{\psi_i : V^{\otimes i} \to V\}_{i \geq 1}$ from Lemma 2.3., where ψ_i is of degree $|\psi_i|$. Then any map $\varrho : T^W V \to W$ given by $\varrho = \sum_{k \geq 0, l \geq 0} \varrho_{k,l}$, with $\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to W$ of degree $|\varrho_{k,l}|$, lifts uniquely to a map $\bar{\varrho} : T^W V \to T^W V$*

$$\begin{array}{ccc} & & T^W V \\ & \overset{\bar{\varrho}}{\nearrow} & \Big\downarrow \text{projection} \\ T^W V & \xrightarrow[\varrho]{} & W \end{array}$$

which makes the following diagram commute (compare diagram (2.12) in Proposition 2.11.)

$$\begin{array}{ccc} T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\ \Big\downarrow \bar{\varrho} & & \Big\downarrow (id \otimes \bar{\varrho} + \psi \otimes id) \oplus (\bar{\varrho} \otimes id + id \otimes \psi) \\ T^W V & \xrightarrow[\Delta^W]{} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \end{array} \quad (2.14)$$

This map is given by taking

$$\tilde{\varrho}(v_1, ..., v_k, w, v_{k+1}, ..., v_{k+l}) :=$$

$$:= \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (-1)^{|\psi_i| \cdot \epsilon_{j-1}} (v_1, ..., \psi_i(v_j, ..., v_{i+j-1}), ..., w, ..., v_{k+l}) +$$

$$+ \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{|\varrho_{i,j}| \cdot \epsilon_{k-i}} (v_1, ..., \varrho_{i,j}(v_{k-i+1}, ..., w, ..., v_{k+j}), ..., v_{k+l}) +$$

$$+ \sum_{i=1}^{l} \sum_{j=1}^{l-i+1} (-1)^{|\psi_i|(|w| + \epsilon_{k+j-1})} (v_1, ..., w, ..., \psi_i(v_{k+j}, ..., v_{k+i+j-1}), ..., v_{k+l}),$$

where $\epsilon_r = |v_1| + ... + |v_r|$.

(Notice that the condition of satisfying diagram (2.14) is not linear, i.e. if χ *and* ψ *both make diagram (2.14) commute, then* $\chi + \psi$ *will not.)*

(b) *There is a one-to-one correspondence between maps* $\sigma : T^W V \to T^W V$ *that make diagram (2.14) commute and maps* $\varrho = \sum \varrho_{k,l}$ *like in (a), given by* $\sigma = \tilde{\varrho}$.

Proof. (a) Again one uses induction on the output-component of $\tilde{\varrho}$. Denote by $\tilde{\varrho}^j$ the component of $\tilde{\varrho}$ mapping $T^W V \to \bigoplus_{k+l=j} V^{\otimes k} \otimes W \otimes V^{\otimes l}$ and by ψ^j the component of ψ mapping $TV \to V^{\otimes}$. Then $\tilde{\varrho}^0, ..., \tilde{\varrho}^{m-1}$ will uniquely determine the component $\tilde{\varrho}^m$.

$$\Delta^W(\tilde{\varrho}(v_1, ..., v_k, w, v_{k+1}, ..., v_{k+l})) =$$

$$
\begin{aligned}
&= (id \otimes \tilde{\varrho} + \psi \otimes id + \tilde{\varrho} \otimes id + id \otimes \psi) \\
&\qquad (\Delta^W (v_1, ..., v_k, w, v_{k+1}, ..., v_{k+l})) = \\
&= (id \otimes \tilde{\varrho} + \psi \otimes id + \tilde{\varrho} \otimes id + id \otimes \psi) \\
&\qquad (\sum_{i=0}^{k} (v_1, ..., v_i) \otimes (v_{i+1}, ..., w, ..., v_{k+l}) + \\
&\qquad + \sum_{i=k}^{k+l} (v_1, ..., w, ..., v_i) \otimes (v_{i+1}, ..., v_{k+l})) = \\
&= \sum_{i=0}^{k} (-1)^{|\tilde{\varrho}| \sum_{r=1}^{i} |v_r|} (v_1, ..., v_i) \otimes \tilde{\varrho}(v_{i+1}, ..., w, ..., v_{k+l}) + \\
&\quad + \sum_{i=0}^{k} \psi(v_1, ..., v_i) \otimes (v_{i+1}, ..., w, ..., v_{k+l}) + \\
&\quad + \sum_{i=k}^{k+l} \tilde{\varrho}(v_1, ..., w, ..., v_i) \otimes (v_{i+1}, ..., v_{k+l}) + \\
&\quad + \sum_{i=k}^{k+l} (-1)^{|\psi|(|w| + \sum_{r=1}^{i} |v_r|)} (v_1, ..., w, ..., v_i) \otimes \psi(v_{i+1}, ..., v_{k+l}).
\end{aligned}
$$

Then projecting both sides to

$$
\bigoplus_{r+s+t=m} (V^{\otimes r} \otimes W \otimes V^{\otimes s}) \otimes V^{\otimes t} + V^{\otimes} \otimes (V^{\otimes s} \otimes W \otimes V^{\otimes t}) \subset
$$

$$
\subset T^W V \otimes TV + TV \otimes T^W V
$$

yields

$$
\Delta^W (\tilde{\varrho}^m (v_1, ..., w, ..., v_k)) =
$$

$$= \sum_{i=0}^{k} \pm(v_1,\dots,v_i)\otimes\tilde{\varrho}^{m-i}(v_{i+1},\dots,w,\dots,v_{k+l}) +$$
$$+\sum_{i=0}^{k}\psi^{m+i-k-l}(v_1,\dots,v_i)\otimes(v_{i+1},\dots,w,\dots,v_{k+l}) +$$
$$+\sum_{i=k}^{k+l}\tilde{\varrho}^{m+i-k-l}(v_1,\dots,w,\dots,v_i)\otimes(v_{i+1},\dots,v_{k+l}) +$$
$$+\sum_{i=k}^{k+l}\pm(v_1,\dots,w,\dots,v_i)\otimes\psi^{m-i}(v_{i+1},\dots,v_{k+l}).$$

So the righthand side depends only on ψ^j's, which are all explicitly known by Lemma 2.3., and $\tilde{\varrho}^j$ with $j < m$, (except for the uninteresting terms $\tilde{\varrho}^m(v_1,\dots,w,\dots,v_{k+l})\otimes 1$ and $1\otimes\tilde{\varrho}^m(v_1,\dots,w,\dots,v_{k+l})$). With this, an induction argument shows that $\tilde{\varrho}^m$ is given by the formula of the Lemma.

(b) Let $X := \{\sigma : T^W V \to T^W V \mid \sigma$ makes diagram (2.14) commute $\}$. Then

$$\alpha : \{\varrho : T^W V \to W\} \to X, \qquad \varrho \mapsto \tilde{\varrho},$$
$$\beta : X \to \{\varrho : T^W V \to W\}, \qquad \sigma \mapsto pr_W \circ \sigma$$

are inverse to each other by (a).

□

Definition 2.15. *Given an A_∞-algebra (A, D). Then an **A_∞-bimodule** (M, D^M) consists of a module M together with a map $D^M : B^M A \to B^M A$ of degree -1, which makes the diagram (2.12) of Proposition 2.11. commute, and satisfies $(D^M)^2 = 0$.*

By Proposition 2.11. one can put the differential $\delta^M : CoDer(TA, T^M A) \to CoDer(TA, T^M A)$, $\delta(f) := D^M \circ f - (-1)^{|f|} f \circ D$ *on the Hochschild-cochain-complex. Now it satisfies* $(\delta^M)^2 = 0$, *because with* $(D^M)^2 = 0$, *one gets* $(\delta^M)^2(f) = D^M \circ D^M \circ f - (-1)^{|f|} D^M \circ f \circ D - (-1)^{|f|+1} D^M \circ f \circ D + (-1)^{|f|+|f|+1} f \circ D \circ D = 0$.

The definition of an A_∞*-bimodule was already given in [9] section 3 and also in [14], and coincides with the one here.*

Proposition 2.16. *Let* (A, D) *be an* A_∞*-algebra with a system of maps* $\{m_i : A^{\otimes i} \to A\}_{i \geq 1}$ *associated to* D *by Proposition 2.4. (where* $m_0 = 0$ *is assumed). Let* (M, D^M) *be an* A_∞*-bimodule over* A *with a system of maps* $\{D^M_{k,l} : sA^{\otimes k} \otimes sM \otimes sA^{\otimes l} \to sM\}_{k \geq 0, l \geq 0}$ *from Lemma 2.13.(b) associated to* D^M. *Let* $b_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to M$ *be the induced map by* $D^M_{k,l} = s \circ b_{k,l} \circ (s^{-1})^{\otimes k+l+1}$.

Then the condition $(D^M)^2 = 0$ *is equivalent to the following system of equations:*

$$
\begin{aligned}
b_{0,0}(b_{0,0}(m)) &= 0, \\
b_{0,0}(b_{0,1}(m, a_1)) - b_{0,1}(b_{0,0}(m), a_1) - (-1)^{|m|} b_{0,1}(m, m_1(a_1)) &= 0, \\
b_{0,0}(b_{1,0}(a_1, m)) - b_{1,0}(m_1(a_1), m) - (-1)^{|a_1|} b_{1,0}(a_1, b_{0,0}(m)) &= 0, \\
b_{0,0}(b_{1,1}(a_1, m, a_2)) - b_{0,1}(b_{1,0}(a_1, m), a_2) + b_{1,0}(a_1, b_{0,1}(m, a_2)) + & \\
+ b_{1,1}(m_1(a_1), m, a_2) + (-1)^{|a_1|} b_{1,1}(a_1, b_{0,0}(m), a_2) + & \\
+ (-1)^{|a_1|+|m|} b_{1,1}(a_1, m, m_1(a_2)) &= 0,
\end{aligned}
$$

...

$$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} \pm b_{k-i+1,l}(a_1, \dots, m_i(a_j, \dots, a_{i+j-1}), \dots, m, \dots, a_{k+l}) +$$
$$+\sum_{i=0}^{k}\sum_{j=0}^{l} \pm b_{k-i,l-j}(a_1, \dots, b_{i,j}(a_{k-i+1}, \dots, m, \dots, a_{k+j}), \dots, a_{k+l}) +$$
$$+\sum_{i=1}^{l}\sum_{j=1}^{l-i+1} \pm b_{k,l-i+1}(a_1, \dots, m, \dots, m_i(a_{k+j}, \dots, a_{k+i+j-1}), \dots, a_{k+l}) = 0$$

$$\dots$$

where the signs are exactly analogous to the ones in Proposition 2.4.

Proof. The result follows immediately from Lemma 2.13., after rewriting $D_{k,l}^{M}$ and D_j by $b_{k,l}$ and m_j. (Notice that this replacement only changes a sign.) In order to get the correct sign, first notice that the lifting described in Lemma 2.13. is exactly the usual lifting as coderivations, except that one has to pick $b_{k,l}$ or m_j according to where the element of M is located. Keeping this in mind, it is possible to redo all the steps from Proposition 2.4. □

Example 2.17. *Let's pick up Example 2.6. Let (A, ∂, μ) be a differential graded algebra with the A_∞-algebra-structure $m_1 := \partial$, $m_2 := \mu$ and $m_k := 0$ for $k \geq 3$. Now, let $(M, \partial', \lambda, \rho)$ be a differential graded bimodule over A, where $\lambda : A \otimes M \to M$ and $\rho : M \otimes A \to M$ denote the left- and right-action. It is possible to make M into an A_∞-bimodule over A by taking $b_{0,0} := \partial'$, $b_{1,0} := \lambda$, $b_{0,1} := \rho$ and $b_{k,l} := 0$ for $k + l > 1$. Then the equations of Proposition 2.16. are the defining conditions of a differential bialgebra over*

A:

$$
\begin{aligned}
(\partial')^2(m) &= 0, \\
\partial'(m.a) &= m.\partial(a) + (-1)^{|m|}\partial'(m).a, \\
\partial'(a.m) &= \partial(a).m + (-1)^{|a|}a.\partial'(m), \\
(a.m).b &= a.(m.b), \\
(m.a).b &= m.(a \cdot b), \\
a.(b.m) &= (a \cdot b).m.
\end{aligned}
$$

There are no higher equations.

For later purposes it is convenient to have the following

Lemma 2.18. *Given an A_∞-algebra (A, D) and an A_∞-bimodule (M, D^M), with system of maps $\{b_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to M\}_{k\geq 0, l\geq 0}$ from Proposition 2.16.*

Then the dual space $M^* := Hom_R(M, R)$ has a canonical A_∞-bimodule-structure given by maps $\{b'_{k,l} : A^{\otimes k} \otimes M^* \otimes A^{\otimes l} \to M^*\}_{k\geq 0, l\geq 0}$,

$$(b'_{k,l}(a_1, \dots, a_k, m^*, a_{k+1}, \dots, a_{k+l}))(m) := (-1)^{\varepsilon} \cdot m^*(b_{l,k}(a_{k+1}, \dots, a_{k+l}, m, a_1, \dots, a_k)),$$

$$\text{where } \varepsilon := (|a_1|+\dots+|a_k|)\cdot(|m^*|+|a_{k+1}|+\dots+|a_{k+l}|+|m|)+|m^*|\cdot(k+l+1).$$

Proof. First, it is easy to see that the degrees work out, because of $|b'_{k,l}| = |b_{l,k}|$.

In order to show $(D^{M^*})^2 = 0$, one can use the criterion from Proposition 2.16. The top and the bottom term in the general sum of Proposition 2.16. convert to

$$(b'_{k-i+1,l}(a_1, \dots, m_i(a_j, \dots, a_{i+j-1}), \dots, m^*, \dots, a_{k+l}))(m) =$$

$$= \pm m^*(b_{l,k-i+1}(a_{k+1},...,a_{k+l},m,a_1,...,m_i(a_j,...,a_{i+j-1}),...,a_k)),$$

and

$$(b'_{k,l-i+1}(a_1,...,m^*,...,m_i(a_{k+j},...,a_{k+i+j-1}),...,a_{k+l}))(m) =$$

$$= \pm m^*(b_{l-i+1,k}(a_{k+1},...,m_i(a_{k+j},...,a_{k+i+j-1}),...,a_{k+l},m,a_1,...,a_k)).$$

So, these terms come from terms of the A_∞-bimodule-structure of M. The same is true for the middle term:

$$(b'_{k-i,l-j}(a_1,...,b'_{i,j}(a_{k-i+1},...,m^*,...,a_{k+j}),...,a_{k+l}))(m) =$$

$$= \pm(b'_{i,j}(a_{k-i+1},...,m^*,...,a_{k+j}))(b_{l-j,k-i}(a_{k+j+1},...,a_{k+l},m,a_1,...,a_{k-i})) =$$
$$= \pm m^*(b_{j,i}(a_{k+1},...,a_{k+j},b_{l-j,k-i}(a_{k+j+1},...,a_{k+l},m,a_1,...,a_{k-i}),a_{k-i+1},...,a_k)).$$

So, the sum from Proposition 2.16. for the A_∞-bimodule M^* has exactly the terms of m^* applied the the sum for the A_∞-bimodule M.
The only remaining question is whether the signs are correct. The proof for this is left to the reader. □

2.3 Morphisms of A_∞ Bimodules

Given two A_∞-bimodules (M, D^M) and (N, D^N) over an A_∞-algebra (A, D). What is the natural notion of morphism between them?

Again a motivation is to have for any A_∞-bimodule-map an induced map of their Hochschild-cochain-complexes.

Proposition 2.19. *Given three modules V, W and Z. Let $F : T^W V \to T^Z V$ be a map. Then the induced map $F^\sharp : CoDer(TV, T^W V) \to CoDer(TV, T^Z V)$.*

given by $F^\sharp(f) := F \circ f$, is well-defined, (i.e. it maps coderivations to coderivations,) if and only if the following diagram commutes:

$$\begin{array}{ccc} T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\ \downarrow F & & \downarrow (id\otimes F)\oplus(F\otimes id) \\ T^Z V & \xrightarrow[\Delta^Z]{} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV) \end{array} \tag{2.20}$$

Proof. If both $f : TV \to T^W V$ and $F \circ f : TV \to T^Z V$ are coderivations then this means that the top diagram and the overall diagram below commute.

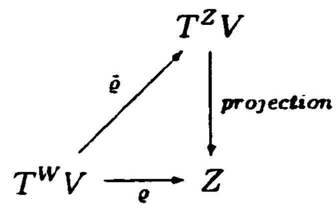
$$\begin{array}{ccc} TV & \xrightarrow{\Delta} & TV \otimes TV \\ \downarrow f & & \downarrow (id\otimes f)+(f\otimes id) \\ T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\ \downarrow F & & \downarrow (id\otimes F)\oplus(F\otimes id) \\ T^Z V & \xrightarrow[\Delta^Z]{} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV) \end{array}$$

But then the lower diagram has to commute if applied to any element in $Im(f) \subset T^W V$. By Lemma 2.10. there are enough coderivations to make this true for all $T^W V$. □

Again let's describe F by a system of maps.

Lemma 2.21. *(a) Given modules V, W and Z and a map $\varrho : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z$ of degree $|\varrho|$, which can be viewed as a map $\varrho : T^W V \to Z$ by letting its only nonzero component be the original ϱ on $V^{\otimes k} \otimes W \otimes V^{\otimes l}$.*

Then ϱ lifts uniquely to a map $\tilde{\varrho} : T^W V \to T^Z V$



which makes the diagram (2.20) in Proposition 2.19. commute (put $\tilde{\varrho}$ instead of F). This map is given by

$$\tilde{\varrho}(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s}) := 0, \qquad \textit{for } r < k \textit{ or } s < l,$$

$$\tilde{\varrho}(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s}) :=$$

$$:= (-1)^{|\varrho| \sum_{i=1}^{r-k} |v_i|}(v_1, \ldots, \varrho(v_{r-k+1}, \ldots, w, \ldots, v_{r+l}), \ldots, v_{r+s}),$$

$$\textit{for } r \geq k \textit{ and } s \geq l.$$

Thus $\tilde{\varrho}\,|_{V^{\otimes r} \otimes W \otimes V^{\otimes s}}: V^{\otimes r} \otimes W \otimes V^{\otimes s} \to V^{\otimes r-k} \otimes Z \otimes V^{\otimes s-l}$.

(b) There is a one-to-one correspondence between maps $\sigma : T^W V \to T^Z V$ making diagram (2.20) commute and systems of maps $\{\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z\}_{k \geq 0, l \geq 0}$, given by $\sigma = \sum_{k \geq 0, l \geq 0} \widetilde{\varrho_{k,l}}$.

Proof. (a) Again one uses induction on the output-component of $\tilde{\varrho}$. Denote by $\tilde{\varrho}^j$ the component of $\tilde{\varrho}$ mapping $T^W V \to \bigoplus_{r+s=j} V^{\otimes r} \otimes Z \otimes V^{\otimes s}$. Then $\tilde{\varrho}^1, \ldots, \tilde{\varrho}^{m-1}$ determine uniquely the component $\tilde{\varrho}^m$.

$$\Delta^Z(\tilde{\varrho}(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s})) =$$

$$
\begin{aligned}
&= (id \otimes \bar{\varrho} + \bar{\varrho} \otimes id)(\Delta^W(v_1, \dots, v_r, w, v_{r+1}, \dots, v_{r+s})) = \\
&= (id \otimes \bar{\varrho} + \bar{\varrho} \otimes id)(\sum_{i=0}^{r}(v_1, \dots, v_i) \otimes (v_{i+1}, \dots, w, \dots, v_{r+s}) + \\
&\qquad + \sum_{i=r}^{r+s}(v_1, \dots, w, \dots, v_i) \otimes (v_{i+1}, \dots, v_{r+s})) = \\
&= \sum_{i=0}^{r}(-1)^{|\bar{\varrho}| \sum_{t=1}^{i} |v_t|}(v_1, \dots, v_i) \otimes \bar{\varrho}(v_{i+1}, \dots, w, \dots, v_{r+s}) + \\
&\qquad + \sum_{i=r}^{r+s} \bar{\varrho}(v_1, \dots, w, \dots, v_i) \otimes (v_{i+1}, \dots, v_{r+s}).
\end{aligned}
$$

Then projecting both sides to

$$
\bigoplus_{r+s+t=m} (V^{\otimes r} \otimes Z \otimes V^{\otimes s}) \otimes V^{\otimes t} + V^{\otimes t} \otimes (V^{\otimes s} \otimes Z \otimes V^{\otimes t}) \subset
$$

$$
\subset T^Z V \otimes TV + TV \otimes T^Z V
$$

yields

$$
\begin{aligned}
&\Delta^Z(\bar{\varrho}^m(v_1, \dots, w, \dots, v_{r+s})) = \\
&\qquad = \sum_{i=0}^{r} \pm(v_1, \dots, v_i) \otimes \bar{\varrho}^{m-i}(v_{i+1}, \dots, w, \dots, v_{r+s}) + \\
&\qquad\quad + \sum_{i=r}^{r+s} \bar{\varrho}^{m+i-r-s}(v_1, \dots, w, \dots, v_i) \otimes (v_{i+1}, \dots, v_{r+s}).
\end{aligned}
$$

So the righthand side depends only on $\bar{\varrho}^j$ with $j < m$, (except for the uninteresting terms $\bar{\varrho}^m(v_1, \dots, w, \dots, v_{r+s}) \otimes 1$ and $1 \otimes \bar{\varrho}^m(v_1, \dots, w, \dots, v_{r+s})$). With this, an induction argument shows that $\bar{\varrho}^m$ is only nonzero on

$V^{\otimes r} \otimes W \otimes V^{\otimes s}$ with $r - k + s - l = m$, where it is given by

$$\tilde{\varrho}^m(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s}) =$$

$$= (-1)^{|\varrho| \sum_{i=1}^{r-k} |v_i|}(v_1, \ldots, \varrho(v_{r-k+1}, \ldots, w, \ldots, v_{r+l}), \ldots, v_{r+s}).$$

(b) Let $X := \{\sigma : T^W V \to T^Z V \mid \sigma$ makes diagram (2.20) commute $\}$. Then

$$\alpha : \{\{\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z\}_{k \geq 0, l \geq 0}\} \to X,$$
$$\{\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z\}_{k \geq 0, l \geq 0} \mapsto \sum_{k \geq 0, l \geq 0} \widetilde{\varrho_{k,l}},$$
$$\beta : X \to \{\{\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z\}_{k \geq 0, l \geq 0}\},$$
$$\sigma \mapsto \{pr_Z \circ \sigma|_{V^{\otimes k} \otimes W \otimes V^{\otimes l}}\}_{k \geq 0, l \geq 0}$$

are inverse to each other by (a).

□

Let's apply this to the Hochschild-space.

Definition 2.22. *Given two A_∞-bimodules (M, D^M) and (N, D^N) over an A_∞-algebra (A, D). Then a map $F : B^M A \to B^N A$ of degree 0 is called an* ***A_∞-bimodule-map*** *:⇔ F makes the diagram*

$$\begin{array}{ccc} B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle (id \otimes F) \oplus (F \otimes id)} \\ B^N A & \xrightarrow[\Delta^N]{} & (BA \otimes B^N A) \oplus (B^N A \otimes BA) \end{array}$$

commute, and in addition it holds that $F \circ D^M = D^N \circ F$.

By Proposition 2.19., every A_∞-bimodule-map induces (by composition F^1 :

$f \mapsto F \circ f$) a map between the Hochschild-spaces, which preserves the differentials, because $(F^\sharp \circ \delta^M)(f) = F^\sharp(D^M \circ f + (-1)^{|f|} f \circ D) = F \circ D^M \circ f + (-1)^{|f|} F \circ f \circ D = D^N \circ F \circ f + (-1)^{|f|} F \circ f \circ D = \delta^N(F \circ f) = (\delta^N \circ F^\sharp)(f)$.

Proposition 2.23. *Let (A, D) be an A_∞-algebra with a system of maps $\{m_i : A^{\otimes i} \to A\}_{i \geq 1}$ from Proposition 2.4. associated to D, (where $m_0 = 0$ is assumed). Let (M, D^M) and (N, D^N) be A_∞-bimodules over A with systems of maps $\{b_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to M\}_{k \geq 0, l \geq 0}$ and $\{c_{k,l} : A^{\otimes k} \otimes N \otimes A^{\otimes l} \to N\}_{k \geq 0, l \geq 0}$ from Proposition 2.16. associated to D^M and D^N respectively. Let $F : T^M A \to T^N A$ be an A_∞-bimodule-map between M and N, and let $\{F_{k,l} : sA^{\otimes k} \otimes sM \otimes sA^{\otimes l} \to sN\}_{k \geq 0, l \geq 0}$ be a system of maps associated to F by Lemma 2.21.(b). Again, rewrite the maps $F_{k,l}$ by $f_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to N$ by using the suspension map: $F_{k,l} = s \circ f_{k,l} \circ (s^{-1})^{\otimes k+l+1}$.*
Then the condition $F \circ D^M = D^N \circ F$ is equivalent to the following system of equations:

$$f_{0,0}(b_{0,0}(m)) = c_{0,0}(f_{0,0}(m)),$$

$$f_{0,0}(b_{0,1}(m,a)) - f_{0,1}(b_{0,0}(m), a) - (-1)^{|m|} f_{0,1}(m, m_1(a)) =$$

$$= c_{0,0}(f_{0,1}(m,a)) + c_{0,1}(f_{0,0}(m), a),$$

$$f_{0,0}(b_{1,0}(a,m)) - f_{1,0}(m_1(a), m) - (-1)^{|a|} f_{1,0}(a, b_{0,0}(m)) =$$

$$= c_{0,0}(f_{1,0}(a,m)) + c_{1,0}(a, f_{0,0}(m)),$$

...

$$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1}(-1)^{\varepsilon} f_{k-i+1,l}(a_1,\ldots,m_i(a_j,\ldots,a_{i+j-1}),\ldots,m,\ldots,a_{k+l+1})+$$
$$+\sum_{j=1}^{k}\sum_{i=k-j+2}^{k+l-j+2}(-1)^{\varepsilon} f_{j,k+l-i-j+3}$$
$$(a_1,\ldots,b_{k-j+1,i+j-k-2}(a_j,\ldots,m,\ldots,a_{i+j-1}),\ldots,a_{k+l+1})+$$
$$+\sum_{i=1}^{l}\sum_{j=k+2}^{k+l-i+2}(-1)^{\varepsilon} f_{k,l-i+1}(a_1,\ldots,m,\ldots,m_i(a_j,\ldots,a_{i+j-1}),\ldots,a_{k+l+1}) =$$
$$= \sum_{j=1}^{k+1}\sum_{i=k-j+2}^{k+l-j+2}(-1)^{\varepsilon'} c_{j,k+l-i-j+3}$$
$$(a_1,\ldots,f_{k-j+1,i+j-k-2}(a_j,\ldots,m,\ldots,a_{i+j-1}),\ldots,a_{k+l+1})$$

In order to simplify notation, it is assume that in $(a_1,\ldots,a_{k+l+1})$ above, only the first k and the last l elements are elements of A and $a_{k+1} = m \in M$. Then the signs are given by

$$\varepsilon = i\cdot\sum_{r=1}^{j-1}|a_r| + (j-1)\cdot(i+1) + (k+l+1) - i,$$

$$\textit{and } \varepsilon' = (i+1)\cdot\left(j+1+\sum_{r=1}^{j-1}|a_r|\right).$$

...

Proof. Up to signs these formulas follow immediately from the explicit lifting properties in Lemma 2.13.(a) and Lemma 2.21.(a). For the sign, the arguments of Proposition 2.4. will be applied.

Let's assume again that in $(a_1,\ldots,a_{k+l+1})$, only the first k and the last l elements are elements of A and $a_{k+1} \in M$. Now notice that just like in Proposition 2.4. one gets

$$F_{k,l}(sa_0,\ldots,sa_{k+l}) = (-1)^{\sum_{j=1}^{k+l+1}(k+l+1-j)\cdot(|a_j|+1)} s\circ f_{k,l}(a_0,\ldots,a_{k+l}).$$

So, when writing out the term $pr_{sN} \circ F \circ D^M(sa_0, ..., sa_{k+l})$, exactly the same signs appear, that were in equation (2.5) of Proposition 2.4. for $pr_{sA} \circ D \circ D$. This is so, because D^M, which has to be applied in the argument of F, has degree -1 just like D, and the application of the suspension map is the same for D or D^M or F. It follows that in this case the signs can simply be taken from Proposition 2.4.

Unfortunately the signs for the term $D^N \circ F$ cannot be taken directly from Proposition 2.4. like above. The difference is that F, which is of degree 0 (and not -1), has to be applied in the argument of D^N. So, when F "jumps" over elements sa_i, no signs are introduced. This means that here one gets a difference in signs compared to Proposition 2.4. given by

$$(-1)^{\sum_{r=1}^{j-1}(|a_r|+1)} = (-1)^{\sum_{r=1}^{j-1}|a_r|+j-1}$$

(compare this with the first equality in (2.5)). Here one has to take the same interpretation for the variables i and j as in Proposition 2.4.; namely $f_{r,s}$ takes exactly i inputs and the first variable in $f_{r,s}$ is given by a_j:

$$(a_1, ..., f_{r,s}(a_j, ..., m, ..., a_{i+j-1}), ..., a_{k+l+1}).$$

Another difference to Proposition 2.4. is given in the last step of equation (2.5), because the interior element $m_i(a_j, ..., a_{j+i-1})$ is replaced by some $f_{r,s}(a_j, ..., a_{j-i-1})$, $(r + s = i - 1)$, with $|m_i| = i - 2$ and $|f_{r,s}| = i - 1$. So, when converting D_{k-i+1} to m_{k-i+1} in (2.5), the suspension map for $a_{j+i}, ..., a_k$ jumps over one degree less. In the given case, this introduces a difference in signs of $(-1)^{k+l+1-i-j+1}$.

Putting this together with the sign in Proposition 2.4. gives

$$\begin{aligned}
&\varepsilon - (\sum_{r=1}^{k+l+1} (k+l+1-r)\cdot(|a_r|+1)) = \\
&\quad = i\cdot\sum_{r=1}^{j-1}|a_r| + (j-1)\cdot(i+1) + (k+l+1) - i + \\
&\quad\quad +(\sum_{r=1}^{j-1}|a_r| + j - 1) - (k+l+1-i-j+1) \equiv \\
&\quad \equiv (i+1)\cdot(j+1+\sum_{r=1}^{j-1}|a_r|) \qquad (mod\ 2)
\end{aligned}$$

Thus, dividing the equation $D^N \circ F = F \circ D^M$ by the sign $(-1)^{\sum_{r=1}^{k+l+1}(k+l+1-r)\cdot(|a_r|+1)}$ yields the result. □

Example 2.24. *Let's pick up the examples 2.6. and 2.17. Let (A, ∂, μ) be a differential graded algebra with the A_∞-algebra-structure $m_1 := \partial$, $m_2 := \mu$ and $m_k := 0$ for $k \geq 3$. Now, let $(M, \partial^M, \lambda^M, \rho^M)$ and $(N, \partial^N, \lambda^N, \rho^N)$ be differential graded bimodules over A, with the A_∞-bialgebra-structures given by $b_{0,0} := \partial^M$, $b_{1,0} := \lambda^M$, $b_{0,1} := \rho^M$ and $b_{k,l} := 0$ for $k + l > 1$, and $c_{0,0} := \partial^N$, $c_{1,0} := \lambda^N$, $c_{0,1} := \rho^N$ and $c_{k,l} := 0$ for $k + l > 1$.*
Given a bialgebra map $f : M \to N$ of degree 0. Then one makes f into a map of A_∞-bialgebras by taking $f_{0,0} := f$ and $f_{k,l} := 0$ for $k + l > 0$. Then the equations of Proposition 2.23. are the defining conditions of a differential bialgebra map from M to N:

$$\begin{aligned}
f \circ \partial^M(m) &= \partial^N \circ f(m) \\
f(m.a) &= f(m).a \\
f(a.m) &= a.f(m)
\end{aligned}$$

There are no higher equations.

2.4 Higher Morphisms of A_∞ Bimodules

In the last section, morphisms of A_∞-bimodules where defined partially by the requirement of having induced maps on the Hochschild-spaces. One could ask for higher morphisms of Hochschild-spaces, like for example a multiplication on a Hochschild-complex or any other higher order maps. For completeness sake, the corresponding discussion for those maps will be given in this section, even though it will not be needed for the rest of this thesis.

Let's start with a generalization of Proposition 2.19.

Proposition 2.25. *Given modules V, W_1, ..., W_n, and Z. Let's use the abbreviation $X := T^{W_1}V \otimes ... \otimes T^{W_n}V$. Let $\Delta^n : TV \to TV^{\otimes n}$, $(v_1, ..., v_k) \mapsto \sum_{0 \le i_1 \le ... \le i_{n-1} \le k} (v_1, ..., v_{i_1}) \otimes (v_{i_1+1}, ..., v_{i_2}) \otimes ... \otimes (v_{i_{n-1}+1}, ..., v_k)$.*
Let $F : X \to T^Z V$ be a map. Then the induced map $F^\sharp : CoDer(TV, T^{W_1}V) \otimes ... \otimes CoDer(TV, T^{W_n}V) \to CoDer(TV, T^Z V)$, given by $F^\sharp(f_1 \otimes ... \otimes f_n) := F \circ (f_1 \otimes ... \otimes f_n) \circ \Delta^n$, is well-defined, (i.e. it maps coderivations to coderivations.) if and only if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\kappa} & (TV \otimes X) \oplus (X \otimes TV) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle (id \otimes F) \oplus (F \otimes id)} \\ T^Z V & \xrightarrow[\Delta^Z]{} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV) \end{array} \tag{2.26}$$

where $\kappa = ((pr_{TV \otimes T^{W_1}V}) \circ \Delta^{W_1}) \otimes id^{\otimes n-1} + id^{\otimes n-1} \otimes ((pr_{T^{W_n}V \otimes TV}) \circ \Delta^{W_n})$ (compare Definition 2.8. for $\Delta^{W_i} : T^{W_i}V \to (T^{W_i}V \otimes TV) \oplus (TV \otimes T^{W_i}V)$).

Proof. Assume $f_i \in CoDer(TV, T^{W_i}V)$, for $i = 1, ..., n$. Now, $F \circ (f_1 \otimes ... \otimes f_n) \circ \Delta^n : TV \to T^Z V$ being a coderivation means that the overall diagram of the following commutes

$$\begin{array}{ccc}
TV & \xrightarrow{\quad\Delta\quad} & TV \otimes TV \\
\Big\downarrow{\scriptstyle \Delta^n} & & \Big\downarrow{\scriptstyle (id\otimes\Delta^n)+(\Delta^n\otimes id)} \\
TV^{\otimes n} & \xrightarrow{\quad\tilde{\kappa}\quad} & (TV \otimes TV^{\otimes n}) \oplus (TV^{\otimes n} \otimes TV) \\
\Big\downarrow{\scriptstyle f_1\otimes\ldots\otimes f_n} & & \Big\downarrow{\scriptstyle (id\otimes f_1\otimes\ldots\otimes f_n)\oplus(f_1\otimes\ldots\otimes f_n\otimes id)} \\
X & \xrightarrow{\quad\kappa\quad} & (TV \otimes X) \oplus (X \otimes TV) \\
\Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle (id\otimes F)\oplus(F\otimes id)} \\
T^Z V & \xrightarrow[\Delta^Z]{\quad\quad} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV)
\end{array}$$

where $\tilde{\kappa} = \Delta \otimes id^{\otimes n-1} + id^{\otimes n-1} \otimes \Delta$, and $\kappa = ((pr_{TV \otimes T^{W_1}V}) \circ \Delta^{W_1}) \otimes id^{\otimes n-1} + id^{\otimes n-1} \otimes ((pr_{T^{W_n}V \otimes TV}) \circ \Delta^{W_n})$.

Now, a simple check makes clear, that the top diagram commutes, because following either path of the diagram, elements of TV are twice broken up in $n+1$ many ways.

Next, also the middle diagram commutes. To this end, project the coderivation property $\Delta^{W_i} \circ f_i = ((id \otimes f_i) + (f_i \otimes id)) \circ \Delta$ for f_1 and f_n to the first and second component respectively, to yield

$$(pr|_{TV \otimes T^{W_1}V}) \circ \Delta^{W_1} \circ f_1 = (id \otimes f_1) \circ \Delta,$$

$$(pr|_{T^{W_n}V \otimes TV}) \circ \Delta^{W_n} \circ f_n = (f_n \otimes id) \circ \Delta.$$

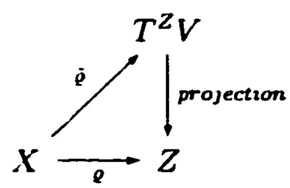
Tensoring the top equation with $f_2 \otimes ... \otimes f_n$ on the right and the bottom equation with $f_1 \otimes ... \otimes f_{n-1}$ on the left gives exactly the desired commutative diagram.

But then the lower diagram has to commute if applied to any element in $Im((f_1 \otimes ... \otimes f_n) \circ \Delta^n) \subset X$. By Lemma 2.10. there are enough coderivations to make this true for all X. □

F can again be described by a system of maps.

Lemma 2.27.

(a) Given modules V, W_1, ..., W_n and Z and denote by $X := T^{W_1}V \otimes ... \otimes T^{W_n}V$, and $X^{k_1,...,k_n}_{l_1,...,l_n} := (V^{\otimes k_1} \otimes W \otimes V^{\otimes l_1}) \otimes ... \otimes (V^{\otimes k_n} \otimes W \otimes V^{\otimes l_n})$. Given a map $\varrho : X^{k_1,...,k_n}_{l_1,...,l_n} \to Z$ of degree $|\varrho|$, which can be viewed as a map $\varrho : X \to Z$ by letting its only nonzero component be the original ϱ on $X^{k_1,...,k_n}_{l_1,...,l_n}$. Then ϱ lifts uniquely to a map $\tilde{\varrho} : X \to T^Z V$



which makes the diagram (2.26) in Proposition 2.25. commute (put $\tilde{\varrho}$ instead of F). This map $\tilde{\varrho}$ is nonzero only on elements in $V^{\otimes r} \otimes X^{k_1,...,k_n}_{l_1,...,l_n} \otimes V^{\otimes s}$, where it is given by

$$\tilde{\varrho}(v_1, ..., v_r, x, v'_1, ..., v'_s) := (-1)^{|\varrho| \sum_{i=1}^{r} |v_i|}(v_1, ..., \varrho(x), ..., v'_s).$$

Thus $\tilde{\varrho} \mid_{X^{r+k_1,...,k_n}_{l_1,...,l_n+s}} : X^{r+k_1,...,k_n}_{l_1,...,l_n+s} \to V^{\otimes r} \otimes Z \otimes V^{\otimes s}$.

(b) There is a one-to-one correspondence between maps $\sigma : X \to T^Z V$ making diagram (2.26) commute and systems of maps $\{\widetilde{\varrho^{k_1,...,k_n}_{l_1,...,l_n}} : X^{k_1,...,k_n}_{l_1,...,l_n} \to Z\}_{k_1,...,k_n,l_1,...,l_n}$, given by $\sigma = \sum_{k_1,...,k_n,l_1,...,l_n} \varrho^{k_1,...,k_n}_{l_1,...,l_n}$.

Proof. (a) Again one uses induction on the output-component of $\tilde{\varrho}$. Denote by $\tilde{\varrho}^j$ the component of $\tilde{\varrho}$ mapping $X \to \bigoplus_{r+s=j} V^{\otimes r} \otimes Z \otimes V^{\otimes s}$. Then $\tilde{\varrho}^1, ..., \tilde{\varrho}^{m-1}$ determine uniquely the component $\tilde{\varrho}^m$.

$$\Delta^Z(\tilde{\varrho}(v_1, ..., v_r, x, v'_1, ..., v'_s)) =$$

$$= (id \otimes \tilde{\varrho} + \tilde{\varrho} \otimes id)(((pr \circ \Delta^{W_1}) \otimes id^{\otimes n-1} + id^{\otimes n-1} \otimes (pr \circ \Delta^{W_n}))$$
$$(v_1, ..., v_r, x, v'_1, ..., v'_s)) =$$
$$= (id \otimes \tilde{\varrho} + \tilde{\varrho} \otimes id)(\sum_{i=0}^{r}(v_1, ..., v_i) \otimes (v_{i+1}, ..., x, ..., v'_s) +$$
$$+ \sum_{i=0}^{s}(v_1, ..., x, ..., v'_i) \otimes (v'_{i+1}, ..., v'_s)) =$$
$$= \sum_{i=0}^{r}(-1)^{|\tilde{\varrho}|\sum_{t=1}^{i}|v_t|}(v_1, ..., v_i) \otimes \tilde{\varrho}(v_{i+1}, ..., x, ..., v'_s) +$$
$$+ \sum_{i=0}^{s} \tilde{\varrho}(v_1, ..., x, ..., v'_i) \otimes (v'_{i+1}, ..., v'_s).$$

Then projecting both sides to

$$\bigoplus_{a+b+c=m} (V^{\otimes a} \otimes Z \otimes V^{\otimes b}) \otimes V^{\otimes c} + V^{\otimes a} \otimes (V^{\otimes b} \otimes Z \otimes V^{\otimes c}) \subset$$
$$\subset T^Z V \otimes TV + TV \otimes T^Z V$$

yields

$$\Delta^Z(\tilde{\varrho}^m(v_1, ..., x, ..., v'_s)) =$$

$$= \sum_{i=0}^{r} \pm(v_1, ..., v_i) \otimes \tilde{\varrho}^{m-i}(v_{i+1}, ..., x, ..., v'_s) +$$
$$+ \sum_{i=0}^{s} \tilde{\varrho}^{m+i-s}(v_1, ..., x, ..., v'_i) \otimes (v'_{i+1}, ..., v'_s).$$

So the righthand side depends only on $\tilde{\varrho}^j$ with $j < m$, (except for the uninteresting terms $\tilde{\varrho}^m(v_1, ..., x, ..., v'_s) \otimes 1$ and $1 \otimes \tilde{\varrho}^m(v_1, ..., x, ..., v'_s)$). With this, an induction argument shows that $\tilde{\varrho}^m$ is nonzero only on $V^{\otimes r} \otimes X^{k_1,...,k_n}_{l_1,...,l_n} \otimes V^{\otimes s}$ with $r + s = m$, where it is given by

$$\tilde{\varrho}^m(v_1, ..., v_r, x, v'_1, ..., v'_s) = (-1)^{|\varrho| \sum_{i=1}^{r-k} |v_i|}(v_1, ..., v_r, \varrho(x), v'_1 ..., v'_s).$$

(b) Let $Y := \{\sigma : X \to T^Z V \mid \sigma$ makes diagram (2.26) commute $\}$. Then

$$\alpha : \{\{\varrho^{k_1,...,k_n}_{l_1,...,l_n} : X^{k_1,...,k_n}_{l_1,...,l_n} \to Z\}_{k_1,...,k_n,l_1,...,l_n}\} \to Y,$$
$$\{\varrho^{k_1,...,k_n}_{l_1,...,l_n} : X^{k_1,...,k_n}_{l_1,...,l_n} \to Z\}_{k_1,...,k_n,l_1,...,l_n} \mapsto \sum_{k_1,...,k_n,l_1,...,l_n} \widetilde{\varrho^{k_1,...,k_n}_{l_1,...,l_n}}.$$
$$\beta : Y \to \{\{\varrho^{k_1,...,k_n}_{l_1,...,l_n} : X^{k_1,...,k_n}_{l_1,...,l_n} \to Z\}_{k_1,...,k_n,l_1,...,l_n}\},$$
$$\sigma \mapsto \{pr_Z \circ \sigma|_{X^{k_1,...,k_n}_{l_1,...,l_n}}\}_{k_1,...,k_n,l_1,...,l_n}$$

are inverse to each other by (a).

□

Definition 2.28. *Given modules V, W_1, ..., W_n and Z, and let again $X := T^{W_1}V \otimes ... \otimes T^{W_n}V$. Denote by*

$$T^{W_1,...,W_n}V := \bigoplus_{k_0,...,k_n} V^{\otimes k_0} \otimes W_1 \otimes V^{\otimes k_1} \otimes W_2 \otimes ... \otimes V^{\otimes k_{n-1}} \otimes W_n \otimes V^{\otimes k_n}.$$

Then a map $F : T^{W_1,...,W_n}V \to Z$ will in the following be understood as a map making diagram (2.26) commute by using the last Lemma 2.27, when *setting* $F^{k_1,0,\dots,0}_{l_1,\dots,l_n} := F$ *and letting all other components vanish. (This gets rid of a certain redundancy of* X*, namely the question of "where to break"* $T^{W_i}V \otimes T^{W_{i+1}}V = \bigoplus V^{\otimes k} \otimes W_i \otimes V^{\otimes l} \otimes V^{\otimes r} \otimes W_{i+1} \otimes V^{\otimes s}$ *for* $l + r = const.$*)*

Remark 2.29. *What is the meaning of the concept of an* <u>A_∞*-structure*</u> *on some Hochschild-complex* $C^*(A, M)$*?*

By Proposition 2.4. it is clear that one needs to construct maps $m_i : (C^*(A, M))^{\otimes i} \to C^*(A, M)$*, satisfying certain properties. But by the above Lemma 2.27. it is enough to define a map* $m_i : T^{M,\dots,M}A \to M$*, where* $T^{M,\dots,M}A$ *contains* i *copies of* M*. Again one can explicitly write down the equations that have to be satisfied by these* m_i*'s.*

An example of this will be given below in Definition 3.4. together with Theorem 3.6. It was shown in [10], that this defines an A_∞*-algebra structure on* $C^*(A, A)$ *and therefore satisfies all the necessary equations.*

2.5 ∞ Poincaré Duality on A_∞ Algebras

There are canonical A_∞-bialgebra-structures on a given A_∞-algebra and its dual. A_∞-bialgebra-maps between them will then be defined to be ∞-inner products.

Lemma 2.30. *Given an* A_∞*-algebra* (A, D)*. Let the coderivation* D *be given by the system of maps* $\{m_i : A^{\otimes i} \to A\}_{i \geq 1}$ *from Proposition 2.4.*

(a) One can define an A_∞*-bimodule-structure on* A *by taking* $b_{k,l} : A^{\otimes k} \otimes A \otimes A^{\otimes l} \to A$ *to be given by*

$$b_{k,l} := m_{k+l+1}.$$

(b) *One can define an A_∞-bimodule-structure on A^* by taking $b_{k,l} : A^{\otimes k} \otimes A^* \otimes A^{\otimes l} \to A^*$ to be given by*

$$(b_{k,l}(a_1, ..., a_k, a^*, a_{k+1}, ..., a_{k+l}))(a) :=$$

$$:= \pm a^*(m_{k+l+1}(a_{k+1}, ..., a_{k+l}, a, a_1, ..., a_k)),$$

where the signs are given in Lemma 2.18.

Proof. (a) First notice that the A_∞-bialgebra extension described in Lemma 2.13.(a) becomes in this case the same as the extension by coderivation described in Lemma 2.3.(a). Now, the equations of Proposition 2.16. become exactly those of Proposition 2.4. and the diagram (2.12) from Proposition 2.11. becomes the usual coderivation diagram for D.

(b) This follows immediately from (a) and Lemma 2.18.

□

Example 2.31. *In the case of a differential algebra (A, ∂, μ), which by Example 2.6. can be seen as an A_∞-algebra, the above A_∞-bialgebra structure on A is exactly the bialgebra structure given by left- and right-multiplication, because then $b_{1,0}(a \otimes b) = m_2(a \otimes b) = a \cdot b$ and $b_{0,1}(a \otimes b) = m_2(a \otimes b) = a \cdot b$, for $a, b \in A$.*
Similar the A_∞-bialgebra structure on A^* is given by right- and left-multiplication in the arguments: $b_{1,0}(a \otimes b^*)(c) = b^*(m_2(c \otimes a)) = b^*(c \cdot a)$ and $b_{0,1}(a^* \otimes b)(c) = a^*(m_2(b \otimes c)) = a^*(b \cdot c)$, for $a, b, c \in A$, and $a^*, b^* \in A^*$.

Definition 2.32. *Given an A_∞-algebra (A, D).*

- *Define an* **∞-inner-product** *on A to be an A_∞-bimodule-map from the A_∞-bimodule A to the A_∞-bimodule A^* given in Lemma 2.30.*

- *The ∞-inner-product is called an* ∞-**Poincaré-duality structure**, *if it induces a quasi-isomorphism of the corresponding Hochschild-complexes* $C^\bullet(A, A) \to C^\bullet(A, A^\bullet)$.

Proposition 2.33. *Given an A_∞-algebra (A, D). Then an ∞-inner product on A is exactly given by a system of inner-products on A, namely $\{< ., ., ... >_{k,l} : A^{\otimes k+l+2} \to R\}_{k \geq 0, l \geq 0}$, that satisfies the following relations:*

$$\sum_{i=1}^{k+l+2} (-1)^{\sum_{j=1}^{i-1}|a_j|} < a_1, ..., \partial(a_i), ..., a_{k+l+2} >_{k,l} = \sum_{i,j,n} \pm < a_i, ..., m_j(a_n, ...), ... >_{r,s},$$

where in the sum on the right side, there is exactly <u>one</u> multiplication m_j $(j \geq 2)$ inside the inner-product $< ... >_{r,s}$ and this sum is taken over all i, j, n subject to the following conditions:

(i) The cyclic order of the $(a_1, ..., a_{k+l+2})$ is preserved.

(ii) a_{k+l+2} is always in the last slot of $< ... >_{r,s}$.

(iii) It might happen that a_{k+l+2} is inside m_j. By ii), this is the only case, when the inner product can start with an $a_i \neq a_1$, (e.g. $< a_{i+1}, ..., m_j(a_n, ..., a_{k+l+2}, a_1, ..., a_i) >_{r,s}$ for $i \geq 1$).

(iv) a_{k+1} and a_{k+l+2} are never inside the m_j together. (This is exactly the significance of the indices k and l.)

(v) r and s are given by looking at which slot the element a_{k+1} ends up in the inner-product. More exactly, a_{k+1} will sit in the $(r+1)$st spot of $< ... >_{r,s}$. s is then determined by saying that $< ... >_{r,s}$ takes exactly $r + s + 2$ arguments.

Proof. Let's use the description given in Proposition 2.23. for A$_\infty$-bimodule-maps. An A$_\infty$-bimodule-map from A to A^* is given by maps $f_{k,l} : A^{\otimes k} \otimes A \otimes A^{\otimes l} \to A^*$, for k, $l \geq 0$. These can clearly be interpreted as maps $A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A \to R$, which are then denoted by the inner-product-symbol $< ... >_{k,l}$ from above:

$$< a_1, ..., a_{k+l+1}, a' >_{k,l} := (-1)^{|a'|}(f_{k,l}(a_1, ..., a_{k+l+1}))(a').$$

Being an A$_\infty$-bimodule-map means that the general equation from Proposition 2.23. is satisfied. This equation is

$$\sum \pm f_{k,l}(..., m_i(...), ..., a, ...) + \sum \pm f_{k,l}(..., b_{i,j}(..., a, ...), ...)+$$
$$+ \sum \pm f_{k,l}(..., a, ..., m_i(...), ...) = \sum \pm c_{i,j}(..., f_{k,l}(..., a, ...), ...).$$

Here $a \in A$ is the $(k+1)$st entry of an element in $A^{\otimes k} \otimes A \otimes A^{\otimes l}$, which means it comes from the A_∞*-bimodule* A, instead of the A_∞*-algebra* A. Now, by Lemma 2.30.(a), $b_{i,j} = m_{i+j+1}$ is just one of the multiplications, and thus the left side of the equation is just $f_{k,l}$ applied to all possible multiplications m_i. As $f_{k,l}$ maps into A^*, one can apply the left side to an element $a' \in A$ and therefore use the maps $< ... >_{k,l}$:

$$\sum \pm (f_{k,l}(..., m_i(...), ...))(a') = \sum \pm < ..., m_i(...), ..., a' >_{k,l} . \qquad (2.34)$$

In order to rewrite the right side of the equation, one uses Lemma 2.30.(b) (with $f_{k,l}(..., a, ...) \in A^*$) and the maps $< ... >_{k,l}$, when evaluating on $a' \in A$:

$$\sum \pm (c_{i,j}(a_1, ..., f_{k,l}(..., a, ...), ..., a_{k+l+1}))(a') = \qquad (2.35)$$
$$= \sum \pm (f_{k,l}(..., a, ...))(m_r(..., a_{k+l+1}, a', a_1, ...)) =$$
$$= \sum \pm < ..., a, ..., m_r(..., a_{k+l+1}, a', a_1, ...) >_{k,l} .$$

Now, with the identities (2.34) and (2.35), it is clear that the inner-products have to satisfy equations with sums over all possibilities of applying one multiplication to the arguments of the inner-product subject to the conditions (i)-(iv). This is of course just what is stated in the equation of the Proposition, when isolating the ∂-terms to the left. For condition (v), notice that the extensions of D and D^A from Lemma 2.3.(a) and Lemma 2.13.(a) record exactly the special entry a in the A_∞-bimodule A. Thus, the A_∞-bimodule element a determines the number k, and then l is determined by the number of arguments of $< ... >_{k,l}$.

In order to see that the signs can be written as in the proposition, one has to insert the signs for the case $m_i = m_1 = \partial$. The important terms in (2.34) are

$$(-1)^{\varepsilon}(f_{k,l}(..., \partial a_j, ...))(a') = (-1)^{(\sum_{r<j}|a_r|)+1+k+l+1+|a'|} < ..., \partial a_j, ..., a' >_{k,l},$$

where ε is the ε from Proposition 2.23. with $i = 1$. In (2.35) one only has to look at one term, namely

$$(-1)^{\varepsilon'}(c_{0,0}(f_{k,l}(a_1, ..., a_{k+l+1}))(a'),$$

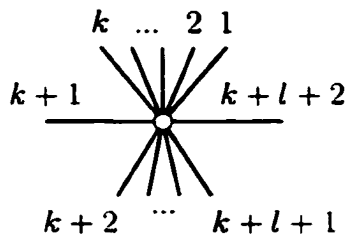
where Proposition 2.23. implies $\varepsilon' \equiv 0 \ (mod\ 2)$, because $i = 1$. So, by Lemma 2.18., this term is given by

$$(-1)^{|f_{k,l}(a_1,...,a_{k+l+1})|-1}(f_{k,l}(a_1, ..., a_{k+l+1}))(\partial a') =$$
$$= (-1)^{k+l+(\sum_{r=1}^{k+l+1}|a_r|)+(|a'|-1)} < a_1, ..., a_{k+l+1}, \partial a' >_{k,l} .$$

Bringing this term to the left side and dividing by $(-1)^{k+l+|a'|}$ yields the result. □

There is a diagrammatic way of picturing Proposition 2.33.

Definition 2.36. *Given an A_∞-algebra (A, D) with the ∞-inner product $\{< \ldots, \ldots >_{k,l} \colon A^{\otimes k+l+2} \to R\}_{k\geq 0, l\geq 0}$ from Proposition 2.33. To the inner-product $< \ldots >_{k,l}$, one associates the symbol*

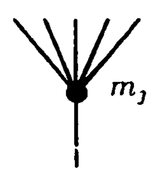


More generally, to any inner-product which has (possibly iterated) multiplications $m_2, m_3, m_4, \ldots$ (but <u>*no*</u> *differential $\partial = m_1$) inside, e.g.*

$$< a_i, \ldots, m_j(\ldots), \ldots, m_p(\ldots, m_q(\ldots), \ldots), \ldots >_{k,l},$$

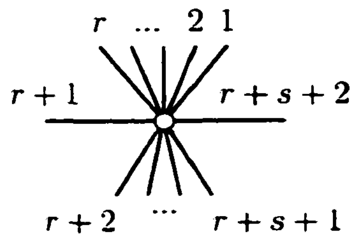
one can associate a diagram like above, by the following rules:

i) To every multiplication m_j, associate a tree with j inputs and one output.



The symbol for the multiplication will also occur in a rotated way. It should always be clear, where the inputs and the output are located.

ii) To the inner product $< ... >_{r,s}$, associate an "evaluation on an open circle":



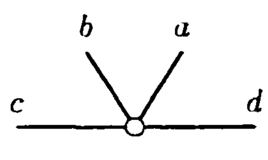
Here there are r elements sitting on top of the circle, s elements are coming in from the bottom of the circle and the two (special) inputs $(r+1)$ and $(r+s+2)$ on the left and right. Thus one gets the required $r+s+2$ inputs.

iii) Around the diagram, one "sticks in" the elements a_i counterclockwise, (where the last element a_{r+s+2} is in the far right slot).
When multiplications m_j of the graph are performed, one uses the counterclockwise orientation of the plane to find the correct order of the arguments a_i in m_j (see examples below).

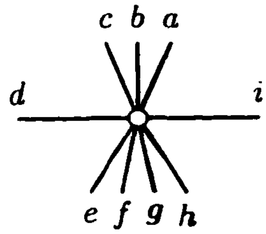
Let's refer to those diagrams as ***inner-product-diagrams.***

Examples: Let $a, b, c, d, e, f, g, h, i, j, k \in A$.

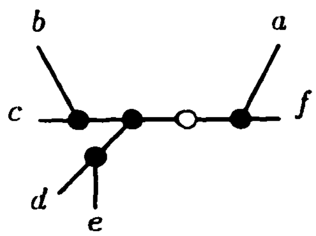
$< a, b, c, d >_{2,0}$, *(deg = 2):*



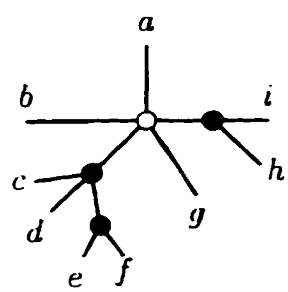
$< a, b, c, d, e, f, g, h, i >_{3,4}$, *(deg = 7):*



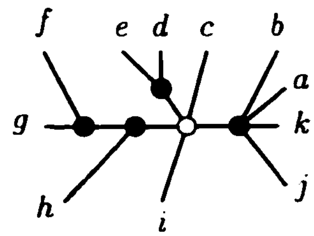
$< m_2(m_2(b, c), m_2(d, e)), m_2(f, a) >_{0,0}$, *(deg = 0):*



$< a, b, m_3(c, d, m_2(e, f)), g, m_2(h, i)) >_{1,2}$, *(deg = 4):*



$< c, m_2(d, e), m_2(m_2(f, g), h), i, m_4(j, k, a, b) >_{2,1}$, *(deg = 5):*



There is a chain-complex associated to the inner-product-diagrams:

i) Degree:

The degree of the inner-product-diagram associated to an inner-product $< \ldots >_{k,l}$ *with the multiplications* $m_{i_1}, \ldots, m_{i_n}$ *inside is defined to be*

$$deg(Diagram) := k + l + \sum_{j=1}^{n} (i_j - 2).$$

Examples are given above.

ii) Chain-complex:

For $n \geq 0$, *let* C_n *be the space generated by inner-product-diagrams of degree* n. *Then let* $C := \bigoplus_{n \geq 0} C_n$.

iii) Differential:

Let's define a differential on the inner-products to be the composition with the operator $\tilde{\partial} := \sum_i id \otimes \ldots \otimes id \otimes \partial \otimes id \otimes \ldots \otimes id$ *(where* $\partial = m_1$ *is being in the i-th spot):*

$$(d(< \ldots . m(\ldots . m(\ldots) . \ldots) . \ldots >))(a_1 . \ldots . a_s) :=$$

$$:= (< \ldots . m(\ldots . m(\ldots) . \ldots) . \ldots >)(\sum_{i=1}^{s} (-1)^{\sum_{j=1}^{i-1} |a_j|} (a_1 . \ldots . \partial(a_i) . \ldots . a_s)).$$

Why is this well-defined and what is its diagrammatic interpretation? *First let's look at the inner-product* $< \ldots >_{k,l}$ *without any multiplications inside. Then by Proposition 2.33. this means that one puts one multiplication into the inner-product-diagram in all possible places, such that the two lines on the far left and on the far right are not being multiplied (see Proposition 2.33. (iv)).*

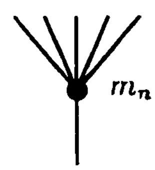
Now, if there are multiplications inside the inner-product, then one can observe from Proposition 2.3., that

$$\sum_i m_n \circ (id \otimes \ldots \otimes \partial \otimes \ldots \otimes id) =$$

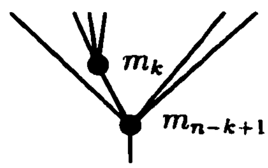
$$= \sum_{k=2}^{n-1} \sum_i m_{n+1-k} \circ (id \otimes \ldots \otimes m_k \otimes \ldots \otimes id) + \quad (2.37)$$

$$+\partial \circ m_n \quad (2.38)$$

(The sum over i on both sides of the above equation means that one has to put ∂ *[or respectively* m_k*] in the i-th spot of the tensor-product.)* *Now, (2.37) "brakes" the given multiplication* m_n



into all possible smaller parts m_{n+1-k} *and* m_k



The last term (2.38) is also important. It makes an inductive argument of the above possible. One gets a term $\partial(m_n(\ldots))$ *being inside the inner-product or possibly another multiplication, which then will have arguments applied to* $\tilde{\partial}$*, so that the above discussion works again.*
So, on the level of graphs, the differential means to take just one more multiplication in all possible places without multiplying the given far left and far right lines (c.f. Example 2.40. below).

Theorem 2.39. $d : C_n \to C_{n-1}$*, and* $d^2 = 0$.

Proof. By the formula for the degrees in Definition 2.36., a multiplication m_n with n inputs contributes by $n-2$. Now, taking the differential means to put in one more multiplication in all possible ways. Let's assume one wants to put m_n into the formula. Then this replaces n arguments with one argument in the higher level of the formula. Thus

$$\begin{aligned} new\ degree &= (old\ degree) - n + 1 + (n-2) = \\ &= (old\ degree) - 1. \end{aligned}$$

One can prove $d^2 = 0$ in two ways:

i) Algebraically:

The definition of d on the inner-products is just a composition with the operator $\bar{\partial} = \sum_i id\otimes ...\otimes id\otimes \partial \otimes id\otimes ...\otimes id$ (∂ being in the i-th spot) on TA. Thus d^2 is composition with

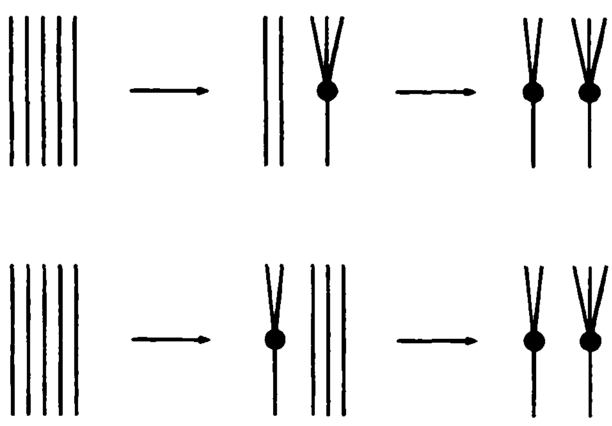
$$\bar{\partial}^2 = \sum_{i,j} \pm id \otimes ... \otimes \partial \otimes ... \otimes \partial \otimes ... \otimes id = 0.$$

This gives zero, because ∂ occurring at the i-th and the j-the spot can be obtained by first taking the one at the i-th and then the one at the j-th, or first taking the one on the j-th and then the one at the i-th spot. These two possibilities cancel each other out, because ∂ is of degree -1 and the first ∂ either has to "jump" over the other ∂, which gives a "$-$" sign, or not.

ii) Diagrammatically (without signs):

If d means to create one new multiplication inside the inner-product-diagram, then d^2 obviously corresponds to creating two new multiplications. For two given multiplications, there are always two ways of obtaining them.

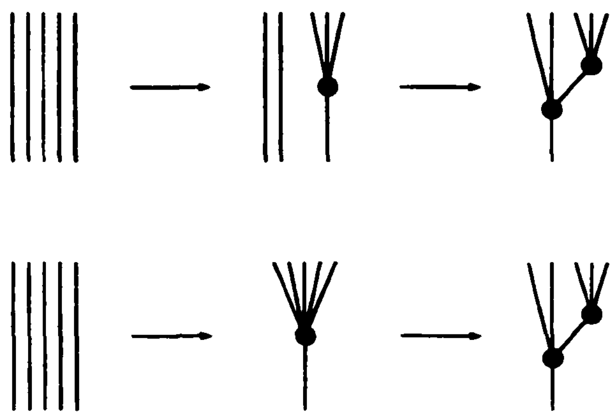
Case 1: The multiplications are on different outputs



Clearly, one can do one first and then the other, or vice versa. In this way one always gets this term cancelling itself.

Case 2: The multiplications are on the same output

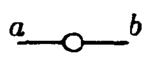
Here are the two ways of obtaining the same picture, and thus cancelling out:



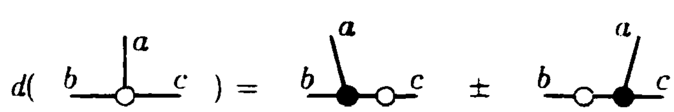
□

Example 2.40. *Let* a, b, $c \in A$.

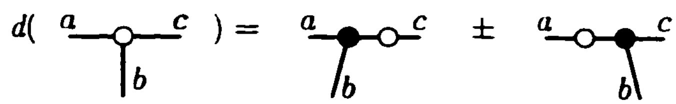
$k = 0,\ l = 0$: $d(< a, b >_{0,0}) = 0$



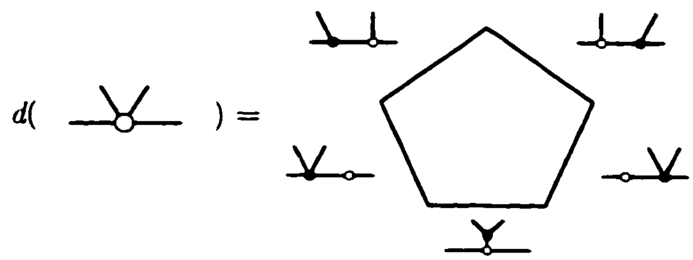
$k = 1,\ l = 0$: $d(< a, b, c >_{1,0}) =< a \cdot b, c >_{0,0} \pm < b, c \cdot a >_{0,0}$



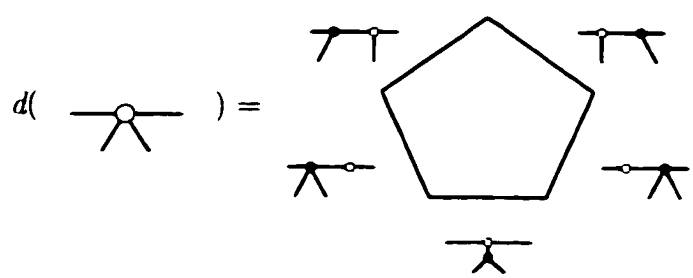
$k = 0,\ l = 1$: $d(< a, b, c >_{0,1}) = < a \cdot b, c >_{0,0} \pm < a, b \cdot c >_{0,0}$



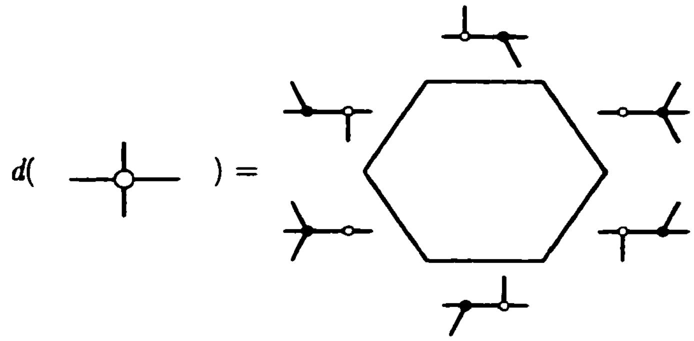
$k = 2,\ l = 0$:



$k = 0,\ l = 2$:



$k = 1$, $l = 1$:



In the last three pictures (with $k + l = 2$) the righthand side is understood to be a sum over the five, or respectively six, little inner-product-diagrams. Then, as $d^2 = 0$, one can arrange these elements according to their corresponding boundaries to give a boundary-free object. In this way one gets certain polyhedra associated to the $< \ldots >_{k,l}$'s.

3 BV-Structure on the Hochschild Complex

3.1 Definition of the Involved Maps

The goal of this section is to define operations on the Hochschild cochain complex, including a multiplication, a bracket and a Δ-operator, that make $H^\bullet(A, A)$ into a BV-algebra.

Let's first briefly recall the definitions of a Gerstenhaber-algebras and BV-algebras.

Definition 3.1. *Given a ring R. A* ***Gerstenhaber-algebra*** $(A, \cdot, \{-, -\})$ *over R consists of*

- *a graded commutative associative algebra $(A, \cdot)$,*
- *a Lie-bracket $\{-, -\} : A \otimes A \to A$ of degree $+1$ satisfying the graded symmetry condition and graded Jacobi-identity:*

$$\{a, b\} = -(-1)^{|a| \cdot |b|}\{b, a\},$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a| \cdot |b|}\{b, \{a, c\}\},$$

- *such that the multiplication and the Lie-bracket satisfy the graded Leibniz rule:*

$$\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{|a| \cdot |b|} b \cdot \{a, c\}$$

There is a more general structure:

Definition 3.2. *Given a ring R. A* ***Batalin-Vilkovisky-algebra****, or short* ***BV-algebra****, $(A, \cdot, \Delta)$ over R consists of*

- *a graded commutative associative algebra* $(A, \cdot)$,
- *a differential* $\Delta : A \to A$ *of degree* -1 *such that* $\Delta \circ \Delta = 0$, *and*
- *such that the deviation of* Δ *from being a graded derivation defines a Gerstenhaber-algebra on* A, *i.e. with the bracket* $\{-,-\} : A \otimes A \to A$ *defined by*

$$\{a, b\} := \Delta(a \cdot b) - \Delta(a) \cdot b - (-1)^{|a|} a \cdot \Delta(b),$$

the triple $(A, \cdot, \{-,-\})$ *forms a Gerstenhaber-algebra.*

This definition immediately implies the following Lemma:

Lemma 3.3. *A BV-algebra structure implies the structure of a Gerstenhaber-algebra.*

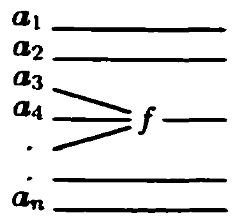
Now, the goal is to define such a structure on the Hochschild-cochain-complex of an A_∞-algebra A. In order to describe the necessary maps. a convenient diagrammatic notation for maps of the Hochschild complex will be used. Let's first rewrite elements of the Hochschild complex in terms of diagrams.

Let A be a graded module. In order to include the signs, it will be convenient to denote degree of an element $a \in A_i$ by $|a| = i$. and the corresponding degree in the shifted space $a \in A_i = (sA)_{i+1}$ by $||a|| = i + 1$.

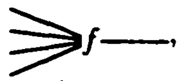
Now, the Hochschild-cochain-complex (Definition 2.7.) consists of lifts of maps $f : (sA)^{\otimes i} \to sA$ to coderivations $\tilde{f} : BA \to BA$ according to Lemma 2.3.(a):

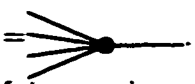
$$\tilde{f}(a_1, \dots, a_n) = \sum_{j=0}^{n-i} (-1)^{||f|| \cdot (||a_1|| + \dots + ||a_j||)} (a_1, \dots, f(a_{j+1}, \dots, a_{j+i}), \dots, a_n).$$

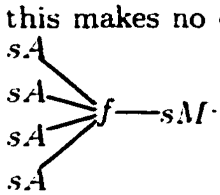
Diagrammatically this can be represented by

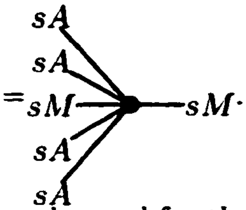
$$\bar{f}(a_1, \ldots, a_n) = \sum_{j=0}^{n-i} \pm$$ 

where the sum moves f from top to bottom.

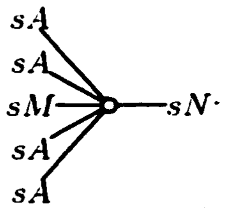
In general $\bar{f}$ consists of a sum of lifts of f's, corresponding to maps starting from different components $(sA)^{\otimes i}$, i.e. taking different amount of arguments. It is useful to represent $\bar{f}$ simply by , keeping in mind, that this represents not only a sum over different amount of arguments, but also a sum corresponding to the lift of the components to BA like above.

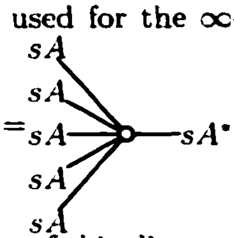
For example if (A, D) is an A_∞-algebra, then part of the structure is given by the coderivation $D : BA \to BA$. This coderivation was denoted by a fat dot: $D =$ .

If M is another graded module, then exactly the same arguments can be applied to coderivations $\bar{f} : BA \longrightarrow B^M A$. The only difference is, that the maps $f : (sA)^{\otimes i} \longrightarrow sM$ now map into sM and therefore lift to maps $(sA)^{\otimes n} \longrightarrow \bigoplus_{k+l=n-i} (sA)^{\otimes k} \otimes (sM) \otimes (sA)^{\otimes l}$. Diagrammatically this makes no difference, when remembering from where to where f maps:


Now, if (M, D^M) is an A_∞-bimodule over the A_∞-algebra (A, D), then, by the same reasoning as above, one can represent the coderivation $D^M : B^M A \to B^M A$, given by maps $(sA)^{\otimes i} \otimes (sM) \otimes (sA)^{\otimes j} \to sM$, in a diagrammatic way: $D^M =$ 

Here the same symbol of a fat dot will be used that was already used for the A_∞-algebra structure of A, because in the applications below, M will be either A or A^*, and in these cases D^M is given by D (; compare Lemma 2.30.).

Finally, one can repeat the arguments above for graded modules A, M, N, and a given bi-comodule-map $F : B^M A \to B^N A$. This means that F is given by maps $(sA)^{\otimes i} \otimes (sM) \otimes (sA)^{\otimes j} \to sN$ for $i, j \geq 0$, which respectively lift to maps $(sA)^{\otimes k} \otimes (sM) \otimes (sA)^{\otimes l} \to (sA)^{\otimes k-i} \otimes (sN) \otimes (sA)^{\otimes l-j}$. The symbol that will be used in this dissertation to denote bi-comodule-maps are open circles: 

This will of course be used for the ∞-inner-product $F : B^A A \to B^{A^*} A$ of an A_∞-algebra A: $F =$ 

(; compare Definition 2.36.). (Note that instead of thinking of this diagram as having an output in sA^*, one can interpret it as a map with one more input-argument on the right.)

With this notation, one can introduce a diagrammatic representation for maps of the Hochschild complex.

Definition 3.4. *Let A be a graded module.*

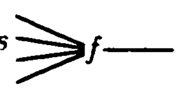
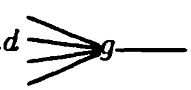
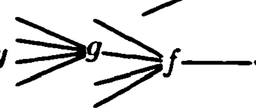
- *The basic operation that can be performed in the Hochschild-complex*

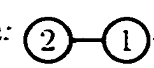
$C^*(A,A)$ is the "composition". This cannot be the usual composition of the coderivations as maps, because the composition of two coderivations is in general not a coderivation. In detail, if $f : (sA)^{\otimes i} \to sA$ and $g : (sA)^{\otimes j} \to sA$ are linear maps, then lifting them to coderivations $\tilde{f}$ and $\tilde{g}$, and composing them as <u>maps</u>, gives the map

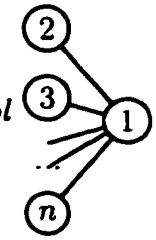
$$\begin{aligned}(a_1, ..., a_n) \mapsto\ & \sum (-1)^{\varepsilon} \cdot (..., f(a_{k+1}, ..., a_{k+i}), ..., g(a_{l+1}, ..., a_{l+j}), ...) + \\ &+ \sum (-1)^{\varepsilon} \cdot (..., f(a_{k+1}, ..., g(a_{l+1}, ..., a_{l+j}), ...), ...) + \\ &+ \sum (-1)^{\varepsilon'} \cdot (..., g(a_{k+1}, ..., a_{k+j}), ..., f(a_{l+1}, ..., a_{l+i}), ...),\end{aligned}$$

where $\varepsilon = ||f|| \cdot \sum_{i=1}^{k} ||a_i|| + ||g|| \cdot \sum_{i=1}^{l} ||a_i||$, and $\varepsilon' = ||f|| \cdot (||g|| + \sum_{i=1}^{l} ||a_i||) + ||g|| \cdot \sum_{i=1}^{k} ||a_i||$. This is (clearly) not a coderivation, because the first and last component of the sum are not of the form of coderivations. So, for the definition of the ***composition****, one takes only the middle term*

$$\tilde{f} \circ \tilde{g}(a_1, ..., a_n) := \sum (-1)^{||f|| \cdot \sum_{i=1}^{k} ||a_i|| + ||g|| \cdot \sum_{i=1}^{l} ||a_i||} \cdot (..., f(a_{k+1}, ..., g(a_{l+1}, ..., a_{l+j}), ..., a_{k+i-j+1}), ...).$$

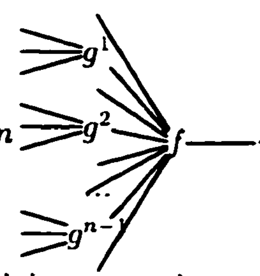
Diagrammatically, for the coderivations  *and*  *the composition $\tilde{f} \circ \tilde{g}$ is denoted by* *, where it is clear again to sum in all possible ways.*

Here is the definition of a symbol for the composition operation: .
It symbolizes to map $\tilde{f} \otimes \tilde{g}$ to $\tilde{f} \circ \tilde{g}$, by plugging the second map g into the first map f in all possible combinations.

- *More generally the symbol* 

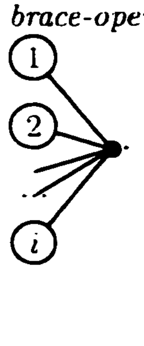
will denote the ***brace****-operation, which is defined as follows. For linear maps* $f : (sA)^{\otimes i} \to sA$ *and* $g^k : (sA)^{\otimes j_k} \to sA$, *with* $k = 1, \ldots, n-1$, *and corresponding lifts* $\tilde{f}$, $\tilde{g}^k$, *one takes*

$$(\tilde{f}\{\tilde{g}^1, ..., \tilde{g}^{n-1}\})(a_1, ..., a_m) := \sum (-1)^{\|f\|\cdot\sum_{i=1}^{k}\|a_i\| + \|g\|\cdot\sum_{i=1}^{l}\|a_i\|} \cdot$$
$$\cdot (..., f_i(..., g^1_{j_1}(...), ..., g^2_{j_2}(...), ..., g^{n-1}_{j_{n-1}}(...), ...).$$

This corresponds to the coderivation with the diagram 

Here, the nodes represent certain coderivations which have an arbitrary amount of inputs. There is an implicit sum with signs over all those possible inputs.

- *If* (A, D) *is an* A_∞*-algebra, then one defines a sequence of multiplications given by* $M_i(\tilde{f}^1, ..., \tilde{f}^i) := D\{\tilde{f}^1, ..., \tilde{f}^i\}$, *where* $\{...\}$ *denotes the brace-operation from above. The symbol for the multiplication* M_i *is*

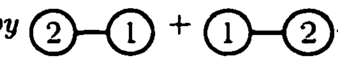


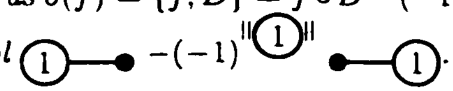
With the help of the composition it is also possible to define the Lie-bracket on the Hochschild complex.

Definition 3.5. *Let A be a graded module.*

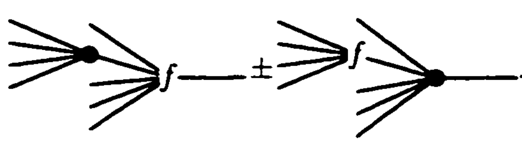
- *Let $f, g \in C^\bullet(A, A)$ Then define the* **bracket**-*operation on $C^\bullet(A, A)$ to be the graded commutator of the composition:*

  $$\{f, g\} := f \circ g - (-1)^{\|f\| \cdot \|g\|} g \circ f.$$

  The symbol for the bracket is clearly given by .
  Note, that this bracket _*is*_ *the same as if one would take the commutator of the coderivations as maps, because the troubling terms in Definition 3.4. cancel each other out. The commutator of coderivations is a coderivation.*

- *Now, if one has an A_∞-algebra (A, D), then the differential on $C^\bullet(A, A)$ can be rewritten as $\delta(f) = \{f, D\} = f \circ D - (-1)^{\|f\|} D \circ f$ and therefore it has the symbol* .

- *More generally for any A_∞-bimodule (M, D^M), the differential on $C^\bullet(A, M)$ is written as $\delta^M(f) = f \circ D - (-1)^{\|f\|} D^M \circ f$, and its diagram is given by*

  $\delta^M(f) = f \circ D - (-1)^{\|f\|} D^M \circ f =$ 

These definitions give the well-known Gerstenhaber-structure on the Hochschild complex.

Theorem 3.6. *Let (A, D) be an A_∞-algebra. Denote by $(M_2)_\bullet$ and $\{-,-\}_\bullet$ the induced maps of (M_2) and $\{-,-\}$ on homology.*
Then, the triple $(H^\bullet(A, A), (M_2)_\bullet, \{-,-\}_\bullet)$ forms a Gerstenhaber algebra.

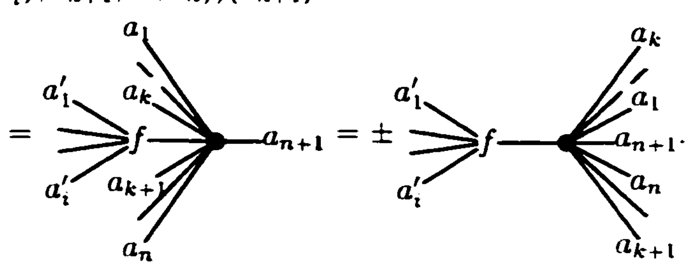
Proof. This is a well known fact, originally proved by Gerstenhaber for associative algebras in [8], and in general by Getzler and Jones in [10]. □

Remark 3.7. *Below, the last definition 3.5. will have to be applied to the case of the A_∞-module $M = A^\bullet$ over an A_∞-algebra (A, D). It will be useful to make this more explicit. In order to do this, one can use the description of $D^{A^\bullet} : B^{A^\bullet}A \to B^{A^\bullet}A$ from Lemma 2.18. and Lemma 2.30. Namely, as $D^{A^\bullet}$ is a coderivation, it is enough to look at the components $D^{A^\bullet} : B^{A^\bullet}A \to A^\bullet$, which are given by*

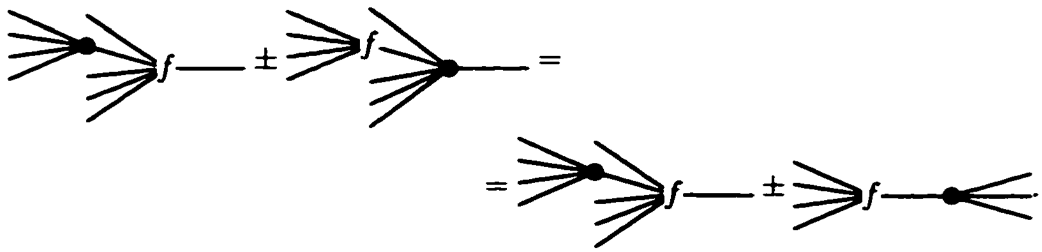
$$(D^{A^\bullet}_{k,n-k-1}(a_1, \dots, a_k, a^\bullet, a_{k+1}, \dots, a_n))(a_{n+1}) =$$
$$= (-1)^{\|a^\bullet\| + (\sum_{i=1}^{k} \|a_i\|)\cdot(\|a^\bullet\| + \sum_{i=k+1}^{n+1} \|a_i\|)} a^\bullet(D_{n+1}(a_{k+1}, \dots, a_n, a_{n+1}, a_1, \dots, a_k)).$$

So, for $f : sA^{\otimes i} \to sA^\bullet$, one has

$$(D^{A^\bullet}(a_1, \dots, a_k, f(a'_1, \dots, a'_i), a_{k+1}, \dots, a_n))(a_{n+1}) =$$



Therefore, the differential $\delta^{A^\bullet}(f) = f \circ D - (-1)^{\|f\|} D^{A^\bullet} \circ f$ is represented by *the diagrams*



Notice that this just means to add a multiplication D in all possible ways around f, which corresponds exactly to the interpretation of Definition 2.36. In other words, if one identifies

$$C^*(A, A^*) = Coder(BA, B^{A^*}A) \cong$$
$$\cong \prod_{i\geq 0} Hom((sA)^{\otimes i}, (sA)^*) \cong \prod_{i\geq 0} Hom((sA)^{\otimes i+1}, R),$$

then the output of a map $f : (sA)^{\otimes i} \to (sA)^*$ is interpreted as another input, and the picture for the differential δ^{A^*} above shows that this is just the dual of applying D to all possible spots in the inputs.

The last ingredient that is needed is a BV-operator. The definition for it will not be given on $C^*(A, A)$, but on a quasi-isomorphic subcomplex of $C^*(A, A^*)$, namely the reduced Hochschild complex.

Definition 3.8. *Given an A_∞-algebra (A, D), and write $D = \sum_{i\geq 0} D_i$ in its components (compare Lemma 2.3.(b)).*

- *Then an element $1 \in A_0 \cong (sA)_{+1} \subset BA$ is called a* ***unit of*** *A, if, D_n applied to any element of the form $(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) \in (sA)^{\otimes n}$ always gives 0, except for the case $n = 2$, where*

$$-D_2(1, a) = a = (-1)^{\|a\|} D_2(a, 1).$$

$$D_n(a_1, ..., a_{i-1}, 1, a_{i+1}, ..., a_n) = 0 \ \ for \ \ n \neq 2.$$

*In this case $(A, D, 1)$ is called a **unital A_∞-algebra**.*

Notice that the signs are given such that for the associated map $m_2 = s^{-1} \circ D_2 \circ (s \otimes s)$ one has $m_2(1, a) = m_2(a, 1) = a$ (compare the proof of Proposition 2.4.).

- *Given a unital A_∞-algebra $(A, D, 1)$ and an A_∞-bimodule (M, D^M) over A. Then define the **reduced Hochschild-cochain-complex** $\overline{C^\bullet}(A, M)$ to be the subspace of $C^\bullet(A, M) = Coder(BA, B^M A)$ given by maps f which map to 0, whenever applied to tensor-products including the unit 1:*

$$f \in \overline{C^\bullet}(A, M) \iff f \in C^\bullet(A, M) \quad and$$
$$\forall\, a_1, ..., a_n \in A : \quad f(a_1, ..., 1, ..., a_n) = 0.$$

Lemma 3.9. *$\overline{C^\bullet}(A, M)$ is a subcomplex of $C^\bullet(A, M)$ with the inclusion being a quasi-isomorphism $\overline{C^\bullet}(A, M) \subset C^\bullet(A, M)$.*

Proof. See Loday [13], p.46, Proposition 1.6.5. □

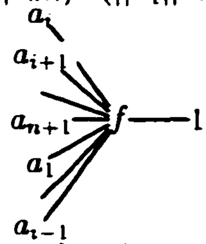
With this, it is finally possible to define the BV-operators. It turns out that there are actually two operations β and Δ, which satisfy $\beta^2 = 0$ and the deviation of Δ from being a derivation is a Gerstenhaber algebra (on homology). In the important case where $\beta_* = \Delta_*$, one gets a BV-algebra.

Definition 3.10.

- *Given a unital A_∞-algebra $(A, D, 1)$. The dual of Connes' B-operator, denoted by β, is an operator $\beta : \overline{C^\bullet}(A, A^*) \to \overline{C^\bullet}(A, A^*)$. For $f \in$*

$\overline{C^\bullet}(A, A^*)$, *let*

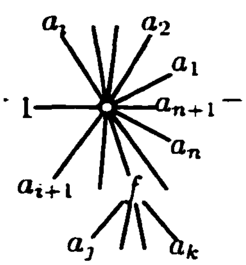
$$((\beta(f))(a_1, ..., a_n))(a_{n+1}) := \sum_{i=1}^{n+1} (-1)^\varepsilon (f(a_i, ..., a_{n+1}, a_1, ..., a_{i-1}))(1),$$

where $\varepsilon = ||f|| + ||a_1|| + ... + ||a_{n+1}|| + (||a_i|| + ... + ||a_n||) \cdot (||a_1|| + ... + ||a_{i-1}||)$. *Diagrammatically, this coderivation is* $\pm$ 

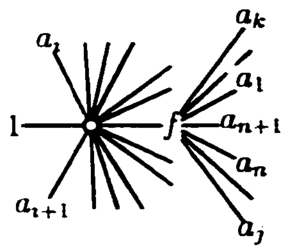
where the sum over i is implicitly assumed. The operation β is denoted (up to sign) by 

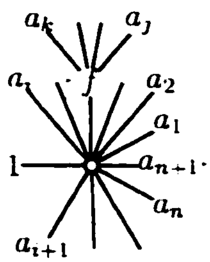
Here, the fat line on the left denotes the position of the last element a_{n+1} plugged into f, which now doesn't have to come from the right any more. Then all the other elements a_j will be inserted in a cyclic way starting from a_1.

- *In order to define the Δ-operator, one needs to assume an ∞-inner-product F on A. With this, $\Delta : \overline{C^\bullet}(A, A) \to C^\bullet(A, A^*)$ is given by the diagrammatic picture*

$$((\Delta(f))(a_1, ..., a_n))(a_{n+1}) := (-1)^{\varepsilon_1} \cdot$$ 

$-$

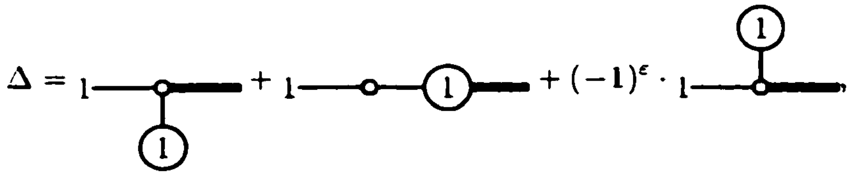
$$-(-1)^{\varepsilon_2} \cdot$$ 

$$+(-1)^{\varepsilon_3} \cdot$$ 

where a sum over all combinations is assumed,

$$
\begin{aligned}
\varepsilon_1 &= \sum_{r=1}^{i} ||a_r|| + ||f|| \cdot (\sum_{r=1}^{j-1} ||a_i||), \\
\varepsilon_2 &= \sum_{r=k+1}^{i} ||a_r|| + ||f|| \cdot (\sum_{r=k+1}^{j-1} ||a_i||) + (\sum_{r=1}^{k} ||a_i||) \cdot (\sum_{r=k+1}^{n+1} ||a_i||), \\
\varepsilon_3 &= \sum_{r=1}^{i} ||a_r|| + ||f|| + ||f|| \cdot (\sum_{r=1}^{j-1} ||a_i||),
\end{aligned}
$$

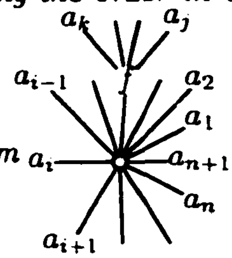
and the ∞-inner-product F is as usual denoted by an open circle (compare Definition 2.36.). The symbol for the Δ-operator is given by



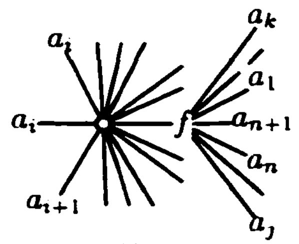
where ε is the degree of the operation to be plugged into (1)*, and the fat line on the right denotes again the position for a_{n+1} to be inserted. The correct sign can be obtained by the following sign convention.*

Sign Convention for symbols: *Given a symbol for an operation on the Hochschild-cochain-complex. Then it is assumed that this represents a sum of all combinations of placing the elements $(a_1, \ldots, a_{n+1})$ into the given picture, with the last element a_{n+1} ending up in to position indicated by the fat line. In particular, there is a unique configuration of doing this for which all $a_1, \ldots, a_{n+1}$ are plugged into the same operation in which a_{n+1} is being plugged into. (This means that one can think of the a_i's as sitting "next to each other" at the fat line.) This configuration will be assumed to have a vanishing sign (i.e. $+1$). Then every other configuration for entering the a_i's can be compared* *to the one described above, and thus obtains a sign given by the usual sign convention, namely when writing down an explicit expression for a diagram, one introduces a sign* $(-1)^{||x||\cdot||y||}$ *for every letter* x *that has to jump over a letter* y.

Here are two examples for determining the order in which the letters would occur in a given diagram. The diagram

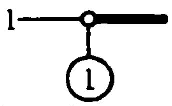


is explicitly written as $F(a_1, ..., a_{j-1}, f(a_j, ..., a_k), a_{k+1}, ..., a_{n+1})$. *The diagram*

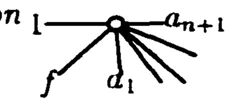


represents the expression $F(a_{k+1}, ..., a_{j-1}, f(a_j, ..., a_{n+1}, a_1, ..., a_k))$.

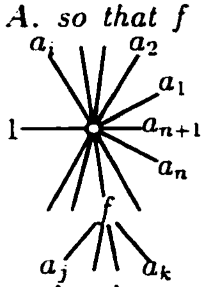
Here is an example for the sign convention. In the above definition of Δ, *the symbol*



was used. It contains the special configuration



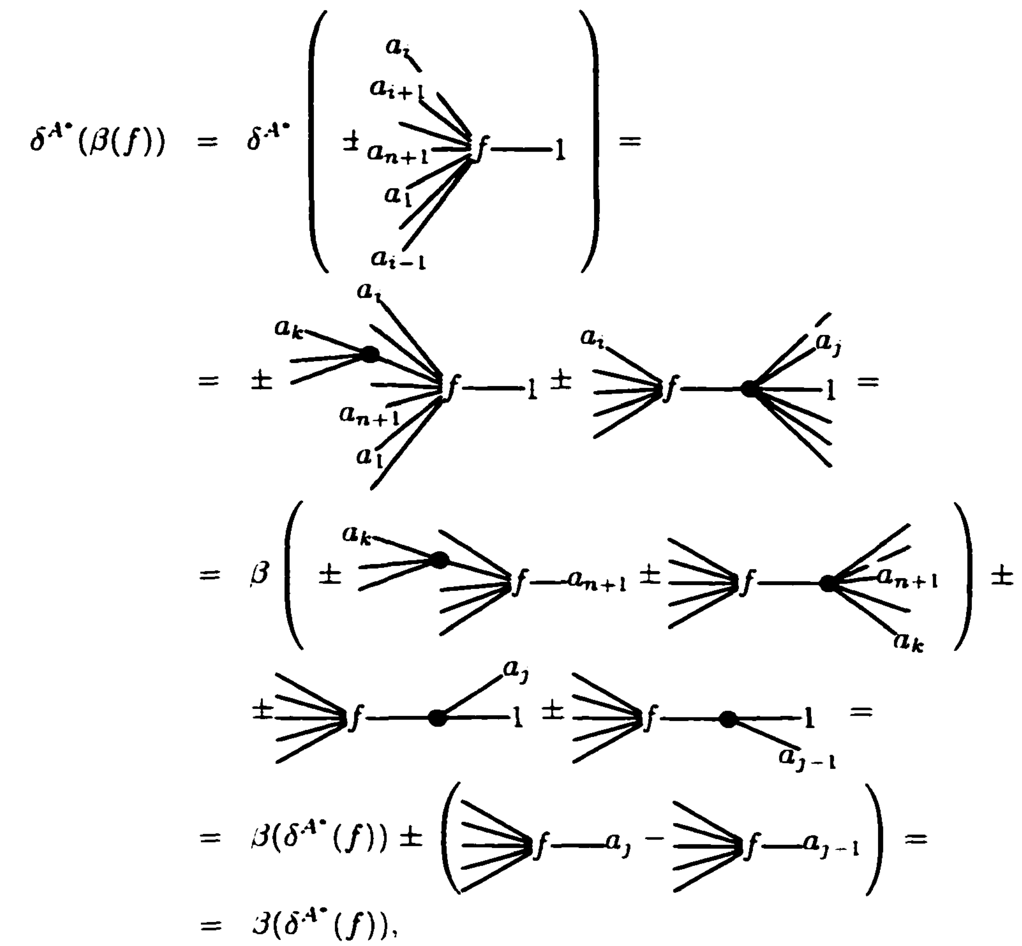
, which has the $+1$ *sign. (Recall that the lowest component of* f *is* $f : R \to A$, *so that* f *without any input makes sense). Therefore the configuration*



obtains the sign $\sum_{r=1}^{i} ||a_r|| + ||f|| \cdot (\sum_{r=1}^{j-1} ||a_r||)$, *which is exactly what was written as* ε_1 *in the definition for* Δ *above.*

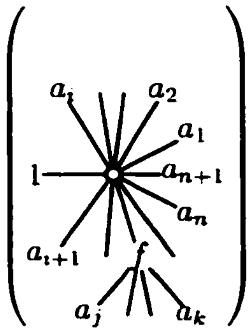
Claim: ∂ *and* Δ *are (graded) chain maps.*

Proof. The proofs will be given twice. First, in order to get a better understanding, the explicit diagrams (without signs) will be used. But the proof becomes much easier, when being performed on the level of symbols. Then, the above sign convention will also proof the correctness of signs.
With Remark 3.7., one gets

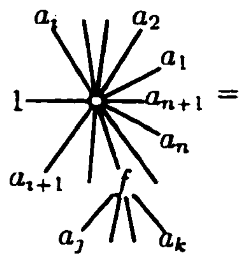


where the second term in the second line comes from multiplying 1 with other elements via D, which, as 1 is a unit, can only be done for $a_j \otimes 1$ and $1 \otimes a_{j-1}$. This gives terms with a_j and a_{j-1}, which cancel after summing over j. (Notice that the expressions with $a_j \otimes 1$ and $1 \otimes a_j$ differ by a sign $(-1)^{\|a_j\|}$, and $D_2(1, a_j) = -(-1)^{\|a_j\|} \cdot D_2(a_j, 1)$.)

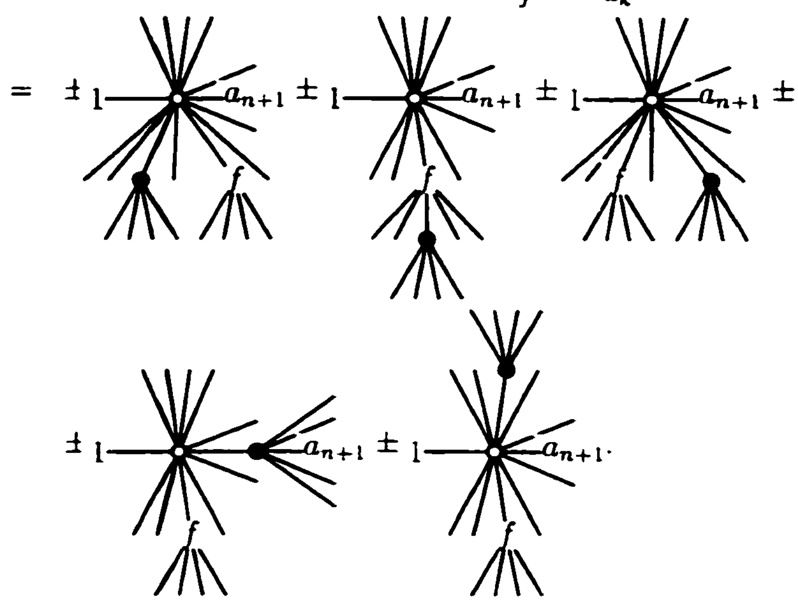
Similarly,

δ^{A^*} 

= sum of all possibilities of placing the

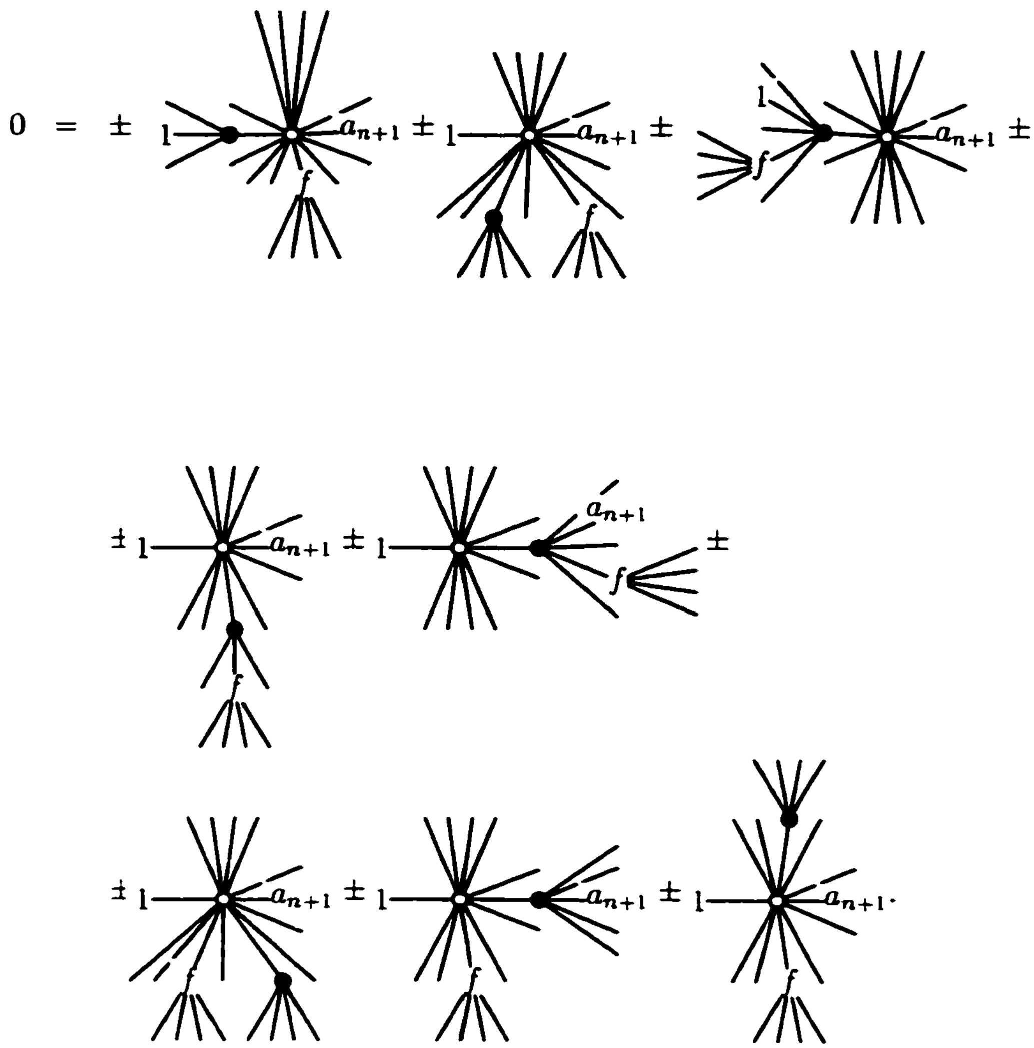
multiplication D around 

=

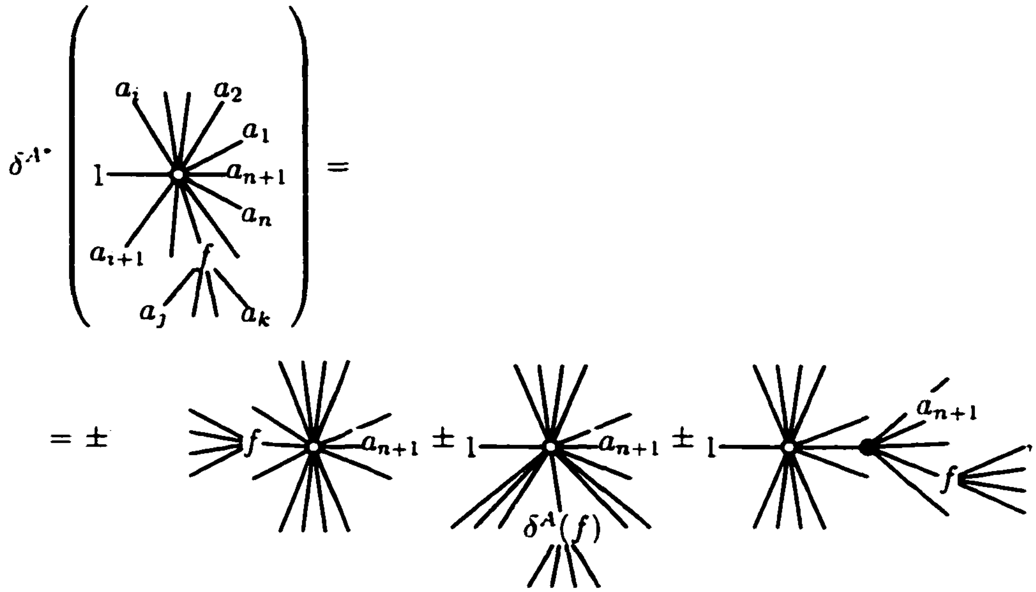


But notice that by the property $D^{A^*} \circ F = F \circ D^A$ of the ∞-inner-product

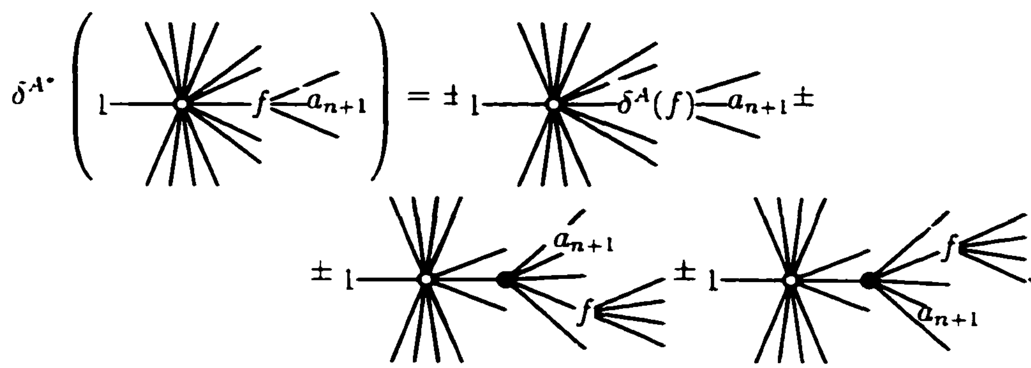
F, one has (compare Definition 2.36.):

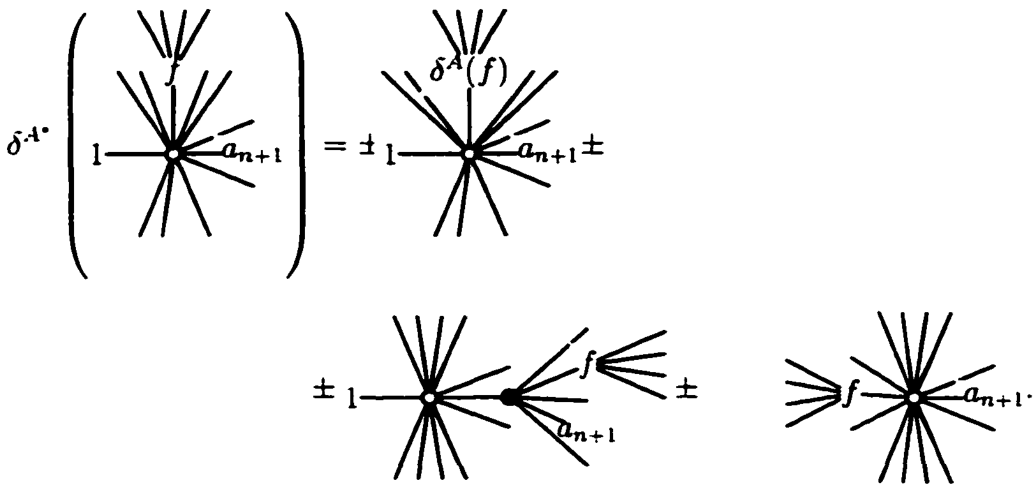


Comparing this with the above expression, one sees that the differential is given by



where the property of the unit 1 was used in a similar way as before. A calculation of the differential of the other two terms from $\Delta(f)$ results in

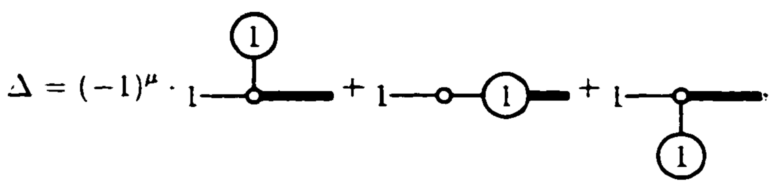




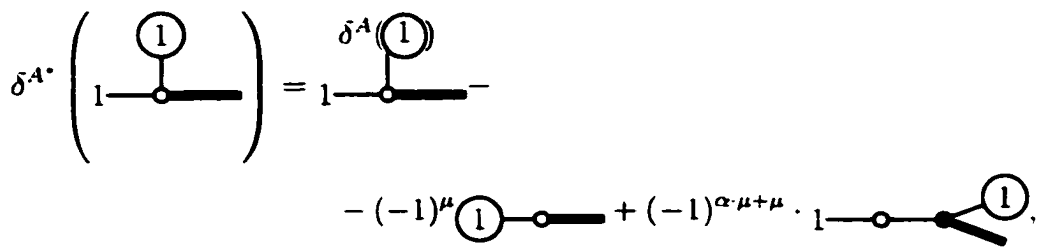
Adding these three terms, one sees that most of them cancel, except for

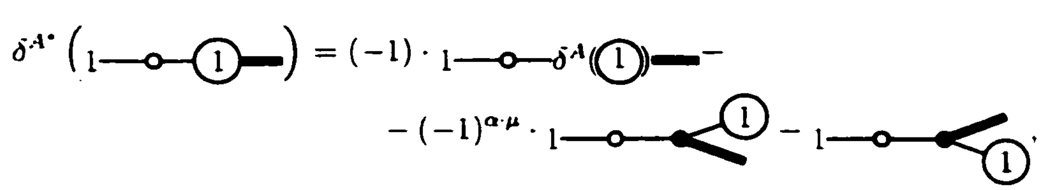
$$\delta^{A^\bullet}(\Delta(f)) = -\Delta(\delta^{A}(f)).$$

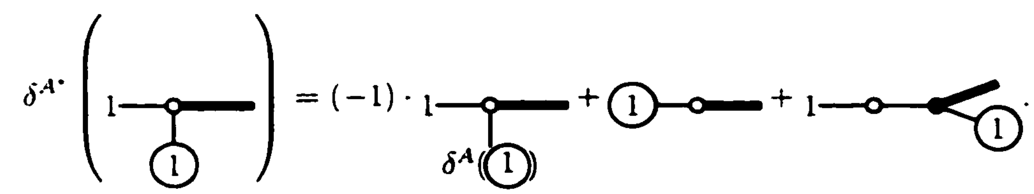
On the level of symbols the proof above is written in the following way: One needs to show that the β and Δ commute (in a graded way) with the differentials δ^{A} and δ^{A}. For this, abbreviate $\alpha := \sum_{i=1}^{n+1} ||a_i||$ the total degree of the input elements a_i, and let $\mu := ||f||$ be the degree of the Hochschild-cochain f that is applied to β and Δ. Then the claim follows from

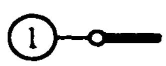
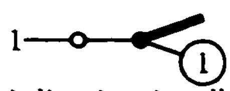
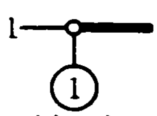
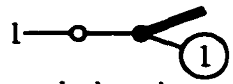
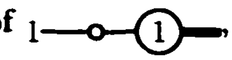


together with

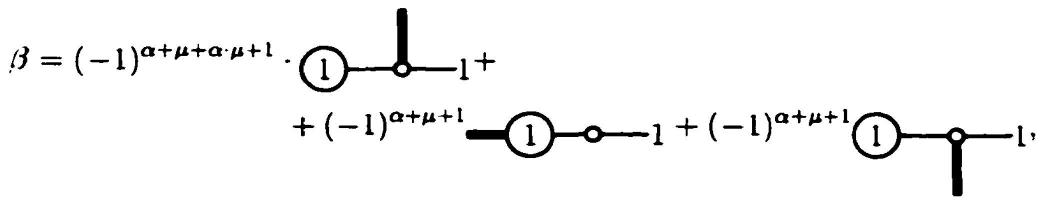




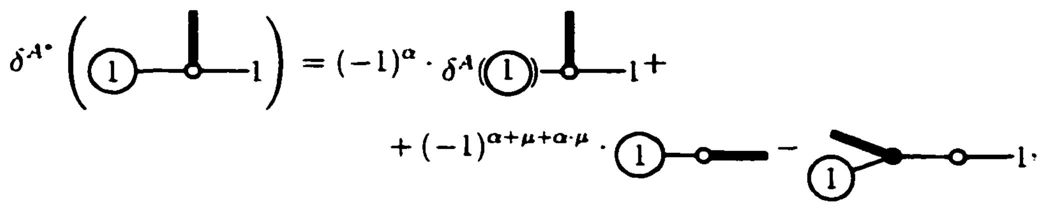


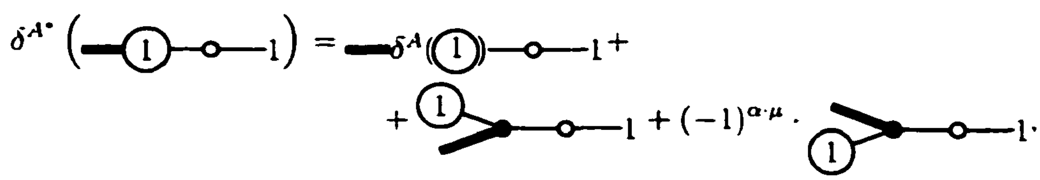
Here are some remarks on how to obtain the signs. All the symbols above have a uniquely given sign by the sign convention stated above. In order to determine the sign one has to see which function-letters are being commuted and introduce the corresponding sign for them. Furthermore, the sign rule for the unit from Definition 3.8. has to be used. Finally, one has to be careful that e.g. the boundary components  and  of  come from a cancellation of placing the multiplication in all possible places around the inner-product (compare above). This means that those terms are being subtracted from the differential in question(- which, in the given case, will come to an overall plus-sign, because the multiplication D jumps over the unit 1, and thus introduces another minus-sign). (Compare also the boundary component  of , which does not come from a cancellation and thus has an overall minus-sign from D jumping over 1.)

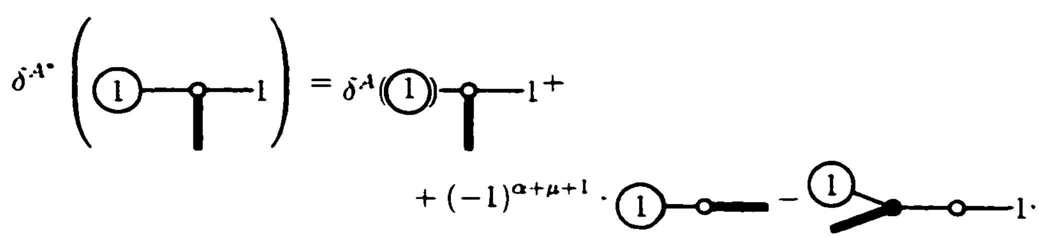
Similarly for β, one has



with







which shows that

$$\delta^{A^*}(\beta(f)) = -\beta(\delta^A(f)).$$

□

So, both β and Δ are (graded) chain maps on the space $\overline{C^\bullet}(A, A^\bullet)$, which, by Lemma 3.9., is quasi-isomorphic to $C^\bullet(A, A^\bullet)$. If one also has an ∞-Poincaré-duality-structure F given on A, then, by definition (Definition 2.32.), $C^\bullet(A, A^\bullet)$ is also quasi-isomorphic to $C^\bullet(A, A)$:

$$C^\bullet(A, A) \approx C^\bullet(A, A^\bullet), \qquad f \mapsto F \circ f.$$

Therefore, the induced maps on homology $\beta_\bullet : H_\bullet(\overline{C^\bullet}(A, A^\bullet)) \to H_\bullet(\overline{C^\bullet}(A, A^\bullet))$ and $\Delta_\bullet : H_\bullet(\overline{C^\bullet}(A, A^\bullet)) \to H_\bullet(C^\bullet(A. A))$ can (and will) also be interpreted as a maps $H^\bullet(A, A) \to H^\bullet(A, A)$.

More explicitly, if $f \in C^\bullet(A, A)$, then $\beta_\bullet([f])$ and $\Delta_\bullet([f])$ are defined by

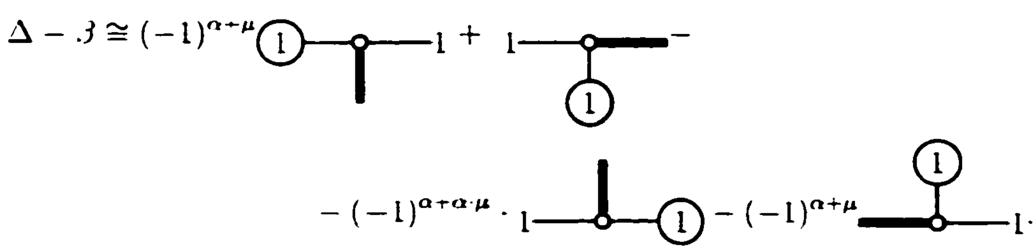
$$\beta_\bullet([f]) = \beta_\bullet([F \circ f]), \quad \text{and} \quad \Delta_\bullet([f]) = \Delta_\bullet([F \circ g]).$$

Definition 3.11. *Given a unital A_∞-algebra $(A, D, 1)$ with ∞-Poincaré-duality-structure F. Then F is called* ***symmetric****, if the operators Δ and β induce the same map on homology:*

$$\Delta_\bullet = \beta_\bullet$$

Here is a lemma that gives a motivation for the word "symmetric" in the last definition.

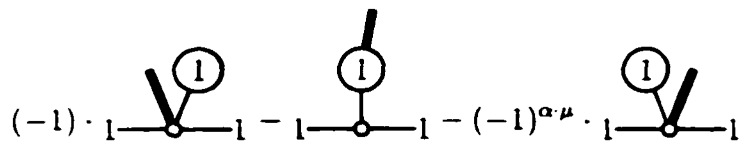
Lemma 3.12. *The difference of Δ and β is given up to homology by the operation*



where α is the total degree of the inputs a_i, and μ is the degree of the Hochschild-cochain plugged into the symbol.

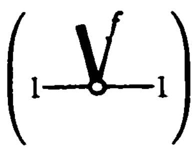
Example 3.13. *The lemma shows that if F is a strict bimodule-isomorphism of $A \to A^*$ and has no higher components, then it is symmetric. This is clear, because as there are no higher components, the righthand-side of the equation above vanishes.*

Proof. The above lemma will be proved by a general method of proving identities in the Hochschild complex. Explicitly, one writes down an element x from the Hochschild complex, whose differential $\delta(x)$ will then be 0 in Hochschild cohomology. because it is a boundary.
Let's look at the following element:



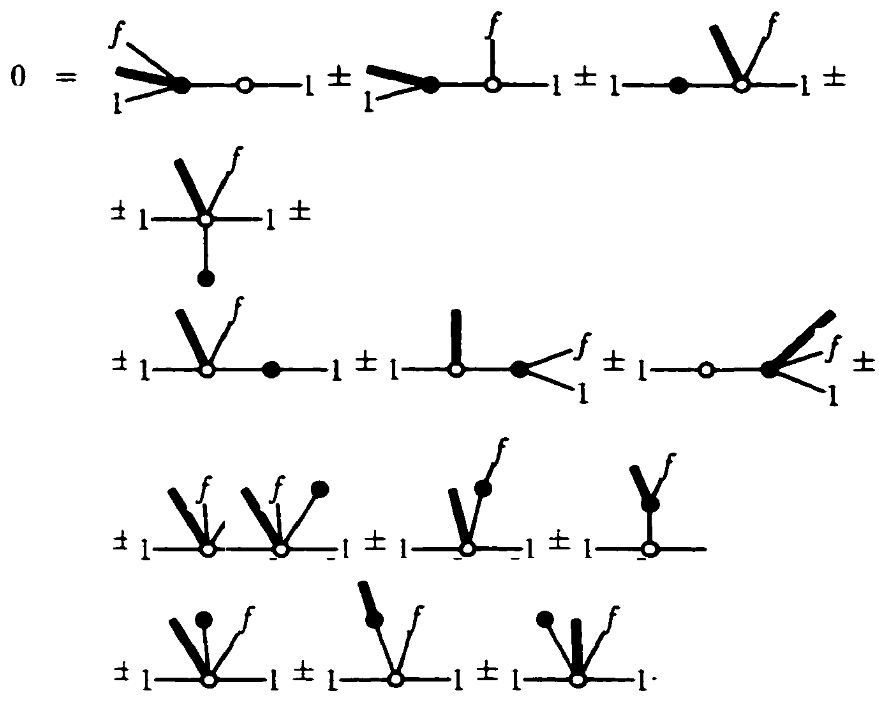
(As before. the number 1 inside the circle indicates that for any given element $f \in C^\bullet(A, A)$. this expression gives an element in $C^\bullet(A, A^\bullet)$ by plugging f into the indicated position. The fat line indicates where one has to start the evaluation when applying the expression to $a_1 \otimes \ldots \otimes a_{n+1}$.)
Let's start with calculating the differential of this element. In order to understand this calculation better, the symbol (1) will be replaced by an explicit element $f \in C^\bullet(A, A)$. and as before it is always assumed that the elements $a_1, \ldots, a_{n+1}$ are placed in all possible spots. Also, the sign will be added later.

when performing the proof with symbols. By Remark 3.7. one has

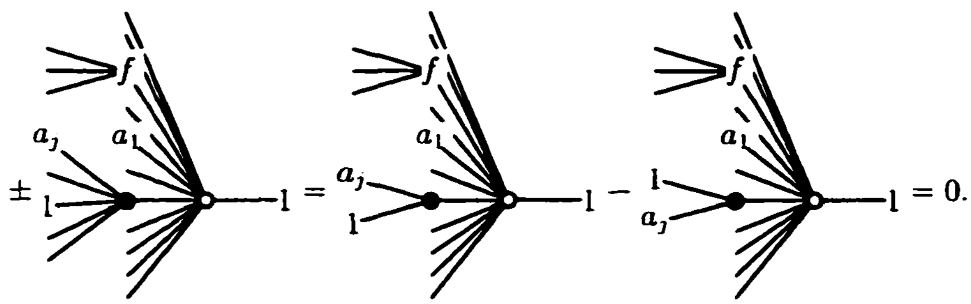
δ^{A^*}  = sum over all combinations of

placing a multiplication D around 

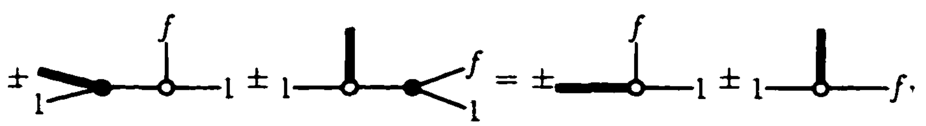
When simplifying this, it will be useful that the ∞-inner-product satisfies $D^{A^*} \circ F - F \circ D^A = 0$. This means that the sum of all combinations of placing D around the open circle (which represents the ∞-inner-product) vanishes (compare Remark 3.7.). Placing the unit, f and the fat line at the places indicated by the above diagram, one sees that the following sum is zero:



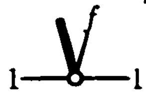
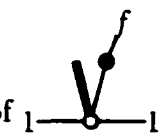
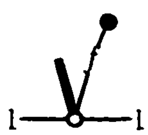
Several terms of this sum are immediately zero. Namely the first terms in the first line and the last term in the third line become zero, as the multiplication of three or more elements which include the unit always vanish. The last term in the first line vanishes, because this element represents a sum of terms

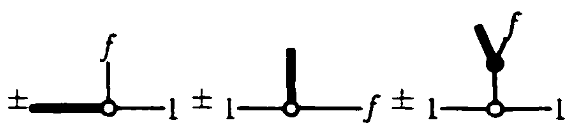


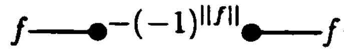
(It was used that the multiplication is nonzero only on two elements, where $m_2(a, 1) = a = m_2(1, a)$.) Similar the first term of the third line becomes zero. The only remaining terms in the first and third line are:

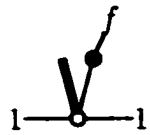
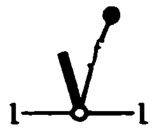


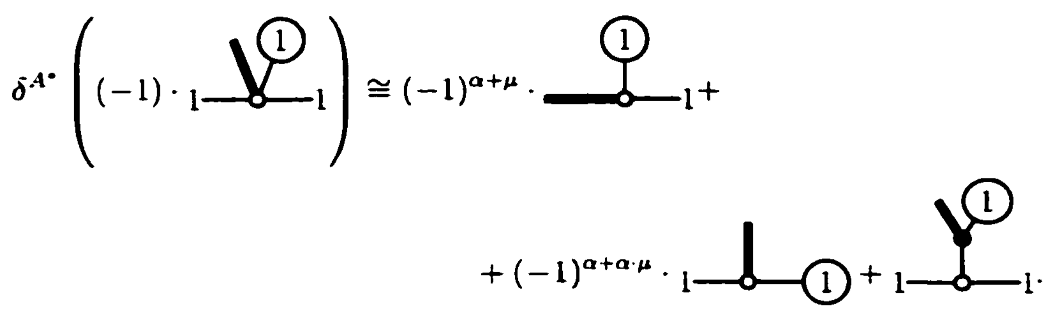
where it was used again, that 1 is a unit.

Going back to the differential that had to be calculated, one has to place the multiplication in all possible places around the expression . After doing this, many of the above terms appear. But instead of , the term  appears, and above one has three additional terms, namely

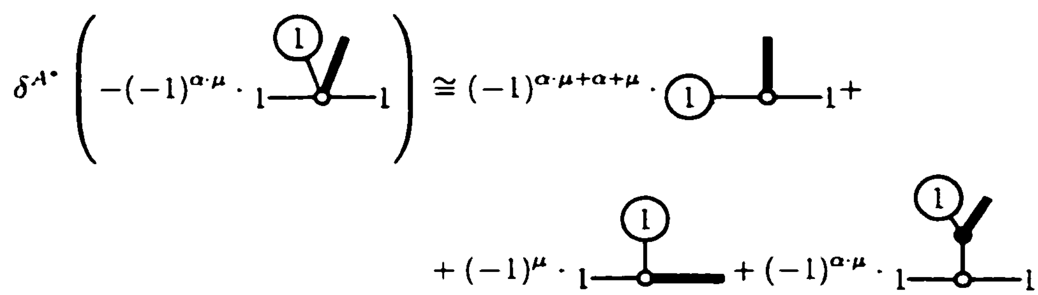


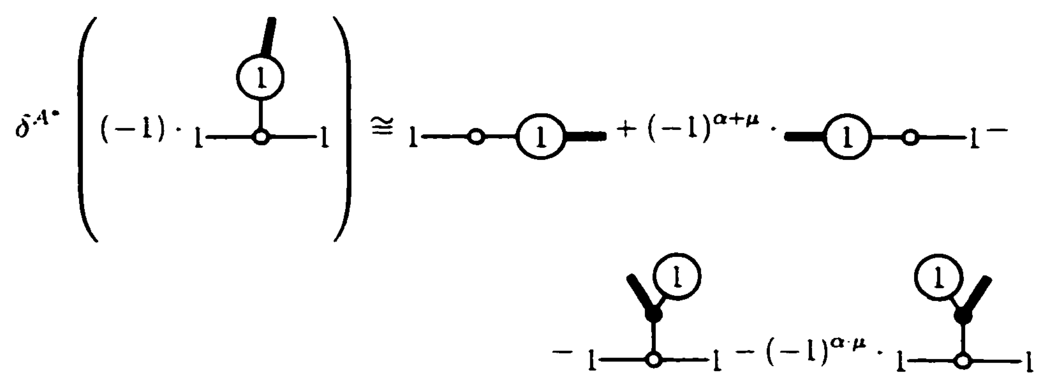
As the claim is supposed to be true on homology, it will be assumed that f represents a closed element and therefore $0 = \delta^A(f) =$ ,

which thus identifies  with  (up to sign). In conclusion:

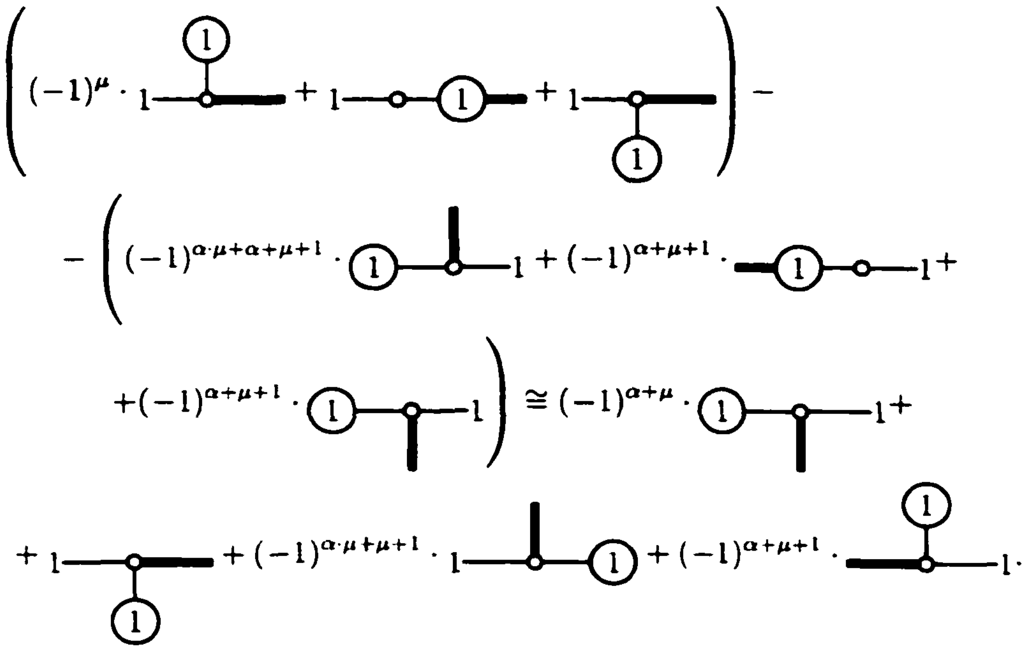


Similarly, one gets the other terms





Adding these terms, one gets



Now, the first bracket clearly corresponds to the Δ-operator, whereas the second bracket was already shown to correspond to the β-operator on homology. More precisely, this term corresponds to $\beta(f) \cong \beta(f$—o$)$ when applied to an element f—o$= F \circ f$ for $f \in C^{\bullet}(A, A)$, because by definition one inserts the unit 1 on the right and lets the arguments $a_1, \ldots, a_n$ rotate around the function f—o $\in C^{\bullet}(A, A^{\bullet})$ (compare Definition 3.10.). □

3.2 Proof of the BV-structure on the Hochschild Complex

Theorem 3.14. *Given a unital A_∞-algebra $(A, D, 1)$ with symmetric ∞-Poincaré-duality-structure F.*

Then, in the notation of definitions 3.5. and 3.10., the space $(H^{\bullet}(A, A), (M_2)_{\bullet}, \Delta_{\bullet})$

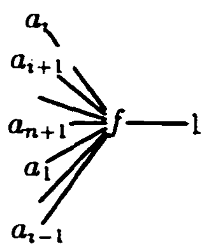
is a BV-algebra, whose induced Gerstenhaber-structure is given by $\{-,-\}_\bullet$.

Proof. The remainder of this section is devoted to the proof of Theorem 3.14.

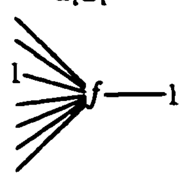
Using Theorem 3.6., it suffices to show that $(\Delta_\bullet)^2 = 0$ and the deviation of $\Delta_\bullet$ from being a graded derivation is given by $\{-,-\}_\bullet$.The first statement comes from the assumption $\Delta_\bullet = \beta_\bullet$ and the following Lemma.

Lemma 3.15. *On the reduced complex* $\overline{C^\bullet}(A, A^\bullet)$, *it is* $\beta^2 = 0$. *It follows that* $(\beta_\bullet)^2 = 0$.

Proof. Let $f \in \overline{C^\bullet}(A, A^\bullet)$. Then

$\beta^2(f) \;=\; \beta(\,\pm$


$) =$

$= \;\pm$


$=$

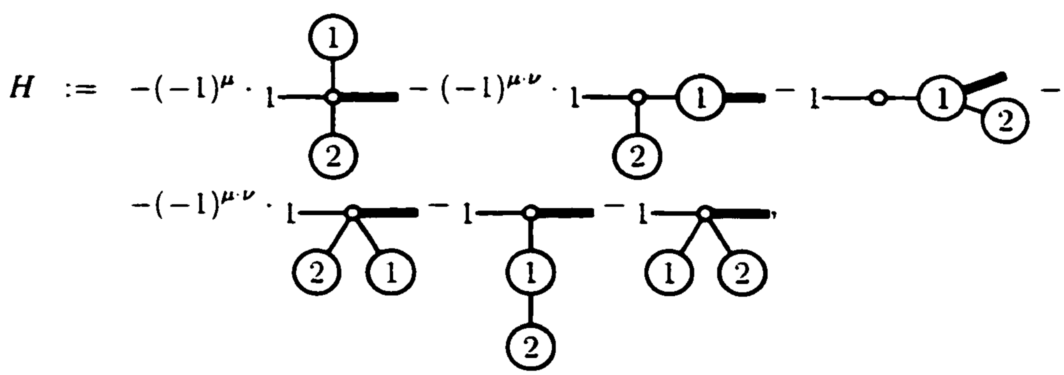
$= \;0,$

because $f \in \overline{C^\bullet}(A, A^\bullet)$ maps any tensor-product of the form $\ldots \otimes 1 \otimes \ldots$ to 0. □

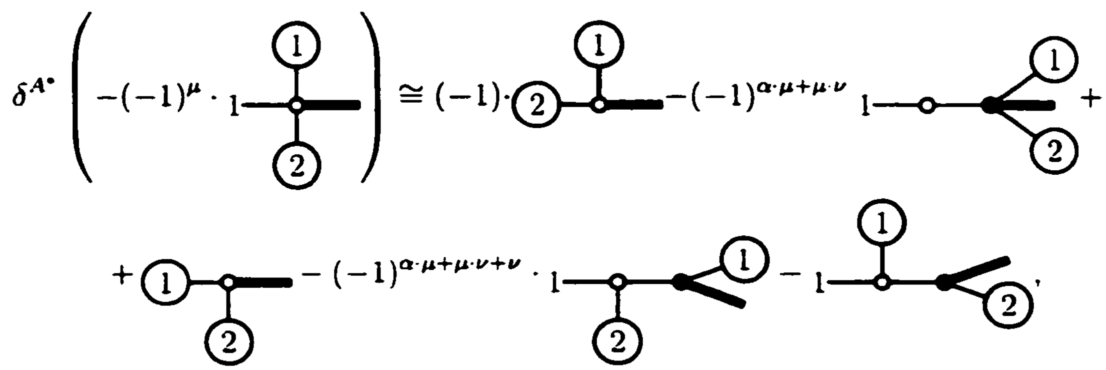
It is left to compare the deviation of $\Delta_\bullet$ from being a graded derivation with $\{-,-\}_\bullet$. These will be shown to be equal by the same method that was used in Lemma 3.12. Namely for two elements $f, g \in C^\bullet(A, A)$, the sum of both terms (applied to f and g) can be seen to be the boundary of some other element $H(f,g) \in C^\bullet(A, A^\bullet)$.

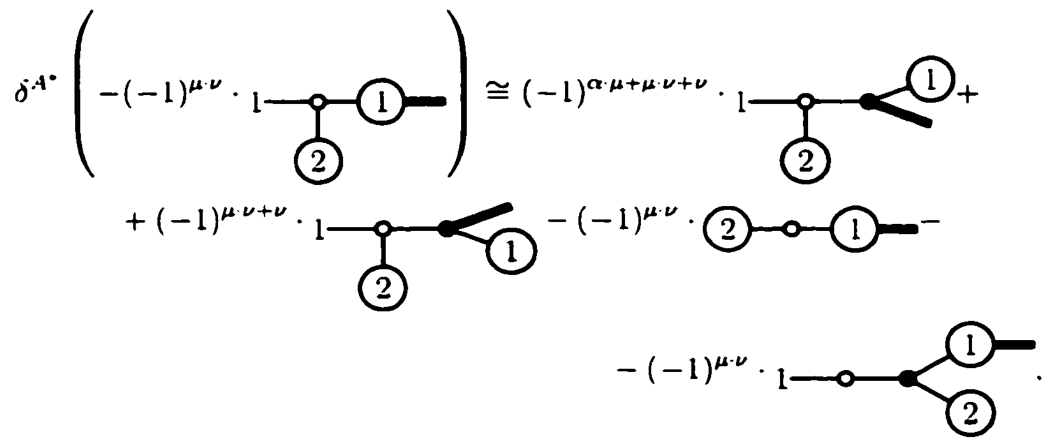
At this point, it seems useful to make a remark about this element $H(f,g)$. There is a graphical way of picturing f, g and $H(f,g)$, which was described in [6], Lemma 5.2. In fact, the proof of the current theorem is nothing but [6], Lemma 5.2 rewritten in the Hochschild-cochain language. The term $H(f,g)$ below will correspond exactly to the homotopy of [6], Figure 7. It turns out that this proof presented here is not the shortest possible, as some of the terms of $H(f,g)$ will be subtracted later on. But in order to have a clear correspondence with the pictures from [6], this proof seems more useful.

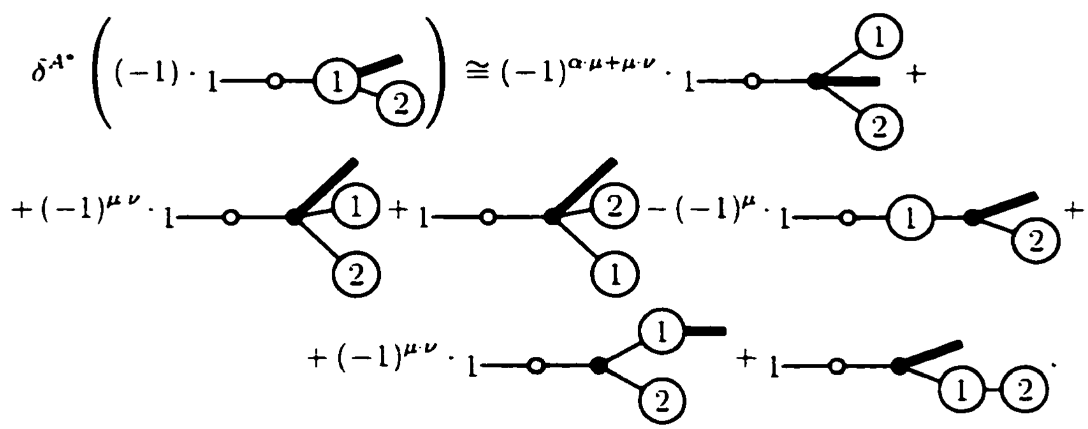
Let's again use the abbreviation α for the total degree of the elements a_i to be applied as arguments, and $\mu = ||f||$ and $\nu = ||g||$, when f and g will be placed into the first, respectively second spot of a symbol. The needed homotopy is given by

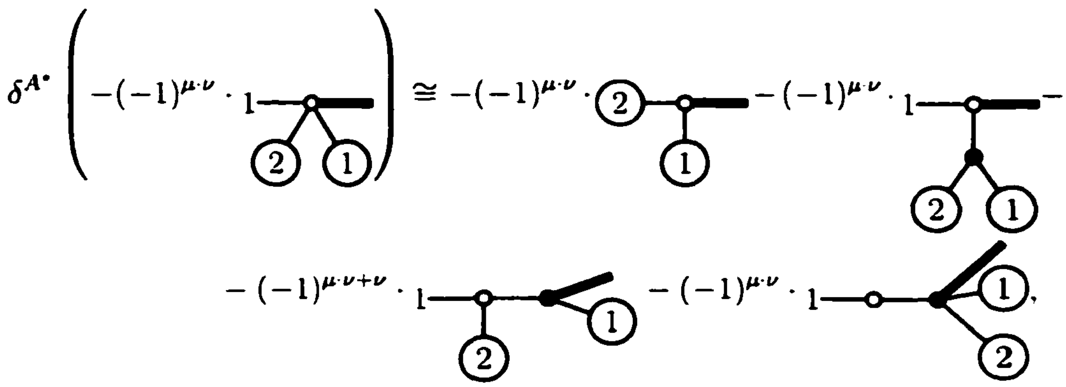


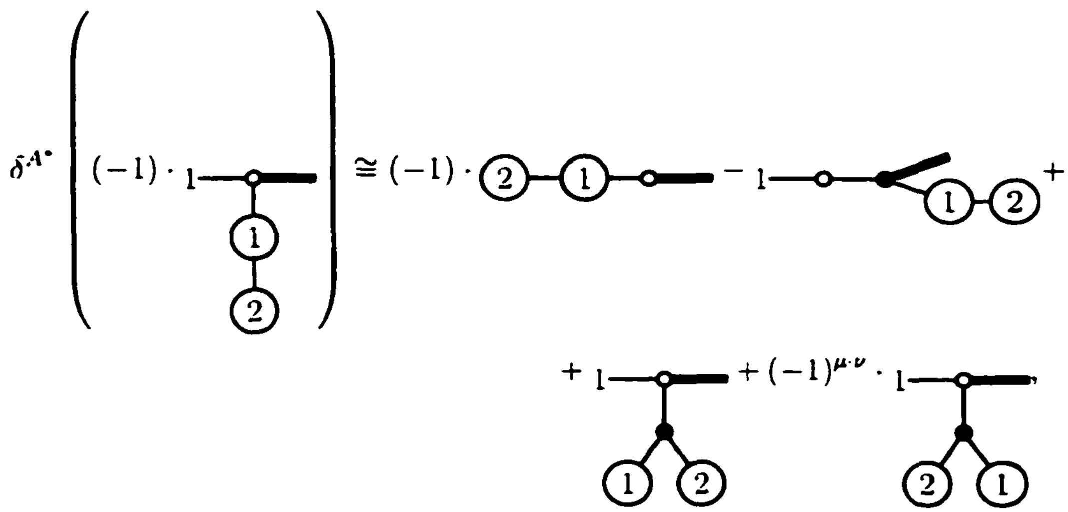
Let's again assume that f and g are closed, so that $D \circ f = f \circ D$ and $D \circ g = g \circ D$. Then one gets the following boundary terms (compare the calculation in Lemma 3.12.):

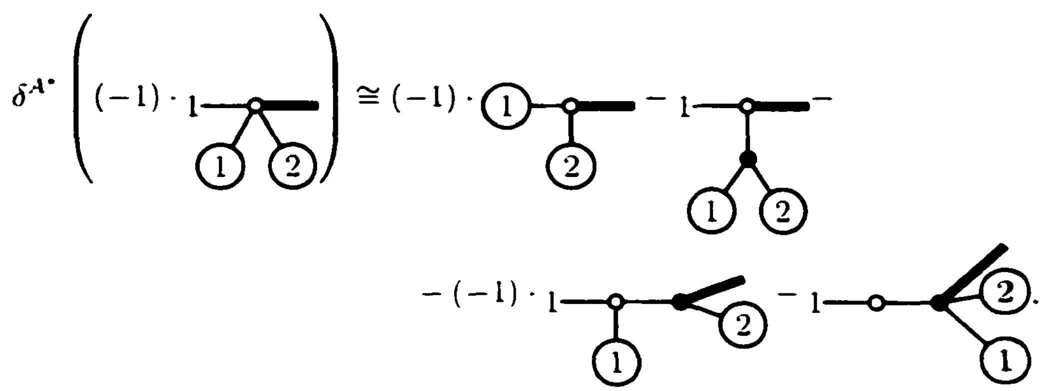






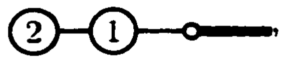


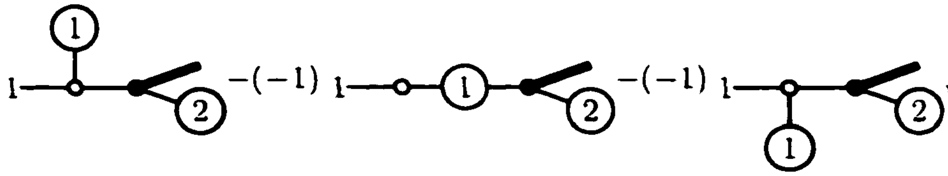


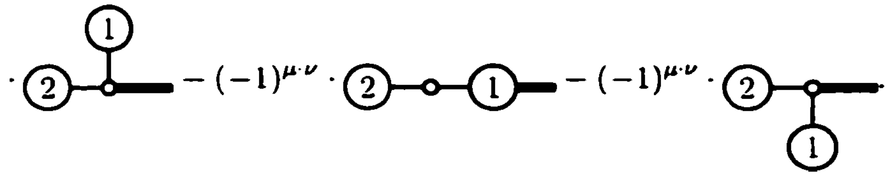


At this point many of the above terms cancel each other out after a summation. and the only remaining terms of $\delta^{A^*}(H)$ are given by $\delta^{A^*}(H) =$

$-X + Y + Z$, with

$X :=$ 

$Y := (-1)$ 

$Z := (-1) \cdot$ 

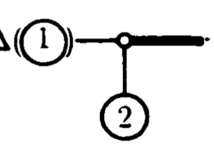
Therefore, the above shows that for any $f, g \in C^{\bullet}(A, A)$

$$X(f, g) \cong Y(f, g) + Z(f, g).$$

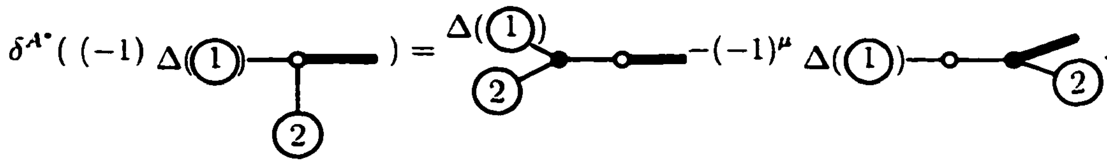
It is clear that $X(f, g) = F \circ (f \circ g)$. Now the claim is that $Y(f, g) \simeq -\Delta(f) \smile g$, whereas the term $Z(f, g) - (-1)^{\|f\| \cdot \|g\|} Z(g, f) \simeq \Delta(f \smile g)$. After this is shown, the theorem follows, because then (using the fact that $\smile$ is graded commutative on homology):

$$\begin{aligned}
\{f, g\} &= f \circ g - (-1)^{\|f\| \cdot \|g\|} g \circ f \cong \\
&\cong X(f, g) - (-1)^{\|f\| \cdot \|g\|} X(g, f) \cong \\
&\cong Y(f, g) - (-1)^{\|f\| \cdot \|g\|} Y(g, f) + Z(f, g) - (-1)^{\|f\| \cdot \|g\|} Z(g, f) \cong \\
&\cong -\Delta(f) \smile g + (-1)^{\|f\| \cdot \|g\|} \Delta(g) \smile f + \Delta(f \smile g) \cong \\
&\cong -\Delta(f) \smile g - (-1)^{\|f\|} f \smile \Delta(g) + \Delta(f \smile g).
\end{aligned}$$

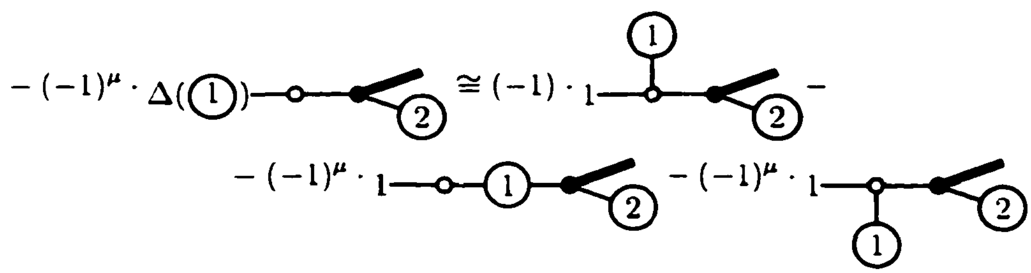
In order to show $Y(f, g) \simeq -\Delta(f) \smile g$, it will be useful to look at the boundary of the following term

$(-1) \cdot \Delta$ 

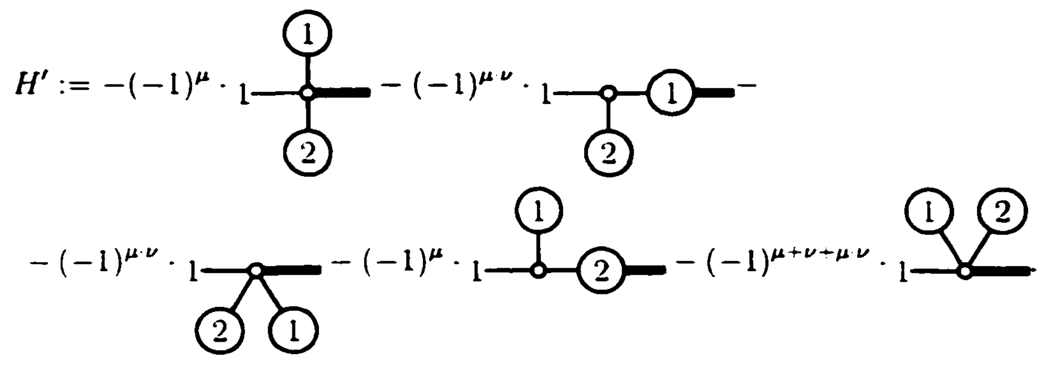
Let's assume again that this symbol will be applied to closed elements f and g, so that $\Delta(f)$ will also be closed. Then the differential of this symbol is given by



The first term on the right hand side is clearly $F \circ (\Delta(f) \smile g)$. The second term is also immediately seen to be $Y(f, g)$ after applying the explicit description for Δ given in Lemma 3.12.:

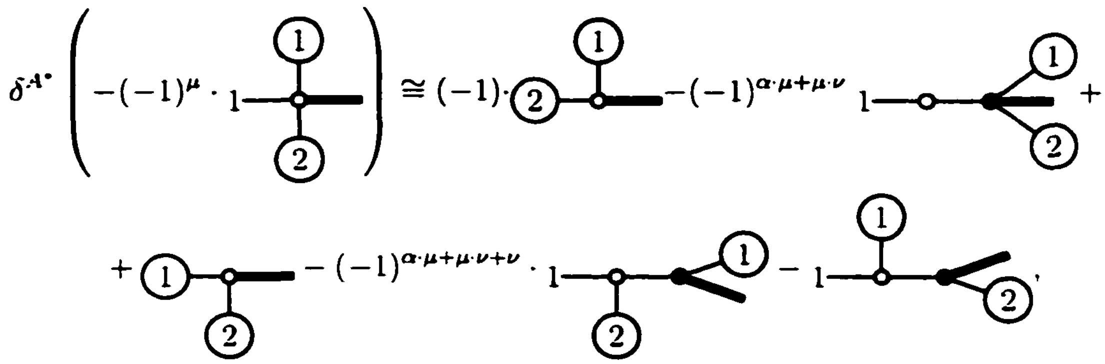


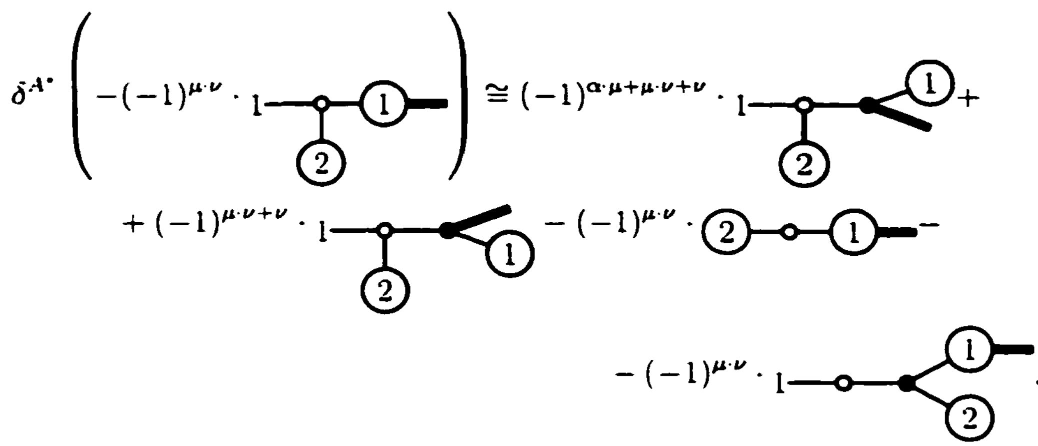
It is left to show that $\Delta(f \smile g) \simeq Z(f, g) - (-1)^{\|f\| \cdot \|g\|} \cdot Z(g, f)$. This will be done again by stating an explicit element whose boundary is the sum $-\Delta(f \smile g) + Z(f, g) - (-1)^{\|f\| \cdot \|g\|} Z(g, f)$. The element in question is

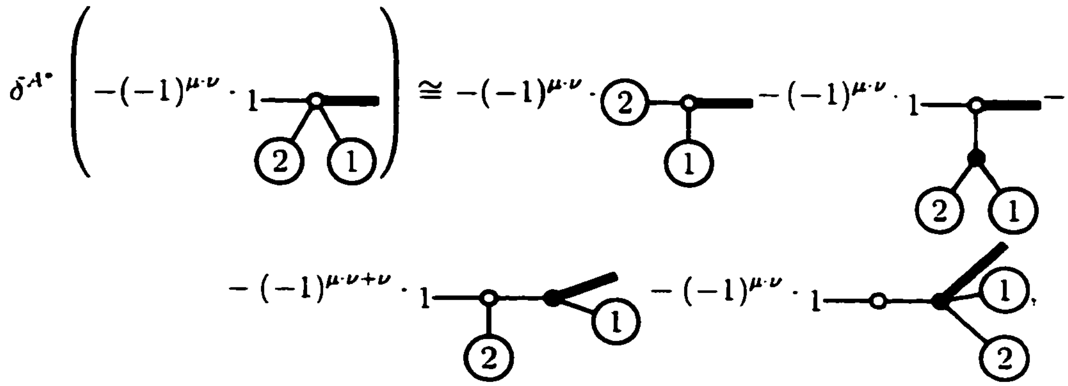


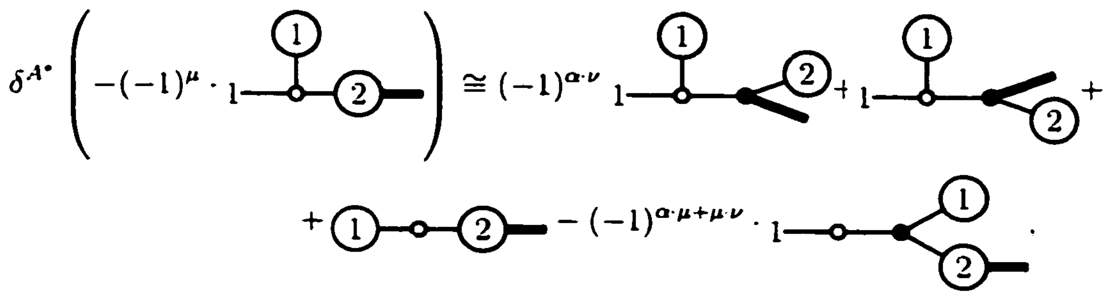
(Notice that the first three terms were already used in the definition of H, and they are now merely subtracted! Clearly, it would have been much faster not to introduce them at all. But as it was mentioned before, when using the faster way the correspondence with pictures of the little disc would be less clear.)

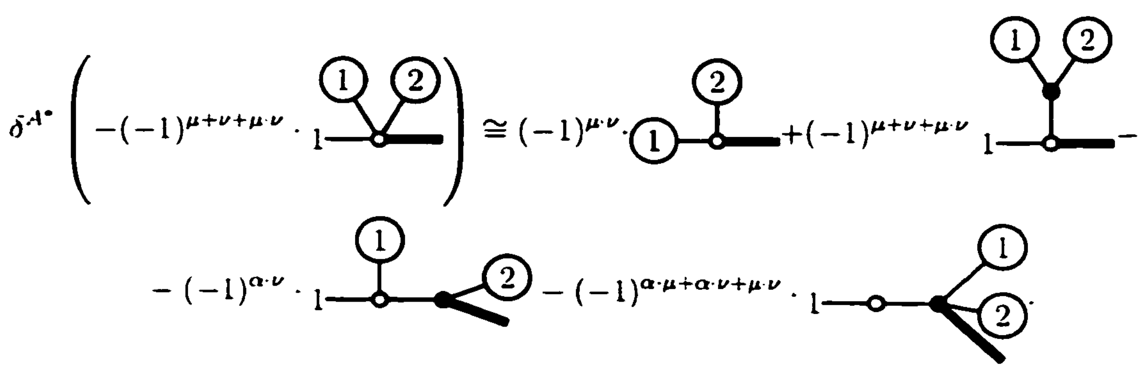
For completeness sake, here are the terms gained by applying the differential:



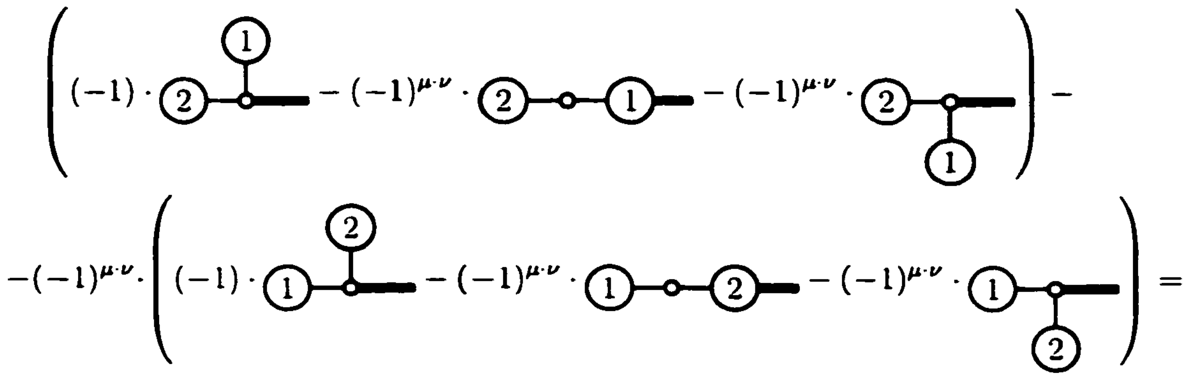



δ^{A•}
−(−1)^{μ·ν} · 1
≅ −(−1)^{μ·ν} ·
− (−1)^{μ·ν} · 1
− (−1)^{μ·ν+ν} · 1
− (−1)^{μ·ν} · 1


δ^{A•}
−(−1)^{μ} · 1
≅ (−1)^{α·ν} 1
+ 1
+
+
− (−1)^{α·μ+μ·ν} · 1

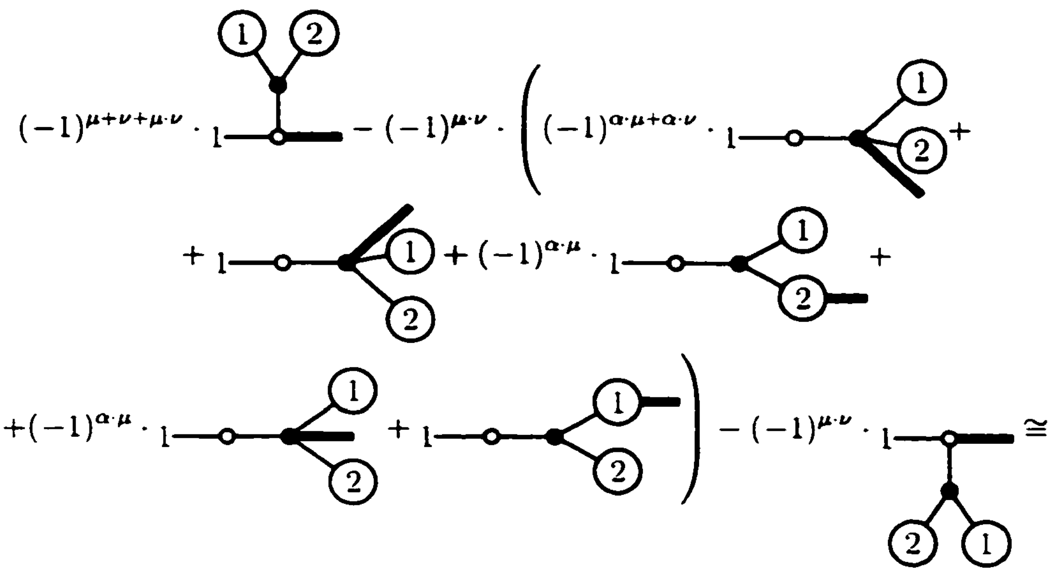

δ^{A•}
−(−1)^{μ+ν+μ·ν} · 1
≅ (−1)^{μ·ν} ·
+(−1)^{μ+ν+μ·ν} 1
− (−1)^{α·ν} · 1
− (−1)^{α·μ+α·ν+μ·ν} · 1

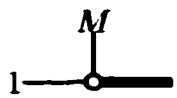
A thorough investigation shows that all terms cancel except

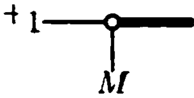


$$= Z(f, g) - (-1)^{\mu \cdot \nu} \cdot Z(g, f),$$

(when applied to elements f and g), and furthermore, one has the remaining terms



$$\cong -(-1)^{\mu\cdot\nu}\cdot\Bigg((-1)^{\mu+\nu+1}\cdot$$  $$+$$  $$+$$

$$+$$  $$\Bigg)\cong -(-1)^{\mu\cdot\nu}\cdot\Delta(M),$$

where $M := \textcircled{2}\smile\textcircled{1}\simeq(-1)^{\mu\cdot\nu}\cdot\textcircled{1}\smile\textcircled{2}$ is abbreviated. □

4 Geometric Construction of ∞ Inner Products on Closed Oriented Manifolds

In this section an ∞-Poincaré-duality will be constructed on a simplicial model of a compact oriented manifold. The idea of the construction will follow a method of Dennis Sullivan for obtaining local infinity structures, see [18]. The correct algebraic concept to which this idea has to be applied will be developed first.

This section is joint work with Mahmoud Zeinalian and will appear in [21].

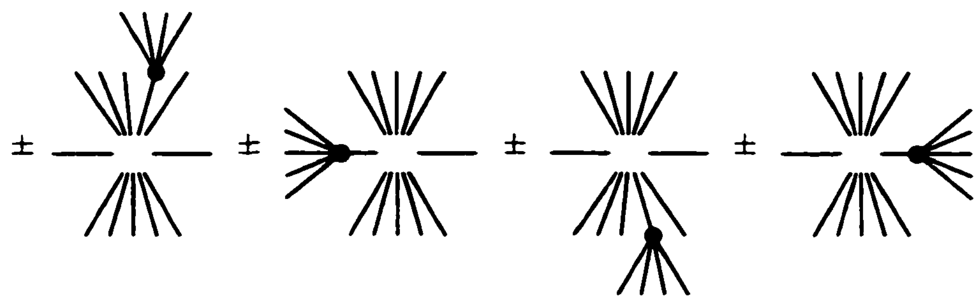
Let's first introduce a complex which is convenient for dealing with ∞-inner-products. It turns out that closed elements of this complex correspond to ∞-inner-products and exact elements correspond to homological trivial maps of the corresponding Hochschild-complexes.

Definition 4.1. *Given graded modules V, W and Z over a ring R. Denote by $T_Z^W V := \bigoplus_{n,m\geq 0} V^{\otimes n} \otimes W \otimes V^{\otimes m} \otimes Z$.*

- *For a given A_∞-algebra (A, D), Proposition 2.33. describes ∞-inner-products as certain elements $F : T_A^A A \to R$. One can put a differential on $T_A^A A$, such that the closed elements of $Hom(T_A^A A, R)$ are exactly ∞-inner-products(, compare Lemma 4.2 below). The differential $\mathfrak{D} : T_A^A A \to T_A^A A$ is given by a sum of applying D at all possible places in $T_A^A A$ in the cyclic way described in Proposition 2.33.*

$$\mathfrak{D}(a_1, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots, a_{n+m+1}, a_{n+m+2}) :=$$
$$:= \sum_{i,j} \pm(\ldots a_{i-1}, D(a_i, \ldots, a_j), a_{j+1}, \ldots).$$

The sign and the condition for applying D are stated in Proposition 2.33. (i)-(v). Basically, these conditions express, that one has to keep track of the two special elements of $T_A^A A$, which can never be multiplied by D. The diagrammatic picture for $\mathfrak{D}$ was described in Definition 2.36.(, where one has to insert the elements a_i cyclically):



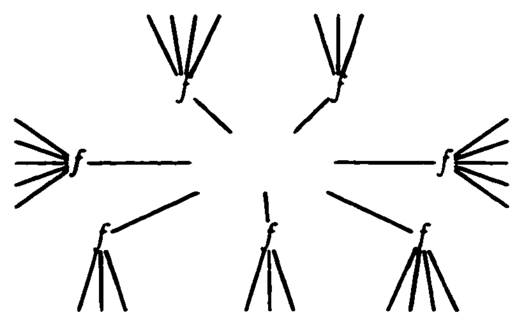
The only difference to 2.33. is that one has to apply all of $D = \partial_1 + \partial_2 + \partial_3 \ldots$ including the differential ∂_1, and not just the higher terms.
Claim: $\mathfrak{D}^2 = 0$.

Proof. The diagrams for $\mathfrak{D}^2$ are given by applying D in two places. There are two cases(; compare Theorem 2.39.). Either one of the two multiplications was "stuck" on the other one, in which case $D^2 = 0$ shows, that the sum of those diagrams vanish. Or the multiplications occur at different places, which cancels out with the same expression but where the order of multiplication was reversed. □

- *Now given another A_∞-algebra (A', D') and an A_∞-algebra-map $f : A \to A'$. Then, one can define a map $\hat{f} : T_A^A A \to T_{A'}^{A'} A'$ by taking the* *sum of applying f at all possible places simultaneously:*

$$\hat{f}(a_1, ..., a_n, a_{n+1}, a_{n+2}, ..., a_{n+m+1}, a_{n+m+2}) :=$$
$$:= \sum \pm(f(a_i, ..., a_j), f(a_{j+1}, ..., a_k), ..., f(a_p, ..., a_q), ..., f(a_r, ..., a_s)).$$

One has to take the same "cyclic" rules for the positions of the elements a_i, that were taken in the definition of $\mathfrak{D}$. Namely, one has to insert the a_i cyclically such that the element a_{n+m+2} ends up in the last spot. Furthermore the two special elements a_{n+1} and a_{n+m+2} of $T_A^A A$ uniquely determine the new special elements of $T_{A'}^{A'} A'$. Therefore a_{n+1} and a_{n+m+2} can never be inside some f with only one output component. Diagrammatically, one has



Claim: *$\hat{f} : (T_A^A A, \mathfrak{D}) \to (T_{A'}^{A'} A', \mathfrak{D}')$ is a chain map.*

Proof. $\mathfrak{D}' \circ \hat{f}$ corresponds to applying f and having one multiplication D' inside in one position. $\hat{f} \circ \mathfrak{D}$ corresponds to applying f and having one multiplication D outside in one position. But the fact that f is an A_∞-algebra-map ($D' \circ f = f \circ D$) means exactly that f commutes with inside and outside multiplication of D' and D. □

Lemma 4.2. *Every element in $Hom(T_A^A A, R)$ can be understood as a map $Hom(T^A A, A^\bullet)$ by dualizing the last component, and thus by Lemma 2.21. and Proposition 2.19. as a map $C^\bullet(A, A) \to C^\bullet(A, A^\bullet)$.*
Closed elements in $(Hom(T_A^A A, R), \mathcal{D}^\bullet)$ correspond to chain maps from $(C^\bullet(A, A), D^A) \to (C^\bullet(A, A^\bullet), D^{A^\bullet})$. Exact elements in $(Hom(T_A^A A, R), \mathcal{D}^\bullet)$ correspond to homological trivial maps $H^\bullet(A, A) \xrightarrow{0} H^\bullet(A, A^\bullet)$.

Proof. Let $F \in Hom(T_A^A A, R)$. The proof of Proposition 2.33. is exactly the interpretation of $\mathcal{D}^\bullet(F) = F \circ \mathcal{D}$ as a map $C^\bullet(A, A) \to C^\bullet(A, A^\bullet)$ given by $D^{A^\bullet} \circ F - (-1)^{||F||} \cdot F \circ D^A$. (In fact, Proposition 2.33. was the guideline for the definition of $\mathcal{D}$.) This immediately implies the claim. □

Next, an inductive construction will result in a closed element in $T_{A^\bullet}^{A^\bullet} A^\bullet \subset Hom(T_A^A A, R)$ of degree 0. The differential $\mathcal{D}^\bullet$ of $T_{A^\bullet}^{A^\bullet} A^\bullet$ is nothing but applying $D^\bullet : T^{A^\bullet} A^\bullet \to T^{A^\bullet} A^\bullet$ in all possible places according to Definition 4.1. By Lemma 4.2. a closed element in $T_{A^\bullet}^{A^\bullet} A^\bullet$ of degree 0 represents of course an ∞-inner-product, where the degree is given in the following way. Let $F = (c_1 \otimes \ldots \otimes c_n) \otimes c' \otimes (c''_1 \otimes \ldots \otimes c''_m) \otimes c''' \in (A^\bullet)^{\otimes n} \otimes A^\bullet \otimes (A^\bullet)^{\otimes m} \otimes A^\bullet$, with $c_i \in (A^\bullet)_{k_i}$, $c' \in (A^\bullet)_{k'}$, $c''_i \in (A^\bullet)_{k''_i}$ and $c''' \in (A^\bullet)_{k'''}$. Then the total degree of F is given by

$$||F|| = \sum_{i=1}^{n}(k_i - 1) + (k' - 1) + \sum_{i=1}^{m}(k''_i - 1) + (1 - k''').$$

Given an A_∞-coalgebra $(C, D^\bullet)$, which means by definition that the dual space $(A := C^\bullet, D)$ is an A_∞-algebra. As the rest of this section mainly deals with A_∞-coalgebras, the notation will be simplified by writing D for $D^\bullet$. From the context it should be clear whether the A_∞-algebra or the A_∞-coalgebra-structure is being studied. The differential of the A_∞-coalgebra is given by a sequence of maps $D = \tilde{\partial}_1 + \tilde{\partial}_2 + \tilde{\partial}_3 + \dots$, given by $\partial_i : C \to C^{\otimes i}$ for $i \geq 1$, which lift as derivations to $\tilde{\partial}_i : TC \to TC$. This is easily seen from Proposition 2.4. Furthermore it is clear that the condition $D^2 = 0$ implies that $\partial_1^2 = 0$, so that (C, ∂_1) is a complex.

Notation 4.3. *Denote by*

$$T_C^C C^k := \bigoplus_{i+j=k} C^{\otimes i} \otimes C \otimes C^{\otimes j} \otimes C,$$

and then for any map $\chi : V \to T_C^C C$, *let*

$$\chi^k := (pr_{T_C^C C^k}) \circ \chi : V \xrightarrow{\chi} T_C^C C \xrightarrow{pr} T_C^C C^k.$$

With this, one can finally use the idea in [18] to construct ∞-inner-products.

Proposition 4.4. *Let* $(C, D = \tilde{\partial}_1 + \tilde{\partial}_2 + \tilde{\partial}_3 + \dots)$ *be an* A_∞*-coalgebra. Assume that* (C, ∂_1) *is a simplicial complex such that the closure of every simplex is contractible, and such that all* $\partial_i : C \to C^{\otimes i}$ *are local. (By "local", it is meant, that a map* $C \longrightarrow TC$ *maps a simplex* σ *into* $T(C(\overline{\sigma})) \subset TC$*, where* $C(\overline{\sigma}) \subset C$ *is the subcomplex of* C *generated by the closure of* σ*.)*
Then there exists a <u>*chain*</u> *map* $\chi : (C, \partial_1) \to (T_C^C C, \mathfrak{D})$ *of degree* 0*, whose lowest component is given by* $\chi^0 = \partial_2 : C \to C \otimes C$.

Proof. Inductively, one wants to define local maps $\chi^j : C \longrightarrow T_C^C C^j$, $j = 0, \dots, n$, of degree 0, such that

$$\left(\bigoplus_{0 \leq i \leq n} \chi^i\right) \circ \partial_1 - \mathfrak{D} \circ \left(\bigoplus_{0 \leq i \leq n} \chi^i\right) = \epsilon_{n+1} + \text{higher order terms}, \tag{4.5}$$

where $\epsilon_{n+1} : C \longrightarrow T_C^C C^{n+1}$ and the higher terms map $C \longrightarrow \bigoplus_{i > n+1} T_C^C C^i$. Here, "local" means, that $\chi^j : \sigma \mapsto T_{C(\overline{\sigma})}^{C(\overline{\sigma})} C(\overline{\sigma})^j \subset T_C^C C^j$, for any simplex σ.

$n = 0:$ Take $\chi^0 := \partial_2$. This map is local by assumption and satisfies

$$\begin{aligned}\chi^0 \circ \partial_1 - \mathcal{D} \circ \chi^0 &= \partial_2 \circ \partial_1 - \\ &\quad -\tilde{\partial}_1 \circ \partial_2 + (higher\ \mathcal{D}\ terms\ than\ \partial_1) \circ \partial_2 = \\ &= higher\ terms,\end{aligned}$$

since ∂_2 is a chain map(; compare the conditions for $D^2 = 0$).

$n > 0:$ Assume one has build local maps χ^j, $j = 0, \dots, n$ satisfying (4.5). Now notice that

$$\begin{aligned}0 &= ((\_\circ \partial_1) - (\mathcal{D} \circ \_))^2 \left(\bigoplus_{0 \le i \le n} \chi^i\right) = \\ &= ((\_\circ \partial_1) - (\mathcal{D} \circ \_)) \circ ((\_\circ \partial_1) - (\mathcal{D} \circ \_)) \left(\bigoplus_{0 \le i \le n} \chi^i\right) = \\ &= ((\_\circ \partial_1) - (\mathcal{D} \circ \_))(\epsilon_{n+1} + \text{higher order terms}) = \\ &= [\partial_1, \epsilon_{n+1}] + \text{higher order terms}.\end{aligned}$$

Therefore $0 = [\partial_1, \epsilon_{n+1}] = \tilde{\partial}_1 \circ \epsilon_{n+1} - \epsilon_{n+1} \circ \partial_1$. Now, the goal is to show that ϵ_{n+1} can be written as $\epsilon_{n+1} = -[\partial_1, \chi^{n+1}]$, where $\chi^{n+1} : C \to T_C^C C^{n+1}$ is a local map, because with this, equation (4.5) will be satisfied:

$$\begin{aligned}((\_\circ \partial_1) - (\mathcal{D} \circ \_)) \left(\bigoplus_{0 \le i \le n+1} \chi^i\right) &= ((\_\circ \partial_1) - (\mathcal{D} \circ \_)) \left(\bigoplus_{0 \le i \le n} \chi^i\right) + \\ &\quad + ((\_\circ \partial_1) - (\mathcal{D} \circ \_))(\chi^{n+1}) = \\ &= \epsilon_{n+1} + \text{higher order terms} + \\ &\quad + [\partial_1, \chi^{n+1}] + \text{higher order terms} = \\ &= \text{higher order terms}.\end{aligned}$$

where the higher order terms now only have components $C \longrightarrow \bigoplus_{i>n+1} T^{C}_{C} C^{i}$.

In order to construct χ^{n+1}, first notice, that ϵ_{n+1} is a local map, because $(\epsilon_{n+1} + \text{higher order terms}) = ((\_ \circ \partial_1) - (\mathcal{D} \circ \_))\,(\bigoplus_{0\le i\le n} \chi^{i})$ which is clearly local, by induction and the locality assumptions about the ∂_i's (; compare Lemma 4.6(a) below).

Thus for every simplex σ, the map ϵ_{n+1} restricts to a map $\epsilon^{\bar\sigma}_{n+1} : C(\bar\sigma) \longrightarrow T^{C(\bar\sigma)}_{C(\bar\sigma)} C(\bar\sigma)$. By assumption $(C(\bar\sigma), \partial_1)$ is acyclic, and therefore, by Lemma 4.6(b) and Lemma 4.2, also the complex $(T^{C(\bar\sigma)}_{C(\bar\sigma)} C(\bar\sigma), \tilde\partial_1)$. Thus, $K^{\bar\sigma} := Hom\left(C(\bar\sigma), T^{C(\bar\sigma)}_{C(\bar\sigma)} C(\bar\sigma)\right)$ is acyclic with the differential given by $\partial := [\partial_1, \_] = (\tilde\partial_1 \circ \_) - (\_ \circ \partial_1)$. Now, $\epsilon^{\bar\sigma}_{n+1} \in K^{\bar\sigma}$ and $[\partial_1, \epsilon^{\bar\sigma}_{n+1}] = 0$, so that $\epsilon^{\bar\sigma}_{n+1}$ is a closed element, and therefore also exact. By Lemma 4.6(c), one can find a canonical element $f^{\bar\sigma}_{n+1} \in K^{\bar\sigma}$ such that $\epsilon^{\bar\sigma}_{n+1} = \partial(f^{\bar\sigma}_{n+1}) = [\partial_1, f^{\bar\sigma}_{n+1}]$.

Then define $\chi^{n+1} : C \longrightarrow T^{C}_{C} C^{n+1}$ by $\chi^{n+1}(\sigma) := f^{\bar\sigma}_{n+1}(\sigma)$. This satisfies

$$
\begin{aligned}
[\partial_1, \chi^{n+1}](\sigma) &= \tilde\partial_1 \circ \chi^{n+1}(\sigma) - \chi^{n+1} \circ \partial_1(\sigma) = \\
&= \tilde\partial_1 \circ f^{\bar\sigma}_{n+1}(\sigma) - f^{\overline{\text{face of } \sigma}}_{n+1} \circ \partial_1(\sigma) = \\
&= \tilde\partial_1 \circ f^{\bar\sigma}_{n+1}(\sigma) - f^{\bar\sigma}_{n+1} \circ \partial_1(\sigma) = \\
&= [\partial_1, f^{\bar\sigma}_{n+1}](\sigma) = \\
&= \epsilon^{\bar\sigma}_{n+1}(\sigma) = \\
&= \epsilon_{n+1}(\sigma).
\end{aligned}
$$

and χ^{n+1} is local by construction.

□

Lemma 4.6.

(a) Given two local derivations that can be composed. Then its composition is also local. Therefore the commutator of two local derivations is a local derivation.

(b) Given two contractible complexes (X, ∂^X) and (Y, ∂^Y). Extend the differentials to $\partial^{X^\bullet} : T^{X^\bullet}X \longrightarrow T^{X^\bullet}X$ and $\partial^Y : T^Y Y \longrightarrow T^Y Y$ as derivations. Then the complex $Hom(T^{X^\bullet}X, T^Y Y)$ together with the differential $\partial(f) := \partial^Y \circ f - (-1)^{|f|} f \circ \partial^{X^\bullet}$ is also contractible.

(c) Given a complex (K, ∂) with non-degenerate inner product, and suppose, that $Im(\partial)$ and $Im(\partial^\dagger)$ are closed subsets of K, where $\partial^\dagger$ is the adjoint operator of ∂. If $x \in K$ is an exact element, then there exists a unique element $z \in K$ such that

$$x = \partial(z) \qquad \text{and} \qquad z \in Im(\partial^\dagger).$$

Proof. The proof of (a) and (b) follows by standard arguments. For (c), Hodge-theory gives $K = H^\bullet K \oplus Im(\partial) \oplus Im(\partial^\dagger)$. As x is exact, one has $x = \partial(y)$, where y decomposes uniquely as $y = y_1 \oplus y_2 \oplus y_3 \in H^\bullet K \oplus Im(\partial) \oplus Im(\partial^\dagger)$. Then take $z := y_3 \in Im(\partial^\dagger)$. □

Proposition 4.7. *Given a closed, triangulated and oriented manifold M. Let (C, ∂_1) be a simplicial model for M and assume that one has extended this to an A_∞-coalgebra (C, D), such that ∂_2 represents the coproduct on C. Assume furthermore that the closure of every simplex is contractible, and that all $\partial_i : C \to C^{\otimes i}$ are local. Finally, denote by $\mu \in C$ the fundamental cycle* *of M.*

Then there exists a an ∞-inner-product ($\in T_C^C C$) such that the lowest element is given by $\partial_2(\mu) \in C \otimes C = T_C^C C^0$.

Proof. By Proposition 4.4, one has a chain map $\chi : (C, \partial_1) \to (T_C^C C, \mathcal{D})$. Then define the ∞-inner-product to be $F := \chi(\mu) \in T_C^C C$. By Lemma 4.2, F is an ∞-inner-product, because F is closed:

$$\mathcal{D}(F) = \mathcal{D}(\chi(\mu)) = \chi(\partial_1(\mu)) = \chi(0) = 0.$$

□

One would like to show that the ∞-inner-product from Proposition 4.7 is non-degenerate on the associated Hochschild-spaces, and thus represents an ∞-Poincaré-duality-structure. Unfortunately, the construction of a quasi-inverse involves the use of $C^\bullet$, where one cannot assume very good local properties for the multiplications $\partial_i^\bullet$ any more. But by assuming a "very fine" triangulation, one can gain an isomorphism on Hochschild-cohomology up to any desired monomial degree:

Proposition 4.8. *Given the assumption from Proposition 4.7, and let F be the constructed ∞-inner-product. Let $N \in \mathbb{N}$. If for every simplex σ in C, the N-th iterated closure of the star of σ is contractible, then one can construct a quasi-inverse to F up to the N-th level. (As usual $A := C^\bullet$ is the dual of C.)*

More precisely, there is an N-the level quasi-inverse $G : T^{A^\bullet} A \to T^A A$ and *homotopies $H : T^A A \to T^A A$ and $K : T^{A^*} A \to T^{A^*} A$ up to the N-th level:*

$$G \circ F - id_{T^A A} = D^A \circ H + H \circ D^A \quad on \quad \bigoplus_{k+l \leq N} A^{\otimes k} \otimes A \otimes A^{\otimes l} \tag{4.9}$$

$$F \circ G - id_{T^{A^*} A} = D^{A^*} \circ K + K \circ D^{A^*} \quad on \quad \bigoplus_{k+l \leq N} A^{\otimes k} \otimes A^* \otimes A^{\otimes l} \tag{4.10}$$

Proof. By Proposition 4.7 one has a map $F : T^A A \to T^{A^*} A$. One can see, that the homotopies H and K have to lift in the same way F and G do (see Definition 2.22.) in order for $D^A \circ H + H \circ D^A$ and $D^{A^*} \circ K + K \circ D^{A^*}$ to be A_∞-bimodule maps. Thus, one only has to construct maps $G : T^{A^*} A \to A$, $H : T^A A \to A$ and $K : T^{A^*} A \to A^*$, which can then be lifted to the correct maps. In the dual picture with $A = C^*$, it is enough to construct $G \in T^{C^*}_{C^*} C$, $H \in T^C_{C^*} C$ and $K \in T^{C^*}_C C$, satisfying certain conditions.

In order to start the construction, one has to choose a local quasi-inverse $G^0 : C \to C^*$ to the given map $F^0 : C^* \to C$, and local homotopies $H^0 : C^* \to C^*$ and $K^0 : C \to C$ with

$$G^0 \circ F^0 - id_{C^*} = \partial_1^* \circ H^0 + H^0 \circ \partial_1^*,$$

$$F^0 \circ G^0 - id_C = \partial_1 \circ K^0 + K^0 \circ \partial_1.$$

This is possible, if one understands the notion of "locality" as mapping a given simplex $\sigma \in C$ into the subcomplex of the closure of the star, denoted by $C(\overline{star(\sigma)})$.

Then, the condition for G to commute with the Hochschild differentials can be described in a similar way to Definition 4.1. But one has to involve the multiplication on $T^{A^*} A$, which was defined in Lemma 2.30.(b). Dualizing this multiplication to the given situation and using the fact that each $\partial_i =$

$m_i^\bullet : C \to C^{\otimes i}$ is local(, i.e. it maps a simplex σ only inside the complex of its closure $\overline{\sigma}$), one sees that the dual multiplications have the locality condition of mapping the dual of a simplex $\sigma^\bullet \in C^\bullet$ only inside the complex of the closure of its star $\overline{star(\sigma)}$, i.e. $b_{k,l}^\bullet(\sigma^\bullet) \in T^{C^\bullet(\overline{star(\sigma^\bullet)})}C(\overline{star(\sigma)})$, where $b_{k,l}$ is taken from Lemma 2.30.(b). Now, the inductive construction of G up to the N-th degree works in the same way as in Proposition 4.4. Namely, one has to be careful that, when constructing G^j out of the ∂_i's, $b_{k,l}^\bullet$'s and lower G_r's, one jumps from the $(j-1)$-st to the j-th closure of the star of a given dual simplex $\sigma^\bullet$. But the subcomplex generated by this is only guaranteed to be contractible up to degree N. The construction of G has to stop at the N-th step.

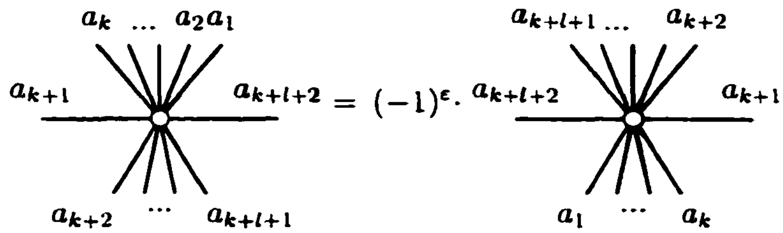
Similar arguments apply to the constructions of H and K. The guiding line for their inductive construction are the homotopy equations (4.9) and (4.10). It is easy to see that they commute with the differentials of $T^A A$ and $T^{A^\bullet} A$, respectively. Thus, the lowest components are closed and by locality condition can be lifted. Again this only works up to the N-th step.

So, for a given N, one can construct the G, H and K up to the N-th part. □

Proposition 4.11. *Given the assumption from Proposition 4.7. and let $F : T^A A \to T^{A^\bullet} A$ be the constructed ∞-inner-product. Assume furthermore that the map $\partial_2 : C \to C \otimes C$ is (graded) cocommutative, and that the A_∞-algebra (A, D) has a unit 1 (in the sense of Definition 3.8.). Then all the maps $F_{k,l} : sA^{\otimes k} \otimes sA \otimes sA^{\otimes l} \to sA^\bullet$ for $k + l \geq 1$ are invariant under the unique cyclic rotation of the arguments, mapping the two special elements into each* *other:*

$$F_{k,l}(a_1,\dots,a_{k+1},\dots,a_{k+l+2}) = (-1)^{\varepsilon}\cdot F_{l,k}(a_{k+2},\dots,a_{k+l+2},a_1,\dots,a_{k+1}), \quad (4.12)$$

where a_{k+1} *and* a_{k+l+2} *are the two special elements, and the sign is given by* $\varepsilon = (\|a_1\| + \dots + \|a_{k+1}\|)\cdot(\|a_{k+2}\| + \dots + \|a_{k+l+2}\|)$. *In diagrams, this is*



It follows from Lemma 3.12., that for this given ∞*-inner-product* F, *the induced* Δ *and* β*-operators from Definition 3.10. coincide, and thus* F *is symmetric in the sense of Definition 3.11.*

Proof. Equation (4.12) will be shown by induction on $n = k + l$.

$n = 1$: One has to check the equation $F_{0,1}(a,b,c) = (-1)^{\|a\|\cdot(\|b\|+\|c\|)}\cdot F_{1,0}(b,c,a)$ for any $a,b,c \in A$.

The condition of $\mathcal{D}(F) = 0$ is of course simply the condition of F being a chain-map of the Hochschild-complexes. But this is also described by Proposition 2.33. By this, the terms $F_{0,1}(a,b,c)$ and $F_{1,0}(b,c,a)$ are homotopies of $F_{0,0}(a\cdot b,c) - F_{0,0}(a,b\cdot c)$ and $F_{0,0}(b\cdot c,a) - (-1)^{\|b\|\cdot(\|a\|+\|c\|)}\cdot F_{0,0}(c,a\cdot b)$ being zero, respectively (compare Example 2.40.). But the definition of $F_{0,0}$ in the dual picture was given by $F_{0,0} = (\partial_2(\mu))^* \in C\otimes C \subset Hom(A\otimes A,R)$. Therefore $F_{0,0}(a,b) = (\partial_2(\mu))^*(a,b) = (a\cdot b)(\mu)$.

This implies that

$$F_{0,0}(a \cdot b, c) - F_{0,0}(a, b \cdot c) = ((a \cdot b) \cdot c - a \cdot (b \cdot c))(\mu), \quad \text{and}$$

$$\begin{aligned} F_{0,0}(b \cdot c, a) - (-1)^{\|b\| \cdot (\|a\| + \|c\|)} \cdot F_{0,0}(c, a \cdot b) = \\ = ((b \cdot c) \cdot a - (-1)^{\|b\| \cdot (\|a\| + \|c\|)} \cdot c \cdot (a \cdot b))(\mu) = \\ = (-1)^{\|a\| \cdot (\|b\| + \|c\|)} \cdot (a \cdot (b \cdot c) - (a \cdot b) \cdot c)(\mu), \end{aligned}$$

because the multiplication was chosen to be graded commutative. As the homotopies $F_{0,1}$ and $F_{1,0}$ were chosen canonically, they have to coincide on the cyclically rotated arguments.

$n > 1$: Again by the description from Proposition 2.33, one knows that $F_{k,l}$, $k + l = n$, is a homotopy of a sum of lower terms $F_{r,s}$ $(r, s \geq 0)$ with one multiplication D_t $(t \geq 2)$ in the argument applied, being zero. Comparing those terms for $F_{k,l}(a_1, ..., a_{k+1}, ..., a_{k+l+2})$ with the terms for $F_{l,k}(a_{k+2}, ..., a_{k+l+2}, a_1, ..., a_{k+1})$, one easily sees that for every $F_{r,s}(..., D_t(...), ...)$ from the boundary of $F_{k,l}(a_1, ...)$ there exists a corresponding term $F_{s,r}(..., D_t(...), ...)$ from the boundary of $F_{l,k}(a_{k+2}, ...)$, because the conditions for inserting a multiplication D_t is cyclically invariant. By induction these terms are equal (up to sign), and thus the canonical definitions of $F_{k,l}(a_1, ...)$ and $F_{l,k}(a_{k+2}, ...)$ have to coincide.

□

Remark 4.13. *In order to construct an* ∞*-inner-product such that the* Δ *operator coincides with the dual of Connes' operator* β*, one needs the following data:*

M has to be a closed, triangulated and oriented manifold. (C, ∂_1) is a simplicial model for M such that the closure of every simplex is contractible. There exists a special element $1 \in C$, called the unit. ∂_2 represent a graded cocommutative coproduct on C, and one has extended ∂_1 and ∂_2 to a local (in the sense of Proposition 4.4) and unital A_∞-coalgebra (C, D).

A construction of this can be performed in the following way. Start with a unital, cocommutative (but not necessarily coassociative) coalgebra map $\partial_2 : C \rightarrow C \otimes C$. This can always be obtained in a field of characteristic 0 by symmetrization of a diagonal approximation. Then by an induction similar to Proposition 4.4 or Proposition 4.8, one can build the higher maps ∂_i, which turn out to vanish on the unit 1. In fact, Dennis Sullivan's construction in [18] is far more general, as it builds a C_∞-structure where all higher maps ∂_i have to satisfy "good" cocommutativity conditions.

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