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ONE RELATOR METABELIAN GROUPS

by

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INTRODUCTION

This thesis had its origin in a short paper, Groups with One Defining Relator, [1] which was presented by my adviser, Gilbert Baumslag, at the Annual Meeting of the Australian Mathematical Society in 1964. The paper surveys the algorithmic solutions to various problems of the group

$$H = (x_1, \dots, x_n; r) .$$

It is natural to ask which of these theorems for groups with a single defining relator have analogues that can be proven for metabelian groups with a single defining relator.

Solutions for metabelian groups, some of them only partial, for all of the problems mentioned, will be presented in this paper:

1) the extended word problem,
2) the extended conjugacy problem,
3) a "Freiheitssatz," i.e., a condition on the relator that ensures that every proper subset of the generators freely generates a free metabelian group,
4) the center,
5) the existence of elements of finite order,
6) the question of whether such groups are residually finite p for all primes p,
7) the question of whether such groups are residually free metabelian.

Solutions to all of these problems, except the last, are complete when the relator lies in the derived group. It is possible that the solutions of these problems for one relator metabelian groups will give much information for all finitely presented metabelian groups. That this might be the case follows from a theorem of P. Hall [2]:

Theorem i) Finitely generated metabelian groups satisfy the maximal condition for normal subgroups.

Let M be a finitely generated free metabelian group. Then any finitely generated metabelian group $G \cong M/_K$ where K is the normal closure of a finite set. This means that Hall's theorem, when restricted to the variety of metabelian groups, can be stated as

Theorem ii) In the variety of metabelian groups, the finitely generated groups are finitely related.

This paper presents a solution to the conjugacy problem, which has been solved by Jane Matthews [8]. The techniques used here for this problem also give a solution for the extended conjugacy problem. With the work of Alice Whittemore concerning Frattini subgroups, this paper completes a rather thorough survey of the properties of one relator metabelian groups [10].

CHAPTER 1

PRELIMINARIES

Section 1. Algebraic Definitions and Notations

In this chapter we establish the general algebraic definitions and notations used in the sequel. Those properties which are later required are discussed in detail.

Definition 1.1: A group F is freely generated by a subset X if

i) X generates F

ii) for any group H, any (set) function from X to H extends as a homomorphism from F to H.

The group F is then called the free group on X, or, briefly, free, if it is freely generated by some (unspecified) subset X. Property ii) is described as a universal mapping property for the set X, with respect to all groups.

Definition 1.2: A word (in the elements of X) is any expression of the form $x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}$, where $x_i \in X, \epsilon_i = \pm 1$. It is a reduced word if it contains no segment of the form $x_i^{\epsilon_i} x_{i+1}^{\epsilon_{i+1}}$, $i = i + 1$, $\epsilon_i + \epsilon_{i+1} = 0$. It is a cyclically reduced word if it is reduced and $x_1 \neq x_n$, of if $x_1 = x_n, \epsilon_1 \neq -\epsilon_n$.

Theorem 1.3:

Any set X is a free generating set for some free group. In particular, we may take F as the set of all reduced words in X (including an empty word), with multiplication defined by juxtaposition, followed by deletion of any segments $x_i^{\epsilon_i} x_{i+1}^{\epsilon_{i+1}}$, $i = i + 1$, $\epsilon_i + \epsilon_{i+1} = 0$, which appear until a reduced word is obtained.

We note that reduced words in F represent the same element of F if and only if they are identical.

For the remainder of this chapter, the letter F denotes the free group, freely generated by the finite set $\{f_i \mid i = 1,\ldots,n\}$ as described in Theorem 1.3 above.

The following notational conventions will be observed. For g,h , elements of a group G , and A,B , subsets of G ,

$g^h = hgh^{-1}$, the conjugate of g by h

$A^h = \{hah^{-1} \mid a \in A\}$.

$[g,h] = ghg^{-1}h^{-1}$, the commutator of g and h

$[A,B]$ = subgroup generated by $[a,b], a \in A,\ b \in B$.

The subgroup $[G,G] = G'$ is called the derived group of G , and $[G',G'] = G''$ is the second derived group of G .

<u>Definition 1.4</u>: The <u>normal closure of S in G</u> , $nm_G(S)$, is the subgroup:

$$nm_G(S) = \cap\{N \mid S \subset N \text{ and } N \text{ a normal subgroup of } G\} .$$

Since the intersection of a family of normal subgroups of G is again a normal subgroup of G , the normal closure of a set is normal in G . Thus $nm_G(S)$ is the unique minimal normal subgroup of G which contains S , and it may be seen that it is the subgroup generated by all conjugates s^g of elements $s \in S$ by elements $g \in G$.

<u>Definition 1.5</u>: The group H is said to be presented by the epimorphism $\eta : G \to H$.

It follows that η induces an isomorphism

$$G/R \cong H$$

where R is the kernel of η. Suppose now that G is a free group, freely generated by $\{x_i | i \in I\}$ and that R is the normal closure in G of the set $\{r_j | j \in J\}$. The expression $(x_i ; r_j)$ is called a <u>presentation</u> for H. The x_i are called generators for H and the r_j are relators for H. We write

$$H = (x_i ; r_j) .$$

We note that we can assign a group H to any sets $\{x_i | i \in I\}$ and $\{r_j | j \in J\}$ where r_j are words in the x_i, once a procedure is established which assigns a unique group G to the $\{x_i | i \in I\}$.

<u>Definition 1.6</u>: A variety $\underline{\underline{V}}$ is a class of groups which is closed with respect to isomorphisms, subgroups, factor groups, and (unrestricted) Cartesion products.

<u>Definition 1.7</u>: A <u>law</u> of a variety $\underline{\underline{V}}$ is any element $f \in F$ which is in the kernel of every homomorphism from F to any group in $\underline{\underline{V}}$. The set of all laws of $\underline{\underline{V}}$ is denoted $\underline{\underline{V}}(F)$. A group $G \in \underline{\underline{V}}$ is termed a free $\underline{\underline{V}}$ group if there is a subset X of G which generates G and which satisfies the universal mapping property with respect to groups in $\underline{\underline{V}}$.

<u>Proposition 1.8</u>: Let $\underline{\underline{V}}$ be a variety and let G be a finitely generated $\underline{\underline{V}}$ group. Then G has a presentation of the form

$$F/\underline{\underline{V}}(F) \to G \text{ with}$$

$$G \cong F/\underline{\underline{V}}(F)R$$

which we indicate as follows: $G = (f_i;R)_{\underline{\underline{V}}}$, or $G = (f_i;r_j)_{\underline{\underline{V}}}$ where r_j are words in the f_i whose normal closure is R .

The class of abelian groups is a variety which we will denote by $\underline{\underline{A}}$. In this case, $\underline{\underline{A}}(F) = F'$, the derived group of F , and F/F' is a free abelian group on the $\underline{\underline{A}}$ free generators $f_1F', f_2F', \ldots$. We shall write $\overline{x_i} = f_iF'$ and $A = F/F'$. We indicate elements of A as small roman letters with upper bars; accordingly $(\overline{x_i};\overline{r_j})$ is the presentation of any finitely generated abelian group as an image of A and A itself has the presentation, $A = (\overline{x_1},\overline{x_2},\ldots)$.

Definition 1.9: If $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are varieties, their product $\underline{\underline{U}}\cdot\underline{\underline{V}}$ is the class of all groups G which contain a normal subgroup $U \in \underline{\underline{U}}$ with factor group $G/U \in \underline{\underline{V}}$.

It can be shown that $\underline{\underline{UV}}$ is a variety, and that $\underline{\underline{UV}}(F) = \underline{\underline{U}}(\underline{\underline{V}}(F))$ [ ].

We shall be concerned with the variety $\underline{\underline{AA}}$, the *metabelian* variety. It follows that $\underline{\underline{AA}}(F) = F''$ and that F/F'' is a free metabelian group on the free $\underline{\underline{AA}}$ generators $f_1F'', f_2F'', \ldots$.

We shall write $x_i = f_iF''$,

$$M = F/F'' .$$

We shall indicate presentations of finitely generated metabelian groups as factor groups of M by double parenthesis; in particular

$$M = ((x_1,x_2,\ldots,x_n)) .$$

In the sequel all metabelian groups will have a finite set of metabelian generators.

We will observe that

$$M/_{M'} \cong {}^{F/F''}/_{F'/F''} \cong {}^{F}/_{F'} = A \ .$$

The isomorphism $M/_{M'} \cong A$ identifies $x_iM' \in M/_{M'}$ with $\overline{x_i} \in A$.

<u>Definition 1.10</u>: A one relator metabelian group is a metabelian group G with some presentation of the form

$$G = ((x_1, \ldots, x_n; r))$$

It follows that $G \cong M/_{nm_M(r)}$, $G' \cong M'/_{M' \cap nm_M(r)}$, and $G/_{G'} \cong M/_{M' \cdot nm_M(r)}$.

We now turn to some concepts needed for the Magnus representation of free metabelian groups. We recall first

<u>Definition 1.11</u>: A representation of a group G is a homomorphism $\rho : G \to H$. It is termed <u>faithful</u> if ρ is injective.

<u>Definition 1.12</u>: The <u>integral group ring</u> $Z(G)$ of a group G is the following:

As a set $Z(G) = \left\{ \sum\limits_{g \in G} n_g \cdot g \,\middle|\, n_g \in Z \ , \ n_g \neq 0 \text{ for only finitely many } g \in G \right\}$.

The ring operations in $Z(G)$ are

$$\sum_{g \in G} n_g \cdot g + \sum_{g \in G} m_g g = \sum_{g \in G} (n_g + m_g) g$$

$$\left(\sum_{g \in G} n_g g \right) \left(\sum_{g \in G} m_g g \right) = \sum_{g \in G} \left(\sum_{hk=g} n_h m_k \right) g$$

We shall identify

$$\sum_{g \in G} n_g g \quad \text{with} \quad \sum_{g \in S} n_g g$$

where $n_g = 0$, $g \in G - S$.

Lemma 1.13: If G is abelian, $Z(G)$ is commutative.

In order to analyze the Magnus representation, we require information about the group ring, $Z(A)$, where A is a free abelian group. This is the content of the next section.

Section 2. The group ring $Z(A)$.

Let $A = (\overline{x}_1, \ldots, \overline{x}_n)$. Each $a \in A$ may be uniquely expressed as $a = \overline{x}_1^{\,i_1} \ldots \overline{x}_n^{\,i_n}$ which we call the normal form for $a \in A$.

Definition 1.14: The element $r \in Z(A)$ is in normal form if

$$r = \sum_{j=1}^{N} n_j a_j$$

where each a_j is in normal form and N is minimal.

Proposition 1.15: Every $r \in Z(A)$ corresponds to a unique normal form $\sum_{j=1}^{N} n_j a_j$, and N is finite.

Proof: Each element of A may be written in normal form, and each element of $Z(A)$ has only finitely many non-zero terms.

Definition 1.16: $r \in Z(A)$ is called non-negative if in the normal form expression $r = \sum_{j=1}^{N} n_j a_j$; $a_j = \overline{x}_1^{\,i_{j,1}} \ldots \overline{x}_n^{\,i_{j,m}}$, the exponents $i_{j,k}$ are non-negative.

Proposition 1.17: The set of non-negative elements in $Z(A)$ is a subring of $Z(A)$ which is isomorphic to the polynomial ring $Z[\overline{x}_1, \ldots, \overline{x}_n]$.

We shall accordingly identify the non-negative elements of $Z(A)$

with $Z[\overline{x_1},\ldots,\overline{x_n}]$.

Lemma 1.18: For every $r \in Z(A)$, there exists a unique element $b_r = \overline{x_1}^{k_1}\ldots\overline{x_n}^{k_n} \in A \cap Z[\overline{x_1},\ldots,\overline{x_n}]$ such that $b_r r \in Z[\overline{x_1},\ldots,\overline{x_n}]$ and each k_i is minimal.

Proof: Write r in normal form

$$r = \sum_{j=1}^{N} n_j a_j \ ; \ a_j = \overline{x_1}^{i_{j,1}}\ldots\overline{x_n}^{i_{j,n}}$$

and set

$$k_\ell = - \min\left\{i_{j,\ell} \text{ or } 0 \,\middle|\, j = 1,\ldots N\right\} .$$

Proposition 1 19: The units of $Z(A)$ are ± the elements of A .

Proof: The units of a ring are, by definition, those elements which have multiplicative inverses. $a \in A$ has inverse a^{-1} , $-a$ has inverse $-a^{-1}$.

Suppose now that $\xi \in Z(A)$, $\xi \notin A$, and ξ has inverse $\eta \in Z(A)$. Let ξ be given in normal form

$$\xi = \sum_{j=1}^{N} n_j a_j,\ a_j \in A$$

Case 1. If $N = 1$, $\xi = n_1 a_1$ for some $n_1 \in Z$, $n_1 \neq 1$, and $a_1 \in A$. Then $\xi a_1^{-1} = n_1$ is also a unit in $Z(A)$, which implies $n_1 = 1$.

Case 2. $N > 1$. Select b,c for ξ,η as in Proposition 1.18. Then

$$(b\xi)(c\eta) = bc\ ,$$

an equation in $Z[\overline{x_1},\ldots,\overline{x_n}]$. But $b\xi$ has more than one term, hence $(b\xi)(c\eta)$ does also. But bc is a product of monomial terms, hence a contradiction, so this case cannot occur either.

<u>Definition 1.20</u>: The element b of lemma 1.18 will be termed the <u>least unit</u> corresponding to r and denoted u_r .

<u>Lemma 1.21</u>: If $u_r \cdot r = a\rho$, where $\rho, a \in Z[\bar{x}_1,\ldots,\bar{x}_n]$ and $a \in A$, then $r = a\sigma$ where $u_r = u_\sigma$.

<u>Proof</u>: Let $\sigma = u_r^{-1}\rho$ where ρ has no negative exponents. Determine u_σ according to lemma 1.18. We claim that $u_\sigma = u_r$.

$$r = a\sigma$$

$$\text{Therefore} \quad u_\sigma r = a u_\sigma \sigma$$

But $a u_\sigma \sigma \in Z[\bar{x}_1,\ldots,\bar{x}_n]$, so $u_\sigma r \in Z[\bar{x}_1,\ldots,\bar{x}_n]$, and therefore $u_r | u_\sigma$ in $Z[\bar{x}_1,\ldots,\bar{x}_n]$.

$u_r \sigma = \rho \in Z[\bar{x}_1,\ldots,\bar{x}_n]$, therefore $u_\sigma | u_r$ in $Z[\bar{x}_1,\ldots,\bar{x}_n]$ and we conclude $u_\sigma = u_r$.

<u>Proposition 1.22</u>: $Z(A)$ is an integral domain.

<u>Proof</u>: Let $r, s \in Z(A)$, $r \neq 0$, $s \neq 0$. Choose b, c , the least units of r and s . $(br)(cs) = bc(rs)$. bc is a unit in $Z(A)$, hence $rs = 0 \Longleftrightarrow br \cdot cs = 0$. But br and cs are elements of $Z[\bar{x}_1,\ldots,\bar{x}_n]$, and the polynomial ring in $\bar{x}_1,\ldots,\bar{x}_n$ over the integers has no zero divisors. Hence $br \neq 0$ and $cs \neq 0$ implies $br \cdot cs \neq 0$, so $r \cdot s \neq 0$.

Thus $Z(A)$ is a commutative ring with identity having no proper zero divisors, so $Z(A)$ is an integral domain.

<u>Definition 1.23</u>: Recall that if $\xi = \eta\epsilon$, where ϵ is a unit, then

ξ and η are said to be <u>associates</u>. The associates of an element and units are <u>improper divisors</u> of the element.

An element $\xi \in Z(A)$ is <u>irreducible</u> if it is not a unit and every divisor is improper. Recall that if ξ is irreducible, so is any associate of ξ.

<u>Proposition 1.24</u>: Every non-unit of $Z(A)$ is a finite product of irreducible factors.

<u>Proof</u>: Let $\xi \in Z(A)$ be a non-unit. Choose u_ξ its corresponding least unit, and $u_\xi \xi \in Z[\overline{x_1}, \ldots, \overline{x_n}]$.

But $Z[\overline{x_1}, \ldots, \overline{x_n}]$ is a unique factorization domain, then $u_\xi \xi$ is a finite product of irreducible factors in $Z[\overline{x_1}, \ldots, \overline{x_n}]$, say

$$u_\xi \xi = \prod_{i=1}^{q} \rho_i$$

where ρ_i are irreducible in $Z[\overline{x_1}, \ldots, \overline{x_n}]$.

We claim that each ρ_i is irreducible in $Z(A)$ or is a unit in $Z(A)$.

Now with $\rho_i = na$, either n is a prime integer and ρ_i is irreducible in $Z(A)$ or $n = \pm 1$ and ρ_i is a unit in $Z(A)$. It thus suffices to show that if ρ is irreducible in $Z[\overline{x_1}, \ldots, \overline{x_n}]$, and in normal form

$$\rho = \sum_{j=1}^{N} n_j a_j , \; N > 1 ,$$

then ρ is irreducible in $Z(A)$.

Suppose $p = q_1 q_2$, $q_i \in Z(A)$. Let u_1, u_2 be the least units corresponding to q_1 and q_2. Then $u_1 u_2 p = (u_1 q_1)(u_2 q_2) \in Z[\overline{x_1}, \ldots, \overline{x_n}]$.

But p is irreducible in $Z[\overline{x_1},\ldots,\overline{x_n}]$ so $p|(u_1q_1)$ or (u_2q_2) in $Z[\overline{x_1},\ldots,\overline{x_n}]$. Hence

$$u_1q_1 = vp \; , v \in Z[\overline{x_1},\ldots,\overline{x_n}]$$

which implies $u_1u_2 = v\,u_2q_2$, hence $u_1 = vq_2$. But u_1 is a product of $\overline{x_i}$'s , which are irreducible in $Z[\overline{x_1},\ldots,\overline{x_n}]$. Utilizing the uniqueness of factorization in $Z[\overline{x_1},\ldots,\overline{x_n}]$, $q_2 \in A \cap Z[\overline{x_1},\ldots,\overline{x_n}]$ and so is a unit in $Z(A)$.

Theorem 1.25:

$Z(A)$ is a unique factorization domain.

Proof: Suppose

$$\xi = \prod_{i=1}^{N} p_i = \prod_{j=1}^{M} q_j$$

are two factorizations of ξ into $Z(A)$ - irreducible factors. Let u_i, v_j be the least units for p_i, q_j . Then

$$\prod_{i=1}^{N} \prod_{j=1}^{M} u_i v_j \xi = \prod_{j=1}^{M} v_j \prod_{i=1}^{N} (u_i p_i) = \prod_{i=1}^{N} u_i \prod_{j=1}^{M} (v_j q_j) \; .$$

Now $\prod_{j=1}^{M} v_j = av$; $\prod_{i=1}^{N} u_i = au$, where v and u are relatively prime elements of $Z[\overline{x_1},\ldots,\overline{x_n}]$ and $a \in Z[\overline{x_1},\ldots,\overline{x_n}]$ and $v \prod_{i=1}^{N} (u_i p_i) = u \prod_{j=1}^{M} (v_j q_j)$.

Suppose $\overline{x_i}$ divides v in $Z[\overline{x_1},\ldots,\overline{x_n}]$. Then $\overline{x_i}$ divides some v_jq_j , so that

$$q_j = \overline{x_i}\sigma_j$$

and the least unit of σ_j is v_j . Continuing in this fashion, obtain

$$\prod_{i=1}^{N} (u_i\tau_i) = \prod_{j=1}^{M} (v_j\sigma_j)$$

where $u_i\tau_i$ is an associate of p_i in $Z(A)$; $u_j\sigma_j$ is an associate of q_j in $Z(A)$; $u_i\tau_i$ and $v_j\sigma_j \in Z[\overline{x}_1,\ldots,\overline{x}_n]$. But $u_i\tau_i, v_j\sigma_j$ are irreducible in $Z[\overline{x}_1,\ldots,\overline{x}_n]$, which is a unique factorization domain.

Recall that in a unique factorization domain, any pair of elements has greatest common divisor, determined up to multiplication by units.

Definition 1.26: Let $\xi, \eta \in Z(A)$. The strong greatest common divisor of ξ and η , written (ξ,η) is a greatest common divisor δ defined as follows: Consider u_ξ, u_η and $u_\xi \cdot \xi$ and $u_\eta \cdot \eta \in Z[\overline{x}_1,\ldots,\overline{x}_n]$. Within $Z[\overline{x}_1,\ldots,\overline{x}_n]$, let d = greatest common divisor u_ξ and u_η and Δ = greatest common divisor $u_\xi \cdot \xi$ and $u_\eta \cdot \eta$. δ is defined to be $d^{-1}\Delta$. δ is unique up to units ± 1 of $Z[\overline{x}_1,\ldots,\overline{x}_n]$.

Definition 1.27: A (two-sided) R-module is an abelian group S (in which the group operation is written additively) together with a function $\mu: R \times S \times R \to S$, which satisfies the following relations:

a) μ is additive in each variable

b) for $\xi, \eta, \tau \in R$, $x \in S$ and $\mu(\xi,x,\tau) = \xi x \tau$, $(\xi\eta)x\tau = \xi(\eta x \tau) = \xi(\eta x)\tau$, $\xi x(\eta\tau) = \xi(x\eta)\tau = (\xi x \eta)\tau$.

We define a two-sided $Z(A)$ module T as follows:

$$T = \left\{ \sum_{i=1}^{n} \xi_i t_i \,\middle|\, \xi_i \in Z(A) \right\} .$$

The operations are

$$\sum_{i=1}^{n} \xi_i t_i + \sum_{i=1}^{n} \eta_i t_i = \sum_{i=1}^{n} (\xi_i + \eta_i) t_i$$

$$\eta\left(\sum_{i=1}^{n} \xi_i t_i \right)\tau = \sum_{i=1}^{n} (\eta \xi_i \tau) t_i \ .$$

Thus T is simply the direct sum of n copies of Z(A) , an abelian group. The action of Z(A) on T is the obvious term by term multiplication.

CHAPTER 2

THE MAGNUS REPRESENTATION

Section 1: Introduction to the Freiheitssatz.

We recall the statement of the Freiheitssatz for groups with one defining relator.

Theorem 2.1: W. Magnus

Let G be a group with the presentation $G = (x_1,\dots,x_n;r)$ in which r is a cyclically reduced word in the x_i. If x_n appears in r, then the subgroup of G generated by $x_1,\dots,x_{n-1}$ is freely generated by them.

It is convenient to observe that Theorem 2.1 is equivalent to

Theorem 2.2:

Let F_n be the free group, freely generated by $\{x_1,\dots,x_n\}$ and let r be a cyclically reduced word in which x_n appears. Then

$$nm_F(r) \cap H = \{1\} ,$$

where H is the subgroup of F generated by $\{x_1,\dots,x_{n-1}\}$.

We shall establish substitutes for the notions "cyclically reduced" and "appears" in the free metabelian group $((x_1,\dots,x_n))$ which will then yield a Freiheitssatz for one-relator metabelian groups. We first present an example which illustrates the need for new definitions.

Example 2.3:

Let $M = ((a,b,c))$ and let $r = c[a,b] = caba^{-1}b^{-1}$, a cyclically reduced word in the symbols a,b, and c .

$$\begin{aligned} r(r^{-1})^{[a,c]} &= c[a,b][a,c][a,b]^{-1}c^{-1}[a,c]^{-1} \\ &= c[a,c]c^{-1}[a,c]^{-1} \\ &= [c,[a,c]] \end{aligned}$$

and so $nm_M(r) \cap ((a,c)) \neq \{1\}$.

This example underlines the fact that in $M = ((x_1,\ldots,x_N))$, there is no uniqueness of expression in terms of $x_1,\ldots,x_N$

We will develop a representation of M that eliminates some of this problem. Before giving the representation we make two further observations concerning words $m \in M$.

Let $m \in M$. Suppose m is given to us in terms of the generators $x_1,\ldots,x_n$ and has the form

$$m = x_{i_1}^{\epsilon_{i_1}} \ldots x_{i_n}^{\epsilon_{i_n}}$$

Then the natural homomorphism $M \to A$, given by $x_i \to \overline{x_i}$ maps m to

$$\bar{m} = \overline{x}_{i_1}^{\epsilon_{i_1}} \ldots \overline{x}_{i_\ell}^{\epsilon_{i_\ell}} = \bar{x}_1^{i_1} \ldots \bar{x}_n^{i_n}$$

where the latter is in normal form. For each $k = 1,\ldots,n$, the exponent i_k is termed the exponent sum of m on x_k . It is the sum of those exponents ϵ_j for which $i_j = k$ and does not depend on the choice of expression for m in terms of $x_1,\ldots,x_n$.

<u>Definition 2.4</u>: The element $\tilde{m} = x_1^{i_1} \ldots x_n^{i_n} \in M$ is called the <u>content</u> of m .

<u>Lemma 2.5</u>: Each $m \in M$ may be rewritten as

$$m = \tilde{m}[m]$$

where $[m] \in M'$ and $\tilde{m}$ is the content of m .

Proof: The element $[m] = \tilde{m}^{-1}m$ belongs to the kernel of the natural homomorphism $M \to A$, which is M' .

Corollary 2.6: If the content of m is 1 , $m \in M'$.

Remark

The rewriting of m as $m = \tilde{m}[m]$ may be accomplished by moves of the following sort

$$xy = yx[x^{-1},y^{-1}] \ .$$

Section 2: A Representation for Metabelian Groups.

We require a theorem of W. Magnus which we state for our situation. [6].

Definition 2.7: Let $\mathfrak{M}$ be the group of 2×2 matrices of the form $\begin{pmatrix} a & \tau \\ 0 & 1 \end{pmatrix}$ where $a \in A$, $\tau \in T$, the free $Z(A)$ module [1.27], and the group operation is the usual matrix multiplication.

We observe that $\mathfrak{M}$ is a metabelian group, for

$$\left[\left[\begin{pmatrix} a_1 & t_1 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} a_2 & t_2 \\ 0 & 1 \end{pmatrix}\right],\left[\begin{pmatrix} a_3 & t_3 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} a_4 & t_4 \\ 0 & 1 \end{pmatrix}\right]\right] = 1 \ .$$

Theorem 2.8: W. Magnus proof of M. Hall Theorem.

Let $F = (x_1,\ldots,x_n)$. Then the map $\mu : F/F'' \to \mathfrak{M}$ defined by $x_i\mu = \begin{pmatrix} \overline{x_i} & t_i \\ 0 & 1 \end{pmatrix}$, $i = 1,\ldots n$ is a faithful representation of $F/F'' = M = ((x_1,\ldots,x_n))$.

Section 3: Properties of the Representation.

It will be convenient now to record several results concerning $M\mu \leq \mathfrak{M}$. Most of them are of a computational nature.

Lemma 2.9: If $M = ((x_1,\ldots,x_N))$ then the matrices $\begin{pmatrix} \bar{x}_i & t_i \\ 0 & 1 \end{pmatrix},\ldots\begin{pmatrix} \bar{x}_n & t_N \\ 0 & 1 \end{pmatrix}$ are the free metabelian generators of $M\mu$.

Proof: Clearly $M\mu$ is generated by $\left\{\begin{pmatrix} \bar{x}_i & t_i \\ 0 & 1 \end{pmatrix} \middle| i = 1,\ldots,n\right\}$.

Let G be any metabelian group and φ a map on $\left\{\begin{pmatrix} \bar{x}_i & t_i \\ 0 & 1 \end{pmatrix} \middle| i = 1,\ldots,N\right\}$ into G . φ extends to a homomorphism from M to G via μ^{-1} since any map on the generators of M extends

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\mu"] & G \\ \left\{\begin{pmatrix} \bar{x}_i & t_i \\ 0 & 1 \end{pmatrix}\right\} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Lemma 2.10: If $m \in M, m = 1 \Longleftrightarrow m\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Proof: μ is monomorphic.

Lemma 2.11: If $m,n \in M, m = n \Longleftrightarrow m\mu = n\mu$. We translate this observation into the following, which will be our standard test for equality.

If $m\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, $n\mu = \begin{pmatrix} \bar{n} & \sum_{i=1}^{N} n_i t_i \\ 0 & 1 \end{pmatrix}$, $m = n \Longleftrightarrow \bar{m} = \bar{n}$ and

$m_i = n_i \ \forall\ i$.

At this point we have asolution of the word problem for M, which we state as

<u>Proposition 2.12</u>: Let $m \in M$ and suppose $m\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$. Then $m = 1 \Leftrightarrow m\mu$ is such that $\bar{m} = 1$ and $m_i = 0 \ \forall i$.

<u>Lemma 2.13</u>: The inverse of $m \in M$ has the following image under μ:

If $m\,\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, then $m^{-1}\mu = \begin{pmatrix} \bar{m}^{-1} & -\bar{m}^{-1}\left(\sum_{i=1}^{N} m_i t_i\right) \\ 0 & 1 \end{pmatrix}$.

In particular observe that if $m = x_i$,

$$x_i^{-1}\mu = \begin{pmatrix} \bar{x}_i^{-1} & -x_i^{-1} t_i \\ 0 & 1 \end{pmatrix} \qquad (i)$$

and that if $m \in M'$, $m\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$,

$$m^{-1}\mu = \begin{pmatrix} 1 & -\sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}. \qquad (ii)$$

<u>Lemma 2.14</u>: Products have the following form under μ. If $m,n \in M$,

$m\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, $n\mu = \begin{pmatrix} \bar{n} & \sum_{i=1}^{N} n_i t_i \\ 0 & 1 \end{pmatrix}$, then

$$mn\mu = \begin{pmatrix} \bar{m}\bar{n} & \sum_{i=1}^{N} m_i t_i + \bar{m}\left(\sum_{i=1}^{N} n_i t_i\right) \\ 0 & 1 \end{pmatrix}.$$

All of the following lemmas can be verified by matrix multiplication. However, if $m \in M$ and m is given in terms of the generators $x_1,\dots,x_n$, then there is a method of calculating $m\mu$ that eliminates the need to compute $m\mu$ as a product of matrices $\begin{pmatrix} \bar{x}_i & t_i \\ 0 & 1 \end{pmatrix}$. This computational aid will provide an easy tool in all the following.

<u>Definition 2.15</u>: Let $m = x_{i_1}^{\epsilon_{i_1}} \dots x_{i_j}^{\epsilon_{i_j}}$, $\epsilon_{i_k} = \pm 1$. The Fox derivative with respect to a letter x_j is denoted $\frac{\partial m}{\partial x_j}$ and is computed as follows:

If x_j appears in m, $\frac{\partial m}{\partial x_j}$ is a sum of $\frac{\partial m}{\partial x_{i_k}}$ for all appearances $x_{i_k} = x_j$, where $\frac{\partial m}{\partial x_{i_k}}$ is computed as follows:

If $\epsilon_{i_k} = +1$, $\frac{\partial m}{\partial x_{i_k}}$ is the initial segment of m up to x_{i_k}.

If $\epsilon_{i_k} = -1$, $\frac{\partial m}{\partial x_{i_k}}$ is the negative of the initial segment of m up to and including x_{i_k}.

If x_j does not appear in m, $\frac{\partial m}{\partial x_j} = 0$.

<u>Lemma 2.16</u>: If $m\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, then each m_i is given by computing the Fox derivative $\frac{\partial m}{\partial x_i}$ and evaluating it modulo M'.

<u>Proof</u>: Lyndon and Fox [3],[4],[5].

Hence the m_i are the abelianized Fox derivatives of the $x_i \in m$. We give an example to illuatrate this type of computation.

If $m = x_3[x_1,x_2] = x_3x_1x_2x_1^{-1}x_2^{-1}$, then

$$m_1 = \overline{\frac{\partial m}{\partial x_1}} = \bar{x}_3(1 - \bar{x}_2)$$

$$m_2 = \overline{\frac{\partial m}{\partial x_2}} = \bar{x}_3(\bar{x}_1 - 1)$$

$$m_3 = \overline{\frac{\partial m}{\partial x_3}} = 1$$

Proposition 2.17: There is an algorithm for determining whether a given element of $\mathfrak{M}$ belongs to $M\mu$ or not.

Proof: Let $y = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in \mathfrak{M}$. Then $\alpha = \bar{x}_1^{\epsilon_1} \dots \bar{x}_n^{\epsilon_n} \in A$ and corresponds to $g = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ under the homomorphism $M \to A$.

$y(g^{-1})\mu \in M\mu$ if and only if $y \in M\mu$, thus we may restrict ourselves to deciding whether matrices of the form $y = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ lie in $M'\mu$.

Hence we focus our attention on the entry $\beta = \sum_i \sum_j n_{ij}a_{ij}t_i$.

Now M' is generated by products of the form $[x_i,x_j]^n$, hence the matrices of $M'\mu$ are characterized by the fact that the upper right hand entry lies in the $Z(A)$ submodule of T generated by $(1 - x_n)t_i + (x_i - 1)t_j$. Observe that it does not happen that the coefficient of only one t_i is non-zero for matrices in $M'\mu$, and that if $\beta = \sum_{i=1}^{N} \sum_j n_{ij}a_{ij}t_i$ does come from $M'\mu$, the $\sum_j n_{ij} = 0$ for each $i = 1,\dots,n$.

Now with $\beta = \sum_{i=1}^{N} \sum_j n_{ij}a_{ij}t_i$, we may, by the above, assume that

$\sum_j n_{ij} = 0$ for each $i = 1,\dots,n$. Take i smallest such that $\sum_j n_{ij}a_{ij} \neq 0$. Rewrite $\sum n_{ij}a_{ij}$ as an element of $Z(\bar{x}_{i+1},\dots,\bar{x}_n)$ with coefficients from the abelian group $(\bar{x}_1,\dots,\bar{x}_i)$.

$$\sum_j n_{ij}a_{ij} = \sum \bar{r}_{ij}z_{ij} ,$$

$$\bar{r} \in (\bar{x}_1,\dots,\bar{x}_i) \ , \ z_{ij} \in Z(\bar{x}_{i+1},\dots,\bar{x}_n) .$$

In z_{ij}, if we replace each $\bar{x}_{i+1},\dots,\bar{x}_n$ by 1 ; z_{ij} must be zero for $y \in M\mu$.

If so, we can rewrite $z_{ij} = \sum_{k=i+1}^{n} \bar{b}_{ik}(1 - \bar{x}_k)$, $\bar{b}_{jk} \in Z(\bar{x}_{i+1},\dots,\bar{x}_n)$.

Now $\beta_i = \sum_{i,j,k} \bar{r}_{ijk}\ \bar{b}_{ijk}[(1 - \bar{x}_k)t_i + (\bar{x}_i - 1)t_k]$ is the upper right hand entry coming from a product of conjugates (by elements of M which correspond to $\bar{r}_{ijk}\ \bar{b}_{ijk}$)of commutators $[x_i,x_k]$, $k > i$.

Replace β by $\beta - \beta_i = \beta^*$. We have now a new element β^* with coefficient of $t_1,\dots,t_i$ equal to zero. Successive moves of this type then yeild our result. We illustrate this procedure:

i) Consider the following element of $\mathcal{M}$,

$$y = \begin{pmatrix} \bar{a}\bar{b} & \begin{array}{l} [1 + \bar{a}\bar{b}(1 - \bar{c})t_a \\ [\bar{a}(1 - \bar{c}) + \bar{a}]t_b \\ [\bar{a}(\bar{b} - 1) + \bar{a}\bar{b}(\bar{a} - 1)t_c \end{array} \\ 0 & 1 \end{pmatrix}$$

$$a\mu = \begin{pmatrix} \bar{a} & t_a \\ 0 & 1 \end{pmatrix} \ , \ b\mu = \begin{pmatrix} \bar{b} & t_b \\ 0 & 1 \end{pmatrix} \ , (ab)^{-1}\mu = \begin{pmatrix} \bar{a}\bar{b} & -\bar{b}^{-1}t_b - (\bar{a}\bar{b})^{-1}t_a \\ 0 & 1 \end{pmatrix}$$

$$y(ab)^{-1}\mu = \begin{pmatrix} 1 & [1 + \bar{a}\bar{b}(1 - \bar{c}) - 1]t_a \\ & [\bar{a}(1 - \bar{c}) + \bar{a} - \bar{a}]t_b \\ & [\bar{a}(\bar{b} - 1) + \bar{a}\bar{b}(\bar{a} - 1)]t_c \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & [\bar{a}\bar{b}(1 - \bar{c})]t_a \\ & [\bar{a}(1 - \bar{c})]t_b \\ & [\bar{a}(\bar{b} - 1) + \bar{a}\bar{b}(\bar{a} - 1)]t_c \\ 0 & 1 \end{pmatrix}$$

$\beta_a = \bar{a}\bar{b}[(1 - \bar{c})t_a + (\bar{a} - 1)t_c$, which corresponds to $([c,a]^{ab})\mu$.

$\beta - \beta_a = \bar{a}(1 - \bar{c})t_b + \bar{a}(\bar{b} - 1)t_c$

$\beta_b = \bar{a}[(1 - \bar{c})t_b + (\bar{b} - 1)t_c]$, which corresponds to $[c,b]^a\mu$.

$\beta - \beta_a - \beta_b = 0$

Therefore $y \in M\mu$ and $y\mu^{-1} = [b,c]^a[a,c]^{ab}ab$

$= a[b,c]b[a,c]$.

ii) Consider

$$y = \begin{pmatrix} 1 & [\bar{a}(\bar{b} - 1)]t_a \\ & [1 - \bar{a}]t_b \\ 0 & 1 \end{pmatrix}$$

$$\beta = \bar{a}(\bar{b} - 1)t_a + (1 - \bar{a})t_b$$

$$\beta_a = \bar{a}[(\bar{b} - 1)t_a + (1 - \bar{a})t_b]$$

$$\beta - \beta_a = (1 - \bar{a} - \bar{a} + \bar{a}^2)t_b$$

hence $y \notin M\mu$.

Lemma 2.18: Conjugates have the following form under μ .

If $m,n \in M$, $m\mu$ and $n\mu$ as above ,

$$m^n\mu = \begin{pmatrix} \bar{m} & \bar{n}\left(\sum_{i}^{N} m_i t_i\right) + \left(1 - \bar{m}\right)\sum_{i}^{N} n_i t_i \\ 0 & 1 \end{pmatrix} .$$

Corollary 2.19: If $m \in M'$, then

$$m^n\mu = \begin{pmatrix} 1 & \bar{n}\sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix} , \text{ and}$$

$$m^{n_1}m^{n_2}\mu = \begin{pmatrix} 1 & (\bar{n}_1 + \bar{n}_2)\left(\sum_{i=1}^{N} m_i t_i\right) \\ 0 & 1 \end{pmatrix}$$

Proposition 2.20: If $r \in M'$, then

$$\left(nm_M(r)\right)\mu = \left(\begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix} \middle| \xi \in Z(A)\right)$$

where $\sum_{i=1}^{N} r_i t_i$ is the upper right entry of $r\mu$.

Proof: $nm_M(r) = \left(\left\{r^g \middle| g \in M\right\}\right)$.

Lemma 2.19 gives that $r^g\mu = \begin{pmatrix} 1 & \bar{g}\left(\sum_{i=1}^{N} (r_i t_i)\right) \\ 0 & 1 \end{pmatrix}$ and products of conjugates have the form $\begin{pmatrix} 1 & \xi\left(\sum r_i t_i\right) \\ 0 & 1 \end{pmatrix}$, $\xi \in Z(A)$.

Therefore $\left(nm_M(r)\right)\mu \le \left\{ \begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix} \middle| \xi \in Z(A) \right\}$. Since the map μ is not onto, we need the following lemma to show that $\left(nm_M(r)\right)\mu \ge \left\{ \begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix} \middle| \xi \in Z(A) \right\}$.

Lemma 2.21: Let $\begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix} \in \mathcal{M}$ where $r\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} r_i t_i \\ 0 & 1 \end{pmatrix}$.

Then $\exists$ elements of M, $m_1, \dots, m_k$, such that $\left(r^{m_1} \dots r^{m_k}\right)\mu = \begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix}$

Proof: Let ξ be expressed in normal form, i.e. $\xi = \sum_{j=1}^{k} n_j \bar{a}_j$ where $\bar{a}_j \in A$. Let $m_j \in M$ such that $\bar{m}_j = \bar{a}_j$ and $\epsilon_j = +1$ if $n_j > 0$ and -1 if $n_j < 0$. Then

$$\Big(\underbrace{r^{\epsilon_1 m_1} \dots r^{\epsilon_1 m_1}}_{|n_1| \text{ terms}} \dots \underbrace{r^{\epsilon_k m_k} \dots r^{\epsilon_k m_k}}_{|n_k| \text{ terms}}\Big)\mu = \begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix}.$$

Lemma 2.22: Powers have the following form under μ .

If $m \in M$ and $m\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, then

$$m^n\mu = \begin{pmatrix} \bar{m}^n & \left(\bar{m}^{\,n-1} + \dots + \bar{m} + 1\right) \sum_{i=1}^{n} m_i t_i \\ 0 & 1 \end{pmatrix} .$$

In particular if $m \in M'$ and $m\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, then

$$m^n\mu = \begin{pmatrix} 1 & n\sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix} .$$

<u>Lemma 2.23</u>: Let $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in \mathcal{M}$, then $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} \alpha^n & \left(\alpha^{n-1} + \dots + \alpha + 1\right)\beta \\ 0 & 1 \end{pmatrix}$

(2.24) If $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \mathcal{M}'$, $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n\beta \\ 0 & 1 \end{pmatrix}$.

<u>Definition</u>: We will refer to 2.24 as the <u>special power form</u> in $\mathcal{M}$.

<u>Proposition 2.25</u>: Suppose $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in \mathcal{M}$ is a power in $\mathcal{M}$, say $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}^n$. Then $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in M\mu \iff \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in M\mu$.

<u>Proof</u>: If $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = m \in M$, then the isomorphism $\mu : m^n \longrightarrow \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Hence $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in M\mu$.

Suppose $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in M\mu$, and $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n\beta \\ 0 & 1 \end{pmatrix}$. Since

$n\beta$ represents the abelianized Fox derivative of a word in M repeated n times, one can easily compute that word m such that $m^n\mu = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $m\mu = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$.

<u>Definition 2.26</u>: Let $\mathcal{M}/\mathcal{M}_{p^m}$ be a matrix group whose elements are $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$ where $\alpha \in A_{p^m}$ and $\beta \in$ the module over $Z_{p^m}(A_{p^m})$ with basis $t_1,\dots,t_n$.

With $M = ((x_1,\dots,x_n))$, let M_{p^m} denote the normal closure in M of the p^mth powers of the generators x_i . Let Ω_{p^m} be the canonical projection

$$\Omega_{p^m} : M \longrightarrow M/M_{p^m} .$$

With $M\mu = \left(\left(\begin{pmatrix} \bar{x}_1 & t_1 \\ 0 & 1 \end{pmatrix},\dots,\begin{pmatrix} \bar{x}_n & t_n \\ 0 & 1 \end{pmatrix}\right)\right)$, let $\tilde{\Omega}_{p^m} : M\mu \longrightarrow \mathcal{M}/\mathcal{M}_{p^m}$ be the obvious homomorphism.

There is an induced map $\bar{\mu} : M/M_{p^m} \longrightarrow \mathcal{M}/\mathcal{M}_{p^m}$, such that the following diagram commutes.

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[d, "\Omega_{p^m}"] & M\mu \arrow[d, "\tilde{\Omega}_{p^m}"] \\
M/M_{p^m} \arrow[r, "\bar{\mu}"] & \mathcal{M}/\mathcal{M}_{p^m}
\end{tikzcd}$$

The existence of $\bar{\mu}$ and the commutativity of the diagram follows from the fact that $x_i^{p^m} \to 1$ in $\mathfrak{M}/\mathfrak{M}_{p^m}$.

$\bar{\Omega}_{p^m}$ is defined on $\begin{pmatrix} \bar{x}_i & t_i \\ 0 & 1 \end{pmatrix}$ and the action of $\bar{\Omega}_{p^m}$ on $M\mu$ is defined on $Z(A)$ as follows:

$$\bar{a} \in A, \bar{a}^{\,p^m} = 1$$

$$\xi \in Z(A), p^m \xi = 0 \text{ . Hence,}$$

$$\begin{pmatrix} \bar{x}_i & t_i^{p^m} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{x}_i^{\,p^m} & (\bar{x}_i^{\,p^m-1} + \dots + \bar{x}_i + 1)t_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

since $\bar{x}_i^{\,p^m} = 1$ and $\bar{x}_i^{\,p^m-1} + \dots + \bar{x}_i + 1 = \dfrac{\bar{x}_i^{\,p^m} - 1}{\bar{x}_i - 1} = 0$.

Section 4: A "Reduced" Word in the Free Metabelian Group.

Definition 2.27: $m \in M'$, m is reduced if in $m\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, the m_i are strongly relatively prime; i.e., the strong greatest common divisor of the m_i is 1 .

The purpose of the condition in the Freiheitssatz that a word be cyclically reduced is to guarantee that a word in the free group is not the conjugate of a shorter word. The concept of a reduced word in the metabelian group plays an analogous role in our principal theorems.

Definition 2.28: If m is reduced, $m \in M', m\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, we say x_n a generator of M is *involved* in m if

i) $m_n \neq 0$ or

ii) $\bar{x}_n$ appears in m_j , that is, if $m_j = \sum_{\substack{n_i \neq 0 \\ a_i \neq a_j, i \neq j}} n_i a_i$,

then there exists an i such that in $a_i = \bar{x}_1^{\epsilon_{i_1}} \ldots \bar{x}_n^{\epsilon_{i_n}}$, $\epsilon_{i_n} \neq 0$.

Proposition 2.29: If $m \neq 0$, m reduced, $m \in M'$, and x_n is involved in m , then x_n is involved in every element of $nm_M(m)$.

Proof: Let $m\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$. Then $g \in nm_M(m)$ is such that

$$g\mu = \begin{pmatrix} 1 & \xi\left(\sum_{1}^{N} m_i t_i\right) \\ 0 & 1 \end{pmatrix}, \xi \in Z(A) .$$

i) If $m_n \neq 0$ then $\xi m_n \neq 0 \ \forall \ \xi \neq 0 \in Z(A)$, since $Z(A)$ has no zero divisors.

ii) If $\bar{x}_n$ appears in m_j for some j , then $\bar{x}_n$ appears in some ξm_j for any ξ .

Since m is reduced, the strong greatest common divisor of $\{m_j\}$ is 1 . Therefore with $m_j = \sum_{n_{i,j}} n_{i,j} \, a_{i,j}$, $\bar{x}_n$ does not appear in all $a_{i,j}$. We view these m_j as elements of the group ring $R(\bar{x}_n)$, where R is the quotient field of the group ring $Z(B)$, B the free abelian group on $\bar{x}_1,\ldots,\bar{x}_{n-1}$. The units of $R(\bar{x}_n)$ are the elements $r\bar{x}_n^{\ell}$, $r \in R$. Thus for some $\xi \in Z(A)$, $\bar{x}_n$ does not appear in any ξm_i , then the ξm_i are units in $R(\bar{x}_n)$. Hence $\xi = \rho\bar{x}_n^{-\ell}$, $m_i = \rho_i\bar{x}_n^{-\ell}$ and therefore in $Z(A)$, $m_i = \tilde{m}_i\bar{x}_n^{\ell}$, $\tilde{m}_i \in Z(B)$ which is a contradiction.

CHAPTER 3

FREE PROPERTIES OF SINGLE RELATOR METABELIAN GROUPS

Section 1: The Freiheitssatz for Metabelian Groups, $G = ((x_1,\dots,x_n;r)), n \geq 3, r \in M'$.

Theorem 3.1:

Let $G = ((x_1,\dots,x_n;r)), n \geq 3, r \in M'$, r reduced and x_n involved in r . Then the subgroup $H = (x_1,\dots,x_{n-1})$ of G is a free metabelian group, freely generated by $x_1,\dots,x_{n-1}$.

Proof: $G = M/_{nm_M(r)} = M/_R$ where $M = ((x_1,\dots,x_n))$.

$$R\mu = \left\{ \begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix} \middle| \xi \in Z(A) \quad \text{and} \quad r\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} r_i t_i \\ 0 & 1 \end{pmatrix} \right\}.$$

If x_n is involved in r , then by Proposition 2.29, x_n is involved in every element of $nm_M(r)$.

H is the subgroup generated by $x_1,\dots,x_{n-1}$. Let $N = ((x_1,\dots,x_{n-1}))$. Then $H = N/_{N \cap R}$.

$$N\mu = \begin{pmatrix} \bar{n}^{\epsilon} & \sum_{i=1}^{N-1} n_i t_i \\ 0 & 1 \end{pmatrix}$$

Therefore $N\mu \cap R\mu = 1$,

Therefore $N \cap R = 1$,

Therefore $H = N/_{N \cap R} \cong N$, a free metabelian group, freely generated by $x_1,\dots,x_{n-1}$.

3.2 Generalizations of the Freiheitssatz.

In order to prove the Freiheitssatz for metabelian groups when $r \in M'$, we had to place two conditions on r, both of which could only be checked by the representation map μ.

This is probably the best possible result because we have no unique way of expressing a word $m \in M = ((x_1,\ldots,x_n)), n \geq 3$. But even though the test is not given in terms of some operations within the group itself, the situation is not unfortunate. Most large and important classes of one relator metabelian groups do satisfy the conditions imposed by the map μ.

In particular for the class of fundamental groups of two dimensional orientable manifolds, modulo the second derived group; i.e.,

$$G = ((x_1,y_1,\ldots,x_n,y_n;[x_1,y_1]\ldots[x_n,y_n])) ,$$

μ shows r to be reduced and each generator is involved in r, so the Freiheitssatz holds.

Section 2: The Cases where there is no Freiheitssatz.

Theorem 3.3:

Let $G = ((x_1,\ldots,x_n)), n \geq 3, r \notin M'$. Let $r = \tilde{r}[r]$, where $\tilde{r}$ is the content of r and $[r] \in M'$. Suppose x_n has zero exponent in the content of r, but x_n is involved in $[r]$, according to Definition 2.28. Then the subgroup $H = ((x_1,\ldots,x_{n-1}))$ of G is not free metabelian.

Proof: Let $M = ((x_1,\ldots,x_n))$, $G = M/_R$ where $R = nm_M(r)$.

Let $N = ((x_1,\ldots,x_{n-1}))$. $H \cong N/_{N \cap R}$. We show $N \cap R \neq 1$.

Choose $n = [x_1, x_2] \in N'$. Then we claim $[r,n] \neq 1$ and $[r,n] \in N \cap R$.

$$[r,n] = [\tilde{r}[r],n] = [[r,n]^{\tilde{r}}[\tilde{r},n] = [\tilde{r},n] \in N .$$

$$[r,n] = r(r^{-1})^n \in R \text{ , hence } [r,n] \in N \cap R .$$

To see that $[r,n] = [\tilde{r},n] \neq 1$, we apply the map μ.

$$n\mu = [x_1,x_2]\mu = \begin{pmatrix} 1 & (1 - \bar{x}_2)t_1 + (\bar{x}_1 - 1)t_2 \\ 0 & 1 \end{pmatrix} .$$

$$\tilde{r} = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_{n-1}^{\epsilon_{n-1}}$$

$$\tilde{r}\mu = \begin{pmatrix} \bar{x}_1^{\epsilon_1} \ldots x^{\epsilon_{n-1}} & \begin{array}{l} (1 + \bar{x}_1 + \ldots + \bar{x}_1^{\epsilon_1 - 1})t_1 \\ \bar{x}^{\epsilon_1}(1 + \ldots + \bar{x}^{\epsilon_2 - 1})t_2 + \ldots \end{array} \\ 0 & 1 \end{pmatrix}$$

$$[\tilde{r},n]\mu = \begin{pmatrix} 1 & \begin{array}{l} [(\tilde{r} - 1)(1 - \bar{x}_2) - (1 + \bar{x}_1 + \ldots + \bar{x}_1^{\epsilon_1 - 1})]t_1 + \\ [(\tilde{r} - 1)(\bar{x}_1 - 1) - (\bar{x}_1^{\epsilon_1}(1 + \ldots + \bar{x}_2^{\epsilon_2 - 1})]t_2 + \ldots \end{array} \\ 0 & 1 \end{pmatrix}$$

It is easy to see that the entry in the upper right corner of this matrix can never be 0. In fact, it suffices to show the coefficient of t_1 cannot be 0. If in $\tilde{r} = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_{n-1}^{\epsilon_{n-1}}$, $\epsilon_i \neq 0, i \geq 3$, the statement is obvious. So consider

$$-(1 - x_2 - x_1^{\epsilon_1} x_2^{\epsilon_2} + x_1^{\epsilon_1} x_2^{\epsilon_2+1}) - (1 + x_1 + \dots + x_1^{\epsilon_1 -1}) \neq 0$$

for any ϵ_1 or ϵ_2 . It is clear that the constant term is either -2 or -3 .

Remark: Suppose $G = ((x_1,\dots,x_n;r))$ and x_n has non-zero exponent in the content of r . If $H = ((x_1,\dots,x_n))$ we cannot decide if $H \cap R = 1$, by these methods.

The following is intended only to show the nature of the problems that arise. If $r \notin M'$, elements of $R = \{r^m | m \in M\}$ are mapped by μ into elements of $M\mu$ of the form

$$\begin{pmatrix} \bar{r}^{\,k} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$$

where $k = 0, \pm 1, \pm 2$ and the m_i are terms in $Z(A)$. There is no "nice" form for the terms m_i . If $k \neq 0$, i.e. for elements of $R \backslash R'$ we observe that $H \cap R \backslash R' = 1$. But when $k = 0$ it cannot be determined if $H \cap R' = 1$.

There is no possibility that there can be a unique way of expressing $g \in R$ in terms of r^m for $m \in M$.

$R \subset M$ is the subgroup of the free metabelian group, but R is not free metabelian. G. Baumslag has shown that a subset of a free metabelian group generates a free metabelian subgroup if and only if the subset generates a free abelian group module M' [9].

3.4: The Case $n = 2$.

Let $M = ((x_1,x_2))$. Then M' = normal subgroup generated by

$[x_1,x_2]$ and every $g \in M'$ can then be expressed as $[x_1,x_2]^{\xi}$ where $\xi \in Z(A)$. If we require that $r \in M'$ for $M = ((x_1,x_2))$, be reduced, $r = [x_1,x_2]$ and $G = ((x_1,x_2;r))$ is the free abelian group. If r is not reduced, the above methods are not applicable. It will be assumed in the sequel that $n \geq 3$ except where noted.

CHAPTER 4

FURTHER PROPERTIES OF SINGLE RELATOR METABELIAN GROUPS

The map μ can be used to prove several other theorems concerning single relator metabelian groups.

Section 1: The Extended Word Problem.

A solution to the word problem for single relator metabelian groups, when the relator lies in the derived group.

Theorem 4.1:

One can decide, in a finite number of steps, whether a word $m \in M$ is in $R = nm_M(r)$ where $r \in M'$, r is reduced.

Proof: Compute $r\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} r_i t_i \\ 0 & 1 \end{pmatrix}$.

$$R\mu = \left\{ \begin{pmatrix} 1 & \xi\left(\sum_{i=1}^{N} r_i t_i\right) \\ 0 & 1 \end{pmatrix} \middle| \xi \in Z(A) \right\} .$$ Compute $m\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$.

If $m\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $m = 1$ in M , hence $m \in R$.

If $m \neq 1, m \in R \Leftrightarrow m\mu \in R\mu \Leftrightarrow \bar{m} = 1$ and $\exists \xi \in Z(A)$ such that $m_i = \xi r_i V_i$. (Proposition 2.20)

To determine such a ξ , if it exists, we proceed as follows: Compute for $r\mu$ and the given $m\mu$, $\{u_{r_i}\}$ and $\{u_{m_i}\}$ as in lemma 1.18.

Then u_{r_i} , u_{m_i} , $u_{r_i} \cdot r_i$ and $u_{m_i} \cdot m_i$ are elements of $Z[\bar{x}_1, \ldots, \bar{x}_n]$.

Compute the greatest common divisor of $\left\{u_{m_i}\cdot m_i\right\}$ and of $\left\{u_{m_i}\right\}$. Then compute the strong greatest common divisor ξ of $\{m_i\}$. This gives us $m_i = \xi n_i$ for all i. We check to see that $n_i = \pm r_i \forall_i$.

Section 2: Conjugacy Problems in Single Relator Metabelian Groups.

Part 1: Conjugacy in M.

Theorem 4.2:

There is an algorithm for determining whether two words, m and $n \in M$, are conjugate in M.

Proof: (1) If $m,n \in M'$, we compute $m\mu$ and $n\mu$. Suppose

$$m\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}, \quad n\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} n_i t_i \\ 0 & 1 \end{pmatrix}.$$

Now $m = n^g$ if and only if $m_i = \bar{g} n_i \forall_i$, $\bar{g} \in A$. That is, m_i and n_i differ by the same unit for all i.

To determine if m_i and n_i differ by the same unit and to determine that unit, we proceed as follows.

Compute $\left\{u_{n_i}\right\}$ and $\left\{u_{m_i}\right\}$. In $Z[\bar{x}_1,\ldots,\bar{x}_n]$, we seek $\bar{g}$ such that $(u_{\bar g})(u_{n_i})(u_{m_i} m_i) = (u_{\bar g}\bar{g})(u_{m_i})(u_{n_i} n_i)$. Solving formally

$$\frac{\left(u_{\bar g}\bar{g}\right)}{u_{\bar g}} = \frac{\left(u_{m_i} m_i\right)\left(u_{n_i}\right)}{\left(u_{n_i} n_i\right)\left(u_{m_i}\right)} = \frac{\left(u_{m_i} m_i\right)\left(u_{m_i}\right)^{-1}}{\left(u_{n_i} n_i\right)\left(u_{n_i}\right)^{-1}}$$

We observe that for m and n to be conjugate, the ratios $\dfrac{\left(u_{m_i} m_i\right)\left(u_{m_i}\right)^{-1}}{\left(u_{n_i} n_i\right)\left(u_{n_i}\right)^{-1}}$ must be the same monomials or the same units in $Z(A)$.

(This is stronger than requiring that $\{u_{m_i} m_i\}$ and $\{u_{n_i} n_i\}$ be associates in $Z(A)$.)

Recall our definition of strong greatest common divisor:

$$\mathrm{stgcd}\{m_i\} = \Delta_{m_i} \delta_{m_i}^{-1}$$

where $\Delta_{m_i} = \gcd\{u_{m_i} m_i\}$ and $\delta_{m_i} = \gcd\{u_{m_i}\}$. Therefore we compute Δ_{m_i} , δ_{m_i} , Δ_{n_i} , and δ_{n_i} in $Z[x_1,\dots,x_n]$. Then

$$\bar{g} = \frac{u_g \bar{g}}{u_{\bar{g}}} = \frac{\Delta_{m_i} \delta_{m_i}^{-1}}{\Delta_{n_i} \delta_{n_i}^{-1}} = \frac{\mathrm{stgcd}\{m_i\}}{\mathrm{stgcd}\{n_i\}} \quad \text{in } Z(A) .$$

(2) If $m,n \notin M'$ and $\bar{m} \neq \bar{n}$, then m and n cannot be conjugate.

Let $m\mu = \begin{pmatrix} \bar{m} & \sum_{i=1}^{N} m_i t_i \\ 0 & 1 \end{pmatrix}$, $n\mu = \begin{pmatrix} \bar{n} & \sum_{i=1}^{N} n_i t_i \\ 0 & 1 \end{pmatrix}$, $s\mu = \begin{pmatrix} \bar{s} & \sum_{i=1}^{N} s_i t_i \\ 0 & 1 \end{pmatrix}$.

$$m^s\mu = \begin{pmatrix} \bar{m} & \bar{s}\left(\sum_{i=1}^{N} m_i t_i\right) + \left(1 - \bar{m}\right)\left(\sum_{i=1}^{N} s_i t_i\right) \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} \bar{n} & \sum_{i=1}^{N} n_i t_i \\ 0 & 1 \end{pmatrix} \quad \text{when}$$

$\bar{m} \neq \bar{n}$.

(3) If $m,n \notin M'$ and $\bar{m} = \bar{n}$, a test for conjugacy is developed in the following set of observations.

Suppose $\exists\, \alpha \in \mathcal{M}, \alpha = \begin{pmatrix} \bar{a} & \sum_{i=1}^{N} \alpha_i t_i \\ 0 & 1 \end{pmatrix}$, such that $(m\mu)^{\alpha} = n\mu$.

$(m\mu)^{\alpha} = n\mu \Leftrightarrow$

$$\begin{pmatrix} \bar{m} & (1 - \bar{m})\left(\sum_{i=1}^{N} \alpha_i t_i\right) + \bar{a}\left(\sum_{i=1}^{N} m_i t_i\right) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{n} & \sum_{i=1}^{N} n_i t_i \\ 0 & 1 \end{pmatrix}$$

$\Leftrightarrow \bar{m} = \bar{n}$ and $*(1 - \bar{m})\alpha_i + \bar{a}(m_i) = n_i \forall_i$.

Therefore m and n are conjugate in M if and only if

i) we can solve $*$ for $\bar{a} \in A$ and $\alpha_i \in Z(A)$ and

ii) any solution $\{\bar{a}, \alpha_i\}$, belongs to $M\mu$.

To find $\bar{a}$, we suppose further that $\exists$ a $\in$ M such that $m^a = n$, or $(m^a)\mu = n\mu$.

Let a be given as $a = \tilde{a}[a]$, where $\tilde{a} = x_1^{\eta_1} \ldots x_n^{\eta_n}$ is the content of a , lexiographically ordered.

Suppose further that $ama^{-1} = n$ is given in exactly this arrangement. We recall that since μ is an isomorphism, any conclusion we make about $(ama^{-1})\mu$ based on the particular arrangement of ama^{-1} is true for any word $n = ama^{-1}$.

We compute $(ama^{-1})\mu = n\mu$ by computing $(\tilde{a}[a]\tilde{m}[m][a]^{-1}\tilde{a}^{-1})\mu = n\mu$. Suppose η_j is the last non-zero exponent in $\tilde{a} = x_1^{\eta_1} \ldots x_n^{\eta_n}$.

If $\eta_j > 0$, then n_j, the coefficient of t_j is given by

$$n_j = \tilde{a}\bar{x}_j^{-\eta_j}(1 + \ldots + \bar{x}_j^{\eta_j - 1}) +$$

$$\tilde{a}\,\frac{\overline{\partial[a]}}{\partial x_j} + \tilde{a}\,\frac{\overline{\partial\tilde{m}}}{\partial x_j} + \tilde{a}\tilde{m}\,\frac{\overline{\partial m}}{\partial x_j} + \tilde{a}\tilde{m}\,\frac{\overline{\partial[a]^{-1}}}{\partial x_j} - \tilde{m}\,.$$

If $\eta_j < 0$, $n_j = \tilde{\bar{a}}x_j^{-\eta_j}(x_j^{-1} + \ldots + x_j^{\eta_j}) + \tilde{a}\,\frac{\overline{\partial[a]}}{\partial x_j} + \tilde{a}\,\frac{\overline{\partial\tilde{m}}}{\partial x_j}$

$$+ \tilde{a}\tilde{m}\,\frac{\overline{\partial m}}{\partial x_j} + \tilde{a}\tilde{m}\,\frac{\overline{\partial[a]^{-1}}}{\partial x_j} - \tilde{m}\,.$$

In either case $n_j \neq 0$ since not all the terms are comparable. So if we let $n_j = \sum_k u_{jk}$, where $u_{jk} = x_1^{\epsilon_{jk_1}} \ldots x_n^{\epsilon_{jk_n}}$ then since $\bar{\tilde{a}} = \bar{a}$, $\bar{a}$ either equals $u_{jk}x_k$ or $\bar{a} = x_1^{d_1} \ldots x_n^{d_n}$ where $|d_i| \leq |\epsilon_{jk_i}|$ for some k.

Thus given $\{n_j\}$ there are a finite number of possibilities for the term $\bar{a}$. We check whether for any such $\bar{a}$, $1 - \bar{m} \mid n_i - \bar{a}m_i$, yielding $\{\alpha_i\} \in Z(A)$.

Finally we check that the set of $\alpha_i = \dfrac{n_i - \bar{a}m_i}{1 - \bar{m}}$ are such that

$$\begin{pmatrix} \bar{a} & \sum_{i=1}^{N} \alpha_i t_i \\ 0 & 1 \end{pmatrix} \mu^{-1} \in M\,.$$

To summarize, we have the following algorithm to determine if m and n are conjugate when $m,n \notin M'$.

Theorem 4.3:

$m,n \notin M'$, m and n are conjugate in M if and only if

i) $\bar{m} = \bar{n}$

ii) In $m\mu$ and $n\mu$ where $n_j = \sum_k u_{jk}$ and $u_{jk} = \bar{x}_1^{\epsilon_{jk_1}} \dots \bar{x}_n^{\epsilon_{jk_n}}$, we consider all possible $\bar{a} = \bar{x}_1^{d_1} \dots \bar{x}_n^{d_n}$ where $|d_i| \le |\epsilon_{jk_i}| + 1$.

Then $1 - \bar{m}$ must divide $n_i - \bar{a}m_i$ for all i for some $\bar{a}$ so determined.

iii) If $\alpha_i = \dfrac{n_i - \bar{a}m_i}{1 - \bar{m}}$, $\begin{pmatrix} \bar{a} & \sum_{i=1}^{N} \alpha_i t_i \\ 0 & 1 \end{pmatrix}$ must lie in $M\mu$.

Part 2: The Extended Conjugacy Problem.

For groups with a single defining relator, the extended conjugacy problem is known only in isolated cases, [7 ,p.400-401]. For metabelian groups with a single defining relator, where the relator lies in the derived group, there is an algorithm for determining whether two words in the derived group are conjugate.

Theorem 4.4:

If $a,b \in G'$ where $G = ((x_1,\dots,x_n;r))$, $r \in G'$, r reduced, we can determine if a and b are conjugate and we can determine the conjugating element.

Proof: Again apply the map μ . Suppose

$$r\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} r_i t_i \\ 0 & 1 \end{pmatrix}, \quad a\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} a_i t_i \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad b\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} b_i t_i \\ 0 & 1 \end{pmatrix} .$$

Suppose also $\exists s \in G$, $s\mu = \begin{pmatrix} \bar{s} & \sum_{i=1}^{N} s_i t_i \\ 0 & 1 \end{pmatrix}$ such that $a^s = b$. Then

$$a^s b^{-1} = 1$$

if and only if

$$\left(a^s b^{-1}\right)\mu = \begin{pmatrix} 1 & \bar{s}\left(\sum_{i=1}^{N} a_i t_i\right) - \left(\sum_{i=1}^{N} b_i t_i\right) \\ 0 & 1 \end{pmatrix}$$

is in $R\mu$. This means $\exists\ \xi \in Z(A)$ such that

$$* \qquad \bar{s}\left(a_i\right) - b_i = \xi r_i V_i \ .$$

We can determine by observation if there exists $\bar{s}$ such that $\bar{s}\left(a_i\right) - b_i = 0$. Further, since $*$ is a set of n equations in two unknowns over $Z(A)$, we know solutions for $\bar{s}$ and ξ are unique if they exist.

$Z(A)$ has been shown to be a unique factorization domain. So solutions may be found by Craemer's rule in a quotient field over $Z(A)$. Solutions will then have the form $\sigma/_\tau$, $\sigma,\tau \in Z(A)$, σ,τ relatively prime. Such a solution is an element of $Z(A)$ if and only if τ is a unit. Further, $\sigma/_\tau$ giving $\bar{s}$ must be a "monomial" or unit.

If $\bar{s},\xi$ are found so that $\bar{s}\left(a_i\right) - b_i = \xi\left(r_i\right)V_i$, a and b are conjugate. Further, $s = \tilde{s}g, g \in G'$ is an element such that

$$a^s = b \ .$$

<u>Section 3</u>: <u>The Center of Single Relator Metabelian Groups</u>.

<u>Theorem 4.5</u>:

Let $G = ((x_1,\dots,x_n;r)), n \geq 3, r \in M'$, r reduced. Then G has a trivial center.

<u>Proof</u>: Represent G via μ as usual. Let

$$r\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} r_i t_i \\ 0 & 1 \end{pmatrix} .$$

We will suppose G has a center and show this leads to a contradiction.

So let $z \in G, z \neq 1$ be such that $[z,a] = 1 \ \forall a \in G$. Recall that $z \neq 1$, where $z\mu = \begin{pmatrix} \bar{z} & \sum_{i=1}^{N} z_i t_i \\ 0 & 1 \end{pmatrix}$, means that in $z\mu$, either $\bar{z} \neq 1$

or if $\bar{z} = 1$, then it is not the case that $\exists\ \xi \in Z(A)$ such that $g_i = \xi r_i V_i$.

Let $a \in G, a\mu = \begin{pmatrix} \bar{a} & \sum_{i=1}^{N} a_i t_i \\ 0 & 1 \end{pmatrix}$. Then

$$[z,a]\mu = \begin{pmatrix} 1 & (\bar{z} - 1)\left(\sum_{i=1}^{N} a_i t_i\right) + (1 - \bar{a})\left(\sum_{i=1}^{N} z_i t_i\right) \\ 0 & 1 \end{pmatrix}. \quad [z,a] = 1 \text{ in } G$$

if and only if in $[z,a]\mu$,

$$* \ (\bar{z} - 1)\left(\sum_{i=1}^{N} a_i t_i\right) + (1 - \bar{a})\left(\sum_{i=1}^{N} g_i t_i\right) = \xi\left(\sum_{i=1}^{N} r_i t_i\right) \text{ for some } \xi \in Z(A) .$$

Suppose first that $z \notin G'$, thus $\bar{z} \neq 1$. Choose $a \in G', a \neq 1$. Then $\bar{a} = 1$ and $\sum_{i=1}^{N} a_i t_i$ is not a multiple of $\sum_{i=1}^{N} r_i t_i$ in $Z(A)$.

Equation * now reads

$$(\bar{z} - 1)\left(\sum_{i=1}^{N} a_i t_i\right) = \xi\left(\sum_{i=1}^{N} r_i t_i\right) ,$$

or

$$(\bar{z} - 1)a_i = \xi r_i \text{ for all } i .$$

But r is reduced, so $\bar{z} - 1$ divides ξ in $Z(A)$. Therefore, $a_i = \frac{\xi}{\bar{z}-1} r_i$, so $a = 1$ in G .

More explicitly we can construct $a \in G'$, $a \neq 1$, such that $[z,a] \neq 1$. Choose $r_m \neq 0$, any non-zero coefficient in $\sum_{i=1}^{N} r_i t_i$ which we know exists.

Let $a \in G'$ be chosen such that $a = x_m b x_m^{-1} b^{-1}$ where $b \in G'$ is any word in the generators not involving x_m. Observe that $\frac{\bar{\partial}_a}{\partial_{x_m}} = a_m = 0$. Hence $(\bar{z} - 1)a_m = 0 \ \forall \bar{z} \neq 1$. Since $r_m \neq 0$, $\xi r_m \neq 0$ for any ξ since Z(A) has no zero divisors. Therefore,

$$(\bar{z} - 1)a_m \neq \xi r_m \quad \text{and} \quad (\bar{z} - 1)\Big(\sum_{i=1}^{N} a_i t_i\Big) \neq \xi\Big(\sum_{i=1}^{N} r_i t_i\Big) .$$

Now suppose $z \in G'$, $z \neq 1$. $z \in G'$ means $\bar{z} = 1$ so $[z,a]\mu = \begin{pmatrix} 1 & (1 - \bar{a}) \sum_{i=1}^{N} z_i t_i \\ 0 & 1 \end{pmatrix}$. Since $z \in$ center of G, z must commute with all $a \in G$, in particular with $a \notin G'$ where $\bar{a} \neq 1$.

Equation * now reads

$$(1 - \bar{a})\Big(\sum_{i=1}^{N} z_i t_i\Big) = \xi\Big(\sum_{i=1}^{N} r_i t_i\Big) \quad \text{or}$$

$$(1 - \bar{a}) z_i = \xi r_i \quad \text{for all } i .$$

But r is reduced, so $1 - \bar{a}$ divides ξ in $Z(A)$. Therefore,

$z_i = \frac{\xi}{1-\bar{a}} r_i$, so $z = 1$ in G .

Section 4: Elements of Finite Order.

Theorem 4.6:

Let $G = ((x_1,\ldots,x_n;r)), n \geq 3, r \in G'$, r reduced. Then G has elements of finite order if and only if r is a proper power.

Proof: Via the representation μ .

$$\text{Let} \quad r\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} r_i t_i \\ 0 & 1 \end{pmatrix} .$$

If r is a power we show first that G has elements of finite order.

Suppose $r = s^n$, then $r\mu = (s^n)\mu = \begin{pmatrix} 1 & n\sum_{i=1}^{N} s_i t_i \\ 0 & 1 \end{pmatrix}$. We produce $g \in G, g \neq 1$, such that $g^n = 1$. Let $g = s$. Then $g\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} s_i t_i \\ 0 & 1 \end{pmatrix}$.

$g \neq 1$ because $\sum_{i=1}^{N} s_i t_i \neq 0$ and $\exists$ no $\xi \in Z(A)$ such that

$\sum_{i=1}^{N} s_i t_i = \xi\left(\sum_{i=1}^{N} r_i t_i\right)$. However, $g^n = r = 1$.

If r is not a proper power, we show G has no elements of finite order. If r is not a power, then $r\mu$ does not have special power form, i.e. in $r\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} r_i t_i \\ 0 & 1 \end{pmatrix}$, there exists no $n \neq 1, n \in Z$, such that $r_i = ns_i \forall_i$.

Suppose $\exists\ a \in G, a \neq 1$, such that $a^m = 1$. We show this assumption leads to a contradiction.

If $a \notin G', \bar{a} \neq m$, hence $\bar{a}^m \neq 1$ in A for any m , so $a^m \neq 1$ in M for any m .

If $a \in G', \bar{a} = 1$, so $a\mu$ has the form $a\mu = \begin{pmatrix} 1 & \sum_{i=1}^{N} a_i t_i \\ 0 & 1 \end{pmatrix}$ and

$a^m\mu = \begin{pmatrix} 1 & m\left(\sum_{i=1}^{N} a_i t_i\right) \\ 0 & 1 \end{pmatrix}$. $a \neq 1$, so $\sum_{i=1}^{N} a_i t_i$ is not a $Z(A)$ multiple of $\sum_{i=1}^{N} r_i t_i$; but $a^m = 1$ means $\exists\ \xi \in Z(A)$ such that $ma_i = \xi r_i \forall_i$.

But r_i is reduced, so m divides ξ in $Z(A)$, which implies that

$\sum_{i=1}^{N} a_i t_i = \left(\frac{\xi}{m}\right) \sum_{i=1}^{N} r_i t_i$ and so $a = 1$ in G .

Section 5: Residually Finite p .

Theorem 4.7:

If $G = ((x_1, \ldots, x_n; r)), r \in G'$, r reduced, $n \geq 3$, and r is not a proper power, then G is residually finite p for all primes p , i.e. $G \in \mathcal{R}\mathfrak{F}_p \ \forall_p$.

Proof: The proof is accomplished by demonstrating that for each $g \in G$, $g \neq 1$, a homomorphism

$$\rho_g : G \longrightarrow G_{p^n}$$

may be chosen so that $g\cdot\rho_g \neq 1$ where G_{p^n} is a finite p group.

We use the representation μ and the maps Ω_{p^m} and $\tilde{\Omega}_{p^m}$ as in Definition 2.26.

We have the following commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\mu"] \arrow[d, "\Omega_{p^m}"] & M\mu \arrow[d, "\tilde{\Omega}_{p^m}"] \\
M/M_{p^m} \arrow[r, "\bar{\mu}"] & (M/M_{p^m})^{\mu}
\end{tikzcd}$$

We shall choose Ω_{p^m} for g such that the matrices representing $g\mu$ and $g\mu\Omega_{p^m}$ and the matrices representing $r\mu$ and $r\mu\Omega_{p^m}$ are identical.

To choose Ω_{p^m} we proceed as follows: Suppose $m \in M$ is given as

$$m = x_{j_1}^{\epsilon_{j_1}} \cdots x_{j_k}^{\epsilon_{j_k}} ,$$

$x_{j_\ell} \in \{x_1, \dots, x_n\}$ the generators of M.

For each x_i appearing in m we compute $S_{x_i} = \sum_{j_\ell = i} |\epsilon_{j_\ell}|$ where ϵ_{j_ℓ} is an exponent of $x_i = x_{j_\ell}$. For m we compute $S_m = \sum_{i=1}^{N} \cdot S_{x_i}$.

Choose m so that p^m is greater than S_g or S_r.

Define Ω_{p^m} and $\tilde{\Omega}_{p^m}$ using the m so determined.

Proposition 4.8: $r\mu\Omega_{p^m} = r\Omega_{p^m}\bar{\mu} = r\mu$ and $r\mu\tilde{\Omega}_{p^m} = r\Omega_{p^m}\bar{\mu} = r\mu$.

Proof: Let $g\mu = \begin{pmatrix} \bar{g} & \sum_{i=1}^{N} g_i t_i \\ 0 & 1 \end{pmatrix}$.

By the choice of m, $\bar{g}$ has no exponent $\geq p^m$. Each g_i is the abelianized Fox derivative of g and each g_i is thus determined by the predecessors of the letter x_i (or the predecessors plus the letter itself if the exponent of x_i is negative.) By our choice of p^m no letter can have as many p^m predecessors. Hence in g_i there occurs no p^m exponents, no $p^m - 1$ exponents, and no p^m multiples of elements of $Z(A)$.

Similarly p^m is chosen for r and we can conclude $g\mu\tilde{\Omega}_{p^*} = g\mu$ and $r\mu\tilde{\Omega}_{p^*} = r\mu$. Therefore since Ω_{p^*} can be determined for any $g \in G$, we conclude $G \in {}_{\mathcal{R}}\mathfrak{F}_p \forall_p$.

CHAPTER 5

RESIDUALLY FREE METABELIAN GROUPS

One would hope that some of the above techniques could be used to show that $G = ((x_1,\ldots,x_n;r)), n \geq 3, r \in G'$, r reduced, is residually free metabelian.

Definition 5.1: A group G is denoted $G \in {}_R\mathfrak{F}\mathfrak{M}$, residually free metabelian, if for every element $g \in G, g \neq 1$, $\exists$ a homomorphism $\varphi_g : G \to M$, a free metabelian group, such that $\varphi_g(g) \neq 1$.

So far there is no general result.

G. Baumslag has shown that if a metabelian group is residually free metabelian, then it is residually two generator free metabelian,[9].

We remark that if $g \notin G'$, we can easily produce a map φ_g as above. However, the problems that arise for $g \in G'$ seem difficult.

The following sections employ techniques not used in the first part of this paper and are presented only as an outline of a likely method to be used to solve this problem. It will be applied in the one case in which a result has been obtained.

Section 1 : A Presentation for G' where $G = ((x_1,\ldots,x_n;r)), n \geq 3, r \in G'$.

In this section the Schreier-Reidemeister rewriting technique will be outlined and applied. [7]

Notation: Let $H = (x_1,\ldots,x_n;r)$ where $G = ((x_1,\ldots,x_n;r))$, for $n \geq 3$. Then $G = H/_{H''}$. Since $r \in G'$, $G/_{G'}$ is a finitely generated free abelian group generated by $\{x_1,\ldots,x_n\}$; $G/_{G'} = H/_{H''}\Big/{H'/_{H''}} \cong H/_{H'}$

$$G' = H'/_{H''}\ .$$

The Schreier-Reidemeister technique gives the generators $\{h_1, h_2, \dots\}$ and the relators of H', hence $\{h_1H'' = \bar{h}_1, h_2H'' = \bar{h}_2, \dots\}$ is the set of abelian generators of $H'/_{H''} \cong G'$.

The technique gives the generators and relators of H' in terms of a Schreier transversal or set of coset representatives for H' in H in which the initial segment of any coset representative is also a representative.

For our Schreier transversal, we choose $\{x_1^{\epsilon_1} \dots x_n^{\epsilon_n}\}$ where $x_1, \dots, x_n$ generate H and where each ϵ_i ranges independently over $\{0, \pm 1, \pm 2, \dots\}$.

The generators of H' in terms of this transversal are $\left\{x_1^{\epsilon_1} \dots x_n^{\epsilon_n}\, x_j\, \overline{x_1^{\epsilon_1} \dots x_n^{\epsilon_n}\, x_j}^{\,-1}\right\}$ where the upper bar denotes the element of the Schreier transversal that represents $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}\, x_j$. The relators of H' are given in terms of this transversal by $\left\{x_1^{\epsilon_1} \dots x_n^{\epsilon_n}\, r\, x_n^{-\epsilon_n} \dots x_1^{-\epsilon_1}\right\}$, rewritten in terms of the generators of H' given above.

We will let K represent a member of our Schreier transversal. We will emphasize a specific member $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ of our transversal by attaching subscripts to K as $K_{\epsilon_1 \dots \epsilon_n}$, when they are needed.

We modify this notation for simplicity so that H is the group generated by

$$\left\{Kx_i\overline{Kx_i}^{-1} = \int_{K;x_i},\ i = 1, \dots, n\right\}$$

and the relators of H are given by

$$\left\{KrK^{-1}\text{ , rewritten as words in } \int_{K;x_i}\right\}.$$

We use a theorem of Schreier which we state in terms of our notation, letting τ represent the rewriting of a word in terms of $\int_{K;x_i}$, $i = 1,\dots,n$.

Theorem 5.3:

If $H = (x_1,x_2,\dots,x_n;r)$, then H' has presentation $*\left(\int_{K;x_1},\int_{K;x_2},\dots,\int_{M;x_i},\dots;\tau(KrK)^{-1}\right)$ where $i = 1,\dots,n,K,M$ elements of a Schreier transversal, but M is a representative such that $Mx_i = \overline{Mx_i}$. Where we have the relation $\int_{M;x_i} = \tau(Mx_i\overline{Mx_i}^{-1})$, both these generators and relators can be deleted.

Further, we can apply Tietze transformations to the presentation $*$ to delete a generator $\int_{K;x_i}$ and a corresponding relator when that relator has the form

$$\int_{K;x_i} = \omega\left(\int_{M;x_j}\right) , \text{ and}$$

$\omega\left(\int_{M;x_j}\right)$ is a word in the generators, not involving $\int_{K;x_i}$. We remark the following lemma, derivable from basic Tietze transformations, that will be used in the sequel, [7] p. 49-50.

Lemma 5.4: If $G = (a_1,a_2,\dots,b_1,b_2;\ a_1 = a_2\omega(b),\ a_2 = a_3\omega(b),\dots,r_j$ not involving a_i , then $G = (a_1,b_1,b_2,\dots;r_j)$.

We shall focus our attention on groups H' which can be shown to be free groups, freely generated by $\left\{\int_{K;x_i}\right\}$, by applying the Schreier-Reidemeister rewriting techniques and Tietze transformations to their presentation. Hence $G' = H'/_{H''}$ is shown to be a free abelian group on an infinite set of generators.

We can show that there are many important classes of one relator metabelian groups for which G' is free abelian. These demonstrations involve explicitly rewriting H' to determine G' using the Schreier-Reidemeister techniques.

Groups for which explicit rewriting processes have been carried out include

$$G = ((x_1,\ldots,x_n;[x_1,x_2] \ldots [x_{n-1},x_n])) \text{ , } n \text{ even.}$$

$$G = ((x_1,\ldots,x_n;[x_1,x_2](x_2,x_3] \ldots [x_{n-1},x_n])) \text{ , } n \text{ odd.}$$

$$G = ((x_1,\ldots,x_n;[x_1,x_2]\omega(x_i) \text{ , where } \omega(x_i) \text{ is a word in } G' \text{ in the generators } x_3,\ldots,x_n)) \text{ .}$$

Because the rewriting involves applying known techniques, we will actually carry it out in only one case, where we show that the group in question is residually free metabelian. We remark that using this technique allows one to show that specific classes of groups are residually finite p for all primes p ; ' + a more general proof was given in IV, §5.

Section 2: A Residually Free Metabelian Group.

Theorem 5.6:

$$G = ((a,b,c;[a,b][b,c])) \in {}_{\mathcal{R}}\mathcal{F}\mathcal{M} \text{ .}$$

Proof: Let $H = (a,b,c;[a,b][b,c])$. $G/_{G'} = H/_{H''} \Big/ {H'/_{H''}} \cong H/_{H'}$ = the free abelian group generated by $\bar{a},\bar{b},\bar{c}$. $G' \cong H'/_{H''}$.

We establish a free set of generators for H ; hence the abelian group $H'/_{H''} = G'$ will be free abelian.

Let $\{K_{i,j,k}\} = \{a^i b^j c^k\}$ be a Schreier transversal for H' in H where $i,j,k = 0,\pm1,\pm2,\ldots$ independently.

If $K_{i,j,k}a\,\overline{K_{i,j,k}a}^{-1} = \int_{K_{i,j,k};a}$ and $K_{i,j,k}b\,\overline{K_{i,j,k}b}^{-1}$ $= \int_{K_{i,j,k};b}$, etc., H' is presented by

$$\left(\int_{K_{i,j,k};a}, \int_{K_{i,j,k};b}, \int_{K_{i,j,k};c};\ \tau(KrK^{-1}),\ \int_{M;x} = \tau(Mx\overline{Mx}^{-1})\right),$$

where $r = [a,b][b,c]$.

$$\tau(KrK^{-1}) = \int_{K_{i,j,k};a} \cdot \int_{K_{k+1,j,k};b} \cdot \int_{K_{i,j+1,k};a}^{-1} \cdot \int_{K_{i,j,k+1};b}^{-1} , \ * \ .$$

Solving each rewritten relator for $\int_{K_{i,j,k+1};b}$: $\int_{K_{i,j,k+1};b} =$ $\int_{K_{i,j,k};a} \cdot \int_{K_{i+1,j,k};b} \cdot \int_{K_{i,j+1,k};a}^{-1}$. We simplify the presentation by the following deletions:

i) $\int_{K_{i,j,k};c} \approx 1$, hence may be deleted.

ii) In the rewritten relators * we have for fixed j and for each i , a sequence of relations:

$$\int_{K_{i,j,0};b} = \int_{K;a} \cdot \int_{K_{i+1,j,-1};b} \cdot \int_{K';a} \cdot$$

$$\int_{K_{i+1,j,-1};b} = \int_{K;a} \cdot \int_{K_{i+2,j,-2};b} \cdot \int_{K';a} \ldots$$

By Lemma 5.4, we can delete all $\int_{K;b}$ generators save $\int_{K_{i,j,0};b}$.

But $\int_{K_{i,j,0};b} \approx 1$, so it can also be deleted.

Therefore H' has presentation

$\left(\left\{\int_{K_{i,j,k};a} \middle| i,j,k = 0,\pm 1,\pm 2,\ldots,\ \text{not both}\ j = k = 0\right\}\right)$. $G' = H'/_{H''}$ is the free abelian group on these generators.

Let $\varphi_\lambda : G \to M = ((a,b))$ be defined by $\varphi_\lambda : a^i \to a^i,\ b^j \to b^j$, and $c^k \to (ab^\lambda)^k$. φ_λ is a homomorphism, mapping commutators to commutators and $\varphi_\lambda([a,b][b \cdot \quad = \varphi_\lambda(aba^{-1}cb^{-1}c^{-1}) = aba^{-1}ab^\lambda b^{-1}b^{-\lambda}a^{-1} = 1$.

Let $g \in G,\ g \notin G'$. Then $\exists\ \lambda$ such that $\varphi_\lambda(g) \neq 1$. Express g as $a^\alpha b^\beta c^\gamma[h]$, where $[h] \in G'$ and $\gamma \neq 0$ or the following is trivially true.

Let $\lambda > |\alpha| + |\beta| + |\gamma|$. Then $\varphi_\lambda(g) = a^\alpha b^\beta (ab^\lambda)^\gamma$. $\varphi_\lambda(g) \neq 1$, for $b^{\beta \pm \lambda\gamma} \neq 1$ since $\beta \pm \lambda\gamma \neq 0$.

Let $g \in G$ and $g \in G'$. Express g in terms of the free generators $\int_{K_{i,j,k};a}$.

If $k = 0$, $\int_{K_{i,j,k}};a$ is expressed in terms of a and b only, so any map $\varphi_\lambda : G \to ((a,b))$ will take $g \neq 1$ in terms of such $\int_{K;a}$ into itself. So we need consider the images of generators $\int_{K_{i,j,k};a}$ where $k \neq 0$.

We also assume that g has been reduced; i.e., $\int_{K_{i,j,k};a} \cdot \int_{K_{i',j',k'};a} \neq 1$ for any pair of generators. This is equivalent to requiring that for any pair of generators, it is not the case that $i = i',\ j = j'\ ,\ k = k'$.

Let $\vartheta = \Sigma|i|$ where i is an exponent of $a \in \int_{K;a} \in g$

$$\mathcal{J} = \Sigma|j| \quad \text{where} \quad j \quad \text{is an exponent of} \quad b \in \int_{K};\ a \in g$$

$$\mathcal{K} = \Sigma|k| \quad \text{where} \quad k \quad \text{is an exponent of} \quad c \in \int_{K};\ a \in g\ .$$

Let $\lambda > \mathcal{I} + \mathcal{J} + \mathcal{K}$ be chosen. We first compute the map φ_λ applied to a single generator $\int_{K_{i,j,k};a}$.

$$\begin{aligned}
\left(\int_{K,a}\right)\varphi_\lambda &= \left(a^i b^j c^k a c^{-k} b^{-j} a^{-(i+1)}\right)\varphi_\lambda \\
&= a^i b^j (ab^\lambda)^k a (ab^\lambda)^{-k} b^{-j} a^{-(i+1)} \\
&= a^i b^j (ab^\lambda)^k a (ab^\lambda)^{-k} a^{-1} a b^{-j} a^{-(i+1)} \\
&= a^i b^j [(ab^\lambda)^k, a] a b^{-j} a^{-(i+1)} \\
&= a^i b^j [(ab^\lambda)^k, a] b^{-j} a^{-i} a^i b^j a\ b^{-j} a^{-(i+1)} \\
&= [ab^\lambda)^k, a]^{b^j a^i} [a^i b^j, a] \\
&= [ab^\lambda, a]^{\frac{(ab^\lambda)^k - 1}{ab^\lambda - 1} a^i b^j} [b, a]^{\left(\frac{b^j - 1}{b-1}\right) a^i} \\
&= [b^\lambda, a]^{a\left(\frac{(ab^\lambda)^k - 1}{ab^\lambda - 1}\right) a^i b^j} [b, a]^{\left(\frac{b^j - 1}{b-1}\right) a^i} \\
&= [a,b]^{\left(\frac{b^\lambda - 1}{b-1}\right)\left(a\left(\frac{(ab^\lambda)^k - 1}{ab^\lambda - 1}\right) a^i b^j\right)} \cdot [a,b]^{-a^i\left(\frac{b^j - 1}{b-1}\right)} \\
&= [a,b]^{-(\lambda)\, a(\lambda,k)\, a^i b^j - a^i (j)}
\end{aligned}$$

where

$$(\lambda) = \left(\frac{b^\lambda - 1}{b-1}\right),\quad (j) = \left(\frac{b^j - 1}{b-1}\right),\quad (\lambda,k) = \left(\frac{(ab^\lambda)^k - 1}{ab^\lambda - 1}\right).$$

We will be concerned with the exponent of $[a,b]$ in $\left(\int_{K;a}\right)\varphi_\lambda$. In the above notation, this has form

$$* \quad -(\lambda)a(\lambda,k)a^{i}b^{j} - a^{i}(j) .$$

We need to show that a product of $\left(\int_{K;a}\right)\varphi_\lambda$ is not 1 for a non-trivial product of $\int_{K;a}$ and suitably chosen λ. This means we must show that the corresponding sum of exponents of the form $*$ is not equal to 0.

Let $\int_{M;a}$ be those elements in the expression for g where in $K_{i,j,k}$, $|k|$ is maximal. Consider $\prod\int_{M;a}\varphi_\lambda = [a,b]^{Exp}$ where

$$\begin{aligned} Exp = {} & n_1[-1(\lambda)a(\lambda,k)a^{i_1}b^{j_1} - a^{i_1}(j_1)] \\ & + n_2[-(\lambda)a(\lambda,k)a^{i_2}b^{j_2} - a^{i_2}(j_2)] \\ & + \dots + n_n[-(\lambda)a(\lambda,k)a^{i_n}b^{j_n} - a^{i_n}(j_n)] . \end{aligned}$$

First note that there are b exponents in this expression which are strictly larger than any other in $\prod\int_{K;a}\varphi_\lambda = g\varphi_\lambda$. So we examine whether it is possible for a sum of terms with maximal b exponent to be zero. This sum has the form

$$n_1[-b^{\lambda+\lambda k-2+j_1}a^{k+i_1}] + n_2[-b^{\lambda+\lambda k-2+j_2}a^{k+i_2}] + \dots + n_n[-b^{\lambda+\lambda k-2+j_n}a^{k+i_n}] .$$

Certainly this sum can be zero if and only if all terms are comparable. This can happen only if all i_m's and all j_m's are equal. But then $\prod\int_{M,a}$ is a product of generators where $M_{i,j,k}$ are equal, hence not reduced.

BIBLIOGRAPHY

1. Baumslag, Gilbert, "Groups With One Defining Relator," *Journal of the Australian Mathematical Society*, Vol. IV, Part 4, p. 385-392.

2. Hall, Philip, "Finiteness Conditions for Soluble Groups, *Proc. London Mathematical Society* (3)4, P. 419-436, 1954.

3. Fox, R.H., "Free Differential Calculus I," *Annals of Mathematics*, 57, p. 547-560, 1953.

4. Fox, R.H., "Free Differential Calculus II," *Annals of Mathematics*, 59, p. 196-210, 1954.

5. Lyndon, R.C., "Identities in Finite Algebras," *Proc. of the American Mathematical Society*, 5, p. 8-9, 1954.

6. Magnus, Wilhelm, "On a Theorem of Marshall Hall," *Annals of Mathematics*, Vol. 40, No. 4, October 1939.

7. Magnus, Wilhelm; Karrass, Abraham; Solitar, Donald, *Combinatorial Group Theory*, Interscience Publishers, John Wiley & Sons, New York, 1966.

8. Matthews, Jane, "The Conjugacy Problem in Wreath Product and Free Metabelian Groups," *Transactions of the American Mathematical Society*, Vol. [illegible]1, No. 2, pp. 329-339, February 1966.

9. Neumann, Hanna, *Varieties of Groups*, Springer-Verlag, New York, 1967.

10. Whittemore, Alice, *The Frattini Subgroup*, Thesis, Department of Mathematics, The City University of New York, 1967.