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City University of New York PH.D. 1980

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FORCE BALANCING MACHINES AND OPTIMIZATION OF DYNAMIC REACTIONS FOR PLANAR MECHANISMS

by

STEPHEN J. TRICAMO

A dissertation submitted to the Graduate Faculty in Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1980

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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The City University of New York

To my wife, Elaine, and our daughters, Leigh, Nancy and Jennifer, whose love made it all possible.

ABSTRACT

The results of certain theoretical and experimental investigations in the field of force balancing of planar mechanisms, which introduce novel computer-aided design methods, are presented.

Experimentation showed that a mechanism designed to be fully force balanced exhibits a residual imbalance, brought about by unavoidable manufacturing tolerances and material nonhomogeneities. A new theory for a group of devices which allow the complete removal of these residual shaking forces of a four-bar linkage and other balanceable mechanisms, is shown.

This theory was subsequently applied to the actual design of a balancing device for a four-bar linkage. It was found that the goal of complete elimination of the residual shaking forces was not attainable and that it was necessary to modify the theory such that the best possible reduction in these forces could be obtained. Subsequent experiments, which use minimax as well as least squares approaches and had most successful outcomes, are described.

Further, a new two-counterweight method of partial

force balancing which makes it possible, by way of what has been called the equipollent circle constraint equation, to prescribe the maximum shaking force of a constant input speed four-bar linkage by way of a wide range of practical counterweight configurations, is given.

Finally, a novel nonlinear optimization formulation which makes it possible to simultaneously minimize the maximum values of the various dynamic reactions of a four-bar linkage while maintaining the ability to prescribe the maximum shaking force, is introduced. This computer-aided design technique, which employs counterweights on all links, was applied to both partial as well as full force balancing and produced gratifying results.

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NOMENCLATURE

a_j = pivot-to-pivot dimension of link j

$a_j'(t)$ = j^{th} time dependent link-to-link length

A, A^o = magnitude of equivalent mass-distance product of input and coupler link for counterweighted and original mechanism, respectively

B, B^o = magnitude of equivalent mass-distance product of output and coupler link for counterweighted and original mechanism, respectively

C, C_1, C_2 = center of smallest circumscribing circle to shaking force hodograph of four-bar mechanism using simple reduction method, two-counterweight reduction method and three-counterweight reduction method, respectively

C_{Sx}, C_{Sy} = x and y components of center of smallest circumscribing circle to shaking force hodograph of four-bar mechanism with input counterweight only

$c_i(\bar{x})$ = i^{th} constraint of optimization problem

$\bar{d}$ = constant vector

d_j = width of link j

F_{M/G_X}, F_{M/G_Y} = x and y components of shaking force

F_{12} = bearing force between input link and coupler link

F_{23} = bearing force between coupler link and output link

F_{34} = bearing force between output link and ground link

F_{41} = bearing force between ground link and input link

F_{CX_i}, F_{CY_i} = i^{th} computed value of x and y components of shaking force

F_{EX_i}, F_{EY_i} = i^{th} experimental value of x and y components of shaking force

F_{X_i}, F_{Y_i} = i^{th} difference of computed and experimental shaking force components

F_{LX}, F_{LY} = x and y components of output of left force platform

F_{RX}, F_{RY} = x and y components of output of right force platform

F_{TOT} = experimental reading of sum of bearing force components in x or y direction

h_j = thickness of link j

h_j^* = thickness of counterweight on link j

h_{rj}, h_{kj} = fixed pivot-to-pivot dimension of link j

K = percentage error of individual bearing force component readings

K_j = kinematic variable relating to link j

m_j, m_j^o, m_j^* = masses of total link, existing link and counterweight, respectively

m_{jp}^c, m_{jq}^c = correction masses required to fine balance a mechanism

m_jp_j, m_jq_j = mass-distance product of link j along x and y axis, respectively

M = total mass of moving links of a mechanism

$M_{M/G}$ = shaking moment

M_{41} = input moment

p_j, q_j = body-fixed coordinates of the position of the center of mass of link j

p_j^c, q_j^c = location of balancing holes on link j along x and y axis, respectively

P_j, Q_j = general mass-distance terms

r_j = body-fixed link dimension locating center of mass of link j

R = radius of smallest circumscribing circle to shaking force hodograph of four-bar mechanism with multiple counterweights

R_E = radius of equipollent circle

R_j^* = radius of counterweight attached to link j

R_S = radius of smallest circumscribing circle to shaking force hodograph of four-bar mechanism with input counterweight only

W_j = weighting factors for reactions used in nonlinear optimization procedure

z = objective function used in certain nonlinear optimization procedures

α, α^o = angle of equivalent mass-distance product of input and coupler link for counterweighted and original mechanism, respectively

β, β^o = angle of equivalent mass-distance product of output and coupler link for counterweighted and original mechanism, respectively

γ_j = kinematic variable

δ_s, δ, δ_1 = angles of inclination with respect to x axis of circumscribing circles to hodograph with centers C_S, C and C_1, respectively

ζ = angular position of center of circumscribing circle

η = factor expressing reduction in unbalanced maximum shaking force

θ_j = angle of inclination of center of mass of link j with respect to body-fixed x axis

θ_j^* = angle to center of counterweight on link j as measured from x axis

μ = proportionality factor relating R to R_S

ρ_j^* = density of counterweight on link j

ϕ_j = angular position of link j taken with respect to x axis

$\dot{\phi}_j$, $\ddot{\phi}_j$ = angular velocity and angular acceleration of link j, respectively

ψ = angle of rotation of shaking force hodograph

I. INTRODUCTION

The elimination or reduction of disturbances transmitted by high speed or heavy duty machinery to their surroundings is of great importance. These disturbances are objectionable because of their negative effects on the productivity of these machines. The resulting vibration, wear and fatigue problems limit their speeds and, with that, their full potential. In addition, the associated noise may represent a serious threat to the health of the operator.

The present work represents an attempt to answer some of the unresolved questions in this field.

Experimentation had shown that if a mechanism was designed to be fully force balanced, it did not necessarily imply that all the shaking forces were removed. This residual imbalance is due to unavoidable manufacturing tolerances and the unpredictable nonhomogeneity of the materials used. Thus, clearly there was a need to devise some procedure by which this residual imbalance could be totally removed or, at least, considerably reduced. To this end, a theory for a group of devices which would allow the complete removal of the residual shaking forces of a four-bar linkage and other balanceable mechanisms, was

developed.

When this theory was applied to a specific design of a balancing device for a four-bar linkage, it was found that this goal of complete elimination of the residual shaking forces was not attainable and that it was necessary to modify the theory such that the best possible reduction in these forces could be achieved. An experiment which uses this new approximating method was most successful and a reduction of as much as 65 percent of the residual shaking force could be obtained.

Secondly, since at the time of the inception of this work G. G. Lowen and F. R. Tepper had just completed a single counterweight method of prescribing the RMS shaking force of a four-bar linkage [80], it was felt that it would be desirable to develop a technique of partial force balancing which allows the limitation of the maximum shaking force of such a mechanism. This goal was realized by the adaptation of a Chebyshev-type partial balancing procedure devised by Ya. L. Gheronimus [25]. The resulting new two-counterweight method makes it possible, with the help of what has been called the equipollent circle constraint equation, to attain the identical prescribed maximum shaking force by way of a considerable range of practical counterweight configurations.

While this work proceeded, and with some insights provided by previous work of G. G. Lowen, F. R. Tepper and R. S. Berkof [51], it became clear that if one extends the above approach by introducing an additional coupler counterweight, it becomes possible to obtain significant reductions in such dynamic reactions as the maximum bearing forces, the maximum input moment and the maximum shaking moment of a four-bar linkage, while maintaining the ability to prescribe the maximum shaking force. Professor Donald Goldfarb, of the Computer Science Department of the City College of New York, suggested that a simultaneous optimization of these reactions may be obtained by the use of an optimization method previously used to solve linear minimax problems. In this formulation, the increases in the dynamic reactions, due to balancing, are controlled by a common minimization parameter which represents the objective function of the problem. This new approach was incorporated into the Harwell Augmented Lagrangian optimization code[1] and it was found that it led to most gratifying

---

[1] It was found that this code could also be used to excellent advantage in connection with the balancing machine as well as the method for prescribing the maximum shaking force of a mechanism. Appendix C serves as a guide for the use of this program.

results. It was possible to keep the increases in the various dynamic reactions of both partially and fully force balanced mechanisms, when compared to those of the unbalanced linkage, to values that previously had not been attained.

A. Background

Since the mechanism balancing literature was reviewed by G. G. Lowen and R. S. Berkof in 1968 [49] and again by the same authors, together with F. R. Tepper, in 1977 [10], only those references which are of importance to the present work, and which have not been mentioned in the previous section, will be briefly discussed.

The Method of Linearly Independent Vectors, first given by R. S. Berkof and G. G. Lowen [8], represents the point of departure for the investigation of a four-bar linkage balancing device. This method was later generalized by F. R. Tepper and G. G. Lowen [76]. The latter work also gave a balanceability criterion which is used to good advantage in the generalization of the theory of full force balancing machines.

In the field of mechanism optimization, J. P. Sadler and R. W. Mayne [63] introduced the use of nonlinear programming techniques for the minimization of the maximum values of certain reactions in a constant input speed general four-bar linkage. They minimized the maximum shaking force while limiting the magnitude of the shaking moment. This optimization was repeated with an additional constraint on the maximum value of the largest bearing force. All

solutions are represented by trade-off charts. These optimizations were accomplished with point mass counterweights which are attached to the input and output links. (While the neglect of the counterweight moments of inertia is of no consequence as far as the shaking force is concerned, it matters to varying degrees when it comes to the shaking moment, the input moment and the various bearing forces.) The mathematical technique consisted of converting constrained optimization problems to unconstrained ones by using the exterior penalty function approach. The subsequent optima were then determined with the help of a variable metric procedure.

J. P. Sadler [64] applied the above optimization procedure to six-bar linkages and individually minimized the maximum values of the shaking force, the shaking moment, the input torque fluctuation as well as a bearing force by both one and two point masses of prescribed magnitude. A trade-off between maximum shaking force and input torque fluctuation was also undertaken, using either two or three balancing masses. For the two balancing mass case, this was accomplished both with and without an additional bearing force constraint. Finally, a 40 percent reduction in both maximum shaking force and shaking moment was accomplished

for a six-bar stone crusher mechanism with the help of a composite objective function. The efficiency of nonlinear programming, when applied to linkage balancing, was demonstrated by duplicating the analytically obtainable full force balance results for both four-bar and six-bar linkages.

In a unique study, combining kinematic synthesis and the dynamic design of four-bar linkages, F. L. Conte, G. R. George, R. W. Mayne and J. P. Sadler [14] showed that nonlinear programming can be successfully applied for the determination of that linkage which not only satisfies the kinematic objectives, but also provides optimum dynamic characteristics.

B. Porter and D. J. Sanger [59] also showed the application of nonlinear programming techniques to the optimization of certain dynamic reactions of an inline four-bar linkage with links of constant cross-section. To this end, an objective function consisting of the sum of the weighted mean square values of the shaking force, the shaking moment, the input torque and the bearing forces was introduced. The minimization of this function is subject to constraints on the two design variables which are represented by the inline location as well as the magnitude of the radius of a circular counterweight attached to the input link. (The

input link counterweight was designed according to the Method of Linearly Independent Vectors and was chosen to have a minimum moment of inertia.) The authors selected the weighting factors in such a way that they only performed the simultaneous optimization of two reactions. Their success was greater for forces than for moments.

S. Schonfeld [65] reported on a computer program which combines the ability to generate the various reactions of planar linkages together with their optimization by means of nonlinear programming techniques. A successful example of a weighted optimization of the shaking moment and the input torque of a six-bar linkage is shown. The same author together with H. Dresig [66] discussed, without example, the simultaneous optimization of all linkage reactions by the application of the usual techniques of linear analysis.

B. Results Of Investigation

The present section reports briefly on the results of the investigation which consists of the following parts:

1. Theory Of Full Force Balancing Machines For Reciprocating Mechanisms
2. A Practical Balancing Machine For Four-Bar Linkages
3. A New Method For Prescribing The Maximum Shaking Force Of A Four-Bar Linkage With Flexibility In Counterweight Design
4. Simultaneous Optimization Of Dynamic Reactions Of A Four-Bar Linkage With Prescribed Maximum Shaking Force

1. Theory Of Full Force Balancing Machines For Reciprocating Mechanisms

The theoretical basis for new balancing fixtures capable of removing the unavoidable residual shaking force found in theoretically balanced mechanisms is presented. This new approach is first applied to a four-bar linkage with a general mass distribution. Subsequently, it is generalized to include all balanceable planar mechanisms.

In both instances, sets of shaking force component

equations are solved for certain general mass-distance products. To obtain these parameters, which form the basis for the design of the required correction masses, the values of the residual shaking force components and of all kinematic properties must be known from experimentation for certain positions of the mechanism.

2. A Practical Balancing Machine For Four-Bar Linkages

The development of an actual force balancing machine along the lines of the new theory and capable of removing a large portion of the residual shaking force found in a theoretically fully force balanced four-bar linkage, is described.

While it was not possible, because of experimental error, to obtain unique answers for the desired correction masses from various sets of balancing equations, a method was devised which makes use of many sets of such expressions. The problem was interpreted as one of finding the best possible solution to an overdetermined system of balancing equations. Both a least squares and a minimax approach were developed.

Although it was not possible to remove all residual imbalance, both methods furnished considerable reductions

of the shaking force components. The minimax method was found to give a somewhat greater reduction than the least squares approach.

3. A New Method For Prescribing The Maximum Shaking Force Of A Four-Bar Linkage With Flexibility In Counterweight Design

This novel two-counterweight method of partial force balancing of a four-bar linkage, which forms the basis for a comprehensive optimization method discussed later, allows the designer to realize a prescribed value for the maximum shaking force anywhere between zero and an inherent upper limit. In addition, by taking advantage of a certain mathematical property, which has been called the equipollent circle constraint equation, it becomes possible to attain the identical prescribed maximum shaking force for a wide range of output and associated input link counterweight configurations.

This technique, which is generally only applicable to constant speed mechanisms, builds upon a Chebyshev-type partial balancing method, first given by Ya. L. Gheronimus, which only uses an input link counterweight. The equipollent circle constraint equation, which allows flexibility in

counterweight design for a prescribed maximum shaking force, represents an adaptation of a parallel concept, used by G. G. Lowen and F. R. Tepper to prescribe the RMS shaking force.

Examples involving both a 50 and a 75 percent reduction in the maximum shaking force are given and the associated ranges of possible counterweight designs are determined. The magnitudes of such maximum dynamic reactions as bearing forces, input moment and shaking moment have been computed for each of the above reductions in the maximum shaking force for a full range of counterweight designs. These reactions show little variation in magnitude over a substantial portion of this range. In light of the above, this flexibility in counterweight configuration offers advantages when certain space limitations have to be observed.

4. Simultaneous Optimization Of Dynamic Reactions Of A Four-Bar Linkage With Prescribed Maximum Shaking Force

This extension of the previously described new partial balancing method makes it possible to simultaneously minimize the maximum dynamic reactions of a constant speed four-bar linkage while maintaining the ability to prescribe the

maximum shaking force. The general reduction of the values of these reactions, when compared to those obtained by the previous method, is based on the introduction of a coupler counterweight in addition to those located on the input and output links. The resulting increase in the number of design variables is reflected in what has now been called the general equipollent circle constraint equation. The simultaneous optimum reduction of the maximum bearing forces, the maximum input moment and the maximum shaking moment now becomes a function of the proper combination of the coupler and output link counterweight parameters.

To this end, an optimization procedure was devised which allows for the determination of these counterweight parameters. This procedure satisfies the general equipollent circle constraint equation while it imposes constraints on the maximum dynamic reactions. The latter inequality constraints are controlled by a minimization parameter z which serves as the objective function of the optimization.

The above optimization procedure was incorporated into the Harwell Augmented Lagrangian optimization code which was used to obtain optimum counterweight designs for a 50 and a 75 percent reduction in the maximum shaking force. It was found that the benefits of this optimization were most

pronounced for the larger reduction in the maximum shaking force.

This three-counterweight method was also adapted to a fully force balanced mechanism. The associated reductions in the dynamic reactions, when compared to those associated with the two-counterweight Method of Linearly Independent Vectors, was found to be especially gratifying.

Finally, the optimization procedure was modified for an input moment minimization of a fully force balanced mechanism. Again, a comparison of the resulting maximum input moment with that associated with the above two-counterweight method shows a considerable reduction.

II. THEORY OF FULL FORCE BALANCING MACHINES FOR RECIPROCATING MECHANISMS

A. Introduction

While it is theoretically possible to design linkages which are fully force balanced, there are practical difficulties in attaining this aim. Due to unavoidable manufacturing tolerances and the unpredictable nonhomogeneity of the materials used, the shaking force of an actual mechanism, built to be theoretically force balanced, generally does not vanish completely. Furthermore, this residual imbalance may be amplified by any variation in the input velocity.

The following describes the first theoretical approach to full force balancing machines for reciprocating mechanisms. It is felt to be unique because it represents the first attempt at such machines for mechanisms in which the counterweights do not all rotate at the speed of a single shaft, as is the case in automotive engines.

The new theory is first applied to a four-bar linkage with a general mass distribution. Subsequently, it is generalized to include all balanceable planar mechanisms [76].

In both instances, sets of shaking force component equations are solved for certain general mass-distance products. To obtain these parameters, which form the basis for the design of the required correction masses, the values of the residual shaking force components and of all kinematic properties must be known from experimentation for certain positions of the mechanism.

B. Theory Of Four-Bar Linkage Balancing Machine

To understand the concept[1] of the balancing machine for a four-bar linkage with a general mass distribution, as shown in Figure 2.1, consider first the following compact form of the shaking force expression, as given by equation (A.7) of Appendix A:

$$\bar{F}_{M/G} = (P_1 + iQ_1)K_1 e^{i\gamma_1} + (P_3 + iQ_3)K_3 e^{i\gamma_3} \qquad (2.1)$$

where

$$P_1 = m_1 p_1 + m_2 a_1 - \frac{a_1}{a_2} m_2 p_2$$

$$Q_1 = m_1 q_1 - \frac{a_1}{a_2} m_2 q_2$$

$$P_3 = m_3 p_3 + \frac{a_3}{a_2} m_2 p_2$$

$$Q_3 = m_3 q_3 + \frac{a_3}{a_2} m_2 q_2$$

$$\left.\begin{aligned} K_j &= \sqrt{\dot{\phi}_j^{\,4} + \ddot{\phi}_j^{\,2}} \\ \gamma_j &= \phi_j - \phi_j' \\ \phi_j' &= \tan^{-1} \frac{\ddot{\phi}_j}{\dot{\phi}_j^{\,2}} \end{aligned}\right\} \quad j = 1,3$$

---

[1] In the search for a force balancing machine for four-bar linkages, it became evident that the present approach represents the only feasible one. Appendix B.1 proves this by showing that one may not uniquely determine the individual link mass-distance products $m_j p_j$ and $m_j q_j$ (j=1,2,3) which could possibly be modified by suitable correction masses.

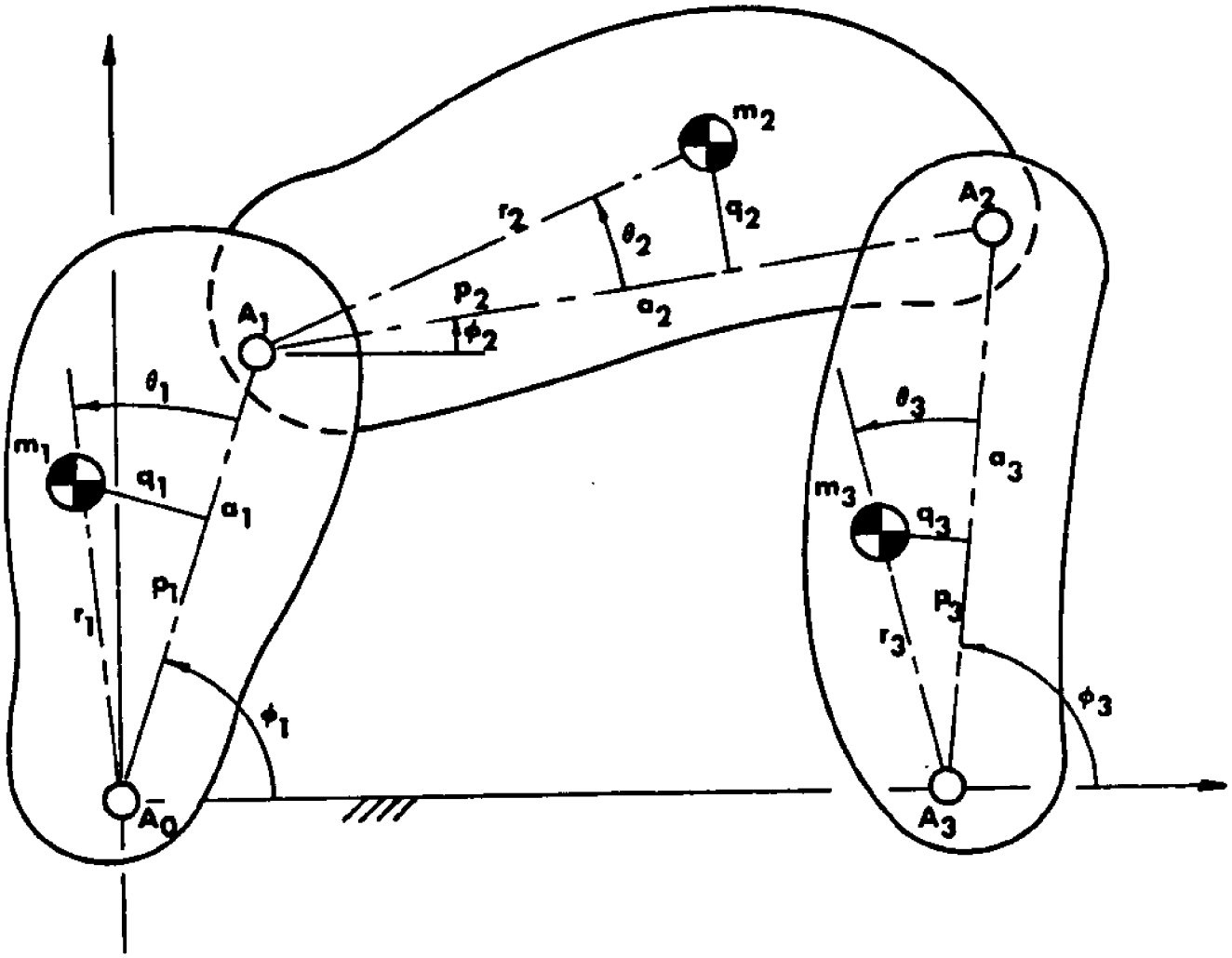


FIG. 2.1 GENERAL FOUR-BAR LINKAGE

For a fully force balanced mechanism, the shaking force $\bar{F}_{M/G}$ vanishes because the general mass-distance terms P_j and Q_j have been made to vanish. For the almost balanced mechanism, these factors are not equal to zero because of the various previously mentioned uncertainties. If their actual magnitudes for a given mechanism can be found experimentally, they can be made to vanish by a suitable adjustment of the mass distribution of the links involved.

For example, with

$$P_1 \neq 0,$$

one may then find a correction quantity defined as

$$\Delta P_1 = m_{1p}{}^c p_1{}^c$$

such that

$$P_1 + \Delta P_1 = 0. \tag{2.2}$$

This corrective mass-distance product ΔP_1 may be positive or negative, depending on the sign of P_1. It can be physically realized by adding or subtracting a correction mass $m_{1p}{}^c$ at a distance $p_1{}^c$ along the body-fixed x-axis of link 1. Similar corrections to the remaining mass-distance

products can be defined as follows:

$$\Delta Q_1 = -Q_1 = m_{1q}{}^{c}q_1{}^{c} \tag{2.3}$$

$$\Delta P_3 = -P_3 = m_{3p}{}^{c}p_3{}^{c} \tag{2.4}$$

$$\Delta Q_3 = -Q_3 = m_{3q}{}^{c}q_3{}^{c} \tag{2.5}$$

Appendix B.2 proves that if one applies the component forms (A.17) and (A.18) of the shaking force equation (2.1) at two mechanism positions for which all kinematic values and shaking force components are known from experimentation, one may solve uniquely for the four general mass-distance products P_j and Q_j (j=1,3). When given in matrix form, this set of expressions becomes (see equation (B.9)):

$$\begin{bmatrix} K_{11}\cos\gamma_{11} & -K_{11}\sin\gamma_{11} & K_{31}\cos\gamma_{31} & -K_{31}\sin\gamma_{31} \\ K_{11}\sin\gamma_{11} & K_{11}\cos\gamma_{11} & K_{31}\sin\gamma_{31} & K_{31}\cos\gamma_{31} \\ K_{12}\cos\gamma_{12} & -K_{12}\sin\gamma_{12} & K_{32}\cos\gamma_{32} & -K_{32}\sin\gamma_{32} \\ K_{12}\sin\gamma_{12} & K_{12}\cos\gamma_{12} & K_{32}\sin\gamma_{32} & K_{32}\cos\gamma_{32} \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \\ P_3 \\ Q_3 \end{bmatrix} = \begin{bmatrix} F_{M/G_{X1}} \\ F_{M/G_{Y1}} \\ F_{M/G_{X2}} \\ F_{M/G_{Y2}} \end{bmatrix} \quad (2.6)$$

where[1]

$$K_{ij} = \sqrt{\dot{\phi}_{ij}^{\,4} + \ddot{\phi}_{ij}^{\,2}} \quad \text{(These terms are always positive.)}$$

$$\gamma_{ij} = \phi_{ij} - \phi_{ij}'$$

$$\phi_{ij}' = \tan^{-1}\frac{\ddot{\phi}_{ij}}{\dot{\phi}_{ij}^{\,2}}$$

$i = 1,3$

$j = 1,2$

(The second subscript in these terms indicate at which position the quantity is evaluated.)

This procedure furnishes the identical results whether or not friction in the joints is considered. This may be shown by the following reasoning: Whenever there is a residual shaking force, the force equal and opposite to it has, according to Newton's law for systems of rigid bodies,

---

[1] The required kinematic relationships are reproduced in Appendix I.

the same magnitude as the vector which represents the product of the total mass of the mechanism and the vectorial acceleration of its total center of mass. This net force of the ground on the mechanism remains the same whether or not friction is present. (Of course, the individual ground pivot forces are changed when cognizance is taken of bearing friction in a mechanism.)

C. Generalization Of Theory Of Full Force Balancing Machines

The following shows that the principle of the four-bar linkage balancing machine may be extended to all planar single degree of freedom linkages in which no group of links produce identical kinematic terms at all times and which meet the balancing criterion given by F. R. Tepper and G. G. Lowen [76]. This criterion states that "A planar mechanism without axisymmetric link groupings can be fully force balanced by internal mass redistribution if, and only if, from each link there is a contour to the ground by way of revolutes only".

The following generalization will follow the same line of reasoning as used in Appendix B where it was shown that the general mass-distance products P_j and Q_j can be found from the repeated application of shaking force component equations which are made up of linearly independent terms. This linear independence is again assured by the use of constraint equations which have their origin in the double differentiation of the applicable mechanism loop equations. As before, the values of the kinematic terms and the shaking force components must be known from experimentation at the applicable mechanism positions.

The generalization of balancing machines will first be shown for linkages which contain only revolutes. Subsequently, it is extended to mechanisms which also contain sliders.

1. Generalization For Mechanisms Containing Revolutes Only

In order to derive the shaking force equation for an n-linked mechanism containing revolutes only (for a typical case, see loop O_1ABO_2 of Figure 2.2), F. R. Tepper and G. G. Lowen [76] showed that the mass-distance product associated with its center of mass is given by:

$$M\bar{r}_s = \sum_{j=1}^{n-1}\left[m_j r_j e^{i\theta_j} + \sum_{k=1}^{k^*}(mhe^{i\alpha})_{kj}\right] e^{i\phi_j} + \bar{d} \qquad (2.7)$$

In the foregoing, there will always be $n-1$ terms of the type $m_j r_j e^{i\theta_j}$, which refer to the mass-distance products of the individual link centers of mass with respect to their body-fixed origins. The k^* terms containing the constant coefficients $(mhe^{i\alpha})_{kj}$ represent the k^* mass-distance products which involve the fixed pivot-to-pivot vectors $he^{i\alpha}$ (such as $b_2e^{i\alpha_2}$ in Figure 2.2). They occur only when the description of the center of mass trajectory of a specific link includes the fixed pivot-to-pivot distances of another link (for example, when the center

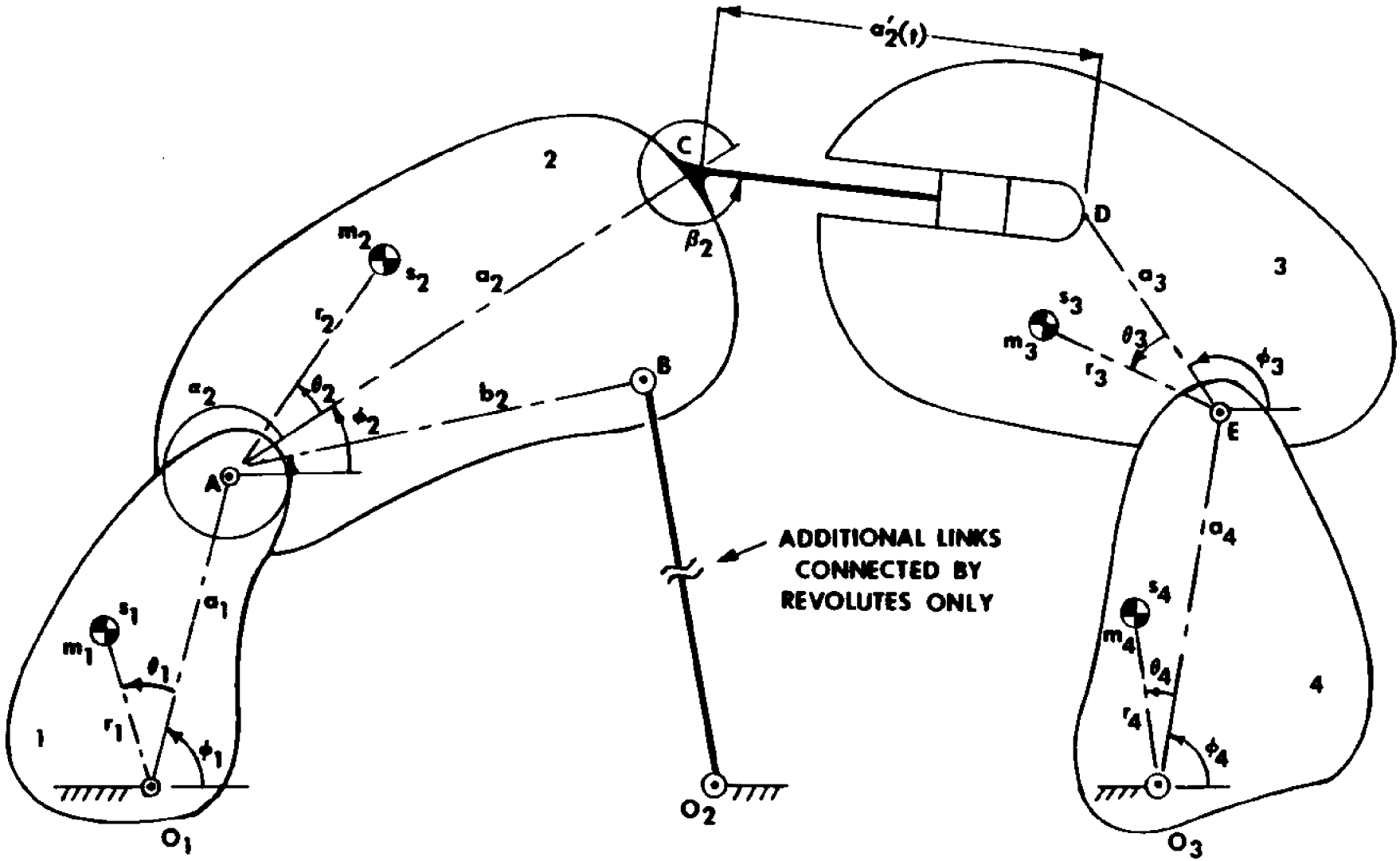


FIGURE 3.4 BALANCEABLE N-LINKED MECHANISM

of mass trajectory $\bar{r}_{s_2}$ is defined by way of the contour O_2BAS_2). Finally, $\bar{d}$ is a constant mass-distance vector.

Double differentiation, change of sign and rearrangement of the above furnishes, according to Newton's law, the following shaking force component equations[1]:

$$\sum_{j=1}^{n-1}\left[(K_j\cos\gamma_j)A_j - (K_j\sin\gamma_j)B_j\right] = -M\ddot{r}_{sx_j} = F_{M/G_{X_j}} \qquad (2.8)$$

and

$$\sum_{j=1}^{n-1}\left[(K_j\sin\gamma_j)A_j + (K_j\cos\gamma_j)B_j\right] = -M\ddot{r}_{sy_j} = F_{M/G_{Y_j}} \qquad (2.9)$$

where

$$A_j = m_j r_j \cos\theta_j + \sum_{k=1}^{k^*}(mh)_{kj}\cos\alpha_{kj}$$

$$B_j = m_j r_j \sin\theta_j + \sum_{k=1}^{k^*}(mh)_{kj}\sin\alpha_{kj}$$

$$K_j = \sqrt{\dot{\phi}_j^{\,4} + \ddot{\phi}_j^{\,2}}$$

$$\gamma_j = \phi_j - \phi_j'$$

$$\phi_j' = \tan^{-1}\frac{\ddot{\phi}_j}{\dot{\phi}_j^{\,2}}$$

---

[1] Equations (2.8) and (2.9) hold regardless of whether or not friction in the various bearings is taken into account. See page 21.

Each of the component equations consists of a sum of $2(n-1)$ terms with constant mass-distance products. Many of these terms are proportional to each other. It will now be shown that this proportionality can be eliminated by making use of the doubly differentiated component forms of the $\frac{n}{2} - 1$ independent loop equations associated with an n-linked mechanism [57].

The r^{th} independent loop equation of such a mechanism containing revolutes only may be resolved into the following component form:

$$\sum_{j=1}^{n} h_{rj}\cos(\alpha_{rj} + \phi_j) = 0 \qquad (2.10)$$

and

$$\sum_{j=1}^{n} h_{rj}\sin(\alpha_{rj} + \phi_j) = 0, \ (r=1,2,\ldots, \tfrac{n}{2} - 1) \qquad (2.11)$$

Double differentiation of the above results in the following component form:

$$-\sum_{j=1}^{n-1} h_{rj}K_j\cos(\alpha_{rj} + \gamma_j) = 0 \qquad (2.12)$$

and

$$-\sum_{j=1}^{n-1} h_{rj}K_j\sin(\alpha_{rj} + \gamma_j) = 0 \qquad (2.13)$$

With $\left(\frac{n}{2} - 1\right)$ loop equations there will be $2\left(\frac{n}{2} - 1\right)$ constraint equations of the above type. The simultaneous application of these constraint equations to each of the shaking force component expressions reduces the number of terms in these expressions to n, i.e.

$$n = 2(n - 1) - 2\left(\frac{n}{2} - 1\right) \tag{2.14}$$

The resulting expressions are:

$$\sum_{j=1}^{n/2}\left[(K_j \cos\gamma_j)P_j - (K_j \sin\gamma_j)Q_j\right] = F_{M/G_{X_j}} \tag{2.15}$$

and

$$\sum_{j=1}^{n/2}\left[(K_j \sin\gamma_j)P_j + (K_j \cos\gamma_j)Q_j\right] = F_{M/G_{Y_j}} \tag{2.16}$$

In the above, the general mass-distance products P_j and Q_j, which again must be determined to obtain the corrective mass-distance products ΔP_j and ΔQ_j, are of the following form:

$$P_j = m_j p_j + \sum_{w=\frac{n}{2}+1}^{n-1} f_{wj} m_w r_w \cos(\sigma_{wj} + \theta_w) + \sum_{k=1}^{k^*} \left[(mh\cos\alpha)_{kj} + \sum_{w=\frac{n}{2}+1}^{n-1} f_{wj} (mh)_{kw} \cos(\sigma_{wj} + \alpha_{kw}) \right] \quad (2.17)$$

and

$$Q_j = m_j q_j + \sum_{w=\frac{n}{2}+1}^{n-1} f_{wj} m_w r_w \sin(\sigma_{wj} + \theta_w) + \sum_{k=1}^{k^*} \left[(mh\sin\alpha)_{kj} + \sum_{w=\frac{n}{2}+1}^{n-1} f_{wj} (mh)_{kw} \sin(\sigma_{wj} + \alpha_{kw}) \right] \quad (2.18)$$

The f_{wj} and σ_{wj} terms represent constants which result from the elimination of the linear dependence.

Because of the use of constraint equations, each of the above force component expressions is made up of a sum of n linearly independent terms. Both expressions are linearly independent of each other since they represent the orthogonal components of a single vector.

In order to solve for the n corrective mass-distance products, it is necessary to use $\frac{n}{2}$ pairs of shaking force component equations, of the form of equations (2.15) and (2.16), for $\frac{n}{2}$ mechanism positions for which all kinematic and

shaking force component values are known from experimentation. The resulting set of simultaneous equations takes the following matrix form:

$$\begin{bmatrix} K_{11}\cos\gamma_{11} & -K_{11}\sin\gamma_{11} & \cdots & K_{1\frac{n}{2}1}\cos\gamma_{1\frac{n}{2}1} & -K_{1\frac{n}{2}1}\sin\gamma_{1\frac{n}{2}1} \\ K_{11}\sin\gamma_{11} & K_{11}\cos\gamma_{11} & \cdots & K_{1\frac{n}{2}1}\sin\gamma_{1\frac{n}{2}1} & K_{1\frac{n}{2}1}\cos\gamma_{1\frac{n}{2}1} \\ \vdots & \vdots & & \vdots & \vdots \\ K_{1\frac{n}{2}}\cos\gamma_{1\frac{n}{2}} & -K_{1\frac{n}{2}}\sin\gamma_{1\frac{n}{2}} & \cdots & K_{\frac{n}{2}\frac{n}{2}}\cos\gamma_{\frac{n}{2}\frac{n}{2}} & -K_{\frac{n}{2}\frac{n}{2}}\sin\gamma_{\frac{n}{2}\frac{n}{2}} \\ K_{1\frac{n}{2}}\sin\gamma_{1\frac{n}{2}} & K_{1\frac{n}{2}}\cos\gamma_{1\frac{n}{2}} & \cdots & K_{\frac{n}{2}\frac{n}{2}}\sin\gamma_{\frac{n}{2}\frac{n}{2}} & K_{\frac{n}{2}\frac{n}{2}}\cos\gamma_{\frac{n}{2}\frac{n}{2}} \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \\ \vdots \\ P_{\frac{n}{2}} \\ Q_{\frac{n}{2}} \end{bmatrix} = \begin{bmatrix} F_{M/G_{X1}} \\ F_{M/G_{Y1}} \\ \vdots \\ F_{M/G_{X\frac{n}{2}}} \\ F_{M/G_{Y\frac{n}{2}}} \end{bmatrix} \tag{2.19}$$

The second subscript in the above expression refers to the $\frac{n}{2}$ mechanism positions.

Each of the resulting n row vectors of the coefficient matrix is linearly independent as long as no groups of the $\frac{n}{2}$ links produce identical kinematic terms for all $\frac{n}{2}$ positions of the mechanism. Thus, the matrix has rank n and may be solved uniquely for the n general mass-distance products P_j and Q_j.

With the P_j and Q_j terms now available, the corrective mass-distance products ΔP_j and ΔQ_j (and therefore the corrective masses m_j^c) required to eliminate the residual imbalance are found from:

$$\left.\begin{aligned} P_j + \Delta P_j &= 0 \\ \text{and} \qquad Q_j + \Delta Q_j &= 0 \end{aligned}\right\} \quad j = 1,2,\ldots, \frac{n}{2} \qquad (2.20)\ (2.21)$$

2. Generalization For Mechanisms Containing Both Revolutes And Sliders

For an n-linked mechanism which contains s sliders in addition to revolutes (see Figure 2.2), the mass-distance product associated with its center of mass is given by:

$$M\bar{r}_s = \sum_{j=1}^{n-1-s} \left[(m_j r_j e^{i\theta_j} + \sum_{k=1}^{k^*} (mhe^{i\alpha})_{kj}) e^{i\phi_j} + \sum_{u=1}^{s} (ma'(t)e^{i\beta})_{uj} e^{i\phi_j} \right] + \bar{d} \qquad (2.22)$$

The s time dependent mass-distance products $(ma'(t)e^{i\beta})_{uj}$ are due to the variable "link lengths" associated with the s sliders (for example, see $a_2'(t)e^{i\beta 2}$ in Figure 2.2). All other terms are identical to those given in equation (2.7).

For each of the s sliders in a balanceable mechanism there is a loop equation which contains only one time dependent term, i.e.:

$$a_r'(t)e^{i(\beta_r + \phi_r)} + \sum_{j=1}^{n} h_{rj}e^{i(\alpha_{rj} + \phi_j)} = 0, \quad (r=1,2,\ldots,s) \tag{2.23}$$

These s loop equations are now used to eliminate the time dependent mass-distance products in equation (2.22). This results in an expression similar to equation (2.7) containing $(n - 1 - s)$ terms with constant mass-distance products in addition to the constant:

$$M\bar{r}_s = \sum_{j=1}^{n-1-s}\left[m_j r_j e^{i\theta_j} + \sum_{k=1}^{k'}(mhe^{i\alpha})_{kj}\right]e^{i\phi_j} + \bar{d} \tag{2.24}$$

(Note that the use of the loop equations leads to a greater number of terms of the type $mhe^{i\alpha}$.)

Again, following the procedure outlined in the previous section, the above expression is doubly differentiated and put into component form. Each of these component expressions now consists of a sum of $2(n - 1 - s)$ linearly dependent terms. These are made linearly independent with the help of $2(\frac{n}{2} - 1 - s)$ component constraint equations of the type

of equations (2.12) - (2.13) which are obtained from the remaining $(\frac{n}{2} - 1 - s)$ independent loop equations. The resulting number of linearly independent terms in the shaking force component equations will again be n, since

$$n = 2(n - 1 - s) - 2(\frac{n}{2} - 1 - s)$$

These final expressions are identical with equations (2.15) and (2.16) and the n corrective mass-distance products may again be obtained with the help of equation (2.19) with its associated matrix of rank n.

III. A PRACTICAL BALANCING MACHINE FOR FOUR-BAR LINKAGES

A. Introduction

The following reports on the development of an actual force balancing machine capable of fine balancing a four-bar linkage.

Because of the experimental errors inherent in the determination of the shaking force components and the various necessary kinematic quantities, it was not possible to proceed completely along the lines of the theory given in Section II-B. Whenever the general mass-distance products P_j and Q_j (j=1,3) were evaluated for a given set of two angular positions, according to equations (2.6), the results differed from those obtained for any other set of two angular positions. Thus, it became clear that a method for determining these quantities, which makes use of many sets of observations, had to be developed.

Section B shows that this problem may be interpreted as one of finding the best possible solution to an overdetermined system of balancing equations. It further outlines how solutions may be obtained by both a minimax and a least squares approach[1].

---

[1] This was accomplished for both cases with the help of an existing Augmented Lagrangian optimization program (Harwell Subroutine Library, VF01AD). While more conventional programs could have been used for these solutions, it was found most convenient to formulate these problems for the above program since it was used in other parts of this dissertation.

Section C gives a description of the experimental apparatus and explains the methods by which data are obtained.

Section D shows examples of both minimax and least squares fine balancing procedures. While both approaches furnished considerable reductions of the shaking force components of a theoretically fully force balanced mechanism, the minimax method led to the greater reduction of these components.

Section E gives a detailed discussion of the sources of the experimental errors and concludes that there is good reason why the minimax method furnishes the more desirable results.

Appendix C shows the use of the Harwell code as well as its adaptation to the minimax approach of finding the best possible solution to the overdetermined system of balancing equations.

Appendix D shows the adaptation of the same program to the least squares approach.

B. Adaptation Of Balancing Machine Theory

Since, as stated earlier, due to experimental errors, more than two sets of data must be employed for the determination of the general mass-distance products P_j and Q_j (j=1,3), the following overdetermined system of balancing equations, based on equation (2.6), results:

$$\begin{bmatrix} K_{11}\cos\gamma_{11} & -K_{11}\sin\gamma_{11} & K_{31}\cos\gamma_{31} & -K_{31}\sin\gamma_{31} \\ K_{11}\sin\gamma_{11} & K_{11}\cos\gamma_{11} & K_{31}\sin\gamma_{31} & K_{31}\cos\gamma_{31} \\ \vdots & \vdots & \vdots & \vdots \\ K_{1m}\cos\gamma_{1m} & -K_{1m}\sin\gamma_{1m} & K_{3m}\cos\gamma_{3m} & -K_{3m}\sin\gamma_{3m} \\ K_{1m}\sin\gamma_{1m} & K_{1m}\cos\gamma_{1m} & K_{3m}\sin\gamma_{3m} & K_{3m}\cos\gamma_{3m} \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \\ P_3 \\ Q_3 \end{bmatrix} = \begin{bmatrix} F_{M/G_{X1}} \\ F_{M/G_{Y1}} \\ \vdots \\ F_{M/G_{Xm}} \\ F_{M/G_{Ym}} \end{bmatrix} \quad (3.1)$$

where m represents the total number of pairs (i.e. angular positions) of data chosen.

The solutions of the above system will furnish values for the P_j and Q_j terms such that the left hand side of the matrix equations approximates the right hand side either in a minimax or a least squares sense at all chosen positions.

Figure 3.1 shows the above from a different point of view. The solid curves represent purely experimental values of the shaking force components F_{M/G_X} and F_{M/G_Y} as plotted against the crank angle ϕ_1. The dashed curves stand for computed values of these forces. They are obtained with the help of experimentally determined angles, angular velocities and accelerations of the input link. The P_j and Q_j terms in these expressions are chosen such that the differences ΔF_x and ΔF_y between the two sets of curves are minimized according to a minimax or a least squares criterion.

Both of these approaches will now be discussed. The results of their actual implementation are shown in Section D. (The use of the Harwell program for this purpose is shown in Appendices C and D.)

1. Minimax Balancing Optimization

The maximum value of the differences ΔF_x and ΔF_y between the sets of shaking force curves is minimized in this approach.

To accomplish this by way of the Augmented Lagrangian

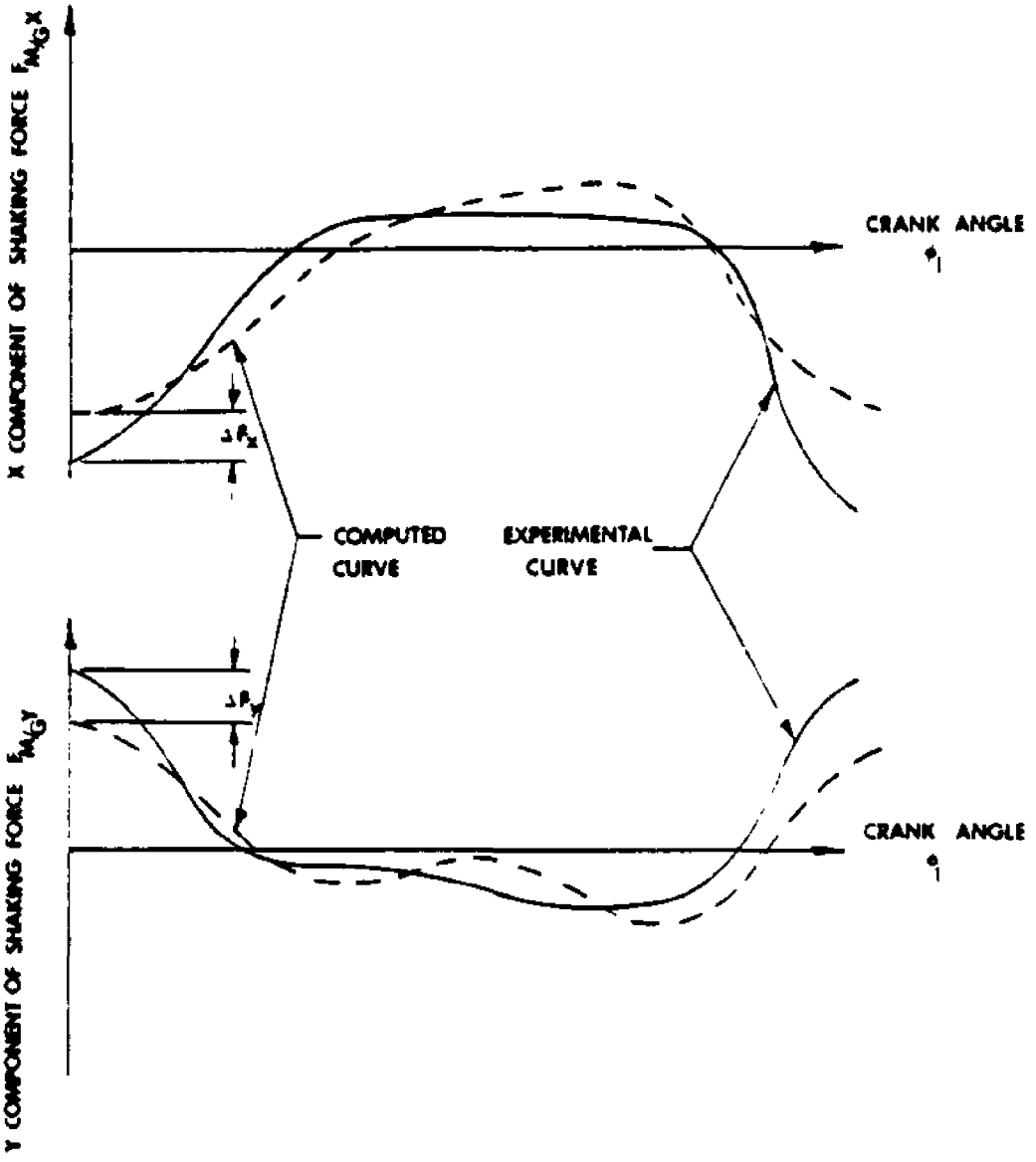


FIGURE 3.1 : COMPARISON OF TYPICAL COMPUTED AND EXPERIMENTAL RESIDUAL SHAKING FORCE COMPONENTS FOR A FOUR-BAR LINKAGE

program, the square of the differences[1] ΔF_x and ΔF_y are constrained at m values of the input angle ϕ_1 to be less than or equal to a new variable z. This variable represents the objective function which is to be minimized.

In equation form, the above becomes

$$\text{minimize} \quad z \tag{3.2}$$

by the optimal choice of P_1, Q_1, P_3 and Q_3, subject to the following inequality constraints:

$$\left.\begin{aligned} (\Delta F_{x_i})^2 &\leq z \\ (\Delta F_{y_i})^2 &\leq z \end{aligned}\right\} \quad (i=1,2,\ldots,m) \tag{3.3}$$

where

$$\Delta F_{x_i} = F_{CX_i} - F_{EX_i}$$

$$\Delta F_{y_i} = F_{CY_i} - F_{EY_i}$$

and

F_{CX_i} = i^{th} computed value of shaking force component F_{M/G_x}

F_{CY_i} = i^{th} computed value of shaking force component F_{M/G_y}

---

[1] The reason for constraining the squares of these differences between the curves, rather than the actual differences or the absolute values of the differences directly, lies in the nature of the optimization program. In essence, the absolute values of the differences are constrained.

F_{EX_i} = i^{th} experimental value of shaking force component F_{M/G_x}

F_{EY_i} = i^{th} experimental value of shaking force component F_{M/G_y}

The subscript i indicates that the constraints are evaluated at the m mechanism positions.

2. Least Squares Balancing Optimization

In the least squares approach, the sum of the squares of the differences ΔF_x and ΔF_y are minimized at the m mechanism positions. This sum also represents the objective function. Thus formally,

$$\text{minimize} \quad \sum_{i=1}^{m} \left[(\Delta F_{X_i})^2 + (\Delta F_{Y_i})^2 \right] \tag{3.4}$$

C. Balancing Machine Design And Instrumentation

The following describes the design of the experimental balancing machine as well as the manner in which the required test data are obtained.

1. Basic Configuration Of Balancing Machine

Figure 3.2 shows an overall view as well as a schematic of the experimental balancing setup while Figure 3.3 gives photographs of the final configuration. All components are listed in Table 3.1.

The variable speed DC motor, whose armature becomes part of the input link of the mechanism, is mounted on one of the force platforms while the output link ground bearing support is mounted on the other. The coupler link is attached to the flywheel portion of the input link. Both input and output links carry circular counterweights which are designed according to the Method of Linearly Independent Vectors [8]. In addition, each of these links contains four symmetrical balancing holes at known distances from their axes of revolution. These serve, depending on need, to receive the correction masses associated with the mass-distance products ΔP_j and ΔQ_j (j=1,3).

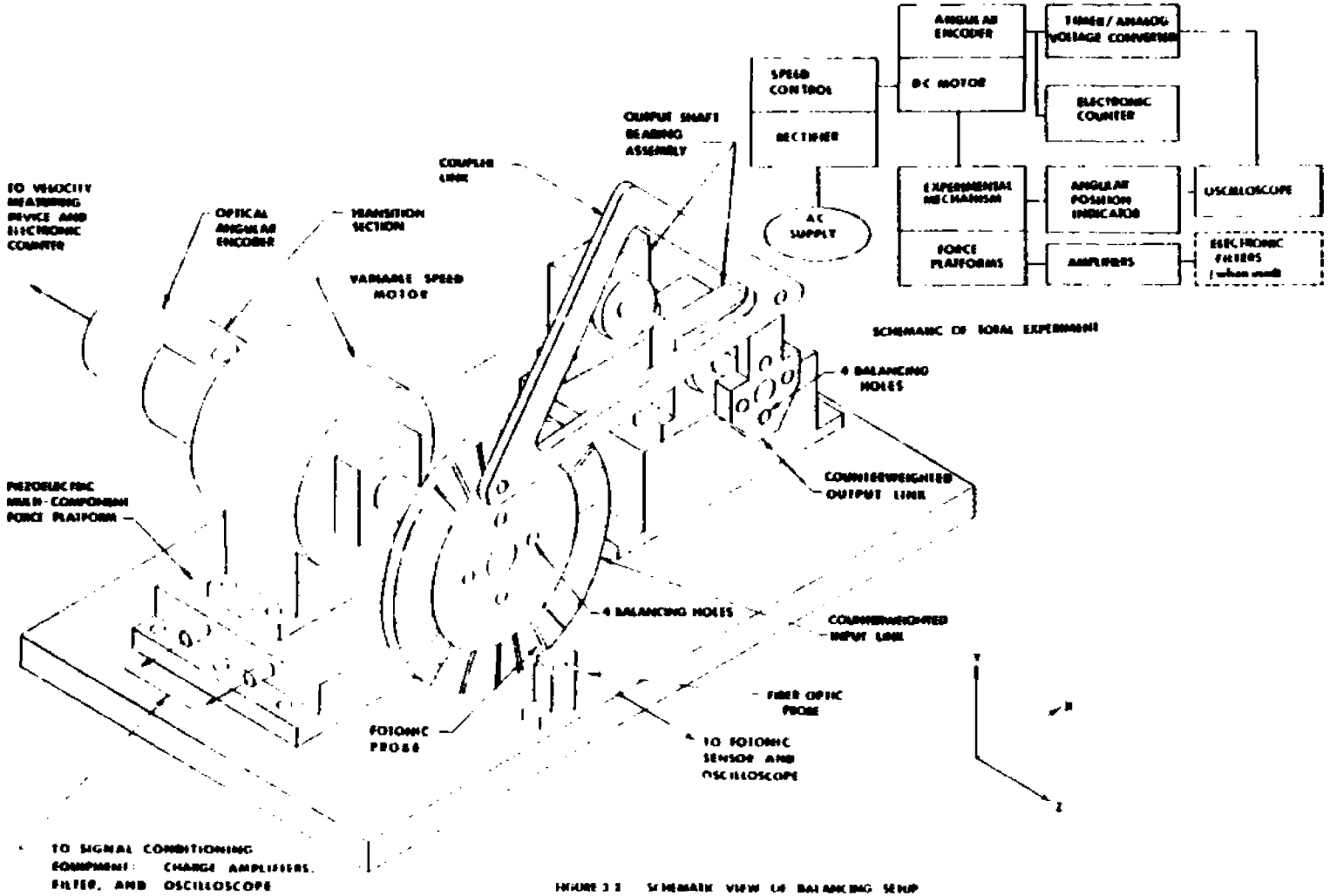
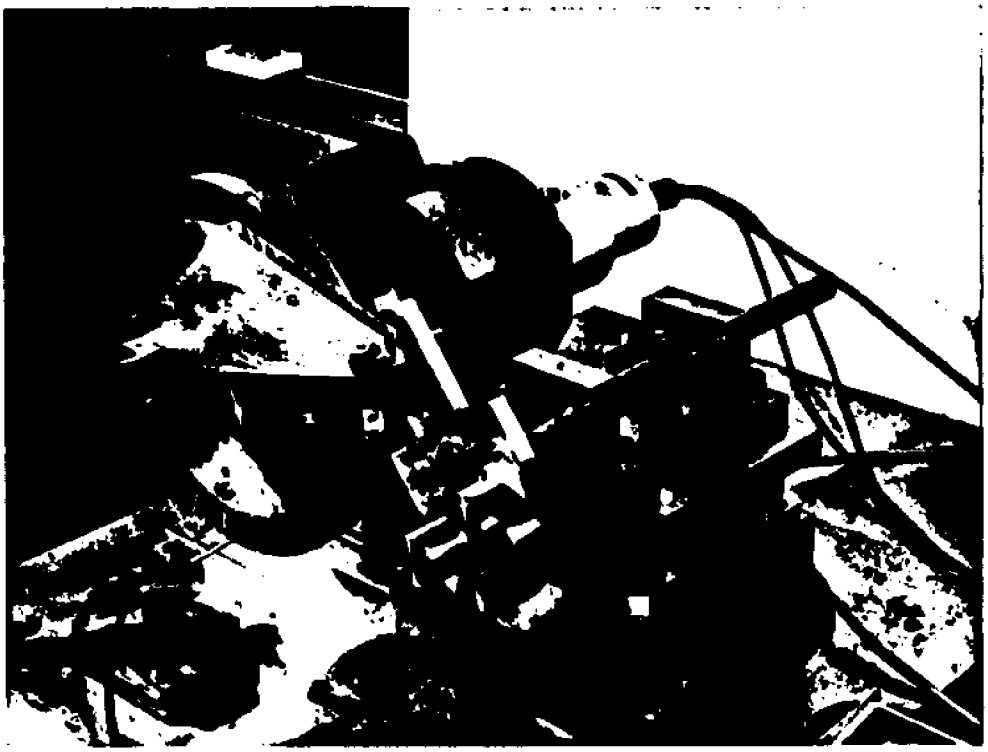


FIGURE 3.1 SCHEMATIC VIEW OF BALANCING SETUP



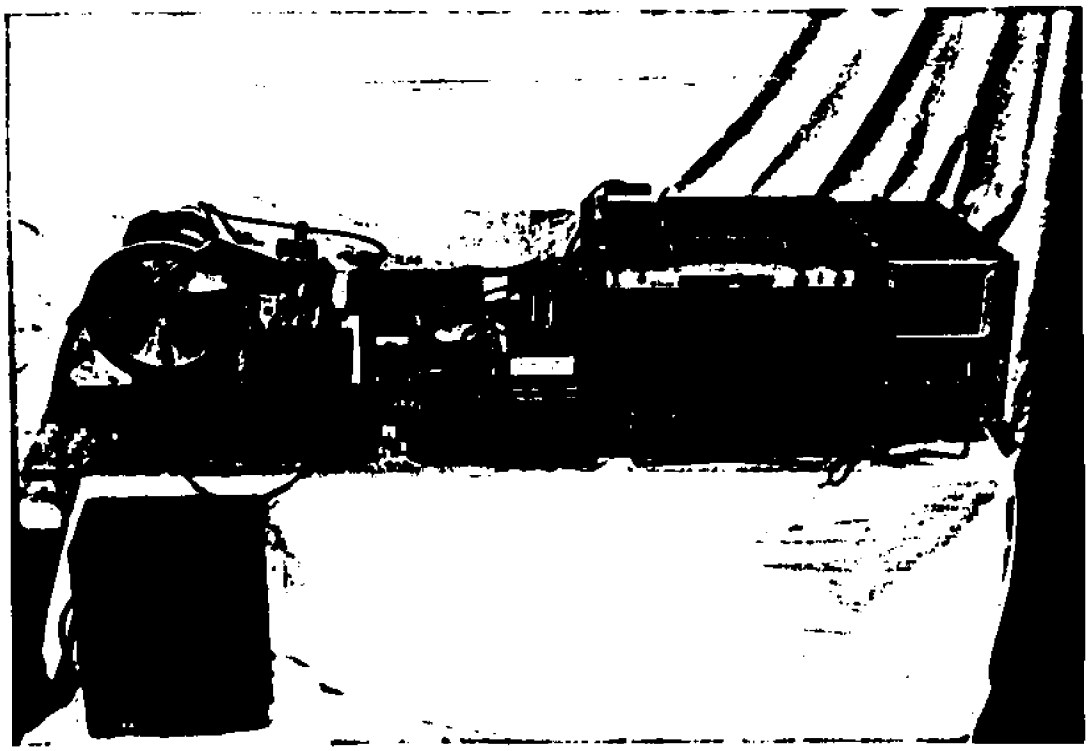


Figure 3.3: Photographs of Balancing Setup

Motor	Westinghouse DC Type FK, 1/6 HP
Motor Speed Control	Boston Gear Radiotrol 1/3 HP Motor Speed Control
Force Platforms	Kistler Instrumente, A.G. Three Component Measuring Platform Type 9257A
Amplifiers	Kistler Instrumente, A.G. Charge Amplifier, Type 5001
Fiber Optic Probe	MTI Instruments Division Fotonic Sensor Model KD-38
Angular Encoder	Theta Instruments Optical Incremental Shaft Encoder Model 05-720-3
Electronic Counter	Hewlett-Packard Timer-Counter Model 5326 A
Electronic Filters	Krohn-Hite Corporation Electronic Filter Model 3750
Oscilloscope	Tektronix Dual Trace Storage Display Model 5103N/D11

Table 3.1: Equipment Used In Experiment

The piezoelectric force platforms furnish the X and Y components of the ground bearing forces of the input and output links.

The fiber optic probe together with the reflecting strips on the flywheel are used for the determination of the angular position of the input link.

The angular encoder, which is driven directly by the armature shaft of the motor, supplies the signal for the instantaneous angular velocity measurement of the input link. It is also used in conjunction with an electronic counter to obtain the average angular velocity for one mechanism cycle.

Figure 3.4 shows the important dimensions of the linkage, including the locations of the balancing holes and of the centers of the circular counterweights. In addition, it indicates the position of the edge of the "zero-degree" reflecting strip with respect to the axis A_0A_1 (see Section 2b below). Table 3.2 lists all link mass parameters before and after the theoretical full force balance.

In order to solve for the correction masses, it is necessary to know, in addition to link dimensions, the magnitudes of the shaking force components together with the corresponding values of the input link angular velocity

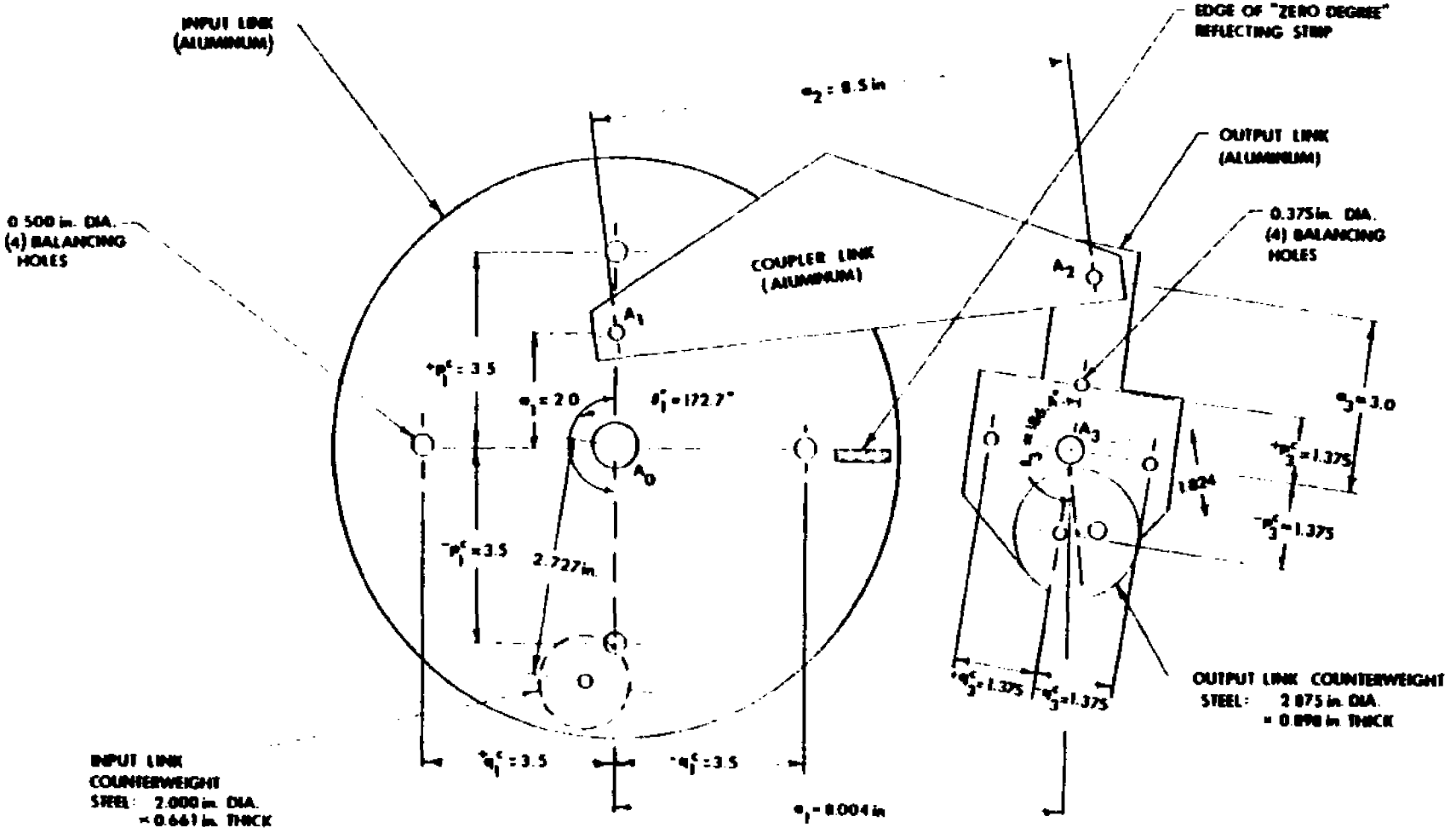


FIGURE 3.4 LINK DIMENSIONS

Table 3.2: Experimental Mechanism Data

Before Theoretical Balance			After Theoretical Balance			
Link 1	Link 2	Link 3	Link 1		Link 3	
p_1^o= .046 in	p_2^o=p_2= 4.950 in	p_3^o= .142 in	p_1^*=-2.702 in	p_1=- .105 in	p_3^*=-1.813 in	p_3=- .545 in
q_1^o= 0.0 in	q_2^o=q_2= 0.647 in	q_3^o= .001 in	q_1^*= 0.346 in	q_1= 0.019 in	q_3^*=-0.204 in	q_3=- .071 in
m_1^o= 274.041X10^{-4} lb-sec^2/in	m_2^o=m_2= 36.503X10^{-4} lb-sec^2/in	m_3^o= 75.855X10^{-4} lb-sec^2/in	m_1^*= 15.933X10^{-4} lb-sec^2/in	m_1= 289.964X10^{-4} lb-sec^2/in	m_3^*= 41.088X10^{-4} lb-sec^2/in	m_3= 116.943X10^{-4} lb-sec^2/in

and acceleration as functions of the input link angle.

The method of measuring these quantities will now be discussed.

2. Instrumentation And Data Evaluation

a. Shaking Force Components

The shaking force components F_{M/G_x} and F_{M/G_y} are obtained by adding the respective X and Y components of the ground bearing forces. These signals, once processed through charge amplifiers, are fed directly into the dual trace oscilloscope. When filtered traces are desired, a low pass electronic filter, adjusted to eliminate signals above 30 Hz, is used. (For typical unfiltered and filtered traces, see Figure 3.7.)

b. Input Link Angle

The flywheel carries 72 equally spaced reflecting strips which return the light of the fotonic sensor. The light beam is of sufficiently small diameter so that the probe is capable of furnishing a clearly defined signal on the oscilloscope as soon as the leading edge of the reflecting tape crosses the beam. (See the spiked signals on the bottom of the photographs in Figures 3.6a and 3.6b.) The

alignment of the reflecting strips was guided by radially engraved lines on the flywheel. The first of these accurately placed 5 degree lines coincides with the axis A_0A_1 which is used to define the input angle ϕ_1 .

When ϕ_1 equals zero, the leading edge of the "zero-degree" reflecting strip coincides with a vertical line through the center point A_0. Figure 3.4 shows this reflecting strip located to the left of this line which is at 90 degrees to the axis A_0A_1. This configuration is necessitated by the counterclockwise rotation of the link. In order to make sure that the signal indicating the zero position is initiated at the correct point, the probe is adjusted while the mechanism is clamped to make line A_0A_1 horizontal.

The angular encoder could not be used for this purpose since no simple alignment for the zero position could be devised.

c. Instantaneous Input Link Angular Velocity

The instantaneous input link angular velocity is obtained by determining the time intervals between the one-half degree signals from the angular encoder with the help of a digital counter. Subsequently, this count is converted to an analog voltage. Thus, the shorter the time,

the lower the voltage. Because of the nature of the counter and the digital-to-analog converter, it was necessary to correlate the voltage range to a given interval of angular velocities. Since the mechanism was run at speeds between 100 and 150 RPM, the 10 volt range had to correspond to time intervals of between 1/1200 to 1/1800 of a second[1]. Figure 3.5 illustrates the appearance of the velocity-voltage trace on the oscilloscope screen in relation to the input link angular position signals. The zero voltage line corresponding to 150 RPM is on top of the figure, while, due to an inverted voltage scale, lower speeds, as expressed by higher voltages, appear further down.

The determination of the angular velocity at point A of the above figure now serves as an example of the conversion of voltage to angular velocity:

a. Measure the voltage V_A with the help of the oscilloscope scale.

b. Convert this voltage to time:

$$t = \frac{1}{1800} + \left(\frac{V_A}{10}\right)\left(\frac{1}{1200} - \frac{1}{1800}\right) \qquad (3.5)$$

---

[1] The time for each of the 720 one-half degree intervals is given by $t = \frac{1}{12 \times RPM}$ (sec).

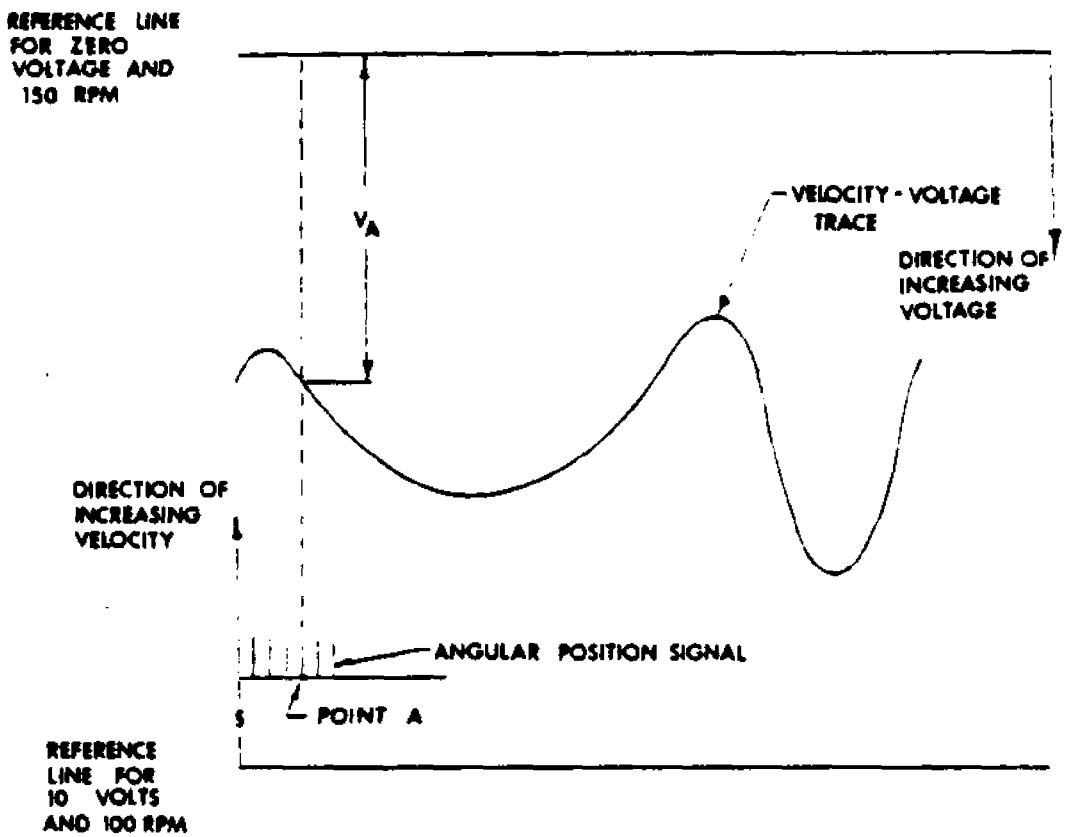


FIGURE 3.5: DETERMINATION OF INSTANTANEOUS ANGULAR VELOCITY

c. The angular velocity corresponding to this time interval between one-half degree signals is given by:

$$RPM = \frac{60}{720t}$$

or

$$\dot{\phi}_1 = \frac{2\pi}{720t} \quad \text{(rad/sec)} \tag{3.6}$$

d. Instantaneous Input Link Angular Acceleration

Once the instantaneous angular velocity $\dot{\phi}_1$ of the input link has been determined experimentally for all of the 72 five degree positions ϕ_1, it is expressed in the form of the Fourier series $\dot{\phi}_1 = f(\phi_1)$. This series is differentiated with respect to time to obtain the instantaneous input link angular acceleration $\ddot{\phi}_1$, i.e.

$$\ddot{\phi}_1 = \frac{df(\phi_1)}{d\phi_1} \cdot \dot{\phi}_1 \tag{3.7}$$

The value for $\dot{\phi}_1$ is taken from the experimental data.

This is accomplished by computer program FOURIE which is reproduced in Appendix C along with the Augmented Lagrangian package.

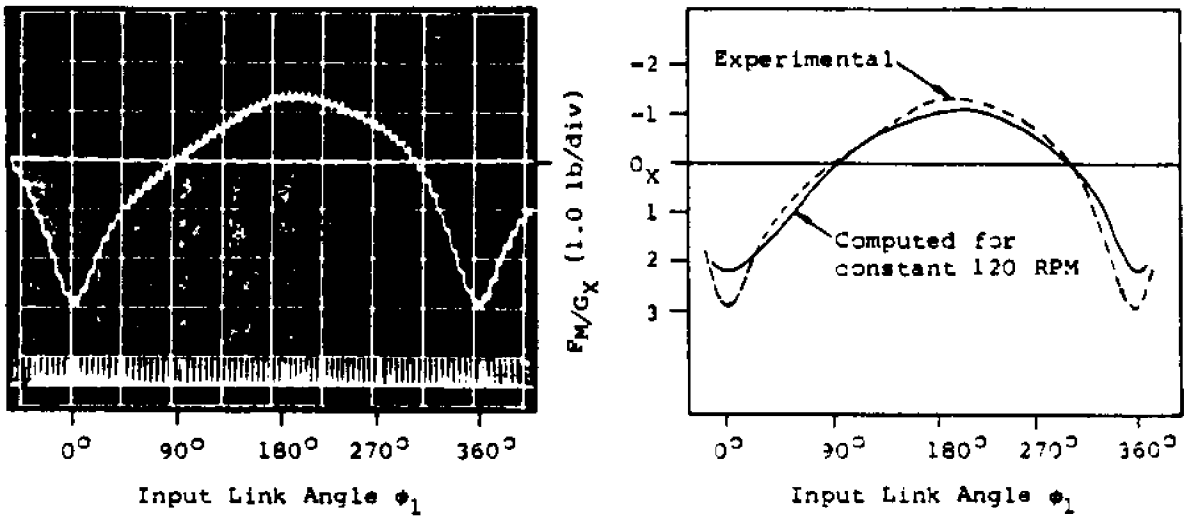
D. Experiment And Application Of Balancing Optimization Techniques

The following section shows how the experimental data, obtained from the theoretically balanced mechanism, are used in both the minimax and least squares fine balancing approaches. It further illustrates that the results of both methods lead to considerable shaking force reductions by way of the associated correction masses. The computational details of the minimax optimization are described in Appendix C. (This appendix also serves as an example of the use of the Augmented Lagrangian optimization program.) Appendix D shows the details of the program modifications for the least squares balancing optimization.

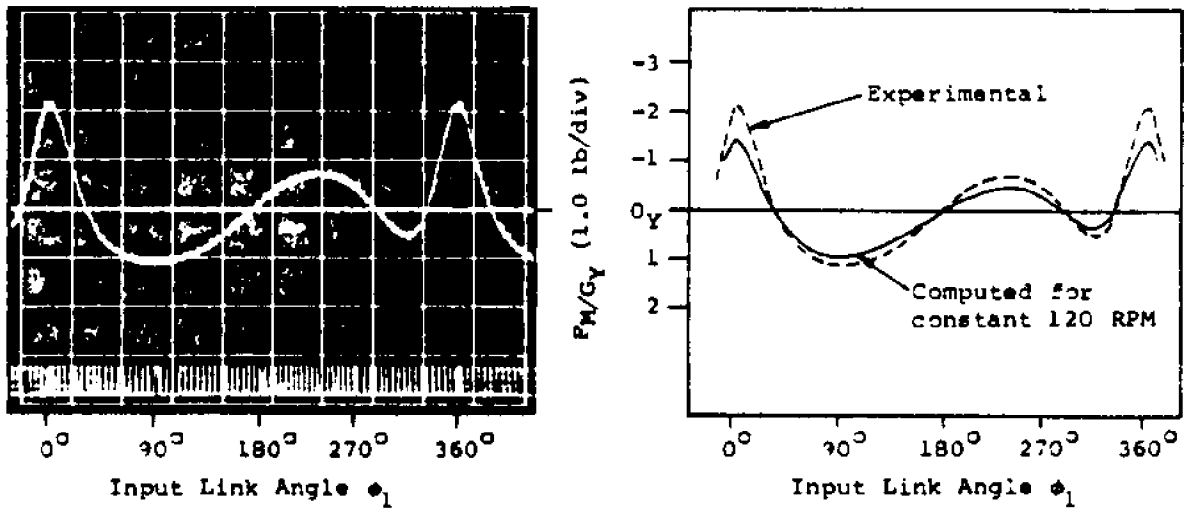
To put certain experimental procedures into proper perspective, the experience gained in connection with the unbalanced mechanism is discussed first.

1. Experience With Unbalanced Mechanism: Motor Vibrations And Speed Variations

Figure 3.6 shows oscillograms of the shaking force components of the unbalanced mechanism running at an average speed of 120 RPM, together with the associated angular



a. X-Component of Shaking Force (F_{M/G_X})



b. Y-Component of Shaking Force (F_{M/G_Y})

Figure 3.6: Comparison of Experimental and Computed Values of Shaking Force Components of Unbalanced Mechanism (Average Input Speed: 120 RPM)

position signals of the input link. (Note that in all shaking force oscillograms, the downward direction is positive.)

The superposed 120 cps wave is due to the vibrations of the DC motor which runs on a rectified 60 cps AC current. This effect, which is due to the magnetic contraction of the motor housing, becomes more pronounced as the force signals require greater amplification with improved force balance. Since this "noise" could not be excluded without totally changing the experimental setup and since electrical filtering distorts the output, it was decided to use graphically faired versions of these experimental curves for computational purposes.

Figures 3.6a and 3.6b also show comparisons of such faired experimental and computed force component curves of the unbalanced mechanism. (The computations were made with the parameters of Table 3.2 for a constant speed of 120 RPM.) These comparisons indicate that the experimental values of the forces exceed the computed ones by as much as 25 and 34 percent for the X and Y components, respectively. Investigation showed that a good part of this difference may be attributed to the variation of angular velocity during one mechanism cycle. This variation is even more pronounced for the

theoretically balanced mechanism, and it became clear early in the investigation that this change in angular velocity had to be determined and included in the computations. (See Section C-2-c for the actual experimental determination of the instantaneous velocity.)

2. Theoretically Balanced Mechanism

Figure 3.7 shows original, filtered as well as faired traces of the theoretically balanced mechanism when run at an average velocity of 120 RPM. (See Table 3.2 and Figure 3.4 for design data.)

The maximum shaking forces are 0.480 pounds and -0.590 pounds in the X and Y direction, respectively. These surprisingly large residual shaking force components furnish the justification for the present investigation. In addition to the shaking force components, Figure 3.7 gives simultaneous five degree angular position signals of the input link as well as the associated voltage-velocity trace. (Note the intentional enlargements of the angular position signals at 0, 90, 180 and 270 degrees.)

Table 3.3 lists force, angular velocity and angular acceleration values for each five degree interval of the input link angle ϕ_1 . The values for angular velocity

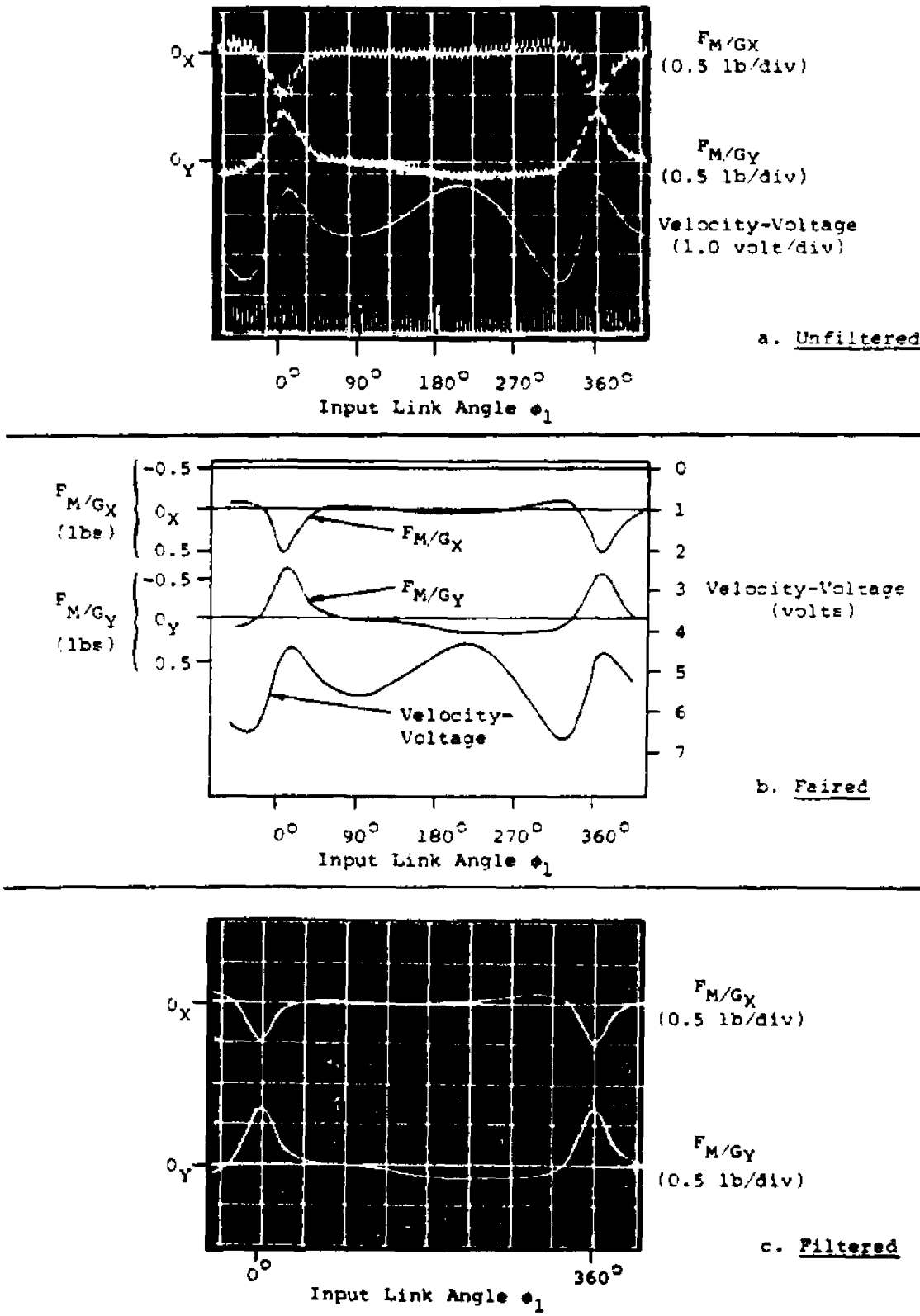


Figure 3.7: Shaking Force Components and Angular Velocity of Theoretically Balanced Mechanism (Average Input Speed: 120 RPM)

INPUT LINK ANGLE ϕ_1 (DEG.)	INPUT LINK ANGULAR VELOCITY $\dot{\phi}_1$ (RAD/SEC)	INPUT LINK ANGULAR ACCELERATION $\ddot{\phi}_1$ (RAD/SEC)	X COMPONENT OF SHAKING FORCE $F_{M/Gx}$ (lbs.)	Y COMPONENT OF SHAKING FORCE $F_{M/Gy}$ (lbs.)
0	12.751	27.136	.479	-.570
5	12.940	19.551	.480	-.590
10	12.965	- 7.477	.449	-.551
15	12.911	- 8.293	.384	-.468
20	12.832	-14.269	.298	-.374
25	12.746	-10.480	.202	-.282
30	12.670	-11.865	.105	-.218
35	12.597	- 9.863	.060	-.173
40	12.537	- 9.254	.030	-.120
45	12.476	- 7.376	.009	-.097
50	12.433	- 5.382	-.003	-.066
55	12.398	- 4.339	-.015	-.054
60	12.372	- 3.152	-.026	-.039
65	12.352	- 2.554	-.029	-.029
70	12.337	- 1.754	-.031	-.024
75	12.327	- 1.060	-.034	-.018
80	12.321	- 0.684	-.035	-.012
85	12.317	- 0.190	-.034	-.009
90	12.320	0.763	-.031	-.006
95	12.325	0.603	-.029	0.0
100	12.331	1.368	-.029	.006
105	12.341	1.403	-.027	.009
110	12.355	2.357	-.024	.015
115	12.371	2.120	-.023	.023
120	12.390	3.726	-.020	.026
125	12.418	3.589	-.015	.030
130	12.442	3.952	-.012	.039
135	12.474	4.908	-.012	.048
140	12.511	5.396	-.015	.058
145	12.552	6.182	-.017	.065
150	12.600	7.747	-.018	.075
153	12.654	7.408	-.021	.083
160	12.706	8.631	-.024	.093
165	12.770	8.968	-.027	.099
170	12.826	7.990	-.030	.116
175	12.891	7.770	-.031	.122
180	12.928	6.303	-.032	.129
185	12.968	5.399	-.033	.132
190	12.997	3.569	-.037	.141
195	13.019	2.973	-.038	.144
200	13.034	1.683	-.038	.150
205	13.043	0.905	-.038	.152
210	13.047	0.435	-.039	.155
215	13.049	0.092	-.041	.156
220	13.046	- 1.026	-.046	.158
225	13.038	- 1.417	-.051	.159
230	13.025	- 2.434	-.053	.160
235	13.006	- 3.212	-.056	.160
240	12.980	- 5.012	-.061	.161
245	12.939	- 6.305	-.063	.161
250	12.891	- 7.551	-.072	.161
255	12.835	- 9.277	-.077	.160
260	12.763	-11.500	-.084	.160
265	12.682	-12.206	-.091	.159
270	12.598	-12.382	-.096	.155
275	12.512	-12.186	-.101	.156
280	12.425	-12.824	-.113	.155
285	12.338	-11.257	-.119	.153
290	12.261	-11.568	-.126	.152
295	12.175	-11.208	-.134	.145
300	12.106	- 8.992	-.143	.144
305	12.037	- 9.297	-.147	.134
310	11.977	- 8.367	-.150	.129
315	11.913	- 7.419	-.144	.116
320	11.890	0.395	-.123	.096
325	11.896	1.444	-.119	.074
330	11.930	7.646	-.081	.044
335	11.989	8.686	-.032	-.023
340	12.077	15.597	.036	-.111
345	12.199	19.405	.122	-.206
350	12.390	33.705	.258	-.345
355	12.600	22.634	.356	-.438
360	12.751	27.136	.479	-.570

TABLE 3.3: EXPERIMENTAL DATA FOR THEORETICALLY BALANCED MECHANISM

and acceleration were determined by the methods of Section C-2-c and C-2-d, respectively. (See also program FOURIE in Appendix C-3-b.) This table furnishes the data for the minimax as well as the least squares balancing optimizations.

3. Results Of Minimax Balancing Optimization

This section gives the results of the minimax balancing optimization. Equations (C.9) in Section 3-d of Appendix C gives the following values for the "minimax" mass-distance products P_j and Q_j (j=1,3):

$$\begin{aligned} P_1 &= -50.85 \times 10^{-5} \quad \text{lb-sec}^2 \\ Q_1 &= -57.56 \times 10^{-5} \quad \text{lb-sec}^2 \\ P_3 &= 181.98 \times 10^{-5} \quad \text{lb-sec}^2 \\ Q_3 &= 21.95 \times 10^{-5} \quad \text{lb-sec}^2 \end{aligned} \tag{3.8}$$

According to equations (2.2) - (2.5), the mass-distance products of the correction masses then become:

$$\begin{aligned} \Delta P_1 &= m_{1p}{}^{c}(p_1{}^{c}) = 50.85 \times 10^{-5} \quad \text{lb-sec}^2 \\ \Delta Q_1 &= m_{1q}{}^{c}(q_1{}^{c}) = 57.56 \times 10^{-5} \quad \text{lb-sec}^2 \\ \Delta P_3 &= m_{3p}{}^{c}(p_3{}^{c}) = -181.98 \times 10^{-5} \quad \text{lb-sec}^2 \\ \Delta Q_3 &= m_{3q}{}^{c}(q_3{}^{c}) = -21.95 \times 10^{-5} \quad \text{lb-sec}^2 \end{aligned} \tag{3.9}$$

With the values for p_i^c and q_i^c (i=1,3) as given in Figure 3.4 the four correction masses are computed to be:

$$m_{1p}^c = \frac{\Delta P_1}{p_1^c} = \frac{50.85\times10^{-5}}{3.500} = 14.53\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the positive p_1^c location

$$m_{1q}^c = \frac{\Delta Q_1}{q_1^c} = \frac{57.56\times10^{-5}}{3.500} = 16.45\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the positive q_1^c location

$$m_{3p}^c = \frac{\Delta P_3}{p_3^c} = \frac{-181.98\times10^{-5}}{1.375} = 132.35\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the negative p_3^c location

$$m_{3q}^c = \frac{\Delta Q_3}{q_3^c} = \frac{-21.95\times10^{-5}}{1.375} = 15.96\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the negative q_3^c location

(3.10)

Figure 3.8 shows a photograph of the installed correction masses as they appear in both the input and output links. Figure 3.9 gives a comparison of the filtered shaking force components before and after minimax fine balancing at the same average input velocity of 120 RPM.

The maximum shaking force components are now $F_{M/G_x} = .174$ lb and $F_{M/G_y} = -.215$ lb. This represents a reduction of approximately 65 percent for each of the components when compared to the theoretically force balanced case. (See Section 2 above.)

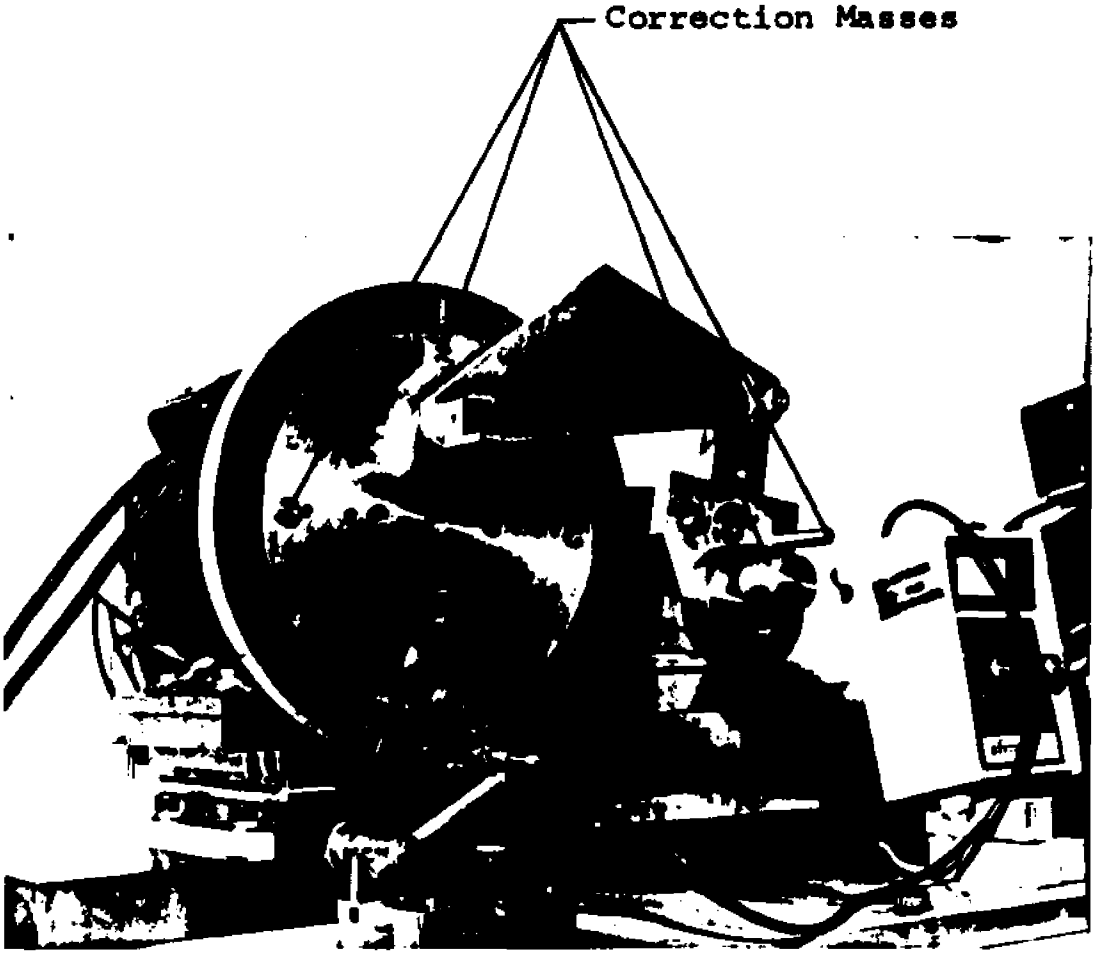
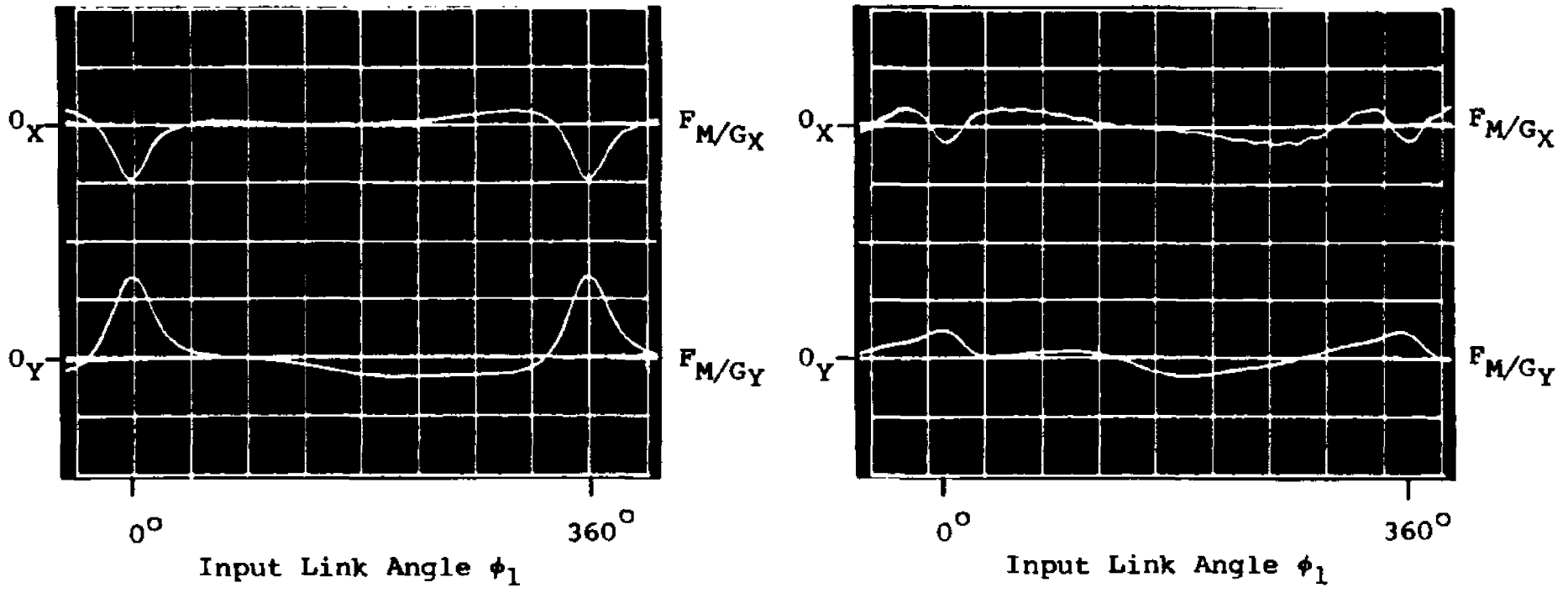


Figure 3.8: Photograph of Minimax Fine-Balanced Experimental Mechanism Showing Correction Masses Installed



a. Theoretically Balanced Mechanism

b. Minimax Fine-Balanced Mechanism

(All Force Scales: 0.5 lb/div)

Figure 3.9: Comparison of Filtered Shaking Force Components Before and After Fine Balancing By Minimax Optimization Method (Average Input Speed: 120 RPM)

4. Results Of Least Squares Balancing Optimization

This section reports on the results of the least squares balancing optimization. Equations (D.6) in Section 3 of Appendix D give the following values for the "least squares" mass-distance products P_j and Q_j (j=1,3):

$$\begin{aligned} P_1 &= -56.25 \times 10^{-5} \quad \text{lb-sec}^2 \\ Q_1 &= -51.81 \times 10^{-5} \quad \text{lb-sec}^2 \\ P_3 &= 158.42 \times 10^{-5} \quad \text{lb-sec}^2 \\ Q_3 &= 0.36 \times 10^{-5} \quad \text{lb-sec}^2 \end{aligned} \tag{3.11}$$

According to equations (2.2)-(2.5), the mass-distance products of the correction masses then become:

$$\begin{aligned} \Delta P_1 &= m_{1p}{}^c(p_1{}^c) = 56.25 \times 10^{-5} \quad \text{lb-sec}^2 \\ \Delta Q_1 &= m_{1q}{}^c(q_1{}^c) = 51.81 \times 10^{-5} \quad \text{lb-sec}^2 \\ \Delta P_3 &= m_{3p}{}^c(p_3{}^c) = -158.42 \times 10^{-5} \quad \text{lb-sec}^2 \\ \Delta Q_3 &= m_{3q}{}^c(q_3{}^c) = -0.36 \times 10^{-5} \quad \text{lb-sec}^2 \end{aligned} \tag{3.12}$$

With the values for $p_i{}^c$ and $q_i{}^c$ (i=1,3) as given in Figure 3.4, the four correction masses are computed to be:

$$m_{1p}{}^{c} = \frac{\Delta P_1}{p_1{}^{c}} = \frac{56.25\times10^{-5}}{3.500} = 16.07\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the positive $p_1{}^{c}$ location

$$m_{1q}{}^{c} = \frac{\Delta Q_1}{q_1{}^{c}} = \frac{51.81\times10^{-5}}{3.500} = 14.80\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the positive $q_1{}^{c}$ location

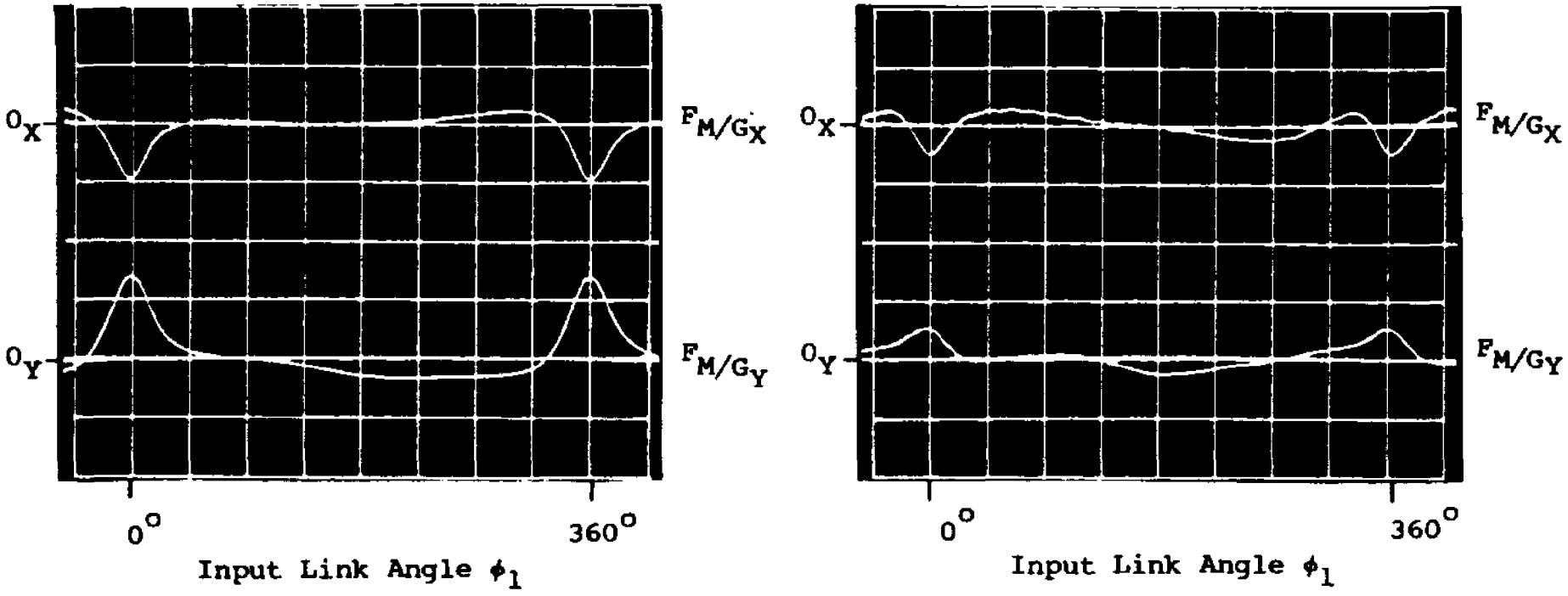
$$m_{3p}{}^{c} = \frac{\Delta P_3}{p_3{}^{c}} = \frac{-158.42\times10^{-5}}{1.375} = 115.21\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the negative $p_3{}^{c}$ location

$$m_{3q}{}^{c} = \frac{\Delta Q_3}{q_3{}^{c}} = \frac{-0.36\times10^{-5}}{1.375} = 0.26\times10^{-5} \left(\frac{\text{lb-sec}^2}{\text{in}}\right)$$
at the negative $q_3{}^{c}$ location

(3.13)

Figure 3.10 gives a comparison of the filtered shaking force components before and after least squares fine balancing at the same average input velocity of 120 RPM.

The maximum shaking force components are now $F_{M/G_x} = .241$ lb and $F_{M/G_y} = -.243$ lb. This represents a reduction of approximately 50 and 60 percent, respectively, for each of the components when compared to the theoretically force balanced case.



a. Theoretically Balanced Mechanism

b. Least Squares Fine-Balanced Mechanism

(All Force Scales: 0.5 lb/div)

Figure 3.10: Comparison of Filtered Shaking Force Components Before and After Fine Balancing By Least Squares Optimization Method (Average Input Speed: 120 RPM)

E. Discussion Of Results

The following sections first discuss the sources of a number of experimental inaccuracies as well as their influence on the magnitudes of the shaking force readouts. Subsequently, the two optimization approaches are compared in the light of these experimental inaccuracies and it is concluded that the minimax method is to be preferred since it furnishes the lowest residual shaking forces after fine balancing.

1. Sources Of Experimental Inaccuracies

a. Motor Vibration

As discussed in Section D-1, a 120 cps vibration of the motor was responsible for the superposed wave of approximately 0.1 lb. in amplitude in the shaking force readouts[1]. The necessary use of faired shaking force component curves clearly represents a source of inaccuracy.

The direct mounting of the motor on one of the force platforms is the result of a design compromise. In an earlier version of the setup, the motor was mounted separately and its moment was transmitted to the input shaft of the

---

[1] This motor vibration due to the rectified AC current was also found in another motor of higher horsepower.

mechanism by way of two universal joints. The input shaft bearing assembly was mounted on the force platform in the manner of the present output shaft bearing assembly (see Figure 3.2). Because of the inaccuracies of the universal joints, certain forces, in addition to the moment, were transmitted. These forces, which were then picked up by force platform, were sufficiently large to obliterate the shaking force component readings.

An improved signal-to-noise ratio might have been obtained if one could have run the mechanism at higher speeds. The clearance in the motor bearings made speeds much above 120 RPM impossible since the resulting impacts at the crank reversal points distorted the force traces even more than the motor vibrations.

In order to exclude the motor vibration as well as the unwanted force transmission from the motor, it would be best to redesign the setup such that the motor is mounted off the force platform and that it is uncoupled from the mechanism while the force readings are taken.

b. Cross Talk In Force Platforms

According to the manufacturer, the component charge outputs of the individual force platforms are subject to

cross talk, i.e. the output due to the force in one direction is influenced by that in the other. The specification sheet shows the following relationships:

$$F_{LX} = F_{14X} - .003F_{14Y} \tag{3.14}$$

$$F_{LY} = F_{14Y} - .002F_{14X} \tag{3.15}$$

$$F_{RX} = F_{34X} + .002F_{34Y} \tag{3.16}$$

$$F_{RY} = F_{34Y} + .009F_{34X} \tag{3.17}$$

where

F_{LX} = X component output of left force platform

F_{LY} = Y component output of left force platform

F_{RX} = X component output of right force platform

F_{RY} = Y component output of right force platform

This type of error may become significant with the present two-platform design since, at certain points in the cycle, the bearing force components become quite large. It might be reduced by mounting the entire mechanism on a single force platform. In such an arrangement, the sums of the bearing force components would be measured directly.

Since these sums are much smaller than the individual components, the effect of the cross talk will also be considerably reduced.

c. Ground Bearing Force Addition

The shaking force components were obtained by the addition of the appropriate individual force platform charge outputs. Because only two charge amplifiers were available, it was necessary to add the respective component signals before they entered these amplifiers. Since each component force pickup of a given platform requires a unique predetermined charge amplifier setting, the addition of the signals makes it impossible to obtain the correct setting for both force gauges simultaneously. On the suggestion of the manufacturer, an average setting was chosen, even though this causes the component outputs of the left (input link) platform to read somewhat higher than the actual value. The opposite effect occurs for the output of the right (output link) platform.

It will now be shown that this condition leads to an error which is proportional to the sum of the absolute values of the individual ground bearing force components.

For this purpose, let

F_L = output of left force platform (X or Y component force)

F_R = output of right force platform (X or Y component force)

and K = percentage error of individual bearing force component readings, assumed to be common to both platforms.

The resulting reading, F_{TOT}, of the sum of the bearing force components in a given direction is then given by:

$$F_{TOT} = (1 + K)F_L + (1 - K)F_R \tag{3.18}$$

If the actual shaking force component is represented by

$$R = F_L + F_R \tag{3.19}$$

equation (3.18) becomes:

$$F_{TOT} = R + K(F_L - F_R) \tag{3.20}$$

Since, near or at full force balance, F_L will always have a sign opposite to that of F_R, the above clearly shows

that the absolute magnitude of the error term $K(F_L - F_R)$ becomes largest when the magnitudes of the bearing force components are at their maxima. Thus, the reading at ϕ_1 = 0 degrees becomes more distorted than at all other positions.

To avoid this distortion, it would again be best to use only a single force platform.

2. Influence Of Cross Talk And Addition Errors On Magnitudes Of Experimental Shaking Force Readouts

Evidence (not shown here) from certain computer runs, in which the judicious variations of mass-distance products were made, indicates that the experimental results near ϕ_1 = 0 degrees are more exaggerated than at all other positions. (See Figure 3.6 for unbalanced mechanism and Figure 3.11, which is used in conjunction with the fine balancing optimizations.) It will now be shown how a combination of cross talk and addition errors contributes to this exaggeration.

The approximate magnitudes of the ground bearing force components of the theoretically balanced mechanism, at ϕ_1 = 0 degrees, are:

$F_{14X} \simeq 10.2$ lb $\quad F_{34X} \simeq -9.8$ lb so that $F_{M/G_X} \simeq 0.4$ lb

$F_{14Y} \simeq 2.0$ lb $\quad F_{34Y} \simeq -2.5$ lb and $F_{M/G_Y} \simeq -0.5$ lb

The absolute values of the above components represent maxima for the cycle.

Due to cross talk alone one obtains, according to equations (3.14)-(3.17):

$$F_{LX} = 10.2 - .003(2.0) = 10.194 \text{ lb} \tag{3.21}$$

$$F_{LY} = 2.0 - .002(10.2) = 1.980 \text{ lb} \tag{3.22}$$

$$F_{RX} = -9.8 + .002(-2.5) = -9.805 \text{ lb} \tag{3.23}$$

$$F_{RY} = -2.5 + .009(-9.8) = -2.588 \text{ lb} \tag{3.24}$$

With these values, the cross error components, C_X and C_Y, become:

$$C_X = (F_{LX} + F_{RX}) - F_{M/G_X}$$
$$= (10.194 - 9.805) - .400 = -.011 \text{ lb} \tag{3.25}$$

and $C_Y = (F_{LY} + F_{RY}) - F_{M/G_Y}$

$$= (1.980 - 2.588) + .500 = -.108 \text{ lb} \tag{3.26}$$

Thus, the cross talk error is larger for the Y component of the shaking force than for the X component.

Now consider how the addition error further influences the magnitudes of the final shaking force components. With the assumption that the percentage error of the individual bearing force component readings is represented by K=.005, substitution of the results shown in equations (3.21)-(3.24) into equation (3.18) yields the following values for the shaking force component readings:

$$F_{TOT_X} = 1.005(10.194) + .995(-9.805) = .489 \text{ lb}$$

$$F_{TOT_Y} = 1.005(1.980) + .995(-2.588) = -.585 \text{ lb}$$

Thus, the X component reads too high by .089 lb (.489 lb vs. .400 lb) while the Y component is too negative by .085 lb (-.585 lb vs. -.500 lb)[1].

Investigation, using equation (3.20), showed that after the addition of correction masses, i.e. at or very near to full force balance, there will be a reading error of essentially the same magnitude as long as the same instrumentation is used (i.e. K remains unchanged).

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[1] For comparison with the oscillograms, recall that the direction of the positive axes is downward.

3. Comparison Of Optimization Methods

In order to compare the relative advantages of the minimax and least squares methods, it is necessary to consider how well the curves associated with each of these methods, and which are used to determine the correction masses, approximate those obtained from experimentation. Further, it must be kept in mind that the experimental values are most exaggerated at ϕ_1 = 0 degrees, as has been discussed in the previous section. Figure 3.11 shows such comparisons for both methods.

The minimax method, as might be expected, makes the differences between the experimental and optimized curves essentially equal. For the least squares approach, these differences in each of the shaking force components are largest near ϕ_1 = 0 degrees. In addition, when compared to the minimax case, the absolute magnitudes of these differences are greater near ϕ_1 = 0 degrees and somewhat smaller near ϕ_1 = 90 and 180 degrees.

These results are consistent with the fact that the exaggeration of the experimental readings are most pronounced near ϕ_1 = 0 degrees. The minimax approach would give equal weight to all the data and attempt to equalize the differences

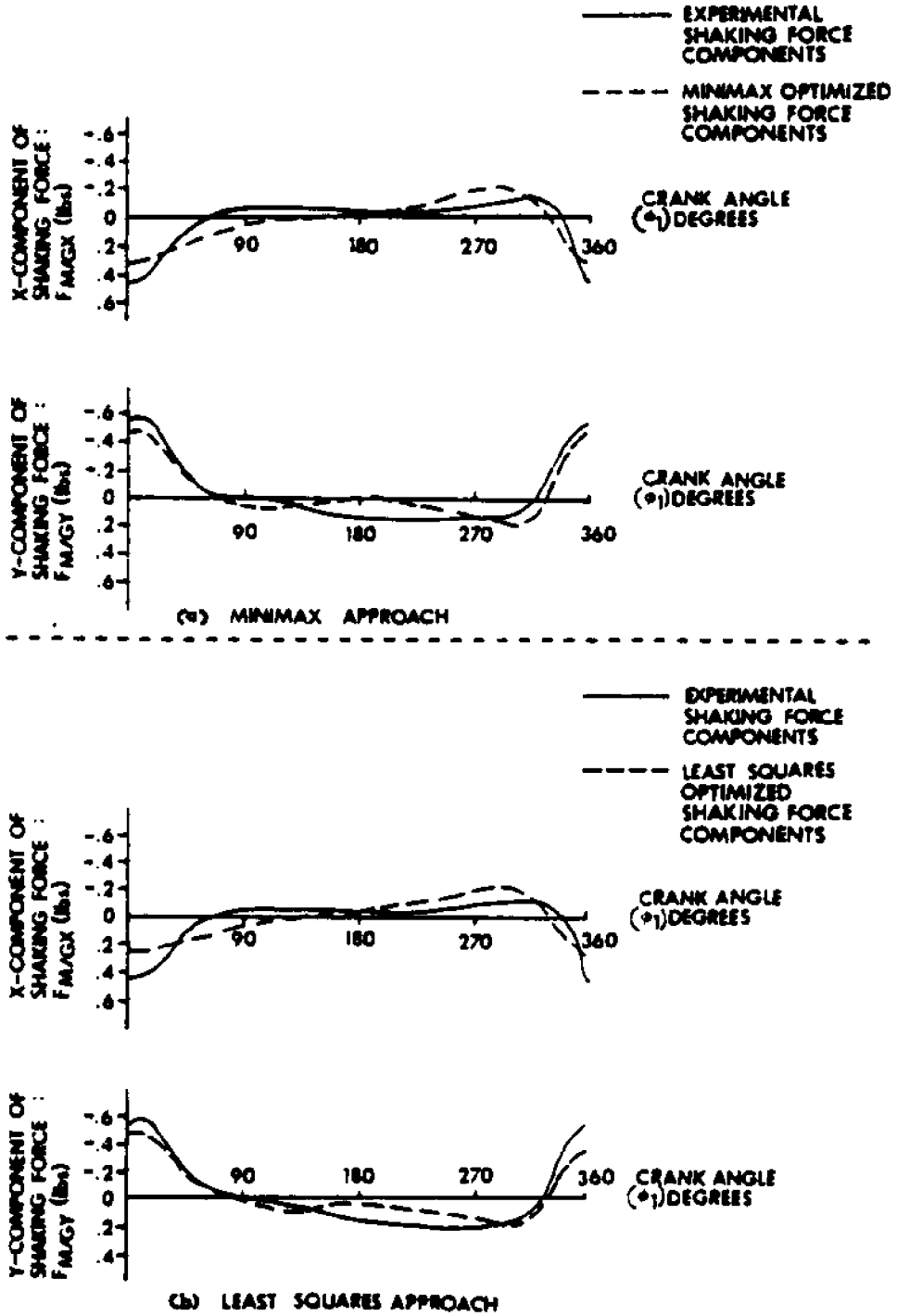

FIGURE 2.11: COMPARISON OF MINIMAX AND LEAST SQUARES OPTIMIZED SHAKING FORCE COMPONENTS WITH CORRESPONDING EXPERIMENTAL CURVES

between the curves. The least squares approach puts less weight on a single area of exaggerated readings and thus furnishes a somewhat better match at the remaining portions of the curves[1].

Even though the minimax method may somewhat overcompensate at $\phi_1 = 0$ degrees due to the greater exaggeration of the experimental values associated with this position of maximum shaking force components, it nevertheless reduces the magnitude of the shaking force components more than the least squares method. In view of the fact that these desirable decreases in the maximum values are not accompanied by substantial increases in the shaking force components at other locations during the cycle, one may conclude that the minimax method is to be preferred. This latter point especially holds in comparison with the mid-cycle results of the least squares method. (The above may best be seen by comparing the oscillograms of Figures 3.9 and 3.10.)

Finally, it should be noted that, because of the various experimental errors, no improvement could be obtained by a second fine balancing run.

Notwithstanding all of the above limitations, the

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[1] Computations which attempted to compensate for the errors at $\phi_1 = 0$ degrees alone resulted in a small improvement in the matching of these curves.

reductions in the maximum shaking force as obtained by both methods are most gratifying.

IV. A NEW METHOD FOR PRESCRIBING THE MAXIMUM SHAKING FORCE OF A FOUR-BAR LINKAGE WITH FLEXIBILITY IN COUNTERWEIGHT DESIGN

A. Introduction

This section presents a new two-counterweight method of partial force balancing of a four-bar linkage which allows the designer to realize a prescribed value for the maximum shaking force anywhere between zero and an inherent upper limit. By taking advantage of a certain mathematical property, which has been called the equipollent circle constraint equation, it is possible to attain the identical prescribed maximum shaking force for a wide range of output and associated input link counterweight configurations.

This technique, which is generally only applicable to constant speed mechanisms, builds upon a Chebyshev-type partial balancing method, first given by Ya. L. Gheronimus [25], which only uses an input link counterweight. (For a brief review of this work of Gheronimus, see Appendix E.) The equipollent circle constraint equation, which allows flexibility in counterweight design for a prescribed maximum shaking force, represents an adaptation of a parallel concept, used by G. G. Lowen and F. R. Tepper [52]

to prescribe the RMS shaking force.

In the method of Gheronimus, the shaking force expression is divided by the instantaneous input angle, which has the effect of projecting the shaking force hodograph onto the input link. The resulting hodograph now consists of a constant vector which represents the shaking force contributions proportional to the square of the input link speed, while the contributions of the coupler and output links are represented by a rotating vector of variable length. In order to reduce the maximum shaking force, a counterweight is added to the input link which is proportional to the distance from the origin to the "center" of the hodograph. To this end, the smallest circumscribing circle to the hodograph is determined. Its center represents the Chebyshev, or best uniform, approximation to the given hodograph by a polynomial of degree zero, i.e. a constant. The resulting maximum shaking force is proportional to the radius of the smallest circumscribing circle.

The present extension of this method addresses itself to that portion of the original hodograph which is due to the contributions of the coupler and the output links. It is reasoned that if the inclusion of an output link counterweight can reduce the size of the smallest circumscribing

circle, the resulting shaking force can also be decreased and, with that, limited in magnitude to a prescribed value.

Such a limiting procedure, which gives unique input and output link counterweights by the direct reduction of the circumscribing circle, is first shown.

Subsequently, the equipollent circle constraint equation is introduced. By offering an invariant relationship between the parameters of the output link counterweight, it presents the possibility of accomplishing the identical reduction in the maximum shaking force by a whole range of counterweight designs.

The determination of the smallest circumscribing circle as well as the coordinates of its center have been mechanized with the help of the Harwell Augmented Lagrangian optimization routine. Other computer programs which are based on the appropriate design equations furnish the counterweight dimensions for both of the aforementioned methods.

Examples involving both a 50 and a 75 percent reduction in the maximum shaking force are given and the associated ranges of possible counterweight designs are determined. The magnitudes of such maximum dynamic reactions as bearing forces, input moment and shaking moment have been computed for each of the above reductions in maximum shaking force for a

full range of counterweight designs. While it is clear that for some of the counterweight designs there are unreasonable increases in these dynamic reactions, there is a substantial range of counterweight configurations for which the variation in these reactions is not very large. In light of the above, this flexibility in counterweight configuration offers advantages when certain space limitations have to be observed.

The present work furnishes the basis for the effort in the next section where, in addition to prescribing the maximum shaking force, a simultaneous optimization of the maximum dynamic reactions is accomplished.

B. Hodograph And Statement Of Problem

Equation (A.26) in Appendix A, when modified for a constant input speed, gives the following expression for the shaking force of an unbalanced four-bar linkage (see Figure 2.1)[1]:

$$\bar{F}_{M/G} = A^{o}K_1 e^{i(\phi_1 + \alpha^{o})} + B^{o}K_3 e^{i(\gamma_3 + \beta^{o})} \tag{4.1}$$

where $A^{o} = \sqrt{(P_1{}^{o})^2 + (Q_1{}^{o})^2}$ (See equation (A.27).)

$B^{o} = \sqrt{(P_3{}^{o})^2 + (Q_3{}^{o})^2}$ (See equation (A.29).)

where the mass-distance products $P_j{}^{o}$ and $Q_j{}^{o}$ (j=1,3) are given by equations (A.31)-(A.34) and

$$K_1 = \dot{\phi}_1^{\,2} \quad \text{(See equation (A.3).)} \tag{4.2}$$

$$K_3 = \sqrt{\dot{\phi}_3^{\,4} + \ddot{\phi}_3^{\,2}} \quad \text{(See equation (A.3).)}$$

$$\gamma_3 = \phi_3 - \tan^{-1}\frac{\ddot{\phi}_3}{\dot{\phi}_3^{\,2}} \quad \text{(See equation (A.3).)}$$

$$\alpha^{o} = \tan^{-1}\frac{Q_1{}^{o}}{P_1{}^{o}} \quad \text{(See equation (A.28).)}$$

$$\beta^{o} = \tan^{-1}\frac{Q_3{}^{o}}{P_3{}^{o}} \quad \text{(See equation (A.30).)}$$

---

[1] The notation for parameters of balanced and unbalanced linkages, as given in Appendix A-3, is used.

The vector associated with the first term of the above expression is of constant magnitude and rotates with the constant angular velocity $\dot{\phi}_1$. The vector corresponding to the second term has a time-dependent magnitude and rotates with a time-dependent angular velocity.

If one now divides both terms of equation (4.1) by the unit vector $e^{i\phi_1}$ associated with the input link, one obtains:

$$\bar{F}_{M/G}e^{-i\phi_1} = A^{o}K_1 e^{i\alpha^{o}} + B^{o}K_3 e^{i(\gamma_3 + \beta^{o} - \phi_1)} \qquad (4.3)$$

This operation leaves the magnitude of the shaking force vector corresponding to any mechanism position intact, while its instantaneous direction is rotated through the angle $-\phi_1$. (See also Appendix E.)

Figure 4.1 shows a hodograph corresponding to equation (4.3). The constant vector $\overline{OD}^{o}$ represents the first term on the right hand side of equation (4.3), while vector $\overline{D^{o}E^{o}}$ stands for the second term. The magnitude of the maximum shaking force of the unbalanced mechanism is given by R^{o}_{MAX}. R_S is the radius of the smallest circumscribing circle to the hodograph. Its center is located by the vector $\overline{OC}_S$ at angle δ_S with respect to the X-axis. As shown in Appendix E, the radius R_S represents the maximum shaking force which

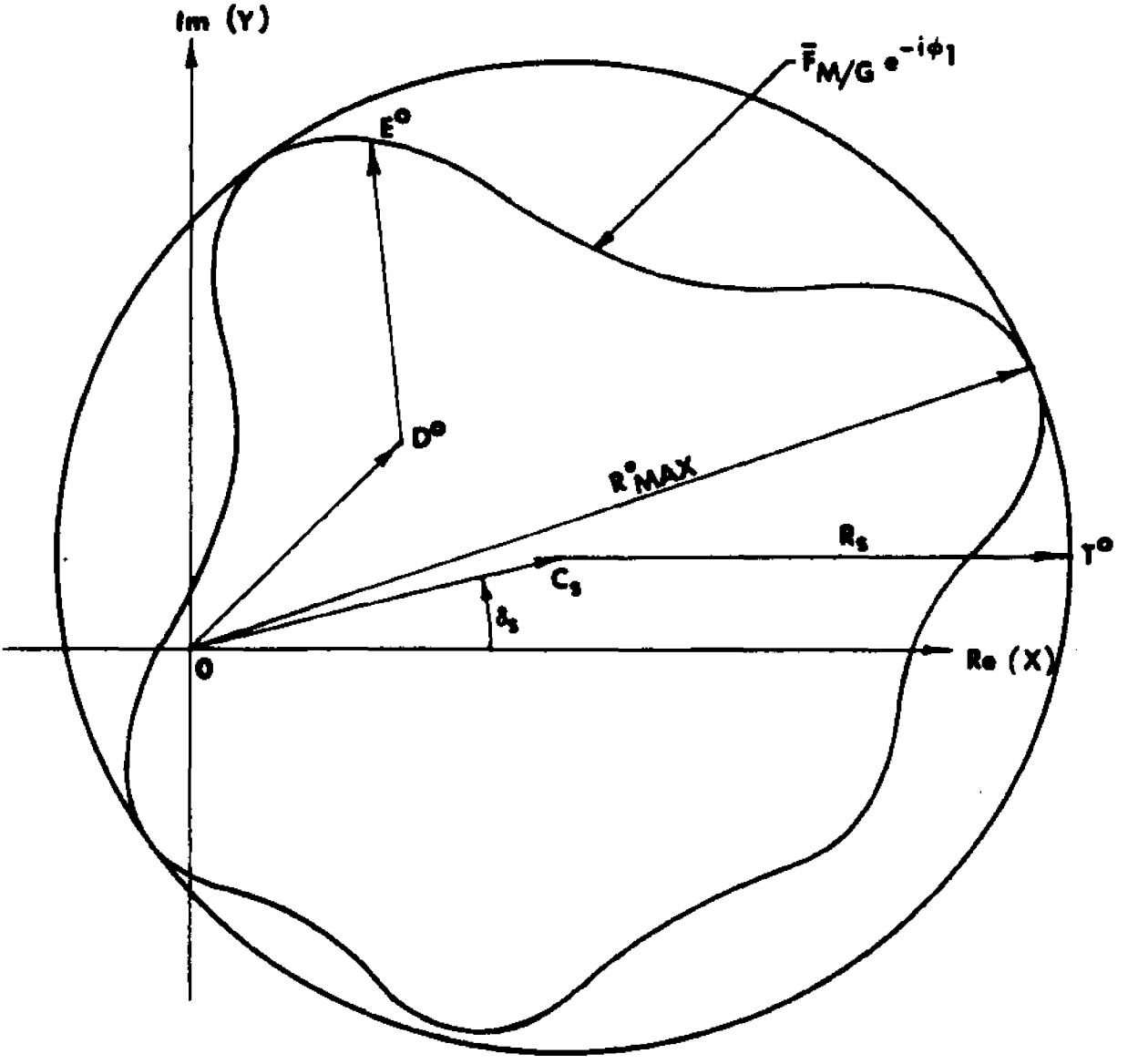


FIG. 4.1 SHAKING FORCE HODOGRAPH OF CONSTANT INPUT SPEED FOUR-BAR LINKAGE WITH ALL VECTORS ROTATED BY NEGATIVE OF INPUT ANGLE

results from a Chebyshev-type optimization involving a single counterweight attached to the input link[1].

If one now introduces an additional output link counterweight, one may reduce the maximum resulting shaking force further to a value corresponding to a radius R, which is smaller than R_S. Such a reduction may be related to the maximum unbalanced shaking force by way of reduction factor η, i.e.

$$\eta = \frac{R}{R^O_{MAX}} \tag{4.4}$$

The following sections show the realization of this goal. For the sake of clarity, a special case of the more general approach will be discussed first. It involves a proportional reduction of the hodograph which leads to unique counterweight parameters.

The more general approach shows that a whole range of counterweight parameters will satisfy the identical prescribed maximum shaking force. This method involves a rotation of the hodograph, in addition to its proportional reduction.

In either case, the radius R_S of the smallest circumscribing circle to the hodograph of the unbalanced

---

[1] This single counterweight method alone causes an approximate 35 percent reduction in maximum shaking force when applied to a four-bar linkage.

mechanism, according to equation (4.3), together with the location of the center C_S of this circle must first be found.

The required Chebyshev criterion is satisfied by the solution of the following minimization problem with inequality constraints:

$$\text{minimize} \quad R_S' \tag{4.5}$$

subject to the inequality constraints

$$(F_{X_i} - C'_{S_x})^2 + (F_{Y_i} - C'_{S_y})^2 \leq (R_S')^2 \tag{4.6}$$

where R_S' = the "trial radius" during the optimization (with units of force)

F_{X_i} = X coordinate of equation (4.3)

F_{Y_i} = Y coordinate of equation (4.3)

C'_{S_x}, C'_{S_y} = coordinates of point C_S', i.e. center of circumscribing circle of radius R_S'.

$i = 1,2,\ldots,360$, which corresponds to the angles $\phi_1 = 0,1,\ldots,359$ degrees.

Appendix F shows how this optimization, which uses the parameters R_S', C'_{S_x} and C'_{S_y} as variables, may be

accomplished for an example linkage with the help of the Harwell Augmented Lagrangian optimization program. (For program details, refer to Appendix C.)

When this optimization is completed, the objective function R_S' equals the desired radius R_S and the associated point C_S' becomes the final center C_S. The angle δ_S may be obtained from

$$\delta_S = \tan^{-1} \frac{C_{Sy}}{C_{Sx}} \tag{4.7}$$

C. Special Case Of General Solution: Simple Reduction Of Hodograph

The prescribed maximum shaking force corresponding to radius R is obtained by proportionally reducing the hodograph in such a way that R becomes the radius of the resulting circumscribing circle. Figure 4.2 shows the reduced hodograph which is traced out by vector

$$\overline{D^{o}E} = \mu \overline{D^{o}E^{o}} \quad . \tag{4.8}$$

The proportionality factor μ relates the radius R of the reduced circumscribing circle to the radius R_S of the original smallest circumscribing circle, i.e.

$$\mu = \frac{R}{R_S} \quad , \quad (\mu < 1) \tag{4.9}$$

where μ may be obtained with the help of equation (4.4):

$$\mu = \eta \frac{R^{o}_{MAX}}{R_S} \tag{4.10}$$

After the reduction, the sides of the quadrilateral $D^{o}ETC$ of the smaller hodograph have the proportionality μ with respect to the sides of the quadrilateral $D^{o}E^{o}T^{o}C_S$ of

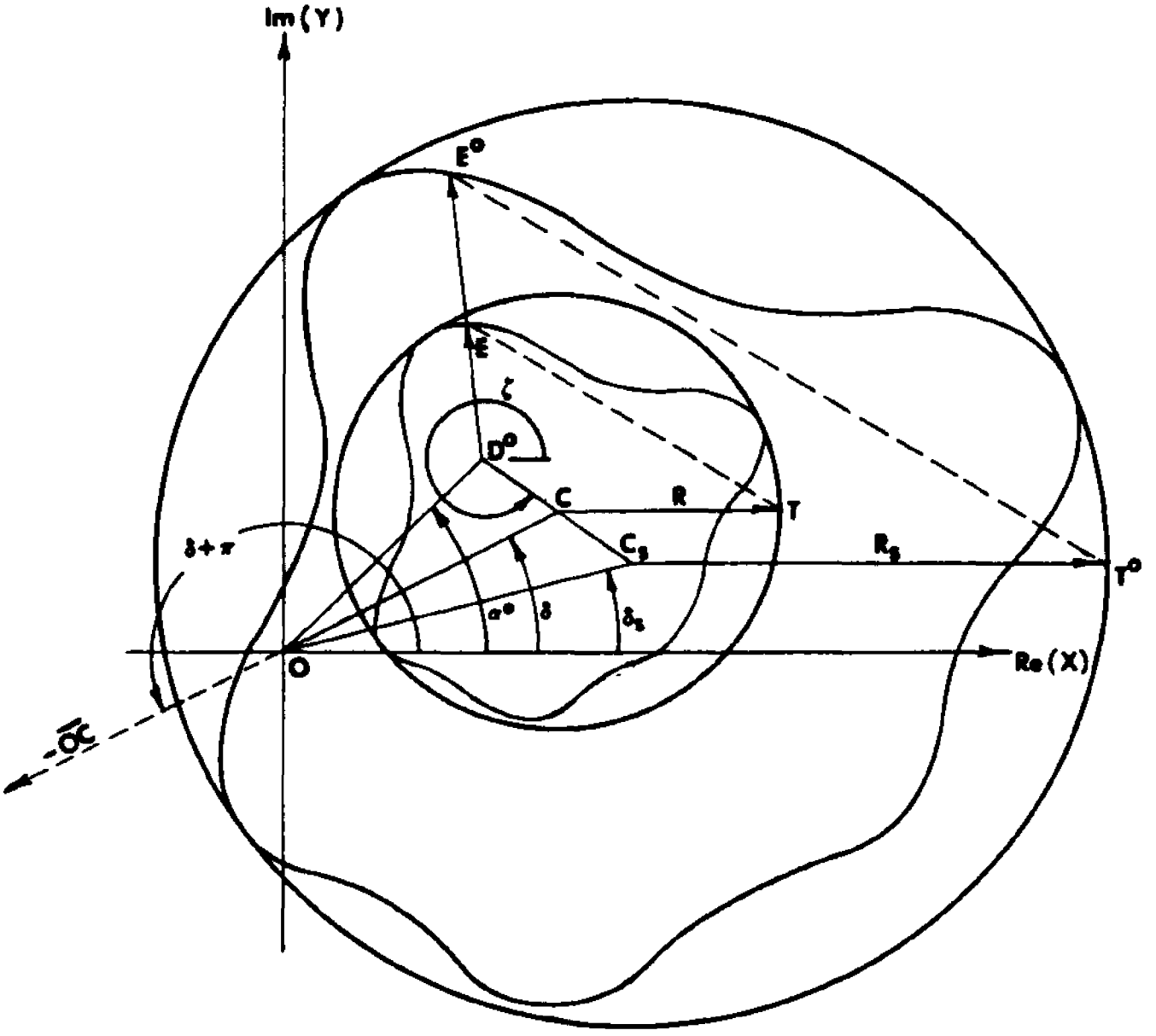


FIG. 4.2 ORIGINAL AND SIMPLY REDUCED SHAKING FORCE HODOGRAPH

the original hodograph. Therefore, the center of the reduced circumscribing circle of radius R is moved along line D^oC_s and may be found by:

$$\overline{D^oC} = \mu \overline{D^oC_s} \tag{4.11}$$

In order to realize the prescribed maximum shaking force, both output as well as input link counterweights must be designed. The output link counterweight must be such that the force corresponding to vector $\overline{D^oE^o}$ is reduced to one corresponding to vector $\overline{D^oE}$. The input link counterweight must be proportional to the vector $-\overline{OC}$.

1. Design Of Output Link Counterweight

The proportional reduction of the magnitude of the vector $\overline{D^oE^o}$ to $\overline{D^oE}$ is accomplished with the help of an output link counterweight. Accordingly, the second term in equation (4.3) becomes:

$$\overline{D^oE} = BK_3 e^{i(\gamma_3 + \beta - \phi_1)} \tag{4.12}$$

where

$$B = \mu B^o = \sqrt{(P_3)^2 + (Q_3)^2} \tag{4.13}$$

(See equation (A.15).)

When the counterweight parameters $m_3^* p_3^*$ and $m_3^* q_3^*$ are included, as shown in equations (A.37) and (A.38), one obtains:

$$P_3 = m_3^* p_3^* + P_3^o \tag{4.14}$$

and

$$Q_3 = m_3^* q_3^* + Q_3^o \tag{4.15}$$

Further, angle β becomes, according to equation (A.16),

$$\beta = \tan^{-1} \frac{Q_3}{P_3} \quad . \tag{4.16}$$

The simple reduction method satisfies the requirement of equation (4.13) by letting

$$P_3 = \mu P_3^o \tag{4.17}$$

and

$$Q_3 = \mu Q_3^o \quad . \tag{4.18}$$

The output link counterweight parameters can now be obtained by substituting equations (4.17) and (4.18) into equations (4.14) and (4.15). This results in:

$$m_3^* p_3^* = -(1 - \mu) P_3^o \tag{4.19}$$

and

$$m_3^* q_3^* = -(1 - \mu) Q_3^o \quad . \tag{4.20}$$

The above may be reduced, with the help of equations (A.21) and (A.29), to give

$$m_3^* r_3^* = (1 - \mu)B^o \qquad (4.21)$$

Use of equations (A.22) and (A.30) leads to the counterweight angle

$$\theta_3^* = \tan^{-1} \frac{q_3^*}{p_3^*} = \beta^o + \pi \quad . \qquad (4.22)$$

(See Figure A.2)

The fact that the reduced hodograph has the same relative position as the original one also reflects itself in the angle β . When equation (4.16) is applied to the present case, one obtains:

$$\beta = \tan^{-1} \frac{\mu Q_3^o}{\mu P_3^o} = \beta^o \qquad (4.23)$$

2. Design Of Input Link Counterweight

The force due to the required input link counterweight is represented by vector $-\overline{OC}$ at angle $\delta + \pi$. The following, which shows how the vector $\overline{OC} = OCe^{i\delta}$ is obtained, forms the basis for the design of the counterweight:

a. The optimization routine supplies the force components C_{S_x} and C_{S_y}, associated with the center point C_S. Thus, the vector $\overline{OC_S}$ is given by:

$$OC_S\, e^{i\delta_S} = \sqrt{(C_{S_x})^2 + (C_{S_y})^2}\; e^{i\delta_S} \qquad (4.24)$$

where $\delta_S = \tan^{-1} \dfrac{C_{Sy}}{C_{S_x}}$

b. The vector $\overline{OD^o}$ is known from equation (4.3), i.e.

$$\overline{OD^o} = A^o K_1 e^{i\alpha^o} \qquad (4.25)$$

c. To obtain the vector $OC\, e^{i\delta}$ it is further necessary to define the vector $\overline{D^oC_S}$ by way of

$$\overline{D^oC_S} = \overline{OC_S} - \overline{OD^o} \qquad (4.26)$$

This becomes, with the help of equations (4.24) and (4.25):

$$D^oC_S\, e^{i\zeta} = \sqrt{(C_{S_x})^2 + (C_{S_y})^2}\, e^{i\delta_S} - A^o K_1 e^{i\alpha^o} \qquad (4.27)$$

where $\zeta = \tan^{-1} \dfrac{D^oC_{Sy}}{D^oC_{S_x}}$

d. The vector $\overline{OC}$ may now be found with the help of equation (4.11) in addition to equations (4.25) and (4.27). Thus,

$$\overline{OC} = OC\,e^{i\delta} = A^{O}K_1 e^{i\alpha^{O}} + \mu D^{O}C_S e^{i\zeta} \tag{4.28}$$

where $\delta = \tan^{-1}\dfrac{OC_Y}{OC_X}$

e. As stated earlier, the force supplied by the input link counterweight must equal

$$-OC\,e^{i\delta} = OC\,e^{i(\delta + \pi)} \tag{4.29}$$

This suggests the following design equations:

$$m_1^{*} r_1^{*} = \frac{|\overline{OC}|}{\dot{\phi}_1^{2}} \tag{4.30}$$

and $$\theta_1^{*} = \delta + \pi \tag{4.31}$$

3. Relationship To Full Force Balancing

The full force balancing results associated with the Method of Linearly Independent Vectors [8] are obtained by first letting $\mu = 0$. Substitution of this value into equation (4.21) and use of equation (4.22) furnishes the

output link counterweight parameters. The radius of the reduced circumscribing circle is now zero and its center C is coincident with point D^{o} (see Figure 4.2 and equation (4.11)). Therefore, the input link counterweight parameters are found by substituting $\overline{OC} = \overline{OD}^{o}$ and $\delta = \alpha^{o}$ into equations (4.30) and (4.31), respectively.

D. General Solution: Equipollent Circle Constraint Equation And Flexibility In Counterweight Design

The following shows that the prescribed maximum shaking force which the mechanism exerts on the ground remains constant as long as the mass-distance products associated with the center of mass of the output link lie on certain circles. This invariance phenomenon gives the opportunity for flexibility in counterweight design since a whole range of counterweight radii and angles produce the identical maximum shaking force.

1. Equipollent Circle Constraint Equation And Resulting Hodograph

When equation (4.13) is rewritten with the help of equations (A.10) and (A.11), one obtains, after squaring both sides of the resulting expression:

$$(m_3 p_3 + \frac{a_3}{a_2} m_2 p_2)^2 + (m_3 q_3 + \frac{a_3}{a_2} m_2 q_2)^2 = (\mu B^{\circ})^2 \qquad (4.32)$$

This form implies that the magnitude of μB°, as well as the absolute magnitude of a force vector analogous to vector $\overline{D^{\circ}E} = \mu \overline{D^{\circ}E^{\circ}}$ in Figure 4.2, remains the same as

long as the mass-distance product terms m_3p_3 and m_3q_3, associated with the center of mass of the balanced output link, lie on a circle of radius

$$R_E = \mu B^o \quad . \tag{4.33}$$

Thus, a range of output link counterweight configurations will lead to the identical prescribed maximum shaking force. Because of this invariance, the circles represented by equation (4.32) have been termed equipollent circles while the expression itself is called the equipollent circle constraint equation. (This terminology was first used in [52].)

For the purpose of counterweight design, one must consider how the above influences the angular position of the reduced hodograph and, with that, the location of the center C_1 of its smallest circumscribing circle.

Equations (4.14)-(4.16) show that a given choice of output link parameters $m_3^*p_3^*$ and $m_3^*q_3^*$ decides the magnitude of the general mass-distance terms P_3 and Q_3 and, with that, the angle β. Consequently, whenever the angle β is not equal to

$$\beta^o = \tan^{-1} \frac{Q_3^o}{P_3^o} \quad ,$$

the reduced shaking force component

$$BK_3 e^{i(\gamma_3 + \beta - \phi_1)}$$

leads the original term

$$B^{o}K_3 e^{i(\gamma_3 + \beta^{o} - \phi_1)}$$

by the constant phase angle

$$\psi = \beta - \beta^{o} \quad . \tag{4.34}$$

Figure 4.3 represents this reduced force component by the vector $\overline{D^{o}E_1}$. It is shown to lead the vector $\overline{D^{o}E^{o}}$ by the angle ψ . As a consequence, the center of the circumscribing circle is rotated into the new position C_1 (with magnitude $D^{o}C_1 = D^{o}C$) and is located by way of the angle δ_1 with respect to the X-axis. The input link counterweight must therefore provide a force proportional to vector $-\overline{OC_1}$ at the angle $\delta_1 + \pi$.

2. Adaptation To Output Link Counterweight Design

a. Determination Of Theoretical Design Range

In order to explore the range of counterweight designs which produce the identical prescribed maximum shaking force,

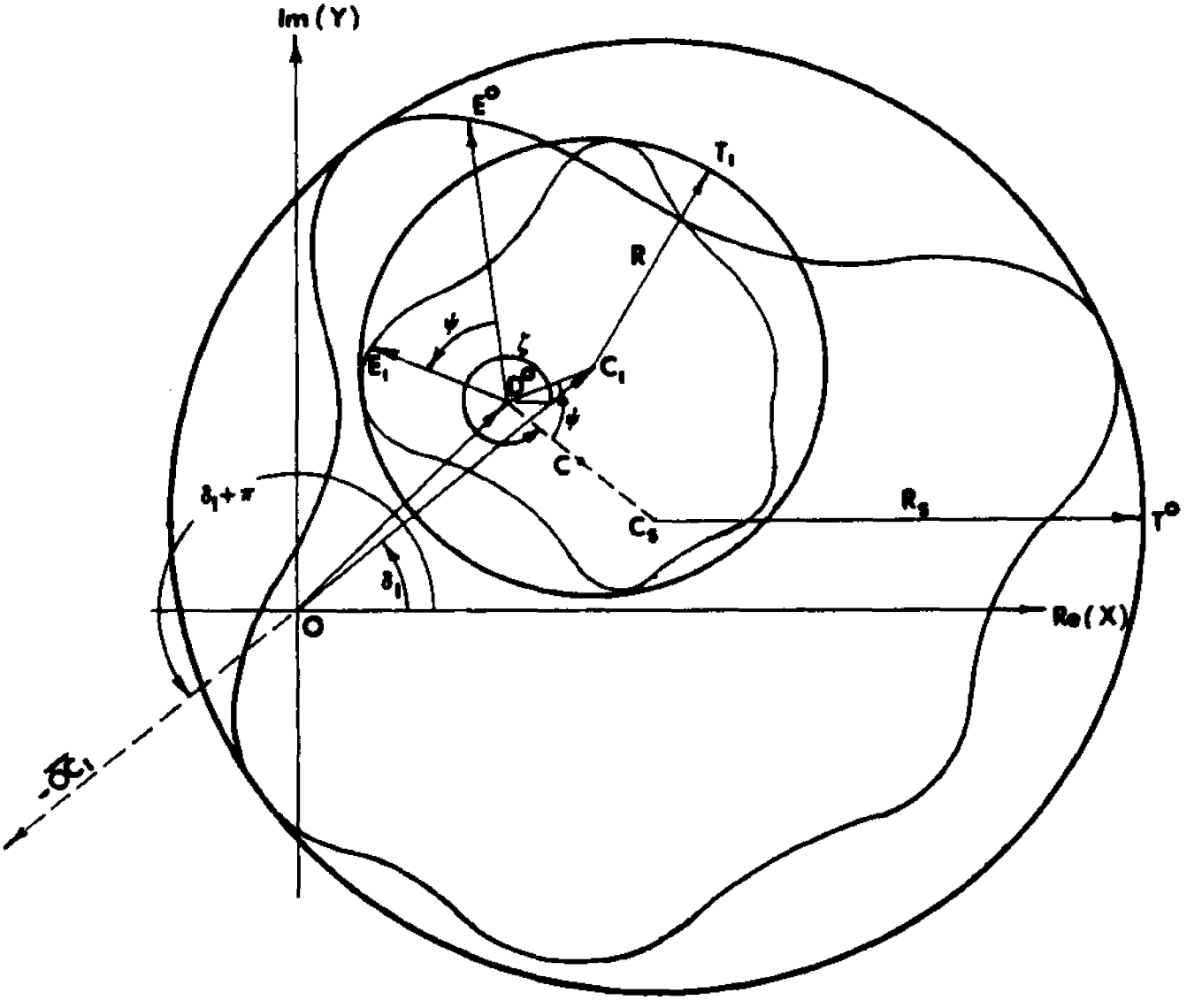


FIG. 4.3 ORIGINAL AND REDUCED SHAKING FORCE HODOGRAPH RESULTING FROM THE FLEXIBLE DESIGN METHOD

equation (4.32) is rewritten in terms of output link counterweight parameters (see equations (4.14), (4.15)):

$$(m_3^* p_3^* + P_3^o)^2 + (m_3^* q_3^* + Q_3^o)^2 = (\mu B^o)^2 \qquad (4.35)$$

For convenience, the above is transformed to the following using equations (A.21), (A.22) and (A.29):

$$(m_3^* r_3^*)^2 + 2(P_3^o \cos\theta_3^* + Q_3^o \sin\theta_3^*) m_3^* r_3^* + (1 - \mu^2)(B^o)^2 = 0 \qquad (4.36)$$

All combinations of $m_3^* r_3^*$ and θ_3^* which satisfy the above expression represent theoretically possible designs for a given maximum shaking force. To determine the limits of $m_3^* r_3^* = m_3^* r_3^*(\theta_3^*)$ within which this flexibility exists, equation (4.36) is first implicitly differentiated with respect to θ_3^* and then $\frac{d(m_3^* r_3^*)}{d\theta_3^*}$ is set equal to zero. Thus,

$$\left[2m_3^* r_3^* + 2(P_3^o \cos\theta_3^* + Q_3^o \sin\theta_3^*)\right]\frac{d(m_3^* r_3^*)}{d\theta_3^*} = 2(P_3^o \sin\theta_3^* - Q_3^o \cos\theta_3^*) m_3^* r_3^* \qquad (4.37)$$

For $\frac{d(m_3^* r_3^*)}{d\theta_3^*} = 0$, one obtains the two extremum conditions:

$$(\theta_3^*)_{EX_{1,2}} = \tan^{-1} \frac{Q_3^o}{P_3^o} \tag{4.38}$$

These two possible solutions for the extreme counterweight angles correspond to angles β^o and $\beta^o + \pi$ (see equation (A.30)).

Solution of equation (4.36) for the associated mass-distance products $(m_3^* r_3^*)_{EX}$ shows that the necessary positive values of these parameters can only be obtained for

$$\theta_3^* = \beta^o + \pi \tag{4.39}$$

This requires substitution of the following into equation (4.36):

$$\sin(\theta_3^*)_{EX} = \frac{-Q_3^o}{B^o} \tag{4.40}$$

and

$$\cos(\theta_3^*)_{EX} = \frac{-P_3^o}{B^o} \tag{4.41}$$

where

$$B^o = \sqrt{(P_3^o)^2 + (Q_3^o)^2}$$

The resulting solution of the quadratic equation furnishes the minimum and maximum counterweight parameters

of the theoretical design range:

$$(m_3^* r_3^*)_{MIN} = (1 - \mu)B^o \tag{4.42}$$

and

$$(m_3^* r_3^*)_{MAX} = (1 + \mu)B^o \tag{4.43}$$

Note that the minimum case is identical to that obtained by the simple reduction of the hodograph (see equations (4.21) and (4.22)).

b. Determination Of Output Link Counterweight Angles θ_3^*

To determine the values of the angles θ_3^* corresponding to any mass-distance product $m_3^* r_3^*$ within the theoretical design range, the desired $m_3^* r_3^*$ is substituted into the following modification of equation (4.36):

$$K_4 + K_5 \cos\theta_3^* = -K_6 \sin\theta_3^* \tag{4.44}$$

where

$$K_4 = (m_3^* r_3^*)^2 + (1 - \mu^2)(B^o)^2$$

$$K_5 = 2P_3^o m_3^* r_3^*$$

$$K_6 = 2Q_3^o m_3^* r_3^*$$

Squaring, simplification and subsequent rearrangement leads to the following quadratic equation:

$$\left[(K_5)^2 + (K_6)^2\right]\cos^2\theta_3^* + 2K_4K_5\cos\theta_3^* + (K_4)^2 - (K_6)^2 = 0 \tag{4.45}$$

Solving for $\cos\theta_3^*$ yields:

$$\cos\theta_3^* = \frac{-K_4K_5 \pm K_6\sqrt{(K_5)^2 + (K_6)^2 - (K_4)^2}}{(K_5)^2 + (K_6)^2} \tag{4.46}$$

Because of the squaring of equation (4.44), there will be four possible solutions. The two valid ones must satisfy equation (4.44).

It is to be noted that for any given mass-distance product $m_3^* r_3^*$ within the theoretical design range, there are two possible angular locations θ_3^*.

3. Design Of Input Link Counterweight

Since there are two possible angles θ_3^* for a given $m_3^* r_3^*$ within the theoretical design range, there must also be two corresponding input link counterweight designs[1].

For a given choice of angle θ_3^*, there will be a vector

$$-\overline{OC}_1 = OC_1 e^{i(\delta_1 + \pi)}$$

---

[1] Recall that for each of the extrema, there is only one input link counterweight.

which forms the basis of the input link counterweight design (see Figure 4.3). This design is accomplished by the following steps:

a. Angle β is found with the help of a modification of equation (4.16), i.e.

$$\beta = \tan^{-1} \frac{Q_3^{\circ} + m_3^* r_3^* \sin\theta_3^*}{P_3^{\circ} + m_3^* r_3^* \cos\theta_3^*} \qquad (4.47)$$

b. The angle of rotation ψ is then obtained by way of equation (4.34):

$$\psi = \beta - \beta^{\circ} \qquad (4.48)$$

where $\beta^{\circ} = \tan^{-1} \frac{Q_3^{\circ}}{P_3^{\circ}}$ (See equation (4.2).)

c. The vector $\overline{D^{\circ}C_1}$, as shown in Figure 4.3, is found by the rotation of vector $\overline{D^{\circ}C}$. Thus, with the help of equation (4.27):

$$\overline{D^{\circ}C_1} = \mu D^{\circ}C_S e^{i(\zeta + \psi)} \qquad (4.49)$$

d. With the above, the vector $\overline{OC_1}$ may be determined

from:

$$OC_1 e^{i\delta_1} = OD^{o} e^{i\alpha^{o}} + \mu D^{o} C_S e^{i(\zeta + \psi)} \qquad (4.50)$$

where $$\delta_1 = \tan^{-1} \frac{OC_{1Y}}{OC_{1X}}$$

e. Finally, the force of the counterweight must equal

$$-OC_1 e^{i\delta_1} = OC_1 e^{i(\delta_1 + \pi)} \qquad (4.51)$$

This leads to the design equations

$$m_1^{*} r_1^{*} = \frac{|\overline{OC}_1|}{\dot{\phi}_1^{2}} \qquad (4.52)$$

and $$\theta_1^{*} = \delta_1 + \pi \ . \qquad (4.53)$$

E. Examples

The following sample calculations apply the previously developed techniques to the four-bar linkage shown in Figure 4.4, which is assumed to run at a constant speed of 500 rpm. While a 50 percent reduction in the maximum shaking force is shown in detail, only the results of a 75 percent reduction are given. The influence of the design flexibility on the dynamic reactions is considered for both cases.

The counterweights are assumed to be circular as well as tangent to the pivots A_0 and A_3, respectively.

1. Parameters Of Unbalanced Mechanism

Table 4.1 lists the parameters of the unbalanced mechanism. All links and counterweights are made of steel (density = 0.283 lb/in^3).

The general mass-distance terms of the unbalanced mechanism are, according to equations (A.31)-(A.34):

$$P_1^{\circ} = 18.777 \times 10^{-4} \quad lb\text{-}sec^2 \tag{4.54}$$

$$Q_1^{\circ} = -1.836 \times 10^{-4} \quad lb\text{-}sec^2 \tag{4.55}$$

$$P_3^{\circ} = 30.913 \times 10^{-4} \quad lb\text{-}sec^2 \tag{4.56}$$

$$Q_3^{\circ} = 2.755 \times 10^{-4} \quad lb\text{-}sec^2 \tag{4.57}$$

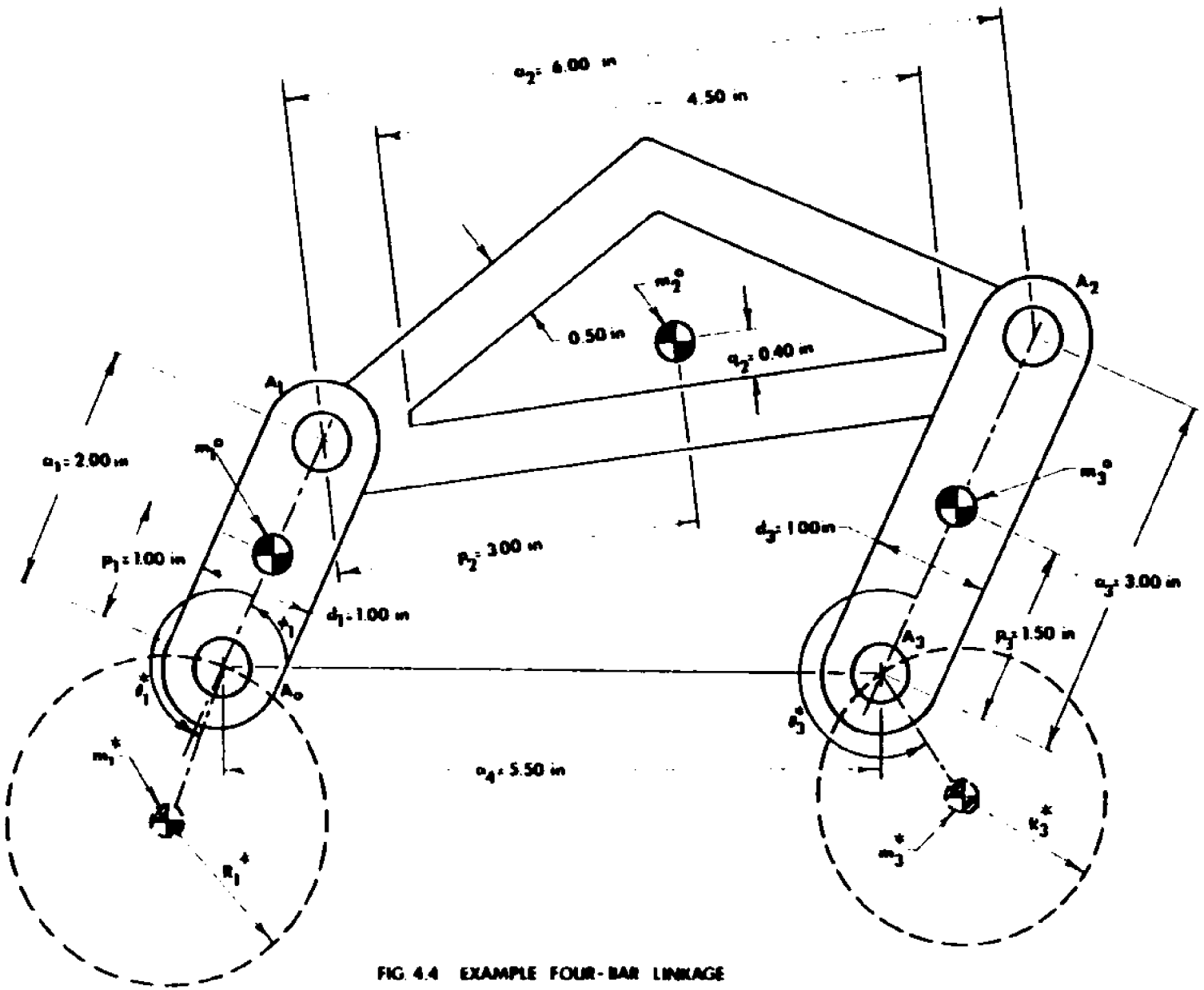


FIG. 4.4 EXAMPLE FOUR-BAR LINKAGE

Table 4.1: Parameters of Unbalanced Linkage

Link j	1	2	3	4
Link length, a_j (in)	2.000	6.000	3.000	5.500
Link width, d_j (in)	1.000	As shown in Figure 4.4	1.000	-
Link thickness, h_j (in)	0.250	0.250	0.250	-
Link mass, m_j^o (lb-sec^2/in)	5.1062×10^{-4}	13.6710×10^{-4}	6.9378×10^{-4}	-
Center of mass location, p_j^o (in)	1.000	3.000	1.500	-
Center of mass location, q_j^o (in)	0.0	0.403	0.0	-
Radius of gyration, k_j^o (in)	Not required	2.0970	1.6747	-

Substitution of the above into equations (A.27)-(A.30) leads to the following coefficients and angles associated with equation (4.3):

$$A^{o} = 18.867 \times 10^{-4} \quad \text{lb-sec}^2 \tag{4.58}$$

$$\alpha^{o} = -5.585 \text{ deg} = -.09747 \text{ rad} \tag{4.59}$$

$$B^{o} = 31.036 \times 10^{-4} \quad \text{lb-sec}^2 \tag{4.60}$$

$$\beta^{o} = 5.093 \text{ deg} = .08889 \text{ rad} \tag{4.61}$$

$$K_1 = 2741.557 \quad \text{rad/sec}^2 \tag{4.62}$$

2. Determination Of Radius And Location Of Center Of Smallest Circumscribing Circle To Rotated Shaking Force Hodograph. Maximum Shaking Force.

Appendix F.1 shows the essential parts as well as the output of program CIRCLE which is used for the determination of the radius and the coordinates of the center of the smallest circumscribing circle to the rotated shaking force hodograph. This program is an adaptation of the Harwell optimization routine and is based on the formulation given by equations (4.5) and (4.6).

Equations (F.7)-(F.9) give the following results:

$$R_S = 15.734 \quad (\text{lbs}) \tag{4.63}$$

$$C_{Sx} = 8.987 \quad (\text{lbs}) \tag{4.64}$$

$$C_{Sy} = -3.464 \quad (\text{lbs}) \tag{4.65}$$

It is to be noted that all "geometric" results are now given in the units of pounds.

From the above, according to equation (4.7):

$$\delta_S = -21.079 \text{ deg} = -.36790 \text{ rad} \tag{4.66}$$

The maximum shaking force is obtained by way of program FMAX. This program, which is given in Appendix F.2, evaluates the absolute magnitude of the shaking force at one degree intervals of the input angle ϕ_1 and chooses the largest value. Equation (F.16) shows the final result:

$$R^{O}_{MAX} = 25.004 \quad (\text{lbs}) \tag{4.67}$$

With $\eta = 0.5$, the proportionality factor μ, as given by equation (4.10), and with equations (4.63) and (4.67), becomes:

$$\mu = \eta \frac{R^{O}_{MAX}}{R_S} = .795 \tag{4.68}$$

3. Determination Of Counterweights For Simple 50 Percent Reduction

a. Output Link Counterweight

When the simple reduction method of Section C is used, the mass-distance product of the output link counterweight is determined according to equation (4.21). With equations (4.60) and (4.68), one obtains:

$$m_3^* r_3^* = (1 - \mu)B^{\circ} = 6.362 \times 10^{-4} \quad \text{lb-sec}^2 \qquad (4.69)$$

The counterweight radius, R_3^*, is then found with the help of the following general design equation for circular and tangent counterweights as shown in Figure 4.4:

$$R_j^* = \left[\frac{m_j^* r_j^*}{\rho_j^* \pi h_j^*}\right]^{\frac{1}{3}}, \quad j = 1,3 \qquad (4.70)$$

where R_j^* = counterweight radius

ρ_j^* = density of counterweight

h_j^* = thickness of counterweight

With equation (4.69) and letting $h_j^* = 0.625$ in., one

obtains from the above:

$$R_3^* = .762 \text{ in} \tag{4.71}$$

Further, with equations (4.22) and (4.61), the counterweight angle is given by

$$\theta_3^* = \beta^\circ + \pi = 185.093 \text{ deg} \tag{4.72}$$

b. Input Link Counterweight (See Section C.2)

The vector $\overline{OC}_S$ is obtained with the help of equations (4.24), (4.64) and (4.65):

$$OC_S e^{i\delta_S} = 9.631\, e^{-.36790\, i} \tag{4.73}$$

The vector $\overline{OD}^\circ$ is determined according to equations (4.25), (4.58), (4.62) and (4.59):

$$\overline{OD}^\circ = 5.172\, e^{-.09747\, i} \tag{4.74}$$

Equation (4.27) gives the vector $\overline{D^\circ C_S}$ with the help of equations (4.73) and (4.74):

$$D^\circ C_S = 4.848 \tag{4.75}$$

and

$$\zeta = -37.638 \text{ deg} = -.65690 \text{ rad} \tag{4.76}$$

The vector $\overline{OC}$ is found with the help of equations (4.28), (4.74), (4.75) and (4.68):

$$OC = 8.683 \tag{4.77}$$

with $$\delta = -19.210 \text{ deg} = -.33527 \text{ rad} \tag{4.78}$$

The input counterweight parameters become, according to equations (4.30) and (4.31):

$$m_1^* r_1^* = 31.672 \times 10^{-4} \text{ lb-sec}^2 \tag{4.79}$$

and $$\theta_1^* = 160.790 \text{ deg} \tag{4.80}$$

The associated circular counterweight is again obtained by way of equation (4.70) with $h_1^* = 0.625$ inches. Thus,

$$R_1^* = 1.301 \text{ in} \quad . \tag{4.81}$$

4. Determination Of Counterweights For 50 Percent Reduction Of Maximum Shaking Force By General Solution

a. Theoretical Design Range

The theoretical range of possible output link counterweight parameters associated with the general solution is found with the help of equations (4.42) and (4.43).

Accordingly, the minimum mass-distance product is given by

$$(m_3^* r_3^*)_{MIN} = (1 - \mu)B^\circ = 6.362 \times 10^{-4} \text{ lb-sec}^2, \quad (4.82)$$

while the maximum mass-distance product is represented by

$$(m_3^* r_3^*)_{MAX} = (1 + \mu)B^\circ = 55.710 \times 10^{-4} \text{ lb-sec}^2. \quad (4.83)$$

(For B° and μ, see equations (4.60) and (4.68), respectively.)

Equation (4.39) indicates the counterweight angle for both extrema as

$$\theta_3^* = \beta^\circ + \pi = 185.093 \text{ deg} \quad (4.84)$$

(See also equation (4.61).)

The associated output link counterweight radii, with $h_3^* = 0.625$ inches, are, according to equation (4.70):

$$(R_3^*)_{MIN} = 0.762 \text{ in} \quad (4.85)$$

and

$$(R_3^*)_{MAX} = 1.570 \text{ in}, \text{ respectively.} \quad (4.86)$$

Note again that the minimum mass-distance solution is identical to that obtained by the simple reduction.

b. Counterweight Design For Arbitrary Point Within Theoretical Design Range

To show the determination of output as well as input counterweights within the theoretical design range, the following example uses the output link parameter

$$m_3^* r_3^* = 24.880 \times 10^{-4} \quad \text{lb-sec}^2 \tag{4.87}$$

which is approximately in the center of the design range.

i. Output Link Counterweights

The angles associated with the above value (which also must satisfy equation (4.44)) are given by

$$\theta_{31}^* = 134.168 \quad \text{deg} \tag{4.88}$$

and

$$\theta_{32}^* = 236.017 \quad \text{deg} \tag{4.89}$$

The corresponding counterweight radius as found with the help of equation (4.70) and with $h_3^* = 0.625$ inches, is

$$R_3^* = 1.200 \quad \text{in} \tag{4.90}$$

The above shows that counterweights of the same mass but with two different angles will produce the identical maximum shaking force.

ii. Input Link Counterweights

The following shows the detailed determination of the input link counterweight parameters corresponding to the output link parameters of equations (4.87) and (4.88) and only gives the results for the input link counterweight design which corresponds to angle θ_{32}^* as given by equation (4.89). Because of these two output link counterweight design possibilities, additional subscripts are introduced where appropriate. The computational sequence indicated in Section D.3 is followed.

a. Angle β_1 becomes, according to equations (4.47), (4.56), (4.57), (4.87) and (4.88):

$$\beta_1 = \tan^{-1} \frac{Q_3^{\circ} + m_3^* r_3^* \sin\theta_{31}^*}{P_3^{\circ} + m_3^* r_3^* \cos\theta_{31}^*}$$

$$= \tan^{-1} \frac{2.755\text{x}10^{-4} + 24.880\text{x}10^{-4}\sin(134.168)}{30.913\text{x}10^{-4} + 24.880\text{x}10^{-4}\cos(134.168)}$$

$$= 56.613 \quad \text{deg} \qquad (4.91)$$

b. The angle of rotation ψ_1 is then obtained by way of equations (4.48) and (4.91):

$$\psi_1 = \beta_1 - \beta^\circ = 56.597 - 5.093$$

$$= 51.520 \text{ deg} = .89919 \text{ rad} \qquad (4.92)$$

where β° is given by equation (4.61).

c. The vector $\overline{D^\circ C_1}$ is found with the help of equations (4.49), (4.68), (4.75), (4.76) and (4.92):

$$\overline{D^\circ C_1} = \mu D^\circ C_S e^{i(\zeta + \psi_1)} = .795 \times 4.848 e^{(-.65690 + .89919)i}$$

$$= 3.854 e^{.24229\ i} \qquad (4.93)$$

d. With the above, the vector $\overline{OC_1}$ may be determined with equations (4.50) and (4.74):

$$OC_1 = 8.899 \qquad (4.94)$$

and $$\delta_1 = 2.714 \text{ deg} = .04737 \text{ rad} \qquad (4.95)$$

e. The input counterweight parameters become, according to equations (4.52) and (4.53):

$$(m_1^* r_1^*)_1 = \frac{|\overline{OC}_1|}{\dot{\phi}_1^2} = \frac{8.899}{2741.557} = 32.460 \times 10^{-4} \text{ lb-sec}^2 \qquad (4.96)$$

and $$\theta_{11}^* = \delta_1 + \pi = 182.714 \text{ deg} \qquad (4.97)$$

The corresponding circular counterweight radius is again found by way of equation (4.70) and with h_1^*=0.625 inches:

$$(R_1^*)_1 = 1.311 \quad \text{in} \tag{4.98}$$

When the output link counterweight of angle $\theta_{32}^* = 236.090$ degrees is chosen, one obtains the following:

$$\beta_2 = -46.427 \quad \text{deg} \tag{4.99}$$

$$\psi_2 = -51.520 \quad \text{deg} \tag{4.100}$$

With the above:

$$(R_1^*)_2 = 1.198 \quad \text{in} \tag{4.101}$$

and

$$\theta_{12}^* = 140.064 \quad \text{deg} \tag{4.102}$$

c. Counterweight Design Flexibility And Its Influence On Various Mechanism Reactions

Figure 4.5 represents a graph of associated output link counterweight parameters R_3^* and θ_3^* for the full design range. The counterweight thickness of 0.625 inches is maintained throughout. In addition, this graph shows input link counterweight parameters R_1^* and θ_1^* for selected output link counterweight configurations. For purposes of

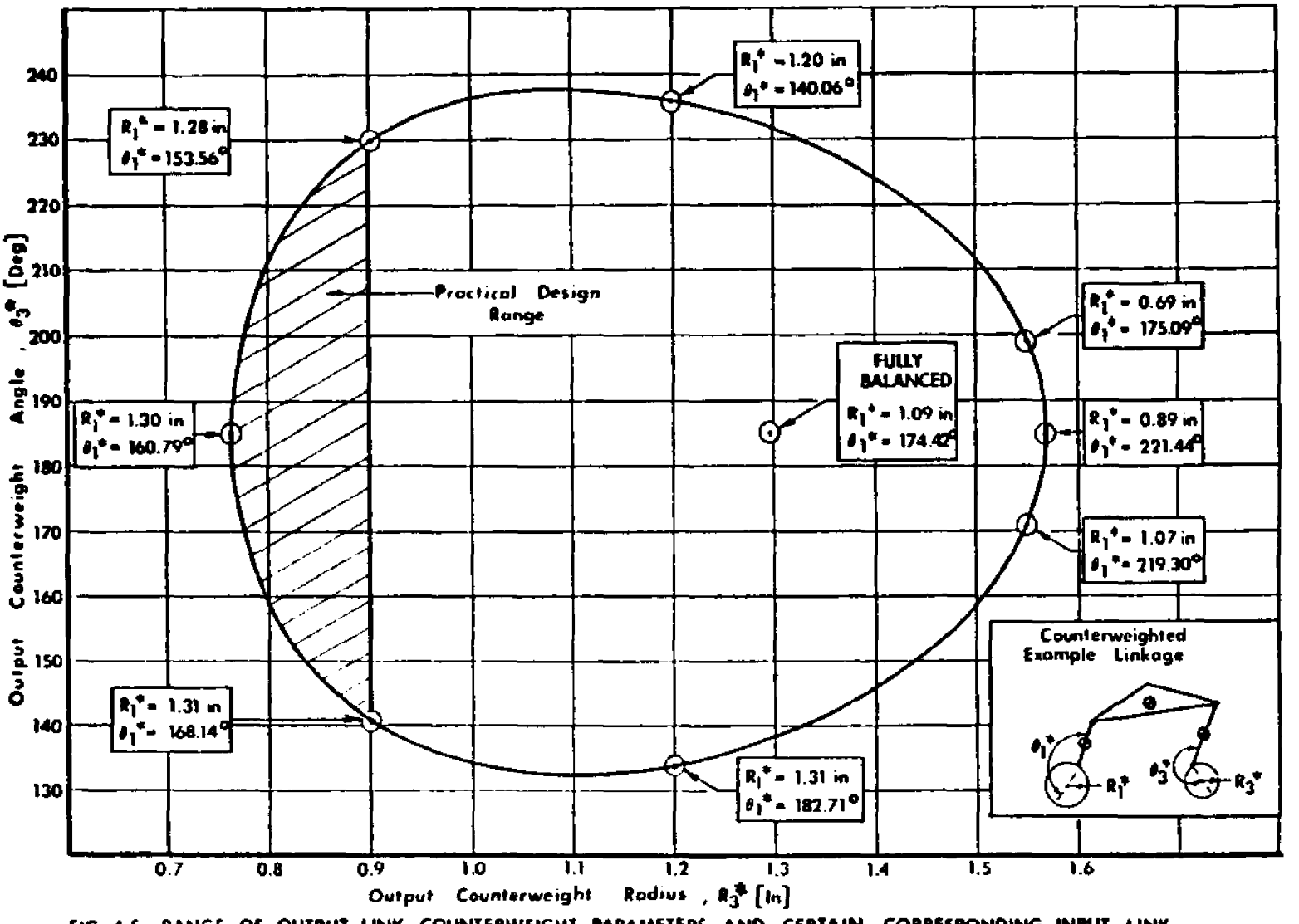


FIG. 4.5 RANGE OF OUTPUT LINK COUNTERWEIGHT PARAMETERS AND CERTAIN CORRESPONDING INPUT LINK COUNTERWEIGHT PARAMETERS FOR A 50 PERCENT REDUCTION IN MAXIMUM SHAKING FORCE [Counterweight Thickness = 0.625 in.]

comparison, the counterweight parameters of a fully force balanced mechanism are also indicated.

To gain perspective concerning the usefulness of this design flexibility, one must also consider the associated variations in the mechanism reactions. Table 4.2 lists the absolute maximum values of all the bearing forces, the input moment and the shaking moment (taken with respect to the midpoint between the ground pivots) for the counterweight configurations of Figure 4.5 and compares them with those of both the unbalanced and the fully balanced mechanism[1]. The table shows that the smallest increases of the maximum dynamic reactions are, for the most part, associated with the minimum mass-distance product of the theoretical design range.

If one considers increases in these reactions of as much as 20 to 30 percent above those of the unbalanced mechanism as an acceptable trade-off for a 50 percent reduction of the maximum shaking force, that portion of the curve in Figure 4.5 which lies to the left of the vertical line at $R_3^* = .90$ inches represents a practical design range. Within this practical range, the output link counterweight angles may be varied between 140 and 230 degrees

---

[1] Expressions for these reactions are given in [75].

Table 4.2: Influence of Partial Balancing (η=0.5) on Various Forces and Moments

Percentage Reduction in Maximum Shaking Force		Input Counterweight Parameters[1]		Output Counterweight Parameters[1]		Maximum Bearing Force				Maximum Input Moment	Maximum Shaking Moment[2]
		R_1^* (in)	θ_1^* (deg)	R_3^* (in)	θ_3^* (deg)	$\lvert F_{41}\rvert$ (lb)	$\lvert F_{21}\rvert$ (lb)	$\lvert F_{23}\rvert$ (lb)	$\lvert F_{43}\rvert$ (lb)	$\lvert M_{41}\rvert$ (in-lb)	$\lvert M_{M/G}\rvert$ (in-lb)
0 Unbalanced $(F_{M/G})_{MAX}$ = 25.00 lb		-	-	-	-	52.12	50.74	40.38	39.27	38.72	90.90
100 Fully Balanced[3] $(F_{M/G})_{MAX}$ = 0 lb		1.09	174.42	1.29	185.09	78.52	82.09	71.23	78.52	64.84	95.45
50 Partially Balanced $(F_{M/G})_{MAX}$ = 12.50 lb	Practical Design Range	1.30	160.79	.76	185.09	49.05	54.47	44.00	43.73	41.83	94.39
		1.31	168.14	.90	140.59	51.75	58.13	47.58	42.91	44.88	84.31
		1.28	153.56	.90	229.60	53.78	58.13	47.58	52.13	44.88	110.59
		1.31	182.71	1.20	134.17	66.69	74.08	63.30	54.39	58.17	72.05
		1.20	140.06	1.20	236.02	71.91	74.08	63.30	76.92	58.17	140.72
		1.07	219.30	1.55	171.06	112.70	115.70	104.64	114.73	92.86	87.25
		.69	175.09	1.55	199.13	115.83	115.70	104.64	127.28	92.86	136.55
		.89	221.44	1.57	185.09	118.16	119.20	108.13	126.05	95.77	111.64

(1) Steel counterweights, 0.625 inches thick (This is 2.5 times link thickness).
(2) Shaking moment is taken with respect to midpoint between ground bearings.
(3) Fully balanced by the Method of Linearly Independent Vectors[8].

while the associated output link counterweight radii remain considerably smaller than those needed for full force balancing. The resulting input link counterweights are somewhat larger than those of the fully balanced linkage.

5. Results For 75 Percent Reduction In Maximum Shaking Force By General Solution

Table 4.3 lists counterweight parameters and maximum dynamic reactions, again corresponding to a constant input speed of 500 rpm, for the extrema of the theoretical design range of a 75 percent reduction of the maximum shaking force by way of the general solution. In addition, the results for the unbalanced mechanism and the fully balanced mechanism are shown together with the limiting values of the 50 percent reductions, discussed in the previous section.

The smallest increases of most of the dynamic reactions are again associated with the smallest possible output link counterweight, i.e. the minima of the design range.

(Table 4.3 will also serve as a basis for certain comparisons presented in the following section on simultaneous optimization of dynamic reactions of a four-bar linkage.)

Table 4.3: Maximum Dynamic Reactions For Extrema Of Design Ranges Corresponding To 50 And 75 Percent Reductions Of Maximum Shaking Force

Percentage Reduction in Maximum Shaking Force	Counterweight Parameters (1)				Maximum Bearing Forces				Maximum Input Moment	Maximum Shaking Moment (2)
	R_1^* (in)	ϕ_1^* (deg)	R_3^* (in)	ϕ_3^* (deg)	$\lvert F_{41}\rvert$ (lb)	$\lvert F_{21}\rvert$ (lb)	$\lvert F_{23}\rvert$ (lb)	$\lvert F_{43}\rvert$ (lb)	$\lvert M_{41}\rvert$ (in-lb)	$\lvert M_{M/G}\rvert$ (in-lb)
0 Unbalanced $(F_{M/G})_{MAX}$ = 25.00 lb	-	-	-	-	52.12	50.74	40.38	39.27	38.72	90.90
100 Fully Balanced (3) $(F_{M/G})_{MAX}$ = 0	1.09	174.42	1.29	185.09	78.52	82.09	71.23	78.52	64.84	95.45
50 $(F_{M/G})_{MAX}$ = 12.50 lb (Minimum R_3^*)	1.30	160.79	.76	185.09	49.05	54.47	44.00	43.73	41.83	94.39
50 $(F_{M/G})_{MAX}$ = 12.50 lb (Maximum R_3^*)	.89	221.44	1.57	185.09	118.16	119.20	108.13	126.05	95.77	111.64
75 $(F_{M/G})_{MAX}$ = 6.25 lb (Minimum R_3^*)	1.20	165.88	1.09	185.09	62.16	66.68	55.99	58.90	52.01	92.84
75 $(F_{M/G})_{MAX}$ = 6.25 lb (Maximum R_3^*)	.98	190.51	1.44	185.09	97.35	99.67	88.69	101.17	79.49	102.43

(1) Steel counterweights .625 inches thick (This is 2.5 times the link thickness.)
(2) Shaking moment is taken with respect to midpoint between ground bearings.
(3) Fully balanced by the Method of Linearly Independent Vectors[8].

V. SIMULTANEOUS OPTIMIZATION OF DYNAMIC REACTIONS OF A FOUR-BAR LINKAGE WITH PRESCRIBED MAXIMUM SHAKING FORCE

A. Introduction

The following section introduces a technique which makes it possible to simultaneously minimize the maximum dynamic reactions of a constant speed four-bar linkage while maintaining the ability to prescribe the maximum shaking force. The general reduction of the values of these reactions when compared to those obtained by the method of the previous section is based on the introduction of a coupler counterweight in addition to those located on the input and output links. The resulting increase in the number of design variables is reflected in what has now been called the general equipollent circle constraint equation, which makes it possible to prescribe the maximum shaking force. The simultaneous optimum reduction of such dynamic reactions as the bearing forces, the input moment and the shaking moment now becomes a function of the proper combination of the coupler and output link counterweight parameters.

To this end, an optimization procedure was devised which allows for the determination of these counterweight

parameters. This procedure satisfies the general equipollent circle constraint equation while it imposes constraints on the maximum dynamic reactions. The latter inequality constraints are controlled by a minimization parameter z which serves as the objective function of the optimization.

The following first gives the form of the new general equipollent circle constraint equation and then describes the formulation of the optimization as well as its adaptation to the Harwell Augmented Lagrangian program. Subsequently, the design procedure for this type of partial force balancing, which also includes the determination of the input link counterweight parameters, is outlined. This procedure is incorporated into program OPTIMIZE, which represents the adaptation of the Harwell code. (Program OPTIMIZE is listed and explained, together with a sample run, in Appendix G.)

The design procedure has been applied to a 50 and a 75 percent reduction in the maximum shaking force. It was found that the reduction in the dynamic reactions due to the simultaneous optimization became most pronounced for the larger reduction in the maximum shaking force.

This new three-counterweight method is also adapted to a fully force balanced mechanism. The associated reductions in the dynamic reactions, when compared to those of the two-

counterweight Method of Linearly Independent Vectors, was found to be especially gratifying.

Finally, the optimization procedure was modified for an input moment minimization of a fully force balanced mechanism. The constraints on all dynamic reactions, with the exception of that for the input moment, were removed. Again, a comparison of the resulting maximum input moment with that associated with the Method of Linearly Independent Vectors shows a considerable reduction. In addition, even though unconstrained, all other reactions were either considerably smaller or, at worst, remained substantially the same as those obtained by the above two-counterweight method of full force balancing.

B. General Equipollent Circle Constraint Equation

Equation (4.32) in Section IV, which is an expanded form of the equipollent circle constraint equation (4.35), shows that a prescribed maximum shaking force can be maintained as long as the following relationship is satisfied:

$$(P_3)^2 + (Q_3)^2 = (\mu B^o)^2 \tag{5.1}$$

where

$$P_3 = m_3 p_3 + \frac{a_3}{a_2} m_2 p_2 \tag{5.2}$$

$$Q_3 = m_3 q_3 + \frac{a_3}{a_2} m_2 q_2 \tag{5.3}$$

If one desires to make changes in the mass distribution of the coupler link in addition to that of the output link, equation (5.1) must be modified to become the following general equipollent circle constraint equation:

$$(P_3^* + P_3^o)^2 + (Q_3^* + Q_3^o)^2 = (\mu B^o)^2 \tag{5.4}$$

where

$$P_3^* = m_3^* p_3^* + \frac{a_3}{a_2} m_2^* p_2^* \tag{5.5}$$

$$Q_3^* = m_3^* q_3^* + \frac{a_3}{a_2} m_2^* q_2^* \tag{5.6}$$

$$P_3^o = m_3^o p_3^o + \frac{a_3}{a_2} m_2^o p_2^o \tag{5.7}$$

$$Q_3^o = m_3^o q_3^o + \frac{a_3}{a_2} m_2^o q_2^o \tag{5.8}$$

(See also equation (A.25).)

In order to be able to optimize mechanism reactions, a specific counterweight configuration has to be adopted. For the present work, circular counterweights which are tangent to their pivots will be used. (See Figure 5.2) Equations (5.5) and (5.6) then become:

$$P_3^* = \rho_3^* \pi h_3^* (R_3^*)^3 \cos\theta_3^* + \frac{a_3}{a_2} \rho_2^* \pi h_2^* (R_2^*)^3 \cos\theta_2^* \qquad (5.9)$$

and $$Q_3^* = \rho_3^* \pi h_3^* (R_3^*)^3 \sin\theta_3^* + \frac{a_3}{a_2} \rho_2^* \pi h_2^* (R_2^*)^3 \sin\theta_2^* \qquad (5.10)$$

where R_j^* = counterweight radius

θ_j^* = counterweight angle

ρ_j^* = density of counterweight

h_j^* = thickness of counterweight

C. Formulation Of Optimization Problem

The basic objective of the optimization consists of choosing the variables R_2^*, θ_2^*, R_3^* and θ_3^* such that the maximum values of all bearing forces, the input moment and the shaking moment are simultaneously minimized while holding the maximum shaking force to a prescribed value by way of the general equipollent circle constraint equation. The magnitudes of the input link counterweight parameters depend entirely on the choice of the coupler and output link counterweight parameters and are attained in a manner similar to that shown in Section IV. (These values, of course, are necessary to compute the mechanism reactions.)

The optimization method is formulated in the following way:

$$\text{minimize} \quad z \tag{5.11}$$

subject to the equality constraints of the general equipollent circle constraint equation (5.4) as well as the following inequality constraints:

$$W_1 \left[\frac{(F_{41})^2 - (F_{41}^{\circ})^2}{(F_{41}^{\circ})^2} \right]_{MAX} \leq z \qquad (5.12)$$

$$W_2 \left[\frac{(F_{12})^2 - (F_{12}^{\circ})^2}{(F_{12}^{\circ})^2} \right]_{MAX} \leq z \qquad (5.13)$$

$$W_3 \left[\frac{(F_{23})^2 - (F_{23}^{\circ})^2}{(F_{23}^{\circ})^2} \right]_{MAX} \leq z \qquad (5.14)$$

$$W_4 \left[\frac{(F_{34})^2 - (F_{34}^{\circ})^2}{(F_{34}^{\circ})^2} \right]_{MAX} \leq z \qquad (5.15)$$

$$W_5 \left[\frac{(M_{M/G})^2 - (M_{M/G}^{\circ})^2}{(M_{M/G}^{\circ})^2} \right]_{MAX} \leq z \qquad (5.16)$$

$$W_6 \left[\frac{(M_{41})^2 - (M_{41}^{\circ})^2}{(M_{41}^{\circ})^2} \right]_{MAX} \leq z \qquad (5.17)$$

where F_{41} = bearing force between ground link and input link

F_{12} = bearing force between input link and coupler link

F_{23} = bearing force between coupler link and output link

F_{34} = bearing force between output link and ground link

$M_{M/G}$ = shaking moment taken with respect to midpoint between ground pivots A_0 and A_3

M_{41} = input moment

W_j = weighting factors for each reaction , (j=1,2,...,6)

The usual notation is used whereby a zero superscript corresponds to the unbalanced linkage while the absence of a superscript represents the balanced linkage. (See Appendix A.3)

D. Computational Sequence Of Optimization Procedure

The following outlines computational procedures for partial as well as full force balancing which are used in program OPTIMIZE (see Appendix G).

1. Determination Of Counterweight Parameters For Partial Force Balancing

To determine all mechanism reactions, each iteration of the optimization procedure requires a complete counterweight design. The following shows the necessary computational sequence. It is different from those used previously because of the changes in the mass distribution of the coupler link:

1. As in Section IV-B, from the dimensions of the unbalanced mechanism as well as for a given input speed and reduction factor η, the parameters R_S, C_{Sx}, C_{Sy}, δ_S and R^o_{MAX} are first obtained. Subsequently, the proportionality factor μ is found according to equation (4.10).

2. The value of $B = \mu B^o$ is then computed with the help of the parameters of the unbalanced linkage for use in equation (5.4).

3. A set of output link and coupler counterweight parameters R_2^*, θ_2^*, R_3^* and θ_3^*, which satisfy the constraint equation (5.4), is chosen. (The thickness h^* and density ρ^* are prescribed.)

4. To devise a method for determining the input link counterweight parameters, one must consider the changes in the shaking force hodograph as indicated by equation (4.3), i.e.

$$\bar{F}_{M/G} e^{-i\phi_1} = AK_1 e^{i\alpha} + BK_3 e^{i(\gamma_3 + \beta - \phi_1)} \quad (5.18)$$

In the previously shown flexible counterweight design (see Section IV-D), only the parameters B and β are different from B^o and β^o, respectively. The added change in the coupler link configuration also results in changes in the magnitudes of the parameters A^o and α^o to A and α. To illustrate how this difference reflects itself in the shaking force hodograph, it is useful to review Figure 4.3 which shows the effects of the changes in the parameters B and β only. Figure 5.1 indicates that, for the present case, there must not only be a rotation of the line D^oC through the angle ψ, but there must also be

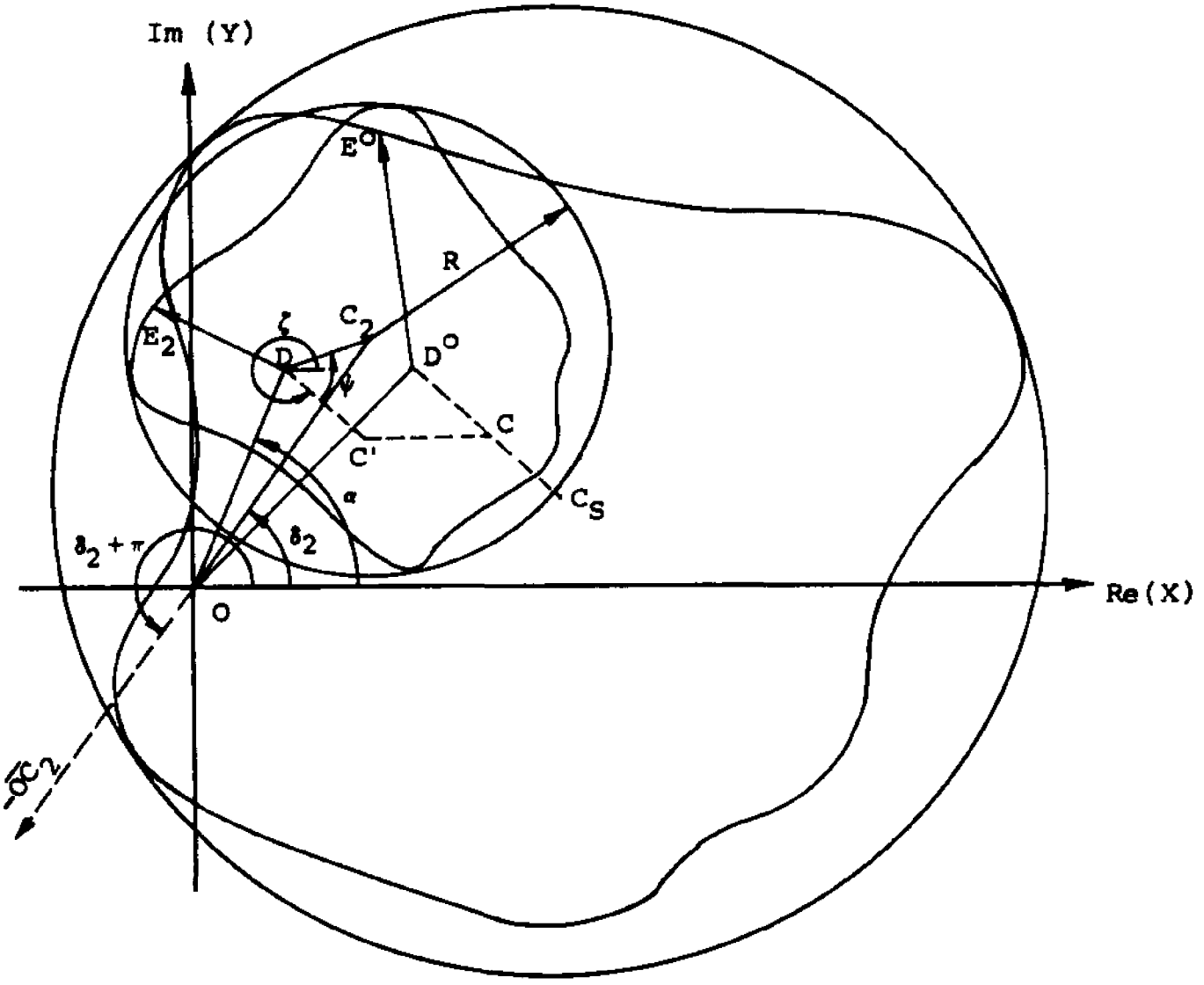


FIGURE 5.1: ORIGINAL AND REDUCED SHAKING FORCE HODOGRAPH RESULTING FROM THREE-COUNTERWEIGHT OPTIMIZATION METHOD

a shift of point D^o to point D. This shift is due to the changes in A and α. The input link counterweight is finally determined with the help of the vector $\overline{OC}_2$ at the angle δ_2. It may be noted that for the same reduction in shaking force, the vector $\overline{DC}_2$ is identical in magnitude and direction to the vector $\overline{D^oC}_1$ of the flexible method of Section IV-D. As a consequence of the above, the computation for the input link counterweight procedes as follows:

a. The angle β is obtained with the help of equation (4.16), i.e.

$$\beta = \tan^{-1} \frac{Q_3}{P_3} \tag{5.19}$$

where
$$P_3 = P_3^{\,o} + \rho_3^{\,*} \pi h_3^{\,*} (R_3^{\,*})^3 \cos\theta_3^{\,*} + \frac{a_3}{a_2} \rho_2^{\,*} \pi h_2^{\,*} (R_2^{\,*})^3 \cos\theta_2^{\,*}$$

$$Q_3 = Q_3^{\,o} + \rho_3^{\,*} \pi h_3^{\,*} (R_3^{\,*})^3 \sin\theta_3^{\,*} + \frac{a_3}{a_2} \rho_2^{\,*} \pi h_2^{\,*} (R_2^{\,*})^3 \sin\theta_2^{\,*}$$

(See equations (5.9), (5.10), (A.33), (A.34), (A.41) and (A.42).)

b. The angle of rotation ψ is then found from

$$\psi = \beta - \beta^{\circ} \tag{5.20}$$

where $$\beta^{\circ} = \tan^{-1} \frac{Q_3^{\circ}}{P_3^{\circ}}$$

(See equations (4.2), (4.34).)

c. The vector $\overline{DC}_2 = BK_3 e^{i(\gamma_3 + \beta - \phi_1)}$ is found with the help of equations (4.27) and (4.49):

$$\overline{DC_2} = \overline{D^{\circ}C_1} = \mu D^{\circ}C_S \, e^{i(\zeta + \psi)} \tag{5.21}$$

where

$$D^{\circ}C_S \, e^{i\zeta} = \sqrt{(C_{Sx})^2 + (C_{Sy})^2} \, e^{i\delta_S} - A^{\circ}K_1 e^{i\alpha^{\circ}}$$

and $$\zeta = \tan^{-1} \frac{D^{\circ}C_{Sy}}{D^{\circ}C_{Sx}}$$

(See equations (4.2), (4.7), (4.10), (5.20).)

d. The vector $\overline{OD}$ is obtained by way of:

$$\overline{OD} = AK_1 e^{i\alpha} \tag{5.22}$$

where

$$A = \left\{ \left(P_1^{\,o} + \rho_2^{*}\pi h_2^{*}(R_2^{*})^2 a_1 - \frac{a_1}{a_2}\rho_2^{*}\pi h_2^{*}(R_2^{*})^3 \cos\theta_2^{*} \right)^2 + \left(Q_1^{\,o} - \frac{a_1}{a_2}\rho_2^{*}\pi h_2^{*}(R_2^{*})^3 \sin\theta_2^{*} \right)^2 \right\}^{\frac{1}{2}}$$

$$\alpha = \tan^{-1}\frac{OD_Y}{OD_X}$$

(See equations (A.31) and (A.32).)

e. With the above, the vector $\overline{OC}_2$, which determines the input link counterweight, is found from:

$$OC_2 e^{i\delta_2} = ODe^{i\alpha} + DC_2 e^{i(\zeta + \psi)} \tag{5.23}$$

where $\delta_2 = \tan^{-1}\frac{OC_{2Y}}{OC_{2X}}$

f. Finally, since $K_1 = \dot{\phi}_1^{\,2}$, the force of the input link counterweight must equal:

$$m_1^{*} r_1^{*} \dot{\phi}_1^{\,2} e^{i\theta_1^{*}} = -OC_2 e^{i\delta_2} = OC_2 e^{i(\delta_2 + \pi)} \tag{5.24}$$

This leads to the design equations:

$$m_1^{*} r_1^{*} = \frac{|\overline{OC}_2|}{\dot{\phi}_1^{\,2}} \tag{5.25}$$

and

$$\theta_1^{*} = \delta_2 + \pi \tag{5.26}$$

2. Determination Of Counterweight Parameters For Full Force Balancing

Just as for partial force balancing, each iteration of the optimization procedure for full force force balancing requires a complete counterweight design. The computational procedure given below represents a simplification of the partial force balancing procedure, since now the radius of the circumscribing circle vanishes:

a. For full force balancing, the proportionality factor μ in equations (5.1) and (5.4) is set equal to zero.

b. A set of coupler and output link counterweight parameters R_2^*, θ_2^*, R_3^* and θ_3^* which satisfy this revised equipollent circle constraint equation is now chosen. (Again, the thickness h^* and density ρ^* are prescribed.)

c. To devise a method[1] for determining the input link

---

[1] The present method for designing the input link counterweight was chosen because it is used in the existing computer program. With the mass of the coupler known, the identical counterweight design may be obtained with the help of equation (8) of [8].

counterweight parameters R_1^* and θ_1^*, one considers the shaking force hodograph equation (4.3), which now has been reduced to:

$$\bar{F}_{M/G} e^{-i\phi_1} = AK_1 e^{i\alpha} \qquad (5.27)$$

This, of course, indicates that the hodograph has been reduced to a single constant vector whose negative furnishes the input link counterweight. This vector corresponds to the vector $\overline{OD}$ at angle α in Figure 5.1.

d. The quantities A and α of equation (5.27) are now obtained with the help of equation (5.22):

$$A = \left\{ \left(P_1^{\circ} + \rho_2^* \pi h_2^* (R_2^*)^2 a_1 - \frac{a_1}{a_2} \rho_2^* \pi h_2^* (R_2^*)^3 \cos\theta_2^* \right)^2 + \left(Q_1^{\circ} - \frac{a_1}{a_2} \rho_2^* \pi h_2^* (R_2^*)^3 \sin\theta_2^* \right)^2 \right\}^{\frac{1}{2}}$$

$$\alpha = \tan^{-1} \frac{OD_Y}{OD_X}$$

e. Since $K_1 = \dot{\phi}_1^2$, the force of the input link counterweight is obtained, in a manner parallel to that represented by equation (5.24), from

$$m_1^* r_1^* \dot{\phi}_1^2 e^{i\theta_1^*} = -\overline{OD} e^{i\alpha} = \overline{OD} e^{i(\alpha + \pi)}$$

The counterweight design equations become:

$$m_1^* r_1^* = \frac{|\overline{OD}|}{\dot{\phi}_1^2}$$

and $$\theta_1^* = \alpha + \pi$$

It is to be noted that the magnitude of the counterweight mass-distance product $m_1^* r_1^*$ is equal to the quantity A in equation (5.27).

The above method also furnishes the correct answer even if the input link does not rotate at constant speed. Since the total mechanism center of mass is stationary in a fully balanced mechanism, the net force by the mechanism on the ground vanishes regardless of input speed. This may be shown in terms of the present method of derivation by considering the form of equation (5.27) when $\ddot{\phi}_1 \neq 0$. According to equation (A.12), equation (5.27) becomes:

$$\bar{F}_{M/G} e^{-i\gamma_1} = AK_1 e^{i\alpha}$$

Again, the quantity A in the above is identical with that in equation (5.27), even though one deals with a vector of variable length (and constant direction). If a counterweight whose mass-distance product equals the magnitude of A is added, the shaking force is reduced to zero.

3. Determination Of Optimum Mechanism Reactions

Once the optimum counterweight parameters R_i^* and θ_i^* (i=1,2,3) have been obtained, the associated maximum mechanism reactions are obtained with the help of program FMAX which is listed in Appendix F.2.

E. Examples

The present section gives the results of several groups of optimization runs and discusses their significance in the light of previously shown methods of limiting the maximum shaking force. The example mechanism is the same as that used in the previous section, with the exception that now an additional coupler counterweight is introduced (see Figure 5.2 and Table 4.1). As before, both the original mechanism and all counterweights are assumed to be made of steel and the mechanism speed is taken to be 500 rpm.

The following groups of optimizations are involved:

1. Reduction Of 50 And 75 Percent In The Maximum Shaking Force

Optimized dynamic reactions were obtained both for a 50 percent and a 75 percent reduction in the maximum shaking force. The weighting factors W_1 to W_6, associated with the constraint equations (5.12)-(5.17), were set equal to unity (see discussion below). The optimum counterweight parameters were obtained with the help of program OPTIMIZE. Appendix G shows the details of the optimization associated with the 50 percent reduction of the maximum shaking force. Program FMAX was subsequently used to obtain the maximum dynamic

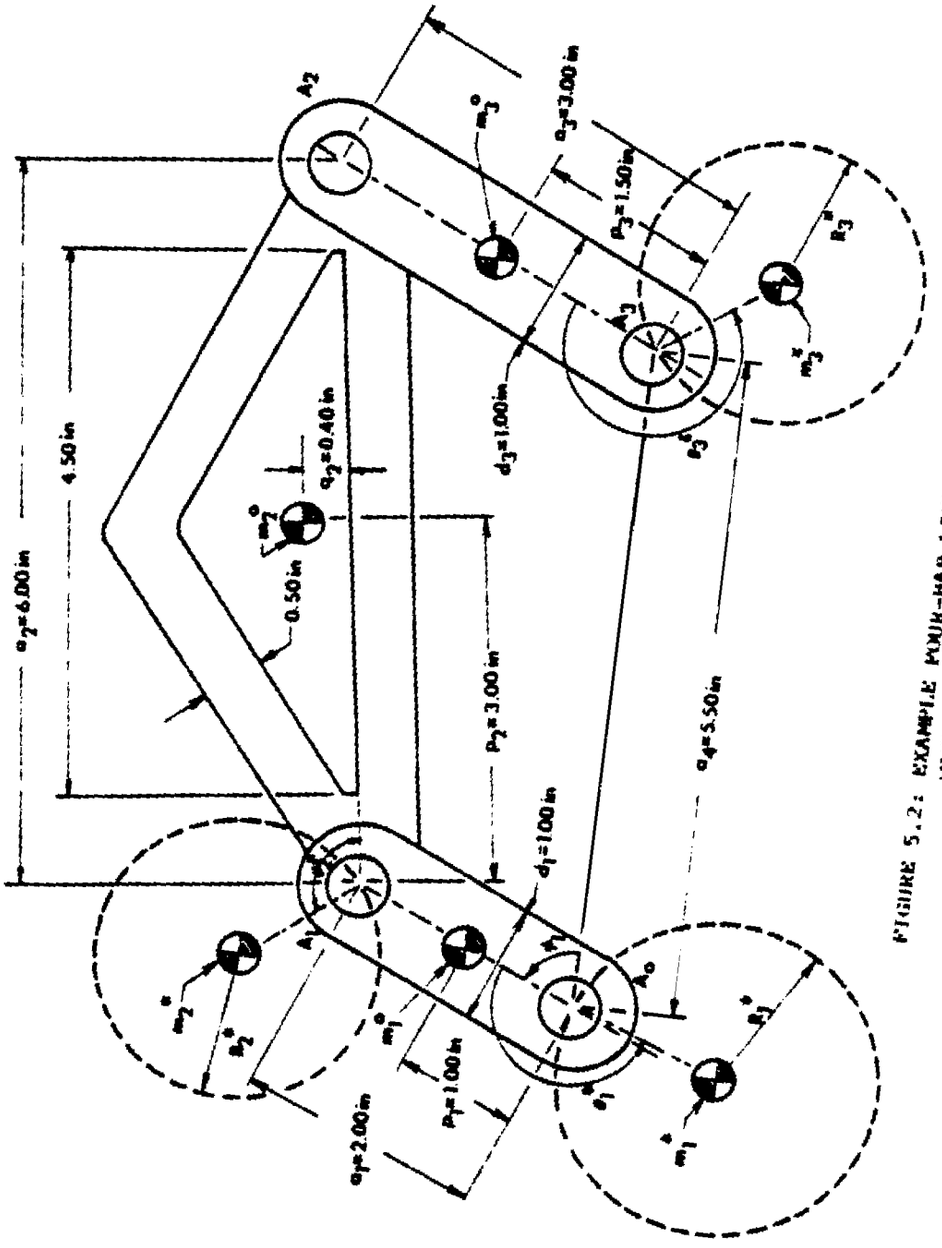


FIGURE 5.2: EXAMPLE FOUR-BAR LINKAGE WITH THREE COUNTERWEIGHTS

reactions associated with the counterweight parameters of program OPTIMIZE serving as the input.

2. Full Force Balance

The optimum dynamic reactions were also obtained for a fully force balanced mechanism employing three counterweights. The methods of optimization are identical to those described above.

3. Input Moment Optimization For A Fully Force Balanced Mechanism

An input moment optimization for a fully balanced mechanism was also performed. Appendix H, which outlines a modification of program OPTIMIZE, was used for this purpose. In order to minimize only the maximum input moment M_{41}, the constraint equations for all other dynamic reactions were removed from the program. This is equivalent to letting $W_6=1$ and $W_1=W_2=W_3=W_4=W_5=0$.

4. Results Of Optimizations And Conclusions

Table 5.1 gives the results of the various optimizations. The three sets of optimized counterweight parameters, the four maximum bearing forces, the maximum input moment, and the maximum shaking moment are listed together with the minimization

Table 5.1: Comparison of Maximum Dynamic Reactions of Unbalanced, Balanced and Optimized Example Mechanism

Run	Percentage Reduction in Maximum Shaking Force	Quantity Optimized	Counterweight Parameters[1] R_1^*(in) ϕ_1^*(deg)	R_2^*(in) ϕ_2^*(deg)	R_3^*(in) ϕ_3^*(deg)	Maximum Bearing Forces $\|P_{41}\|$ (lb)	$\|P_{21}\|$ (lb)	$\|P_{23}\|$ (lb)	$\|P_{43}\|$ (lb)	Maximum Input Moment $\|M_{41}\|$ (in-lb)	Maximum Shaking Moment[2] $\|M_{M/G}\|$ (in-lb)	Optimization Parameter z (lb^2)
1	0 Unbalanced $(P_{M/G})_{MAX}$=25.00 lb	None	-	-	-	52.12	50.74	40.38	39.27	38.72	90.90	-
2	50 $(P_{M/G})_{MAX}$=12.50 lb	Best Two-Cwt Case[5]	1.30 160.79	-	0.76 185.09	49.05	54.47	44.00	43.73	41.83	94.39	-
3	50 $(P_{M/G})_{MAX}$=12.50 lb	All Reactions	1.65 172.57	1.07 230.99	0.42 170.95	40.08	55.07	40.30	39.12	33.42	92.57	0.178
4	75 $(P_{M/G})_{MAX}$= 6.25 lb	Best Two-Cwt Case[5]	1.20 165.88	-	1.09 185.09	62.16	66.68	55.99	58.90	52.01	92.84	-
5	75 $(P_{M/G})_{MAX}$= 6.25 lb	All Reactions	1.81 179.10	1.37 214.67	0.71 153.29	41.44	62.29	45.30	43.00	38.53	91.72	0.507
6	100 Fully Balanced[3] $(P_{M/G})_{MAX}$= 0 lb	None	1.09 174.42	-	1.29 185.09	78.52	82.09	71.23	78.52	64.84	95.45	-
7	100 $(P_{M/G})_{MAX}$= 0 lb	All Reactions	1.88 182.50	1.48 202.92	0.92 146.43	49.10	72.68	53.23	49.10	51.19	92.40	1.052
8	100 $(P_{M/G})_{MAX}$= 0 lb	Input Moment Only	1.86 187.10	1.49 236.17	1.19 135.63	55.82	79.70	64.43	55.82	44.03	75.40	0.293[4]

(1) Steel counterweights, 0.625 inches thick. (This is 2.5 times the link thickness.)
(2) Shaking moment is taken with respect to midpoint between ground bearings.
(3) Fully balanced by the two-counterweight Method of Linearly Independent Vectors [8].
(4) Applies only to input moment constraint.
(5) Using the method of Section IV with minimum radius output link counterweight. (See Table 4.3)

parameter z for each of the optimization groupings discussed above. For purposes of comparison, similar data are given for the unbalanced mechanism, as well as a mechanism which has been fully balanced by the two-counterweight Method of Linearly Independent Vectors. In addition, the results associated with the minimum radius output link counterweight for the 50 percent and the 75 percent reductions in the maximum shaking force by the method of Section IV are reproduced from Table 4.3.

The following conclusions may be drawn:

a. The optimizations associated with the 50 percent and the 75 percent reductions in the maximum shaking force are of interest from two points of view. First, it may be seen that when the values of the maximum dynamic reactions of the new optimization technique are compared to those of the minimum radius output link counterweight method of the previous section, it becomes clear that the value of the present optimization method increases as the prescribed reduction in the maximum shaking force increases. While run 3 of Table 5.1 shows some improvement in the maximum reactions over those of run 2, a

comparison of runs 4 and 5 clearly shows the advantages of the present optimization method. For this latter case, the majority of the dynamic reactions are from approximately 19 to 33 percent smaller than those obtained from the two-counterweight method.

Secondly, when comparing runs 3 and 5 with run 1, which gives the maximum dynamic reactions of the unbalanced mechanism, one finds decreases as well as suprisingly small increases in many of these reactions.

b. The present three-counterweight optimization is especially effective when applied to the fully force balanced mechanism. Comparison with the two-counterweight method (see runs 6 and 7) shows a substantial decrease in all reactions, with the exception of the shaking moment, which remains essentially unchanged. This most advantageous result is obtained at the cost of the additional coupler counterweight as well as an increase in the size of the input counterweight. (This is associated with some decrease in the size of the output link counterweight.)

c. The input moment optimization of the fully force balanced mechanism also shows favorable results when compared to the mechanism balanced by the Method of Linearly Independent Vectors (see runs 6 and 8). The maximum input moment is decreased by approximately 32 percent. In addition, all other reactions are either considerably decreased or, at worst, remain substantially the same.

While only a limited effort in this direction has been made, it has been found that the outcome of the various optimizations cannot be greatly influenced by modifying the weighting factors of the controlling constraint equations.

If desired, this controlling expression may be found by substituting the maximum reactions into the appropriate constraint equations (5.12)-(5.17) and by determining that result which is closest to the governing optimization parameter z. For example, in the optimization associated with the 50 percent reduction in maximum shaking force (see run 3 in Table 5.1), the constraint equations (5.12) and (5.13), pertaining to forces F_{41} and F_{21}, respectively, furnish the following values:

$$W_1 \left[\frac{(F_{41})^2 - (F_{41}{}^{o})^2}{(F_{41}{}^{o})^2}\right] = (1)\left[\frac{(40.08)^2 - (52.12)^2}{(52.12)^2}\right] = -.41$$

and

$$W_2 \left[\frac{(F_{21})^2 - (F_{21}{}^{o})^2}{(F_{21}{}^{o})^2}\right] = (1)\left[\frac{(55.07)^2 - (50.74)^2}{(50.74)^2}\right] = .18$$

Clearly, the second value is closest to $z=.178$ and therefore, the constraint on the force F_{21} controls.

VI. APPENDICES

APPENDIX A

SHAKING FORCE EXPRESSIONS FOR A FOUR-BAR LINKAGE WITH GENERAL LINK MASS DISTRIBUTION

For the purpose of this work it is sometimes advantageous to use the general shaking force expressions as functions of the link angles ϕ_1, ϕ_2 and ϕ_3, while at other times it is more desirable to have them as functions of ϕ_1 and ϕ_3 only (see Figure A.1). Both approaches are outlined below and expressions, both in exponential and Cartesian component form, are given. Finally, the previously introduced notation [8], which allows the distinction between the mass parameters of balanced and unbalanced mechanism, is reviewed.

1. Shaking Force Expression in Terms of ϕ_1, ϕ_2 and ϕ_3

a. Exponential Form

The mass-distance product associated with the center of mass trajectory of the four-bar linkage shown in Figure A.1 may be expressed as follows [8]:

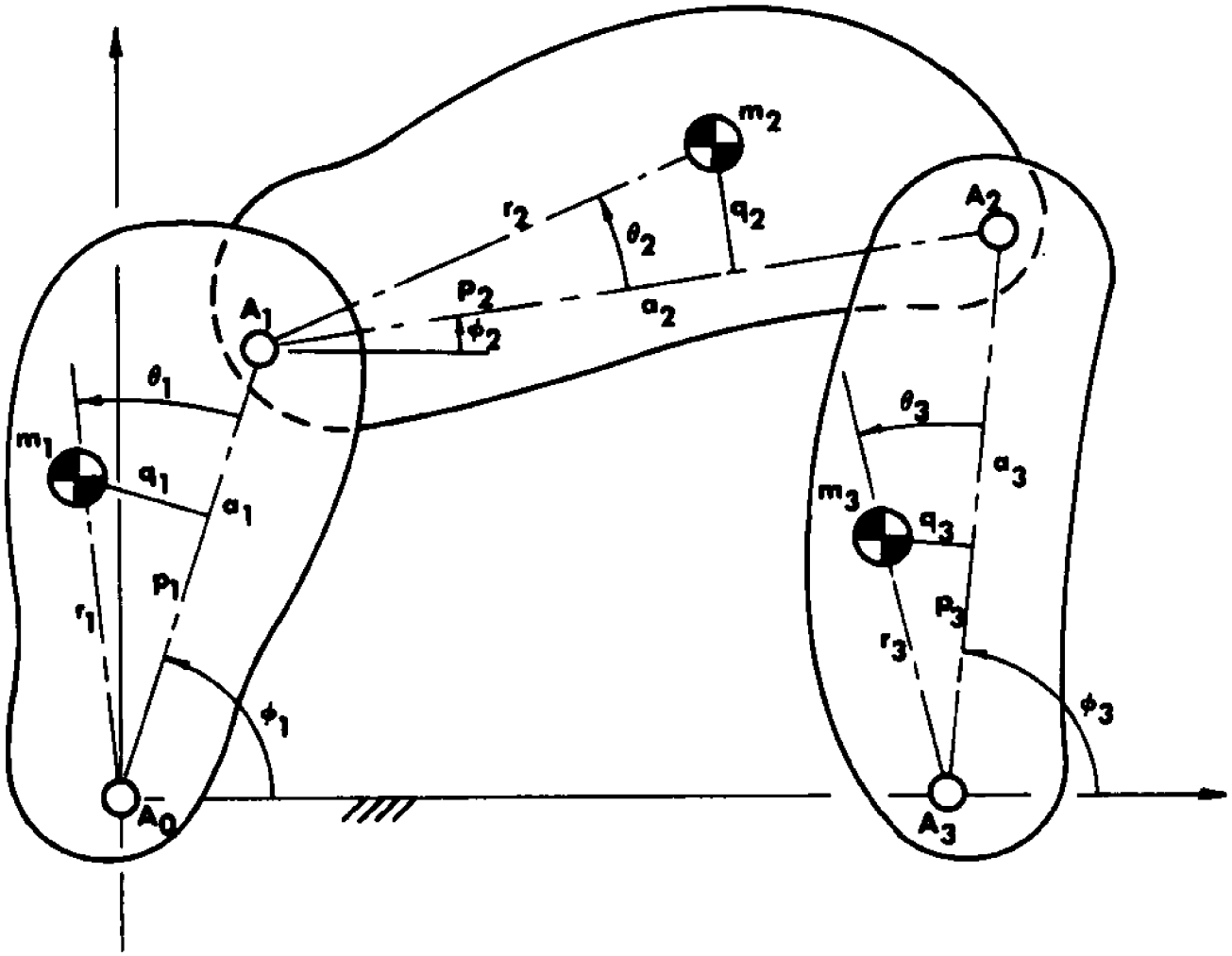


FIG. A.1 GENERAL FOUR-BAR LINKAGE

$$M\bar{r}_s = (m_1 r_1 e^{i\theta_1} + m_2 a_1)e^{i\phi_1} + m_2 r_2 e^{i\theta_2} e^{i\phi_2} + m_3 r_3 e^{i\theta_3} e^{i\phi_3} + m_3 a_4 \qquad (A.1)$$

Since the shaking force $\bar{F}_{M/G}$ is equal to the sum of the D'Alembert forces of the individual links, it may be obtained by double differentiation of the above and the subsequent change of sign of the resulting expression:

$$\bar{F}_{M/G} = -M\ddot{\bar{r}}_s = \left[m_1(p_1 + iq_1) + m_2 a_1\right] K_1 e^{i\gamma_1} + m_2(p_2 + iq_2)K_2 e^{i\gamma_2} + m_3(p_3 + iq_3)K_3 e^{i\gamma_3} \qquad (A.2)$$

where

$$\left.\begin{aligned} p_j &= r_j \cos\theta_j \\ q_j &= r_j \sin\theta_j \\ \gamma_j &= \phi_j - \phi'_j \\ \phi'_j &= \tan^{-1}\frac{\ddot{\phi}_j}{\dot{\phi}_j^{\,2}} \\ K_j &= \sqrt{\dot{\phi}_j^{\,4} + \ddot{\phi}_j^{\,2}} \end{aligned}\right\} \quad j = (1,2,3) \qquad (A.3)$$

b. Cartesian Component Form

Transformation of equation (A.2) yields the following X and Y components of the shaking force:

$$F_{M/G_X} = (K_1\cos\gamma_1)m_1p_1 - (K_1\sin\gamma_1)m_1q_1 + (K_2\cos\gamma_2)m_2p_2 - (K_2\sin\gamma_2)m_2q_2 + (K_3\cos\gamma_3)m_3p_3 - (K_3\sin\gamma_3)m_3q_3 + m_2a_1K_1\cos\gamma_1 \quad (A.4)$$

and $$F_{M/G_Y} = (K_1\sin\gamma_1)m_1p_1 + (K_1\cos\gamma_1)m_1q_1 + (K_2\sin\gamma_2)m_2p_2 + (K_2\cos\gamma_2)m_2q_2 + (K_3\sin\gamma_3)m_3p_3 + (K_3\cos\gamma_3)m_3q_3 + m_2a_1K_1\sin\gamma_1 \quad (A.5)$$

2. Shaking Force Expression in Terms of ϕ_1 and ϕ_3

a. Exponential Form

When the loop equation is used to express the angle ϕ_2 in terms of ϕ_1 and ϕ_3 [8], the expression for the mass-distance product associated with the center of mass trajectory becomes[1]:

$$M\bar{r}_s = \left[(m_1p_1 + m_2a_1 - \frac{a_1}{a_2}m_2p_2) + i(m_1q_1 - \frac{a_1}{a_2}m_2q_2)\right]e^{i\phi_1} + \left[(m_3p_3 + \frac{a_3}{a_2}m_2p_2) + i(m_3q_3 + \frac{a_3}{a_2}m_2q_2)\right]e^{i\phi_3} + \left[(m_3a_4 + \frac{a_4}{a_2}m_2p_2) + i\frac{a_4}{a_2}m_2q_2\right] \quad (A.6)$$

---

[1] Equation (A.6) can also be expressed in terms of other combinations of individual crank angles ϕ_i, such as ϕ_1 and ϕ_2 or ϕ_2 and ϕ_3.

Double differentiation of the above and change of sign again furnishes an expression for the shaking force $\bar{F}_{M/G}$:

$$\bar{F}_{M/G} = (P_1 + iQ_1)K_1 e^{i\gamma_1} + (P_3 + iQ_3)K_3 e^{i\gamma_3} \tag{A.7}$$

where the following "general mass-distance products" are defined:

$$P_1 = m_1 p_1 + m_2 a_1 - \frac{a_1}{a_2} m_2 p_2 \tag{A.8}$$

$$Q_1 = m_1 q_1 - \frac{a_1}{a_2} m_2 q_2 \tag{A.9}$$

$$P_3 = m_3 p_3 + \frac{a_3}{a_2} m_2 p_2 \tag{A.10}$$

$$Q_3 = m_3 q_3 + \frac{a_3}{a_2} m_2 q_2 \tag{A.11}$$

The K_j and γ_j terms are identical with those of equation (A.3) (now j = 1,3).

Under certain circumstances it proves useful to modify equation (A.7) further, i.e.

$$\bar{F}_{M/G} = AK_1 e^{i(\gamma_1 + \alpha)} + BK_3 e^{i(\gamma_3 + \beta)} \tag{A.12}$$

where

$$A = \sqrt{(P_1)^2 + (Q_1)^2} \tag{A.13}$$

$$\alpha = \tan^{-1} \frac{Q_1}{P_1} \tag{A.14}$$

$$B = \sqrt{(P_3)^2 + (Q_3)^2} \tag{A.15}$$

$$\beta = \tan^{-1} \frac{Q_3}{P_3} \tag{A.16}$$

b. Cartesian Component Form

Transformation of equation (A.7) gives the following X and Y components of the shaking force:

$$F_{M/G_X} = (K_1\cos\gamma_1)P_1 - (K_1\sin\gamma_1)Q_1 + (K_3\cos\gamma_3)P_3 - (K_3\sin\gamma_3)Q_3 \qquad (A.17)$$

and

$$F_{M/G_Y} = (K_1\sin\gamma_1)P_1 + (K_1\cos\gamma_1)Q_1 + (K_3\sin\gamma_3)P_3 + (K_3\cos\gamma_3)Q_3 \qquad (A.18)$$

3. Notation Allowing Distinction Between Balanced and Unbalanced Mechanisms

In order to distinguish between the parameters of the links of balanced and unbalanced mechanisms as well as those of counterweights, the following notation is used:

a) the superscript zero refers to parameters of the unbalanced mechanism
b) the superscript asterisk refers to counterweight parameters
c) the absence of any superscript refers to parameters of a balanced linkage.

Accordingly, the distances to the various centers of

mass, as shown in Figure A.2, and their corresponding angles are given by:

$$r_j^{\circ} = \sqrt{(p_j^{\circ})^2 + (q_j^{\circ})^2} \tag{A.19}$$

$$\theta_j^{\circ} = \tan^{-1} \frac{q_j^{\circ}}{p_j^{\circ}} \tag{A.20}$$

$$r_j^{*} = \sqrt{(p_j^{*})^2 + (q_j^{*})^2} \tag{A.21}$$

$$\theta_j^{*} = \tan^{-1} \frac{q_j^{*}}{p_j^{*}} \tag{A.22}$$

$$r_j = \sqrt{p_j^2 + q_j^2} \tag{A.23}$$

$$\theta_j = \tan^{-1} \frac{q_j}{p_j} \tag{A.24}$$

$$(j = 1,2,3)$$

The mass-distance products for a balanced linkage become:

$$m_j p_j = m_j^{\circ} p_j^{\circ} + m_j^{*} p_j^{*}$$

$$m_j q_j = m_j^{\circ} q_j^{\circ} + m_j^{*} q_j^{*} \qquad (j = 1,2,3) \tag{A.25}$$

When equation (A.12) represents an unbalanced mechanism, it becomes:

$$\bar{F}_{M/G} = A^{\circ} K_1 e^{i(\gamma_1 + \alpha^{\circ})} + B^{\circ} K_3 e^{i(\gamma_3 + \beta^{\circ})} \tag{A.26}$$

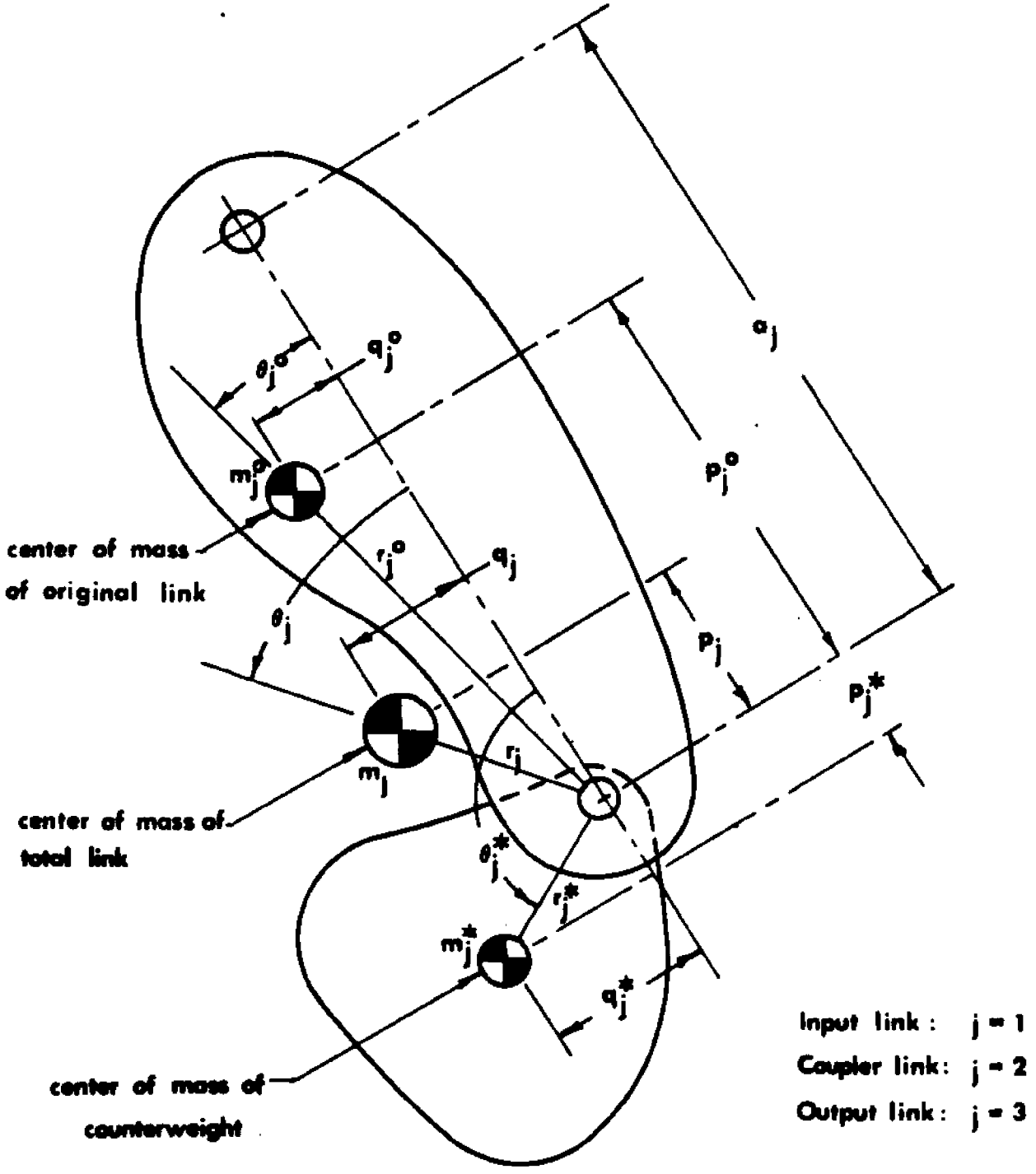


FIG. A.2 NOMENCLATURE ASSOCIATED WITH LINKS AND COUNTERWEIGHTS

where $$A^{o} = \sqrt{(P_1{}^{o})^2 + (Q_1{}^{o})^2} \tag{A.27}$$

$$\alpha^{o} = \tan^{-1} \frac{Q_1{}^{o}}{P_1{}^{o}} \tag{A.28}$$

$$B^{o} = \sqrt{(P_3{}^{o})^2 + (Q_3{}^{o})^2} \tag{A.29}$$

$$\beta^{o} = \tan^{-1} \frac{Q_3{}^{o}}{P_3{}^{o}} \tag{A.30}$$

and $$P_1{}^{o} = m_1{}^{o}p_1{}^{o} + m_2{}^{o}a_1 - \frac{a_1}{a_2} m_2{}^{o}p_2{}^{o} \tag{A.31}$$

$$Q_1{}^{o} = m_1{}^{o}q_1{}^{o} - \frac{a_1}{a_2} m_2{}^{o}q_2{}^{o} \tag{A.32}$$

$$P_3{}^{o} = m_3{}^{o}p_3{}^{o} + \frac{a_3}{a_2} m_2{}^{o}p_2{}^{o} \tag{A.33}$$

$$Q_3{}^{o} = m_3{}^{o}q_3{}^{o} + \frac{a_3}{a_2} m_2{}^{o}q_2{}^{o} \tag{A.34}$$

When equation (A.12) describes the shaking force of a mechanism with counterweights on links 1 and 3, the general mass-distance products become:

$$P_1 = m_1{}^{*}p_1{}^{*} + P_1{}^{o} \tag{A.35}$$

$$Q_1 = m_1{}^{*}q_1{}^{*} + Q_1{}^{o} \tag{A.36}$$

$$P_3 = m_3{}^{*}p_3{}^{*} + P_3{}^{o} \tag{A.37}$$

$$Q_3 = m_3{}^{*}q_3{}^{*} + Q_3{}^{o} \tag{A.38}$$

Finally, when counterweights are attached to all three moving links, the general mass-distance products become:

$$P_1 = P_1^{\circ} + P_1^{*} \tag{A.39}$$

$$Q_1 = Q_1^{\circ} + Q_1^{*} \tag{A.40}$$

$$P_3 = P_3^{\circ} + P_3^{*} \tag{A.41}$$

$$Q_3 = Q_3^{\circ} + Q_3^{*} \tag{A.42}$$

where

$$P_1^{*} = m_1^{*}p_1^{*} + m_2^{*}a_1 - \frac{a_1}{a_2} m_2^{*}p_2^{*} \tag{A.43}$$

$$Q_1^{*} = m_1^{*}q_1^{*} - \frac{a_1}{a_2} m_2^{*}q_2^{*} \tag{A.44}$$

$$P_3^{*} = m_3^{*}p_3^{*} + \frac{a_3}{a_2} m_2^{*}p_2^{*} \tag{A.45}$$

$$Q_3^{*} = m_3^{*}q_3^{*} + \frac{a_3}{a_2} m_2^{*}q_2^{*} \tag{A.46}$$

APPENDIX B

ON THE NUMBER OF MASS-DISTANCE PARAMETERS OF A FOUR-BAR LINKAGE WHICH MAY BE DETERMINED WITH THE HELP OF THE SHAKING FORCE COMPONENTS

When the investigation concerning a force balancing machine for four-bar linkages was first started, it was felt desirable to determine the actual values of the individual link mass-distance products $m_j p_j$ and $m_j q_j$ (j=1,2,3) in order to be able to modify them, by whatever force balancing method chosen, in the direction of full force balance.

It was subsequently found that this is not possible because the resulting set of shaking force expressions, which have to be based on the repeated use of equations (A.4) and (A.5), produce a singular coefficient matrix due to the linear dependence of certain coefficients.

The present approach resulted once this linear dependence was removed with the help of the component loop equations. In this formulation, the repeated use of equations (A.17) and (A.18) allows for the determination of the general mass-distance products P_j and Q_j (j=1,3) from a resulting nonsingular matrix.

The following illustrates the above in detail.

1. Proof Of Singularity Of Matrix Originally Proposed For Solution Of Individual Link Mass-Distance Products

If it were possible to solve for the six $m_j p_j$ and $m_j q_j$ terms, a set of six simultaneous equations would have to be generated by writing the two shaking force component equations (A.4) and (A.5) for three different positions of the input crank. For each of these three sets of expressions, the associated shaking force components, together with the applicable kinematic quantities as well as the mass m_2, would have to be available.

To demonstrate that the solution matrix of such a set of component shaking force equations is singular, consider the only available loop equation [8] for the given mechanism:

$$a_1 e^{i\phi_1} + a_2 e^{i\phi_2} - a_3 e^{i\phi_3} - a_4 e^{i\phi_4} = 0 \qquad (B.1)$$

Double differentiation of the above leads to:

$$K_2 e^{i\gamma_2} = -\frac{a_1}{a_2} K_1 e^{i\gamma_1} + \frac{a_3}{a_2} K_3 e^{i\gamma_3} \qquad (B.2)$$

Equating real and imaginary parts, one obtains the following constraint equations:

$$K_2 \cos\gamma_2 = -\frac{a_1}{a_2} K_1 \cos\gamma_1 + \frac{a_3}{a_2} K_3 \cos\gamma_3 \qquad (B.3)$$

$$K_2 \sin\gamma_2 = -\frac{a_1}{a_2} K_1 \sin\gamma_1 + \frac{a_3}{a_2} K_3 \sin\gamma_3 \qquad (B.4)$$

If these relationships are now substituted into the component expressions (A.4) and (A.5), one obtains:

$$(K_1\cos\gamma_1)m_1p_1 - (K_1\sin\gamma_1)m_1q_1 + (-\frac{a_1}{a_2}K_1\cos\gamma_1 + \frac{a_3}{a_2}K_3\cos\gamma_3)m_2p_2$$

$$- (-\frac{a_1}{a_2}K_1\sin\gamma_1 + \frac{a_3}{a_2}K_3\sin\gamma_3)m_2q_2 + (K_3\cos\gamma_3)m_3p_3 - (K_3\sin\gamma_3)m_3q_3$$

$$= F_{M/G_X} - m_2K_1a_1\cos\gamma_1 \qquad (B.5)$$

and

$$(K_1\sin\gamma_1)m_1p_1 + (K_1\cos\gamma_1)m_1q_1 + (-\frac{a_1}{a_2}K_1\sin\gamma_1 + \frac{a_3}{a_2}K_3\sin\gamma_3)m_2p_2$$

$$+ (-\frac{a_1}{a_2}K_1\cos\gamma_1 + \frac{a_3}{a_2}K_3\cos\gamma_3)m_2q_2 + (K_3\sin\gamma_3)m_3p_3 + (K_3\cos\gamma_3)m_3q_3$$

$$= F_{M/G_Y} - m_2K_1a_1\sin\gamma_1 \qquad (B.6)$$

It may now be seen that the coefficients of m_2p_2 and m_2q_2 in each of the above expressions are linear combinations of the coefficients of the remaining terms.

This linear dependence will cause the resulting 6 x 6 matrix, obtained by writing equations (B.5) and (B.6) for three positions of the input angle ϕ_1 and needed for the solution of the six m_jp_j and m_jq_j parameters, to be singular.

2. Proof Of Nonsingularity Of Balancing Equation Matrix Used To Solve For General Mass-Distance Products

When the linear dependence of terms in equations (B.5) and (B.6) is removed with the help of the component loop equations (B.3) and (B.4), equations (A.17) and (A.18) result, i.e.

$$F_{M/G_X} = (K_1 \cos\gamma_1)P_1 - (K_1 \sin\gamma_1)Q_1 + (K_3 \cos\gamma_3)P_3 - (K_3 \sin\gamma_3)Q_3 \qquad (B.7)$$

and $$F_{M/G_Y} = (K_1 \sin\gamma_1)P_1 + (K_1 \cos\gamma_1)Q_1 + (K_3 \sin\gamma_3)P_3 + (K_3 \cos\gamma_3)Q_3 \qquad (B.8)$$

The linear independence of terms in each expression is assured since there are no further mechanism loop equations. Furthermore, both expressions are linearly independent of each other since they represent the orthogonal components of a single vector.

In order to solve for the four P_j and Q_j parameters, two sets of shaking force component equations of type (B.7) and (B.8) are now written for two mechanism positions[1]. In matrix form this becomes:

---

[1] Of course, all kinematic values and shaking force components must be known from experimentation for these two positions.

$$\begin{bmatrix} K_{11}\cos\gamma_{11} & -K_{11}\sin\gamma_{11} & K_{31}\cos\gamma_{31} & -K_{31}\sin\gamma_{31} \\ K_{11}\sin\gamma_{11} & K_{11}\cos\gamma_{11} & K_{31}\sin\gamma_{31} & K_{31}\cos\gamma_{31} \\ K_{12}\cos\gamma_{12} & -K_{12}\sin\gamma_{12} & K_{32}\cos\gamma_{32} & -K_{32}\sin\gamma_{32} \\ K_{12}\sin\gamma_{12} & K_{12}\cos\gamma_{12} & K_{32}\sin\gamma_{32} & K_{32}\cos\gamma_{32} \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \\ P_3 \\ Q_3 \end{bmatrix} = \begin{bmatrix} F_{M/G_{X1}} \\ F_{M/G_{Y1}} \\ F_{M/G_{X2}} \\ F_{M/G_{Y2}} \end{bmatrix} \quad (B.9)$$

The second subscripts in these terms indicate at which position the quantity is evaluated.

Each of the four row vectors of the coefficient matrix is linearly independent as long as no pairs of links produce identical kinematic terms for the two chosen positions of the mechanism. Thus, the matrix has rank four and may be solved uniquely for the four general mass-distance products. (The only known exception to this is the parallel-bar linkage in which the first and third terms as well as the second and fourth terms of equations (B.7) and (B.8) will be identical.)

APPENDIX C

DESCRIPTION OF AUGMENTED LAGRANGIAN OPTIMIZATION PROGRAM AND APPLICATION TO MINIMAX BALANCING OPTIMIZATION

This appendix deals with the Augmented Lagrangian program which is used in various parts of this dissertation.

Section 1 briefly outlines the theory which forms the basis of this optimization method. Section 2 describes the associated computer program and details the manner in which it is used. Section 3 shows the minimax solution of the overdetermined system of balancing equations as an example.

1. Theory Of Optimization Method

The optimization program is based upon the minimization of the Augmented Lagrangian type of penalty function:

$$\Phi(\bar{x},\bar{\sigma},\bar{\theta}) = F(\bar{x}) + \frac{1}{2}\sum_i \sigma_i \left(c_i(\bar{x}) - \theta_i\right)^2 \qquad (C.1)$$

In the above, $F(\bar{x}) = F(x_1,x_2,\ldots,x_n)$ is the function to be minimized, which may be subject to the equality constraints

$$c_i(\bar{x}) = 0 \qquad (i = 1,2,\ldots,k),\ k \le n, \qquad (C.2)$$

as well as the inequality constraints

$$c_i(\bar{x}) \ge 0 \qquad (i = k+1,\ldots,m). \qquad (C.3)$$

The optimization variables are represented by x_i $(i = 1,2,\ldots,n)$. The negative subscript following the penalty terms implies that a specific term of this type only then becomes active when the associated constraint is violated.

In each inner iteration, with fixed values of the parameters $\bar{\sigma}$ and $\bar{\theta}$, the function $\Phi(\bar{x},\bar{\sigma},\bar{\theta})$ is then minimized by the variable metric method of Davidon, Fletcher and Powell [21]. Outer iterations vary the values of the parameters $\bar{\sigma}$ and $\bar{\theta}$ such that the sequence of minima $\bar{x}_{\sigma,\theta}$ tends to the solution of the constrained problem.

2. Description Of Computer Program

The core of the computer program consists of the five subroutines VF01AD, VF01ZD, MC11AD, VA09AD and VE04AD[1]. In addition, the user must supply a main program and an additional subroutine VF01BD. The latter serves to define the specific objective function $F(\bar{x})$ and the associated constraints $c_i(\bar{x})$ together with the derivatives of the objective function and the constraints with respect to all the variables involved.

The following gives sequential instructions for the

---

[1]All subroutines are written in Fortran with double precision arithmetic.

formulation of the main program and the subroutine VF01BD. The core program is reproduced in conjunction with the example in Section 3 of this appendix.

a. Main Program

The main program begins with the following COMMON statements which define the array sizes for the constraints C and their derivatives GC (the latter is only necessary if any of the derivatives are constant):

```
COMMON/VF01ED/C(3M)
COMMON/VF01FD/GC(N,M).
```

Subsequently, the following arguments are defined:

N - An integer, set to the number of variables n ($N \geq 2$)

M - An integer, set to the total number of constraints m ($M \geq 1$)

K - An integer, set to the total number of equality constraints k

X - A real array of N elements representing the initial estimate of the solution set. (This estimate need not be feasible, i.e. satisfy the constraints.) After the first iteration, VF01AD returns the solution $\bar{x}$ in X.

AKMIN - A real number in which the relative error tolerance required in the constraint residuals must be set. (For example, AKMIN = $1x10^{-3}$ indicates that no further outer iterations will take place if the largest scaled constraint residual changes by less than $1x10^{-3}$. (See below.))

EPS - A real array of N elements in which the tolerances for the unconstrained minimizations must be set (i.e. if all the variables do not change by more than these values in the successive inner iterations, this inner iteration is terminated). EPS(I) should be set so that $EPS(I)/X(I) \approx AKMIN$.

DFN - A real number in which the likely reduction in $F(\bar{x})$ from its initial value as obtained from X must be set. Since DFN is used only in the first iteration, an order of magnitude estimate suffices.

MAXFN - An integer in which the maximum number of calls of VF01BD (the user supplied subroutine) on any one unconstrained minimization, i.e. inner iteration, must be set. The number 500 has proven to be reasonable for this purpose.

IPR1 - An integer controlling the frequency of the printing of the outer iterations. Printing occurs every IPR1 outer iterations except for details of increases to the σ_i which are always printed. No printing at all occurs (except for diagnostics) if IPR1=0.

IPR2 - An integer controlling the frequency of the printing of the inner iterations. Its use is similar to that of IPR1.

IW - An integer giving the amount of storage available in COMMON/VF01LD/W(IW). Set to 2500 unless the restrictions that $N \leq 25$ and $M \leq 50$ are violated.

MODE - An integer controlling the mode of operation of VF01AD. A usual setting is MODE=1. (See specification sheet for Harwell Subroutine Library.)

Next, the constraint scale factors, also designated C(1),C(2),...,C(M), are set by substituting the initial approximations of x_i into the constraint equations and using the absolute values of the results.

If any of the constraints are linear functions of any of the variables, their derivatives with respect to these

variables, i.e. the resulting constants, must be given by way of the array GC(N,M), where N represents the variable number and M stands for the constraint number.

Finally, the subroutine is called by:

CALL VF01AD(N,M,K,X,EPS,AKMIN,DFN,MAXFN,IPR1,IPR2,IW,MODE).

b. Subroutine VF01BD

This subroutine starts by way of:

```
SUBROUTINE VF01BD(N,M,X)
REAL X(N)
COMMON/VF01CD/F
COMMON/VF01DD/G(2N)
COMMON/VF01ED/C(3M)
COMMON/VF01FD/GC(N,M)
```

where F is the objective function $F(\bar{x})$ in terms of variables x

G represents the derivatives of the objective function with respect to the variables

C represents the constraints

GC represents the derivatives of the nonlinear constraints with respect to the variables. For use of derivatives of linear constraints, see previous section.

The subroutine must now define the terms listed below. Data as well as expressions required to obtain these terms may be introduced where convenient.

$F = F(x_1, x_2, \ldots, x_n)$

$G = G(1), G(2), \ldots, G(N) = \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n}$

$C = C(1), C(2), \ldots, C(M)$. (These constraints must be given such that the resulting expression is equal to or greater than zero. For an example, see Section 3b.)

$GC = GC(1,1), GC(1,2), \ldots, GC(N,M) = \frac{\partial C(1)}{\partial x_1}, \frac{\partial C(2)}{\partial x_1}, \ldots, \frac{\partial C(M)}{\partial x_n}$

3. Example: Minimax Solution Of Overdetermined System Of Balancing Equations

The following shows how the computer program is adapted to the minimax solution of the overdetermined system of balancing equations according to the formulation given by equations (3.2) and (3.3). Section 3a shows how the various parameters are defined in the main program according to Section 2a above. Subsequently, a listing of the actual routine is given. Section 3b gives the applicable subroutine

VF01BD. Again, the listing of the final program is preceded by a detailed discussion. This subroutine is also used to call on subprograms FOURIE and MOTION which furnish certain kinematic terms of the mechanism. Section 3c shows the Augmented Lagrangian subroutines VF01AD, VF01ZD, MC11AD, VA09AD and VE04AD. Section 4d gives the most important parts of the output of the program and interprets the results.

a. Main Program

The main program, which is listed at the end of this section, begins with comment cards which are designed to serve as additional explanations. These are followed by the COMMON statement which defines the array size for the constraint C. The following variables are then defined:

N = 5; stands for the five variables P_1, Q_1, P_3, Q_3 and the objective function z.

M = 144; stands for the number of constraints, representing the square of the difference of the two shaking force components evaluated every 5 degrees between 0 and 355 degrees.

K = 0, since there are no equality constraints.

X(1) to X(5), the starting point estimates of the variables, are made 0.5 and $1x10^{-3}$ since previous experience has shown these values to be of this order of magnitude. (Such accurate estimates are not necessary. They only serve to decrease the number of iterations.)

With AKMIN chosen arbitrarily to be $1x10^{-3}$, EPS(1) to EPS(5) result from X(1) to X(5).

DFN is chosen to be smaller than X(1) = 0.5, i.e it is made 0.4.

MAXFN = 500, as discussed earlier.

IPR1 = 1, to print every outer iteration.

IPR2 = 1, to print every inner iteration.

IW = 10,000, since there are more than 50 constraints.

MODE = 1, as indicated earlier.

Subsequently, a DO loop is set up to set the scale factors C(I) for all 144 constraints equal to 0.3. These are estimates of the magnitudes of the constraints when evaluated with the starting point values of the variables.

Since there are no linear constraints, the derivatives GC(N,M) do not enter this routine.

MAIN

```
C   THIS PROGRAM WILL CHOOSE THE VALUES OF THE GENERAL MASS DISTANCE
C   PRODUCTS P1,P2,P3 AND P3 WHICH WILL RESULT IN THE BEST MINIMAX FIT
C   OF THE COMPUTED SHAKING FORCE CURVES IN THE X AND Y DIRECTIONS TO
C   A KNOWN EXPERIMENTAL CURVE.  THIS WILL BE ACCOMPLISHED BY THE
C   USE OF THE AUGMENTED LAGRANGIAN OPTIMIZATION ROUTINE.
C   THE VARIABLES FOR THIS PROBLEM ARE
C        X(1) = F = FUNCTION TO BE MINIMIZED
C        X(2) = P1
C        X(3) = P2
C        X(4) = P3
C        X(5) = P3
C
C   THE CONSTRAINTS ARE
C          C(I) = X(1) - (FSX(I) - FEX(I))**2
C        C(I+72) = X(1) - (FSY(I) - FEY(I))**2,  I=1,2,....,72
0001          IMPLICIT REAL*8(A-H,O-Z)
0002          COMMON/FOIED/C(432)
0003          REAL*8 X(5),EPS(5)
0004          N = 5
0005          M = 144
0006          K=0
0007          PI=3.14159300
C   THE STARTING POINT IS ESTIMATED TO BE
0008          X(1)=0.500
0009          X(2)=1.00-3
0010          X(3)=1.00-3
0011          X(4)=1.00-3
0012          X(5)=1.00-3
0013          AKMIN=10-3
0014          EPS(1)=0.50-3
0015          EPS(2)=1.00-6
0016          EPS(3)=1.00-6
0017          EPS(4)=1.00-6
0018          EPS(5)=1.00-6
0019          DFN=.400
0020          MAXF=500
0021          IPR1=1
0022          IPR2=1
C   SET IW=10000.  IF MORE STORAGE IS REQUIRED, VFOIAD WILL
C   STOP AND PRINT OUT NEW REQUIREMENT.
0023          IW=10000
0024          MODE=1
C   THE CONSTRAINT SCALE FACTORS ARE ALL ESTIMATED TO BE 0.3
0025          DO 10 I=1,144
0026       10 C(I)=.3300
0027          CALL VFOIAD(N,M,K,X,EPS,AKMIN,DFN,MAXF,IPR1,IPR2,IW,MODE)
0028          STOP
0029          END
```

b. Subroutine VF01BD (See program, followed by subroutines FOURIE and MOTION, at the end of this section.)

After allocating storage for the various subscripted variables, the subroutine VF01BD starts with the COMMON statements indicated in Section 2a. The subsequent steps are:

1. F = X(1), i.e. the objective function z is declared as the variable x_1 (see equation (3.2)).

2. G(1) = 1, represents the derivative $\frac{\partial F}{\partial x_1}$. Since all other derivatives of F vanish, they must be set equal to zero (i.e., G(2) to G(5) = 0).

3. A1 to A4 represent the link dimensions of links 1 to 4.

4. DELPHI = 5, stands for the 5 degree increments of the input angle ϕ_1.

5. P_1, Q_1, P_3 and Q_3 are defined as the remaining variables x_2, x_3, x_4 and x_5.

6. FEX(1) to FEY(72) are the 144 experimentally obtained shaking force components at 5 degree intervals from 0 to 355 degrees. (See Table 3.3)

7. A DO loop is set up to compute the constraints (see equation (3.3)) as well as their derivatives. Since the constraints must be set up such that they are equal to or larger than zero, the constraints of the type

$$(F_{CX} - F_{EX})^2 \leq z \qquad (C.4)$$

or $(F_{CY} - F_{EY})^2 \leq z$ (see equation (C.3)), (C.5)

become

$$C(I) = z - (F_{CX} - F_{EX})^2 \qquad (C.6)$$

or $$C(I) = z - (F_{CY} - F_{EY})^2 \qquad (C.7)$$

The derivatives of the constraints are derivatives with respect to the variables z, P_1, Q_1, P_3, Q_3, i.e. x_1, x_2, x_3, x_4, x_5, respectively. Both the constraints and their derivatives are obtained with the help of expressions for the computed shaking force components F_{CX} and F_{CY}. The following first shows how these specific 144 quantities are computed. Subsequently, the formulation of the constraint equations and their derivatives will be discussed:

i) Subroutine FOURIE starts from the experimentally obtained angular velocities, read in in terms of RPM, and converts these to a Fourier series in which the angular velocity $\dot{\phi}_1$ is expressed in terms of the angle ϕ_1. The associated angular accelerations are subsequently obtained from

$$\ddot{\phi}_1 = \frac{d\dot{\phi}_1}{d\phi_1} \cdot \dot{\phi}_1 \qquad (C.8)$$

ii) Subroutine MOTION computes ϕ_3, $\dot{\phi}_3$ and $\ddot{\phi}_3$ as functions of ϕ_1, $\dot{\phi}_1$ and $\ddot{\phi}_1$.

iii) The coefficients of the shaking force component expressions (see equations (A.3), (A.17) and (A.18)) are computed in the following manner:

```
VAR1(I) = KAY1(I) x COS(GMA1(I))
VAR2(I) = KAY1(I) x SIN(GMA1(I))
VAR3(I) = KAY3(I) x COS(GMA3(I))
VAR4(I) = KAY3(I) x SIN(GMA3(I))
VAR5(I) = KAY1(I) x SIN(GMA1(I))
VAR6(I) = KAY1(I) x COS(GMA1(I))
VAR7(I) = KAY3(I) x SIN(GMA3(I))
VAR8(I) = KAY3(I) x COS(GMA3(I))
```

where

$$KAY1(I) = \sqrt{(\dot{\phi}_{1I})^4 + (\ddot{\phi}_{1I})^2}$$

$$KAY3(I) = \sqrt{(\dot{\phi}_{3I})^4 + (\ddot{\phi}_{3I})^2}$$

$$GMA1(I) = \phi_{1I} - \tan^{-1}\frac{\ddot{\phi}_{1I}}{(\dot{\phi}_{1I})^2}$$

$$GMA3(I) = \phi_{3I} - \tan^{-1}\frac{\ddot{\phi}_{3I}}{(\dot{\phi}_{3I})^2}$$

iv) The shaking force components F_{CXi} and F_{CYi} are evaluated in terms of P_1, Q_1, P_3 and Q_3. These are given as:

$$F_{CXi} = P_1 xVAR1(I) - Q_1 xVAR2(I) + P_3 xVAR3(I) - Q_3 xVAR4(I)$$

$$F_{CYi} = P_1 xVAR5(I) + Q_1 xVAR6(I) + P_3 xVAR7(I) + Q_3 xVAR8(I)$$

v) The constraint equations are written in the form of equations (C.6) and (C.7), i.e.

$$C(I) = X(1) - (F_{CXi} - F_{EXi})^2$$

and $C(I+72) = X(1) - (F_{CYi} - F_{EYi})^2$

vi) Finally, differentiation of the constraints with

respect to the variables gives:

$$GC(1,I) = 1$$

$$GC(1,I+72) = 1$$

$$GC(2,I) = AAxVAR1(I)$$

$$GC(2,I+72) = BBxVAR5(I)$$

$$GC(3,I) = -AAxVAR2(I)$$

$$GC(3,I+72) = BBxVAR6(I)$$

$$GC(4,I) = AAxVAR3(I)$$

$$GC(4,I+72) = BBxVAR7(I)$$

$$GC(5,I) = -AAxVAR4(I)$$

$$GC(5,I+72) = BBxVAR8(I)$$

where $AA = -2(F_{CXi} - F_{EXi})$

$BB = -2(F_{CYi} - F_{EYi})$

```
0001                SUBROUTINE VPOINT(N,M,X)
0002                IMPLICIT REAL*8(A-H,O-Z)
0003                REAL*8 X(5)
0004                INTEGER PHI, PHI01, DELPHI
0005                REAL*8 XAY1(360),XAY3(360)
0006                DIMENSION SMA1(360),SMA3(360),FCY(360),FCY(360)
0007                DIMENSION XAR1(360),XAR2(360),XAR3(360),XAR4(360)
0008                DIMENSION XAR5(360),XAR6(360),XAR7(360),XAR8(360)
0009                DIMENSION PSI(360),PSIV(360)
0010                DOUBLE PRECISION DCOS,DSIN,DSQRT,DATAN2
0011                DOUBLE PRECISION FE(72),FEY(72)
0012                COMMON/VPOINT/F
0013                COMMON/VPOINT/G(10)
0014                COMMON/VPOINT/EDIED/C(432)
0015                COMMON/VPOINT/COS(5,144)
0016                P(=3.14159D00
                C   DEFINE OBJECTIVE FUNCTION
0017                F=X(1)
                C     G(I)=PARTIAL DERIVATIVES OF OBJECTIVE FUNCTION WITH
                C          RESPECT TO VARIABLES X(1) THRU X(5)
0018                G(1)=1D0
0019                G(2)=0D0
0020                G(3)=0D0
0021                G(4)=0D0
0022                G(5)=0D0
                C     THE LINK DIMENSIONS ARE
0023                A1=200
0024                A2=8.500
0025                A3=300
0026                A4=8.20400
0027                DELPHI = 5
                C   DEFINE REMAINING VARIABLES
0028                P1 = X(2)
0029                Q1 = X(3)
0030                P3 = X(4)
0031                Q3 = X(5)
                C   THE EXPERIMENTAL SHAKING FORCE COMPONENTS, DETERMINED
                C   EVERY 5 DEGREES, ARE GIVEN AS FOLLOWS, WHERE THE
                C   FIRST VALUE CORRESPONDS TO THE READING AT 0 DEGREES
                C   AND THE SEVENTY SECOND VALUE TO THE READING AT 355 DEGREES.
0032                FE(1)  =  .477
0033                FE(2)  =  .483
0034                FE(3)  =  .449
0035                FE(4)  =  .394
0036                FE(5)  =  .258
0037                FE(6)  =  .202
0038                FE(7)  =  .105
0039                FE(8)  =  .060
0040                FE(9)  =  .030
0041                FE(10) =  .007
0042                FE(11) = -.003
0043                FE(12) = -.015
0044                FE(13) = -.026
0045                FE(14) = -.029
0046                FE(15) = -.031
0047                FE(16) = -.034
```

```
0048                FEX(17) = -.035
0049                FEX(18) = -.034
0050                FEX(19) = -.031
0051                FEX(20) = -.029
0052                FEX(21) = -.025
0053                FEX(22) = -.027
0054                FEX(23) = -.024
0055                FEX(24) = -.023
0056                FEX(25) = -.020
0057                FEX(26) = -.015
0058                FEX(27) = -.012
0059                FEX(28) = -.012
0060                FEX(29) = -.015
0061                FEX(30) = -.017
0062                FEX(31) = -.018
0063                FEX(32) = -.021
0064                FEX(33) = -.024
0065                FEX(34) = -.027
0066                FEX(35) = -.030
0067                FEX(36) = -.031
0068                FEX(37) = -.032
0069                FEX(38) = -.033
0070                FEX(39) = -.037
0071                FEX(40) = -.039
0072                FEX(41) = -.038
0073                FEX(42) = -.039
0074                FEX(43) = -.039
0075                FEX(44) = -.041
0076                FEX(45) = -.044
0077                FEX(46) = -.051
0078                FEX(47) = -.053
0079                FEX(48) = -.056
0080                FEX(49) = -.061
0081                FEX(50) = -.063
0082                FEX(51) = -.072
0083                FEX(52) = -.077
0084                FEX(53) = -.084
0085                FEX(54) = -.091
0086                FEX(55) = -.096
0087                FEX(56) = -.101
0088                FEX(57) = -.113
0089                FEX(58) = -.119
0090                FEX(59) = -.126
0091                FEX(60) = -.134
0092                FEX(61) = -.143
0093                FEX(62) = -.147
0094                FEX(63) = -.150
0095                FEX(64) = -.144
0096                FEX(65) = -.123
0097                FEX(66) = -.110
0098                FEX(67) = -.081
0099                FEX(68) = -.032
0100                FEX(69) =  .035
0101                FEX(70) =  .122
0102                FEX(71) =  .258
0103                FEX(72) =  .356
```

```
0104              FEY(1)  = -.570
0105              FEY(2)  = -.590
0106              FEY(3)  = -.551
0107              FEY(4)  = -.463
0108              FEY(5)  = -.374
0109              FEY(6)  = -.292
0110              FEY(7)  = -.218
0111              FEY(8)  = -.173
0112              FEY(9)  = -.120
0113              FEY(10) = -.087
0114              FEY(11) = -.066
0115              FEY(12) = -.054
0116              FEY(13) = -.039
0117              FEY(14) = -.029
0118              FEY(15) = -.024
0119              FEY(16) = -.018
0120              FEY(17) = -.012
0121              FEY(18) = -.009
0122              FEY(19) = -.006
0123              FEY(20) = 0.0
0124              FEY(21) =  .006
0125              FEY(22) =  .009
0126              FEY(23) =  .015
0127              FEY(24) =  .023
0128              FEY(25) =  .026
0129              FEY(26) =  .030
0130              FEY(27) =  .034
0131              FEY(28) =  .043
0132              FEY(29) =  .054
0133              FEY(30) =  .065
0134              FEY(31) =  .075
0135              FEY(32) =  .083
0136              FEY(33) =  .091
0137              FEY(34) =  .099
0138              FEY(35) =  .116
0139              FEY(36) =  .122
0140              FEY(37) =  .127
0141              FEY(38) =  .132
0142              FEY(39) =  .141
0143              FEY(40) =  .144
0144              FEY(41) =  .150
0145              FEY(42) =  .153
0146              FEY(43) =  .155
0147              FEY(44) =  .156
0148              FEY(45) =  .158
0149              FEY(46) =  .159
0150              FEY(47) =  .160
0151              FEY(48) =  .160
0152              FEY(49) =  .161
0153              FEY(50) =  .161
0154              FEY(51) =  .161
0155              FEY(52) =  .160
0156              FEY(53) =  .160
0157              FEY(54) =  .159
0158              FEY(55) =  .159
0159              FEY(56) =  .156
```

```
0160              FEY(57) =  .155
0161              FEY(58) =  .153
0162              FEY(59) =  .150
0163              FEY(60) =  .146
0164              FEY(61) =  .144
0165              FEY(62) =  .134
0166              FEY(63) =  .129
0167              FEY(64) =  .116
0168              FEY(65) =  .096
0169              FEY(66) =  .074
0170              FEY(67) =  .044
0171              FEY(68) = -.023
0172              FEY(69) = -.111
0173              FEY(70) = -.206
0174              FEY(71) = -.345
0175              FEY(72) = -.458
0176              Z=PI/1.3002
                C
0177              DO 120 PHIDI=1.356,DELPHI
0178              PHID=PHIDI - 1
0179              PHI=Z * PHID
0180              I = PHID/DELPHI + 1
                C   THE ANGULAR VELOCITIES AND ACCELERATIONS ARE DETERMINED
                C   IN SUBROUTINE FOURIE
0181                CALL FOURIE (I,PHIVEL,PHIACC)
                C   THE KINEMATICS OF THE LINKAGE ARE DETERMINED IN
                C   SUBROUTINE MOTION
0182                CALL MOTION (PHI,A1,A2,A3,A4,PSI,PHIVEL,PHIACC,
                   1              PSIV(I),PSIA(I))
0183                RAY1(I) = DSQRT(PHIVEL**4 + PHIACC**2)
0184                RAY3(I) = DSQRT(PSIV(I)**4 + PSIA(I)**2)
0185                GMA1(I) = PHI - DATAN2(PHIACC,PHIVEL**2)
0186                GMA3(I) = PSI - DATAN2(PSIV(I),PSIV(I)**2)
0187                VAR1(I) = RAY1(I)*DCOS(GMA1(I))
0188                VAR2(I) = RAY1(I)*DSIN(GMA1(I))
0189                VAR3(I) = RAY3(I)*DCOS(GMA3(I))
0190                VAR4(I) = RAY3(I)*DSIN(GMA3(I))
0191                VAR5(I) = RAY1(I)*DSIN(GMA1(I))
0192                VAR6(I) = RAY1(I)*DCOS(GMA1(I))
0193                VAR7(I) = RAY3(I)*DSIN(GMA3(I))
0194                VAR8(I) = RAY3(I)*DCOS(GMA3(I))
                C   THE COMPUTED SHAKING FORCES ARE
0195                FCX(I) = P1*VAR1(I)-Q1*VAR2(I)+P3*VAR3(I)-Q3*VAR4(I)
0196                FCY(I) = P1*VAR5(I)+Q1*VAR6(I)+P3*VAR7(I)+Q3*VAR8(I)
                C     THE CONSTRAINTS ARE
0197                C(I) = X(I) - (FCX(I) - FEX(I))**2
0198                C(I+72) = X(I) - (FCY(I) - FEY(I))**2
                C  THE TERMS GC(N,N) REPRESENT THE DERIVATIVES OF THE
                C  N CONSTRAINTS WITH RESPECT TO THE N VARIABLES
0199                GC(1,I) = 100
0200                GC(1,I+72) = 100
0201                AA = -200*(FCX(I) - FEX(I))
0202                BB = -200*(FCY(I) - FEY(I))
0203                GC(2,I) = AA*VAR1(I)
0204                GC(2,I+72) = BB*VAR5(I)
0205                GC(3,I) = -AA*VAR2(I)
```

```
0206              GC(3,I+72) = BB*VAR6(I)
0207              GC(4,I) = AA*VAR3(I)
0208              GC(4,I+72) = BB*VAR7(I)
0209              GC(5,I) = -AA*VAR4(I)
0210              GC(5,I+72) = BB*VAR8(I)
0211          120 CONTINUE
0212              RETURN
0213              END
```

```
0001            SUBROUTINE FOURIE (K,PHIVEL,PHIACC)
0002            IMPLICIT REAL*8(A-Z)
              C   PROGRAM TO TAKE DISCRETE EXPERIMENTAL READINGS OF RPM AND
              C   CONVERT IT TO A CONTINUOUS CURVE OF ANGULAR VELOCITY AS A
              C   FUNCTION OF THE INPUT LINK ANGLE.  THIS FOURIER SERIES IS THEN
              C   DIFFERENTIATED TO GIVE AN EXPRESSION FOR THE ANGULAR ACCELERATION
0003            INTEGER I,J,K
0004            DIMENSION PHI(72),PHID(72),A(36),B(36)
0005            DOUBLE PRECISION DCOS,DSIN
0006            DOUBLE PRECISION RPM(72)
              C     INSERT DATA FROM AUTOMATED VELOCITY MEASUREMENT
0007              RPM(1)  = 121.90
0008              RPM(2)  = 123.56
0009              RPM(3)  = 123.92
0010              RPM(4)  = 123.78
0011              RPM(5)  = 122.54
0012              RPM(6)  = 121.71
0013              RPM(7)  = 121.07
0014              RPM(8)  = 120.29
0015              RPM(9)  = 119.72
0016              RPM(10) = 119.13
0017              RPM(11) = 118.73
0018              RPM(12) = 118.34
0019              RPM(13) = 118.15
0020              RPM(14) = 117.95
0021              RPM(15) = 117.81
0022              RPM(16) = 117.71
0023              RPM(17) = 117.66
0024              RPM(18) = 117.62
0025              RPM(19) = 117.67
0026              RPM(20) = 117.69
0027              RPM(21) = 117.76
0028              RPM(22) = 117.85
0029              RPM(23) = 117.98
0030              RPM(24) = 118.13
0031              RPM(25) = 118.32
0032              RPM(26) = 118.52
0033              RPM(27) = 118.82
0034              RPM(28) = 119.12
0035              RPM(29) = 119.47
0036              RPM(30) = 119.86
0037              RPM(31) = 120.32
0038              RPM(32) = 120.83
0039              RPM(33) = 121.34
0040              RPM(34) = 121.94
0041              RPM(35) = 122.49
0042              RPM(36) = 123.10
0043              RPM(37) = 123.45
0044              RPM(38) = 123.93
0045              RPM(39) = 124.11
0046              RPM(40) = 124.32
0047              RPM(41) = 124.47
0048              RPM(42) = 124.55
0049              RPM(43) = 124.57
0050              RPM(44) = 124.51
0051              RPM(45) = 124.59
```

```
C052              RPM(46) = 124.50
C053              RPM(47) = 124.38
C054              RPM(48) = 124.20
C055              RPM(49) = 123.95
C056              RPM(50) = 123.56
C057              RPM(51) = 123.10
C058              RPM(52) = 122.56
C059              RPM(53) = 121.95
C060              RPM(54) = 121.10
C061              RPM(55) = 120.30
C062              RPM(56) = 119.43
C063              RPM(57) = 118.60
C064              RPM(58) = 117.82
C065              RPM(59) = 117.79
C066              RPM(60) = 116.26
C067              RPM(61) = 115.60
C068              RPM(62) = 114.95
C069              RPM(63) = 114.37
C070              RPM(64) = 113.76
C071              RPM(65) = 113.54
C072              RPM(66) = 113.60
C073              RPM(67) = 113.92
C074              RPM(68) = 114.49
C075              RPM(69) = 115.32
C076              RPM(70) = 116.50
C077              RPM(71) = 118.30
C078              RPM(72) = 120.35
C079              PI = 3.1415?3?00
C080              Z = PI/1.802
C081              DELPHI = 5.00*Z
C082              PHI(1) = 0.00
C083              DO 10 I = 1,71
C084              PHI(I+1) = PHI(I) + DELPHI
C085           10 CONTINUE
C086              DO 20 I= 1,72
C087              PHID(I) = RPM(I)*200*PI/501
C088           20 CONTINUE
C089              SUMA0 = 000
C090              DO 30 I = 1,72
C091              SUMA0 = SUMA0 + PHID(I)
C092           30 CONTINUE
C093              A0 = (1.00/3.601)*SUMA0
C094              DO 40 N=1,36
C095              SUMAN = 000
C096              SUMBN = 000
C097              DO 50 I=1,72
C098              SUMAN = SUMAN +PHID(I)*DCOS(N)*PHI(I)
C
C099              SUMBN = SUMBN +PHID(I)*DSIN(N)*PHI(I)
C
C100           50 CONTINUE
C101              A(N) = (1.00/3.601)*SUMAN
C102              B(N) = (1.00/3.601)*SUMBN
C103           40 CONTINUE
C104              F = 000
C105              DO 60 N=1,36
```

```
C     THE FOURIER SERIES OF THE ANGULAR VELOCITY IS
0106        F = F + A(N)*DCOS(N*PHI(K)) + B(N)*DSIN(N*PHI(K))
0107     80 CONTINUE
0108        PHIVEL = F + A0/2.00
0109        G=C00
0110        DO 21 N=1,36
C     DIFFERENTIATING THE FOURIER SERIES OF THE ANGULAR VELOCITY WILL
C     YIELD THE FOLLOWING EXPRESSION FOR THE ANGULAR ACCELERATION
0111        G=G + (-N*A(N)*DSIN(N*PHI(K)) + N*B(N)*DCOS(N*PHI(K)))*PHID(K)
0112     21 CONTINUE
0113        PHIACC = G
0114        RETURN
0115        END
```

```
0001            SUBROUTINE MOTION (PHI,A1,A2,A3,A4,PSI,PHIVEL,PHIACC,
               1                   PSIVEL,PSIACC)
C
C   THIS SUBROUTINE CALCULATES THE OUTPUT LINK ANGLE,
C   ANGULAR VELOCITY AND ANGULAR ACCELERATION AS FUNCTIONS OF
C   THE INPUT LINK ANGLE, ANGULAR VELOCITY AND ANGULAR ACCELERATION
C
0002            IMPLICIT REAL*8(A-H,O-Z)
0003            DOUBLE PRECISION DCOS, DSIN,DSQRT,DATAN2,DABS
0004            REAL*8 LAMBDA, LAMVEL, LAMACC
0005            PI=3.14159D0
0006            TWOPI = 2D0*PI
0007            A =  DSIN(PHI)
0008            B =  DCOS(PHI)  -A4/A1
0009            C = (A1*A1 - A2*A2 + A3*A3 + A4*A4)/(2.D0*A1*A3) - A4* DCOS(PHI)/A3
0010            ROOT = DSQRT (A*A + B*B - C*C)
0011            PSI = 2D0 * DATAN2(A-ROOT, B+C)
0012            IF (PSI) 10,20,20
0013         10  PSI = PSI + TWOPI
0014         20 SA=DSIN(PHI)
0015            CA = DCOS (PHI)
0016            SB = DSIN(PSI)
0017            CB = DCOS (PSI)
0018            LAMBDA = DATAN2 (A3*SB-A1*SA,A3*CB+A4-A1*CA)
0019            IF (LAMBDA) 30,40,40
0020         30 LAMBDA = LAMBDA + 2D0*PI
0021         40 TAU1 = DSIN(PHI-LAMBDA)
0022            TAU3 = DSIN(PSI-LAMBDA)
C
C       CASE IN WHICH TAU3 GOES TO ZERO
0023            IF( DABS(TAU3)-1D-3) 60,50,50
0024         50 TAU7 = DSIN(PHI-PSI)
0025            PSIVEL = PHIVEL*A1*TAU1/(A3*TAU3)
0026            LAMVEL = PHIVEL*A1*TAU7/(A2*TAU3)
0027            TAUDT1 = DCOS (PHI-LAMBDA) * (PHIVEL-LAMVEL)
0028            TAUDT3 = DCOS (PSI-LAMBDA) * (PSIVEL-LAMVEL)
0029            TAUDT7 = DCOS (PHI-PSI) * (PHIVEL-PSIVEL)
0030            PSIACC = PHIACC*PSIVEL/PHIVEL + PHIVEL*A1*(TAUDT1-TAUDT3*TAU1/
               1          TAU3)/(A3*TAU3)
0031            LAMACC = PHIACC*LAMVEL/PHIVEL + PHIVEL*A1*(TAUDT7-TAUDT3*TAU7/
               1          TAU3)/(A2*TAU3)
0032            GO TO 999
0033         60 LAMVEL = 0D0
0034            LAMACC = 0D0
0035            PSIVEL = 0D0
0036            PSIACC = 0D0
0037         999 RETURN
0038            END
```

c. Augmented Lagrangian Subroutines

The following lists the Augmented Lagrangian subroutines VF01AD, VF01ZD, MC11AD, VA09AD and VE04AD for the convenience of the reader.

```
0001            SUBROUTINE VF01AD(N,M,C,X,EPS,AKMIN,DFN,MAXFN,IPR1,IPR2,IW,MODE)
0002            IMPLICIT REAL*8(A-H,O-Z)
0003            REAL*8 X(1),EPS(1)
0004            COMMON/VF01CD/F,MM,KL,IS,MK,NU
0005            COMMON/VF01DD/S(10)
0006            COMMON/VF01ED/C(1317)
0007            COMMON/VF01FD/GC(2105)
0008            COMMON/VF01GD/T(1317)
0009            COMMON/VF01HD/GP(439)
0010            COMMON/VF01ID/G2P(15)
0011            COMMON/VF01JD/V(439)
0012            COMMON/VF01KD/MM(1317)
0013            COMMON/VF01LD/W(10000)
0014            COMMON/VF01MD/ZZ(873)
0015            COMMON/VF01ND/LT(879)
0016            EXTERNAL VF01ZD
0017          1000 FORMAT(3014)
0018          1001 FORMAT(3015.7)
0019            NU=MAX0(25,N)
0020            IF(N.GT.50)NU=N
0021            IX=N
0022            ICS=M
0023            ICB=M+M
0024            IS=M
0025            IL=IS+M
0026            IP=M
0027            ILT=M
0028            NN=N*(N+1)/2
0029            MM=M
0030            KL=K
0031            MINS=0
0032            AK=1D60
0033            R=100
0034            MK=0
0035            DO 1 I=1,M
0036          1 C(ICS+I)=C(I)
0037            CALL VF01BJ(N,M,X)
              C
0038            DF=DFN
0039            IF(DFN.LT.0D0)DF=DABS(DFN*F)
0040            IF(DFN.EQ.0D0)DF=F
0041            IF(DF.LE.0D0)DF=100
0042            DO 2 I=1,M
0043            CC=C(I)
0044            IF(I.GT.K)CC=DMIN1(CC,0D0)
0045            IF(DABS(CC).GT.C(ICS+I))C(ICS+I)=DABS(CC)
0046          2 CONTINUE
0047            IF(IPR1.EQ.0)GOTO4
0048            IF(MOD(MINS,IPR1).NE.0)GOTO4
0049            PRINT 1002
0050          1002 FORMAT('1ENTRY TO VF01A'///'0CONSTRAINT SCALE PARAMETERS ARE')
0051            PRINT 1001,(C(ICS+I),I=1,M)
0052          4 CONTINUE
0053            IF(MODE.LT.0)GOTO5
0054            DO 3 I=1,M
0055            T(IS+I)=2D0*DF/C(ICS+I)**2
```

```
0056            3 T(I)=0D0
0057            5 CONTINUE
0058              MD=IABS(MODE)
0059            9 CONTINUE
0060              MINS=MINS+1
0061              DO 9 I=1,NN
0062            9 W(I)=G2P(I)
0063              IF(IPR1.EQ.0)GOTO7
0064              IF(MOD(MINS,IPR1).NE.0)GOTO7
0065              PRINT 1003,MINS
0066         1003 FORMAT(////'0OUTER ITERATION NUMBER IS',I3)
0067              PRINT 1004
0068         1004 FORMAT('0X(I)')
0069              PRINT 1001,(X(I),I=1,N)
0070              PRINT 1005
0071         1005 FORMAT('0THETA(I)')
0072              PRINT 1001,(T(I),I=1,M)
0073              PRINT 1006
0074         1006 FORMAT('0S(GMA(I)')
0075              PRINT 1001,(T(IS+I),I=1,M)
0076            7 CONTINUE
0077              CALL VA09AD(VF01ZD,N,X,PHI,GP,W,WW,DF,EPS,MD,MAXFN,IPR2,IEXIT)
0078              CALL VF01BD(N,M,X)
0079              MD=3
0080              AKK=0D0
0081              DO 10 I=1,M
0082              CC=C(I)
0083              IF(I.GT.K.AND.C(I).GE.T(I))CC=DMIN1(CC,0D0)
0084              T(I)=T(I)*T(IS+I)
0085              WW(I)=DABS(CC)/C(IC5+I)
0086              IF(WW(I).GT.AKK)AKK=WW(I)
0087           10 CONTINUE
0088              IF(IPR1.EQ.0)GOTO16
0089              IF(MOD(MINS,IPR1).NE.0)GOTO16
0090              PRINT 1007
0091         1007 FORMAT('0EXIT FROM VA09AD'//'0LAGRANGE MULTIPLIER ESTIMATES')
0092              PRINT 1001,(T(I),I=1,M)
0093              PRINT 1008
0094         1008 FORMAT('0LARGEST SCALED CONSTRAINT VIOLATION'/
                    1' THIS ITERATION, BEST ITERATION')
0095              PRINT 1001,AKK,AK
0096              PRINT 1009
0097         1009 FORMAT('0CONSTRAINT RESIDUALS')
0098              PRINT 1001,(C(I),I=1,M)
0099              PRINT 1010
0100         1010 FORMAT('0SCALED CONSTRAINT VIOLATIONS')
0101              PRINT 1001,(WW(I),I=1,M)
0102           16 CONTINUE
0103              IF(IEXIT.EQ.0.OR.IEXIT.EQ.3)GOTO20
0104              IF(AKK.LE.AKMIN)GOTO20
0105              IF(AKK.GE.AK)GOTO11
0106              DO 15 I=1,NN
0107           15 G2P(I)=W(I)
0108              DO 17 I=1,M
0109              IF(I.GT.K.AND.T(I).EQ.0D0.AND.C(I).GT.0D0)GOTO17
0110              ZZ(IP+I)=-T(IS+I)*C(I)
```

```
0111            IF(I.GT.K.AND.ZZ(IP+I).LT.-T(I))ZZ(IP+I)=-T(I)
0112          17 CONTINUE
0113            IF(NNS.EQ.1)GOTO40
0114            GOTO18
0115          11 CONTINUE
0116            DO 14 I=1,*
0117            IF(WW(I).LE.AK.OR.C(ICB+I).GE.400*WW(I))GOTO14
0118            OS=900*(1+S+I)
0119            T(IS+I)=ADI*T(IS+I)
0120            IF(IPRI.NE.0)PRINT 1011,I,T(IS+I)
0121       1011 FORMAT('0SIGMA(',I3,')  INCREASED TO ',D15.7)
0122            DO 12 J=1,N
0123          12 V(J)=GG(I*(I-1)+NJ+J)
0124            CALL MCIIA(G2P,N,V,OS,V,N,N,OS)
0125          14 CONTINUE
0126          18 CONTINUE
0127            DO 13 I=1,N
0128            IF(DABS(X(I)-G(IX+I)).GT.EPS(I))GOTO21
0129          13 CONTINUE
0130            PRINT 1013
0131       1013 FORMAT('0REQUESTED ACCURACY NOT OBTAINED')
0132          20 CONTINUE
0133            IF(IEXIT.EQ.0)PRINT 2000
0134       2000 FORMAT('0MATRIX SET IN G2P BY USER IS NOT POSITIVE DEFINITE')
0135            IF(IPRI.EQ.0)RETURN
0136            PRINT 1012
0137       1012 FORMAT('0BEST SOLUTION OBTAINED'/'0F,(3(1),I=1,N)')
0138            PRINT 1001,F,(G(I),I=1,N)
0139            RETURN
0140          21 CONTINUE
0141            IF(KK.LT.AK)GOTO40
0142            DO 32 I=1,*
0143          32 V(I)=T(IL+I)
0144            GOTO70
0145          40 CONTINUE
0146            MK=0
0147            KK=0
0148            DO 41 I=1,*
0149            T(IL+I)=T(I)
0150            C(IC3+I)=WW(I)
0151            IF(I.GT.K.AND.T(IL+I).EQ.000.AND.C(I).GT.0.0)GOTO41
0152            KK=KK+1
0153            LT(1LT+KK)=I
0154            GP(KK)=-1030
0155            IF(I.GT.K)GP(KK)=-T(IL+I)
0156            V(KK)=1030
0157            ZZ(KK)=-C(I)
0158          41 CONTINUE
0159            IF(KK.EQ.0)GOTO20
0160            DO 42 I=1,N
0161          42 G(IX+I)=X(I)
0162            KKK=KK*(KK+1)/2
0163            II=MAX0(KK+NN,KK*KK)
0164            IF(II.LE.I4)GOTO50
0165            PRINT 2001,II
0166       2001 FORMAT('0INCREASE STORAGE IN COMMON/VPOILOT/',I7,'ELEMENTS')
```

```
0167            RETURN
0168         50 CONTINUE
0169            II=IW-KKK
0170            DO 53 I=1,KK
0171            LI=LT(ILT+I)
0172            DO 51 JJ=1,N
0173         51 X(JJ)=GG((LI-1)*NU+JJ)
0174            CALL MG11B(W,N,X,X,N)
0175            DO 53 J=1,I
0176            LJ=LT(ILT+J)
0177            Z=0D0
0178            DO 52 JJ=1,N
0179         52 Z=Z+X(JJ)*GG((LJ-1)*NU+JJ)
0180            II=II+1
0181         53 W(II)=Z
0182            JJ=IW-KKK
0183            II=0
0184            DO 56 I=1,KK
0185            DO 55 J=1,I
0186            JJ=JJ+1
0187            II=II+1
0188         55 W(II)=W(JJ)
0189         56 II=II+KK
0190            CALL VE04AD(KK,W,KK,ZZ,GP,V,T,Z,LT,JJ,WW)
0191            IF(IPRL.EQ.0)GOTO59
0192            IF(MOD(MING,IPRL).NE.0)GOTO59
0193            PRINT 1020,KK
0194      1020 FORMAT(14,' ACTIVE CONSTRAINTS, NUMBERED')
0195            PRINT 1000,(LT(ILT+I),I=1,KK)
0196            PRINT 1021
0197      1021 FORMAT('0LAGRANGE MULTIPLIER CORRECTIONS FOR ACTIVE CONSTRAINTS')
0198            PRINT 1001,(T(I),I=1,KK)
0199         59 CONTINUE
0200            DO 60 I=1,4
0201         60 V(I)=T(ILT+I)
0202            DO 62 I=1,KK
0203            LI=LT(ILT+I)
0204            V(LI)=V(LI)+T(I)
0205            Z=400*DABS((T(I)-ZZ(IP+LI))/ZZ(IP+LI))
0206            IF(Z.LE.100)GOTO62
0207            US=(Z-100)*T(IIS+LI)
0208            T(IIS+LI)=Z*T(IIS+LI)
0209            IF(IPRL.NE.0)PRINT 1011,LI,T(IIS+LI)
0210            DO 61 J=1,N
0211         61 GP(J)=GG((LI-1)*NU+J)
0212            CALL MG11A(G2P,N,GP,DS,GP,N,N,DS)
0213         62 CONTINUE
0214            AK=8*KK
0215         70 CONTINUE
0216            DO 71 I=1,4
0217         71 T(I)=V(I)/T(IIS+I)
0218            DO 72 I=1,N
0219         72 X(I)=G(IX+I)
0220            DF=1050
0221            GOTO9
0222            END
```

```
0001            SUBROUTINE VF012D(N,X,PHI,GPHI)
0002            IMPLICIT REAL*8(A-H,O-Z)
0003            REAL*8 X(1),GPHI(1)
0004            COMMON/VF01CD/F,M,K,IS,MK,NU
0005            COMMON/VF01DD/G(10)
0006            COMMON/VF01ED/C(1317)
0007            COMMON/VF01FD/CC(2175)
0008            COMMON/VF01GD/T(131)
0009            IF(MK.EQ.1)CALL VF018D(N,M,X)
0010            MK=1
0011            PHI=.DD0
0012            DO 10 I=1,N
0013         10 GPHI(I)=G(I)
0014            DO 12 I=1,M
0015            CC=C(I)-T(I)
0016            IF(I.GT.K)CC=DMIN1(CC,.DD)
0017            Y=T(IS+I)*CC
0018            IF(Y.EQ.DD)GOTO12
0019            PHI=PHI+Y*CC
0020            DO 11 J=1,N
0021         11 GPHI(J)=GPHI(J)+Y*CC((I-1)*NU+J)
0022         12 CONTINUE
0023            PHI=.5DD*PHI+F
0024            RETURN
0025            END
```

```
0001              SUBROUTINE MCLLAD(A,V,Z,SIG,W,IR,MK,EPS)
0002              IMPLICIT REAL*8(A-H,O-Z)
0003              DIMENSION A(1),Z(1),W(1)
                C   UPDATE FACTORS GIVEN IN A BY   SIG*Z*ZTRANSPOSE
0004              IF(N.GT.1)GOTO1
0005              A(1)=A(1)+SIG *Z(1)**2
0006              IR=1
0007              IF(A(1).GT.0.0)RETURN
0008              A(1)=0.0
0009              IR=0
0010              RETURN
0011            1 CONTINUE
0012              NP=N+1
0013              IF(SIG.GT.0.0)GOTO40
0014              IF(SIG.EQ.0.0.OR.IR.EQ.0)RETURN
0015              TI=1.0/SIG
0016              IJ=1
0017              IF(MK.EQ.0)GOTO10
0018              DO 7 I=1,N
0019              IF(A(IJ).NE.0.0)TI=TI+W(I)**2/A(IJ)
0020            7 IJ=IJ+NP-I
0021              GOTO20
0022           10 CONTINUE
0023              DO 11 I=1,N
0024           11 W(I)=Z(I)
0025              DO 15 I=1,N
0026              IP=I+1
0027              V=W(I)
0028              IF(A(IJ).GT.0.0)GOTO12
0029              W(I)=0.0
0030              IJ=IJ+NP-I
0031              GOTO15
0032           12 CONTINUE
0033              TI=TI+V**2/A(IJ)
0034              IF(I.EQ.N)GOTO14
0035              DO 13 J=IP,N
0036              IJ=IJ+1
0037           13 W(J)=W(J)-V*A(IJ)
0038           14 IJ=IJ+1
0039           15 CONTINUE
0040           20 CONTINUE
0041              IF(IR.LE.0)GOTO21
0042              IF(TI.GT.0.0)GOTO22
0043              IF(MK-1)40,40,23
0044           21 TI=0.0
0045              IR=-IR-1
0046              GOTO23
0047           22 TI=EPS/SIG
0048              IF(EPS.EQ.0.0)IR=IR-1
0049           23 CONTINUE
0050              MM=1
0051              TIM=TI
0052              DO 30 I=1,N
0053              J=NP-I
0054              IJ=IJ-1
0055              IF(A(IJ).NE.0.0)TIM=TI-W(J)**2/A(IJ)
```

```
0056              W(J)=TI
0057           30 TI=TIM
0058              GOTO41
0059           40 CONTINUE
0060              MM=0
0061              TIM=100/SEC
0062           41 CONTINUE
0063              IJ=1
0064              DO 66 I=1,N
0065              IP=I+1
0066              V=Z(I)
0067              IF(A(IJ).GT.0.0001GOTO53
0068              IF(IR.GT.0.OR.SIG.LT.0.0.AND.V.EQ.0.0001GOTO52
0069              IR=1-IR
0070              A(IJ)=V**2/TIM
0071              IF(I.EQ.N)RETURN
0072              DO 51 J=IP,N
0073              IJ=IJ+1
0074           51 A(IJ)=Z(J)/V
0075              RETURN
0076           52 CONTINUE
0077              TI=TIM
0078              IJ=IJ+NP-I
0079              GOTO56
0080           53 CONTINUE
0081              AL=V/A(IJ)
0082              IF(MM)54,54,55
0083           54 TI=TIM+V*AL
0084              GOTO56
0085           55 TI=W(I)
0086           56 CONTINUE
0087              R=TI/TIM
0088              A(IJ)=A(IJ)*R
0089              IF(R.EQ.0.0)GOTO70
0090              IF(I.EQ.N)GOTO70
0091              B=AL/TI
0092              IF(R.GT.4.0)GOTO62
0093              DO 61 J=IP,N
0094              IJ=IJ+1
0095              Z(J)=Z(J)-V*A(IJ)
0096           61 A(IJ)=A(IJ)+B*Z(J)
0097              GOTO64
0098           62 GM=TIM/TI
0099              DO 63 J=IP,N
0100              IJ=IJ+1
0101              Y=A(IJ)
0102              A(IJ)=B*Z(J)+Y*GM
0103           63 Z(J)=Z(J)-V*Y
0104           64 CONTINUE
0105              TIM=TI
0106              IJ=IJ+1
0107           66 CONTINUE
0108           70 CONTINUE
0109              IF(IR.LT.0)IR=-IR
0110              RETURN
                C   FACTORIZE A MATRIX GIVEN IN A
```

```
0111              ENTRY MCLL90(A,N,IR)
0112              IR=N
0113              IF(N.GT.1)GOTO100
0114              IF(A(1).GT.000)RETURN
0115              A(1)=0.00
0116              IR=0
0117              RETURN
0118            100 CONTINUE
0119              NP=N+1
0120              II=1
0121              DO 104 I=2,N
0122              AA=A(II)
0123              NI=II+NP-I
0124              IF(AA.GT.000)GOTO101
0125              A(II)=0.00
0126              IR=IR-1
0127              II=NI+1
0128              GOTO104
0129            101 CONTINUE
0130              IP=II+1
0131              II=NI+1
0132              JK=II
0133              DO 103 IJ=IP,NI
0134              V=A(IJ)/AA
0135              DO 102 IK=IJ,NI
0136              A(JK)=A(JK)-A(IK)*V
0137            102 JK=JK+1
0138            103 A(IJ)=V
0139            104 CONTINUE
0140              IF(A(II).GT.000)RETURN
0141              A(II)=0.00
0142              IR=IR-1
0143              RETURN
                C   MULTIPLY OUT THE FACTORS GIVEN IN A
0144              ENTRY MCLLDO(A,N)
0145              IF(N.EQ.1)RETURN
0146              NP=N+1
0147              II=N*NP/2
0148              DO 202 NIP=2,N
0149              JK=II
0150              NI=II-1
0151              II=II-NIP
0152              AA=A(II)
0153              IP=II+1
0154              IF(AA.GT.000)GOTO203
0155              DO 204 IJ=IP,NI
0156            204 A(IJ)=0.00
0157              GOTO202
0158            203 CONTINUE
0159              DO 201 IJ=IP,NI
0160              V=A(IJ)*AA
0161              DO 200 IK=IJ,NI
0162              A(JK)=A(JK)+A(IK)*V
0163            200 JK=JK+1
0164            201 A(IJ)=V
0165            202 CONTINUE
```

```
0166                RETURN
                C   MULTIPLY A VECTOR Z BY THE FACTORS GIVEN IN A
0167                ENTRY MCLLSD(A,N,Z,W)
0168                IF(N.GT.1)GOTO100
0169                Z(1)=Z(1)*A(1)
0170                W(1)=Z(1)
0171                RETURN
0172              100 CONTINUE
0173                NP=N+1
0174                II=1
0175                N1=N-1
0176                DO 303 I=1,N1
0177                Y=Z(I)
0178                IF(A(II).EQ.0.0D0)GOTO302
0179                IJ=II
0180                IP=I+1
0181                DO 301 J=IP,N
0182                IJ=IJ+1
0183              301 Y=Y+Z(J)*A(IJ)
0184              302 Z(I)=Y*A(II)
0185                W(I)=Z(I)
0186              303 II=II+NP-I
0187                Z(N)=Z(N)*A(II)
0188                W(N)=Z(N)
0189                DO 311 K=1,N1
0190                I=N-K
0191                II=II-NP+I
0192                IF(Z(I).EQ.0.0D0)GOTO311
0193                IP=I+1
0194                IJ=II
0195                Y=Z(I)
0196                DO 310 J=IP,N
0197                IJ=IJ+1
0198              310 Z(J)=Z(J)+A(IJ)*Z(I)
0199              311 CONTINUE
0200                RETURN
                C   MULTIPLY A VECTOR Z BY THE INVERSE OF THE FACTORS GIVEN IN A
0201                ENTRY MCLLSD(A,N,Z,W,IR)
0202                IF(IR.LT.N)RETURN
0203                W(1)=Z(1)
0204                IF(N.GT.1)GOTO400
0205                Z(1)=Z(1)/A(1)
0206                RETURN
0207              400 CONTINUE
0208                DO 402 I=2,N
0209                IJ=I
0210                II=I-1
0211                V=Z(I)
0212                DO 401 J=1,II
0213                V=V-A(IJ)*Z(J)
0214              401 IJ=IJ+N-J
0215                W(I)=V
0216              402 Z(I)=V
0217                Z(N)=Z(N)/A(IJ)
0218                NP=N+1
0219                DO 411 III=2,N
```

```
0220              I=NP-NIP
0221              II=IJ-NIP
0222              V=Z(I)/A(II)
0223              IP=I+1
0224              IJ=II
0225              DO 410 J=IP,N
0226              II=II+1
0227          410 V=V-A(II)*Z(J)
0228          411 Z(I)=V
0229              RETURN
                C   COMPUTE THE INVERSE MATRIX FROM FACTORS GIVEN IN A
0230              ENTRY MCLIFO(A,N,IR)
0231              IF(IR.LT.N)RETURN
0232              A(1)=1DO/A(1)
0233              IF(N.EQ.1)RETURN
0234              NP=N+1
0235              NI=N-1
0236              II=2
0237              DO 511 I=2,N
0238              A(II)=-A(II)
0239              IJ=II+1
0240              IF(I.EQ.N)GOTO502
0241              DO 501 J=I,NI
0242              IK=II
0243              JK=IJ
0244              V=A(IJ)
0245              DO 500 K=1,J
0246              JK=JK+NP-K
0247              V=V+A(IK)*A(JK)
0249          500 IK=IK+1
0249              A(IJ)=-V
0250          501 IJ=IJ+1
0251          502 CONTINUE
0252              A(IJ)=1DO/A(IJ)
0253              II=IJ+1
0254              AA=A(IJ)
0255              IJ=1
0256              IP=I+1
0257              NI=N-I
0258              DO 511 J=2,I
0259              V=A(IJ)*AA
0260              IK=IJ
0261              K=IJ-IP+J
0262              II=IJ-1
0263              NIP=NI+IJ
0264              DO 510 JK=K,II
0265              A(JK)=A(JK)+V*A(IK)
0266          510 IK=IK+NIP-JK
0267              A(IJ)=V
0268          511 IJ=IJ+NP-J
0269              RETURN
0270              END
```

```
0001            SUBROUTINE VA09AD(FUNCT,N,X,F,G,H,W,DFN,EPS,MODE,MAXFN,IPRINT,
               1      IEXIT)
0002            IMPLICIT REAL*8 (A-H,O-Z)
0003            REAL*8 X(1),G(1),H(1),W(1),EPS(1)
0004            IF(IPRINT.NE.0)PRINT 1000
0005          1000 FORMAT('1ENTRY TO VA09AD'/)
0006            NN=N*(N+1)/2
0007            IG=N
0008            IGG=N+N
0009            IS=IGG
0010            IEXIT=0
0011            IR=N
0012            IF(MODE.EQ.3)GOTO15
0013            IF(MODE.EQ.2)GOTO10
0014            IJ=NN+1
0015            DO 5 I=1,N
0016            DO 6 J=1,I
0017            IJ=IJ-1
0018          6 H(IJ)=0.D0
0019          5 H(IJ)=1.D0
0020            GOTO15
0021         10 CONTINUE
0022            CALL MC11BD(H,N,IR)
0023            IF(IR.LT.N)RETURN
0024         15 CONTINUE
0025            Z=F
0026            ITN=0
0027            CALL FUNCT(N,X,F,G)
0028            IFN=1
0029            DF=DFN
0030            IF(DFN.EQ.0.D0)DF=F-Z
0031            IF(DFN.LT.0.D0)DF=DABS(DF*F)
0032            IF(DF.LE.0.D0)DF=1.D0
0033         20 CONTINUE
0034            IF(IPRINT.EQ.0)GOTO21
0035            IF(MOD(ITN,IPRINT).NE.0)GOTO21
0036            PRINT 1001,ITN,IFN
0037          1001 FORMAT(2I5)
0038            PRINT 1002,F
0039          1002 FORMAT(5D24.16)
0040            IF(IPRINT.LT.0)GOTO21
0041            PRINT 1002,(X(I),I=1,N)
0042            PRINT 1002,(G(I),I=1,N)
0043         21 CONTINUE
0044            ITN=ITN+1
0045            DO 22 I=1,N
0046         22 W(IG+I)=G(I)
0047            CALL MC11ED(H,N,G,W,IR)
0048            GS=0.D0
0049            DO 23 I=1,N
0050            W(IS+I)=-G(I)
0051         23 GS=GS-G(I)*W(IG+I)
0052            IEXIT=2
0053            IF(GS.GE.0.D0)GOTO32
0054            GS0=GS
0055            ALPHA=-2.D0*DF/GS
```

```
0056              IF(ALPHA.GT.100)ALPHA=100
0057              DF=F
0058              TOT=0.00
0059           30 CONTINUE
0060              IEXIT=3
0061              IF(IFN.EQ.MAXFN)GOTO92
0062              ICON=0
0063              IEXIT=1
0064              DO 31 I=1,N
0065              Z=ALPHA*W(IS+I)
0066              IF(DABS(Z).GE.EPS(I))ICON=1
0067           31 X(I)=X(I)+Z
0068              CALL FUNCTN(N,X,FY,G)
0069              IFN=IFN+1
0070              GYS=0.00
0071              DO 32 I=1,N
0072           32 GYS=GYS+G(I)*W(IS+I)
0073              IF(FY.GE.F)GOTO40
0074              IF(DABS(GYS/GS0).LE..900)GOTO50
0075              IF(GYS.GT.000)GOTO40
0076              TOT=TOT+ALPHA
0077              Z=10.00
0078              IF(GS.LT.GYS)Z=GYS/(GS-GYS)
0079              IF(Z.GT.10.00)Z=10.00
0080              ALPHA=ALPHA+Z
0081              F=FY
0082              GS=GYS
0083              GOTO30
0084           40 CONTINUE
0085              DO 41 I=1,N
0086           41 X(I)=X(I)-ALPHA*W(IS+I)
0087              IF(ICON.EQ.0)GOTO92
0088              Z=3.00*(F-FY)/ALPHA+GYS+GS
0089              ZZ=DSQRT(Z**2-GS*GYS)
0090              Z=100-(GYS+ZZ-Z)/(200*ZZ+GYS-GS)
0091              ALPHA=ALPHA*Z
0092              GOTO30
0093           50 CONTINUE
0094              ALPHA=TOT+ALPHA
0095              F=FY
0096              IF(ICON.EQ.0)GOTO70
0097              DF=DF-F
0098              DGS=GYS-GS0
0099              DO 51 I=1,N
0100              W(IGS+I)=G(I)
0101           51 G(I)=-W(IG+I)
0102              IF(DGS+ALPHA*GS0.GT.0.00)GOTO60
              C   COMPLEMENTARY DFP FORMULA
0103              SIG=100/GS0
0104              IR=-IR
0105              CALL MCLLA2(N,N,G,SIG,W,IR,1,.000)
0106              DO 52 I=1,N
0107           52 G(I)=W(IGG+I)-W(IG+I)
0108              SIG=100/(ALPHA*DGS)
0109              IR=-IR
0110              CALL MCLLA2(N,N,G,SIG,W,IR,0,.000)
```

```
0111              GOTO70
0112           60 CONTINUE
               C   OFP FORMULA
0113              ZZ=ALPHA/(DGS-ALPHA*SSD)
0114              SIG=-ZZ
0115              CALL MCILAD(H,N,G,SIG,W,IR,1,1D-16)
0116              Z=DGS*ZZ-1.DD
0117              DO 61 I=1,N
0118           61 G(I)=W(IGG+I)+Z*W(IG+I)
0119              SIG=1.DD/(ZZ*DGS**2)
0120              CALL MCILAD(H,N,G,SIG,W,IR,0,DDD)
0121           70 CONTINUE
0122              DO 71 I=1,N
0123           71 G(I)=W(IGG+I)
0124              GOTO20
0125           92 CONTINUE
0126              DO 91 I=1,I
0127           91 G(I)=W(IG+I)
0128           90 CONTINUE
0129              IF(IPRINT.EQ.0)RETURN
0130              PRINT 1001,ITN,IFN,IEXIT
0131              PRINT 1002,F
0132              PRINT 1002,(X(I),I=1,N)
0133              PRINT 1002,(G(I),I=1,N)
0134              RETURN
0135              END
```

```
0001            SUBROUTINE VE04AD(N,A,IA,B,BL,BU,X,O,LT,K,D)
0002            IMPLICIT REAL*8 (A-H,O-Z)
0003            DIMENSION A(IA,1),B(1),BL(1),BU(1),X(1),LT(1),G(1)
0004            IS=N
0005            IAS=N
0006            IV=1
0007            ICAC=N+N
0008            ID=ICAC
0009            DO 9 I=1,N
0010          9 G(I)=-B(I)
0011            DO 10 I=1,N
0012            X(I)=0.00
0013            LT(I)=I
0014            G(ICAC+I)=A(I,I)
0015            IF(O.GE.BL(I).AND.O.LE.BU(I))GOTO10
0016            IF(O.LT.BL(I))X(I)=BL(I)
0017            IF(O.GT.BU(I))X(I)=BU(I)
0018            DO 12 J=1,I
0019         12 G(J)=G(J)+A(J,I)*X(I)
0020            IF(I.EQ.N)GOTO10
0021            II=I+1
0022            DO 11 J=II,N
0023         11 G(J)=G(J)+A(I,J)*X(I)
0024         10 CONTINUE
0025            K=0
0026            K1=1
0027         20 CONTINUE
0028            IOUT=0
0029            DEL=0.00
0030            DO 21 I=K1,N
0031            LI=LT(I)
0032            IF(X(LI).EQ.BL(LI).AND.G(I).GE.0.0D)GOTO21
0033            IF(X(LI).EQ.BU(LI).AND.G(I).LE.0.0D)GOTO21
0034            IF(G(I).LT.0.0D)GOTO22
0035            Z=X(LI)-BL(LI)
0036            J=1
0037            GOTO23
0038         22 CONTINUE
0039            Z=BU(LI)-X(LI)
0040            J=0
0041         23 CONTINUE
0042            IF(G(ICAC+I).LE.0.0D)GOTO24
0043            BETA=DABS(G(I))/G(ICAC+I)
0044            IF(BETA.GE.Z)GOTO24
0045            Z=BETA
0046            D=.500*Z*DABS(G(I))
0047            J=-1
0048            GOTO26
0049         24 CONTINUE
0050            D=Z*(DABS(G(I))-.500*Z*G(ICAC+I))
0051         26 CONTINUE
0052            IF(D.LT.DEL)GOTO21
0053            DEL=D
0054            ALPHA=Z
0055            IOUT=I
0056            IIN=I
```

```
0057              IF(J.LT.0)IIN=0
0058              LB=J
0059           21 CONTINUE
0060              IF(IOUT.NE.0)GOTO29
0061           27 CONTINUE
0062              Q=0.00
0063              DO 28 I=1,4
0064              LI=LT(I)
0065           28 Q=Q+X(LI)*(G(I)-B(LI))
0066              Q=.500*Q
0067              RETURN
0068           29 CONTINUE
0069              SIG=1.00
0070              IF(G(IOUT).GT.0D0)SIG=-1.00
0071              LIOUT=LT(IOUT)
0072              LIIN=LIOUT
0073           25 CONTINUE
0074              SAS=G(ICAC+IOUT)
0075              IF(K.EQ.0)GOTO31
0076              DO 30 I=1,K
0077           30 G(IS+I)=G(ID+I)*A(IOUT,I)
0078           31 CONTINUE
0079              DO 37 I=K1,N
0080              LI=LT(I)
0081              IF(LI-LIOUT)32,37,33
0082           32 Z=A(LI,LIOUT)
0083              GOTO34
0084           33 Z=A(LIOUT,LI)
0085           34 CONTINUE
0086              IF(K.EQ.0)GOTO36
0087              DO 35 J=1,K
0088           35 Z=Z-A(I,J)*G(IS+J)
0089           36 G(IS+I)=Z
0090           37 CONTINUE
0091              G(IS+IOUT)=SAS
0092              IF(K.EQ.0)GOTO42
0093              G(IS+K)=-A(IOUT,K)
0094              IF(K.EQ.1)GOTO42
0095              I=K
0096              DO 41 II=2,K
0097              I=I-1
0098              Z=-A(IOUT,I)
0099              I1=I+1
0100              DO 40 J=I1,K
0101           40 Z=Z-G(IS+J)*A(J,I)
0102           41 G(IS+I)=Z
0103           42 CONTINUE
0104              IF(SIG.LT.1.001)GOTO51
0105              DO 50 I=1,4
0106           50 G(IS+I)=-G(IS+I)
0107           51 CONTINUE
0108              IF(K.EQ.0)GOTO62
0109              DO 61 I=1,K
0110              IF(G(IS+I).EQ.0001)GOTO61
0111              LI=LT(I)
0112              J=I
```

```
0113              Z=BL(LI)-X(LI)
0114              IF(G(IS+I).LT.0.0)GOTO60
0115              J=0
0116              Z=BU(LI)-X(LI)
0117           60 CONTINUE
0118              Z=Z/G(IS+I)
0119              IF(Z.GE.ALPHA)GOTO61
0120              ALPHA=Z
0121              LO=J
0122              IIN=I
0123              LIIN=LI
0124           61 CONTINUE
0125           62 CONTINUE
0126              X(LI)OUT)=X(LI)OUT)+SIG*ALPHA
0127              IF(K.EQ.0)GOTO71
0128              DO 70 I=1,K
0129              LI=LT(I)
0130           70 X(LI)=X(LI)+ALPHA*G(IS+I)
0131           71 CONTINUE
0132              DO 72 I=K1,N
0133           72 G(I)=G(I)+ALPHA*G(IAS+I)
0134              IF(IIN.EQ.0)GOTO90
0135              X(LIIN)=BL(LIIN)
0136              IF(LO.EQ.0)X(LIIN)=BU(LIIN)
0137              IF(IIN.EQ.1)OUT)GOTO20
0138              K2=K-1
0139              SG=G(ID+IIN)
0140              II=IIN+1
0141              DO 80 I=II,N
0142           80 G(IV+I)=A(I,IIN)
0143              IF(IIN.EQ.K)GOTO86
0144              I2=IIN+2
0145              SO=DD/SG
0146              DO 85 I=IIN,K2
0147              V=G(IV+I)
0148              VD=V/G(ID+I)
0149              S1=SO+V*VD
0150              R=S1/SO
0151              G(ID+I)=G(ID+I)*R
0152              BETA=VD/S1
0153              IF(R.GT.4.0)GOTO841
0154              DO 81 J=I2,N
0155           81 G(IV+J)=G(IV+J)-V*A(J,I)
0156              IF(I.GT.K2)GOTO83
0157              DO 82 J=I1,K2
0158              J1=J+1
0159           82 A(J,I)=A(J1,I)+BETA*G(IV+J)
0160           83 CONTINUE
0161              A(K,I)=BETA
0162              DO 84 J=K1,N
0163           84 A(J,I)=A(J,I)+BETA*G(IV+J)
0164              GOTO949
0165          841 CONTINUE
0166              IF(I.GT.K2)GOTO843
0167              DO 842 J=I1,K2
0168              J1=J+1
```

```
0169          842 A(J,I)=BETA*G(IV+JL)+A(JL,II)/R
0170          843 CONTINUE
0171              A(K,I)=BETA
0172              DO 844 J=K1,N
0173          844 A(J,I)=BETA*G(IV+J)+A(J,II)/R
0174              DO 845 J=I2,N
0175          845 G(IV+J)=G(IV+J)-V*A(J,II)
0176          847 CONTINUE
0177              LT(I)=LT(II)
0178              SO=S1
0179              II=I2
0180           85 I2=I2+1
0181              SG=1.00/S1
0182              LT(K)=LIN
0183              G(IC+K)=SG
0184              IF(IIN.EQ.1)GOTO851
0185              II=IIN-1
0186              DO 852 I=1,II
0187              Z=A(IIN,I)
0188              DO 853 J=IIN,K2
0189          853 A(J,I)=A(J+1,I)
0190          852 A(K,I)=Z
0191          851 CONTINUE
0192           36 CONTINUE
0193              DO 87 I=K1,N
0194           87 G(IC+I)=G(IC+I)+SG*G(IV+I)**2
0195              K1=K
0196              K=K2
0197              IIN=0
0198              ALPHA=1075
0199              SAS=G(IC+IOUT)
0200              IF(SAS.GT.0.00)ALPHA=DABS(G(IOUT))/SAS
0201              IF(G(IOUT).LT.0.001)GOTO898
0202              J=1
0203              Z=X(LIOUT)-BL(LIOUT)
0204              GOTO899
0205          898 CONTINUE
0206              J=0
0207              Z=BU(LIOUT)-X(LIOUT)
0208          899 CONTINUE
0209              IF(Z.GE.ALPHA)GOTO25
0210              ALPHA=Z
0211              L9=J
0212              IIN=IOUT
0213              LIN=LIOUT
0214              GOTO25
0215           90 CONTINUE
0216              K2=K1+1
0217              IF(SIG.EQ.1.00)GOTO91
0218              DO 901 I=K1,N
0219          901 G(IAS+I)=-G(IAS+I)
0220           91 CONTINUE
0221              IF(IOUT.EQ.K1)GOTO97
0222              LT(IOUT)=LT(K1)
0223              LT(K1)=LIOUT
0224              G(IAS+IOUT)=G(IAS+K1)
```

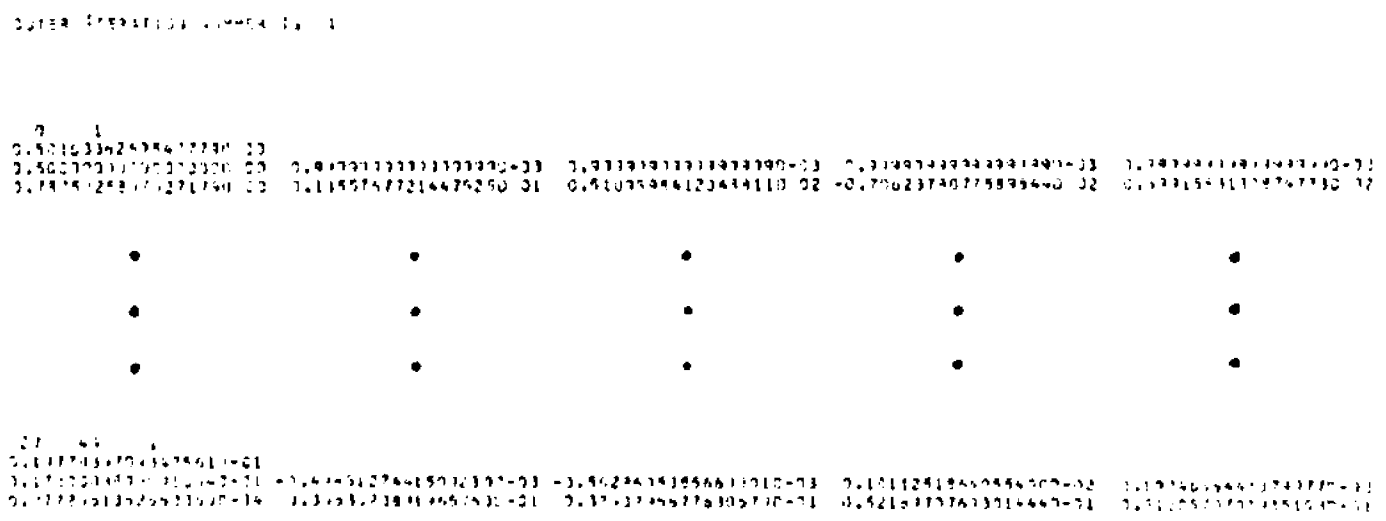
```
0225            G(ICAC+IOUT)=G(ICAC+K1)
0226            G(ICAC+K1)=SAS
0227            G(IOUT)=G(K1)
0228            IF(K.EQ.0)GOTO93
0229            DO 92 I=1,K
0230            Z=A(K1,I)
0231            A(K1,I)=A(IOUT,I)
0232         92 A(IOUT,I)=Z
0233         93 CONTINUE
0234            IF(K2.EQ.IOUT)GOTO95
0235            II=IOUT-1
0236            DO 94 I=K2,II
0237         94 A(IOUT,I)=A(I,K1)
0238         95 CONTINUE
0239            IF(IOUT.EQ.N)GOTO97
0240            II=IOUT+1
0241            DO 96 I=II,N
0242         96 A(I,IOUT)=A(I,K1)
0243         97 CONTINUE
0244            G(K1)=0.00
0245            K=K1
0246            IF(K.EQ.N)GOTO27
0247            DO 98 I=K2,N
0248            Z=G(IAS+I)/SAS
0249            A(I,K1)=Z
0250         98 G(ICAC+I)=G(ICAC+I)-Z*G(IAS+I)
0251            K1=K2
0252            GOTO20
C
0253            ENTRY MS043D(N,A,IA,J,K)
0254            IF(K.EQ.0)RETURN
0255            ID=N+N
0256            G(N+1)=100/G(ID+1)
0257            IF(K.EQ.1)RETURN
0258            N1=K-1
0259            DO 111 I=1,N1
0260            II=I+1
0261            A(II,I)=-A(II,I)
0262            IF(I.EQ.N1)GOTO102
0263            II=I+2
0264            DO 101 J=II,K
0265            Z=A(J,I)
0266            J1=J-1
0267            DO 100 L=II,J1
0268        100 Z=Z+A(J,L)*A(L,I)
0269        101 A(J,I)=-Z
0270        102 CONTINUE
0271            AA=100/G(I)*(1)
0272            G(N+I)=AA
0273            DO 111 J=1,I
0274            Z=A(I,J)*AA
0275            G(N+J)=G(N+J)+Z*A(I,J)
0276            IF(I.EQ.1)GOTO111
0277            J1=J+1
0278            DO 110 L=J1,I
0279        110 A(L,J)=A(L,J)+A(I,L)*Z
```

```
0290          111 A(I1,J1)=Z
0291              RETURN
0292              END
```

d. Essential Elements Of Program Output

The output of the program contains a series of outer iterations each of which consists of a number of inner ones. Each outer iteration is made for a chosen set of the parameters σ and θ, while the variables x_i are changed in each inner iteration. The following shows composites of the essential portions of the first and fourth (last) outer iteration and gives all necessary explanations. The listings of the σ's, the θ's, the Lagrange multipliers, etc. are omitted.

i) First Outer Iteration



OUTER ITERATION NUMBER I, 1

P	L			
0.5016334254564773E 00				
0.5021793111703373E 00	0.0000000000000000E+00	0.0000000000000000E+00	0.0000000000000000E+00	0.0000000000000000E+00
0.7875429870327174E 00	0.1155757721447925E 01	0.6107599461234441E 02	-0.7902374077589944E 02	0.1331554131787747E 02

• • • • •
• • • • •
• • • • •

P	L			
0.4377913470467561E+01				
0.1700214473E+01	0.4403612764E+00	-0.5625631956611010E-03	0.1011251954555420E-02	0.1976619641743777E-01
0.4777301337E+01	0.3438138114E+01	0.3743279577E+00	0.5215377613514467E-01	0.1105137715516147E-01

The above shows the first and last inner iteration associated with outer iteration number 1. The following explains the format of the inner iterations.

First line: The first number stands for the total number of previously completed inner iterations. The second number represents the number of times the subroutine VF01ZD has been called. (Only the last iteration has a third number in this line. It represents a code number for stopping the iteration.)

Second line: This number represents the current value of the applicable composite function Φ according to equation (C.1).

Third line: This line gives the current values of the variables z, P_1, Q_1, P_3 and Q_3 (in the given order). In the first iteration, the initial values are reproduced. (It is to be recalled that the solution of the problem becomes better as the value of z decreases.)

Fourth line: These numbers represent the derivatives of the composite function Φ with respect to the variables in the order of their subscripts.

ii) Fourth (Last) Outer Iteration

The last group of inner iterations in the above contains the results of the minimax optimization, i.e.

$$z = .0235 \quad lb^2$$

$$P_1 = -50.85 \times 10^{-5} \quad lb\text{-}sec^2$$

$$Q_1 = -57.56 \times 10^{-5} \quad lb\text{-}sec^2$$

$$P_3 = 181.98 \times 10^{-5} \quad lb\text{-}sec^2$$

$$Q_3 = 21.95 \times 10^{-5} \quad lb\text{-}sec^2$$

The computation was terminated at this point since no further significant decrease in the value of z was possible, i.e. the prescribed value of AKMIN was reached.

APPENDIX D

APPLICATION OF AUGMENTED LAGRANGIAN OPTIMIZATION PROGRAM TO LEAST SQUARES BALANCING OPTIMIZATION

The present appendix shows how the Augmented Lagrangian program is adapted to the least squares solution of the overdetermined system of balancing equations[1]. (For formulation, see equation (3.4).) Section 1 first outlines how the various parameters must be defined in the main program according to Section 2a of Appendix C. Subsequently, a listing of the actual routine is shown. Section 2 gives the applicable subroutine VF01BD and discusses it. Section 3 contains the important parts of the output of the program and interprets the results.

1. Main Program

The main program, which is listed at the end of this section, begins with comment cards which are designed to serve as additional explanations. These are followed by the COMMON statements which define the array size for the constraints C and their derivatives GC. The following variables are then defined:

---

[1] Since the Harwell program requires constraints and the least squares optimization represents an unconstrained problem, it becomes necessary to introduce a dummy constraint. This constraint is chosen in such a way that it will never be violated.

N = 4; stands for the four variables P_1, Q_1, P_3 and Q_3.

M = 1; refers to the single dummy constraint required for the program to operate. (See Section 2 below.)

K = 0, since there are no equality constraints.

X(1) to X(4), the starting point estimates of the variables, are made $1x10^{-3}$ since previous experience has shown these values to be of this order of magnitude. (While accurate estimates are not required, they decrease the number of iterations.)

With AKMIN chosen arbitrarily to be $1x10^{-3}$, EPS(1) to EPS(4) result from X(1) to X(4).

DFN is estimated to be 0.4.

MAXFN = 500 (See Section 2a of Appendix C.)

IPR1 = 1, to print every outer iteration.

IPR2 = 1, to print every inner iteration.

IW = 2500, since the restrictions that $N \leq 25$ and $M \leq 50$ are met.

MODE = 1 (See Section 2a of Appendix C.)

The scale factor C(1) = 10.001 represents the magnitude of the constraint when evaluated at the starting point X(1) = $1x10^{-3}$. G(1,1) to G(4,1) represent the derivatives of the single linear dummy constraint with respect to the four variables X(1) to X(4).

MAIN

```
C     THIS PROGRAM WILL CHOOSE THE VALUES OF THE GENERAL MASS DISTANCE
C     PRODUCTS P1,J1,P3 AND J3 WHICH WILL RESULT IN THE BEST LEAST SQUARES
C     FIT OF THE COMPUTED SHAKING FORCE CURVES IN THE X AND Y DIRECTIONS
C     TO A K40&1 EXPERIMENTAL CURVE.  THIS WILL BE ACCOMPLISHED BY THE
C     USE OF THE AUGMENTED LAGRANGIAN OPTIMIZATION ROUTINE.
C     THE VARIABLES FOR THIS PROBLEM ARE
C          X(1) = P1
C          X(2) = J1
C          X(3) = P3
C          X(4) = J3
C
C     THE CONSTRAINT IS
C            C(1) = X(1) + 1.0      (DUMMY CONSTRAINT)
C
0001            IMPLICIT REAL*8(A-H,O-Z)
0002            COMMON/VF01ED/C(150)
0003            COMMON/VF01FD/CC(25,50)
0004            REAL*8 X(4),EPS(4)
0005            N = 4
0006            M = 1
0007            K=0
0008            PI=3.14159300
C     THE STARTING POINT IS ESTIMATED TO BE
0009            X(1)=1.00-3
0010            X(2)=1.00-3
0011            X(3)=1.00-3
0012            X(4)=1.00-3
C
0013            AKMIN=10-3
0014            EPS(1)=1.00-5
0015            EPS(2)=1.00-6
0016            EPS(3)=1.00-6
0017            EPS(4)=1.00-6
C
0018            DFN=.400
0019            MAXFN=500
0020            IPR1=1
0021            IPR2=1
0022            IW=2500
0023            MODE=1
C     THE CONSTRAINT SCALE FACTOR IS
0024            C(1)=10.00100
C     THE DERIVATIVES OF THE LINEAR CONSTRAINT WITH RESPECT
C     TO THE VARIABLES ARE
0025            CC(1,1) = 100
0026            CC(2,1) = 000
0027            CC(3,1) = 000
0028            CC(4,1) = 000
C
0029            CALL VF01AD(N,M,K,X,EPS,AKMIN,DFN,MAXFN,IPR1,IPR2,IW,MODE)
0030            STOP
0031            END
```

2. Subroutine VF01BD (See program at the end of this section.)

After allocating storage for the various subscripted variables, the subroutine VF01BD starts with the COMMON statements indicated in Section 2b of Appendix C. The subsequent steps are:

a. A1 to A4 represent the link dimensions of links 1 to 4.

b. DELPHI = 5, stands for the 5 degree increments of the input angle ϕ_1.

c. P_1, Q_1, P_3 and Q_3 are variables x_1, x_2, x_3 and x_4.

d. FEX(1) to FEY(72) are the 144 experimentally obtained shaking force components at 5 degree intervals from 0 to 355 degrees. (See Table 3.3)

e. The dummy constraint C(1) = X(1) + 10 was chosen such that it never will become negative for the anticipated range of the variable X(1). (See equation (C.3).)

f. SUMF, SUMX1 to SUMX4 represent the initialization of the objective function F and its derivatives.

g. A DO loop, encompassing the 72 positions of the mechanism, is set up to compute the objective function as well as its derivatives. The objective function becomes, according to equation (3.4):

$$F = \sum_{i=0^\circ}^{355^\circ} \left[(F_{CX_i} - F_{EX_i})^2 + (F_{CY_i} - F_{EY_i})^2 \right] \qquad (D.1)$$

The derivatives of F with respect to the four variables are given by:

$$\frac{\partial F}{\partial P_1} = \sum_{i=0^\circ}^{355^\circ} \left[AA \times PFCX1 + BB \times PFCY1 \right] \qquad (D.2)$$

$$\frac{\partial F}{\partial Q_1} = \sum_{i=0^\circ}^{355^\circ} \left[AA \times PFCX2 + BB \times PFCY2 \right] \qquad (D.3)$$

$$\frac{\partial F}{\partial P_3} = \sum_{i=0^\circ}^{355^\circ} \left[AA \times PFCX3 + BB \times PFCY3 \right] \qquad (D.4)$$

$$\frac{\partial F}{\partial Q_3} = \sum_{i=0^\circ}^{355^\circ} \left[AA \times PFCX4 + BB \times PFCY4 \right] \qquad (D.5)$$

where

AA = 2(FCX(I) - FEX(I))

BB = 2(FCY(I) - FEY(I))

$$PFCX1 = m_1(\dot{\phi}_{1i}^2\cos\phi_{1i} + \ddot{\phi}_{1i}\sin\phi_{1i})$$

$$PFCY1 = m_1(\dot{\phi}_{1i}^2\sin\phi_{1i} - \ddot{\phi}_{1i}\cos\phi_{1i})$$

$$PFCX2 = -m_1(\dot{\phi}_{1i}^2\sin\phi_{1i} - \ddot{\phi}_{1i}\cos\phi_{1i})$$

$$PFCY2 = m_1(\dot{\phi}_{1i}^2\cos\phi_{1i} + \ddot{\phi}_{1i}\sin\phi_{1i})$$

$$PFCX3 = m_3(\dot{\phi}_{3i}^2\cos\phi_{3i} + \ddot{\phi}_{3i}\sin\phi_{3i})$$

$$PFCY3 = -m_3(-\dot{\phi}_{3i}^2\sin\phi_{3i} + \ddot{\phi}_{3i}\cos\phi_{3i})$$

$$PFCX4 = m_3(-\dot{\phi}_{3i}^2\sin\phi_{3i} + \ddot{\phi}_{3i}\cos\phi_{3i})$$

$$PFCY4 = m_3(\dot{\phi}_{3i}^2\cos\phi_{3i} + \ddot{\phi}_{3i}\sin\phi_{3i})$$

All force and motion computations are identical to those in Section 3b of Appendix C.

```
0001            SUBROUTINE VFOLDIN,M,X)
0002            IMPLICIT REAL*8(A-H,O-Z)
0003            REAL*8 X(4)
0004            INTEGER PHID, PHI01, DELPHI
0005            REAL*8 KAY1(360),KAY3(360)
0006            DIMENSION GM41(360),GM43(360),FCX(360),FCY(360)
0007            DIMENSION VAR1(360),VAR2(360),VAR3(360),VAR4(360)
0008            DIMENSION VAR5(360),VAR6(360),VAR7(360),VAR8(360)
0009            DIMENSION PFCX1(360),PFCX2(360),PFCX3(360),PFCY4(360)
0010            DIMENSION PFCY1(360),PFCY2(360),PFCY3(360),PFCY4(360)
0011            DIMENSION PS1(360),PS(360)
0012            DOUBLE PRECISION DCOS,DSIN,DSQRT,DATAN2
0013            DOUBLE PRECISION FEX(72),FEY(72)
0014            COMMON/VFOICO/F
0015            COMMON//FOLDO/G(10)
0016            COMMON/VFOLEO/C(150)
0017            COMMON/VFOIFO/GC(25,50)
0018            PI=3.14159200
                C     THE LINK DIMENSIONS ARE
0019            A1=200
0020            A2=3.900
0021            A3=300
0022            A4=9.00+00
                C
0023            DELPHI = 5
                C   DEFINE  VARIABLES
0024            P1 = X(1)
0025            Q1 = X(2)
0026            P3 = X(3)
0027            Q3 = X(4)
                C   THE EXPERIMENTAL SHAKING FORCE COMPONENTS, DETERMINED
                C   EVERY 5 DEGREES, ARE GIVEN AS FOLLOWS, WHERE THE
                C   FIRST VALUE CORRESPONDS TO THE READING AT 0 DEGREES
                C   AND THE SEVENTY SECOND VALUE TO THE READING AT 355 DEGREES.
0028            FEX(1)  =  .479
0029            FEX(2)  =  .480
0030            FEX(3)  =  .440
0031            FEX(4)  =  .384
0032            FEX(5)  =  .258
0033            FEX(6)  =  .202
0034            FEX(7)  =  .105
0035            FEX(8)  =  .060
0036            FEX(9)  =  .030
0037            FEX(10) =  .009
0038            FEX(11) = -.003
0039            FEX(12) = -.015
0040            FEX(13) = -.026
0041            FEX(14) = -.029
0042            FEX(15) = -.031
0043            FEX(16) = -.034
0044            FEX(17) = -.035
0045            FEX(18) = -.034
0046            FEX(19) = -.031
0047            FEX(20) = -.029
0048            FEX(21) = -.028
0049            FEX(22) = -.027
```

```
0050            FEX(23) = -.024
0051            FEX(24) = -.023
0052            FEX(25) = -.020
0053            FEX(26) = -.015
0054            FEX(27) = -.012
0055            FEX(28) = -.012
0056            FEX(29) = -.015
0057            FEX(30) = -.017
0058            FEX(31) = -.019
0059            FEX(32) = -.021
0060            FEX(33) = -.024
0061            FEX(34) = -.027
0062            FEX(35) = -.030
0063            FEX(36) = -.031
0064            FEX(37) = -.032
0065            FEX(38) = -.033
0066            FEX(39) = -.037
0067            FEX(40) = -.038
0068            FEX(41) = -.038
0069            FEX(42) = -.038
0070            FEX(43) = -.039
0071            FEX(44) = -.041
0072            FEX(45) = -.046
0073            FEX(46) = -.051
0074            FEX(47) = -.053
0075            FEX(48) = -.056
0076            FEX(49) = -.061
0077            FEX(50) = -.063
0078            FEX(51) = -.072
0079            FEX(52) = -.077
0080            FEX(53) = -.084
0081            FEX(54) = -.091
0082            FEX(55) = -.096
0083            FEX(56) = -.101
0084            FEX(57) = -.113
0085            FEX(58) = -.118
0086            FEX(59) = -.125
0087            FEX(60) = -.134
0088            FEX(61) = -.143
0089            FEX(62) = -.147
0090            FEX(63) = -.150
0091            FEX(64) = -.144
0092            FEX(65) = -.123
0093            FEX(66) = -.110
0094            FEX(67) = -.081
0095            FEX(68) = -.032
0096            FEX(69) =  .034
0097            FEX(70) =  .122
0098            FEX(71) =  .259
0099            FEX(72) =  .356
0100            FEY(1)  = -.570
0101            FEY(2)  = -.590
0102            FEY(3)  = -.551
0103            FEY(4)  = -.468
0104            FEY(5)  = -.374
0105            FEY(6)  = -.282
```

```
      FEY(7)  = -.219
      FEY(8)  = -.173
      FEY(9)  = -.120
      FEY(10) = -.087
      FEY(11) = -.065
      FEY(12) = -.054
      FEY(13) = -.037
      FEY(14) = -.029
      FEY(15) = -.024
      FEY(16) = -.018
      FEY(17) = -.012
      FEY(18) = -.009
      FEY(19) = -.006
      FEY(20) = 0.0
      FEY(21) =  .006
      FEY(22) =  .009
      FEY(23) =  .015
      FEY(24) =  .023
      FEY(25) =  .026
      FEY(26) =  .030
      FEY(27) =  .037
      FEY(28) =  .049
      FEY(29) =  .059
      FEY(30) =  .065
      FEY(31) =  .075
      FEY(32) =  .083
      FEY(33) =  .093
      FEY(34) =  .097
      FEY(35) =  .115
      FEY(36) =  .122
      FEY(37) =  .129
      FEY(38) =  .132
      FEY(39) =  .141
      FEY(40) =  .144
      FEY(41) =  .150
      FEY(42) =  .153
      FEY(43) =  .155
      FEY(44) =  .156
      FEY(45) =  .158
      FEY(46) =  .159
      FEY(47) =  .160
      FEY(48) =  .160
      FEY(49) =  .161
      FEY(50) =  .161
      FEY(51) =  .161
      FEY(52) =  .160
      FEY(53) =  .160
      FEY(54) =  .159
      FEY(55) =  .158
      FEY(56) =  .156
      FEY(57) =  .155
      FEY(58) =  .153
      FEY(59) =  .150
      FEY(60) =  .146
      FEY(61) =  .144
      FEY(62) =  .134
```

```
0162              FEY(63) =  .127
0163              FEY(64) =  .116
0164              FEY(65) =  .096
0165              FEY(66) =  .074
0166              FEY(67) =  .044
0167              FEY(68) = -.023
0168              FEY(69) = -.111
0169              FEY(70) = -.206
0170              FEY(71) = -.345
0171              FEY(72) = -.458
                C
                C     THE CONSTRAINT IS
0172              C(I) = X(I) + 1000
                C
0173              Z=PI/1.0D2
                C
0174              SUMF  = 000
0175              SUMX1 = 000
0176              SUMX2 = 000
0177              SUMX3 = 000
0178              SUMX4 = 000
0179              DO 120 PHID1=1.356,DELPHI
0180              PHID=PHID1 - 1
0181              PHI=Z * PHID
0182              I = PHID/DELPHI + 1
                C   THE ANGULAR VELOCITIES AND ACCELERATIONS ARE DETERMINED
                C   IN SUBROUTINE FOURIE
0183              CALL FOURIE (I,PHIVEL,PHIACC)
                C   THE KINEMATICS OF THE LINKAGE ARE DETERMINED IN
                C   SUBROUTINE MOTION
0184              CALL MOTION (PHI,A1,A2,A3,A4,PSI,PHIVEL,PHIACC,
                1              PSIV(I),PSIA(I))
0185              RAY1(I) = DSQRT(PHIVEL**4 + PHIACC**2)
0186              RAY3(I) = DSQRT(PSIV(I)**4 + PSIA(I)**2)
0187              GMA1(I) = PHI - DATAN2(PHIACC,PHIVEL**2)
0188              GMA3(I) = PSI - DATAN2(PSIV(I),PSIA(I)**2)
0189              VAR1(I) = RAY1(I)*DCOS(GMA1(I))
0190              VAR2(I) = RAY1(I)*DSIN(GMA1(I))
0191              VAR3(I) = RAY3(I)*DCOS(GMA3(I))
0192              VAR4(I) = RAY3(I)*DSIN(GMA3(I))
0193              VAR5(I) = RAY1(I)*DSIN(GMA1(I))
0194              VAR6(I) = RAY1(I)*DCOS(GMA1(I))
0195              VAR7(I) = RAY3(I)*DSIN(GMA3(I))
0196              VAR8(I) = RAY3(I)*DCOS(GMA3(I))
                C   THE COMPUTED SHAKING FORCES ARE
0197              FCX(I) = P1*VAR1(I)-Q1*VAR2(I)+P3*VAR3(I)-Q3*VAR4(I)
0198              FCY(I) = P1*VAR5(I)+Q1*VAR6(I)+P3*VAR7(I)+Q3*VAR8(I)
                C
0199              AA = 2D0*(FCX(I) - FEX(I))
0200              BB = 2D0*(FCY(I) - FEY(I))
                C
0201              SP = DSIN(PHI)
0202              CP = DCOS(PHI)
0203              SS = DSIN(PSI)
0204              CS = DCOS(PSI)
0205              PHIV2 = PHIVEL**2
```

```
0206              PSIV2 = (PSIV(I))**2
                C
0207              PFCX1(I) = M1*(PHIV2*CP + PHIACC*SP)
0208              PFCY1(I) = M1*(PHIV2*SP - PHIACC*CP)
0209              PFCX2(I) = -M1*(PHIV2*SP - PHIACC*CP)
0210              PFCY2(I) = M1*(PHIV2*CP + PHIACC*SP)
0211              PFCX3(I) = M3*(PSIV2*CS + PSIA(I)*SS)
0212              PFCY3(I) = -M3*(-PSIV2*SS + PSIA(I)*CS)
0213              PFCX4(I) = M3*(-PSIV2*SS + PSIA(I)*CS)
0214              PFCY4(I) = M3*(PSIV2*CS + PSIA(I)*SS)
                C
0215              SUMF = SUMF + (FCX(I) - FCX(I))**2 + (FCY(I) - FEY(I))**2
0216              SUMX1 = SUMX1 + AA*PFCX1(I) + BB*PFCY1(I)
0217              SUMX2 = SUMX2 + AA*PFCX2(I) + BB*PFCY2(I)
0218              SUMX3 = SUMX3 + AA*PFCX3(I) + BB*PFCY3(I)
0219              SUMX4 = SUMX4 + AA*PFCX4(I) + BB*PFCY4(I)
0220            120 CONTINUE
                C   THE SUM OF THE SQUARES IS
0221              F = SUMF
                C   THE DERIVATIVES OF THE SUM OF THE SQUARES WITH RESPECT
                C   TO THE VARIABLES ARE
0222              G(1) = SUMX1
0223              G(2) = SUMX2
0224              G(3) = SUMX3
0225              G(4) = SUMX4
                C
0226              RETURN
0227              END
```

3. Essential Elements Of Program Output

The output format of the least squares optimization shown below is similar to that given in Section 3d of Appendix C. For the present optimization, only one outer iteration was required.

```
 OUTER ITERATION NUMBER IS  1

   0    1
-3.1288e+04 2.8106e+02
 0.0000000000000000D-03  0.0000000000000000D-03  0.0000000000000000D-03  0.0000000000000000D-03
-0.7929187961e9849468D-13 -0.1022428960347808D-14 -3.3395923068153384D-11 -0.8194315087153593D-13
 .          .          .          .
 .          .          .          .
 .          .          .          .
 .          .          .          .
 19  179    1
 1.1107902323715e9860D-31
-3.5629447e4944e54730D-03 -7.5181319774991e40D-03  0.1944212873367189D-02  0.3412709444e506190D-35
 0.1020537e6236242 3D-11  2.1333492270321113D-11  0.7756462763592470D-38  0.1336e7e3617e121240D-11
```

Again, a set of four lines represent an inner iteration where the first number in the first line stands for the total number of iterations. The second number in this line stands for the number of times the subroutine VF01ZD has been called upon.

The number in the second line represents the current value of the composite function Φ according to equation (C.1) of Appendix C.

The third line gives the current values of P_1, Q_1, P_3 and Q_3. In the first inner iteration the initial values are reproduced.

The fourth line shows the derivatives of the composite function with respect to the four variables.

The last group of inner iterations in the above furnishes the results of the optimization, i.e.

$$\begin{aligned} P_1 &= -56.25 \times 10^{-5} \quad \text{lb-sec}^2 \\ Q_1 &= -51.81 \times 10^{-5} \quad \text{lb-sec}^2 \\ P_3 &= 158.42 \times 10^{-5} \quad \text{lb-sec}^2 \\ Q_3 &= \quad 0.36 \times 10^{-5} \quad \text{lb-sec}^2 \end{aligned} \tag{D.6}$$

The computation was terminated at this point since the value for AKMIN of 1×10^{-3} was reached and, with that, no significant further reduction in the composite function could be obtained.

APPENDIX E

REVIEW OF A CHEBYSHEV-TYPE SINGLE COUNTERWEIGHT SHAKING FORCE OPTIMIZATION OF CONSTANT SPEED PLANAR LINKAGES

This appendix briefly reviews a Chebyshev-type minimization of the maximum shaking force of a constant input speed mechanism which is accomplished with the help of a single counterweight attached to the input link. This approach was first described by Ya. L. Gheronimus [25].

Figure E.1 shows the shaking force hodograph of such a mechanism as it appears when it is rotated by the negative of the angle corresponding to the instantaneous position of the input link.

If vector $\overline{OD}$ represents what is before the above rotation a constant magnitude shaking force component which revolves with the constant angular velocity of the input link, it must be stationary in this figure. Vector $\overline{DE}$, which is due to a component of the shaking force which originally is of variable magnitude and rotates with an angular velocity different from that of the input link, actually traces out the hodograph.

Assume now that a counterweight is attached to the input link which produces a force equal in magnitude but

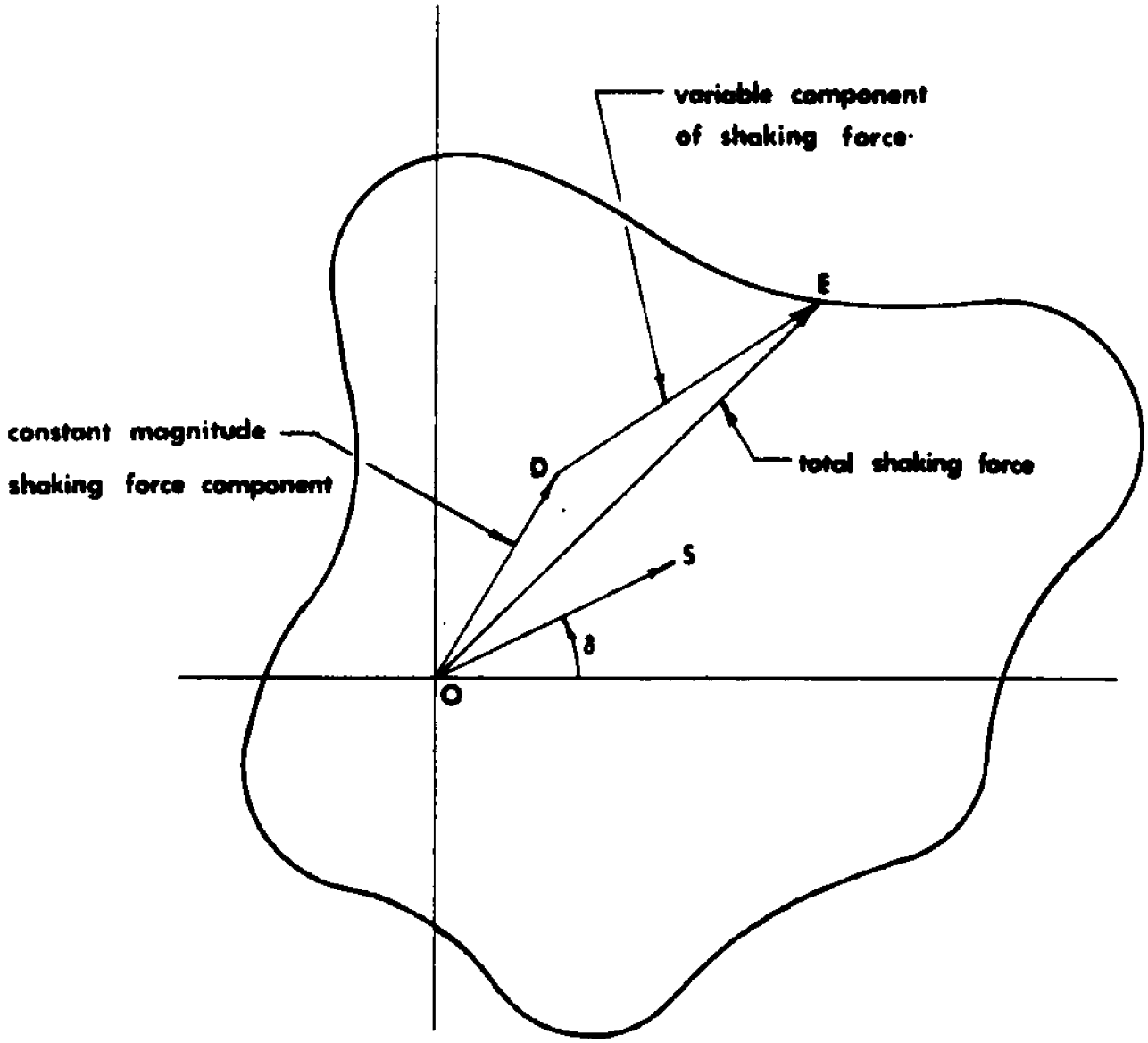


FIG. E.1 SHAKING FORCE HODOGRAPH OF CONSTANT INPUT SPEED MECHANISM ROTATED BY NEGATIVE OF INPUT ANGLE

opposite in direction to that represented by the arbitrary vector $\overline{OS}$, at angle δ . Figure E.2 shows that this operation causes a shift of the hodograph with respect to the coordinate system such that point S of the hodograph now coincides with the origin O. If point S is appropriately chosen, a reduction of the maximum shaking force may be realized.

Ya. L. Gheronimus has shown that the maximum shaking force can be minimized if point S is chosen as the center C_S of the smallest circle which circumscribes the hodograph (see Figure E.3). The center of the circumscribing circle represents the Chebyshev, or best uniform, approximation to the given hodograph by a polynomial of degree zero, i.e. a constant. The maximum modulus of difference between the hodograph and its Chebyshev approximation is represented by the radius R_S of the smallest circumscribing circle.

When this point C_S is now moved to the origin in the manner indicated above, the resulting shaking force can never be larger than the force which is represented by the length of the radius R_S of the circumscribing circle.

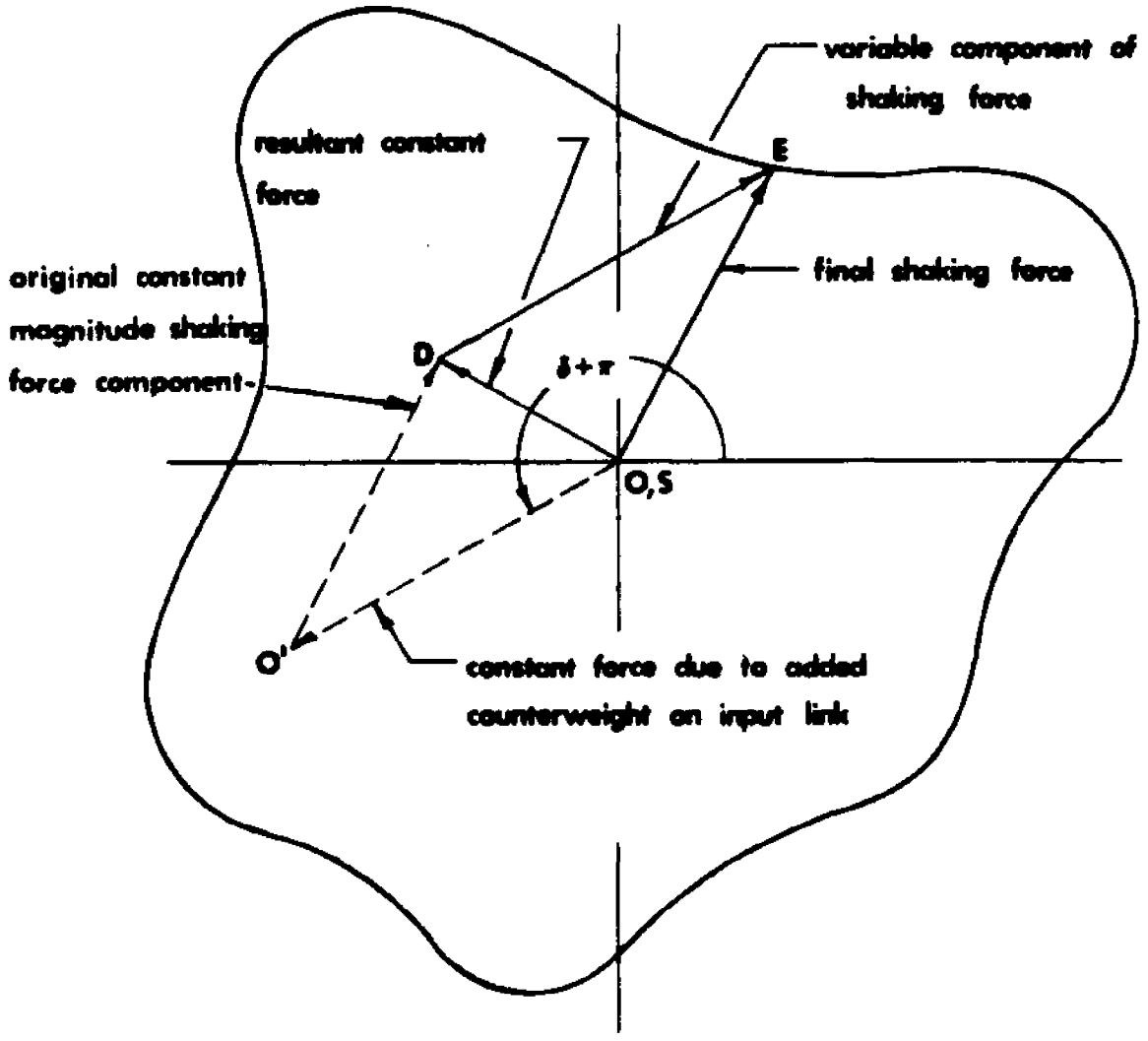


FIG. E.2 SHAKING FORCE HODOGRAPH OF FIG. E.1 MODIFIED BY ADDITION OF INPUT LINK COUNTERWEIGHT

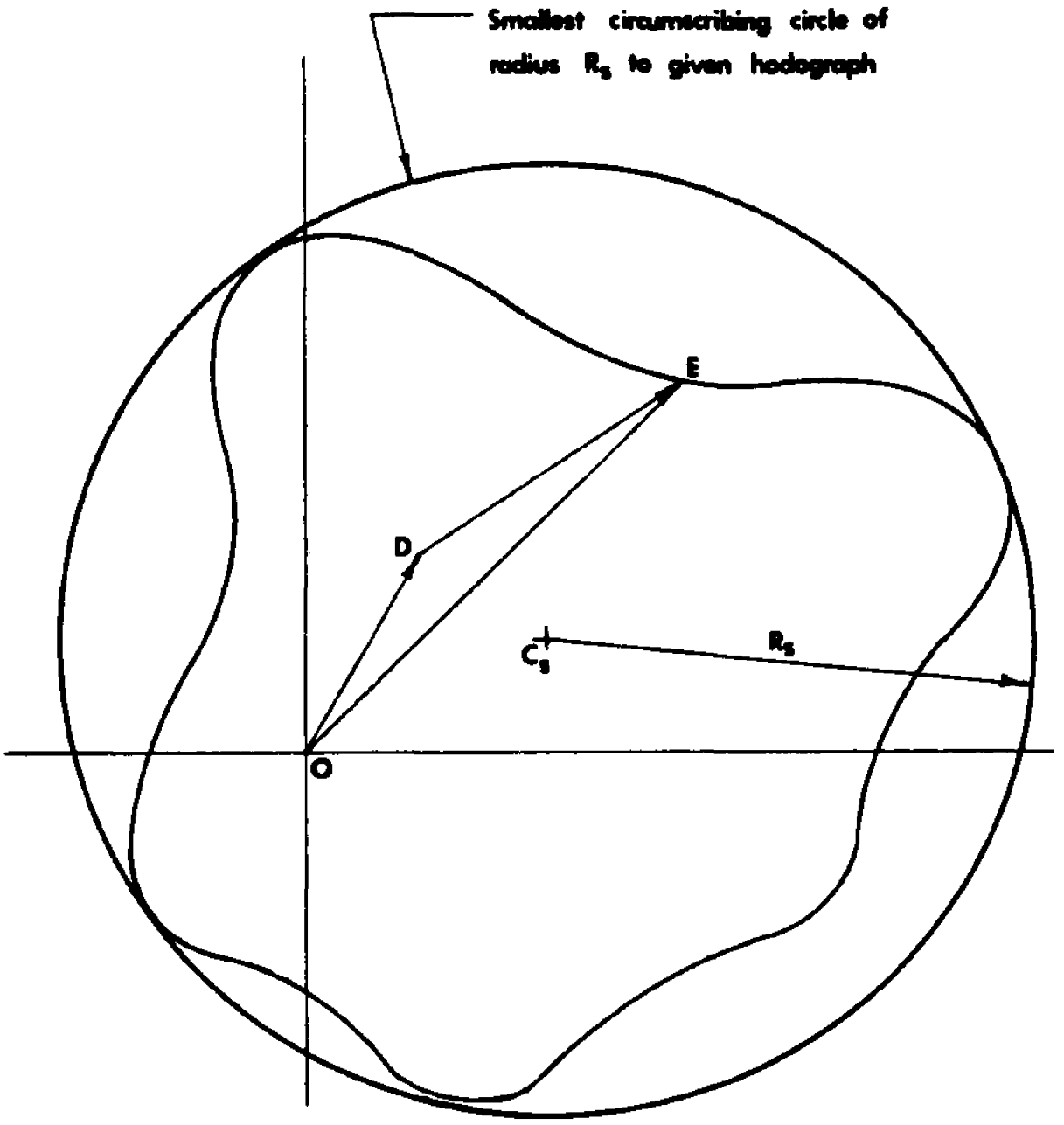


FIG. E.3 SMALLEST CIRCUMSCRIBING CIRCLE TO HODOGRAPH SHOWN IN FIG. E.1

APPENDIX F

COMPUTER PROGRAMS CIRCLE AND FMAX

This appendix gives the essential parts of program CIRCLE which represents an adaptation of the Harwell optimization code to the determination of the radius R_S and the coordinates C_{SX} and C_{SY} of the center of the smallest circumscribing circle to a given rotated shaking force hodograph. This program is based upon the formulation of equations (4.5) and (4.6). With the exception of the specific main program and subroutine VF01BD, the Harwell code is used as given in Appendix C.

In addition, the program FMAX, which finds the maximum values of all the reactions of a given mechanism, is shown. Outputs of programs CIRCLE and FMAX are required for the work of Appendix G.

1. Program CIRCLE: Determination Of Smallest Circumscribing Circle To A Given Shaking Force Hodograph

a. Main Program

The main program, which is listed at the end of this section, begins with the required COMMON statement which describes the array size for the constraints C. The

following variables are then defined:

N = 3; stands for the three variables X(1) = R_S', X(2) = C_{SX}' and X(3) = C_{SY}'.

M = 360; stands for the number of constraints. (These constraints are formed by the use of equation (4.6) at 1 degree intervals of the input angle ϕ_1. See subroutine VF01BD below.)

K = 0, since there are no equality constraints.

X(1), X(2) and X(3), the starting point estimates of the variables, are made equal to 25, 6 and 1, respectively.

With AKMIN chosen arbitrarily to be 1×10^{-3}, EPS(1) to EPS(3) result from X(1) to X(3).

DFN is estimated to be 10.

MAXFN = 500 (See Section 2a of Appendix C.)

IPR1 = 1, to print every outer iteration.

IPR2 = 1, to print every inner iteration.

IW = 10000, since there are more than 50 constraints.

MODE = 1 (See Section 2a of Appendix C.)

Subsequently, a DO loop is set up to make the constraint scale factors, C(I), equal to 500 for all 360 constraints. (See Appendix C.)

MAIN

```
C   THIS PROGRAM WILL CIRCUMSCRIBE A SHAKING FORCE HODOGRAPH
C   ROTATED BY THE NEGATIVE OF THE INPUT ANGLE PHI
C   WITH THE SMALLEST POSSIBLE CIRCLE.
C   THE PROGRAM EXPANDS THE AUGMENTED LAGRANGIAN OPTIMIZATION
C   ROUTINE TO HANDLE AS MANY AS 360 CONSTRAINTS
C   THE VARIABLES FOR THIS PROBLEM ARE
C        X(1) = RS = DISTANCE FROM TEST POINT TO POINTS ON HODOGRAPH
C        X(2) = CSX = X COORDINATE OF TEST POINT
C        X(3) = CSY = Y COORDINATE OF TEST POINT
C   CSX AND CSY WILL REPRESENT THE CENTER OF THE CIRCUMSCRIBING
C   CIRCLE AFTER THE OPTIMIZATION IS COMPLETE.
C
C   THE CONSTRAINTS ARE
C        C(I) = X(1)**2 - (FSX*R(I) - X(2))**2 - (FSYR(I) - X(3))**2.
C
C   THE HODOGRAPH WILL BE APPROXIMATED BY DISCRETE POINTS
C   TAKEN AT 1 DEGREE INTERVALS OF THE INPUT ANGLE PHI.
C   THE STARTING POINT IS
C        X(1)=25. X(2)=6 ,X(3)=1
0001      IMPLICIT REAL*8(A-H,O-Z)
0002      COMMON/VFOLED/C(1080)
0003      REAL*8 X(3),EPS(3)
0004      N = 3
0005      M = 360
0006      K=0
0007      PI=3.14159300
C   THE STARTING POINT IS ESTIMATED TO BE
0008      X(1)=2.501
0009      X(2)=6.000
0010      X(3)=1.000
C
0011      AKMIN=1.D-3
0012      EPS(1)=2.5D-2
0013      EPS(2)=6.0D-3
0014      EPS(3)=1.0D-3
C
0015      OFN=101
0016      MAXFN=500
0017      IPR1=1
0018      IPR2=1
C     SET IW=10000.  IF MORE STORAGE IS REQUIRED, VF01AD WILL
C     STOP AND PRINT OUT NEW REQUIREMENT.
0019      IW=10000
0020      MODE=1
C     THE CONSTRAINT SCALE FACTORS ARE ALL ESTIMATED TO BE 500
0021      DO 10 I=1,360
0022   10 C(I)=502
0023      CALL VF01AD(N,M,K,X,EPS,AKMIN,OFN,MAXFN,IPR1,IPR2,IW,MODE)
0024      STOP
0025      END
```

b. Subroutine VF01BD (See listing of program at the end of this section.)

After allocating storage for the various subscripted variables, subroutine VF01BD lists the COMMON statements described in Section 2a of Appendix C. The subsequent steps are:

1. F = X(1), i.e the objective function R_S' is declared as the variable x_1. (See equation (4.5).)

2. G(1) = 1, represents the derivative $\frac{\partial F}{\partial x_1}$. Since all other derivatives of F vanish, they must be set equal to zero (i.e. G(2)=G(3)=0).

3. A1 to A4 represent the link dimensions of the sample mechanism of Section IV.E.

4. RPM = 500, stands for the constant input rpm.

5. PHIVEL converts RPM to the input angular velocity $\dot{\phi}_1$.

6. PHIACC = 0, represents the zero input link angular acceleration $\ddot{\phi}_1$.

7. DELPHI = 1, represents the 1 degree increments of the input link angle ϕ_1.

8. The constant general mass-distance products are defined in P1, Q1, P3 and Q3 according to equations (4.54) to (4.57).

A DO loop is then set up to compute the constraints as well as their derivatives for each of the 360 positions of the mechanism. To this end, the shaking force components F_{SX} and F_{SY}, given in lines 50 and 51 of the program, must first be found.

The required kinematic values are supplied by way of subroutine MOTION (see Section 3b of Appendix C both for subroutine MOTION and the determination of F_{SX} and F_{SY}).

The shaking force components are then rotated by the negative of the input angle ϕ_1, according to equation (4.3), and one obtains:

$$F_{SX_R}(I) = F_{SX}(I)\cos\phi_1 + F_{SY}(I)\sin\phi_1 \qquad (F.1)$$

$$F_{SY_R}(I) = -F_{SX}(I)\sin\phi_1 + F_{SY}(I)\cos\phi_1 \qquad (F.2)$$

Finally, the constraint equation (4.6) and its derivatives with respect to the three variables are evaluated. Thus,

$$C(I) = \left(X_1\right)^2 - \left(F_{SX_R}(I) - X_2\right)^2 - \left(F_{SY_R}(I) - X_3\right)^2 \qquad (F.3)$$

$$GC(1,I) = 2X_1 \qquad (F.4)$$

$$GC(2,I) = 2\left(F_{SX_R}(I) - X_2\right) \qquad (F.5)$$

$$GC(3,I) = 2\left(F_{SY_R}(I) - X_3\right) \qquad (F.6)$$

```
0001                SUBROUTINE VF01BD(N,M,X)
0002                IMPLICIT REAL*8(A-H,O-Z)
0003                REAL*8 X(3)
0004                INTEGER PHID, PHIDL, DELPHI
0005                REAL*8 KAY1(360),KAY3(360)
0006                DIMENSION GMA1(360),GMA3(360),FSX(360),FSY(360)
0007                DIMENSION VAR1(360),VAR2(360),VAR3(360),VAR4(360)
0008                DIMENSION VAR5(360),VAR6(360),VAR7(360),VAR8(360)
0009                DIMENSION PSIA(360),PSIV(360),FSXR(360),FSYR(360)
0010                DOUBLE PRECISION DCOS,DSIN,DSQRT,DATAN2
0011                COMMON/VF01CD/F
0012                COMMON/VF01DD/G(6)
0013                COMMON/VF01ED/C(1080)
0014                COMMON/VF01FD/GG(3,360)
0015                PI=3.14159000
                C   THE OBJECTIVE FUNCTION IS
0016                F=X(1)
                C   G(I)=PARTIAL DERIVATIVES OF OBJECTIVE FUNCTION (F(X(I))) WITH
                C        RESPECT TO VARIABLES X(1),X(2),X(3).
0017                G(1)=100
0018                G(2)=000
0019                G(3)=000
                C   THE LINK DIMENSIONS ARE
0020                A1=200
0021                A2=600
0022                A3=300
0023                A4=5.500
                C   THE ANGULAR VELOCITY IS
0024                RPM = 502
0025                Z=PI/1.3002
0026                PHIVEL=RPM*Z*600
                C   THE ANGULAR ACCELERATION OF THE INPUT LINK IS
0027                PHIACC=000
                C   THE ANGULAR INCREMENT FOR THE HODOGRAPH FORCES IS
0028                DELPHI = 1
                C   THE GENERAL MASS-DISTANCE PRODUCTS ARE
0029                P1 = .1877720-2
0030                D1 =-.1836710-3
0031                P3 = .3089320-2
0032                D3 = .2754706S0-3
                C
0033                DO 120 PHID1=1,361,DELPHI
0034                PHID=PHID1 - 1
0035                PHI=Z * PHID
0036                I = PHID/DELPHI + 1
                C   THE KINEMATICS OF THE LINKAGE ARE DETERMINED IN
                C   SUBROUTINE MOTION
0037                CALL MOTION (PHI,A1,A2,A3,A4,PSI,PHIVEL,PHIACC,
                   1             PSIV(I),PSIA(I))
0038                KAY1(I) = DSQRT(PHIVEL**4 + PHIACC**2)
0039                KAY3(I) = DSQRT(PSIV(I)**4 + PSIA(I)**2)
0040                GMA1(I) = PHI - DATAN2(PHIACC,PHIVEL**2)
0041                GMA3(I) = PSI - DATAN2(PSIA(I),PSIV(I)**2)
0042                VAR1(I) = KAY1(I)*DCOS(GMA1(I))
0043                VAR2(I) = KAY1(I)*DSIN(GMA1(I))
0044                VAR3(I) = KAY3(I)*DCOS(GMA3(I))
```

```
0045                VAR4(I) = KAY3(I)*DSIN(SIGMA3(I))
0046                VAR5(I) = KAY1(I)*DSIN(SIGMA1(I))
0047                VAR6(I) = KAY1(I)*DCOS(SIGMA1(I))
0048                VAR7(I) = KAY3(I)*DSIN(SIGMA3(I))
0049                VAR8(I) = KAY3(I)*DCOS(SIGMA3(I))
                C   THE SHAKING FORCES ARE
0050                FSX(I) = P1*VAR1(I)-Q1*VAR2(I)+P3*VAR3(I)-Q3*VAR4(I)
0051                FSY(I) = P1*VAR5(I)+Q1*VAR6(I)+P3*VAR7(I)+Q3*VAR8(I)
                C       THE COMPONENTS OF THE SHAKING FORCE ROTATED BY THE NEGATIVE
                C       OF THE INPUT ANGLE PHI ARE)
0052                FSXR(I)=FSX(I)*DCOS(PHI) + FSY(I)*DSIN(PHI)
0053                FSYR(I)=-FSX(I)*DSIN(PHI) + FSY(I)*DCOS(PHI)
                C       THE CONSTRAINTS ARE
0054                C(I) = (X(1))**2 - (FSXR(I) - X(2))**2 - (FSYR(I) - X(3))**2
                C   THE DERIVATIVES OF THE CONSTRAINTS WITH RESPECT TO THE VARIABLES ARE
0055                GC(1,I) = 200*X(I)
0056                GC(2,I) = 200*(FSXR(I) - X(2))
0057                GC(3,I) = 200*(FSYR(I) - X(3))
0058              120 CONTINUE
0059                RETURN
0060                END
```

c. Essential Elements Of Program Output

The output format shown below is similar to that described in Section 3d of Appendix C. For this optimization, three outer iterations were required.

```
 OUTER ITERATION NUMBER IS  1

    0     1
 0.29C0000000000000D 02
 0.2500000000000000D 02  0.6000000000000000D 01  0.1000000000000000D 01
 0.1CC0000000000000D 01  0.0                     0.0

          .                 .                 .
          .                 .                 .
          .                 .                 .

   13    23     1
 0.1525570665822470D 02
 0.1492048641567714D 02  0.9931565874076651D 01 -0.3336708099616710D 01
 0.5274963414078111D-04 -0.4347786343139065D-04  0.1053163331210654D-04

 OUTER ITERATION NUMBER IS  3

    0     1
 0.1583585192761493D 02
 0.1582988404001558D 02  0.8997745252409256D 01 -0.3456607304034218D 01
 0.7409455618873912D 00 -0.1131211040782911D 00  0.7445374409062790D-01
    1     2
 0.1580162064463309D 02
 0.1574398086514367D 02  0.8963375050235685D 01 -0.3448197499607065D 01
 0.8576697897043320D-01 -0.4322091377915103D-01  0.3887911465901826D-01
    2     3
 0.1580101124431529D 02
 0.1573414654775762D 02  0.8978550309981297D 01 -0.3472957518274054D 01
-0.1603753325081790D-03 -0.7543329580955150D-02  0.5271259102975419D-02
    3     4
 0.1590100102757823D 02
 0.1573427310944936D 02  0.8956691708263036D 01 -0.3464663690417748D 01
-0.9331546025764161D-03 -0.1073078838518631D-02  0.8172520259265422D-03
    4     5     1
 0.1590100079435326D 02
 0.1573443123122720D 02  0.8979463715702490D 01 -0.3464060078896583D 01
-0.4378581827024426D-04 -0.6522649782460052D-05  0.6683729493363313D-05
```

Again, a set of four lines represent an inner iteration

where the first number in the first line stands for the total number of inner iterations within the given outer iteration. The second number in this line stands for the number of times the subroutine VF01ZD has been called upon.

The number in the second line represents the current value of the composite function Φ according to equation (C.1).

The third line gives the current values of R_S', C_{SX}' and C_{SY}'. In the first iteration, the initial values are reproduced.

The fourth line shows the derivatives of the composite function with respect to the three variables.

The last group of inner iterations shown represents the final results, i.e.

$$R_S = 15.734 \quad \text{(lbs)} \tag{F.7}$$

$$C_{SX} = 8.987 \quad \text{(lbs)} \tag{F.8}$$

$$C_{SY} = -3.464 \quad \text{(lbs)} \tag{F.9}$$

2. Program FMAX: Determination Of Maximum Bearing Forces, Shaking And Input Moments Of A Given Mechanism (See listing of program at the end of this section.)

This program, which is specifically written for the example mechanism of Section IV.E, first determines the X and Y components of the mechanism reactions with the help of the following subroutines:

MOTION: Finds the coupler and output link angles, angular velocities and angular accelerations. (See Appendix C for a listing of the program.)

KINMAT: Evaluates the positions, velocities and accelerations of the mass centers on the input, coupler and output links.

REACTS: Defines the X and Y components of each bearing force, the shaking force and the input moment as functions of the input angle ϕ_2.

Programs KINMAT and REACTS are also reproduced below.

The absolute magnitudes of each of the reactions are then found for all 72 mechanism positions by way of:

$$F_{M/G}(I) = \left(\left[F_{SX}(I) \right]^2 + \left[F_{SY}(I) \right]^2 \right)^{\frac{1}{2}}$$

$$F_{41}(I) = \left(\left[F_{41X}(I) \right]^2 + \left[F_{41Y}(I) \right]^2 \right)^{\frac{1}{2}}$$

$$F_{43}(I) = \left(\left[F_{43X}(I) \right]^2 + \left[F_{43Y}(I) \right]^2 \right)^{\frac{1}{2}}$$

$$F_{21}(I) = \left(\left[F_{21X}(I) \right]^2 + \left[F_{21Y}(I) \right]^2 \right)^{\frac{1}{2}}$$

$$F_{23}(I) = \left(\left[F_{23X}(I) \right]^2 + \left[F_{23Y}(I) \right]^2 \right)^{\frac{1}{2}}$$

$$M_{M/G}(I) = \left(\left[2.75F_{41Y}(I) - 2.75F_{43Y}(I) - M_{41}(I) \right]^2 \right)^{\frac{1}{2}}$$

$$M_{41}(I) = \left(\left[M_{41}(I) \right]^2 \right)^{\frac{1}{2}}$$

The maximum values of each of the above is obtained by continuous comparison with the previously established largest value. The results are given by:

$$(F_{41})_{MAX} = 52.122 \quad \text{lbs} \qquad \text{(F.10)}$$

$$(F_{21})_{MAX} = 50.743 \quad \text{lbs} \qquad \text{(F.11)}$$

$$(F_{23})_{MAX} = 40.377 \quad \text{lbs} \qquad \text{(F.12)}$$

$$(F_{43})_{MAX} = 39.273 \quad \text{lbs} \qquad \text{(F.13)}$$

$$(M_{M/G})_{MAX} = 90.902 \quad \text{lb-in} \qquad \text{(F.14)}$$

$$(M_{41})_{MAX} = 38.720 \quad \text{lb-in} \qquad \text{(F.15)}$$

and $$(F_{M/G})_{MAX} = R^{O}_{MAX} = 25.004 \quad \text{lbs} \qquad \text{(F.16)}$$

MAIN

```
C   THIS PROGRAM WILL FIND THE MAXIMUM ABSOLUTE VALUES OF THE
C   BEARING REACTIONS, SHAKING MOMENT, INPUT MOMENT AND SHAKING
C   FORCE FOR A GIVEN MECHANISM.
0001      IMPLICIT REAL*8(A-H,O-Z)
0002      INTEGER PHID, PHIOI, DELPHI
0003      REAL*8 K2,K3,M1,M2,M3,LAMBDA,LAMV(73),LAMA(73)
0004      REAL*8 MMG(73),M41(73),M61A(73),MMGMAX,M41MAX
0005      DIMENSION F41X(73),F41Y(73),F21X(73),F21Y(73),F23X(73),
         1F23Y(73),F43X(73),F43Y(73),FSX(73),FSY(73)
0006      DIMENSION F43(73),F41(73),F21(73),F23(73)
0007      DIMENSION PSIV(73),PSIA(73),FMG(73)
0008      DIMENSION X1(73),Y1(73),X2(73),Y2(73),X3(73),Y3(73)
0009      DIMENSION VX1(73),VY1(73),VX2(73),VY2(73),VX3(73),VY3(73)
0010      DIMENSION AX1(73),AY1(73),AX2(73),AY2(73),AX3(73),AY3(73)
0011      DOUBLE PRECISION DCOS,DSIN,DSQRT,DATAN2
0012      PI=3.14159300
0013      Z = PI/1.802
C         THE LINK DIMENSIONS ARE
0014      A1=200
0015      A2=600
0016      A3=300
0017      A4=5.500
C         THE ANGULAR VELOCITY OF THE INPUT LINK IS
0018      RPM = 502
0019      PHIVEL = RPM*Z*600
C         THE ANGULAR ACCELERATION OF THE INPUT LINK IS
0020      PHIACC = 000
C         THE ANGULAR INCREMENT IS
0021      DELPHI = 5
C         THE CENTER OF MASS LOCATIONS FOR EACH LINK ARE
0022      P1 = 100
0023      Q1 = 000
0024      P2 = 300
0025      Q2 = .40300
0026      P3 = 1.500
0027      Q3 = 000
C         THE RADII OF GYRATION OF COUPLER AND OUTPUT LINKS ARE
0028      K2 = 2.097000
0029      K3 = 1.574700
C         THE LINK MASSES ARE
0030      M1 = 5.10620-4
0031      M2 = 13.67100-4
0032      M3 = 6.23780-4
C
0033      F41MAX = 000
0034      F21MAX = 000
0035      F23MAX = 000
0036      F43MAX = 000
0037      MMGMAX = 000
0038      M41MAX = 000
0039      FMGMAX = 000
0040      DO 120 PHIOI=1,356,DELPHI
0041      PHID=PHIOI - 1
0042      PHI=Z*PHID
0043      I = PHID/DELPHI + 1
0044      CALL MOTION(PHI,A1,A2,A3,A4,PSI,PHIVEL,PHIACC,PSIV(I),PSIA(I),
```

```
     1          LAMBDA,LAMV(),LAMA())
C045            CALL KINMAT(PHI,PHIVEL,PHIACC,PSI,PSIV(),PSIA(),A1,A2,A3,A4,
     1          P1,Q1,P2,Q2,P3,Q3,X1(),Y1(),X2(),Y2(),X3(),
     2          Y3(),VX1(),VY1(),VX2(),VY2(),VX3(),VY3(),
     3          AX1(),AY1(),AX2(),AY2(),AX3(),AY3())
C046            CALL REACTS(PHI,PHIVEL,PHIACC,LAMBDA,LAMV(),LAMA(),PSI,PSIV(),
     1          PSIA(),A1,A2,A3,A4,PL,Q1,P2,Q2,P3,Q3,K2,K3,M1,M2,M3,
     2          F41X(),F41Y(),F21X(),F21Y(),F23X(),F23Y(),
     3          F43X(),F43Y(),FSX(),FSY(),M41())
C047            FMG()=DSQRT(FSX()*FSX() + FSY()*FSY())
C048            F41()=DSQRT(F41X()*F41X() + F41Y()*F41Y())
C049            F43()=DSQRT(F43X()*F43X() + F43Y()*F43Y())
C050            F21()=DSQRT(F21X()*F21X() + F21Y()*F21Y())
C051            F23()=DSQRT(F23X()*F23X() + F23Y()*F23Y())
0052            MMG() = DSQRT(2.7500*F41Y() - 2.7500*F43Y() - M41()**2)
C053            M41A() = DSQRT(M41()**2)
              C TEST FOR MAXIMUM VALUES
C054            IF (F41().GT.F41MAX)F41MAX=F41()
C055            IF (F43().GT.F43MAX)F43MAX=F43()
C056            IF (F21().GT.F21MAX)F21MAX=F21()
C057            IF (F23().GT.F23MAX)F23MAX=F23()
C058            IF (MMG().GT.MMGMAX)MMGMAX=MMG()
C059            IF (M41A().GT.M41MAX)M41MAX=M41A()
C060            IF (FMG().GT.FMGMAX)FMGMAX=FMG()
C061        120 CONTINUE
0062            WRITE (6,130)
0063        130 FORMAT (5X,'F41MAX',9X,'F21MAX',9X,'F23MAX',9X,'F43MAX',9X,
     1          'MMGMAX',9X,'M41MAX')
0064            WRITE (6,140) F41MAX,F21MAX,F23MAX,F43MAX,MMGMAX,M41MAX
C065        140 FORMAT (' ',6D15.7)
C066            WRITE (6,150)
0067        150 FORMAT ('0',2X,'THE MAXIMUM SHAKING FORCE IS')
0068            WRITE (6,160) FMGMAX
C069        160 FORMAT (D15.7)
C070            STOP
C071            END
```

```
0001      SUBROUTINE KINMAT (PHI,PHIVEL,PHIACC,PSI,PSIVEL,PSIACC,A1,A2,A3,
     1            A4,P1,Q1,P2,Q2,P3,Q3,X1,Y1,X2,Y2,X3,Y3,VX1,VY1,VX2,VY2,
     2            VX3,VY3,AX1,AY1,AX2,AY2,AX3,AY3)
C     CALCULATION OF X AND Y COMPONENTS OF POSITION, VELOCITY, AND
C     ACCELERATION OF ARBITRARY POINTS ON CRANK, COUPLER, AND OUTPUT
C     LINK OF A FOUR BAR LINKAGE.
C     GIVEN INPUT AND OUTPUT ANGLES, LINKAGE DIMENSIONS.
0002      IMPLICIT REAL*8(A-H,O-Z)
0003      DOUBLE PRECISION DCOS, DSIN,DSQRT,DATAN2,DABS
0004      SA = DSIN (PHI)
0005      CA = DCOS (PHI)
0006      SB = DSIN (PSI)
0007      CB = DCOS (PSI)
0009      X1 = P1*CA - Q1*SA
0009      Y1 = P1*SA + Q1*CA
0010      VX1 = -Y1*PHIVEL
0011      VY1 =  X1*PHIVEL
0012      AX1 = -Y1*PHIACC - VY1*PHIVEL
0013      AY1 =  X1*PHIACC + VX1*PHIVEL
0014      TA = (A2 - P2) * A1/A2
0015      TB = Q2 * A1/A2
0016      TC = P2 * A3/A2
0017      TD = Q2 * A3/A2
0018      APH = TA*CA + TB*SA
0019      BPH = TA*SA - TB*CA
0020      APS = TC*CB - TD*SB
0021      BPS = TC*SB + TD*CB
0022      X2 = APH + APS + A4*P2/A2
0023      Y2 = BPH + BPS + A4*Q2/A2
0024      VX2 = -BPH*PHIVEL - BPS*PSIVEL
0025      VY2 =  APH*PHIVEL + APS*PSIVEL
0026      AX2 = -BPH*PHIACC - APH*PHIVEL**2 - BPS*PSIACC - APS*PSIVEL**2
0027      AY2 =  APH*PHIACC - BPH*PHIVEL**2 + APS*PSIACC - BPS*PSIVEL**2
0028      X3 = P3*CB - Q3*SB + A4
0029      Y3 = P3*SB + Q3*CB
0030      VX3 = -Y3*PSIVEL
0031      VX3 =  Y3*PSIVEL
0032      VY3 = (X3 - A4) * PSIVEL
0033        VY3 =  (X3 - A4) * PSIVEL
0034      AX3 = -Y3*PSIACC - VY3*PSIVEL
0035      AY3 = (X3-A4)*PSIACC + VX3*PSIVEL
0036      RETURN
0037      END
```

```
0001      SUBROUTINE REACTS(PHI,PHIVEL,PHIACC,LAMBDA,LAMVEL,LAMACC,PSI,
     1PSIVEL,PSIACC,A1,A2,A3,A4,P1,Q1,P2,Q2,P3,Q3,K2,K3,M1,M2,M3,F41X
     2,F41Y,F21X,F21Y,F23X,F23Y,F43X,F43Y,FSX,FSY,M41)
0002      IMPLICIT REAL*8(A-H,O-Z)
0003      DOUBLE PRECISION DCOS, DSIN,DSQRT,DATAN2,DABS
0004      REAL*8 LAMBDA,LAMVEL,M1,M2,M3,M41,LAMV2,I2,I3,LAMACC,K2,K3
0005      SP = DSIN(PHI)
0006      CP = DCOS(PHI)
0007      SL = DSIN(LAMBDA)
0008      CL = DCOS(LAMBDA)
0009      SS = DSIN(PSI)
0010      CS = DCOS(PSI)
0011      TAU1 = DSIN(PHI - LAMBDA)
0012      TAU11 = DCOS(PHI - LAMBDA)
0013      TAU3 = DSIN(PSI - LAMBDA)
0014      PHIV2 = PHIVEL*PHIVEL
0015      LAMV2 = LAMVEL*LAMVEL
0016      PSIV2 = PSIVEL*PSIVEL
0017      I2 = M2*(P2*P2 + Q2*Q2 + K2*K2)
0018      I3 = M3*(P3*P3 + Q3*Q3 + K3*K3)
0019      D2 = I2*LAMACC/(A2*TAU3)
0020      D3 = I3*PSIACC/(A3*TAU3)
0021      A2T3 = A2*TAU3
0022      F41X = -M2*A1*PHIV2*CP - M1*P1*PHIV2*CP + M1*Q1*PHIV2*SP
     1        - M2*P2*(LAMACC*SL + LAMV2*CL - A1*CS*PHIV2*TAU1/A2T3)
     2        - M2*Q2*(LAMACC*CL - LAMV2*SL + A1*CS*PHIV2*TAU11/A2T3)
     3        - D2*CS - D3*CL
0023      F41Y = -M2*A1*PHIV2*SP - M1*P1*PHIV2*SP - M1*Q1*PHIV2*CP
     1        + M2*P2*(LAMACC*CL - LAMV2*SL + A1*SS*PHIV2*TAU1/A2T3)
     2        - M2*Q2*(LAMACC*SL + LAMV2*CL + A1*SS*PHIV2*TAU11/A2T3)
     3        - D2*SS - D3*SL
0024      F21X = M2*A1*PHIV2*CP + M2*P2*(LAMACC*SL + LAMV2*CL - A1*CS
     1        *PHIV2*TAU1/A2T3) + M2*Q2*(LAMACC*CL - LAMV2*SL
     2        + A1*CS*PHIV2*TAU11/A2T3) + D2*CS + D3*CL
0025      F21Y = M2*A1*PHIV2*SP + M2*P2*(-LAMACC*CL + LAMV2*SL - A1*SS
     1        *PHIV2*TAU1/A2T3) + M2*Q2*(LAMACC*SL + LAMV2*CL
     2        + A1*SS*PHIV2*TAU11/A2T3) + D2*SS + D3*SL
0026      F23X = M2*P2*A1*CS*PHIV2*TAU1/A2T3 - M2*Q2*A1*CS*PHIV2*TAU11/A2T3
     1        - D2*CS - D3*CL
0027      F23Y = M2*P2*A1*SS*PHIV2*TAU1/A2T3 - M2*Q2*A1*SS*PHIV2*TAU11/A2T3
     1        - D2*SS - D3*SL
0028      F43X = -M2*P2*A1*CS*PHIV2*TAU1/A2T3 + M2*Q2*A1*CS*PHIV2*TAU11/A2T3
     1        - M3*P3*(PSIACC*SS + PSIV2*CS) - M3*Q3*(PSIACC*CS - PSIV2
     2        *SS) + D2*CS + D3*CL
0029      F43Y = -M2*P2*A1*SS*PHIV2*TAU1/A2T3 + M2*Q2*A1*SS*PHIV2*TAU11/A2T3
     1        + M3*P3*(PSIACC*CS - PSIV2*SS) - M3*Q3*(PSIACC*SS + PSIV2
     2        *CS) + D2*SS + D3*SL
0030      FSX = -(F41X + F43X)
0031      FSY = -(F41Y + F43Y)
0032      M41 = M2*P2*(A1*(LAMACC*TAU11 + LAMV2*TAU1)
     1        - A1*A1*DSIN(PHI - PSI   )*PHIV2*TAU1/A2T3)
     2        + M2*Q2*(A1*(LAMACC*TAU1 - LAMV2*TAU11)
     3        + A1*A1*DSIN(PHI - PSI   )*PHIV2*TAU11/A2T3)
     4        + D2*A1*DSIN(PHI - PSI   ) + D3*A1*TAU1
0033      RETURN
0034      END
```

Output from program FMAX:

```
     F41MAX          F21MAX          F23MAX          F43MAX          MMGMAX          M41MAX
 0.52122020 02   0.50743290 02   0.40376540 02   1.39273270 02   0.90901020 02   0.38720070 02

 THE MAXIMUM SHAKING FORCE IS
 0.25C03760 02
```

APPENDIX G

COMPUTER PROGRAM OPTIMIZE

This appendix describes program OPTIMIZE which adapts the Harwell optimization code to the problem of simultaneously minimizing the maximum bearing forces, input moment and shaking moment associated with a prescribed maximum shaking force of a given four-bar linkage. This program is based on the formulation of equations (5.11) - (5.17) and follows the computational sequence outlined in Sections V-D-1 and V-D-2. The main program and the subroutine VF01BD, which are written for the sample mechanism of Section IV-E and a 50 percent reduction in the maximum shaking force[1], are first described. Both of the above routines follow the outline of Appendix C which shows the use of the Harwell code. Subsequently, essential elements of the output of the program are given.

1. Main Program

The main program, which is listed at the end of this section, begins with COMMON statements which define the array size for the constraints C and their derivatives GC.

---

[1] Both the main program and the subroutine VF01BD may be easily adapted to other four-bar linkages and reductions in maximum shaking force, including full force balance.

The following variables are then defined:

N = 5; stands for the five variables X(1)=z, X(2)=R_3^*, X(3)=θ_3^*, X(4)=R_2^*, and X(5)=θ_2^*.

M = 439; stands for the number of constraints. (See comment cards at beginning of program for definition of constraints.)

K = 1; refers to the single equality constraint as represented by the general equipollent circle equation. (See equation (5.4).)

X(1) to X(5), the starting point estimates of the variables, are made equal to 1, .705, 3.23, .559, and 3.23, respectively (a guess based on the two-counterweight method of Section IV).

With AKMIN chosen to be 1×10^{-4}, EPS(1) to EPS(5) result from X(1) to X(5). (This value results in sufficiently close adherence to the equipollent circle equality constraint to insure an accurate prescribed maximum shaking force.)

DFN is estimated to be 0.5.

MAXFN = 900 was specified, rather than the usual 500, because of the increased number of constraints.

IPR1 = 1, to print each outer iteration.

IPR2 = 1, to print each inner iteration.

IW = 10000; since there are more than 50 constraints.

MODE = 1

After setting scale factors for the constraints C(1) to C(7) equal to their starting point estimates, a DO loop is used to make the remaining 432 constraints equal to 0.5. (This again represents an arbitrary estimate.) Subsequently, all derivatives of constraints which result in a constant term are defined by the parameters GC(N,M). (See line 37 in Section 2.) Finally, the subroutine VF01AD is called.

MAIN

```
C     THIS PROGRAM OPTIMIZES THE MAXIMUM BEARING FORCES, MAXIMUM
C     SHAKING MOMENT AND MAXIMUM INPUT MOMENT FOR A PRESCRIBED
C     MAXIMUM SHAKING FORCE FOR A GIVEN FOUR-BAR LINKAGE.
C     CIRCULAR COUNTERWEIGHTS, TANGENT TO PIVOT POINTS, ARE USED ON
C     ALL MOVING LINKS.   EACH REACTION IS EVALUATED AT FIVE DEGREE
C     INCREMENTS OF THE INPUT LINK ANGLE.
C
C     VARIABLES ARE X(1)=Z, X(2)=R3STAR=OUTPUT COUNTERWEIGHT
C     RADIUS, X(3)=THETA3STAR=OUTPUT CWT. ANGLE, X(4)=R2STAR=
C     COUPLER COUNTERWEIGHT RADIUS, X(5)=THETA2STAR=COUPLER CWT.
C     ANGLE.
C
C     THE CONSTRAINTS ARE
C     C(1) = EQUIPOLLENT CIRCLE CONSTRAINT EQUATION
C     C(2) = X(2) (RADIUS R3STAR GREATER THAN ZERO)
C     C(3) = X(3) (ANGLE THETA3STAR GREATER THAN ZERO)
C     C(4) = 2*PI - X(3) (ANGLE THETA3STAR LESS THAN 2*PI)
C     C(5) = X(4) (RADIUS R2STAR GREATER THAN ZERO)
C     C(6) = X(5) (ANGLE THETA2STAR GREATER THAN ZERO)
C     C(7) = 2*PI - X(5) (ANGLE THETA2STAR LESS THAN 2*PI)
C     C(8) - C(79) = INPUT GROUND BEARING CONSTRAINTS
C     C(80) -C(151)= OUTPUT GROUND BEARING CONSTRAINTS
C     C(152)-C(223)= INPUT-TO-COUPLER BEARING CONSTRAINTS
C     C(224)-C(295)= COUPLER-TO-OUTPUT BEARING CONSTRAINTS
C     C(296)-C(367)= SHAKING MOMENT CONSTRAINTS WITH RESPECT TO
C                    MIDPOINT BETWEEN GROUND PIVOTS
C     C(368)-C(439)= INPUT MOMENT CONSTRAINTS
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/VFOLED/C(13L7)
      COMMON//VFOLFD/GC(5,439)
      REAL*8 X(5),EPS(5)
      PI=3.14159300
C     THE TOTAL NUMBER OF VARIABLES IS DESIGNATED BY N
      N=5
C     THE TOTAL NUMBER OF CONSTRAINTS IS DESIGNATED BY M
      M=439
C     THE TOTAL NUMBER OF EQUALITY CONSTRAINTS IS DESIGNATED BY K
      K=1
C     THE STARTING POINT IS ESTIMATED TO BE
      X(1)=100
      X(2)=.70500
      X(3)=3.23000
      X(4)=.55900
      X(5)=3.23000
      AKM[N=10-4
      EPS(1)=10-4
      EPS(2)=.7000-4
      EPS(3)=3.2300-4
      EPS(4)=.5500-4
      EPS(5)=3.2300-4
C
      DFN=0.500
      MAXFN=900
      IP1=1
      IP2=1
C     SET IW=10300.  IF MORE STORAGE IS REQUIRED, VF01AD WILL
```

```
C     STOP AND PRINT OUT NEW REQUIREMENT
      IW=10000
      MODE=1
C     THE CONSTRAINT SCALE FACTORS ARE SET AS FOLLOWS
      C(1)=10-5
      C(2)=.70500
      C(3)=3.23000
      C(4)=3.100
      C(5)=.55900
      C(6)=3.23000
      C(7)=3.100
      DO 10 I=9,439
   10 C(I)=.500
C     THE DERIVATIVES OF LINEAR CONSTRAINTS ARE
      GC(1,1)=000
      GC(1,2)=000
      GC(2,2)=100
      GC(3,2)=000
      GC(4,2)=000
      GC(5,2)=000
      GC(1,3)=000
      GC(2,3)=000
      GC(3,3)=100
      GC(4,3)=000
      GC(5,3)=000
      GC(1,4)=000
      GC(2,4)=000
      GC(3,4)=-100
      GC(4,4)=000
      GC(5,4)=000
      GC(1,5)=000
      GC(2,5)=000
      GC(3,5)=000
      GC(4,5)=100
      GC(5,5)=000
      GC(1,6)=000
      GC(2,6)=000
      GC(3,6)=000
      GC(4,6)=000
      GC(5,6)=100
      GC(1,7)=000
      GC(2,7)=000
      GC(3,7)=000
      GC(4,7)=000
      GC(5,7)=-100
      DO 11 I=8,437
   11 GC(1,I)=100
      DO 12 I=8,79
   12 GC(3,I)=000
      DO 13 I=152,295
   13 GC(3,I)=000
      DO 14 I=363,439
   14 GC(3,I)=000
      CALL VF01A(N,M,K,X,EPS,ARMIN,DFN,MAXFN,IPR1,IPR2,IW,MODE)
      STOP
      END
```

2. Subroutine VF01BD (See listing of program at the end of this section.)

After allocating storage for the various subscripted variables, subroutine VF01BD lists the COMMON statements described in Section 2a of Appendix C. The subsequent steps satisfy program requirements and introduce necessary functions and input data:

1. F = X(1); the objective function z is declared as the variable X(1). (See equation (5.11).)

2. G(1) to G(5); represent the derivatives of the objective function F with respect to the five variables X(1) to X(5).

3. A1 to A4 represent the link dimensions of the sample mechanism as given in Table 4.1

4. RHO1ST, RHO2ST, and RHO3ST represent the densities of the steel counterweights used on the input, coupler and output links. (All are made 7.3316×10^{-4} lb-sec^2/in^4.)

5. H1STAR, H2STAR, and H3STAR define the counterweight thicknesses as 0.625 inches.

6. R^{O}_{MAX} = 25.004 lbs; this maximum shaking force of the unbalanced linkage is obtained with the help of program FMAX (See Appendix F.2)

7. R_S = 15.734 lbs, C_{SX} = 8.987 lbs, C_{SY} = -3.464 lbs; the radius and coordinates of the center of the smallest circumscribing circle to the rotated shaking force hodograph are found with the help of program CIRCLE. (See Appendix F.1)

8. OC_S = magnitude of vector $\overline{OC}_S$ (see equation (4.24)).

9. DELTAS = angle δ_S of vector $\overline{OC}_S$ (see equation (4.24)).

10. ETA = 0.5; defines reduction factor η .

11. MU; proportionality factor μ , as given by equation (4.10).

12. RPM = 500; stands for the constant input rpm.

13. PHIVEL converts RPM to the input angular velocity $\dot{\phi}_1$

14. PHIACC = 0, represents the zero input link angular acceleration $\ddot{\phi}_1$

15. DELPHI = 5, represents the 5 degree increments of the input link angle ϕ_1

16. W1 to W6, stand for the weighting factors of equations (5.12) to (5.17). They are set equal to unity.

17. $(F_{41}{}^{o})_{MAX}$ = 52.12 lbs, $(F_{43}{}^{o})_{MAX}$ = 39.27 lbs, $(F_{21}{}^{o})_{MAX}$ = 50.74 lbs, $(F_{23}{}^{o})_{MAX}$ = 40.38 lbs, $(M_{M/G}{}^{o})_{MAX}$ = 90.90 in-lbs, and $(M_{41}{}^{o})_{MAX}$ = 38.72 in-lbs represent the maximum values of the reactions of the unbalanced linkage as found with the help of program FMAX (see Appendix F.2) and used in equations (5.12) - (5.17).

18. At this point, the masses, center of mass locations and the radii of gyration with respect to the centers of mass of the input, coupler and output links of the unbalanced mechanism are defined according to Table 4.1:

$$m_1{}^{o} = 5.1062 \times 10^{-4} \quad \text{lb-sec}^2/\text{in}$$

$$m_2{}^{o} = 13.6710 \times 10^{-4} \quad \text{lb-sec}^2/\text{in}$$

$$m_3{}^{o} = 6.9378 \times 10^{-4} \quad \text{lb-sec}^2/\text{in}$$

$$p_1{}^{o} = 1.0 \quad \text{in}$$

$$q_1{}^{o} = 0.0 \quad \text{in}$$

$$p_2^{\circ} = 3.0 \quad \text{in}$$

$$q_2^{\circ} = 0.403 \quad \text{in}$$

$$p_3^{\circ} = 1.5 \quad \text{in}$$

$$q_3^{\circ} = 0.0 \quad \text{in}$$

$$k_2^{\circ} = 2.0970 \quad \text{in}$$

$$k_3^{\circ} = 1.6747 \quad \text{in}$$

Subsequently, the following computations are made:

$$P_1^{\circ} = m_1^{\circ} p_1^{\circ} + m_2^{\circ} a_1 - \frac{a_1}{a_2} m_2^{\circ} p_2^{\circ} \qquad \text{(See equation (A.31).)}$$

$$Q_1^{\circ} = m_1^{\circ} q_1^{\circ} - \frac{a_1}{a_2} m_2^{\circ} q_2^{\circ} \qquad \text{(See equation (A.32).)}$$

$$P_3^{\circ} = m_3^{\circ} p_3^{\circ} + \frac{a_3}{a_2} m_2^{\circ} p_2^{\circ} \qquad \text{(See equation (A.33).)}$$

$$Q_3^{\circ} = m_3^{\circ} q_3^{\circ} + \frac{a_3}{a_2} m_2^{\circ} q_2^{\circ} \qquad \text{(See equation (A.34).)}$$

$$A^{\circ} = \sqrt{(P_1^{\circ})^2 + (Q_1^{\circ})^2} \qquad \text{(See equation (A.26).)}$$

$$\alpha^{\circ} = \tan^{-1} \frac{Q_1^{\circ}}{P_1^{\circ}} \qquad \text{(See equation (A.27).)}$$

$$B^{\circ} = \sqrt{(P_3^{\circ})^2 + (Q_3^{\circ})^2} \qquad \text{(See equation (A.28).)}$$

$$\beta^{\circ} = \tan^{-1} \frac{Q_3^{\circ}}{P_3^{\circ}} \qquad \text{(See equation (A.29).)}$$

19. The coupler counterweight, which is circular and tangent to point A_1 (see Figure 5.2), has the following parameters:

$$m_2^* = \rho_2^* \pi h_2^* (R_2^*)^2$$

$$p_2^* = R_2^* \cos\theta_2^* \qquad \text{(See Figure A.2)}$$

$$q_2^* = R_2^* \sin\theta_2^* \qquad \text{(See Figure A.2)}$$

20. The parameters of the balanced coupler link are given by:

$$m_2 = m_2^o + m_2^*$$

$$p_2 = (m_2^o p_2^o + m_2^* p_2^*)/m_2 \qquad \text{(See equation (A.25).)}$$

$$q_2 = (m_2^o q_2^o + m_2^* q_2^*)/m_2 \qquad \text{(See equation (A.25).)}$$

21. The radius of gyration of link 2 with respect to its center of mass is defined as:

$$k_2 = \left[\left(m_2^o \left\{ (p_2^o)^2 + (q_2^o)^2 + (k_2^o)^2 \right\} + 1.5 m_2^* (R_2^*)^2 \right)/m_2 - (p_2^2 + q_2^2) \right]^{\frac{1}{2}}$$

22. The output link counterweight, which is circular and tangent to point A_3 (see Figure 5.2), has the parameters:

$$m_3^* = \rho_3^* \pi h_3^* (R_3^*)^2$$

$$p_3^* = R_3^* \cos\theta_3^*$$

$$q_3^* = R_3^* \sin\theta_3^*$$

23. The balanced output link parameters are given by:

$$m_3 = m_3^o + m_3^*$$

$$p_3 = (m_3^o p_3^o + m_3^* p_3^*)/m_3$$

$$q_3 = (m_3^o q_3^o + m_3^* q_3^*)/m_3$$

24. The radius of gyration of link 3 with respect to its center of mass is defined as:

$$k_3 = \left[(m_3^o \{(p_3^o)^2 + (q_3^o)^2 + (k_3^o)^2\} + 1.5 m_3^* (R_3^*)^2)/m_3 - (p_3^2 + q_3^2) \right]^{\frac{1}{2}}$$

25. The parameters of the input link are now found according to the computational sequence outlined in Section V-D-1:

a. $$\beta = \tan^{-1} \frac{m_3 q_3 + \frac{a_3}{a_2} m_2 q_2}{m_3 p_3 + \frac{a_3}{a_2} m_2 p_2}$$ (See equation (5.19).)

b. $\Psi = \beta - \beta^{\circ}$, where Ψ equals the angle of rotation of the shaking force hodograph.

c. Vector $\overline{D^{\circ}C_S}$ is found with the help of equation (4.27):

$$D^{\circ}C = \left[(D^{\circ}C_{S_X})^2 + (D^{\circ}C_{S_Y})^2\right]^{\frac{1}{2}}$$

where

$$D^{\circ}C_{SX} = OC_S\cos\delta_S - A^{\circ}\dot{\phi}_1^2\cos\alpha^{\circ}$$

$$D^{\circ}C_{SY} = OC_S\sin\delta_S - A^{\circ}\dot{\phi}_1^2\sin\alpha^{\circ}$$

and angle ζ is given by

$$\zeta = \tan^{-1}\frac{D^{\circ}C_{SY}}{D^{\circ}C_{SX}}$$

d. Vector $\overline{DC}_2$ is then found by way of equation (5.22), i.e.

$$DC_2 = \left[(DC_{2X})^2 + (DC_{2Y})^2\right]^{\frac{1}{2}}$$

where

$$DC_{2X} = \mu D^{\circ}C_S\cos(\zeta + \Psi)$$

$$DC_{2Y} = \mu D^{\circ}C_S\sin(\zeta + \Psi)$$

e. The term A is found according to

equation (5.22):

$$A = (P_1^2 + Q_1^2)^{\frac{1}{2}}$$

where $P_1 = m_1{}^{o}p_1{}^{o} + m_2a_1 - \frac{a_1}{a_2} m_2p_2$

$$Q_1 = m_1{}^{o}q_1{}^{o} - \frac{a_1}{a_2} m_2q_2$$

and the angle α is given by

$$\alpha = \tan^{-1}\frac{Q_1}{P_1}$$

f. The resulting location of the rotated center of the circumscribing circle is given by vector $\overline{OC}_2$ at angle δ_2 (see equation (5.23)):

$$OC_2 = \left[(OC_{2X})^2 + (OC_{2Y})^2\right]^{\frac{1}{2}}$$

where $OC_{2X} = A\dot{\phi}_1^2\cos\alpha + DC_2\cos(\zeta + \Psi)$

$$OC_{2Y} = A\dot{\phi}_1^2\sin\alpha + DC_2\sin(\zeta + \Psi)$$

at angle

$$\delta_2 = \tan^{-1}\frac{OC_{2Y}}{OC_{2X}}$$

g. The input counterweight radius and angle is then found with the help of equations (4.70),

(5.25) and (5.26):

$$R_1^* = \left[\frac{OC_2}{\rho_1^* h_1^* \pi \dot{\phi}_1^2}\right]^{\frac{1}{3}}$$

$$\text{and } \theta_1^* = \delta_2 + \pi$$

26. The input counterweight parameters are given by:

$$m_1^* = \rho_1^* h_1^* \pi (R_1^*)^2$$

$$p_1^* = R_1^* \cos\theta_1^*$$

$$q_1^* = R_1^* \sin\theta_1^*$$

27. Finally, the input link parameters are given by:

$$m_1 = m_1^o + m_1^*$$

$$p_1 = (m_1^o p_1^o + m_1^* p_1^*)/m_1$$

$$q_1 = (m_1^o q_1^o + m_1^* q_1^*)/m_1$$

28. Constraints C(1) to C(7) are now set along with the derivatives of C(1) with respect to variables X(2) to X(5):

    a. C(1); general equipollent circle constraint equation (see equations (5.4), (5.9) and (5.10)).

b. C(2) = X(2); the constraint on the output counterweight radius, i.e. $R_3^* \geq 0$.

c. C(3), C(4); the constraints on the output link counterweight angle, i.e. $\theta_3^* \geq 0$ and $2\pi - \theta_3^* \geq 0$.

d. C(5) = X(3); the constraint on the coupler counterweight radius, i.e. $R_2^* \geq 0$.

e. C(6), C(7); the constraints on the coupler counterweight angle, i.e. $\theta_2^* \geq 0$ and $2\pi - \theta_2^* \geq 0$.

f. GC(2,1) to GC(5,1); represent the partial derivatives of the equipollent circle constraint equation with respect to the variables X(2) to X(5). (Note that GC(1,1), the derivative of C(1) with respect to X(1), is zero and is given in the main program because it is a constant.)

29. With all appropriate mechanism parameters now defined, a DO loop is set up to evaluate the remaining constraints and their derivatives. The mechanism reactions corresponding to each input angle ϕ_1 are first obtained by way of the following subroutines which are listed in Appendix F:

MOTION: evaluates coupler and output link angles, angular velocities and angular accelerations.

KINMAT: finds positions, velocities and accelerations of mass centers on input, coupler and output links.

REACTS: determines the X and Y components of each bearing force, the shaking moment as well as the input moment as functions of the input angle ϕ_1.

30. SHAKM(I) = $2.75F_{41Y} - 2.75F_{43Y} - M_{41}$; defines the shaking moment $M_{M/G}$ with respect to the midpoint between the ground bearings for the given configuration.

31. $F_{41}(I) = (F_{41X}^2 + F_{41Y}^2)^{\frac{1}{2}}$; the absolute value of bearing force F_{41}.

32. $F_{43}(I) = (F_{43X}^2 + F_{43Y}^2)^{\frac{1}{2}}$; the absolute value of bearing force F_{43}.

33. $F_{21}(I) = (F_{21X}^2 + F_{21Y}^2)^{\frac{1}{2}}$; the absolute value of bearing force F_{21}.

34. $F_{23}(I) = (F_{23X}^2 + F_{23Y}^2)^{\frac{1}{2}}$; the absolute value of bearing force F_{23}.

35. The constraints are now evaluated according to equations (5.12) - (5.17), where I = 1,2,...,72:

a. $C(I + 7) = X(1) - W1\left[\frac{(F_{41})^2 - (F_{41}^{o})^2}{(F_{41}^{o})^2}\right]$

b. $C(I + 79) = X(1) - W4\left[\frac{(F_{43})^2 - (F_{43}^{o})^2}{(F_{43}^{o})^2}\right]$

c. $C(I + 151) = X(1) - W2\left[\frac{(F_{21})^2 - (F_{21}^{o})^2}{(F_{21}^{o})^2}\right]$

d. $C(I + 223) = X(1) - W3\left[\frac{(F_{23})^2 - (F_{23}^{o})^2}{(F_{23}^{o})^2}\right]$

e. $C(I + 295) = X(1) - W5\left[\frac{(M_{M/G})^2 - (M_{M/G}^{o})^2}{(M_{M/G}^{o})^2}\right]$

f. $C(I + 367) = X(1) - W6\left[\frac{(M_{41})^2 - (M_{41}^{o})^2}{(M_{41}^{o})^2}\right]$

36. The following partial derivatives are then defined so that the derivatives of the constraints with respect to the five variables X(1) to X(5) may be found. (The subscript K is introduced here to distinguish between these variables.):

a. $PF43X(K) = \frac{\partial F_{43X}}{\partial X_K}$

b. $PF43Y(K) = \frac{\partial F_{43Y}}{\partial X_K}$

$K = 2,3,4,5$

c. $PF41X(K) = \frac{\partial F_{41X}}{\partial X_K}$

d. $PF41Y(K) = \frac{\partial F_{41Y}}{\partial X_K}$

$K = 2,4,5$

e. $PF21X(K) = \frac{\partial F_{21X}}{\partial X_K}$

f. $PF21Y(K) = \frac{\partial F_{21Y}}{\partial X_K}$

$K = 2,4,5$

g. $PF23X(K) = \frac{\partial F_{23X}}{\partial X_K}$

h. $PF23Y(K) = \frac{\partial F_{23Y}}{\partial X_K}$

$K = 2,4,5$

i. $PM41(K) = \frac{\partial M_{41}}{\partial X_K}$, $K = 2,4,5$

j. $PSHAKM(K) = \frac{\partial M_{M/G}}{\partial X_K}$, $K = 2,3,4,5$

37. Finally, GC(N,M), the partial derivatives of constraint M with respect to variable N, are found. Note that the partial derivatives of F_{41X}, F_{41Y}, F_{21X}, F_{21Y}, F_{23X}, F_{23Y} and M_{41} with respect to X_3 are zero. This is the reason

why GC(3,I+7), GC(3,I+151), GC(3,I+223) and GC(3,I+367) have been set equal to zero in the main program for I = 1,2,...,72:

a. $GC(K,I+7) = -\frac{2W_1}{(F_{41}^{o})^2}\left(F_{41X}\frac{\partial F_{41X}}{\partial X_K} + F_{41Y}\frac{\partial F_{41Y}}{\partial X_K}\right)$, K = 2,4,5

b. $GC(K,I+79) = -\frac{2W_4}{(F_{43}^{o})^2}\left(F_{43X}\frac{\partial F_{43X}}{\partial X_K} + F_{43Y}\frac{\partial F_{43Y}}{\partial X_K}\right)$, K = 2,3,4,5

c. $GC(K,I+151) = -\frac{2W_2}{(F_{21}^{o})^2}\left(F_{21X}\frac{\partial F_{21X}}{\partial X_K} + F_{21Y}\frac{\partial F_{21Y}}{\partial X_K}\right)$, K = 2,4,5

d. $GC(K,I+223) = -\frac{2W_3}{(F_{23}^{o})^2}\left(F_{23X}\frac{\partial F_{23X}}{\partial X_K} + F_{23Y}\frac{\partial F_{23Y}}{\partial X_K}\right)$, K = 2,4,5

e. $GC(K,I+295) = -\frac{2W_5}{(M_{M/G}^{o})^2}M_{M/G}\frac{\partial M_{M/G}}{\partial X_K}$, K = 2,3,4,5

f. $GC(K,I+367) = -\frac{2W_6}{(M_{41}^{o})^2}M_{41}\frac{\partial M_{41}}{\partial X_K}$, K = 2,4,5

```
      SUBROUTINE VF01DD(N,M,X)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 X(5)
      INTEGER PHI0, PHI01, DELPHI
      REAL*8 LAMD,K2,K3,ML,M2,M3,JJ,KK,LL,PM,LAMBDA,MU,MIZERO
      REAL*8 M2ZERO,M2STAR,M3ZERO,M3STAR,M1STAR,K2ZERO,K3ZERO
      REAL*8 M4I(73),M4IO,LAMV(73),LAMA(73),LAMD(73),LAMV2,I2,I3
      DIMENSION SHAKM(73)
      DIMENSION F41X(73),F41Y(73),F21X(73),F21Y(73),F23X(73),
     1F23Y(73),F43X(73),F43Y(73),FSX(73),FSY(73)
      DIMENSION F43(73),F41(73),F21(73),F23(73)
      DIMENSION PSIO(73),PSIV(73),PSI(73)
      DIMENSION X1(73),Y1(73),X2(73),Y2(73),X3(73),Y3(73)
      DIMENSION VX1(73),VY1(73),VX2(73),VY2(73),VX3(73),VY3(73)
      DIMENSION AX1(73),AY1(73),AX2(73),AY2(73),AX3(73),AY3(73)
      COUBLE PRECISION DCOS,DSIN,DSQRT,DATAN2
      COMMON/VF01CO/F
      COMMON/VF01DD/G(10)
      COMMON//F01ED/C(1317)
      COMMON//F01FD/GC(5,439)
      PI=3.14159300
C     THE OBJECTIVE FUNCTION IS
      F=X(1)
C     G(I)=PARTIAL DERIVATIVES OF OBJECTIVE FUNCTION (F(X(1)) WITH
C          RESPECT TO VARIABLES X(1),X(2),X(3),X(4) AND X(5)
      G(1)=1D0
      G(2)=0D0
      G(3)=0D0
      G(4)=0D0
      G(5)=0D0
C     THE LINK DIMENSIONS ARE
      A1=2D0
      A2=6D0
      A3=3D0
      A4=5.5D0
C     THE COUNTERWEIGHT DENSITIES AND THICKNESS FOR LINKS 1, 2 AND 3 ARE
      RHO1ST=7.3315040-4
      RHO2ST=7.3316060-4
      RHO3ST=7.3316060-4
      H1STAR=.62500
      H2STAR=.62500
      H3STAR=.62500
C     THE MAXIMUM SHAKING FORCE FOR THE UNBALANCED LINKAGE IS
      RQMAX = .2500376002
C     THE RADIUS AND COORDINATES OF THE CENTER OF THE SMALLEST
C     CIRCUMSCRIBING CIRCLE TO THE ROTATED SHAKING FORCE HODOGRAPH ARE
      RS = .1573443002
      CSY = .8987464001
      CSY =-.3464040001
C     THE MAGNITUDE OF VECTOR OCS IS
      OCS = DSQRT(CSX**2 + CSY**2)
C     AT ANGLE
      DELTAS = DATAN2(CSY,CSX)
C     THE REDUCTION FACTOR IS
      ETA = .500
C     THE PROPORTIONALITY FACTOR IS GIVEN BY
```

```
0044              MU = EPS*RJMAX/RS
                C     THE ANGULAR VELOCITY OF THE INPUT LINK IS
0045              IP=0.503
0046              Z=PI/1.0002
0047              PHIVEL=IP*Z*600
                C     THE ANGULAR ACCELERATION OF THE INPUT LINK IS
0048              PHIACC=100
                C     DELPHI=INCREMENT OF INPUT ANGLE PHI
0049              DELPHI=3
                C SET WEIGHTING FACTORS FOR CONSTRAINTS
0050              W1= 100
0051              W2= 100
0052              W3= 100
0053              W4= 100
0054              W5= 100
0055              W6= 100
                C     THE MAXIMUM VALUES OF REACTIONS FOR UNBALANCED LINKAGE ARE
0056              F410=.5212202002
0057              F430=.3327327002
0058              F210=.5074326002
0059              F230=.4037654002
0060              SHAKM0=.7090192002
0061              M410=.3472007002
                C
                C     DEFINE INPUT TO SUBROUTINE USED TO DETERMINE MECHANISM REACTIONS
                C     R1,U1=X AND Y COMPONENTS OF CENTER OF MASS OF INPUT LINK
                C     R2,U2=X AND Y COMPONENTS OF CENTER OF MASS OF COUPLER LINK
                C     R3,U3=X AND Y COMPONENTS OF CENTER OF MASS OF OUTPUT LINK
                C     M1,M2,M3=MASS OF INPUT,COUPLER AND OUTPUT LINK, RESPECTIVELY
                C     K2,K3=RADIUS OF GYRATION OF COUPLER AND OUTPUT LINKS
                C
                C     THE PARAMETERS OF THE UNBALANCED LINKAGE ARE
0062              M1ZERO=5.1262D-4
0063              M2ZERO=13.6710D-4
0064              M3ZERO=6.9378D-4
0065              P10 = 100
0066              Q10 = 000
0067              P20=300
0068              Q20=.40300
0069              P30=1.500
0070              Q30 = 000
0071              K2ZERO = 2.7970)
0072              K3ZERO= 1.6747D0
0073              PE10 = M1ZERO*P10 + M2ZERO*A1 - A1*M2ZERO*P20/A2
0074              QU10 = M1ZERO*Q10 - A1*M2ZERO*Q20/A2
0075              P3ZERO = M3ZERO*P30 + A3*M2ZERO*P20/A2
0076              Q3ZERO = M3ZERO*Q30 + A3*M2ZERO*Q20/A2
0077              AZERO = DSQRT(PE10**2 + QU10**2)
0078              ALPHA0 = DATAN2(QU10,PE10)
0079              BZERO = DSQRT(P3ZERO**2 + Q3ZERO**2)
0080              BETA0 = DATAN2(Q3ZERO,P3ZERO)
                C
                C     THE COUPLER COUNTERWEIGHT PARAMETERS ARE
                C
0081              M2STAR = RHO2ST*PI*M2STAR*(X(4)**2)
0082              P2STAR = X(4)*DCOS(X(5))
```

```
      C2STAR = X(4)*DSIN(X(5))
C
C     THE COUPLER LINK PARAMETERS ARE
      M2 = M2ZERO + M2STAR
      P2 = (M2ZERO*P20 + M2STAR*P2STAR)/M2
      Q2 = (M2ZERO*Q20 + M2STAR*Q2STAR)/M2
C     THE RADIUS OF GYRATION OF LINK 2 IS
      K2 = DSQRT((M2ZERO*(P20**2 + Q20**2 + K2ZERO**2) +
     1           1.5D0*M2STAR*(X(4)**2))/M2 - (P2**2 + Q2**2))
C
C     THE OUTPUT LINK COUNTERWEIGHT PARAMERERS ARE
      M3STAR=RHO3ST*PI*M3STAR*(X(2)**2)
      P3STAR=X(2)*DCOS(X(3))
      Q3STAR=X(2)*DSIN(X(3))
C
C     THE OUTPUT LINK PARAMETERS ARE THEN GIVEN BY
      M3= M3ZERO + M3STAR
      P3 = (M3ZERO*P30 + M3STAR*P3STAR)/M3
      Q3 = (M3ZERO*Q30 + M3STAR*Q3STAR)/M3
C
C     THE RADIUS OF GYRATION OF THE OUTPUT LINK IS
      K3 = DSQRT((M3ZERO*(P30**2 + Q30**2 + K3ZERO**2)
     1           + 1.5D0*M3STAR*(X(2)**2))/M3 - (P3**2 +Q3**2))
C
C     THE PARAMETERS OF THE INPUT CWT. MUST NOW BE FOUND WHICH WILL
C     RESULT IN THE PRESCRIBED MAXIMUM SHAKING FORCE.
C
C     ANGLE BETA0 IS GIVEN BY
C     THE ANGLE BETA IS FOUND BY WAY OF
      PL3 = M3*P3 + (A3/A2)*M2*P2
      QU3 = M3*Q3 + (A3/A2)*M2*Q2
      BETA = DATAN2(QU3,PL3)
C     THE ANGLE OF ROTATION PSI MAY NOW BE FOUND AS
      ROTPSI = BETA - BETA0
C     THE COMPONENTS OF VECTOR DOCS ARE
      DOCSX = OC3*DCOS(DELTA3)-AZERO*(PHIVEL**2)*DCOS(ALPHA0)
      DOCSY = OC3*DSIN(DELTA3)-AZERO*(PHIVEL**2)*DSIN(ALPHA0)
C     THE MAGNITUDE OF DOCS MAY NOW BE FOUND AS
      DOCS = DSQRT(DOCSX**2 + DOCSY**2)
C     AT ANGLE
      ZETA = DATAN2(DOCSY,DOCSX)
C     THE COMPONENTS OF VECTOR OC2 MAY NOW BE FOUND
      OC2X = MU*DOCS*DCOS(ZETA + ROTPSI)
      OC2Y = MU*DOCS*DSIN(ZETA + ROTPSI)
C     THE MAGNITUDE OF OC2 IS
      OC2 = DSQRT(OC2X**2 + OC2Y**2)
C     DUE TO THE ADDITION OF A COUPLER COUNTERWEIGHT, THE TERM AZERO
C     BECOMES A, WHICH IS FOUND BY WAY OF
      PE1 = M1ZERO*P10 + M2*A1 - (A1/A2)*M2*P2
      QU1 = M1ZERO*Q10 - (A1/A2)*M2*Q2
      A = DSQRT(PE1**2 + QU1**2)
C     THE ANGLE ALPHA0 BECOMES ALPHA, WHERE
      ALPHA = DATAN2(QU1,PE1)
C
C     THE RESULTING LOCATION OF THE ROTATED CENTER OF THE CIRCUMSCRIBING
C     CIRCLE IS THEN GIVEN AS OC2 AT ANGLE DELTA2, WHERE
```

```
C110            OC2X =   A*(PHIVEL**2)*DCOS(ALPHA) +
               1       OC2*DCOS(ZETA + ROT*PI)
C111            OC2Y =   A*(PHIVEL**2)*DSIN(ALPHA) +
               1       OC2*DSIN(ZETA + ROT*PI)
C112            OC2 = DSQRT(OC2X**2 + OC2Y**2)
C113            DELTA2 = DATAN2(OC2Y,OC2X)
C               THE INPUT LINK CWT. RADIUS IS GIVEN BY
C114            RISTAR = (OC2/(RHOIST*RISTAR*PI*(PHIVEL**2)))**(1.DO/3.DO)
C               AT ANGLE
C115            THIST = DELTA2 + PI
C
C               THE INPUT CWT. PARAMETERS ARE
C116            RISTAR = RHOIST*RISTAR*PI*(RISTAR**2)
C117            PISTAR = RISTAR*DCOS(THIST)
C118            QISTAR = RISTAR*DSIN(THIST)
C
C               THE REQUIRED INPUT LINK PARAMETERS ARE
C119            RL= RIZERO + RISTAR
C120            PL = (RIZERO*PIO + RISTAR*PISTAR)/RL
C121            QL = (RIZERO*QIO + RISTAR*QISTAR)/RL
C               CONSTRAINTS C(1) TO C(7) ARE NOW SET ALONG WITH THE
C               DERIVATIVES OF C(1) WITH RESPECT TO VARIABLES X(2) TO X(5)
C
C               C(1)=EQUALITY CONSTRAINT (GENERAL EQUIPOLLENT CIRCLE CONSTR. EQ.)
C122            RPH2=RHO2ST*P(PH2STAR*(X(4)**3)
C123            RPH=RHO3ST*P(PH3STAR*(X(2)**3)
C124            C(1)=(RPH*DCOS(X(3)) + (A3/A2)*RPH2*DCOS(X(5)) + P3ZERO)**2
               1  + (RPH*DSIN(X(3)) + (A3/A2)*RPH2*DSIN(X(5)) + Q3ZERO)**2
               2  - (RHO*BZERO)**2
C               C(2)=CONSTRAINT ON OUTPUT COUNTERWEIGHT RADIUS
C125            C(2)= X(2)
C               C(3),C(4)=CONSTRAINTS ON OUTPUT COUNTERWEIGHT ANGLE
C126            C(3)= X(3)
C127            C(4)= 2.DO*PI - X(3)
C               C(5)=CONSTRAINT ON COUPLER COUNTERWEIGHT RADIUS
C128            C(5)= X(4)
C               C(6),C(7)=CONSTRAINTS ON COUPLER COUNTERWEIGHT ANGLE
C129            C(6)=X(5)
C130            C(7)=2.DO*PI - X(5)
C               GC(R,S)=PARTIAL OF C(S) WITH RESPECT TO VARIABLE R
C131            GC(1,1)=RPH*DCOS(X(3)) + (A3/A2)*RPH2*DCOS(X(5)) + P3ZERO
C132            GC(1,2)=RPH*DSIN(X(3)) + (A3/A2)*RPH2*DSIN(X(5)) + Q3ZERO
C133            GC(2,1) = 6.DO*GC(1,1)*RPH*DCOS(X(3))/X(2) +
               1          6.DO*GC(1,2)*RPH*DSIN(X(3))/X(2)
C134            GC(3,1) = -2.DO*GC(1,1)*RPH*DSIN(X(3)) +
               1           2.DO*GC(1,2)*RPH*DCOS(X(3))
C135            GC(4,1) = 6.DO*GC(1,1)*(A3/A2)*(RPH2*DCOS(X(5)))/X(4) +
               1          6.DO*GC(1,2)*(A3/A2)*(RPH2*DSIN(X(5)))/X(4)
C136            GC(5,1) = -2.DO*GC(1,1)*(A3/A2)*RPH2*DSIN(X(5)) +
               1           2.DO*GC(1,2)*(A3/A2)*RPH2*DCOS(X(5))
C
C137            DO 120 PHIDI=1,356,DELPHI
C138            PHID=PHIDI - 1
C139            PHI=2*PHID
C140            I = PHID/DELPHI + 1
C141            CALL MOTION(PHI,A1,A2,A3,A4,PSI,PHIVEL,PHIACC,PSIV(I),PSIA(I),
```

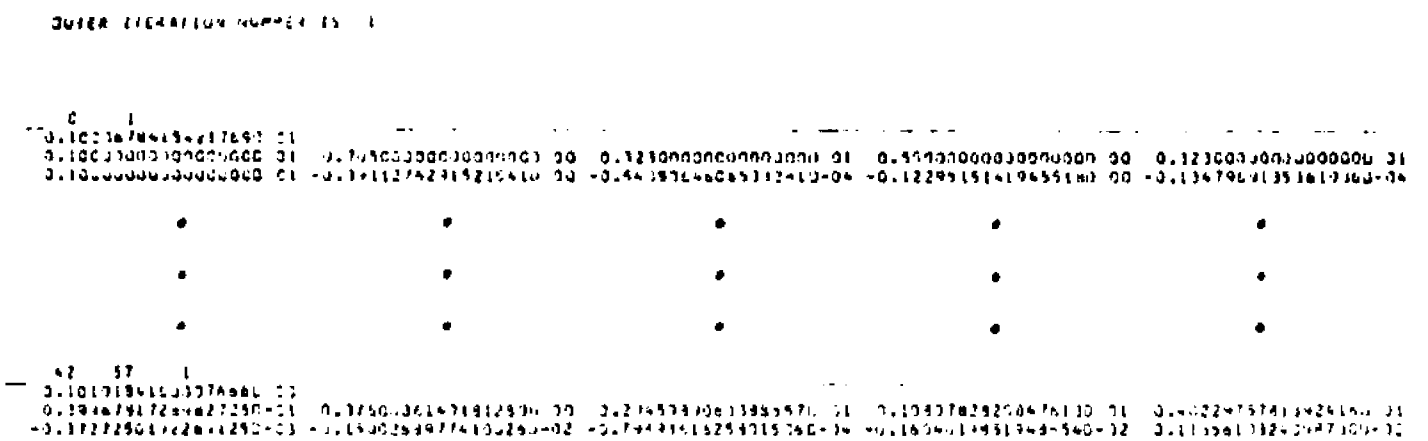
```
                     1            LAMBDA,LAMV(I),LAMA(I))
0142                 CALL KE(MAT(PHI,PHIVEL,PHIACC,PSI,PSIV(I),PSIA(I),AL,A2,A3,A4,
                     1            P1,Q1,P2,Q2,P3,Q3,X1(I),Y1(I),X2(I),Y2(I),X3(I),
                     2            Y3(I),VX1(I),VY1(I),VX2(I),VY2(I),VX3(I),VY3(I),
                     3            AX1(I),AY1(I),AX2(I),AY2(I),AX3(I),AY3(I))
0143                 CALL REACTS(PHI,PHIVEL,PHIACC,LAMBDA,LAMV(I),LAMA(I),PSI,PSIV(I),
                     1     PSIA(I),A1,A2,A3,A4,P1,Q1,P2,Q2,P3,Q3,K2,K3,M1,M2,M3,
                     2       F1X(I),F1Y(I),F21X(I),F21Y(I),F23X(I),F23Y(I),
                     3       F43X(I),F43Y(I),FSX(I),FSY(I),M41(I))
0144                 SHAKM(I) = 2.7500*F41Y(I) - 2.7500*F43Y(I) - M41(I)
0145                 F41(I)=DSQRT(F41X(I)*F41X(I) + F41Y(I)*F41Y(I))
0146                 F43(I) = DSQRT(F43X(I)*F43X(I) + F43Y(I)*F43Y(I))
0147                 F21(I)=DSQRT(F21X(I)*F21X(I) + F21Y(I)*F21Y(I))
0148                 F23(I)=DSQRT(F23X(I)*F23X(I) + F23Y(I)*F23Y(I))
               C     BEARING FORCE CONSTRAINTS MAY NOW BE SET
0149                 C(I+7)=X(I) - W1*((F41(I))**2 - (F410)**2)/(F410)**2)
0150                 C(I+79)=X(I) - W4*((F43(I))**2 - (F430)**2)/(F430)**2)
0151                 C(I+151)=X(I) - W2*((F21(I))**2 - (F210)**2)/(F210)**2)
0152                 C(I+223)=X(I) - W3*((F23(I))**2 - (F230)**2)/(F230)**2)
0153                 C(I+295)=X(I) - W5*((SHAKM(I))**2 - (SHAKM0)**2)/(SHAKM0)**2)
0154                 C(I+367)=X(I) - W6*((M41(I))**2 - (M410)**2)/(M410)**2)
               C
               C
               C     DEFINE PARTIAL DERIVATIVES DC(P,S). RZ=VARIABLE, S=CONSTRAINT
               C     NUMBER
               C
               C     DEFINE PARAMETERS NECESSARY TO FIND PARTIAL DERIVATIVES
               C
0155                 SP = DSIN(PHI)
0156                 CP = DCOS(PHI)
0157                 SS= DSIN(PSI)
0158                 CS= DCOS(PSI)
0159                 PSIV2 = (PSIV(I))**2
0160                 CC = PSIA(I)*SS + PSIV2*CS
0161                 DD = PSIA(I)*CS - PSIV2*SS
0162                 CL = DCOS(LAMBDA)
0163                 TAU1 = DSIN(PHI - LAMBDA)
0164                 TAU11 = DCOS(PHI - LAMBDA)
0165                 TAU3 = DSIN(PSI - LAMBDA)
0166                 PHIV2 = PHIVEL*PHIVEL
0167                 LAMV2 = (LAMV(I))**2
0168                 A2T3 = A2*TAU3
0169                 FF = (PSIA(I)*CL)/(A3*TAU3)
0170                 JJ = DD
0171                 KK = CC
0172                 SL = DSIN(LAMBDA)
0173                 MM = (PSIA(I)*SL)/(A3*TAU3)
0174                 PP = LAMA(I)/A2T3
0175                 QQ = LAMA(I)*SL + LAMV2*CL - A1*CS*PHIV2*TAU1/A2T3
0176                 RR = LAMA(I)*CL - LAMV2*SL + A1*CS*PHIV2*TAU11/A2T3
0177                 TT = LAMA(I)*CL - LAMV2*SL + A1*SS*PHIV2*TAU1/A2T3
0178                 UU = LAMA(I)*SL + LAMV2*CL + A1*SS*PHIV2*TAU11/A2T3
0179                 VV = A1*CS*PHIV2*TAU1/A2T3
0180                 WW = A1*CS*PHIV2*TAU11/A2T3
0181                 XX = A1*SS*PHIV2*TAU1/A2T3
0182                 YY = A1*SS*PHIV2*TAU11/A2T3
```

```
C     THE PARTIAL OF F43X WITH RESPECT TO X(2) IS
      PF43X2 = -300*RPH*DCOS(X(3))*CC/X(2)
     1         - 300*RPH*DSIN(X(3))*DD/X(2)
     2         + 600*RPH*FF
C     THE PARTIAL OF F43Y WITH RESPECT TO X(2) IS
      PF43Y2 = 300*RPH*DCOS(X(3))*JJ/X(2)
     1        -300*RPH*DSIN(X(3))*KK/X(2)
     2        + 600*RPH*MM
C  THE PARTIAL OF F43X AND F43Y WITH RESPECT TO X(3) ARE
      PF43X3 = RPH*DSIN(X(3))*CC - RPH*DCOS(X(3))*DD
      PF43Y3 = -RPH*DSIN(X(3))*JJ - RPH*DCOS(X(3))*KK
C  THE PARTIAL OF F43X W/R TO X(4) AND X(5) ARE
      PF43X4 = -300*(RPH2/X(4))*DCOS(X(5))*VV
     1         + 300*(RPH2/X(4))*DSIN(X(5))*WW
     2         + 600*(LAMA(I)*CS/A2T3)*RPH2
      PF43X5 = RPH2*DSIN(X(5))*VV + RPH2*DCOS(X(5))*WW
C  THE PARTIAL OF F43Y W/R TO X(4) AND X(5) ARE
      PF43Y4 = -300*(RPH2/X(4))*DCOS(X(5))*XX
     1         + 300*(RPH2/X(4))*DSIN(X(5))*YY
     2         + 600*(LAMA(I)*SS/A2T3)*RPH2
      PF43Y5 = RPH2*DSIN(X(5))*XX + RPH2*DCOS(X(5))*YY
C  THE PARTIAL OF F41 W/R TO X(4) AND X(5) ARE
      PF41X4 = -200*(RPH2/X(4)**2)*A1*PHIV2*CP
     1         - 300*(RPH2/X(4))*DCOS(X(5))*QQ
     2         - 300*(RPH2/X(4))*DSIN(X(5))*RR
     3         - 600*(LAMA(I)*CS/A2T3)*RPH2
      PF41X5 = RPH2*DSIN(X(5))*QQ - RPH2*DCOS(X(5))*RR
C THE PARTIALS OF F41Y W/R TO X(4) AND X(5) ARE
      PF41Y4 = -200*(RPH2/X(4)**2)*A1*PHIV2*SP
     1         + 300*(RPH2/X(4))*DCOS(X(5))*TT
     2         - 300*(RPH2/X(4))*DSIN(X(5))*UU
     3         - 600*(LAMA(I)*SS/A2T3)*RPH2
      PF41Y5 = -RPH2*DSIN(X(5))*TT - RPH2*DCOS(X(5))*UU
C THE PARTIALS OF F21X AND F21Y W/R TO X(4) AND X(5) ARE
      PF21X4 = -PF41X4
      PF21X5 = -PF41X5
      PF21Y4 = -PF41Y4
      PF21Y5 = -PF41Y5
C THE PARTIALS OF F23X AND F23Y W/R TO X(4) AND X(5) ARE
      PF23X4 = -PF43X4
      PF23X5 = -PF43X5
      PF23Y4 = -PF43Y4
      PF23Y5 = -PF43Y5
C THE PARTIALS OF M41 ARE FOUND FROM (W/R TO X(4) AND X(5))
      AAA = A1*(LAMA(I)*TAUII + LAMV2*TAUI)
     1     -A1*A1*DSIN(PHI-PS(I))*PHIV2*TAUI/A2T3
      BBB = A1*(LAMA(I)*TAUI - LAMV2*TAUII)
     1     +A1*A1*DSIN(PHI-PS(I))*PHIV2*TAUII/A2T3
      PM414 = 300*(RPH2/X(4))*DCOS(X(5))*AAA
     1        + 300*(RPH2/X(4))*DSIN(X(5))*BBB
     2        + 600*(LAMA(I)*A1*DSIN(PHI-PS(I))/A2T3)*RPH2
      PM415 = -RPH2*DSIN(X(5))*AAA + RPH2*DCOS(X(5))*BBB
C  THE PARTIALS OF THE SHAKING MOMENT W/R TO X(4)  AND X(5) ARE
      PSHKM4 = 2.7500*PF41Y4 - 2.7500*PF43Y4 - PM414
      PSHKM5 = 2.7500*PF41Y5 - 2.7500*PF43Y5 - PM415
C   THE PARTIAL OF F41X WITH RESPECT TO X(2) IS
```

```
0209            PF41X2 = -6D0*RPH*FF
                C   THE PARTIAL OF F41Y WITH RESPECT TO X(2) IS
0210            PF41Y2 = -6D0*RPH*MM
                C   THE PARTIAL OF F21X WITH RESPECT TO X(2) IS
0211            PF21X2 = -PF41X2
                C   THE PARTIAL OF F21Y WITH RESPECT TO X(2) IS
0212            PF21Y2 = -PF41Y2
                C   THE PARTIAL OF F23X WITH RESPECT TO X(2) IS
0213            PF23X2 = PF41X2
                C   THE PARTIAL OF F23Y WITH RESPECT TO X(2) IS
0214            PF23Y2 = PF41Y2
                C
                C  THE PARTIALS OF F41X,F41Y,F21X,F21Y,F23X,F23Y AND M41
                C  WITH RESPECT TO X(3) ARE ZERO.  THEREFORE, GC(3,5)-GC(3,9),
                C  GC(3,15)-GC(3,19),GC(3,20)-GC(3,24) AND GC(3,30)-GC(3,34)
                C  ARE SET TO ZERO IN MAIN PROGRAM
                C THE PARTIAL OF M41 WITH RESPECT TO X(2) IS
0215            PM412 = 6D0*RPH*FF*A1*TAU1/CL
                C  THE PARTIALS OF THE SHAKING MOMENT W/R TO X(2) AND X(3) ARE
0216            PSHKM2 = 2.7500*PF41Y2 - 2.7500*PF43Y2 - PM412
0217            PSHKM3 =-2.7500*PF43Y3
                C   NOW THE GRADIENTS MAY BE FOUND
0218            GC(2,I+7)= -200*W1*(F41X(I)*PF41X2 + F41Y(I)*PF41Y2)/(F410)**2
0219            GC(4,I+7)= -200*W1*(F41X(I)*PF41X4 + F41Y(I)*PF41Y4)/(F410)**2
0220            GC(5,I+7)= -200*W1*(F41X(I)*PF41X5 + F41Y(I)*PF41Y5)/(F410)**2
0221            GC(2,I+23)= -200*W4*(F43X(I)*PF43X2 + F43Y(I)*PF43Y2)/(F430)**2
0222            GC(3,I+23)= -200*W4*(F43X(I)*PF43X3 + F43Y(I)*PF43Y3)/(F430)**2
0223            GC(4,I+23)= -200*W4*(F43X(I)*PF43X4 + F43Y(I)*PF43Y4)/(F430)**2
0224            GC(5,I+23)= -200*W4*(F43X(I)*PF43X5 + F43Y(I)*PF43Y5)/(F430)**2
                C
0225            GC(2,I+15)= -200*W2*(F21X(I)*PF21X2 + F21Y(I)*PF21Y2)/(F210)**2
0226            GC(4,I+15)= -200*W2*(F21X(I)*PF21X4 + F21Y(I)*PF21Y4)/(F210)**2
0227            GC(5,I+15)= -200*W2*(F21X(I)*PF21X5 + F21Y(I)*PF21Y5)/(F210)**2
0228            GC(2,I+22)= -200*W3*(F23X(I)*PF23X2 + F23Y(I)*PF23Y2)/(F230)**2
0229            GC(4,I+22)= -200*W3*(F23X(I)*PF23X4 + F23Y(I)*PF23Y4)/(F230)**2
0230            GC(5,I+22)= -200*W3*(F23X(I)*PF23X5 + F23Y(I)*PF23Y5)/(F230)**2
0231            GC(2,I+29)= -200*W5*(SHAKM(I)*PSHKM2)/(SHAKM0)**2
0232            GC(3,I+29)= -200*W5*(SHAKM(I)*PSHKM3)/(SHAKM0)**2
0233            GC(4,I+29)= -200*W5*(SHAKM(I)*PSHKM4)/(SHAKM0)**2
0234            GC(5,I+29)= -200*W5*(SHAKM(I)*PSHKM5)/(SHAKM0)**2
0235            GC(2,I+36)= -200*W6*(M41(I)*PM412)/(M410)**2
0236            GC(4,I+36)= -200*W6*(M41(I)*PM414)/(M410)**2
0237            GC(5,I+36)= -200*W6*(M41(I)*PM415)/(M410)**2
0238        120 CONTINUE
0239            RETURN
0240            END
```

3. Essential Elements Of Program Output

The output reproduced below shows the first inner iteration of outer iteration number 1 as well as the two inner iterations of the last outer iteration number 5. (For format details, see Section 3d of Appendix C.)



Again, an inner iteration consists of a set of four lines. The first number in the first line stands for the total number of inner iterations within the given outer iteration. The second number in this line stands for the number of times

the subroutine VF01ZD has been called upon in the specific outer iteration.

The number in the second line represents the current value of the composite function Φ according to equation (C.1).

The third line gives the current values of z, R_3^*, θ_3^*, R_2^*, and θ_2^*. In the first iteration, the initial values are reproduced.

The fourth line shows the derivatives of the composite function with respect to the five variables.

The last group of inner iterations shown represents the final results, i.e.

$$z = 0.178 \quad lb^2 \tag{G.1}$$

$$R_3^* = 0.42 \quad in \tag{G.2}$$

$$\theta_3^* = 2.984 \text{ rad} = 170.95 \text{ deg} \tag{G.3}$$

$$R_2^* = 1.07 \quad in \tag{G.4}$$

$$\theta_2^* = 4.032 \text{ rad} = 230.99 \text{ deg} \tag{G.5}$$

The computational sequence given in Section V-D-1 is then followed to determine the final input link counterweight

parameters (see equations (4.70), (5.19) - (5.26)):

$$R_1^* = 1.65 \quad \text{in} \tag{G.6}$$

$$\theta_1^* = 3.012 \text{ rad} = 172.57 \text{ deg} \tag{G.7}$$

APPENDIX H

COMPUTER PROGRAM OPTM41

The present appendix shows how program OPTIMIZE is modified for the exclusive optimization of the maximum input moment M_{41} of a fully force balanced four-bar linkage. To this end, the constraints on the forces F_{41}, F_{12}, F_{23}, F_{34} as well as on the moment $M_{M/G}$ are removed. Thus, the number of variables remains the same as in program OPTIMIZE while the number of constraints is reduced from 439 to 79.

Only the modifications to the main program and subroutine VF01BD are detailed below. These are followed by the essential elements of the program output.

1. Main Program

The main program, which is listed at the end of this section, begins with COMMON statements which define the array size for the constraints C and their derivatives GC. The remainder of the program is identical to that given in Appendix G except for the following modifications:

M = 79; stands for the number of constraints. (See comment cards at beginning of program for definition of constraints. Note that only the input moment appears.)

GC(1,I) = 1 and GC(3,I) = 0 for I = 8,9,...,79. These changes are a result of the fewer number of constraints.

MAIN

```
C   THIS PROGRAM OPTIMIZES THE MAXIMUM INPUT MOMENT FOR A PRESCRIBED
C   MAXIMUM SHARING FORCE FOR A FIVE-BAR LINKAGE.
C   CIRCULAR COUNTERWEIGHTS, TANGENT TO PIVOT POINTS, ARE USED ON
C   ALL MOVING LINKS.  THE INPUT MOMENT IS EVALUATED AT FIVE DEGREE
C   INCREMENTS OF THE INPUT LINK ANGLE.
C
C   VARIABLES ARE X(1)=Z, X(2)=R3STAR*OUTPUT COUNTERWEIGHT
C   RADIUS, X(3)=THETA3STAR=OUTPUT CWT. ANGLE, X(4)=R2STAR=
C   COUPLER COUNTERWEIGHT RADIUS, X(5)=THETA2STAR=COUPLER CWT.
C   ANGLE.
C
C   THE CONSTRAINTS ARE
C   C(1)  = EQUIPOLLENT CIRCLE CONSTRAINT EQUATION
C   C(2) = X(2) (RADIUS R3STAR GREATER THAN ZERO)
C   C(3) = X(3) (ANGLE THETA3STAR GREATER THAN ZERO)
C   C(4) = 2*PI - X(3) (ANGLE THETA3STAR LESS THAN 2*PI)
C   C(5) = X(4) (RADIUS R2STAR GREATER THAN ZERO)
C   C(6) = X(5) (ANGLE THETA2STAR GREATER THAN ZERO)
C   C(7) = 2*PI - X(5) (ANGLE THETA2STAR LESS THAN 2*PI)
C   C(8)-C(79)=  INPUT MOMENT CONSTRAINTS
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/VFDIED/C(237)
      COMMON/VFDIFD/GC(5,79)
      REAL*8 X(5),EPS(5)
      PI=3.14159300
C   THE TOTAL NUMBER OF VARIABLES IS DESIGNATED BY N
      N=5
C   THE TOTAL NUMBER OF CONSTRAINTS IS DESIGNATED BY M
      M=79
C   THE TOTAL NUMBER OF EQUALITY CONSTRAINTS IS DESIGNATED BY K
      K=1
C   THE STARTING POINT IS ESTIMATED TO BE
      X(1)=.00
      X(2)=.70500
      X(3)=3.23000
      X(4)=.55900
      X(5)=3.23000
      ARMI(1)=-4
      EPS(1)=10-4
      EPS(2)=.7050-4
      EPS(3)=3.2300-4
      EPS(4)=.5590-4
      EPS(5)=3.2300-4
C
      DFN=0.500
      MAXFN=300
      IPR1=1
      IPR2=1
C   SET IW=10000.  IF MORE STORAGE IS REQUIRED, VF01AD WILL
C   STOP AND PRINT OUT NEW REQUIREMENT
      IW=10000
      MODE=1
C   THE CONSTRAINT SCALE FACTORS ARE SET AS FOLLOWS
      C(1)=10-6
      C(2)=.70500
      C(3)=3.23000
```

```
      C141=3.100
      C[illegible]
      C[illegible]
      C171=3.100
      DO 10 I=9,79
   10 C(I)=.500
C     THE DERIVATIVES OF LINEAR CONSTRAINTS ARE
      GC(1,1)=000
      GC(1,2)=000
      GC(2,2)=100
      GC(3,2)=000
      GC(4,2)=000
      GC(5,2)=000
      GC(1,3)=000
      GC(2,3)=000
      GC(3,3)=100
      GC(4,3)=000
      GC(5,3)=000
      GC(1,4)=000
      GC(2,4)=000
      GC(3,4)=-1.0
      GC(4,4)=000
      GC(5,4)=000
      GC(1,5)=000
      GC(2,5)=000
      GC(3,5)=000
      GC(4,5)=100
      GC(5,5)=000
      GC(1,6)=000
      GC(2,6)=000
      GC(3,6)=000
      GC(4,6)=000
      GC(5,6)=100
      GC(1,7)=000
      GC(2,7)=000
      GC(3,7)=000
      GC(4,7)=000
      GC(5,7)=-100
      DO 11 I=9,79
   11 GC(1,I)=100
      DO 12 I=3,79
   12 GC(3,I)=000
      CALL VFOLAJIN,M,K,X,EPS,AKMIN,DFN,MAXFN,IP1,IP2,W,MODE)
      STOP
      END
```

2. Subroutine VF01BD (See listing at end of this section.)

After redefining the COMMON statements, the following modifications are made to the subroutine detailed in Appendix G:

a. ETA = 0; defines the reduction factor η associated with full force balance.

b. W6 = 1; only the weighting factor associated with the constraint on the input moment M_{41} need be defined.

c. $(M_{41}{}^{o})_{MAX}$ = 38.72; the maximum bearing reactions and shaking moment are not needed.

d. SHAKM(I), $F_{41}(I)$, $F_{43}(I)$, $F_{21}(I)$ and $F_{23}(I)$ are not required.

e. $C(I + 7) = X(1) - W6\left[\dfrac{(M_{41})^2 - (M_{41}{}^{o})^2}{(M_{41}{}^{o})^2}\right]$, $I = 1,2,\ldots,72.$

The input moment constraints are designated as C(8) - C(79). All higher numbered constraints are eliminated.

f. All partial derivatives which do not involve M_{41} are eliminated. This leaves

$$PM41(K) = \frac{\partial M_{41}}{\partial X_K} \quad , \quad K = 2,4,5$$

g. Finally, the derivatives of the constraints dealing with the input moment are defined as:

$$GC(K,I+7) = - \frac{2W6}{(M_{41}{}^{o})^2} M_{41} \frac{\partial M_{41}}{\partial X_K} \quad , \quad K = 2,4,5$$

All other derivatives are deleted.

```
      SUBROUTINE VF01DD(N,M,X)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 X(5)
      INTEGER PHID, PHID1, DELPHI
      REAL*8 LAMD,K2,K3,M1,M2,M3,LAMBDA,MU,M1ZERO
      REAL*8 M2ZERO,M2STAR,M3ZERO,M3STAR,M1STAR,K2ZERO,K3ZERO
      REAL*8 M41(73),M41D,LAMV(73),LAMA(73),LAMD(73),LAMV2,I2,I3
      DIMENSION F41X(73),F41Y(73),F21(73),F21X(73),F23X(73),
     1F23Y(73),F43X(73),F43Y(73),FSX(73),FSY(73)
      DIMENSION F43(73),F41(73),F21(73),F23(73)
      DIMENSION PSID(73),PSIV(73),PSIA(73)
      DIMENSION X1(73),Y1(73),X2(73),Y2(73),X3(73),Y3(73)
      DIMENSION VX1(73),VY1(73),VX2(73),VY2(73),VX3(73),VY3(73)
      DIMENSION AX1(73),AY1(73),AX2(73),AY2(73),AX3(73),AY3(73)
      DOUBLE PRECISION DCOS,DSIN,DSQRT,DATAN2
      COMMON/VF01CD/F
      COMMON/VF01DD/G(10)
      COMMON/VF01ED/C(237)
      COMMON/VF01FD/CC(5,7))
      PI=3.14159300
C     THE OBJECTIVE FUNCTION IS
      F=X(1)
C     G(I)=PARTIAL DERIVATIVES OF OBJECTIVE FUNCTION (F(X(I))) WITH
C          RESPECT TO VARIABLES X(1),X(2),X(3),X(4) AND X(5)
      G(1)=100
      G(2)=000
      G(3)=000
      G(4)=000
      G(5)=000
C     THE LINK DIMENSIONS ARE
      A1=200
      A2=600
      A3=300
      A4=5.500
C     THE COUNTERWEIGHT DENSITIES AND THICKNESS FOR LINKS 1, 2 AND 3 ARE
      RHO1ST=7.3316060-4
      RHO2ST=7.3316060-4
      RHO3SY=7.3316060-4
      H1STAR=.62500
      H2STAR=.62500
      H3STAR=.62500
C     THE MAXIMUM SHAKING FORCE FOR THE UNBALANCED LINKAGE IS
      RQMAX = .2500376002
C     THE RADIUS AND COORDINATES OF THE CENTER OF THE SMALLEST
C     CIRCUMSCRIBING CIRCLE TO THE ROTATED SHAKING FORCE HODOGRAPH ARE
      RS = .1079443002
      CSX = .4987464001
      CSY = -.3466060001
C     THE MAGNITUDE OF VECTOR OCS IS
      OCS = DSQRT(CSX**2 + CSY**2)
C     AT ANGLE
      DELTAS = DATAN2(CSY,CSX)
C     THE REDUCTION FACTOR IS
      ETA = .000
C     THE PROPORTIONALITY FACTOR IS GIVEN BY
      MU = ETA*RQMAX/RS
```

```
C      THE ANGULAR VELOCITY OF THE INPUT LINK IS
0044          RPM=0.503
0045          Z=PI/1.0002
0046          PHIVEL=RPM*Z*6.0
C      THE ANGULAR ACCELERATION OF THE INPUT LINK IS
0047          PHIACC=.000
C      DELPHI=INCREMENT OF INPUT ANGLE PHI
0048          DELPHI=5
C SET WEIGHTING FACTOR FOR CONSTRAINTS
0049          W6= 100
C      THE MAXIMUM VALUE OF INPUT MOMENT FOR UNBALANCED LINKAGE IS
0050          M10=.3972007002
C
C      DEFINE INPUT TO SUBROUTINE USED TO DETERMINE MECHANISM REACTIONS
C      P1,Q1=X AND Y COMPONENTS OF CENTER OF MASS OF INPUT LINK
C      P2,Q2=X AND Y COMPONENTS OF CENTER OF MASS OF COUPLER LINK
C      P3,Q3=X AND Y COMPONENTS OF CENTER OF MASS OF OUTPUT LINK
C      M1,M2,M3=MASS OF INPUT,COUPLER AND OUTPUT LINK, RESPECTIVELY
C      K2,K3=RADII OF GYRATION OF COUPLER AND OUTPUT LINKS
C
C      THE PARAMETERS OF THE UNBALANCED LINKAGE ARE
0051          M1ZERO=5.17620-4
0052          M2ZERO=13.67100-4
0053          M3ZERO=6.79740-4
0054          P10 = 100
0055          Q10 = 000
0056          P20=300
0057          Q20=.40100
0058          P30=1.500
0059          Q30 = 000
0060          K2ZERO = 2.09700
0061          K3ZERO= 1.674700
0062          PE10 = M1ZERO*P10 + M2ZERO*A1 - A1*M2ZERO*P20/A2
0063          QE10 = M1ZERO*Q10 - A1*M2ZERO*Q20/A2
0064          P3ZERO = M3ZERO*P30 + A3*M2ZERO*P20/A2
0065          Q3ZERO = M3ZERO*Q30 + A3*M2ZERO*Q20/A2
0066          AZE10 = DSQRT(PE10**2 + QE10**2)
0067          ALPHA0 = DATAN2(QE10,PE10)
0068          BZERO = DSQRT(P3ZERO**2 + Q3ZERO**2)
0069          BETA0 = DATAN2(Q3ZERO,P3ZERO)
C
C      THE COUPLER COUNTERWEIGHT PARAMETERS ARE
C
0070          M2STAR = RHO2ST*PI*R2STAR*(X(4)**2)
0071          P2STAR = X(4)*DCOS(A(5))
0072          Q2STAR = X(4)*DSIN(X(5))
C
C      THE COUPLER LINK PARAMETERS ARE
0073          M2 = M2ZERO + M2STAR
0074          P2 = (M2ZERO*P20 + M2STAR*P2STAR)/M2
0075          Q2 = (M2ZERO*Q20 + M2STAR*Q2STAR)/M2
C      THE RADIUS OF GYRATION OF LINK 2 IS
0076          K2 = DSQRT((M2ZERO*(P20**2 + Q20**2 + K2ZERO**2) +
     1                 1.900*M2STAR*(X(4)**2))/M2 - (P2**2 + Q2**2))
C
C      THE OUTPUT LINK COUNTERWEIGHT PARAMETERS ARE
```

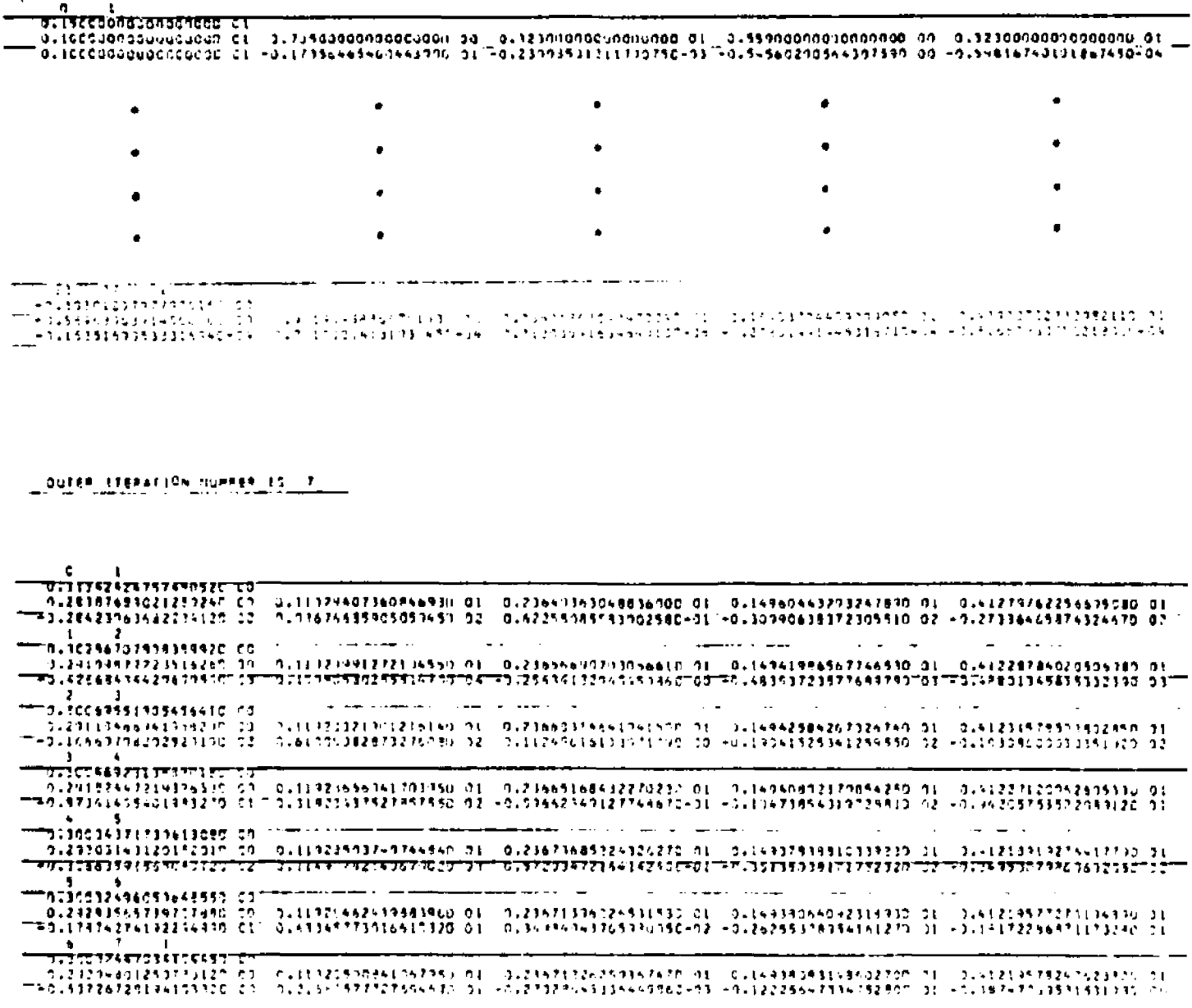
```
0077                M3STAR=(1.0D3ST*PI*M3STAR*(X(2)**2)
0078                P3STAR=X(2)*DCOS(X(3))
0079                Q3STAR=(2)*DSIN(X(3))
                C
                C       THE OUTPUT LINK PARAMETERS ARE THEN GIVEN BY
0080                M3= M3ZERO + M3STAR
0081                P3 = (M3ZERO)*P30 + M3STAR*P3STAR)/M3
0082                Q3 = (M3ZERO*Q30 + M3STAR*Q3STAR)/M3
                C
                C       THE RADIUS OF GYRATION OF THE OUTPUT LINK IS
0083                K3 = DSQRT((M3ZERO*(P30**2 + Q30**2 + K3ZERO**2)
                1           + 1.5D0*M3STAR*(X(2)**2)))/M3 - (P3)**2 +Q3**2))
                C
                C       THE PARAMETERS OF THE INPUT CWT. MUST NOW BE FOUND WHICH WILL
                C       RESULT IN THE PRESCRIBED MAXIMUM SHAKING FORCE.
                C
                C       THE ANGLE BETA IS FOUND BY WAY OF
0084                PE3 = M3*P3 + (A3/A2)*M2*P2
0085                QU3 = M3*Q3 + (A3/A2)*M2*Q2
0086                BETA = DATAN2(QU3,PE3)
                C       THE ANGLE OF ROTATION PST MAY NOW BE FOUND AS
0087                ROTPSI = BETA - BETA0
                C       THE COMPONENTS OF VECTOR OOCS ARE
0088                OOCSX = OCS*DCOS(DELTAS+AZERO*(PHIVEL**2)*DCOS(ALPHA0)
0089                OOCSY = OCS*DSIN((DELTAS)-AZERO*(PHIVEL**2)*DSIN(ALPHA0)
                C       THE MAGNITUDE OF OOCS MAY NOW BE FOUND AS
0090                OOCS = DSQRT(OOCSX**2 + OOCSY**2)
                C       AT ANGLE
0091                ZETA = DATAN2(OOCSY,OOCSX)
                C       THE COMPONENTS OF VECTOR OC2 MAY NOW BE FOUND
0092                OC2X = AO*OOCS*DCOS(ZETA + ROTPSI)
0093                OC2Y = AO*OOCS*DSIN(ZETA + ROTPSI)
                C       THE MAGNITUDE OF OC2 IS
0094                OC2 = DSQRT(OC2X**2 + OC2Y**2)
                C       DUE TO THE ADDITION OF A COUPLER COUNTERWEIGHT, THE TERM AZERO
                C       BECOMES A1, WHICH IS FOUND BY WAY OF
0095                PE1 = M1ZERO*P10 + M2*A1 - (A1/A2)*M2*P2
0096                QU1 = M1ZERO*Q10 - (A1/A2)*M2*Q2
0097                A = DSQRT(PE1**2 + QU1**2)
                C       THE ANGLE ALPHA0 BECOMES ALPHA, WHERE
0098                ALPHA = DATAN2(QU1,PE1)
                C
                C       THE RESULTING LOCATION OF THE ROTATED CENTER OF THE CIRCUMSCRIBING
                C       CIRCLE IS THEN GIVEN AS OC2 AT ANGLE DELTA2, WHERE
0099                CC2X =    A*(PHIVEL**2)*DCOS(ALPHA) +
                1           OC2*DCOS(ZETA + ROTPSI)
0100                CC2Y =    A*(PHIVEL**2)*DSIN(ALPHA) +
                1           OC2*DSIN(ZETA + ROTPSI)
0101                CC2 = DSQRT(OC2X**2 + OC2Y**2)
0102                DELTA2 = DATAN2(OC2Y,OC2X)
                C       THE INPUT LINK CWT. RADIUS IS GIVEN BY
0103                RISTAR = (OC2/((3.0D0*ISTAR*PI*(PHIVEL**2)))**(1.0D0/3.0D0)
                C       AT ANGLE
0104                THIST = DELTA2 + PI
                C
                C       THE INPUT CWT. PARAMETERS ARE
```

```
0105            MISTAR = RHOIST*HISTAR*PI*(RISTAR**2)
0106            PISTAR = RISTAR*DCOS(THIST)
0107            QISTAR = RISTAR*DSIN(THIST)
                C
                C     THE REQUIRED INPUT LINK PARAMETERS ARE
0108            MI= MIZERO + MISTAR
0109            PI = (MIZERO*PIO + MISTAR*PISTAR)/MI
0110            QI = (MIZERO*QIO + MISTAR*QISTAR)/MI
                C     CONSTRAINTS C(1) TO C(7) ARE NOW SET ALONG WITH THE
                C     DERIVATIVES OF C(1) WITH RESPECT TO VARIABLES X(2) TO X(5)
                C
                C     C(1)=EQUALITY CONSTRAINT (GENERAL EQUIPOLLENT CIRCLE CONSTR. EQ.)
0111            RPH2=RHO2ST*PI*HIS*AR*(X(4)**3)
0112            RPR=RHOJST*PI*HISTAR*(X(2)**3)
0113            C(1)=(RPH*DCOS(X(3)) + (A3/A2)*RPH2*DCOS(X(5)) + P3ZERO)**2
                1   + (RPH*DSIN(X(3)) + (A3/A2)*RPH2*DSIN(X(5)) + Q3ZERO)**2
                2   - (MU*B3ZERO)**2
                C     C(2)=CONSTRAINT ON OUTPUT COUNTERWEIGHT RADIUS
0114            C(2)= X(2)
                C     C(3),C(4)=CONSTRAINTS ON OUTPUT COUNTERWEIGHT ANGLE
0115            C(3)= X(3)
0116            C(4)= 2.0D0*PI - X(3)
                C     C(5)=CONSTRAINT ON COUPLER COUNTERWEIGHT RADIUS
0117            C(5)= X(4)
                C     C(6),C(7)=CONSTRAINTS ON COUPLER COUNTERWEIGHT ANGLE
0118            C(6)=X(5)
0119            C(7)=2.0D0*PI - X(5)
                C     GG(R,S)=PARTIAL OF C(S) WITH RESPECT TO VARIABLE R
0120            GG(1,1)=RPH*DCOS(X(3)) + (A3/A2)*RPH2*DCOS(X(5)) + P3ZERO
0121            GG(1,2)=RPH*DSIN(X(3)) + (A3/A2)*RPH2*DSIN(X(5)) + Q3ZERO
0122            GG(2,1) = 2.0D0*GG(1,1)*RPH*DCOS(X(3))/X(2) +
                1           2.0D0*GG(1,2)*RPH*DSIN(X(3))/X(2)
0123            GG(3,1) = -2.0D0*GG(1,1)*RPH*DSIN(X(3)) +
                1           2.0D0*GG(1,2)*RPH*DCOS(X(3))
0124            GG(4,1) = 6.0D0*GG(1,1)*(A3/A2)*(RPH2*DCOS(X(5)))/X(4) +
                1           6.0D0*GG(1,2)*(A3/A2)*(RPH2*DSIN(X(5)))/X(4)
0125            GG(5,1) = -2.0D0*GG(1,1)*(A3/A2)*RPH2*DSIN(X(5)) +
                1           2.0D0*GG(1,2)*(A3/A2)*RPH2*DCOS(X(5))
                C
0126            DO 120 PHID=1.D0,359.D0,DELPHI
0127            PHID=PHID1 - 1
0128            PHI=2*PHID
0129            I = PHID/DELPHI + 1
0130            CALL MOTION(PHI,A1,A2,A3,A4,PSI,PHIVEL,PHIACC,PSIV(I),PSIA(I),
                1           LAMDA,LAMV(I),LAMA(I))
0131            CALL KINMAT(PHI,PHIVEL,PHIACC,PSI,PSIV(I),PSIA(I),A1,A2,A3,A4,
                1           P1,Q1,P2,Q2,P3,Q3,X1(I),Y1(I),X2(I),Y2(I),X3(I),
                2           Y3(I),VX1(I),VY1(I),AX2(I),AY2(I),AX3(I),AY3(I),
                3           AX1(I),AY1(I),AX2(I),AY2(I),AX3(I),AY3(I))
0132            CALL RESULTS(PHI,PHIVEL,PHIACC,LAMDA,LAMV(I),LAMA(I),PSI,PSIV(I),
                1       PSIA(I),A1,A2,A3,A4,P1,Q1,P2,Q2,P3,Q3,X2,X3,M1,M2,M3,
                2       F41X(I),F41Y(I),F21X(I),F21Y(I),F23X(I),F23Y(I),
                3       F43X(I),F43Y(I),FSX(I),FSY(I),M41(I))
                C     INPUT MOMENT CONSTRAINTS MAY NOW BE SET
0133            C(7)=X(1) - MU*((M41(I))**2 - (M41O)**2)/(M41O)**2)
                C
```

```
C
C     DEFINE PARTIAL DERIVATIVES DG(R,S), R=VARIANCE, S=CONSTRAINT
C     NUMBER
C
C     DEFINE PARAMETERS NECESSARY TO FIND PARTIAL DERIVATIVES
      CL = DCOS(LAMBDA)
      TAU1 = DSIN(PHI - LAMBDA)
      TAU11 = DCOS(PHI - LAMBDA)
      TAU3 = DSIN(PSI - LAMBDA)
      PHIV2 = PHIVEL*PHIVEL
      LAMV2 = (LAMV(1))**2
      A2T3 = A2*TAU3
      FF = (PSIA(1)*CL1/(A3*TAU3)
C THE PARTIAL OF M41 WITH RESPECT TO X(2) IS
      PM412 = 600*RPH*FF*A1*TAU1/CL
C THE PARTIALS OF M41 WITH RESPECT TO X(4) AND X(5) ARE FOUND FROM
      AAA = A1*(LAMA(1)*TAU11 + LAMV2*TAU1)
     1    -A1*A1*DSIN(PHI-PSI)*PHIV2*TAU1/A2T3
      BBB = A1*(LAMA(1)*TAU1 - LAMV2*TAU11)
     1    +A1*A1*DSIN(PHI-PSI)*PHIV2*TAU11/A2T3
      PM414 = 300*(RPH2/X(4))*DCOS(X(5))*AAA
     1      + 300*(RPH2/X(4))*DSIN(X(5))*BBB
     2      + 600*(LAMA(1)*A1*DSIN(PHI-PSI)/A2T3)*RPH2
      PM415 = -RPH2*DSIN(X(5))*AAA + RPH2*DCOS(X(5))*BBB
C     THE GRADIENTS MAY NOW BE FOUND
      GG(2,I+7)= -200*W6*(M41(1))*PM412/(M41(1)**2
      GG(4,I+7)= -200*W6*(M41(1))*PM414/(M41(1)**2
      GG(5,I+7)= -200*W6*(M41(1))*PM415)/(M41(1)**2
  120 CONTINUE
      RETURN
      END
```

3. Essential Elements Of Program Output

The output reproduced below shows the first inner iteration of outer iteration number 1 as well as the seven inner iterations of outer iteration number 7. The format is identical to that described in Section 3 of Appendix G.



The last group of inner iterations shown represents the final results, i.e.

$$z = 0.293 \quad lb^2 \tag{H.1}$$

$$R_3^* = 1.19 \quad in \tag{H.2}$$

$$\theta_3^* = 2.367 \text{ rad} = 135.63 \text{ deg} \tag{H.3}$$

$$R_2^* = 1.49 \quad in \tag{H 4}$$

$$\theta_2^* = 4.122 \text{ rad} = 236.17 \text{ deg} \tag{H.5}$$

Again, the corresponding final input link counterweight parameters are found with the help of equations (4.70) and (5.19)-(5.26):

$$R_1^* = 1.86 \quad in \tag{H.6}$$

$$\theta_1^* = 3.266 \text{ rad} = 187.10 \text{ deg} \tag{H.7}$$

APPENDIX I

KINEMATIC EQUATIONS OF A FOUR-BAR LINKAGE

The present appendix reproduces a collection of kinematic equations for the four-bar linkage shown in Figure 2.1 [4] .

1. Link Angles

$$\phi_3 = 2\tan^{-1}\left(\frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C}\right) \tag{I.1}$$

where

$$\left.\begin{aligned} A &= \sin\phi_1 \\ B &= \cos\phi_1 - \frac{\nu}{\lambda} \\ C &= \frac{\lambda^2 + \mu^2 + \nu^2 - 1}{2\mu\lambda} - \frac{\nu}{\mu}\cos\phi_1 \\ \lambda &= \frac{a_1}{a_2} \\ \mu &= \frac{a_3}{a_2} \\ \nu &= \frac{a_4}{a_2} \end{aligned}\right\} \tag{I.2}$$

$$\phi_2 = \tan^{-1}\left(\frac{\lambda\sin\phi_1 - \mu\sin\phi_3}{\lambda\cos\phi_1 - \mu\cos\phi_3 - \nu}\right) \tag{I.3}$$

2. Link Angular Velocities

$$\dot{\phi}_2 = \frac{\lambda}{r_3} \sin(\phi_1 - \phi_3)\, \dot{\phi}_1 \tag{I.4}$$

$$\dot{\phi}_3 = \frac{\lambda r_1}{\mu r_3}\, \dot{\phi}_1 \tag{I.5}$$

where
$$\left.\begin{aligned} r_1 &= \mu \sin(\phi_1 - \phi_3) + \nu \sin\phi_1 \\ r_3 &= \lambda \sin(\phi_1 - \phi_3) + \nu \sin\phi_3 \end{aligned}\right\} \tag{I.6}$$

3. Link Angular Accelerations

$$\ddot{\phi}_2 = \left(\frac{\dot{\phi}_2}{\dot{\phi}_1}\right)\ddot{\phi}_1 + \frac{\nu\lambda}{r_3^2}\left[\cos(\phi_1 - \phi_3)\dot{\phi}_1 \sin\phi_3 - \dot{\phi}_3 \sin\phi_1\right]\dot{\phi}_1 \tag{I.7}$$

$$\ddot{\phi}_3 = \left(\frac{\dot{\phi}_3}{\dot{\phi}_1}\right)\ddot{\phi}_1 + \frac{\lambda}{r_3}\cos(\phi_1 - \phi_3)(\dot{\phi}_1 - \dot{\phi}_3)^2 + \frac{\nu}{\mu r_3}(\lambda \dot{\phi}_1^{\,2}\cos\phi_1 - \mu \dot{\phi}_3^{\,2}\cos\phi_3) \tag{I.8}$$

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