Moduli Spaces of Compact Riemann Surfaces Admitting Automorphisms

by

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Abstract

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By focussing on hyperelliptic surfaces, a geometric picture of a large subvariety of the moduli space of compact Riemann surfaces in arbitrary genus $g \geq 2$ is obtained. Since the hyperelliptic involution is central in any automorphism group containing it, and since the quotient modulo the action of the group generated by the hyperelliptic involution is a sphere, any automorphism group of a hyperelliptic surface which contains the hyperelliptic involution is a central extension of Z_2 by one of the finite automorphism groups of the Riemann sphere, namely, Z_n, D_{2n}, A_4, S_4, or A_5. There are eight infinite families of such groups, plus another eight individual groups. The possible branching data for the actions of these groups are determined from a knowledge of the orbits of the corresponding spherical automorphism group on the sphere. The actions of these groups are classified, up to topological equivalence, using finite group theory, and some of the theory of Fuchsian groups. A geometric picture of the moduli space of hyperelliptic Riemann surfaces is obtained, using the subgroup lattices of the groups. The full picture is obtained in genus 3. A generalization of the notion of hyperelliptic surface leads to clear lines of future research.

To Ravi S. Kulkarni, who continues to be a source of fascinating ideas

and

To Luciana

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Introduction

The set of conformal equivalence classes of compact Riemann surfaces of genus $g \geq 2$ is a quasi-projective normal complex algebraic variety of complex dimension $3g-3$ called the moduli space (M_g). The singular set $S_g \subseteq M_g$ consists of those surface classes which admit non-trivial automorphism groups. In this thesis we obtain a complete geometric understanding of a large (dimension $2g-1$) subvariety $H_g \subseteq S_g$, called the moduli space of hyperelliptic Riemann surfaces. In particular, we show, for arbitrary $g \geq 2$, how to decompose H_g into a nested union of connected subvarieties of successively smaller dimension, each labelled with a finite group action. The group action corresponding to a particular subvariety is the largest action contained in the full automorphism group of every surface class in the subvariety; and is the full automorphism group of at least one point (surface class) in the subvariety. Every group which acts as the full automorphism group of a hyperelliptic Riemann surface of genus g occurs as the label of one of the subvarieties.

The full automorphism group of a hyperelliptic Riemann surface contains a unique central element of order 2, called the hyperelliptic involution, which has the maximum possible number of fixed points $(2g+2)$. In Section 2, we prove the following theorem.

Theorem 1. *Let G be an automorphism group of a hyperelliptic Riemann surface which contains the hyperelliptic involution. In particular, G could be the full automorphism group of a hyperelliptic Riemann surface. Then G is isomorphic to one of the groups listed in Table 1. These groups are precisely the central extensions of Z_2 by one of the groups Z_n, D_{2n}, A_4, S_4 or A_5.*

Table 1. *Central extensions of Z_2 by Z_n, D_{2n}, A_4, S_4 and A_5* [(a),(b)]

Z_{2n}	$n = 1, 2, 3, \ldots$	
$Z_2 \times Z_n$	$n = 2, 4, 6, \ldots$ [(c)]	
$Z_2 \times D_{2n}$	$n = 2, 4, 6, \ldots$ [(d)]	
$D_{4,n,-1}$	$n = (2), 4, 6, \ldots$ [(e)]	$(D_{4,2,-1} \simeq Z_2 \times Z_4)$
G_{4n}	$n = (2), 4, 6, \ldots$ [(f)]	$(G_8 \simeq D_8)$
D_{4n}	$n = (1), 2, 3, \ldots$	$(D_4 \simeq Z_2 \times Z_2)$
Q_{4n}	$n = 2, 3, 4, 5, \ldots$	
SD_{4n}	$n = (2), 4, 6, \ldots$ [(f)]	$(SD_8 \simeq Z_2 \times Z_4)$
$SL_2(Z_3)$		
$A_4 \times Z_2$		
$GL_2(Z_3)$		
O^*		
$S_4 \times Z_2$		
$SL_2(Z_4)$		
$SL_2(Z_5)$		
$A_5 \times Z_2$		

Notes on Table 1.

(a) Group Nomenclature: D_{4n} is the dihedral group of order $4n$. Q_{4n} and SD_{4n} are the generalized quaternion and semidihedral groups of order $4n$, respectively. $D_{4,n,-1}$ and G_{4n} have order $4n$ and semidirect product structures $Z_4 \ltimes Z_n$ and $Z_2 \ltimes (Z_2 \times Z_n)$, respectively. O^* is the binary octahedral group. Other nomenclature is standard. For presentations, see §3.

(b) Low-order isomorphisms (and corresponding values of n) are listed in parentheses (see Table 4b).

(c) $Z_2 \times Z_n$ is defined for odd n, but is isomorphic to Z_{2n}.

(d) $Z_2 \times D_{2n}$ is defined for odd n, but is isomorphic to D_{4n}.

(e) $D_{4,n,-1}$ is defined for odd n, but is isomorphic to Q_{4n} (see Table 4b).

(f) G_{4n} and SD_{4n} are not defined for odd n (see Lemma 3.2.1).

In Section 3 we determine the possible branching data for hyperelliptic actions, using the known orbit structures for the actions of Z_{2n}, $Z_2 \times Z_n$, D_{4n} A_4, A_5 and S_4 on the Riemann sphere. The branching data are given in Tables 7, 8, 12, 13 and 14. In Section 4 we match the groups in Table 1 with the appropriate branching data, and specify the hyperelliptic actions, up to topological equivalence, by giving a representative of an equivalence class of generating vectors. The results are given in Tables 9, 10, 15, 16, and 17. One of the reasons for classifying the actions up to topological equivalence is that the set of surface classes admitting a topological equivalence class of action is a *connected* subvariety of the moduli space (see Section 5).

In Section 6 we show how to obtain a list of maximal hyperelliptic actions in each genus $g \geq 2$ in a routine manner. (Maximal actions are actions that are the full automorphism group of at least one point in the moduli space.) For the first eight genera ($2 \leq g \leq 9$) we present the maximal actions organized into lattices with respect to inclusion of one action in another (Figures 3 through 10). These lattices correspond precisely to the geometric decomposition of H_g spoken of above. Figures 11 and 12 can be used as a guides for constructing the lattices in genera greater than 9. They indicate the major features of the lattice in odd and even genera, respectively.

Subvarieties of dimension 0 corresponding to a topological type of group action are called exceptional points. In Section 6.3 we obtain the following general result:

Theorem 2. *H_g, $g \geq 2$, contains three distinct exceptional points whose full automorphism groups are Z_{4g+2}, G_{8g+8}, and SD_{8g} (the latter group extends to $GL_2(Z_3)$ in genus 2). If $g > 30$ these are the only exceptional points. If $g \leq 30$, there are other exceptional points. These are given in Table 20.*

Finally, in Section 7 we give the complete lattice of maximal actions in genus 3. This extends the lattice of maximal hyperelliptic actions, and gives a complete geometric picture of the singular set in the moduli space in genus 3. In Section 8 we indicate some lines of future research. In particular, we show that a hyperelliptic surface is a special case of a γ-hyperelliptic surface ($\gamma \geq 0$), and indicate how to obtain the lattice of maximal γ-hyperelliptic actions in genera > 3. This will be a further step toward obtaining the complete lattice of maximal actions in genera > 3.

1. Preliminaries

1.1. Hyperelliptic Riemann Surfaces

Let Σ_g be a compact Riemann surface of genus $g \geq 2$, and let P^1 be the Riemann sphere. If there exists a meromorphic function $z : \Sigma_g \to P^1$ having either a single pole of order 2, or two simple poles, Σ_g is said to be hyperelliptic, and z is said to be a hyperelliptic function. The field of meromorphic functions of Σ_g is generated by z and another meromorphic function w which satisfy the equation

$$w^2 = (z - e_1) \cdots (z - e_{2g+2}),$$

where $e_1, \ldots, e_{2g+2}$ are distinct points in P^1. z determines the surface Σ_g as a double covering of P^1, branched over the $2g+2$ points $e_1, \ldots, e_{2g+2}$. The mapping $(w, z) \mapsto (-w, z)$ which interchanges the sheets of the covering is a conformal self-homeomorphism having order 2 and $2g+2$ fixed points, called the hyperelliptic involution. It is unique in the full group of conformal self- homeomorphisms of Σ_g. It will be denoted t in the rest of this paper.

1.2. Group Actions.

We now recall some terminology and general facts about group actions on Riemann surfaces. A group G is said to act on a Riemann surface Σ if there exists a monomorphism $\Psi : G \to \mathrm{Aut}(\Sigma)$, where $\mathrm{Aut}(\Sigma)$ is the group of conformal self-homeomorphisms of Σ. Such an action will be denoted (G, Ψ). It is well-known ([Hu]) that if Σ is a compact Riemann surface of genus $g \geq 2$, then G is a finite group of order $d \leq 84(g-1)$. When such a group G acts on a compact Riemann surface Σ_g, $g \geq 2$ with orbits O_i , $i = 1, \ldots, k$, of length $m_i < d$, then m_i divides d, and the isotropy subgroup of any point in O_i is isomorphic to Z_{r_i}, where $r_i = d/m_i$. (The generic orbit has length d.) The space of orbits has a topology and holomorphic structure which makes it a compact Riemann surface Σ_h of genus $h \leq g$, with respect to which the orbit map is a regular branched covering. The numbers r_i, $i = 1, \ldots, k$ are called the branching indices of the action of G. The k points of Σ_h corresponding to the orbits O_i are called the *branch locus* of the action. They are labelled with the branching indices r_i since any point in the fiber over the ith point has isotropy subgroup cyclic of order r_i. We say that the $k+1$ -tuple $(h; r_1, \ldots, r_k)$ is the *branch data* for the action of G. When the indices are listed in weakly increasing order, this $k+1$ -tuple is unique. (It is shown in [MB] that the ordering of the branching indices is immaterial.) If a particular index r_j is repeated n times, it is listed as r_j^n for brevity. If $h = 0$, as will almost always be the case in this paper, it is omitted from the branch data. Finally, the numbers $d, g, h, r_1, \ldots, r_k$ must satisfy the well-known Riemann-Hurwitz relation:

$$2g - 2 = d\left\{2h - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{r_i}\right)\right\}. \tag{1.1}$$

The following well-known theorem (see [H1], [MB], [M], [B]) contains a necessary and sufficient condition — the existence of a *generating vector* — which guarantees that a group isomorphic to G actually acts on on a compact Riemann surface of genus g.

Theorem 1.1. *A group G of order d acts on a compact Riemann surface Σ_g, with branch data $(h; r_1, \ldots, r_k)$, if and only if $d, g, h, r_1, \ldots, r_k$ satisfy the Riemann-Hurwitz relation (1.1), and there is an ordered set of elements (an $(h; r_1, \ldots, r_k)$- generating vector) of the form $(a_1, b_1, \ldots, a_h, b_h, c_1, \ldots c_k)$ such that*
(i) the elements $a_1, b_1, \ldots, a_h, b_h, c_1, \ldots c_k$ generate G,
(ii) the order of c_i is r_i, and
(iii) $\prod_{i=1}^{h}[a_i, b_i] \prod_{j=1}^{k} c_j = e$, where $[a, b] = aba^{-1}b^{-1}$ and e is the identity in G.

Remark. Because of (iii), G is in fact generated by any subset of the generating vector obtained by removing one of the elements c_i.

The proof of Theorem 1.1 is essentially topological; it stems from combinatorial arguments due originally to Hurwitz [Hu] and Wiman [Wi]. It was later shown that every action of a finite group with branch data $(h; r_1, \ldots, r_k)$ on a surface of genus $g \geq 2$ corresponds to the existence of a Fuchsian group with signature $(h; r_1, \ldots, r_k)$ containing a torsion-free normal subgroup isomorphic to the fundamental group of the surface.

1.3. Relative Branching

Let $K \subseteq G$ be an inclusion of finite groups. If both K and G act on a compact Riemann surface, and the K-action is the restriction of the G-action, we say that the K-action is *included in* or *extends to* the G-action.

Let S_1 be the quotient space Σ_g / K and S_2 the orbit space Σ_g / G. Let $B_K \subset S_1$ be the branch locus of the K-action, and $B_G \subset S_2$ the branch locus of the G-action. The map Φ taking a K-orbit (point in S_1) to the G-orbit (point in S_2) containing it is a branched covering map with respect to the holomorphic structures induced on S_1 and S_2 by the G- and K-actions, respectively. Φ is called the *relative projection* between S_1 and S_2, and the branching behavior of Φ is called the *relative branching.* Let $B'_K = \Phi^{-1}(B_G) \subset S_1$. Note that $B'_K \supseteq B_K$, and the inclusion may be strict. Outside of B'_K, Φ is a regular $d : 1$ covering map, where $d = [G : K]$. Thus all the branching of Φ occurs on B'_K.

We now develop a notation for recording the relative branching. (This notation was suggested by a similar one in [B].) Label the points in B_G by their branching indices with respect to the G-action. List them in some order, say, weakly increasing, seperated by semicolons. Thus one obtains a list of the form $[r_1; r_2; \ldots; r_k]$, where $r_1 \leq r_2 \leq \ldots \leq r_k$, and $k = \#(B_G)$. Label the points in B_K by their branching indices with respect to the K-action, and the points in $B'_K - B_K$ by 1's. To each r_j there corresponds the subset $B_j = \Phi^{-1}(r_j) \subseteq B'_K$. List the labelled elements of B_j in weakly increasing order. Concatenate the lists so obtained, in order of increasing j, seperating adjacent lists by a semi-colon. The restriction of Φ to B'_K (the relative branching) may then be symbolized by the mapping

$$[s_{11},\ldots,s_{1y_1};\ s_{21},\ldots,s_{2y_2};\ \ldots;\ s_{k1},\ldots,s_{ky_k}] \longmapsto [r_1;\ \ldots;\ r_k], \tag{1.3.1}$$

where the s_{ji} are the elements of B_j. There are k "slots" (pairs of corresponding spaces between semicolons). The s_{ji} are either branching indices of the K-action, or 1.

If $s_{ji} = s_{j(i+1)} = \cdots = s_{j(q)}$, where $1 \le i < q \le y_j$, we write $(s_{ji})^{q-i}$ for brevity. If $r_j = r_{j+1} = \cdots = r_q$, where $1 \le j < q \le k$, and if the elements of the corresponding sets $B_j, B_{j+1}, \ldots B_q$ are the same (up to ordering), we abbreviate as follows. The elements of B_j are $s_{j1},\ldots,s_{jy_j}$. By hypothesis, these are also the elements of $B_{j+1},\ldots,B_q$. We write

$$[\ \ldots;\ s_{j1}^{q-j}, s_{j2}^{q-j},\ldots,s_{jy_j}^{q-j};\ \ldots\ ] \mapsto [\ \ldots;\ (r_j)^{q-j};\ \ldots\ ].$$

In this way we avoid filling $q-j$ consecutive "slots" with the same data.

If, for each j, all the s_{ji} are equal, the branching is said to be *uniform,* and the notation simplifies to:

$$[(s_1)^{y_1};\ (s_2)^{y_2};\ \ldots;(s_k)^{y_k}] \mapsto [r_1;\ \ldots;\ r_k]. \tag{1.3.2}$$

Uniform branching occurs if K is a normal subgroup of G. For then Φ is the orbit map for the action of the group G/K on S_1, hence a *regular* branched covering. The sets $\Phi^{-1}(r_j)$ are then G/K-orbits, all of whose points must have conjugate isotropy subgroups. Such subgroups are isomorphic to G'/K, where G' is a subgroup of G which normalizes K. The G-orbit corresponding to r_j therefore consists of $[G : G']$ K-orbits of equal length $(|\ G\ |\ /r_j)\ /\ [G : G']$. These K-orbits are therefore points in B'_K having equal branching indices (which may all be 1).

Example 1. $G = GL_2(3)$ (a group of order 48) has Sylow 2-subgroup $K = SD_{16}$ of index 3. We shall see that on surfaces of genus $g \equiv 2 \bmod 12$ there is a hyperelliptic action of G with branch data $(2^{(g-2)/12}, 2, 3, 8)$. This action contains a hyperelliptic action of K with branch data $(2^{(g-2)/4}, 2, 4, 8)$. The $3:1$ relative branching is

$$[2^{(g-2)/4};\ 1,2;\ 1;\ 4,8] \mapsto [2^{(g-2)/12};\ 2;\ 3;\ 8].$$

The notation decodes as follows: over the point in B_G labelled 8 lie the points in B_K labelled 4 and 8; Over the point in B_G labelled 3 lies a generic point (with respect to the K-action) labelled 1; etc. The first slot is actually an abbreviation for $(g-2)/12$ slots of the form

$$[\ \ldots;\ 2^3;\ \ldots\ ] \mapsto [\ \ldots;\ 2\ ;\ \ldots\ ].$$

Example 2. $G = GL_2(3)$ contains $K = Q_8$ as a normal subgroup of index six. The hyperelliptic action of G on surfaces of genus $g \equiv 2 \bmod 12$ in the previous example contains a hyperelliptic action of Q_8 with branch data $(2^{(g-2)/2}, 4, 4, 4)$. The $6:1$ uniform relative branching is

$$[2^{(g-2)/2};\ 1^3;\ 1^2;\ 4^3] \mapsto [2^{(g-2)/12};\ 2;\ 3;\ 8].$$

Lemma 1.3. *Let $K \subseteq G$ act on on a compact Riemann surface Σ_g, with the K-action included in the G-action. Let (1.3.1) be the relative branching from $S_1 = \Sigma_g/K$ to $S_2 = \Sigma_g/G$. Then for each j, the following relation is satisfied:*

$$\frac{|K|}{s_{j1}} + \frac{|K|}{s_{j2}} + \cdots + \frac{|K|}{s_{jy_j}} = \frac{|G|}{r_j}. \tag{1.3.3}$$

If the relative branching is uniform (1.3.2) this relation simplifies to:

$$y_j \left\{ \frac{|K|}{s_j} \right\} = \frac{|G|}{r_j}. \tag{1.3.4}$$

As a consequence, for a uniform relative branching, r_j is a multiple of s_j, for all j.

Proof. Each side of relation (1.3.3) is a count of the number of points in Σ_g lying over the point in B_G labelled r_j. Relation (1.3.4) follows immediately from the definition of uniform branching. The last statement follows from (1.3.4).

1.4. Topological Equivalence of Actions

Let $\alpha \in \text{Aut}(\Sigma_g)$ have order n, and $H = \langle \alpha \rangle$. Suppose H acts with branch data $(r_1, r_2, \ldots, r_k)$, so that the quotient $\Sigma_g/H = S$ is a sphere. (This is always the case for a hyperelliptic action). Let $p : \Sigma_g \to S$ be the orbit projection, and $B = \{s_1, s_2, \ldots, s_k\} \subset S$ the branch locus of the action. In a neighborhood of each point $x \in p^{-1}(s_i) \subset \Sigma_g$, α acts as rotation about x through some angle $2c_i\pi/n$, where $1 \leq c_i \leq n-1$. By the Riemann-Hurwitz relation, α^{c_i} has order r_i, and $(\alpha^{c_1}, \alpha^{c_2}, \ldots, \alpha^{c_k})$ is a generating vector for the action of H. (This implies that $c_1 + c_2 + \ldots c_k \equiv 0 \pmod{n}$.) The numbers $c_i, i = 1, \ldots, k$ depend on the choice of the generator for H. Two generating vectors for H determine *topologically equivalent* actions if there exist generators of H such that the two sets of c_i's coincide.

The notion of equivalence of generating vectors for cyclic group actions (due originally to Nielsen ([N])) can be extended to actions of arbitrary automorphism groups of Σ_g. This involves the theory of Fuchsian groups in an essential way (see, e.g., [B] or [H]). Two actions (G, Φ) and (G, Φ') on surfaces Σ and Σ', respectively, are defined to be topologically equivalent if there exist an orientation-preserving homeomorphism $h : \Sigma \to \Sigma'$ and an automorphism ω of G such that

$$\Phi'(g) = h\Phi(\omega(g))h^{-1} \quad \text{for all } g \in G. \qquad (1.4.1)$$

Let a finite group G and branching data $(h; r_1, \ldots, r_k)$ be given. Assume that the order of G, and the numbers $h, r_1, \ldots, r_k$

satisfy the Riemann-Hurwitz relation (1.1) for some $g \geq 2$. Then there is at least one surface class Σ_g on which G acts with the branching data above. G may be represented as a factor group Γ/K_g, where Γ is a Fuchsian group with signature $(h; r_1, r_2, \ldots, r_k)$, and $K_g \subseteq \Gamma$ is torsion-free subgroup with signature $(g; -)$. (K_g is called a surface group with orbit-genus g.) Γ acts properly discontinuously on the Poincarè upper half plane D; K_g acts freely on the same space. The surface Σ_g may be represented as the orbit space D/K_g. The action of $G \simeq \Gamma/K_g$ on D/K_g is as follows: a coset $[\gamma] \in \Gamma/K_g$ maps the K_g-orbit of $x \in D$ to the K_g-orbit of γx ($\gamma \in \Gamma$). The orbit space of this action is a compact Riemann surface of genus h which may be represented as $\Sigma_h = D/\Gamma$.

It follows that to each G-action with branch data $(h; r_1, \ldots, r_k)$ there corresponds an exact sequence

$$1 \rightarrow K \rightarrow \Gamma \xrightarrow{\eta} G \rightarrow 1,$$

in which η is an epimorphism with kernel K isomorphic to K_g. Since h is always 0 for a hyperelliptic action, we will assume this is the case to simplify notation. Γ has a presentation of the form

$$\langle \gamma_1, \ldots, \gamma_k \mid \gamma_1^{r_1} = \gamma_2^{r_2} = \ldots = \gamma_k^{r_k} = \gamma_1\gamma_2\cdots\gamma_k = 1 \rangle.$$

In some fixed presentation of G, let $g_1, \ldots, g_k$ be the images under η of $\gamma_1, \ldots, \gamma_k$, respectively. Because the kernel of η is torsion-free, the order of g_i in G is r_i ($i = 1, \ldots, k$). The product $g_1 \cdots g_k$ equals the identity in G; and, since η is surjective, the elements $g_1, \ldots, g_k$ generate G. Thus $(g_1, \ldots, g_k)$ is a generating vector for the action of G. The set of possible surjections $\Gamma \rightarrow G$ evidently corresponds, in one to one fashion, with the set of possible generating vectors for G-actions with branch data $(r_1, \ldots, r_k)$.

Let η be a surjection of Γ onto G, with kernel K isomorphic to K_g. Define

$$\eta' = \omega \circ \eta,$$

where ω is an automorphism of G. Then η' is another epimorphism which also has kernel K. Take $\Phi : G \rightarrow \Gamma/K$ to be the canonical isomorphism. We may associate to η the action (G, Φ) on $\Sigma_g = D/K$ defined by $\Phi(g) : Kx \mapsto K\Phi(g)x$, where $x \in D$. Similarly. we may associate to η' the action (G, Φ') defined by $\Phi'(g) : Kx \mapsto K\Phi(\omega(g))x$. That is, the monomorphism $\Phi' : G \to \Gamma/K$ is just $\Phi \circ \omega$. Taking h to be the identity in (1.4.1), we see that the actions (G, Φ) and (G, Φ') are topologically equivalent.

Lemma 1.4. *Let $\overline{v} = (g_1, \ldots, g_k)$ and $\overline{v}' = (g_1', \ldots g_k')$ be two $(r_1, \ldots, r_k)$-generating vectors for the action of a finite group G on a compact Riemann surface of genus $g \geq 2$. If there exists $\omega \in Aut(G)$ such that*

$$g_i' = \omega(g_i), \quad i = 1, \ldots k,$$

then $\overline{v}$ and $\overline{v}'$ determine topologically equivalent actions of G.

Knowledge of the topological equivalence classes of group actions in a given genus $g \geq 2$ is desirable since each topological type of group action corresponds to a connected subvariety of the singular set of the moduli space in the given genus (see Section 5). In addition, there is a one-to-one correspondence between topological types of group actions and conjugacy classes of finite subgroups of the mapping class group in the given genus.

1.5. The Reduced Type of a Hyperelliptic Group Action

Theorem 1.5. *(Schwarz) Let A be an element of $Aut(\Sigma_g)$, where Σ_g is a hyperelliptic Riemann surface. If A fixes more than four points, it is either the identity or the hyperelliptic involution t.*

Proof. Let $z : \Sigma_g \to P^1$ be the double branched covering for which the involution t is the sheet interchange. Suppose $A \in \text{Aut}(\Sigma_g)$ has more than four fixed points. Consider the map $f : \Sigma_g \to P^1$ defined by $f = z - z \circ A$. f has more than four zeroes, since A has more than four fixed points. On the other hand, f has at most four poles: Suppose z has simple poles at p_1 and $p_2 \in \Sigma_g$. By changing the coordinate on P^1, if necessary, we may arrange that p_1 and p_2 are not among the fixed points of A. If $A(p_1) \neq p_2$ and $A(p_2) \neq p_1$, f has distinct poles at $p_1, p_2, A^{-1}(p_1)$ and $A^{-1}(p_2)$. If $A(p_1) = p_2$ or $A(p_2) = p_1$, (or both), then f may lose one or more poles. Similarly, if z has a double pole at a point $p \in \Sigma_g$, and, as before, the coordinate on P^1 is such that p is not a fixed point of A, then f has two poles at p and $A^{-1}(p)$. In any case, f has no more than four poles. Now, since a nonconstant meromorphic function between compact Riemann surfaces has as many zeroes as poles, counting multiplicities, f must be constant, and hence identically equal to 0. So $z = z \circ A$. A must therefore fix the $2g + 2$ fixed points of z, and permute the set $z^{-1}(x)$, for each $x \in P^1$. If A is not the identity, it must be the sheet interchange, t. (The essence of this argument is in [KN].)

Corollary 1.5. *Let G be a group of automorphisms of a hyperelliptic surface of genus ≥ 2 which contains the hyperelliptic involution t. Then G centralizes $\langle t \rangle$.*

Proof. Let $g \in G$. Then $g^{-1}tg$ has $2g+2 > 4$ fixed points and thus must be t itself, by Theorem 2.1. So t commutes with all elements of G, i.e., $\langle t \rangle$ is central.

Let (G, Φ) be an action of a finite group G on a hyperelliptic surface Σ_g. We shall say that (G, Φ) is a *hyperelliptic action* if $\Phi(G)$ contains the hyperelliptic involution t. Since $\langle t \rangle$ is a normal subgroup of $\Phi(G)$ by Corollary 2.2, the relative projection between $S_1 = \Sigma_g/\langle t \rangle$ and $S_2 = \Sigma_g/\Phi(G)$ is the orbit map for the action of the group $\Phi(G)/\langle t \rangle$ on S_1. Now S_1 is is conformally equivalent to P_1 (the Riemann sphere), hence $\Phi(G)/\langle t \rangle$ must be a (finite) group of automorphisms of the Riemann sphere. It is well known that the only such groups are cyclic, dihedral, or isomorphic to the group of symmetries of the regular tetrahedron, octahedron or icosahedron. (The last three are isomorphic to A_4, S_4 and A_5, respectively.) These groups will be referred to collectively as the *finite spherical automorphism groups*.

By the preceeding argument, any hyperelliptic action (G, Φ) has associated with it a particular finite spherical automorphism group H. We shall say that (G, Φ) has *reduced type* H (the terminology is used in [KN]). Furthermore, there is a surjective homomorphism $h : \Phi(G) \to H$ with kernel $\langle t \rangle$. This places a strong restriction on the structure of G as an abstract group: it has a distinguished element t' of order 2 (namely, $\Phi^{-1}(t)$), and there is a surjective homomorphism (namely, $h \circ \Phi$) of G onto H with kernel $\langle t' \rangle \simeq Z_2$. In the language of group theory, G is a *central extension* of Z_2 by H.

2. Central Extensions of Z_2 by Finite Spherical Automorphism Groups

We use the standard presentations of the finite spherical automorphism groups given in Table 2 below. They have the minimal number of generators in each case. They are easily derived from a knowledge of the branching data of the actions of these groups on P^1 (given in Table 6). For example, Since S_4 acts on P^1 with branching data $(2, 3, 4)$, it has a generating vector (X, Y, Z), where X, Y and Z have orders 2, 3 and 4, respectively, and $XYZ = E$, where E is the identity. By the last relation, $Z = (XY)^{-1}$, so a presentation of S_4 with two generators is $\langle X,\ Y \mid X^2 = Y^3 = (XY)^4 = E \rangle$.

Table 2. *The Finite Spherical Automorphism Groups.*

Z_n	$\langle X \mid X^n = E \rangle$
D_{2n}	$\langle X, Y \mid Y^2 = X^n = (YX)^2 = E \rangle$
A_4	$\langle X, Y \mid X^3 = Y^3 = (XY)^2 = E \rangle$
S_4	$\langle X, Y \mid X^2 = Y^3 = (XY)^4 = E \rangle$
A_5	$\langle X, Y \mid X^2 = Y^3 = (XY)^5 = E \rangle$

Let G be an arbitrary central extension of Z_2 by H, where H has one of the presentations given in the Table. We have seen that G has a distinguished central element of order 2. For simplicity, we denote this element t instead of t' as in the last section. The notation is meant to be suggestive, since we are interested in those

G-actions in which this element is mapped to the hyperelliptic involution.

There is an exact sequence

$$\{e\} \longrightarrow Z_2 \hookrightarrow G \xrightarrow{\Phi} H \longrightarrow \{E\}, \qquad (2.1)$$

where Φ is a surjective homomorphism, e the identity in G and $Z_2 = \langle t \rangle \subset G$. Pick $x \in \Phi^{-1}(X) \subset G$ and $y \in \Phi^{-1}(Y) \subset G$, if Y is a generator of H. Since Φ is $2:1$ with kernel $\langle t \rangle$, $\Phi^{-1}(X) = \{x, tx\}$, and $\Phi^{-1}(Y) = \{y, ty\}$. G is generated by x, y and t, and $\Phi(t) = E$.

Lemma 2.1. *The preimage under Φ of a generator of odd order in H may always be assumed to have minimal (odd) order.*

Proof. Let $Z \in H$ be a generator of odd order q. Pick $z \in \Phi^{-1}(Z)$. z^q is either e or t, since it lies in the kernel of Φ. Suppose $z^q = t$. Then z has order 2q. G is generated by z, t, and possibly other elements. But G is equally well generated by tz, t, and the other elements. So renaming tz as z does no harm. But now things are arranged so that z has minimal order q.

Let us examine the relations which hold in $G = \langle x, y, t \rangle$. First, since t is central, the relation $t^2 = [x,t] = [y,t] = e$ will always hold. Second, a relation such as $X^n = E$ in H will lift to the relation $x^n = \alpha$ in G, where $\alpha \in \{e, t\}$. If there are 3 such relations in H, each lifting to two possible relations in G, then there are as many as $2^3 = 8$ different presentations of G. This number can be reduced by making use of Lemma 2.1.

2.1. Central extensions of Z_2 by Z_n

A central extension of Z_2 by a cyclic group Z_n has presentation $G = \langle x, t \mid x^n = \alpha,\ t^2 = [x,t] = e \rangle$, where α, as above, is either e or t. We obtain

Table 3. *Central Extensions of Z_2 by Z_n*

Name	*Presentation*
$Z_2 \times Z_n$	$\langle x, t \mid x^n = t^2 = [x,t] = e \rangle$
Z_{2n}	$\langle x \mid x^{2n} = e \rangle$

In the second presentation, the generator t has been eliminated by renaming the generating element xt as x. When n is odd the two groups are isomorphic.

2.2. Central Extensions of Z_2 by D_{2n}

It will be convenient to alter slightly the presentation of D_{2n} given in Table 2, so as to obtain the well-known presentation

$$D_{2n} \simeq \langle X, Y \mid X^n = Y^2 = E, \quad YXY^{-1} = X^{-1} \rangle.$$

(The third relation is equivalent to $(YX)^2 = E$.) A central extension of Z_2 by D_{2n} is then a group G of order $4n$ with presentation

$$\begin{aligned} \langle x, y, t \mid x^n = \alpha, \quad y^2 = \beta, \quad yxy^{-1} = x^{-1}\gamma, \\ t^2 = [x,t] = [y,t] = e \rangle \end{aligned} \qquad (2.2.1)$$

where $\alpha, \beta, \gamma \in \{e, t\}$.

In Table 4a below we list the eight possible presentations of G, simplified by eliminating the generator t where possible. A name is supplied only for those groups which, for some values of n, are not isomorphic to any other group in the table. (Isomorphisms between the presentations for certain values of n are given in Table 4b.) Nomenclature is the same as that in Table 1.

Table 4a. *Central Extensions of Z_2 by D_{2n}*

	(α,β,γ)	Name[(a)]	Presentation[(b)]	
(i)	(e,e,e)	$Z_2 \times D_{2n}$	$\langle x,y,t \mid x^n = y^2 = t^2 = e,\ yxy^{-1} = x^{-1}, \ldots \rangle$	
(ii)	(e,t,e)	$D_{4,n,-1}$	$\langle x,y \mid x^n = y^4 = e,\ yxy^{-1} = x^{-1} \rangle$	$(t=y^2)$
(iii)	(e,e,t)	G_{4n}	$\langle x,y,t \mid x^n = y^2 = t^2 = e,\ yxy^{-1} = x^{-1}t, \ldots \rangle$	
(iv)	(e,t,t)	—	$\langle x,y \mid x^n = y^4 = e,\ yxy^{-1} = x^{-1}y^2, \ldots \rangle$	$(t=y^2)$
(v)	(t,e,e)	D_{4n}	$\langle x,y \mid x^{2n} = y^2 = e,\ yxy^{-1} = x^{-1} \rangle$	$(t=x^n)$
(vi)	(t,t,e)	Q_{4n}	$\langle x,y \mid x^{2n} = e,\ y^2 = x^n,\ yxy^{-1} = x^{-1} \rangle$	$(t=x^n)$
(vii)	(t,e,t)	SD_{4n}	$\langle x,y \mid x^{2n} = y^2 = e,\ yxy^{-1} = x^{n-1} \rangle$	$(t=x^n)$
(viii)	(t,t,t)	—	$\langle x,y \mid x^{2n} = e,\ y^2 = x^n,\ yxy^{-1} = x^{n-1} \rangle$	$(t=x^n)$

(Notes appear on the following page.)

Notes on Table 4a.

(a) Nomenclature is explained in Note (a) of Table 1. Unnamed groups are isomorphic to the ones immediately preceding them in the Table.

(b) "..." signifies the further relations $[x,t]=[y,t]=e$.

Table 4b. *Isomorphisms among the groups in Table 4a.*

$$(iv) \simeq G_{4n} \tag{2.2.2}$$

$$(viii) \simeq SD_{4n} \tag{2.2.3}$$

$$Z_2 \times D_{2n} \simeq D_{4n} \quad \text{if n odd} \tag{2.2.4}$$

$$D_{4,n,-1} \simeq Q_{4n} \quad \text{if n odd} \tag{2.2.5}$$

$$G_8 \simeq D_8 \tag{2.2.6}$$

$$D_{4,2,-1} \simeq SD_8 \simeq Z_2 \times Z_4 \tag{2.2.7}$$

Construction of the isomorphisms. To construct the first four isomorphisms in Table 4b, assume that the presentation of the left-hand member in each isomorphism is the same as that in Table 4a, but with the generators in uppercase letters. (2.2.2) can then be defined by $X \mapsto x$, $Y \mapsto yx$ with inverse $x \mapsto X$, $y \mapsto YX^{-1}$, $t \mapsto Y^2$. (2.2.3) can be defined by $X \mapsto x$, $Y \mapsto yx$ with inverse $x \mapsto X$, $y \mapsto YX^{-1}$. (2.2.4) can be defined by $X \mapsto x^{n+1}$, $Y \mapsto y$, $T \mapsto x^n$, with inverse $x \mapsto TX$, $y \mapsto Y$. (2.2.5) can be defined by $X \mapsto y^2x$, $Y \mapsto y$ with inverse $x \mapsto Y^2X$, $y \mapsto Y$. For (2.2.6), we note that G_8 has relations $x^2 = y^2 = t^2 = e$, and $yxy = xt$. From the latter, we derive $(yx)^2 = t$, and using this, that yx has order 4. It is easy to see that the group is generated by y and yx, and that conjugation by y inverts yx. Thus $G_8 \simeq D_8$. (2.2.7) can be defined by $X \mapsto y$, $Y \mapsto yx$, with inverse $x \mapsto YX$, $y \mapsto X$.

Lemma 2.2.1. *Let G be a central extension of Z_2 by D_{2n}, with one of the presentations (2.2.1.) If $\gamma = t$, then n is even.*

Proof. We prove the contrapositive statement: if n is odd, $\gamma = e$. Suppose n is odd and $\gamma = t$. α is either e or t. If $\alpha = e$, then, since conjugate elements have equal order in any finite group, $2n =$

$O(tx) = O(ytxy^{-1}) = O(x^{-1}) = n$, a contradiction. On the other hand if $\alpha = t$, similar reasoning implies $n = O(tx) = O(ytxy^{-1}) = O(x^{-1}) = 2n$, another contradiction.

The lemma implies that the groups SD_{4n} and G_{4n} can exist only if n is even. This fact, together with isomorphisms (2.2.4) and (2.2.5), shows that, up to isomorphism, there are only two central extensions of Z_2 by D_{2n}, if n is odd.

We summarize the results of this section in

Theorem 2.2.2. *The central extensions of Z_2 by D_{2n}, up to isomorphism, are:*
(i) D_{4n} and Q_{4n}, if n is odd;
(ii) the six named groups in Table 4a, if n is even, $n \neq 2$;
(iii) $(Z_2)^3 (\simeq Z_2 \times D_4)$, D_8, Q_8, and $Z_2 \times Z_4$, if $n = 2$.

2.3. Central Extensions of Z_2 by A_4, A_5 and S_4

Before proceeding, we note that there are always at least two central extensions of Z_2 by H, where H is any finite spherical automorphism group (indeed any subgroup of $\mathrm{Aut}(P^1) = PSL_2(C)$). These are the trivial extension $Z_2 \times H$, and the *binary group* H^*. The binary groups arise as follows. $PSL_2(C)$ has universal covering group $SL_2(C)$. If H is one of the finite spherical automorphism groups, H^* is defined to be the preimage in $SL_2(C)$ of H under this 2 : 1 covering map. The kernel of the covering is the center $\{\pm I_2\} \simeq Z_2 \subset SL_2(C)$. The restriction of the covering map to H^* is a homomorphism onto H with kernel $\simeq Z_2$. So H^* is a central extension of Z_2 by H. Since $-I_2$ is the unique element of order two in $SL_2(C)$, the binary groups likewise contain a unique element of order two.

The binary cyclic group Z_n^* is Z_{2n}. The binary dihedral group D_{2n}^* is Q_{4n} ($\simeq D_{4,n,-1}$ if n is odd). The binary polyhedral groups (see [W]) are: the binary tetrahedral group $T^* = SL_2(Z_3)$; the binary icosahedral group $I^* = SL_2(Z_5)$; and the binary octahedral group O^*. (O^* is a subgroup of index 15 of the group $SL_2(F_9)$, where F_9 is the field with 9 elements. Note that $SL_2(F_9)$ contains a subgroup K isomorphic to $SL_2(Z_3)$. O^* is the subgroup generated by K and the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$).

A central extension of Z_2 by A_4, where A_4 has the presentation given in Table 2, is a group of order 24 with two possible presentations

$$\begin{aligned}\langle x, y, t \mid x^3 = e, \quad y^3 = e, \quad (xy)^2 = \gamma, \\ t^2 = [x, t] = [y, t] = e\rangle,\end{aligned} \tag{2.3.1}$$

where $\gamma \in \{e, t\}$. (We have assumed $x^3 = y^3 = e$, by Lemma 3.1.) Of the two possible presentations, the one corresponding to $\gamma = e$ is clearly the trivial extension $Z_2 \times A_4$; the second must therefore be the binary tetrahedral group $SL_2(Z_3)$. Similarly, a central extension of Z_2 by A_5, where A_5 has the presentation given in Table 2, is a group of order 120 with two presentations

$$\begin{aligned}\langle x, y, t \mid x^2 = \gamma, \quad y^3 = e, \quad (xy)^5 = e, \\ t^2 = [x, t] = [y, t] = e\rangle\end{aligned} \tag{2.3.2}$$

where $\gamma \in \{e, t\}$. $\gamma = e$ gives $Z_2 \times A_5$, so $\gamma = t$ must give the binary icosahedral group $SL_2(Z_5)$.

A central extension of Z_2 by S_4, where S_4 has the presentation given in Table 2, is a group of order 48 with four possible presentations

$$\begin{aligned}\langle x, y, t \mid x^2 = \alpha, \quad y^3 = e, \quad (xy)^4 = \beta, \\ t^2 = [x, t] = [y, t] = e\rangle,\end{aligned} \tag{2.3.3}$$

where $\alpha, \beta \in \{e, t\}$. $S_4 \times Z_2$ clearly corresponds to $(\alpha, \beta) = (e, e)$. It is easy see that the binary group O^* corresponds to $(\alpha, \beta) = (t, t)$: this is the only presentation in which there is a unique element of order two. To identify the two remaining presentations, we make use of the isomorphisms

$$S_4 \simeq PGL_2(Z_3) \tag{2.3.4}$$

$$S_4 \simeq PSL_2(Z_4). \tag{2.3.5}$$

(2.3.4) arises from the fact that $GL_2(Z_3)$ acts transitively on the projective line $P^1(Z_3)$, a set containing four elements. This gives a homomorphism $GL_2(Z_3) \longrightarrow S_4$ whose kernel is easily seen to be $\{\pm I\}$. Thus $GL_2(Z_3)/\{\pm I\} \simeq PGL_2(Z_3)$ injects into S_4; since the orders of the two groups are equal, the injection is also an epimorphism. (2.3.5) comes from the fact that both S_4 and $PSL_2(Z_4)$ are homomorphic images, in natural ways, of the modular group $PSL_2(Z)$. Since the kernels of both homomorphisms are the same, namely, the principal congruence subgroup of level 4 of the modular group, the isomorphism (2.3.5) is induced (See [JS] §6.9).

From these two isomorphisms, we see that the other two central extensions of Z_2 by S_4 are $GL_2(Z_3)$ and $SL_2(Z_4)$. In both cases, $-I = t$ in (2.3.3). The presentation determined by $(\alpha, \beta) = (e, t)$ must correspond to $GL_2(Z_3)$, since the relation $(xy)^4 = t$ implies that xy is an element of order 8 in G, and there are no elements of order 8 in $SL_2(Z_4)$. The remaining presentation, determined by $(\alpha, \beta) = (t, e)$, must therefore correspond to $SL_2(Z_4)$.

We summarize the results of this section in

Table 5. *Central Extensions of Z_2 by A_4, A_5 and S_4*

	Group	*Presentation*†
by A_4:	$A_4 \times Z_2$	$\langle x,y,t \mid x^3=e,\ y^3=e,\ (xy)^2=e,\ \ldots\rangle$
	$SL_2(Z_3)$	$\langle x,y,t \mid x^3=e,\ y^3=e,\ (xy)^2=t,\ \ldots\rangle$
by A_5:	$A_5 \times Z_2$	$\langle x,y,t \mid x^2=e,\ y^3=e,\ (xy)^5=e,\ \ldots\rangle$
	$SL_2(Z_5)$	$\langle x,y,t \mid x^2=t,\ y^3=e,\ (xy)^5=e,\ \ldots\rangle$
by S_4:	$S_4 \times Z_2$	$\langle x,y,t \mid x^2=e,\ y^3=e,\ (xy)^4=e,\ \ldots\rangle$
	O^*	$\langle x,y,t \mid x^2=t,\ y^3=e,\ (xy)^4=t,\ \ldots\rangle$
	$GL_2(Z_3)$	$\langle x,y,t \mid x^2=e,\ y^3=e,\ (xy)^4=t,\ \ldots\rangle$
	$SL_2(Z_4)$	$\langle x,y,t \mid x^2=t,\ y^3=e,\ (xy)^4=e,\ \ldots\rangle$

† "..." signifies the further relations $t^2=[x,t]=[y,t]=e$.

We have now obtained, up to isomorphism, all the central extensions of Z_2 by the finite spherical automorphism groups (Tables 3, 4a, and 5). Together with the argument in § 1.5, this proves

Theorem 1, stated in the introduction. It remains to determine the hyperelliptic actions of these groups, up to topological equivalence, on (hyperelliptic) surfaces of genus $g \geq 2$.

3. Branching Data for Hyperelliptic Actions

The branching data for a hyperelliptic action (G, Φ) of reduced type H on a hyperelliptic surface Σ_g is determined by the uniform relative branching $S_1 \rightarrow S_2$, where $S_1 = \Sigma_g/\langle t \rangle$ and $S_2 = \Sigma_g/G$. This is just the branching associated with the action of H on S_1. Since the branching is uniform, each $\Phi(G)$-orbit is a union of $\langle t \rangle$-orbits of equal length. (see §1.3.) Now a $\langle t \rangle$-orbit can have length 1 or 2; the branch locus $B \subseteq S_1$ of the $\langle t \rangle$-action consists of the $2g+2$ $\langle t \rangle$-orbits of length 1. So an arbitrary $\Phi(G)$-orbit is either a union of $\langle t \rangle$-orbits contained in B, or a union of $\langle t \rangle$-orbits disjoint from B. Since each H-orbit is also a G-orbit (when both are considered as subsets of Σ_g), an arbitrary H-orbit is either contained in B, or disjoint from it.

The orbits of H on $S_1 = P^1$ are given in Table 6 below. In each case there are generic orbits of length $|H|$, and finitely many smaller orbits. The lengths of the smaller orbits are listed first, followed by the length of a generic orbit in parentheses. The branching data for each action is also given.

Table 6. *Actions of the Finite Spherical Automorphism Groups*

H	*Branch Data*	*Orbit Lengths*
Z_n	(n, n)	$1, 1, (n)$
D_{2n}	$(2, 2, n)$	$n, n, 2, (2n)$
A_4	$(2, 3, 3)$	$6, 4, 4, (12)$
S_4	$(2, 3, 4)$	$12, 8, 6, (24)$
A_5	$(2, 3, 5)$	$30, 20, 12, (60)$

Suppose there is a hyperelliptic action (G, Φ) of reduced type H on the hyperelliptic surface Σ_g. Let the orbit-lengths of H on S_1 (taken from Table 6) be $a_1, a_2, a_3, (d)$, where $d = |H|$. (If $H = Z_n$,

a_3 is omitted.) Let A_i be the orbit of length a_i. Define

$$\epsilon_i = \begin{cases} 1 & \text{if } A_i \subseteq B \\ 0 & \text{if } A_i \not\subseteq B. \end{cases}$$

Let the number of generic (length d) H-orbits contained in B be q. Then

$$a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + dq = 2g + 2. \tag{3.1}$$

Equation (3.1) will be known as the *orbit equation* for hyperelliptic actions of reduced type H on surfaces of genus g. The eight possible ordered triples $(\epsilon_1, \epsilon_2, \epsilon_3)$ determine the possible branching data, as follows.

Suppose $\epsilon_j = 1$. Then the orbit A_j is contained in B and consists of a_j $\langle t\rangle$-orbits of length 1. The corresponding $\Phi(G)$-orbit has the same length. Since $|G| = 2d$, this orbit corresponds to a point in S_2 having branch index $2d/a_j$. On the other hand, if $\epsilon_j = 0$, A_j lies outside of B and consists of a_i $\langle t\rangle$-orbits of length 2. The corresponding $\Phi(G)$-orbit has length $2a_j$; the associated point in S_2 has branch index d/a_j. The $d:1$ relative branching in the general case is therefore

$$\begin{aligned}&[1^{a_1(1-\epsilon_1)};\ 1^{a_2(1-\epsilon_2)};\ 1^{a_3(1-\epsilon_3)};\ 2^{a_1\epsilon_1};\ 2^{a_2\epsilon_2};\ 2^{a_3\epsilon_3};\ 2^{dq}]\\ &\qquad\longrightarrow [(d/a_1)^{(1-\epsilon_1)};\ (d/a_2)^{(1-\epsilon_2)};\ (d/a_3)^{(1-\epsilon_3)};\\ &\qquad\qquad (2d/a_1)^{\epsilon_1};\ (2d/a_2)^{\epsilon_2};\ (2d/a_3)^{\epsilon_3};\ 2^q].\end{aligned} \tag{3.3}$$

Since for each i, either ϵ_i or $(1-\epsilon_i) = 0$, there are in fact only four slots in any particular case (three if $H = Z_n$). (If an "exponent" on either side is 0, the corresponding slot is omitted.) Note that the last slot is an abbreviation for q slots of the form

$$[\ldots;\ 2^d;\ \ldots] \to [\ldots;\ 2;\ \ldots].$$

The right-hand side of (3.3) is (up to ordering) a possible branching data for a hyperelliptic action (G, Φ) of reduced type H. In general, there are eight different types of branching data, corresponding to the eight possible triples $(\epsilon_1, \epsilon_2, \epsilon_3)$. (If $H = Z_n$ there are at most four types of branching data, corresponding to four ordered pairs (ϵ_1, ϵ_2).) However, if $a_i = a_j$ for some $i \neq j$, certain triples give the same branching data, after reordering.

Example. We consider hyperelliptic actions of reduced type D_{2n}, $n \geq 2$. From Table 6, $a_1 = a_2 = n$, $a_3 = 2$, and $d = 2n$. The orbit equation (3.1) is therefore

$$n\epsilon_1 + n\epsilon_2 + 2\epsilon_3 + 2nq = 2g + 2.$$

The particular triple $(\epsilon_1, \epsilon_2, \epsilon_3) = (1, 0, 0)$ gives (using (3.3)) the $n : 1$ relative projection

$$[1^n;\ 1^2;\ 2^n;\ 2^{2nq}] \rightarrow [2;\ n;\ 4;\ 2^q].$$

Reordering the right-hand side, we obtain the branch data $(2^q, 2, 4, n)$. In this case the orbit equation reduces to

$$n + 2nq = 2g + 2,$$

which yields

$$n = \frac{2g+2}{2q+1}.$$

This implies that n must be a divisor of $2g+2$; moreover, n must be even, since it is the (integral) quotient of an even integer by an odd integer. q may be any nonnegative integer. All this information is summarized in Table 7 as type II.

The other possible types of branching data for hyperelliptic actions of reduced type D_{2n} are obtained from the other ordered triples $(\epsilon_1, \epsilon_2, \epsilon_3)$. Since $a_1 = a_2$, the triple $(0, 1, 0)$ yields the same branching data as $(1, 0, 0)$, and $(0, 1, 1)$ yields the same branching data as $(1, 0, 1)$. So we obtain only six distinct types of branching data (cases I – VI in Table 7).

Table 7. *Possible Branching Data for Hyperelliptic Actions of Reduced Type D_{2n}*

Type	$(\epsilon_1, \epsilon_2, \epsilon_3)$	n	*Branch Data*	
I	$(0,0,0)$	$\frac{g+1}{q}$	$(2^q, 2, 2, n)$	$q \neq 0$
II	$(1,0,0)$	$\frac{2g+2}{2q+1}$	$(2^q, 2, 4, n)$	(n even)
III	$(1,1,0)$	$\frac{g+1}{q+1}$	$(2^q, 4, 4, n)$	
IV	$(0,0,1)$	$\frac{g}{q}$	$(2^q, 2, 2, 2n)$	$q \neq 0$
V	$(0,1,1)$	$\frac{2g}{2q+1}$	$(2^q, 2, 4, 2n)$	(n even)
VI	$(1,1,1)$	$\frac{g}{q+1}$	$(2^q, 4, 4, 2n)$	

Note that the admissible values of n for a given triple are found among the divisors of simple expressions involving g. This is also true of group actions of reduced type Z_n (see Table 9). In contrast, for actions of reduced type A_4, S_4 and A_5, the absence of the parameter n allows the orbit equation to be solved for q explicitly in terms of g for each particular triple. This puts restrictions on g in the form of linear congruences mod 2, 12 or 30 (see Tables 12, 13 and 14).

4. Determination of the Hyperelliptic Actions

4.1 Hyperelliptic Actions of Reduced Type Z_n

The orbit equation for a hyperelliptic action of reduced type Z_n on a surface of genus g is

$$\epsilon_1 + \epsilon_2 + np = 2g + 2, \quad \epsilon_1, \epsilon_2 \in \{0, 1\}, \quad p = 0, 1, 2, \ldots,$$

where p is the number of generic (length n) Z_n-orbits contained in the branch locus $B \subset \Sigma_g/\langle t \rangle$ of the $\langle t \rangle$-action. Corresponding to the four possible pairs (ϵ_1, ϵ_2), we obtain three distinct types of branching data (A, B and C in Table 8). (The pairs $(0,1)$ and $(1,0)$ give the same branching data). From the expressions for n it is evident that $p \neq 0$ in all three cases.

Table 8. *Possible Branching Data for Hyperelliptic Actions of Reduced Type Z_n*

Type	(ϵ_1, ϵ_2)	n	*Branch Data*
A	$(0,0)$	$\frac{2g+2}{p}$	$(2^p, n, n)$
B	$(1,0)$	$\frac{2g+1}{p}$	$(2^p, n, 2n)$
C	$(1,1)$	$\frac{2g}{q}$	$(2^p, 2n, 2n)$

The possible groups are $Z_2 \times Z_n$ (n even), and Z_{2n}. Their hyperelliptic actions on surfaces of genus g are specified in Table 9 below. Each row of the Table corresponds to a family of group actions indexed by the parameter p; for a given genus g there are obviously only finitely many values of p making n an integer. An expression such as $(x^n)^{[p]}$ in a generating vector means the element x^n is repeated p times. We show shortly that the given generating vectors are unique up to equivalence.

Table 9. *Hyperelliptic Actions of Reduced Type Z_n*

	Group	n	Presentation	Branch Data	Gen. Vector(s)
A	$Z_2 \times Z_n$ $n>1^{*,**}$	$\frac{2g+2}{p}$	$\langle t,x \mid t^2=x^n=e \rangle$	$(2^p,n,n)$	$(t^{[p]},x^{-1},x)$ p even; $(t^{[p]},tx^{-1},x)$ p odd
B	Z_{2n} n odd†	$\frac{2g+1}{p}$	$\langle x \mid x^{2n}=e \rangle$	$(2^p,n,2n)$	$((x^n)^{[p]},x^{n-1},x)$ p odd
C	Z_{2n} $n>1^*$	$\frac{2g}{p}$	$\langle x \mid x^{2n}=e \rangle$	$(2^p,2n,2n)$	$((x^n)^{[p]},x^{-1},x)$ p even; $((x^n)^{[p]},x^{n-1},x)$ p odd

Notes on Table 9.

* If n=1 in (A) or (C), the action is Z_2 (2^{2g+2}), which is (B) for n=1.

** If n=2 in (A), the group is $\simeq D_4$ (see also Table 10.)

† The expression for n in (B) shows that both n and p must be odd. The element x^{n-2} in the generating vector has order $2n$ since g.c.d. $(n-2,2n)=1$ for odd n.

We now show, using Lemma 1.4, that all possible generating vectors for each of the actions of type (A), (B) and (C) are equivalent to the ones given in Table 9.

In any generating vector of type (A), the first p elements must be t (the only other possibility is $x^{n/2}$, if n is even), since the points in Σ_g lying over the p points with branch index 2 are, according to the orbit equation, fixed points of t. Any point fixed by t must have an isotropy subgroup in $Z_2 \times Z_n$ which is a cyclic group containing $\langle t \rangle$. But t is not contained in any larger cyclic subgroup of $Z_2 \times Z_n$. Generating vectors of type (A) must therefore have the form

$$(\mathrm{t}^{[p]}, c_1, c_2) \quad \text{if } p \text{ is even}$$
$$(\mathrm{t}^{[p]}, d_1, d_2) \quad \text{if } p \text{ is odd,}$$

where $c_1c_2 = e$ and $d_1d_2 = t$. The possibilities are

$$\begin{aligned} c_1 &= x^{-j}, \quad & c_2 &= x^j \\ d_1 &= tx^{-j}, \quad & d_2 &= x^j, \end{aligned}$$

where $(j, n) = 1$. The case $j = 1$ is the one given in the Table. Generating vectors corresponding to other values of j are equivalent to this one under the automorphism of $Z_2 \times Z_n$ which fixes t and sends $x \mapsto x^j$.

In any generating vector of type (B) or (C), the first p elements must be x^n, since this is the only element of order 2 in Z_{2n}. Generating vectors therefore have the form

$$\begin{aligned} &((x^n)^p, c_1, c_2) \quad \text{if } p \text{ is even} \\ &((x^n)^p, d_1, d_2) \quad \text{if } p \text{ is odd,} \end{aligned}$$

where $c_1c_2 = e$ and $d_1d_2 = x^n$. The possibilities are

$$\begin{aligned} c_1 &= x^{-j}, \quad & c_2 &= x^j \\ d_1 &= x^{n-j}, \quad & d_2 &= x^j, \end{aligned}$$

where $(j, n) = 1$. The case $j = 1$ is the one given in the Table. Generating vectors corresponding to other values of j are equivalent to this one under the automorphism of Z_{2n} taking $x \mapsto x^j$.

4.2. Hyperelliptic Actions of Reduced Type D_{2n}

The hyperelliptic actions of reduced type D_{2n} are given in Table 10, below. The six types branching data are taken from Table 7. The six groups are the ones named in Table 4a, with the presentations given there.

Table 10. *Hyperelliptic Actions of Reduced Type* D_{2n}

Group	n	Branch Data	Generating Vectors	
D_{4n}	$\frac{g}{q}$	$(2^q,2,2,2n)$	$((x^n)^{[q]},\ yx,\ y,\ x)$ $((x^n)^{[q]},\ y,\ yx^{n-1},\ x)$	q even, $q\neq0$ q odd
$Z_2\times D_{2n}$ $n>1$	$\frac{g+1}{q}$	$(2^q,2,2,n)$	$(t^{[q]},\ yx,\ y,\ x)$ $(t^{[q]},\ tyx,\ y,\ x)$	q even, $q\neq0$ q odd
Q_{4n} $n>1$	$\frac{g}{q+1}$	$(2^q,4,4,2n)$	$((x^n)^{[q]},\ y,\ yx^{n-1},\ x)$ $((x^n)^{[q]},\ y,\ yx^{-1},\ x)$	q even or 0 q odd
$D_{4,n,-1}$ $n>2$	$\frac{g+1}{q+1}$	$(2^q,4,4,n)$	$((y^2)^{[q]},\ yx,\ y^{-1},\ x)$ $((y^2)^{[q]},\ yx,\ y,\ x)$	q even or 0 q odd
G_{4n} n even, $n>2$	$\frac{2g+2}{2q+1}$	$(2^q,2,4,n)$	$(t^{[q]},\ y,\ yx^{-1},\ x)$ $(t^{[q]},\ y,\ tyx^{-1},\ x)$	q even or 0 q odd
SD_{4n} n even	$\frac{2g}{2q+1}$	$(2^q,2,4,2n)$	$((x^n)^{[q]},\ y,\ yx^{-1},\ x)$ $((x^n)^{[q]},\ y,\ yx^{n-1},\ x)$	q even or 0 q odd

Notes on Table 10.

(i) When $n=1$, all the groups are isomorphic to either Z_4 or D_4, or not defined. The action of Z_4 is given in (C) of Table 9. The action of D_4 is given in the first row of the present Table, with $n=1$, and also in (A) of Table 9, with $n=2$.

(ii) The actions of $D_{4,2,-1}\simeq SD_8\simeq Z_2\times Z_4$ are topologically equivalent, but topologically distinct from the action of $Z_2\times Z_4$ in (A) of Table 9. See Part (i) in this section.

(iii) The action of $G_8 \simeq D_8$ is topologically equivalent to the action of D_8 given in the first row, with $n=2$. See Part (i) in this section.

The rest of this rather long section is devoted to showing that all possible generating vectors for the hyperelliptic actions in Table 10 are equivalent to the given ones. We verify each row of the table in turn, first for even values of n, and then, where applicable, for odd values of n. In **(i)** we deal seperately with the groups D_8 and $Z_2 \times Z_4$, which occur twice each in the table, due to isomorphisms 2.2.6 and 2.2.7.

Induced Generating Vectors. We will make extensive use of the following technique for determining the possible generating vectors for the action of a group, given the generating vector for the (included) action of a subgroup. Each of the six groups in Table 10 contains a unique index 2 subgroup isomorphic to Z_{2n} or $Z_2 \times Z_n$, whose hyperelliptic actions and generating vectors have been determined in the previous section (Table 9).

We illustrate with an example. Suppose we wish to determine whether a hyperelliptic action of D_{4n}, n even, can have branch data $(2^q, 2, 2, 2n)$ (type IV from Table 7). In any hyperelliptic action of D_{4n}, n even, on a surface Σ_g, the index 2 subgroup $\langle x \rangle \simeq Z_{2n}$ must have the hyperelliptic action of type (C) $(2^p, 2n, 2n)$ from Table 9. Equating the expressions for n in (C) of Table 9 and IV of Table 7, we find that $p = 2q$. There must be a $2:1$ relative projection

$$\Sigma_g/Z_{2n} \to \Sigma_g/D_{4n},$$

uniformly branched, which maps the branch locus of the Z_{2n} action with indices $(2^{2q}, 2n, 2n)$ to the branch locus of the D_{4n} action with indices $(2^q, 2, 2, 2n)$. It is easy to establish that the only possibile relative branching is

$$[2^{2q}; 1; 1; 2n, 2n] \to [2^q; 2; 2; 2n]. \qquad (4.2.1)$$

The branching data on the left-hand side are associated with the generating vector $((x^n)^{[2q]}, x^{-1}, x)$ for Z_{2n}, taken from Table 9

$(p = 2q)$. We rewrite the generating vector in the form

$$((x^n)^{[2q]}; -; -; x^{-1}, x)$$

to emphasize the connection. The dashes represent the identity element in Z_{2n}, corresponding to points in Σ_g whose isotropy subgroups in Z_{2n} are trivial. The branching data on the right-hand side of (4.2.1) are associated with an unknown possible generating vector for D_{4n} of the form

$$(b_1, b_2, \ldots, b_q; c_1; c_2; c_3),$$

where the elements b_i have order 2, and c_1, c_2 and c_3 have orders 2, 2 and 2n, respectively.

The relative branching (4.2.1) "induces" a map

$$((x^n)^{[2q]}; -; -; x^{-1}, x) \to (b_1, b_2, \ldots, b_q; c_1; c_2; c_3) \qquad (4.2.2)$$

on generating vectors which narrows down the possibilities for the b_i and the c_j. The general principle is that each element on the right-hand side of (4.2.2) must generate a subgoup of D_{4n} which contains a conjugate of the (cyclic) subgroup(s) generated by the corresponding element(s) of Z_{2n} on the left-hand side, but no larger cyclic subgroup of Z_{2n}. In this example, $b_1, \ldots b_q$ must all be x^n (in the presentation of D_{4n} given in Table 4a), since x^n (the hyperelliptic involution) is the only element of D_{4n} which generates $\langle x^n \rangle \subset Z_{2n}$. The restrictions placed on c_1 and c_2 are: $\langle c_1 \rangle$ and $\langle c_2 \rangle$ must have trivial intersection with $\langle x \rangle = Z_{2n}$. The restriction on c_3 is: $\langle c_3 \rangle$ must contain $\langle x, x^{-1} \rangle = \langle x \rangle \subset Z_{2n}$. That is, $c_3 = x^j$ or x^{-j}, where $(j, 2n) = 1$. Further restrictions are placed on the elements c_i by the parity of q, and the requirement that the product of all the elements in any generating vector be the identity. (See **(a)** for the conclusion of this example.)

For ease of reference, we say that the 2 : 1 uniform relative branching (4.2.1) has type C-IV, meaning that the overgroup has branch data of type IV from Table 7, and the index 2 subgroup has branching data and generating vector of type (C) from Table 9. Of the 18 possible relative branchings, only six (A-I, A-II, A-III, C-IV, C-V and C-VI) are numerically possible for all positive values of n. We shall refer to these as the unrestricted relative branchings. For convenience, they are collected in Table 11. They are calculated, as in the example above, by equating the expressions for n in the appropriate rows of Tables 7 and 8, solving for p in terms of q, and using Lemma 1.3.

Table 11. *Unrestricted Relative Branchings*

A-I	$[2^{2q}; 1; 1; n, n]$	$\rightarrow$	$[2^q; 2; 2; n]$
A-II	$[2^{2q}; 1; 2; n, n]$	$\rightarrow$	$[2^q; 2; 4; n]$
A-III	$[2^{2q}; 2; 2; n, n]$	$\rightarrow$	$[2^q; 4; 4; n]$
C-IV	$[2^{2q}; 1; 1; 2n, 2n]$	$\rightarrow$	$[2^q; 2; 2; 2n]$
C-V	$[2^{2q}; 1; 2; 2n, 2n]$	$\rightarrow$	$[2^q; 2; 4; 2n]$
C-VI	$[2^{2q}; 2; 2; 2n, 2n]$	$\rightarrow$	$[2^q; 4; 4; 2n]$

(An example of a restricted relative branching is A-IV: p and q are integers only if n is a divisor of both g and $2g + 2$; this requires that $n \leq 2$. Other examples are: all relative branchings of type B-$*$, which are possible only if $n = 1$. The restricted relative branchings are treated in section **(j)**.)

In sections **(a)** and **(b)** we treat the groups D_{4n} and $Z_2 \times Z_n$, for even values of n, and in section **(c)** the same two groups for odd values of n (when they are isomorphic). In sections **(d)** and **(e)** we treat the groups Q_{4n} and $D_{4,n,-1}$, for even values of n, and in section **(f)** the same two groups for odd values of n (when they are isomorphic). Finally in sections **(g)** and **(h)** we treat G_{4n} and SD_{4n}, which are defined only for even values of n. For each group, we give the conjugacy classes of elements of order 2, 4, n and (where they exist) $2n$, since these are the only elements which can belong to generating vectors for the branch data from Table 7. Elements enclosed between curly braces ({ }) form a conjugacy class. There is usually a simple rule, easily derivable from the presentation of the group, which facilitates computation in the group. This is given for reference. The reader may find it useful to consult Appendix B, which gives partial subgroup lattices of the groups in question.

(a) $\mathbf{D_{4n} = \langle x, y \mid x^{2n} = y^2 = e,\ yxy^{-1} = x^{-1} \rangle}$, n even, $\mathbf{(t = x^n)}$.

Rule:

$$yx^jyx^k = x^{k-j}$$

The relevant conjugacy classes are

Order 2:	$\{x^n\}$
	$\{yx^{2k} \mid k = 0, 1, \ldots, n-1\}$
	$\{yx^{2k+1} \mid k = 0, 1, \ldots, n-1\}$
Order 4:	$\{x^{n/2}, x^{-n/2}\}$
Order n:	$\{x^j, x^{-j}\}\quad (j, 2n) = 2$
Order $2n$:	$\{x^j, x^{-j}\}\quad (j, 2n) = 1.$

Any hyperelliptic action of D_{4n} must contain the hyperelliptic action of the index 2 subgroup $Z_{2n} = \langle x \rangle$. This action must be of type (C) from Table 9, if n is even. The unrestricted relative branchings which must be considered are therefore C-IV, C-V and C-VI.

C-IV: This is the example used in the introduction to this section. The 2 : 1 uniform relative branching

$$[2^{2q}; 1; 1; 2n, 2n] \to [2^q; 2; 2; 2n],$$

induces a map on generating vectors

$$((x^n)^{[2q]}; -; -; x^{-1}, x) \to ((x^n)^{[q]}; c_1; c_2; c_3), \qquad \text{if } q \text{ even}$$
$$((x^n)^{[2q]}; -; -; x^{-1}, x) \to ((x^n)^{[q]}; d_1; d_2; d_3), \qquad \text{if } q \text{ odd},$$

where c_1, c_2, d_1, d_2 belong to one of the two conjugacy classes

$$\{yx^j \mid j \text{ even or } 0\}, \quad \{yx^j \mid j \text{ odd}\}.$$

Note: We may assume $c_3 = d_3 = x$, since a generating vector with, say, $c_3 = x^j$, where $(j, 2n) = 1$, is equivalent under the automorphism $x^j \mapsto x$, $y \mapsto y$ to one in which $c_3 = x$. This automorphism may alter c_2 and c_3, but has no effect on x^n, which is invariant under all such automorphisms. (Arguments of this type will be used tacitly in the rest of this section, since the map sending $x \mapsto x^j$, where j is relatively prime to the order of x, and fixing the other generators, is an automorphism in each of the central extensions of Z_2 by D_{2n}.) Since the product of the elements in a generating vector must be the identity, we then have $c_1c_2 = x^{-1}$, and $d_1d_2 = x^{n-1}$. The possibilities are

$$c_1 = yx^{j+1}, \quad c_2 = yx^j$$

and

$$d_1 = yx^j, \quad d_2 = yx^{j+n-1},$$

where $j = 0, 1, \ldots 2n-1$. In both cases we may take $j = 0$, since generating vectors corresponding to other values of j are equivalent to this one under an automorphism of D_{4n} defined by conjugation by a power of x, possibly preceded by the automorphism defined by $x \mapsto x, \quad y \mapsto yx$ (with inverse $x \mapsto x, \quad y \mapsto yx^{-1}$). (The latter automorphism maps elements in the conjugacy class of y to those in the conjugacy class of yx and conversely.)

C-V: The uniform 2 : 1 relative branching

$$[2^{2q}; 1; 2; 2n, 2n] \rightarrow [2^q; 2; 4; 2n]$$

induces the map on generating vectors

$$((x^n)^{[2q]}; -; x^n; x^{n-1}, x) \rightarrow ((x^n)^{[q]}; c_1; c_2; c_3) \quad \text{if } q \text{ even}$$
$$((x^n)^{[2q]}; -; x^n; x^{n-1}, x) \rightarrow ((x^n)^{[q]}; d_1; d_2; d_3) \quad \text{if } q \text{ odd.}$$

c_2 and d_2 must be elements of order 4 in D_{4n} whose square is x^n. But there are no such elements.

C-VI: The 2 : 1 uniform relative branching

$$[2^{2q}; 2; 2; 2n, 2n] \rightarrow [2^q; 4; 4; 2n] \tag{4.2.3}$$

induces the map on generating vectors

$$\begin{aligned} ((x^n)^{[2q]}; x^n; x^n; x^{-1}, x) &\to ((x^n)^{[q]}; c_1; c_2; c_3) \quad \text{if } q \text{ even} \\ ((x^n)^{[2q]}; x^n; x^n; x^{-1}, x) &\to ((x^n)^{[q]}; d_1; d_2; d_3) \quad \text{if } q \text{ odd.} \end{aligned} \tag{4.2.4}$$

For the same reason as in C-V above, c_i and d_i $(i = 1, 2)$ are not possible elements of D_{4n}.

(b) $\mathbf{Z_2 \times D_{2n} = \langle x, y, t | x^n = y^2 = t^2 = [t, x] = [t, y] = e,}$ $\mathbf{yxy^{-1} = x^{-1}\rangle}$**, n even**

The relevant conjugacy classes are

Order 2:	$\{t\}$, $\{x^{n/2}\}$, $\{tx^{n/2}\}$		
	$\{yx^{2k} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$		
	$\{tyx^{2k} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$		
	$\{yx^{2k+1} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$		
	$\{tyx^{2k+1} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$		
Order 4:	$\{x^j, x^{-j}\}$, $\{tx^j, tx^{-j}\}$	$(j, 2n) = \frac{n}{4}$	(if $n \equiv 0(4)$)
Order n:	$\{x^j, x^{-j}\}$, $\{tx^j, tx^{-j}\}$	$(j, n) = 1$	

Since there are no elements of order $2n$, only branch data of types I, II and III are possible. The index 2 subgroup $\langle t, x\rangle = Z_2 \times Z_n$ can have only the hyperelliptic action (A) in Table 9. So we must examine the unrestricted relative branchings A-I, A-II and A-III.

A-I: The $2:1$ uniform relative branching

$$[2^{2q}; 1; 1; n, n] \to [2^q; 2; 2; n]$$

induces the map on generating vectors

$$\begin{aligned} (t^{[2q]}; -; -; x^{-1}, x) &\to (t^{[q]}; c_1; c_2; c_3) \quad \text{if } q \text{ is even} \\ (t^{[2q]}; -; -; x^{-1}, x) &\to (t^{[q]}; d_1; d_2; d_3) \quad \text{if } q \text{ is odd.} \end{aligned}$$

Using the note in **(a)**, we may assume that $c_3 = d_3 = x$. Then $c_1 c_2 = x^{-1}$ and $d_1 d_2 = tx^{-1}$. $c_i, d_i (i = 1, 2)$ must belong to one of the four conjugacy classes of elements of order 2 not contained in $\langle t, x\rangle$. There are two sets of possibilities:

$$\begin{array}{llll} \text{(i)} & c_1 = yx^{j+1} & c_2 = yx^j \\ & d_1 = tyx^{j+1} & d_2 = yx^j \\ \text{(ii)} & c_1 = tyx^{j+1} & c_2 = tyx^j \\ & d_1 = yx^{j+1} & d_2 = tyx^j, \end{array}$$

where $j = 0, 1, 2, \ldots, n-1$ in all cases. The generating vectors in (ii) are equivalent to the corresponding ones in (i) via the automorphism which sends y to ty and fixes the other generators. In (i), we may take $j = 0$, since all other values of j correspond to generating vectors which are equivalent to this one by conjugation by powers of x, possibly preceded by the automorphism which sends y to yx and fixes the other generators.

A-II: The 2 : 1 uniform relative branching is

$$[2^{2q}; 1; 2; n, n] \rightarrow [2^q; 2; 4; n]$$

with corresponding map on generating vectors

$$\begin{array}{ll} (t^{[2q]}; -; t; tx^{-1}, x) \rightarrow (t^{[q]}; c_1; c_2; c_3) & \text{if } q \text{ is even} \\ (t^{[2q]}; -; t; tx^{-1}, x) \rightarrow (t^{[q]}; d_1; d_2; d_3) & \text{if } q \text{ is odd.} \end{array}$$

However, c_2 and d_2 are not possible elements: there is no element of order 4 in $Z_2 \times D_{2n}$ whose square is t.

A-III: This is not possible by an argument similar to the one above.

(c) $\mathbf{D_{4n} \simeq Z_2 \times D_{2n}}$, n odd

The relevant conjugacy classes are the same as those in (a), except that there are no elements of order 4. Because of this fact, we need consider only branch data of type I and IV. The possible unrestricted relative branchings are A-I and C-IV.

A-I: The argument is almost exactly the same as the corresponding one in **(b)**. The only difference is that the automorphism mentioned in the last sentence is now an inner automorphism (defined as conjugation by the element $x^{-(n+1)/2}$). Hence besides $\{t\}$, there are only two conjugacy classes of elements of order 2, namely, $\{yx^j, j = 0, 1, 2, \ldots n-1\}$ and $\{tyx^j, j = 0, 1, 2, \ldots n-1\}$. But this has no effect on the outcome of the argument.

C-IV: The argument is exactly the same as the corresponding one in **(a)**.

(d) $\mathbf{Q_{4n} = \langle\ x, y \mid x^{2n} = e,\ y^2 = x^n,\ yxy^{-1} = x^{-1} \rangle}$, **n even** $\mathbf{(t = x^n = y^2)}$

Rule:

$$yx^j yx^k = y^2 x^{k-j} = x^{n+k-j}.$$

The relevant conjugacy classes are

Order 2:	$\{x^n\}$	
Order 4:	$\{yx^{2k} \mid k = 0, 1, \ldots, n-1\}$	
	$\{yx^{2k+1} \mid k = 0, 1, \ldots, n-1\}$	
	$\{x^j, x^{-j}\}$	$(j, 2n) = \frac{n}{2}$
Order n:	$\{x^j, x^{-j}\}$	$(j, 2n) = 2$
Order $2n$:	$\{x^j, x^{-j}\}$	$(j, 2n) = 1.$

In any hyperelliptic action of Q_{4n}, the subgroup $\langle x \rangle = Z_{2n}$ must have the hyperelliptic action (C) from Table 9. Since Q_{4n} is not generated by the unique element of order 2, branch data of type I and IV are not possible. Similarly, since the square of every element of order 4 is the element of order 2, the group is not generated by the unique element of order 2 and a single element of order 4. Hence branch data of type II and V are also not possible. (We have used the Remark following Theorem 1.1.) Thus C-VI is the only possible unrestricted relative branching.

C-VI: The 2 : 1 uniform relative branching and induced map on generating vectors are (4.2.3) and (4.2.4), respectively. As in previous arguments, we assume that $c_1 = c_3 = x$ in (4.2.4). Then c_1, c_2 are elements of order 4 satisfying $c_1 c_2 = x^{-1}$, and d_1, d_2 are elements of order 4 satisfying $d_1 d_2 = x^{n-1}$. The possibilities are

$$c_1 = yx^j, \quad c_2 = yx^{j+n-1}$$
$$d_1 = yx^j, \quad d_2 = yx^{j-1},$$

where $j = 0, 1, \ldots, 2n-1$. We may take $j = 0$ in both cases, since generating vectors corresponding to other values of j are equivalent under conjugation by powers of x, possibly preceded by the automorphism which sends $y \mapsto yx$ and fixes x.

(e) $\mathbf{D_{4,n,-1} = \langle\, x, y \mid x^n = y^4 = e,\; yxy^{-1} = x^{-1} \rangle}$
n even $\quad \mathbf{(t = y^2)}$

Rule:

$$yx^j \cdot yx^k = y^2 x^{k-j} = y^{-1}x^j \cdot y^{-1}x^k$$

The relevant conjugacy classes are

Order 2:	$\{y^2\}, \quad \{x^{n/2}\}, \quad \{y^2 x^{n/2}\}$
Order 4:	$\{yx^{2k} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$
	$\{y^{-1}x^{2k} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$
	$\{yx^{2k+1} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$
	$\{y^{-1}x^{2k+1} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$
	$\{x^j, x^{-j}\}, \quad \{y^2x^j, y^2x^{-j}\} \quad (j, 2n) = \frac{n}{4}$ (if $n \equiv 0(4)$)
Order n:	$\{x^j, x^{-j}\}, \quad \{y^2x^j, y^2x^{-j}\} \quad (j, 2n) = 2$

Since there are no elements of order $2n$, branch data of type IV, V and VI are not possible. I is also not possible since the elements of order 2 do not generate the group. In any hyperelliptic action of the group, the index 2 subgroup $\langle y^2, x\rangle = Z_2 \times Z_n$ must have the hyperelliptic action (A) from Table 9. So the possible unrestricted relative branchings are A-II and A-III.

A-II: The 2 : 1 uniform relative branching is

$$[2^{2q}; 1; 2; n, n] \to [2^q; 2; 4; n].$$

The induced map on generating vectors is

$$((y^2)^{[2q]}; -; y^2; y^2x^{-1}, x) \to ((y^2)^{[q]}; c_1; c_2; c_3) \quad \text{if } q \text{ is even}$$
$$((y^2)^{[2q]}; -; y^2; y^2x^{-1}, x) \to ((y^2)^{[q]}; d_1; d_2; d_2) \quad \text{if } q \text{ is odd.}$$

(Since the subgroup $Z_2 \times Z_n = \langle y^2, x\rangle$, we have used y^2 instead of t in the generating vector on the left-hand side.) However, c_1 and d_1 are not possible elements: there are no elements of order 2 which do not belong to $\langle y^2, x\rangle$.

A-III: The $2:1$ uniform relative branching is

$$[2^{2q}; 2; 2; n, n] \to [2^q; 4; 4; n].$$

The induced map on generating vectors is

$$((y^2)^{[2q]}; y^2; y^2; x^{-1}, x) \to ((y^2)^{[q]}; c_1; c_2; c_3) \quad \text{if } q \text{ is even}$$
$$((y^2)^{[2q]}; y^2; y^2; x^{-1}, x) \to ((y^2)^{[q]}; d_1; d_2; d_3) \quad \text{if } q \text{ is odd.}$$

Assuming $c_3 = d_3 = x$, we have: c_i, d_i, $i = 1, 2$ are elements of order 4 with $c_1c_2 = x^{-1}$ and $d_1d_2 = y^2x^{-1}$. None of these elements can be of the form $x^{\pm j}$ or $y^2x^{\pm j}$, where $(j, n) = n/4$, since the product of any two such elements, or an element of this form with one of the other elements of order 4, cannot equal x^{-1} or y^2x^{-1}. There are two sets of possibilities:

$$\begin{array}{lll} \text{(i)} & c_1 = yx^{2k+1} & c_2 = y^{-1}x^{2k} \\ & d_1 = yx^{2k+1} & d_2 = yx^{2k} \\ \text{(ii)} & c_1 = yx^{2k} & c_2 = y^{-1}x^{2k-1} \\ & d_1 = yx^{2k} & d_2 = yx^{2k-1}, \end{array}$$

where $k = 0, 1, 2, \ldots (n-2)/2$. Generating vectors in (ii) are equivalent to the corresponding ones in (i) by the automorphism which sends $y \mapsto yx$ and fixes x. In (i) we may take $k = 0$, since generating vectors corresponding to other values of k are equivalent under conjugation by a power of x.

(f) $\mathbf{Q_{4n} \simeq D_{4,n,-1}}$, n odd

We may eliminate branch data of types I,II, IV and V for the reasons given at the beginning of **(d)**. The index 2 subgroup $Z_{2n} \simeq \langle x \rangle \subseteq Q_{4n}$, ($\simeq \langle y^2, x \rangle \subseteq D_{4,n,-1}$), can have hyperelliptic action of type A, B or C from Table 9. The possible unrestricted relative $Z_{2n} \simeq \langle x \rangle \subseteq Q_{4n}$, ($\simeq \langle y^2, x \rangle \subseteq D_{4,n,-1}$), can have hyperelliptic action of type A, B or C from Table 9. The possible unrestricted relative branchings are A-III and C-VI.

A-III: The argument is almost exactly the same as the one given in **(e)**. The only difference is that the automorphism which sends y to yx and fixes x is now an inner automorphism. So there are only two conjugacy classes of elements of order 4, namely, $\{yx^j \mid j = 0, 1, 2 \ldots n-1\}$ and $\{y^{-1}x^j \mid j = 0, 1, 2 \ldots n-1\}$. But this has no effect on the outcome of the argument.

C-VI: The argument is exactly the same as the one given in **(d)**.

(g) $\mathbf{G_{4n} = \langle x, y, t \mid x^n = y^2 = t^2 = e,\ yxy^{-1} = tx^{-1}, [x, t] = [y, t] = e \rangle}$, n even.

Rule:

$$yx^j yx^k = \begin{cases} tx^{k-j} & \text{if } j \text{ odd;} \\ x^{k-j} & \text{if } j \text{ even.} \end{cases} \tag{4.2.5}$$

The rule implies that yx^j and tyx^j have order 4 if j is odd (their square is t), and order 2 if j is even.

The relevant conjugacy classes depend on whether $n \equiv 0(4)$ or $n \equiv 2(4)$.

$$n \equiv 0(4)$$

Order 2:	$\{t\}, \quad \{x^{n/2}\}, \quad \{tx^{n/2}\}$
	$\{yx^{4k}, tyx^{4k+2} \mid k = 0, 1, \ldots, \frac{n-4}{4}\}$
	$\{yx^{4k+2}, tyx^{4k} \mid k = 0, 1, \ldots, \frac{n-4}{4}\}$
Order 4:	$\{yx^{4k+1}, tyx^{4k+3} \mid k = 0, 1, \ldots, \frac{n-4}{4}\}$
	$\{yx^{4k+3}, tyx^{4k+1} \mid k = 0, 1, \ldots, \frac{n-4}{4}\}$
	$\{x^{n/4}, x^{-n/4}\}, \quad \{tx^{n/4}, tx^{-n/4}\} \quad (n \equiv 0(8))$
	$\{x^{n/4}, tx^{-n/4}\} \quad \{tx^{n/4}, x^{-n/4}\} \quad (n \equiv 4(8))$
Order n:	$\{x^j, tx^{-j}\} \quad (j, n) = 1.$

$$n \equiv 2(4)$$

Order 2:	$\{t\}, \quad \{x^{n/2}, tx^{n/2}\}$
	$\{yx^{2k}, tyx^{2k} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$
Order 4:	$\{yx^{2k+1}, tyx^{2k+1} \mid k = 0, 1, \ldots, \frac{n-2}{2}\}$
Order n:	$\{x^j, tx^{-j}\} \quad (j, n) = 1$
	$\{tx^j, tx^{-j}\} \quad \{x^j, x^{-j}\} \quad (j, n) = 2.$

Since G_{4n} contains no elements of order $2n$, we need not consider branch data of type IV, V and VI. The index 2 subgroup $\langle t, x\rangle \simeq Z_2 \times Z_n$ can only have the hyperelliptic action (A) from Table 9. The possible unrestricted relative branchings are therefore A-I, A-II and A-III.

A-I: The 2:1 uniform relative branching is

$$[2^{2q}; 1; 1; n, n] \rightarrow [2^q; 2; 2; n],$$

which induces the map on generating vectors

$$(t^{[2q]}; -; -; x^{-1}, x) \rightarrow (t^{[q]}; c_1; c_2; c_3) \qquad \text{if q is even}$$
$$(t^{[2q]}; -; -; x^{-1}, x) \rightarrow (t^{[q]}; d_1; d_2; d_3) \qquad \text{if q is odd.}$$

If $n \equiv 2(4)$, $c_3, d_3 \neq tx^j$, $(j,n) = 2$ since x, $x^{-1} \notin \langle tx^j \rangle$. Thus c_3 and d_3 belong to one of the conjugacy classes $\{x^j, tx^{-j}\}$, $(j,n) = 1$. By the usual argument, we may assume $j = 1$. A generating vector with, say, c_3 equal to tx^{-1} is equivalent under the automorphism conjugation by y to one in which $c_3 = x$. Thus we may assume $c_3 = d_3 = x$. This implies $c_1c_2 = x^{-1}$ and $d_1d_2 = tx^{-1}$. However, there are no two elements of order 2 whose product is x^{-1} or tx^{-1}

A-II: Equating the expressions for n in (A) of Table 8 and II of Table 7, we obtain $p = 1 + 2q$. The 2:1 uniform relative branching is

$$[2^{2q}; 1; 2; n, n] \to [2^q; 2; 4; n].$$

The induced map on generating vectors is of the form

$$(t^{[2q]}; -; t; tx^{-1}, x) \to (t^{[q]}; c_1; c_2; c_3) \qquad \text{if q is even}$$
$$(t^{[2q]}; -; t; tx^{-1}, x) \to (t^{[q]}; d_1; d_2; d_3) \qquad \text{if q is odd,}$$

where c_1, d_1 are elements of the form yx^j, tyx^j, j even or 0, and c_2 and d_2 are elements of the same form with j odd. By the same argument as in A-I, we may assume $c_3 = d_3 = x$. Then $c_1c_2 = x^{-1}$ and $d_1d_2 = tx^{-1}$. We then have two possibilities:

(i)	$c_1 = yx^j$	$c_2 = yx^{j-1}$	j even or 0
	$d_1 = yx^j$	$d_2 = tyx^{j-1}$	j even or 0
(ii)	$c_1 = tyx^j$	$c_2 = tyx^{j-1}$	j even or 0
	$d_1 = tyx^j$	$d_2 = yx^{j-1}$	j even or 0.

Any generating vector of type (ii) is equivalent to the corresponding one in (i) under the order 2 automorphism of G_{4n} which takes $y \mapsto ty$ and fixes the other generators. Finally, we may take $j = 0$ in (i), since generating vectors corresponding to other values of j may be obtained from it by conjugation by a power of x, possibly preceded by the outer automorphism which sends $y \mapsto yx^2$ and fixes the other generators.

A-III: The 2:1 uniform relative branching

$$[2^{2q}; 2; 2; n, n] \mapsto [2^q; 4; 4; n].$$

induces the map on generating vectors

$$(t^{[2q]}, t, t, x^{-1}, x) \to (t^{[q]}, c_1, c_2, c_3) \qquad \text{if q is even}$$
$$(t^{[2q]}, t, t, x^{-1}, x) \to (t^{[q]}, d_1, d_2, d_3) \qquad \text{if q is odd.}$$

We assume as above that $c_3 = d_3 = x$, which implies $c_1c_2 = x^{-1}$ and $d_1d_2 = tx^{-1}$ Now c_i, d_i $(i = 1, 2)$ are elements of order 4. We exclude the possibility that some of these elements are conjugates of $x^{n/4}$ or $tx^{n/4}$ (if $n \equiv 0(4)$), since it must be true that $c_i^2 = d_i^2 = t$. c_i, d_i must therefore be of the form yx^j or tyx^j, j even or 0. However, no product of elements of this form can equal x or tx^{-1}.

(h) $\mathbf{SD_{4n} = \langle x, y \mid x^{2n} = y^2 = e,\ yxy^{-1} = x^{n-1}\rangle}$, n even $\mathbf{(t{=}x^n)}$

Rule:

$$yx^jyx^k = \begin{cases} x^{k-j} & \text{if } j \text{ even} \\ x^{n+k-j} & \text{if } j \text{ odd.} \end{cases}$$

The relevant conjugacy classes of elements of SD_{4n}, n even are:

Order 2:	$\{x^n\}$, $\{yx^j \mid j \text{ even or } 0\}$
Order 4:	$\{yx^j \mid j \text{ odd}\}$
	$\{x^{n/2}, x^{3n/2}\}$ if n $\equiv 0(4)$
	$\{x^{n/2}\}$, $\{x^{3n/2}\}$ if n $\equiv 2(4)$
Order n:	$\{x^j,\ x^{-j} \mid (j, 2n) = 2\}$
Order $2n$:	$\{x^j,\ x^{n-j} \mid (j, 2n) = 1\}$

Since the elements of order 2 do not generate the group (they generate the subgroup $\langle y, x^2\rangle \simeq D_{2n}$), we may eliminate branching data of types I and IV (see the Remark following Theorem 1.1). In any hyperelliptic action of SD_{4n}, the subgroup $\langle x\rangle \simeq Z_{2n}$ must have the hyperelliptic action (C) from Table 9. We consider the unrestricted relative branchings C-V and C-VI.

C-V: The 2 : 1 uniform relative branching

$$[2^{2q}; 1; 2; 2n, 2n] \to [2^q; 2; 4; 2n]$$

induces the map on generating vectors

$$((x^n)^{[2q]}; -; x^n; x^{n-1}, x) \to ((x^n)^{[q]}; c_1; c_2; c_3) \quad \text{if } q \text{ is even}$$
$$((x^n)^{[2q]}; -; x^n; x^{n-1}, x) \to ((x^n)^{[q]}; d_1; d_2; d_3) \quad \text{if } q \text{ is odd.}$$

We assume $c_3 = d_3 = x$. Then we have $c_1c_2 = x^{-1}$ and $d_1d_2 = x^{n-1}$. c_1 and d_1 belong to the conjugacy class $\{yx^j \mid j \text{ even or } 0\}$; c_2 and d_2 to the conjugacy class $\{yx^j \mid j \text{ odd}\}$. (It is not possible that c_2 or $d_2 = x^{n/2}$, since no product of the form $yx^j \cdot x^{n/2}$ can equal x or x^{-1}.) The possibilities are:

$$c_1 = yx^j, \quad c_2 = yx^{j-1},$$
$$d_1 = yx^j, \quad d_2 = yx^{j+n-1},$$

where j is even or 0. We may take $j = 0$, since generating vectors corresponding to other values of j are equivalent under conjugation by powers of x.

C-VI: The $2:1$ uniform relative branching and induced map on generating vectors are (4.2.3) and (4.2.4), respectively. Assuming $c_3 = d_3 = x$ in (4.2.4), we have $c_1c_2 = x^{-1}$ and $d_1d_2 = x^{n-1}$. c_1, c_2, d_1 and d_2 are either $x^{n/2}$, or elements of the form yx^j, where j is odd. However, no product of two such elements can equal x or x^{-1}. So C-VI is not a possible relative branching for SD_{4n}.

(i) The Groups $\mathbf{D_8}$ and $\mathbf{Z_2 \times Z_4}$

Setting $n = 2$ in first and fifth rows of Table 10, and solving for q, one obtains, in both cases, $q = g/2$, and branch data $(2^q, 2, 2, 4)$. Because of isomorphism 2.2.6, the groups are both isomorphic to D_8. It is evident (after correcting for the different presentations of the two groups), that the two generating vectors are not the same. Nonetheless, they are equivalent: the argument in **(a)** can be adapted to show that there are exactly eight possible $(2^q, 2, 2, 4)$-generating vectors for a hyperelliptic action of D_8. $\mathrm{Aut}(D_8) \simeq D_8$ acts freely on the set of $(2^q, 2, 2, 4)$ generating vectors, since an automorphism which fixes a generating set must be trivial. Since the number of automorphisms equals the number of generating vectors, all the generating vectors are equivalent. Hence there is only one topological type of hyperelliptic D_8 action.

The group $Z_2 \times Z_4$ appears in Table 10 as SD_8 and $D_{4,2,-1}$ (see isomorphism 2.2.7). Setting $n = 2$ in the appropriate rows of Table 10 and solving for q shows that both actions occur on surfaces of odd genus with branch data $(2^{(g-1)/2}, 4, 4, 2)$. After correcting for the different presentations of the two groups, and reordering (which we may do since the group is abelian), one sees that the two generating vectors are equivalent by the automorphism sending an element of order 4 to its inverse. $Z_2 \times Z_4$ also occurs in Table 9 (A) with the same branch data. However, this action is not equivalent to the preceding one. The generating vectors differ in the number of occurrences of the hyperelliptic involution, since $p \neq q$. By a result of Broughton ([B], Proposition 2.6), two generating vectors for equivalent actions of an abelian group with quotient a sphere, can differ, up to automorphisms of the group, only by a permutation of their elements. But the hyperelliptic involution is invariant under all automorphisms.

Note. To distinguish the two topologically distinct hyperelliptic actions of $Z_2 \times Z_4$, we will henceforth use the notation $(Z_2) \times Z_4$ for the action of reduced type D_4 ($D_{4,2,-1}$ and SD_8) and $Z_2 \times (Z_4)$ for the action of reduced type Z_4. The parentheses indicate the factor that does not contain t.

(j) Restricted Relative Branching Types.

We now show that the relative branching types which are defined only when $n = 1$ or 2 give no hyperelliptic actions other than those already obtained.

The case $n = 1$ can be interpreted if we define D_2 (the dihedral group of order 2) to be Z_2. But the central extensions of Z_2 by D_2 are just the groups $Z_2 \times Z_2$ and Z_4, whose hyperelliptic actions have already been obtained and and shown to be topologically unique.

The 2 : 1 uniform relative branchings that are defined when $n = 2$ but for no larger values of n are

C-II, n=2	$[2^g;\ 1;\ 1;\ 4,4]$	$\rightarrow$	$[2^{g/2}; 2; 2; 4]$	g even
C-III, n=2	$[2^{g-1};\ 1;\ 2;\ 4,4]$	$\rightarrow$	$[2^{(g-1)/2}; 2; 4; 4]$	g odd
A-IV, n=2	$[2^g;\ 2,2;\ 1;\ 2]$	$\rightarrow$	$[2^{g/2}; 2; 2; 4]$	g even
A-V, n=2	$[2^{g-1};\ 2,2;\ 2;\ 2]$	$\rightarrow$	$[2^{(g-1)/2}; 2; 4; 4]$	g odd.

The last three are just repeats of the unrestricted relative branchings C-V, A-II and A-III, respectively, with $n = 2$ (cf. Table 11). If C-II is possible, the branching data on the right-hand side is the branching data for one of Q_8, D_8, $Z_2 \times Z_4$ or $(Z_2)^3$. We may eliminate the first three since these groups are not generated by elements of order 2 (see the Remark following Theorem 1.1); $(Z_2)^3$ is eliminated since it contains no element of order 4.

There are two other restricted relative branchings, C-I and A-VI, $n = 2$, which are numerically but not topologically possible : for C-I, the branching would be of the form $(2^p, 4, 4) \rightarrow (2^q, 2, 2, 2)$, which is not possible since there are no branching indices on the right which are multiples of 4 (see Lemma 1.3). For A-VI, the branching would be of the form $(2^p, 2, 2) \rightarrow (2^q, 4, 4, 4)$, which is not possible since it implies the existence of an orientation-preserving Z_2 action on the Riemann sphere which fixes three points.

4.3 Hyperelliptic Actions of Reduced Type $\mathbf{A_4}$, $\mathbf{A_5}$, $\mathbf{S_4}$

The possible branching data for actions of reduced type A_4, A_5 and S_4 are given in Tables 12, 13 and 14. They are calculated from the given orbit equations. Recall that $\epsilon_1, \epsilon_2, \epsilon_3$ are either 0 or 1, and q is a nonnegative integer (see §3). (In the case of reduced type A_4, the eight possible ordered triples $(\epsilon_1, \epsilon_2, \epsilon_3)$ give rise to only six distinct types of branching data.) The possible groups are the ones listed in Table 5, §2. We use the presentations given there. Tables 15, 16 and 17 give the hyperelliptic actions of these groups.

Table 12. *Possible Branching Data for Hyperelliptic Actions of Reduced Type A_4*

(*Orbit Equation:* $6\epsilon_1 + 4\epsilon_2 + 4\epsilon_3 + 12q = 2g + 2$)

Type	$(\epsilon_1, \epsilon_2, \epsilon_3)$	*Branch Data*	q	*Restr. on* g
(a)	$(0,0,0)$	$(2^q, 3, 3, 2)$	$\frac{g+1}{6}$	$g \equiv 5(6)$
(b)	$(0,1,0)$	$(2^q, 3, 6, 2)$	$\frac{g-1}{6}$	$g \equiv 1(6)$
(c)	$(0,1,1)$	$(2^q, 6, 6, 2)$	$\frac{g-3}{6}$	$g \equiv 3(6)$
(d)	$(1,0,0)$	$(2^q, 3, 3, 4)$	$\frac{g-2}{6}$	$g \equiv 2(6)$
(e)	$(1,1,0)$	$(2^q, 3, 6, 4)$	$\frac{g-4}{6}$	$g \equiv 4(6)$
(f)	$(1,1,1)$	$(2^q, 6, 6, 4)$	$\frac{g-6}{6}$	$g \equiv 0(6)$

Table 13. *Possible Branching Data for Hyperelliptic Actions of Reduced Type A_5*

(*Orbit Equation:* $30\epsilon_1 + 20\epsilon_2 + 12\epsilon_3 + 60q = 2g + 2$)

Type	$(\epsilon_1, \epsilon_2, \epsilon_3)$	*Branch Data*	q	*Restr. on* g
(a)	$(0,0,0)$	$(2^q, 2, 3, 5)$	$\frac{g+1}{30}$	$g \equiv 29(30)$
(b)	$(0,0,1)$	$(2^q, 2, 3, 10)$	$\frac{g-5}{30}$	$g \equiv 5(30)$
(c)	$(0,1,0)$	$(2^q, 2, 6, 5)$	$\frac{g-9}{30}$	$g \equiv 9(30)$
(d)	$(0,1,1)$	$(2^q, 2, 6, 10)$	$\frac{g-15}{30}$	$g \equiv 15(30)$
(e)	$(1,0,0)$	$(2^q, 4, 3, 5)$	$\frac{g-14}{30}$	$g \equiv 14(30)$
(f)	$(1,0,1)$	$(2^q, 4, 3, 10)$	$\frac{g-20}{30}$	$g \equiv 20(30)$
(g)	$(1,1,0)$	$(2^q, 4, 6, 5)$	$\frac{g-24}{30}$	$g \equiv 24(30)$
(h)	$(1,1,1)$	$(2^q, 4, 6, 10)$	$\frac{g-30}{30}$	$g \equiv 0(30)$

Table 14. *Possible Branching Data for Hyperelliptic Actions of Reduced Type S_4*

(*Orbit Equation:* $12\epsilon_1 + 8\epsilon_2 + 6\epsilon_3 + 24q = 2g + 2$)

Type	$(\epsilon_1, \epsilon_2, \epsilon_3)$	*Branch Data*	q	*Restr. on* g
(a)	$(0,0,0)$	$(2^q, 2, 3, 4)$	$\frac{g+1}{12}$	$g \equiv 11(12)$
(b)	$(0,1,0)$	$(2^q, 2, 6, 4)$	$\frac{g-3}{12}$	$g \equiv 3(12)$
(c)	$(1,0,0)$	$(2^q, 4, 3, 4)$	$\frac{g-5}{12}$	$g \equiv 5(12)$
(d)	$(1,1,0)$	$(2^q, 4, 6, 4)$	$\frac{g-9}{12}$	$g \equiv 9(12)$
(e)	$(0,0,1)$	$(2^q, 2, 3, 8)$	$\frac{g-2}{12}$	$g \equiv 2(12)$
(f)	$(0,1,1)$	$(2^q, 2, 6, 8)$	$\frac{g-6}{12}$	$g \equiv 6(12)$
(g)	$(1,0,1)$	$(2^q, 4, 3, 8)$	$\frac{g-8}{12}$	$g \equiv 8(12)$
(h)	$(1,1,1)$	$(2^q, 4, 6, 8)$	$\frac{g-12}{12}$	$g \equiv 0(12)$

Table 15. *Hyperelliptic Actions of Reduced Type A_4*

	Group	Branch Data	q	Generating Vectors	
(a)	$A_4 \times Z_2$	$(2^q,3,3,2)$	$\frac{g+1}{6}$	$(t^{[q]},x,y,xy)$	$g \equiv 11(12)$
				$(t^{[q]},x,y,txy)$	$g \equiv 5(12)$
(b)	$A_4 \times Z_2$	$(2^q,3,6,2)$	$\frac{g-1}{6}$	$(t^{[q]},x,ty,txy)$	$g \equiv 1(12)$
				$(t^{[q]},x,ty,xy)$	$g \equiv 7(12)$
(c)	$A_4 \times Z_2$	$(2^q,6,6,2)$	$\frac{g-3}{6}$	$(t^{[q]},tx,ty,xy)$	$g \equiv 3(12)$
				$(t^{[q]},tx,ty,txy)$	$g \equiv 9(12)$
(d)	$SL_2(Z_3)$	$(2^q,3,3,4)$	$\frac{g-2}{6}$	$(t^{[q]},x,y,(xy)^{-1})$	$g \equiv 2(12)$
				$(t^{[q]},x,y,xy)$	$g \equiv 8(12)$
(e)	$SL_2(Z_3)$	$(2^q,3,6,4)$	$\frac{g-4}{6}$	$(t^{[q]},x,ty,(xy)^{-1})$	$g \equiv 10(12)$
				$(t^{[q]},x,ty,xy)$	$g \equiv 4(12)$
(f)	$SL_2(Z_3)$	$(2^q,6,6,4)$	$\frac{g-6}{6}$	$(t^{[q]},tx,ty,(xy)^{-1})$	$g \equiv 6(12)$
				$(t^{[q]},tx,ty,xy)$	$g \equiv 0(12)$

Table 16. *Hyperelliptic Actions of Reduced Type A_5*

	Group	Branch Data	q	Generating Vectors
(a)	$A_5 \times Z_2$	$(2^q,2,3,5)$	$\frac{g+1}{30}$	$(t^{[q]},x,y,(xy)^{-1})$ $g\equiv 59(60)$ $(t^{[q]},tx,y,(xy)^{-1})$ $g\equiv 29(60)$
(b)	$A_5 \times Z_2$	$(2^q,2,3,10)$	$\frac{g-5}{30}$	$(t^{[q]},tx,y,t(xy)^{-1})$ $g\equiv 5(60)$ $(t^{[q]},x,y,t(xy)^{-1})$ $g\equiv 35(60)$
(c)	$A_5 \times Z_2$	$(2^q,2,6,5)$	$\frac{g-9}{30}$	$(t^{[q]},tx,ty,(xy)^{-1})$ $g\equiv 9(60)$ $(t^{[q]},x,ty,(xy)^{-1})$ $g\equiv 39(60)$
(d)	$A_5 \times Z_2$	$(2^q,2,6,10)$	$\frac{g-15}{30}$	$(t^{[q]},x,ty,t(xy)^{-1})$ $g\equiv 15(60)$ $(t^{[q]},tx,ty,t(xy)^{-1})$ $g\equiv 45(60)$
(e)	$SL_2(Z_5)$	$(2^q,4,3,5)$	$\frac{g-14}{30}$	$(t^{[q]},x,y,(xy)^{-1})$ $g\equiv 14(60)$ $(t^{[q]},tx,y,(xy)^{-1})$ $g\equiv 44(60)$
(f)	$SL_2(Z_5)$	$(2^q,4,3,10)$	$\frac{g-20}{30}$	$(t^{[q]},tx,y,t(xy)^{-1})$ $g\equiv 20(60)$ $(t^{[q]},x,y,t(xy)^{-1})$ $g\equiv 50(60)$
(g)	$SL_2(Z_5)$	$(2^q,4,6,5)$	$\frac{g-24}{30}$	$(t^{[q]},tx,ty,(xy)^{-1})$ $g\equiv 24(60)$ $(t^{[q]},x,ty,(xy)^{-1})$ $g\equiv 54(60)$
(h)	$SL_2(Z_5)$	$(2^q,4,6,10)$	$\frac{g-30}{30}$	$(t^{[q]},x,ty,t(xy)^{-1})$ $g\equiv 30(60)$ $(t^{[q]},tx,ty,t(xy)^{-1})$ $g\equiv 0(60)$

Table 17. *Hyperelliptic Actions of Reduced Type S_4*

	Group	Branch Data	q	Generating Vectors
(a)	$S_4 \times Z_2$	$(2^q,2,3,4)$	$\frac{g+1}{12}$	$(t^{[q]},x,y,(xy)^{-1})$ $g \equiv 23(24)$ $(t^{[q]},x,y,t(xy)^{-1})$ $g \equiv 11(24)$
(b)	$S_4 \times Z_2$	$(2^q,2,6,4)$	$\frac{g-3}{12}$	$(t^{[q]},x,ty,t(xy)^{-1})$ $g \equiv 3(24)$ $(t^{[q]},x,ty,(xy)^{-1})$ $g \equiv 15(24)$
(c)	$SL_2(Z_4)$	$(2^q,4,3,4)$	$\frac{g-5}{12}$	$(t^{[q]},x,y,(xy)^{-1})$ $g \equiv 5(24)$ $(t^{[q]},x,y,t(xy)^{-1})$ $g \equiv 17(24)$
(d)	$SL_2(Z_4)$	$(2^q,4,6,4)$	$\frac{g-9}{12}$	$(t^{[q]},x,ty,t(xy)^{-1})$ $g \equiv 9(24)$ $(t^{[q]},x,ty,(xy)^{-1})$ $g \equiv 21(24)$
(e)	$GL_2(Z_3)$	$(2^q,2,3,8)$	$\frac{g-2}{12}$	$(t^{[q]},x,y,(xy)^{-1})$ $g \equiv 2(24)$ $(t^{[q]},x,y,t(xy)^{-1})$ $g \equiv 14(24)$
(f)	$GL_2(Z_3)$	$(2^q,2,6,8)$	$\frac{g-6}{12}$	$(t^{[q]},x,ty,t(xy)^{-1})$ $g \equiv 6(24)$ $(t^{[q]},x,ty,(xy)^{-1})$ $g \equiv 18(24)$
(g)	O^*	$(2^q,4,3,8)$	$\frac{g-8}{12}$	$(t^{[q]},x,y,(xy)^{-1}))$ $g \equiv 8(24)$ $(t^{[q]},x,y,t(xy)^{-1})$ $g \equiv 20(24)$
(h)	O^*	$(2^q,4,6,8)$	$\frac{g-12}{12}$	$(t^{[q]},x,ty,t(xy)^{-1})$ $g \equiv 12(24)$ $(t^{[q]},x,ty,(xy)^{-1})$ $g \equiv 0(24)$

The rest of this section is devoted to showing that all possible generating vectors for hyperelliptic actions of reduced type A_4, A_5 and S_4 are equivalent to the ones given in Tables 15, 16 and 17. (One may easily verify from the presentations in Table 5, §2, that the vectors given in the tables do generate the corresponding groups, and have the required form.) We use an approach, different from the previous cases, which is due to Broughton ([B], [B1]). Let G be a finite group and $\mathcal{V}$ the set of $(r_1, \ldots, r_k)$-generating vectors for an action of G. $\mathrm{Aut}(G)$ acts freely on $\mathcal{V}$, since an automorphism which fixes a generating set must be trivial. This implies that for $\alpha_1, \alpha_2 \in \mathrm{Aut}(G)$ and $v \in \mathcal{V}$, $\alpha_1 v = \alpha_2 v \iff \alpha_1 = \alpha_2$. Hence, $|\mathrm{Aut}(G)| \leq |\mathcal{V}|$. To show that the generating vectors in $\mathcal{V}$ are equivalent, it suffices to show that $|\mathrm{Aut}(G)| \geq |\mathcal{V}|$. The following formula from the character theory of finite groups proves extremely useful (a related formula, from which the one below is easily derivable, is proved in [Go], Theorem 2.12). Let K_1, K_2, ..., K_s be conjugacy classes in G, and let $X(K_1, \ldots, K_s)$ be the set of distinct s-tuples $(g_1, \ldots, g_s)$ of elements of G with $g_i \in K_i$, such that the product $g_1 g_2 \cdots g_s = 1$. An element of $X(K_1, \ldots, K_s)$ is called a $(K_1, \ldots, K_s)$-vector. Then

$$|X(K_1, \ldots, K_s)| =$$

$$\frac{|G|^{s-1}}{|\mathrm{Cent}\ (g_1)| \cdot \cdots \cdot |\mathrm{Cent}\ (g_s)|} \sum_{\chi} \frac{\chi(g_1) \cdot \cdots \cdot \chi(g_s)}{\chi(1)^{s-2}}, \qquad (4.3.1)$$

where the sum is over the irreducible characters of G. We shall use this formula to determine $|\mathcal{V}|$.

$\mathbf{A_4 \times Z_2}$

$A_4 \times Z_2$ cannot have branch data of type (d) (e) or (f) from Table 12, since it contains no elements of order 4. Any $(2^q, 3, 3, 2)$-generating vector for a hyperelliptic action of $A_4 \times Z_2$ must be of the form

$$\begin{array}{lll} (t^{[q]}, c_1, c_2, c_3) & \text{if } q \text{ even} & (g \equiv 11\ (12)) \\ (t^{[q]}, c_1, c_2, tc_3) & \text{if } q \text{ odd} & (g \equiv 5\ (12)), \end{array} \qquad (4.3.2)$$

where (c_1, c_2, c_3) is a $(3,3,2)$-generating vector for A_4. Note that all $(3,3,2)$-vectors in A_4 are generating vectors, since A_4 contains no subgroups of order 6. Let 3_1 and 3_2 denote the two conjugacy classes of elements of order 3 in A_4, and 2_1 the single conjugacy class of elements of order 2. Using (4.3.1) and the character table for A_4 (see Appendix A), we calculate

$$|X(3_1, 3_1, 2_1)| = |X(3_2, 3_2, 2_1)| = 0$$

and

$$|X(3_1, 3_2, 2_1)| = |X(3_2, 3_1, 2_1)| = 12.$$

Thus there are 24 $(3,3,2)$ generating vectors in A_4, and hence 24 $(2^q, 3, 3, 2)$-generating vectors for $A_4 \times Z_2$ of the form (4.3.2). Now $\mathrm{Aut}(A_4) \simeq S_4$ has order 24, so all $(3,3,2)$ vectors in A_4 are equivalent. Every automorphism of A_4 extends to a unique automorphism of $A_4 \times Z_2$ which fixes t. Thus $|\mathrm{Aut}(A_4 \times Z_2)| \geq 24$, and so all 24 $(2^q, 3, 3, 2)$-generating vectors for $A_4 \times Z_2$ of the form (4.3.2) are equivalent.

The argument above is easily adapted to branching data of types (b) $(2^q, 3, 6, 2)$ and (c) $(2^q, 6, 6, 2)$. For example, any $(2^q, 3, 6, 2)$-generating vector for a hyperelliptic action of $A_4 \times Z_2$ must be of the form

$$\begin{aligned} &(t^{[q]}, c_1, tc_2, tc_3) \quad \text{if } q \text{ even} \quad (g \equiv 1 \ (12)) \\ &(t^{[q]}, c_1, tc_2, c_3) \quad \text{if } q \text{ odd} \quad (g \equiv 7 \ (12)), \end{aligned}$$

where, as in the previous case, (c_1, c_2, c_3) is a $(3,3,2)$-generating vector for A_4. The argument proceeds exactly as before.

$\mathbf{SL_2(Z_3)}$

$SL_2(Z_3)$ cannot have branching data of type (a), (b) or (c) from Table 12 since any generating vector of one of these types would have the form

$$(t^{[q+1]}, c_1, c_2),$$

where c_1c_2 equals t or e, depending on the parity of q (t being the unique element of order 2). Let K be the group generated by t, c_1 and c_2. We claim that K cannot be all of $SL_2(Z_3)$. Let C_1 and C_2 be the images of c_1 and c_2 under the surjective homomorphism ϕ in the exact sequence

$$\langle t\rangle \hookrightarrow SL_2(Z_3) \xrightarrow{\phi} A_4.$$

C_1 and C_2 have order 3 in A_4, and their product is the identity; that is, $C_2 = C_1^{-1}$. Thus K is the preimage under a 2 : 1 homomorphism of a cyclic group of order 3, and so has order at most 6.

Any vector in $SL_2(Z_3)$ of the form (d), (e) or (f) from Table 12 generates the whole group, since there are no subgroups of order 12. There are two conjugacy classes (3_1 and 3_2) of elements of order 3 in $SL_2(Z_3)$, two of order 6 (6_1 and 6_2), and one of order 4 (4_1). Using the character table of $SL_2(Z_3)$ (see Appendix A) and (4.3.1) we derive

$$|X((2_1)^q, 3_1, 3_2, 4_1)| = |X((2_1)^q, 3_2, 3_1, 4_1)| = 12$$

and

$$|X((2_1)^q, 3_1, 3_1, 4_1)| = |X((2_1)^q, 3_2, 3_2, 4_1)| = 0.$$

Thus there are 24 $(2^q, 3, 3, 4)$-generating vectors in $SL_2(Z_3)$. Now $GL_2(Z_3)$ acts by conjugation on $SL_2(Z_3)$ with kernel $\pm I$, so

$$|\mathrm{Aut}(SL_2(Z_3))| \geq |PGL_2(Z_3)| = |S_4| = 24,$$

and hence all the $(2^q, 3, 3, 4)$-generating vectors are equivalent. (In fact this argument shows that

$$\mathrm{Aut}(SL_2(Z_3)) = S_4,$$

since $\mathrm{Aut}(SL_2(Z_3))$ acts freely on the set of $(2^q, 3, 3, 4)$-generating vectors.)

Similarly one shows that there are 24 $(2^q, 3, 6, 4)$- and 24 $(2^q, 6, 6, 4)$-generating vectors in $SL_2(Z_3)$; the arguments in these cases conclude exactly as above.

$\mathbf{A_5 \times Z_2}$

(The arguments for $A_5 \times Z_2$ and $SL_2(Z_3)$ are analogous to the ones given for $A_4 \times Z_2$ and $SL_2(Z_3)$.)

$A_5 \times Z_2$ contains no elements of order 4, and so cannot have branching data of types (e) - (h) from Table 13. Any $(2^q, 2, 3, 5)$-generating vector for a hyperelliptic action of $A_4 \times Z_2$ must be of the form

$$\begin{array}{lll} (t^{[q]}, c_1, c_2, c_3) & \text{if } q \text{ even} & (g \equiv 59\ (60)) \\ (t^{[q]}, tc_1, c_2, c_3) & \text{if } q \text{ odd} & (g \equiv 29\ (60)), \end{array} \qquad (4.3.3)$$

where (c_1, c_2, c_3) is a $(2,3,5)$-generating vector for A_5. All $(2,3,5)$-vectors in A_5 are generating vectors, since A_5 contains no subgroup of order 30. Let 5_1 and 5_2 denote the two conjugacy classes of elements of order 5 in A_5, 3_1 and 2_1 the single conjugacy classes of elements of order 3 and 2, respectively. Using (4.3.1) and the character table for A_5 (see Appendix A), we calculate

$$|X(2_1, 3_1, 5_1)| = |X(2_1, 3_1, 5_1)| = 60.$$

Thus there are 120 $(2,3,5)$-generating vectors in A_5, and hence 120 $(2^q, 2, 3, 5)$-generating vectors for $A_5 \times Z_2$ of the form (4.3.3). Now $\mathrm{Aut}(A_5) \simeq S_5$ has order 120, so all $(2,3,5)$-vectors in A_5 are equivalent. Every automorphism of A_5 extends to a unique automorphism of $A_5 \times Z_2$ which fixes t. Thus all 120 $(2^q, 2, 3, 5)$-generating vectors for $A_5 \times Z_2$ of the form (4.3.3) are equivalent.

The argument above is easily adapted to branching data of types (b), (c) and (d) from Table 13. For example, any $(2^q, 2, 6, 5)$-generating vector for a hyperelliptic action of $A_5 \times Z_2$ must be of the form

$$\begin{array}{lll} (t^{[q]}, tc_1, tc_2, c_3) & \text{if } q \text{ even} & (g \equiv 9\ (60)) \\ (t^{[q]}, c_1, tc_2, c_3) & \text{if } q \text{ odd} & (g \equiv 39\ (60)), \end{array}$$

where, as in the previous case, (c_1, c_2, c_3) is a $(2,3,5)$-generating vector for A_5. The argument proceeds exactly as before.

$\mathbf{SL_2(Z_5)}$

$SL_2(Z_5)$ cannot have branching data of type (a) – (d) from Table 13, since any generating vector of one of these types would have the form

$$(t^{[q+1]}, c_1, c_2),$$

where c_1c_2 equals t or e, depending on the parity of q (t being the unique element of order 2). c_1 has order 3 or 6, and c_2 has order 5 or 10. Let C_1 and C_2 be the images of c_1 and c_2 in A_5 under the surjective homomorphism ϕ in the exact sequence

$$\langle t \rangle \hookrightarrow SL_2(Z_5) \xrightarrow{\phi} A_5.$$

C_1 and C_2 have order 3 and 5, respectively, and their product is the identity; there are obviously no such elements in A_5.

Any vector in $SL_2(Z_5)$ of the form (e), (f), (g) or (h) from Table 13 generates the whole group, since there are no subgroups of order 60. There are two conjugacy classes (5_1 and 5_2) of elements of order 5 in $SL_2(Z_5)$, and one each of order 3 and 4 (3_1 and 4_1). Using the character table in Appendix A and (4.3.1) we derive

$$|X((2_1)^q, 4_1, 3_1, 5_1)| = |X((2_1)^q, 4_1, 3_1, 5_2)| = 60.$$

Thus there are 120 $(2^q, 4, 3, 5)$-generating vectors in $SL_2(Z_5)$. Now $GL_2(Z_5)$ acts by conjugation on $SL_2(Z_5)$ with kernel $\pm I$, so

$$|\mathrm{Aut}(SL_2(Z_5))| \geq |PGL_2(Z_5)| = 120,$$

and hence all the $(2^q, 4, 3, 5)$-generating vectors are equivalent. (In fact $\mathrm{Aut}(SL_2(Z_5)) = PGL_2(Z_5)$, since $\mathrm{Aut}(PGL_2(Z_5))$ acts freely on the set of $(2^q, 4, 3, 5)$-generating vectors.)

Similarly one shows that there are 120 each of $(2^q, 4, 3, 10)$-, $(2^q, 4, 6, 5)$- and $(2^q, 4, 6, 10)$-generating vectors in $SL_2(Z_5)$; the arguments in these cases conclude exactly as above.

$\mathbf{GL_2(Z_3)}$

A hyperelliptic action of $GL_2(Z_3)$ can only have branch data of types (e) through (h) in Table 14, since its subgroup $SL_2(Z_3)$ has hyperelliptic actions only in even genera. All vectors of type (e) through (h) generate since the only subgroup of order 24 is $SL_2(Z_3)$ which contains no element of order 8. Using 4.3.1 and the character table in Appendix A, we calculate

$$|X((2_1)^q, 2_1, 3_1, 8_1)| = |X((2_1)^q, 2_1, 3_1, 8_2)| = 0$$
$$|X((2_1)^q, 2_2, 3_1, 8_1)| = |X((2_1)^q, 2_2, 3_1, 8_2)| = 24.$$

Thus there are 48 $(2^q, 2, 3, 8)$-generating vectors in $GL_2(Z_3)$. Since $\left|\frac{GL_2(Z_3)}{\text{Center}}\right| = |PGL_2(Z_3)| = |S_4| = 24$, there are 24 non-trivial inner automorphisms. In addition, the outer automorphism

$$x \mapsto tx, \qquad y \mapsto y, \qquad t \mapsto t$$

has order 2, commutes with all the inner automorphisms, and interchanges the conjugacy classes 8_1 and 8_2. Thus $|\text{Aut}(GL_2(Z_3))| \geq 48$ and all $(2^q, 2, 3, 8)$-generating vectors are equivalent. (This argument shows that $\text{Aut}(GL_2(Z_3)) \simeq S_4 \times Z_2$.) One shows similarly that there are 48 equivalent $(2^q, 2, 6, 8)$-generating vectors, and no generating vectors of the other two types.

O*

Because the author could not locate the character table for O^* in the literature, it is included below.

Table 18. *Character Table of* O^*

Class:	1	2_1	3_1	6_1	4_1	4_2	8_1	8_2
$\lvert \mathrm{Cl}(g)\rvert$:	1	1	8	8	6	12	6	6
χ_1:	1	1	1	1	1	1	1	1
χ_2:	1	1	1	1	1	-1	-1	-1
χ_3:	2	2	-1	-1	2	0	0	0
χ_4:	2	-2	-1	1	0	0	$\sqrt{2}$	$-\sqrt{2}$
χ_5:	2	-2	-1	1	0	0	$-\sqrt{2}$	$\sqrt{2}$
χ_6:	3	3	0	0	-1	-1	1	1
χ_7:	3	3	0	0	-1	1	-1	-1
χ_8:	4	-4	1	-1	0	0	0	0

Note that this table is identical (except for the labelling of the classes) to the character table of $GL_2(Z_3)$ given in Appendix A. This illustrates the fact that a group is not determined uniquely up to isomorphism by its character table.

Since O^* contains $SL_2(Z_3)$, which has hyperelliptic actions only in even genera, it cannot have a hyperelliptic action with branch data of types (a) through (d) from Table 14. Using the character table and (4.3.1) we calculate

$$|X((2_1)^q, 2_1, 3_1, 8_1)| = |X((2_1)^q, 2_1, 3_1, 8_2)| = 0$$
$$|X((2_1)^q, 4_1, 3_1, 8_1)| = |X((2_1)^q, 4_1, 3_1, 8_2)| = 0$$
$$|X((2_1)^q, 4_2, 3_1, 8_1)| = |X((2_1)^q, 4_2, 3_1, 8_2)| = 24.$$

Thus there are 48 $(2^q, 4, 3, 8)$-vectors in O^*. They all generate since the only subgroup of order 24 is $SL_2(Z_3)$, and it contains no element of order 8. Since $\left|\frac{O^*}{\text{Center}}\right| = |S_4| = 24$, there are 24 nontrivial inner automorphisms. In addition, the outer automorphism

$$x \mapsto tx, \qquad y \mapsto y, \qquad t \mapsto t$$

has order 2, commutes with all the inner automorphisms, and interchanges the conjugacy classes 8_1 and 8_2. Thus $|\mathrm{Aut}(O^*)| \geq 48$ and all $(2^q, 4, 3, 8)$-generating vectors are equivalent. One shows similarly that there are 48 $(2^q, 4, 6, 8)$-generating vectors, and that they are all equivalent. (This argument shows that $\mathrm{Aut}(O^*) \simeq S_4 \times Z_2$.)

$\mathbf{S_4 \times Z_2}$

$S_4 \times Z_2$ contains no element of order 8, and so cannot have branching data of types (e) through (h) from Table 14. (c) and (d) are also not possible since the Sylow 2-subgroup $Z_2 \times D_8$ has a hyperelliptic action $\iff \frac{g+1}{q} = 4 \iff g \equiv 3(4)$ (see Table 10). Thus we need only consider branch data of type (a) and (b). We first consider type (a). The character table of $S_4 \times Z_2$ is easily calculated, but for our purpose it is only necessary to know the character table for S_4. Let (c_1, c_2, c_3) be a $(2,3,4)$-generating vector for S_4. All $(2,3,4)$-vectors in S_4 generate the whole group, since the only subgroup of order 12, A_4, contains no element of order 4. Any $(2^q, 2, 3, 4)$-generating vector for a hyperelliptic action of $S_4 \times Z_2$ must have one of the two forms

$$\left.\begin{array}{r}(t^{[q]}, c_1, c_2, c_3)\\ (t^{[q]}, tc_1, c_2, tc_3)\end{array}\right\} \quad \text{if } q \text{ even} \qquad (\mathrm{g} \equiv 23\ (24)) \tag{4.3.4}$$

$$\left.\begin{array}{r}(t^{[q]}, c_1, c_2, tc_3)\\ (t^{[q]}, tc_1, c_2, c_3)\end{array}\right\} \quad \text{if } q \text{ odd} \qquad (\mathrm{g} \equiv 11\ (24)). \tag{4.3.5}$$

Note that since $q \neq 0$ for branch data of type (a), the vectors generate $S_4 \times Z_2$ in all possible cases. Let 3_1 and 4_1 denote the conjugacy classes of elements of S_4 having order 3 and 4, respectively, 2_1 the class of transpositions, and 2_2 the class of even permutations of order 2. Using the character table for S_4 (see Appendix A) and (4.3.1), we calculate

$$|X(2_1, 3_1, 4_1)| = 24$$

and

$$|X(2_2, 3_1, 4_1)| = 0.$$

The 24 $(2,3,4)$-generating vectors in S_4 give rise to 48 $(2^q,2,3,4)$-generating vectors for $S_4 \times Z_2$ of the forms (4.3.4) and (4.3.5). S_4 acts effectively on itself by conjugation, since the center is trivial. Thus all $(2,3,4)$-vectors in S_4 are equivalent. Every automorphism of S_4 extends to a unique automorphism of $S_4 \times Z_2$ which fixes t, so in (4.3.4) and (4.3.5) we may take $(c_1, c_2, c_3) = \left(x, y, (xy)^{-1}\right)$ (using the presentation in Table 5, §2). The map

$$t \longmapsto t, \qquad x \longmapsto tx, \qquad y \longmapsto y$$

is an additional (outer) automorphism of $S_4 \times Z_2$ of order two which commutes with the inner automorphisms. When (c_1, c_2, c_3) is taken as $\left(x, y, (xy)^{-1}\right)$ in (4.3.4) and (4.3.5), this automorphism interchanges the upper and lower generating vectors. Thus all 48 $(2^q,2,3,4)$-generating vectors for $S_4 \times Z_2$ of the form (4.3.4) and (4.3.5) are equivalent. (This argument shows that $\mathrm{Aut}(S_4 \times Z_2) \simeq S_4 \times Z_2$.)

The argument is easily adapted to branching data of type (b) from Table 14. One simply replaces c_2 by tc_2 and c_3 by tc_3 in (4.3.4) and (4.3.5).

$\mathbf{SL_2(Z_4)}$

Because the author could not locate the character table for $SL_2(Z_4)$ in the literature, it is included below.

Table 19. *Character Table of* $SL_2(Z_4)$

Class:	1	2_1	2_2	2_3	4_1	4_2	4_3	4_4	3_1	6_1
$\|Cl(g)\|$:	1	1	3	3	6	6	6	6	8	8
χ_1:	1	1	1	1	1	1	1	1	1	1
χ_2:	1	1	1	1	-1	-1	-1	-1	1	1
χ_3:	1	-1	-1	1	i	$-i$	$-i$	i	1	-1
χ_4:	1	-1	-1	1	$-i$	i	i	$-i$	1	-1
χ_5:	2	2	2	2	0	0	0	0	-1	-1
χ_6:	2	-2	-2	2	0	0	0	0	-1	1
χ_7:	3	-3	1	-1	$-i$	i	$-i$	i	0	0
χ_8:	3	-3	1	-1	i	$-i$	i	$-i$	0	0
χ_9:	3	3	-1	-1	-1	-1	1	1	0	0
χ_{10}:	3	3	-1	-1	1	1	-1	-1	0	0

A hyperelliptic action of $SL_2(Z_4)$ can only have branch data of types (a) through (d) from Table 17, since the subgroup $A_4 \times Z_2$ has a hyperelliptic action only in odd genera. Moreover, (a) and (b) are not possible since the Sylow 2-subgroup G_{16} has a hyperelliptic action $\iff \frac{2g+2}{2q+1} = 4 \iff g \equiv 1(4)$ (see Table 10). So we need only look for generataing vectors of types (c) and (d) from Table 14.

Using the character table and (4.3.1), we find that for each value of q there are $8 \times 24 = 192$ each of $(2^q, 4, 3, 4)$-vectors and $(2^q, 4, 6, 4)$-vectors in $SL_2(Z_4)$. Not all of these generate the whole group, since $SL_2(Z_4)$ contains the subgroup $Q_{12} \simeq D_{4,3,-1}$ which has hyperelliptic actions with branch data of both types (see Table 10). A little experimentation shows that exactly half the elements of order 4 in $SL_2(Z_4)$ (4_3 and 4_4 in the character table), have square equal to $t = -I$. Since Q_{12} contains t as its unique element of order 2, the $(2^q, 4, 3, 4)$- and $(2^q, 4, 6, 4)$-vectors which generate Q_{12} are the ones of the form

$$(2_1^q, 4_A, 3_1, 4_B) \quad \text{and} \quad (2_1^q, 4_A, 6_1, 4_B),$$

where A, B $\in \{3, 4\}$. For each value of q there are 48 such vectors of both types in $SL_2(Z_4)$. The remaining 6×24 vectors generate $SL_2(Z_4)$, since the only other subgroups of order divisible by 12 are A_4 and $A_4 \times Z_2$, neither of which contains an element of order 4. Not all of these, however, can be generating vectors for a *hyperelliptic* action of $SL_2(Z_4)$.

Any hyperelliptic action of $SL_2(Z_4)$ must contain a hyperelliptic action of the index 2 subgroup $A_4 \times Z_2$. If a hyperelliptic $SL_2(Z_4)$ action has branch data of type (c) from Table 14, then, since $g \equiv 5(12)$, the hyperelliptic action of $A_4 \times Z_2$ must be of type (a) from Table 15. Similarly, if a hyperelliptic $SL_2(Z_4)$ action has branch data of type (d) from Table 14, then, since $g \equiv 9(12)$, the hyperelliptic action of $A_4 \times Z_2$ must be of type (c) from Table 15. In either case, the 2 : 1 relative projection

$$\Sigma_g/(A_4 \times Z_2) \to \Sigma_g/(SL_2(Z_4))$$

maps the branch locus of the $A_4 \times Z_2$ action to the branch locus of the $SL_2(Z_4)$ action. The resulting 2 : 1 uniform relative branchings are easily calculated to be:

$$[2^{(g-5)/6};\ 2;\ 3,3;\ 2] \to [2^{(g-5)/12};\ 4;\ 3;\ 4]$$

or

$$[2^{(g-9)/6};\ 2;\ 6,6;\ 2] \to [2^{(g-9)/12};\ 4;\ 6;\ 4].$$

If t, x, y are the generators for $A_4 \times Z_2$ (in the presentation used in Table 15), the induced maps on generating vectors are

$$(t^{(g-5)/6};\ t;\ x,y;\ txy) \mapsto (t^{(g-5)/12};\ c_1;\ c_2;\ c_3)$$

or

$$(t^{(g-9)/6};\ t;\ tx,ty;\ txy) \mapsto (t^{(g-9)/12};\ d_1;\ d_2;\ d_3),$$

respectively, where c_i and d_i are elements of $SL_2(Z_4)$ of the appropriate orders. The maps show that c_1 and d_1 must be elements of order 4 whose square equals t, and that c_3 and d_3 must be elements of order 4 whose square is not equal to t.

Thus, generating vectors for *hyperelliptic* actions of $SL_2(Z_4)$ of types (c) and (d) from Table 14 can only be the ones of the form

$$(2_1^q, 4_A, 3_1, 4_B) \quad \text{and} \quad (2_1^q, 4_A, 6_1, 4_B),$$

respectively, where A $\in \{3,4\}$ and B $\in \{1,2\}$. Let us examine the four possibilities for type (c). Calculations using the character table show that

$$|X(2_1^q, 4_3, 3_1, 4_1)| = \begin{cases} 24 & \text{if } q \text{ even} \\ 0 & \text{if } q \text{ odd} \end{cases}$$

$$|X(2_1^q, 4_3, 3_1, 4_2)| = \begin{cases} 0 & \text{if } q \text{ even} \\ 24 & \text{if } q \text{ odd} \end{cases}$$

$$|X(2_1^q, 4_4, 3_1, 4_1)| = \begin{cases} 0 & \text{if } q \text{ even} \\ 24 & \text{if } q \text{ odd} \end{cases}$$

$$|X(2_1^q, 4_4, 3_1, 4_2)| = \begin{cases} 24 & \text{if } q \text{ even} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

Thus for a given value of q, there are 48 generating vectors of type (c) for $SL_2(Z_4)$. To show that they are all equivalent, we must show that they lie in the same $\text{Aut}(SL_2(Z_4))$ orbit. Suppose q is odd. The 24 $(2_1^q, 4_3, 3_1, 4_2)$-generating vectors are equivalent by inner automorphisms (since $\left|\frac{SL_2(Z_4)}{\text{Center}}\right| = |PSL_2(Z_4)| = |S_4| = 24$, there are 24 non-trivial inner automorphisms). Similarly, the 24 $(2_1^q, 4_4, 3_1, 4_1)$-generating vectors are equivalent by inner automorphisms. Finally, the two sets of generating vectors are equivalent by the outer automorphism

$$x \mapsto tx, \qquad y \mapsto y, \qquad t \mapsto t,$$

which may be realized as conjugation by $\pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2(Z_4)$. This automorphism has order 2, commutes with all the inner automorphisms, and permutes the conjugacy classes in the pairs $(4_1, 4_2)$ and $(4_3, 4_4)$. (It is not difficult to show that $\mathrm{Aut}(SL_2(Z_4)) \simeq Z_2 \times S_4$.) Similar arguments apply if q is even, and to generating vectors of type (d) from Table 14.

5. The Moduli Space and the Lattice of Maximal Actions

Let X be a fixed compact oriented topological surface of genus g. Riemann observed that the set M_g of conformal structures (up to conformal equivalence) that can be placed on X has complex dimension $3g-3$ if $g \geq 2$, one if $g=1$, and zero if $g=0$. (M_0 is a single point since, up to conformal equivalence, there is only one compact Riemann surface of genus 0, namely, the Riemann sphere.) M_g, called the moduli space in genus g, is a singular complex algebraic variety whose singular set $S_g \subseteq M_g$ corresponds to those surface classes of genus g which admit non-trivial automorphism groups. (When $g=2$, "non-trivial" means "strictly containing Z_2"; all genus 2 surfaces admit the action of the hyperelliptic involution.) In [H2] it is shown that S_g is a finite non-disjoint union of smooth subvarieties, each corresponding to a topological type of action of some finite group. In [BSS], Theorem 7, it is shown that each of these subvarieties is connected (see also [E]). Their dimension can be determined from Theorem 5.1 below. Not all the subvarieties arising from group actions in a given genus are distinct. Theorem 5.2 gives a crucial example of a situation in which two subvarieties actually coincide. This leads us to define *maximal* actions, which correspond bijectively to *distinct* subvarieties. The arrangement (with respect to inclusion and intersection) of the distinct subvarieties arising from group actions in a given genus is mirrored exactly in the lattice (with respect to inclusion of one action in another) of maximal actions.

We first determine the dimension of the subvariety associated to a group action. Suppose G acts on Σ_g with branch data $(h; r_1, \ldots, r_k)$. Let $B \subset \Sigma_h$ be the branch locus of the action, and S the preimage of B under the orbit map. The holomorphic structure on $\Sigma_h \setminus B$ can be pulled back to $\Sigma_g \setminus S$ via the restriction of the orbit map, and canonically completed to a holomorphic structure on all of Σ_g. Thus the holomorphic structure on Σ_g is determined up to conformal equivalence by the equivalence class of holomorphic structure that exists on the quotient surface Σ_h with

k punctures.

The space of such equivalence classes of structures has complex dimension $\dim(M_h)+k-q$, where q is the transitivity of the group of automorphisms of a surface of genus h. If $h \geq 2$, $q=0$; if $h=1$, $q=1$; and if $h=0$, $q=3$. One may verify that the dimension is given by the formula $3h-3+k$ in all cases. (Note that when $h=0$, and $g \geq 2$, the Riemann-Hurwitz relation implies that $k \geq 3$, so one never obtains a negative dimension.)

We summarize:

Theorem 5.1. *Let (G, α) be a topological type of G-action which has branch data $(h, r_1, \ldots, r_k)$ in some fixed genus $g \geq 2$. Let $V_{G,\alpha} \subseteq S_g$ be the set of conformal equivalence classes of compact Riemann surfaces admitting (G, α). Then $V_{G,\alpha}$ is a smooth connected subvariety of S_g having dimension $(3h-3+k)$. In particular, if $h=0$ and $k=3$, $V_{G,\alpha}$ is a single point, i.e., there is a unique Riemann surface (up to conformal equivalence) admitting (G, α).*

Let (G, α) and (H, α) be topological types of actions in some fixed genus $g \geq 2$.

Definition. (H, α) is *included in* (G, α) ($(H, \alpha) \subseteq (G, \alpha)$) if H is a subgroup of G, and if, on surfaces admitting (G, α), (H, α) is the restriction of (G, α).

If $(G, \alpha) \subseteq (H, \alpha)$, it is obvious that $V_{G,\alpha} \subseteq V_{H,\alpha}$. If the quotient of both actions is a sphere with three branch points, the reverse inclusion follows immediately, since then $V_{G,\alpha}$ and $V_{H,\alpha}$, being connected and 0-dimensional by Theorem 5.1, must be the same point. A theorem of Singerman ([S2], Theorem 1) implies that if $d = \dim(V_{H,\alpha}) = \dim(V_{G,\alpha}) \neq 0$, and the quotient of both actions is a sphere, then $d \leq 1$, and, if $d=1$, there are only two possible sets of branching data for the actions, according to whether the index $[G:H]$ is 2 or 4. We shall need only the special result given in Theorem 5.2. Our proof of this result uses methods different from those of Singerman. We first prove

Lemma 5.2. *If P_1, P_2, P_3 and P_4 are four distinct points on the Riemann sphere, there exists a unique involution $T \in PSL(2, C)$ such that T permutes the points in the pair (P_1, P_2) and the points in the pair (P_3, P_4).*

Proof. It is easy to show that the involutions in $PSL_2(C)$ (the group of Möbius transformations) correspond precisely to equivalence classes (mod $\pm I$) of traceless matrices in $SL_2(C)$. By the three-transitivity of $PSL_2(C)$, we may assume that P_1, P_2, P_3, P_4 are $0, 1, \infty, \lambda$, respectively, where $\lambda \in C$, $\lambda \neq 0, 1$. We seek a Möbius transformation T such that $T^2 = \text{id}$ and T permutes the elements of the pairs $(0, 1)$ and (λ, ∞). Suppose

$$T : z \mapsto \frac{az + b}{cz + d}$$

is such a transformation. By the trace condition on involutions, $d = -a$. Since $T : 0 \mapsto 1$, $b = d$ and hence $b = -a$. Since $T : \lambda \mapsto \infty$, $c = -d/\lambda$, and hence $c = a/\lambda$. Thus T must be of the form

$$T : z \mapsto \frac{az - ac}{\frac{az}{\lambda} - a}.$$

Since the determinant of the matrix

$$\begin{pmatrix} a & -a \\ \frac{a}{\lambda} & -a \end{pmatrix}$$

must equal 1, we must have

$$a = \pm\sqrt{\frac{\lambda}{1 - \lambda}}.$$

Since $\lambda \neq 1$ or 0, $a \neq 0$ or ∞ and thus the matrix belongs to $SL_2(C)$. Since the matrix is unique up to multiplication by $\pm I$, T is a unique Möbius transformation. One may verify that it permutes the elements of the pairs $(0, 1)$ and (λ, ∞).

◇

Theorem 5.2. *Let $(H, \alpha) \subseteq (G, \alpha)$ be actions in some genus $g \geq 2$, with $[G : H] = 2$ and branching data of the form*

$$\begin{array}{cc} (H, \alpha) & (G, \alpha) \\ (a, a, b, b) & (2, 2, a, b), \end{array}$$

where a and b are integers > 1. Then $V_{H,\alpha} = V_{G,\alpha}$.

Proof. We need only show that $V_{H,\alpha} \subseteq V_{G,\alpha}$. Let Σ_g be a compact Riemann surface admitting (H, α), with branch data of the form (a, a, b, b). Let $B_H = \{P_1, \ldots, P_4\} \subset \Sigma_g/H$ be the branch locus of the H-action. Assume that a is the branching index at P_1 and P_2, and b the branching index at P_3 and P_4. The sphere Σ_g/H has a complex structure induced from the complex structure on Σ_g. We take the unique involution $T \in \text{Aut}\,(\Sigma_g/H)$ given by the lemma to be conformal with respect to this structure. It is then evident that there exists a group K of automorphisms of Σ_g which is an extension of H by $\langle T \rangle$ (i.e., $K/H \simeq \langle T \rangle$), and whose action on Σ_g is (H, α) followed by the action of T on the quotient. Call this action (K, α). Since T permutes the points in the pairs (P_1, P_2) and (P_3, P_4), the relative projection

$$\Sigma_g/H \rightarrow \Sigma_g/K$$

(which is the orbit map for the $\langle T \rangle$-action), has relative branching

$$[1; 1; a, a; b, b] \mapsto [2; 2; a; b].$$

In other words, (K, α) has branch data $(2, 2, a, b)$. It is clear that $(H, \alpha) \subseteq (K, \alpha)$. Thus any surface admitting (H, α) admits (K, α). K may be any extension of H by Z_2 which acts with branch data $(2, 2, a, b)$. In particular, K may be taken to be G. ◇

Definition. (G, α) is said to be a *maximal action* if it is the full automorphism group of at least one point in the moduli space. Equivalently, (G, α) is a maximal action if $V_{G,\alpha} \neq V_{G',\beta}$ for any action (G', β) strictly containing (G, α).

Note that every action is contained in a unique maximal action. The point of the definition is that the *maximal* actions correspond bijectively to the *distinct* subvarieties of S_g arising from group actions. It is convenient to label the distinct subvarieties with the names of the corresponding maximal actions. Thus the subvariety arising from a maximal action (G, α) is called a (G, α)-maximal variety. The maximal actions fall naturally into a lattice with respect to inclusion of actions. Moreover, inclusion of one maximal action in another is precisely reflected in the inclusion (in reverse order) of the corresponding subvarieties. A complete geometric picture of S_g is thus implicit in the lattice of maximal actions.

The following facts are immediate consequences of the definition:

(i) (G, α) is a maximal action $\Longleftrightarrow$ (G, α) is the largest action contained in the full automorphism group of every point in $V_{G,\alpha}$.

(ii) (G, α) is a non-maximal action $\Longleftrightarrow$ (G, α) is not the full automorphism group of any point in S_g $\Longleftrightarrow$ (G, α) extends automatically to a larger group action.

(iii) The full automorphism group of an arbitrary point $X \in S_g$ is just the label (group name) attached to the maximal subvariety of lowest dimension which contains X.

6. The Lattice of Maximal Hyperelliptic Actions

We are in a position to determine a substantial portion of the lattice of maximal actions in all genera $g \geq 2$, namely, the lattice of maximal hyperelliptic actions. This in turn gives us a geometric understanding of a large subvariety of S_g, namely, the moduli space of hyperelliptic Riemann surfaces (H_g). By Theorem 5.1, H_g is a connected subvariety of M_g of dimension $2g - 1$. $H_g \subset S_g \subset M_g$ for all $g > 2$, and (in accordance with the special definition of S_2) $S_2 \subset H_2 = M_2$.

The list of hyperelliptic actions in a given genus $g \geq 2$ is obtained from Tables 9, 10, 15, 16 and 17. Recall that the only group which has hyperelliptic actions of two distinct topological types is $Z_2 \times Z_4$. We use $(Z_2) \times Z_4$ to denote the action of reduced type D_4, and $Z_2 \times (Z_4)$ to denote the action of reduced type Z_4. (The factor in parentheses does not contain t.) In all other cases the name of the group suffices to specify its hyperelliptic action uniquely.

6.1 Non-maximal Hyperelliptic Actions

Theorem 6.1. *For a given genus $g \geq 2$, the non-maximal hyperelliptic actions are:*

$$Z_{2g}, \quad Z_{4g}, \quad Z_2 \times (Z_{g+1}), \quad Z_2 \times Z_{2g+2}, \quad Q_{4g}, \quad D_{4,g+1,-1};$$

and, in addition,

$$\begin{array}{ll} \text{if } g = 2: & SD_{16} \\ & SL_2(Z_3) \\ \text{if } g = 3: & (Z_2) \times Z_4 \\ & A_4 \times Z_2 \\ \text{if } g = 6: & SL_2(Z_3). \end{array}$$

Proof. Figure 1 shows partial subgroup lattices of G_{8g+8} and SD_{8g}. The accompanying branching data are obtained by setting $p = 1$ and $p = 2$ in rows A and C of Table 9, and setting $q = 0$ (if possible) or (otherwise) $q = 1$ in each of the rows of Table 10. It follows from Theorem 5.2 that Z_{2g} extends automatically to D_{4g} and that $Z_2 \times Z_{g+1}$ extends automatically to $Z_2 \times D_{2g+2}$. It is also evident (since the number of branching indices is three) that Z_{4g} and Q_{4g} extend automatically to SD_{8g}, and that $Z_2 \times Z_{2g+2}$ and $D_{4,g+1,-1}$ extend automatically to G_{8g+8}. In genus 6, $SL_2(Z_3)$ $(6,6,4)$ (Table 15 (f)) extends automatically to $GL_2(Z_3)$ $(2,6,8)$ (Table 17 (f)). In genus 3, $(Z_2) \times Z_4$ $(2,2,4,4)$ extends automatically to $Z_2 \times D_8$ $(2,2,2,4)$; $A_4 \times Z_2$ $(6,6,2)$ (Table 15 (c)) extends automatically to $S_4 \times Z_2$ $(2,6,4)$ (Table 17 (b)). The special situation in genus 2 is summarized in Figure 2, which replaces the upper portion of Figure 1 when $g = 2$. $SL_2(Z_3)$ $(3,3,4)$ (Table 15 (d)) extends automatically to $GL_2(Z_3)$ $(2,6,8)$ (Table 17 (e)). So does SD_{16} $(2,4,8)$, although in this case the index is 3, and the inclusion is not normal. The $3:1$ non-uniform relative branching is:

$$[1,2;1;4,8] \mapsto [2;3;8].$$

(see Example 1, §1.3).

To show that the actions in the Theorem are the only non-maximal hyperelliptic actions, one must verify that there are no other instances of a proper inclusion of hyperelliptic actions having the same number of branching indices. We need only look for inclusions between actions having three branch points, and, according to the Theorem of Singerman mentioned before Theorem 5.2, inclusions of index 2 or 4 between actions having 4 branch points. We first note that none of the inclusions can be between groups belonging to the same family, e.g., $G_{16} \subsetneq G_{48}$ or $Z_4 \subsetneq Z_{16}$, since the number of branching indices of the smaller and larger actions are $q+3$ and $q'+3$, respectively, with $q < q'$ (if the actions are from Table 10), or $p+2$ and $p'+2$, respectively, with $p < p'$ (if the actions are from Table 9).

We begin by examining index 2 inclusions of actions from Table 9 in actions from Table 10. The six possibilities, together with their relative branching types (see Table 11, § 4.2) are

$Z_2 \times Z_n \subsetneq Z_2 \times D_{2n}$	(A-I)	$Z_{2n} \subsetneq D_{4n}$	(C-IV)
$Z_2 \times Z_n \subsetneq G_{4n}$	(A-II)	$Z_{2n} \subsetneq SD_{4n}$	(C-V)
$Z_2 \times Z_n \subsetneq D_{4,n,-1}$	(A-III)	$Z_{2n} \subsetneq Q_{4n}$	(C-VI)

The relative branchings are the ones given in Table 11, § 4.2. Equating the number of branching indices on the left with the number of branching indices on the right in each case (recall that 1 is not counted as a branching index), and solving for q yields

$$\begin{aligned} q &= -1 \quad \text{(A-III and C-VI)} \\ q &= 0 \quad \text{(A-II and C-V)} \\ q &= 1 \quad \text{(A-I and C-IV).} \end{aligned}$$

$q = -1$ is obviously not possible. $q = 0$ and $q = 1$ correspond to inclusions we have already obtained, namely, those illustrated in Figure 1. We need not consider inclusions of this type having larger index, since any such inclusion implies an inclusion of actions belonging to the same family. For example, if $Z_2 \times Z_6 \subset G_{48}$ extends automatically to G_{48} then $Z_2 \times Z_6$ extends automatically to $Z_2 \times Z_{12} \subset G_{48}$.

Next we examine inclusions of actions from Tables 9 or 10 in actions from Table 15. Those of smallest index are

$$\begin{array}{lcl} Z_6 & \subseteq & A_4 \times Z_2 \\ (Z_2)^3 & \subseteq & A_4 \times Z_2 \end{array} \qquad \begin{array}{lcl} Z_6 & \subseteq & SL_2(Z_3) \\ Q_8 & \subseteq & SL_2(Z_3) \end{array}$$

We need not consider inclusions of larger index for the same reasons as above. From Table 9, there are three types of branching data for Z_6 and from Table 15, three types of branching data for each of $A_4 \times Z_2$ and $SL_2(Z_3)$. In a given genus, the relative branching depends on the congruence class mod (6) of g. For example, if $g \equiv 5(6)$, the relative branching must be of type A-(a), i.e., Z_6 must have branch data of type (A) from Table 10, and $A_4 \times Z_2$ must have branch data of type (a) from Table 15. The six possible relative branchings of this type are listed below, together with corresponding values of the parameters p and q.

	p	q		p	q
A-(a)	$\frac{2g+2}{3}$	$\frac{g+1}{6}$	A-(d)	$\frac{2g+2}{3}$	$\frac{g-2}{6}$
B-(b)	$\frac{2g+1}{3}$	$\frac{g-1}{6}$	B-(e)	$\frac{2g+1}{3}$	$\frac{g-4}{6}$
C-(c)	$\frac{2g}{3}$	$\frac{g-3}{6}$	C-(f)	$\frac{2g}{3}$	$\frac{g-6}{6}$

The number of branching indices of the Z_6-action is always $p+2$, and the number of branching indices of the $A_4 \times Z_2$ or $SL_2(Z_3)$ action is always $q+3$. Thus for equality to hold we must have $p = q+1$. Substituting the six pairs of values of p and q into this equation and solving for g shows that in all six cases, $g < 2$.

Table 10 indicates that Q_8 acts on surfaces of even genus with branch data $(2^{(g-2)/2}, 4, 4, 4)$, so that the number of branching indices is $3 + (g-2)/2$. Setting this equal to $q+3$, where q takes the values corresponding to (d), (e) and (f) in the table above, and solving for g, one obtains $g = 2$ in case (d), and $g < 2$ in the other two cases. Thus, in genus 2, Q_8 $(4,4,4)$ extends automatically to $SL_2(Z_3)$ $(3,3,4)$ (d). However, Q_8 has already been shown to be non-maximal in genus 2, since it also extends to SD_{16} $(2,4,8)$ and $GL_2(Z_3)$ $(2,3,8)$ (see Figure 2).

Table 10 indicates that $(Z_2)^3$ ($\simeq Z_2 \times D_4$) acts on surfaces of odd genus with branch data $(2^{(g+1)/2}, 2, 2, 2)$, so that the number of branching indices is $3 + (g+1)/2$. Setting this equal to $q+3$, where q takes the values corresponding to (a), (b) and (c) in the table above, and solving for g, one obtains $g < 2$ in all three cases.

Inclusions of actions from Table 15 in actions from Table 17 are all of index 2, and the number of branching indices is $q+3$ for all actions in both tables. Hence, if the number of branching indices of the smaller and larger actions is equal, $q = 0$ or $q = 1$ in both sets of branching data simultaneously. It is easy to see that this is only possible if $g = 2$, 3 or 6. The corresponding non-maximal actions are ones we have already obtained.

Of the remaining possible inclusions, those of smallest index are

$$\begin{array}{rcl}
Q_{16} & \subseteq & O^* \\
Q_{12} & \subseteq & O^* \\
\\
G_{16} & \subseteq & SL_2(Z_4) \\
Q_{12} & \subseteq & SL_2(Z_4) \\
\\
A_4 \times Z_2 & \subseteq & A_5 \times Z_2 \\
D_{20} & \subseteq & A_5 \times Z_2
\end{array}
\qquad
\begin{array}{rcl}
SD_{16} & \subseteq & GL_2(Z_3) \\
D_{12} & \subseteq & GL_2(Z_3) \\
\\
Z_2 \times D_8 & \subseteq & S_4 \times Z_2 \\
D_{12} & \subseteq & S_4 \times Z_2 \\
\\
SL_2(Z_3) & \subseteq & SL_2(Z_5) \\
Q_{20} & \subseteq & SL_2(Z_5)
\end{array}$$

As before, we need not consider inclusions of larger index since they imply inclusions between actions belonging to the same family. We have already noted that SD_{16} is non-maximal in genus 2 because it extends automatically to $GL_2(Z_3)$. To show that there are no other automatic extensions, one equates the number of branching indices for each pair of actions. This amounts to equating the expressions for the parameter q in each pair. In each case one obtains $g < 2$, or if $g \geq 2$, q is not an integer. $\diamond$

6.2 Construction of the Lattices

The list of maximal hyperelliptic actions in any given genus $g \geq 2$ is now easily obtained. One simply lists all the actions from Tables 9, 10, 15, 16, and 17, and crosses out those listed in Theorem 6.1. The inclusion relations among the actions are obtained from the subgroup lattices of the various groups (see Appendix B). Figures 3 through 10 give the lattices of maximal hyperelliptic actions for the first eight genera ≥ 2. The numbers running down the right-hand side of the lattices indicate the dimensions of the subvarieties at that level, calculated from Theorem 5.1. The lattice in genus 2 (Figure 3) appears in [KN]; the others are new. The lattices are drawn so as to bring out some regularities which depend on the parity of g. We shall have more to say about this shortly (see Section 6.4).

In genus 2 it is possible to convert the lattice into an actual picture of H_2. This is done in the lower portion of Figure 3. The lattice indicates that H_2 is 3-dimensional, and that S_2 is the disjoint union of a 0-dimensional Z_{10}-maximal variety and a 2-dimensional D_4-maximal variety. The D_4-maximal variety contains a 1-dimensional D_8-maximal variety and a 1-dimensional $Z_2 \times D_6$-maximal variety, which intersect in two distinct points: a 0-dimensional G_{24}-maximal variety and a 0-dimensional $GL_2(Z_3)$-maximal variety.

In higher dimensions, the lattice is the only way to fully represent the geometry. For example, H_3 is 5-dimensional. However, the lattice in Figure 4 indicates that most of the interesting (in the sense of admitting "large" automorphism groups) hyperelliptic surfaces of genus 3 live on a 3-dimensional D_4-maximal variety. The geometry of this variety is quite complex; the lower portion of Figure 4 is a picture of part of it.

6.3 Exceptional Points in $\mathbf{H_g}$

Let us call a 0-dimensional subvariety of S_g corresponding to a group action an *exceptional point.* We now show that there are always at least three exceptional points in H_g; if $g > 30$ there are exactly three. Considerations of order alone show that G_{8g+8} and (except in genus 2) SD_{8g} are not contained in any larger hyperelliptic actions on a surface of genus g. Hence the exceptional points in H_g determined by these actions are distinct, and their full automorphism groups are G_{8g+8} and (except in genus 2) SD_{8g}. Let us denote these points X_G^g and X_{SD}^g.

Putting $p = 0$ in row B of Table 9, one obtains the action of Z_{4g+2} with branch data $(2, 2g+1, 4g+2)$. Since $4g+2$ does not divide $4g$, $4g+4$, $8g+8$ or $8g$, this action is not inlcuded in any other action from Tables 9 or 10. In particular, it is not included in G_{8g+8} or SD_{8g}. So we have located an exceptional point distinct from X_G^g and X_{SD}^g. We shall call it X_Z^g, since its full automorphism group is Z_{4g+2}: the order of the largest cyclic subgroup of the groups in Tables 15, 16 and 17 is ten ($Z_{10} \subset A_5 \times Z_2$), which is less than $4g+2$ for $g > 2$; when $g = 2$, $A_5 \times Z_2$ does not act. Thus Z_{4g+2} is not contained in any larger hyperelliptic action on a surface of genus g.

For certain $g \leq 30$, there are actions from Tables 15, 16 and 17 having three branching indices. Excluding those which are non-maximal, we obtain the exceptional points listed in Table 20. They are identified by genus and full automorphism group.

Table 20. *Other Exceptional Points in H_g*

Genus	*Full Automorphism Group*	*Branch Data*
2	$GL_2(Z_3)$	$(2,3,8)$
3	$S_4 \times Z_2$	$(2,6,4)$
4	$SL_2(Z_3)$	$(3,6,4)$
5	$SL_2(Z_4)$	$(4,3,4)$
5	$A_5 \times Z_2$	$(2,3,10)$
6	$GL_2(Z_3)$	$(2,6,8)$
8	O^*	$(4,3,8)$
9	$SL_2(Z_4)$	$(4,6,4)$
9	$A_5 \times Z_2$	$(2,6,5)$
12	O^*	$(4,6,8)$
14	$SL_2(Z_5)$	$(4,3,5)$
15	$A_5 \times Z_2$	$(2,6,10)$
20	$SL_2(Z_5)$	$(4,3,10)$
24	$SL_2(Z_5)$	$(4,6,5)$
30	$SL_2(Z_5)$	$(4,6,10)$

With one exception, these surfaces are distinct from each of X^g_G, X^g_{SD} and X^g_Z, and, when two or more are of the same genus, from eachother. The exception is in genus 2, where the exceptional point whose full automorphism group is $GL_2(Z_3)$ is X^2_{SD}. We have now proved Theorem 2, stated in the introduction.

Historical Remarks. The families $\{X^g_Z\}$ and $\{X^g_{SD}\}$, $g \geq 2$, were known to Wiman ([Wi]), who showed them to be hyperelliptic, and unique in each genus. Wiman also showed that $4g+2$ is the maximal order of a cyclic group of automorphisms of a compact Riemann surface of genus $g \geq 2$. Kulkarni ([K6]) showed that the full automorphism group of X_{SD} is SD_{8g}. Accola and Machlachlan discovered independently ([Ac], [M2]) that, in a given genus $g \geq 2$, the maximum order of the full automorphism group of a surface is at least $8g+8$; and that in each genus there exists a surface whose full automorphism group has order exactly $8g+8$. Kulkarni showed

([K4]) that, for $g \equiv 0, 1$ or $2(4)$, this surface is unique, and that its full automorphism group is isomorphic to our G_{8g+8}, although he derived a different presentation (in fact, the presentation (iv) in our Table 4a). Kulkarni also pointed out that, for $g \equiv 3(4)$, there is, in addition to X_G^g, another surface whose full automorphism group is a different group of order $8g + 8$, also acting with branch data $(2, 4, 2g + 2)$. Our work shows that this surface cannot be hyperelliptic. (This action is that of $Z_2 \ltimes D_{2,8,5}$. See Table 22.)

6.4. Regular Features of the Lattices

Figures 11 and 12 indicate some of the regular features in the lattices of maximal hyperelliptic actions for $g > 3$ in odd and even genera, respectively. The figures are constructed using Theorem 6.4 and Lemma 6.5 below, and other general results proved elsewhere in this thesis. Except where noted, all the actions in the Figures are maximal. Most of the lines have been drawn lightly to indicate places where the lattice may be incomplete. Using Figures 11 and 12 as guidelines, in conjunction with Tables 10, 15, 16, 17 and Theorem 6.1, it is a routine matter to construct the lattices of maximal hyperelliptic actions in arbitrary genera $g \geq 2$.

Theorem 6.4. *The exceptional point X_G^g lies on the intersection of the $(Z_2) \times Z_4$-, $Z_2 \times (Z_4)$- and $(Z_2)^3$-maximal varieties if g is odd, and on the D_8-maximal variety if g is even. The exceptional point X_{SD}^g lies on the intersection of the Z_8-, Q_8- and D_8-maximal varieties if g is even, and on the $(Z_2) \times Z_4$-maximal variety if g is odd.*

Theorem 6.4 is a special case of Lemma 6.5 below.

The hyperelliptic actions of groups of order 8 are given below, together with the dimensions of the corresponding subvarieties. It is evident that the parity of g determines which groups of order 8 act.

Action	*Dimension*
$(Z_2)^3$	$(g+1)/2$
$(Z_2) \times Z_4$	$(g-1)/2$
$Z_2 \times (Z_4)$	$(g-1)/2$
D_8	$g/2$
Q_8	$(g-2)/2$
Z_8	$(g-2)/2$

Lemma 6.5. *Any hyperelliptic action of $G_{4n} \subseteq G_{8g+8}$ on surfaces of odd genus includes the hyperelliptic actions of $(Z_2)^3$, $Z_2 \times (Z_4)$ and $(Z_2) \times Z_4$. Any hyperelliptic action of $SD_{4n} \subseteq SD_{8g}$ on surfaces of odd genus includes the hyperelliptic actions of D_8, Z_8 and Q_8.*

Proof. Recall that G_{8g+8} has presentation

$$\langle x, y, t \mid x^{2g+2} = y^2 = t^2 = [x,t] = [y,t] = e,\ yxy^{-1} = x^{-1}t\rangle. \tag{6.3.1}$$

According to Table 10, the action of G_{8g+8} includes the actions of the subgroups G_{4n}, where $n = (2g+2)/k$, and k is an odd divisor of $2g+2$ satisfying $1 \leq k \leq g+1$ (if $k = g+1$, the group is isomorphic to D_8). Such groups have the presentations

$$G_{4n} = \langle x^k,\ y,\ t\rangle,$$

using the generators in (6.3.1). It is not difficult to verify that the subgroups of order 8 of $G_{4n} \subseteq G_{8g+8}$ containing t are all conjugate to one of:

		g odd	g even
$\langle t,\ x^{n/2},\ y\rangle$	$\simeq$	$(Z_2)^3$	D_8
$\langle t,\ x^{n/4}\rangle$	$\simeq$	$Z_2 \times (Z_4)$	—
$\langle yx,\ x^{n/2}\rangle$	$\simeq$	$(Z_2) \times Z_4$	D_8.

(The reader may find it useful to consult the subgroup lattices given in Appendix B.) The different isomorphism types for odd and even g are explained as follows. If g is even, $n \equiv 2(4)$, i.e., $n/2$ odd. In this case, $yx^{n/2}$ has order 4 (see §4.2 (g)). Putting $b = yx^{n/2}$, $a = y$, one may verify that $aba^{-1} = b^{-1}$, and that

$$\langle t, x^{n/2},\ y\rangle = \langle a, b\rangle \simeq D_8.$$

Still assuming g is even, and putting $b = yx$, $a = x^{n/2}$, one may verify that $aba^{-1} = b^{-1}$ and that

$$\langle yx, x^{n/2}\rangle = \langle a, b\rangle \simeq D_8.$$

The isomorphisms types for odd g are clear.

SD_{8g} has presentation

$$\langle x, y \mid x^{4g} = y^2 = e,\ yxy^{-1} = x^{2g-1}\rangle \qquad (t = x^{2g}), \qquad (6.3.2)$$

and the action of SD_{8g} includes the actions of the subgroups

$$SD_{4n} = \langle x^k,\ y\rangle,$$

where k is an odd divisor of $2g$ and $n = (2g/k)$. The subgroups of order 8 of $SD_{4n} \subseteq SD_{8g}$ which contain t are conjugate to one of:

		g odd	g even
$\langle x^{n/2}, y\rangle$	$\simeq$	$(Z_2) \times Z_4$	D_8
$\langle x^{n/4}\rangle$	$\simeq$	—	Z_8
$\langle yx, x^{n/2}\rangle$	$\simeq$	$(Z_2) \times Z_4$	Q_8

The isomorphism types for odd g follow from: g odd $\iff n/2 \equiv 2(4) \implies x^{n/2}$ is central in SD_{4n} (see §4.2 (h)).

◇

7. The Full Lattice of Maximal Actions in Genus 3

For convenience, we list the hyperelliptic actions in genus 3 below. The maximal actions (marked with an asterisk) were determined using Theorem 6.1.

Table 21. *Hyperelliptic Actions in Genus 3*

	Group	*Br. Data*	*Order*	*Dim.*
*	Z_2	(2^8)	2	5
	D_4	(2^6)	4	3
*	Z_4	$(2^3, 4^2)$	4	2
	Z_6	$(2^2, 6^2)$	6	1
	$Z_2 \times (Z_4)$	$(2^2, 4^2)$	8	1
	$(Z_2) \times Z_4$	$(2^2, 4^2)$	8	1
*	$(Z_2)^3$	(2^5)	8	2
*	D_{12}	$(2^3, 6)$	12	1
	Q_{12}	$(4, 4, 6)$	12	0
	Z_{12}	$(2, 12, 12)$	12	0
*	Z_{14}	$(2, 7, 14)$	14	0
	$Z_2 \times Z_8$	$(2, 8, 8)$	16	0
*	$Z_2 \times D_8$	$(2^3, 4)$	16	1
	$D_{4,4,-1}$	$(4, 4, 4)$	16	0
*	SD_{24}	$(2, 4, 12)$	24	0
	$A_4 \times Z_2$	$(6, 6, 2)$	24	0
*	G_{32}	$(2, 4, 8)$	32	0
*	$S_4 \times Z_2$	$(2, 6, 4)$	48	0

(Actions with asterisks are maximal.)

Table 22 below lists the non-hyperelliptic actions on surfaces of genus 3. With the exception of the group K_{48}, this information is taken from a table that appears in Broughton's paper ([B], Table 5). Actions with an asterisk are maximal, although this information was not included in Broughton's paper.

Table 22. *Non-hyperelliptic Actions in Genus 3*

	Group	Br. Data	Top. Type	Order	Dim.
*	$[Z_2]$	$(1;2^4)$		2	4
	Z_2	$(2;-)$		2	4
*	Z_3	(3^5)		3	2
	Z_3	$(1;3^2)$		3	2
	Z_4	(4^4)	$[1,1,1,1]$	4	1
	Z_4	(4^4)	$[1,1,3,3]$	4	1
	Z_4	$(1;2^2)$		4	2
*	$[D_4]$	(2^6)		4	3
	D_4	$(1;2^2)$		4	2
*	Z_6	$(2,3,3,6)$		6	1
*	D_6	$(2^4,3)$		6	2
	D_6	$(1;3)$		6	1
	Z_7	$(7,7,7)$	$[1,1,5]$	7	0
	Z_7	$(7,7,7)$	$[1,2,4]$	7	0
	Z_8	$(4,8,8)$	$[1,1,6]$	8	0
	Z_8	$(4,8,8)$	$[1,2,5]$	8	0
	$[Z_2 \times Z_4]$	$(2^2,4^2)$		8	1
*	D_8	(2^5)		8	2
	D_8	$(2^2,4^2)$		8	1
	D_8	$(1;2)$		8	1
	Q_8	$(1;2)$		8	2
*	Z_9	$(3,9,9)$		9	0
	Z_{12}	$(3,4,12)$		12	0
	A_4	$(2^2,3^2)$		12	1
	$D_{2,8,5}$	$(2,8,8)$		16	0
	$Z_4 \times Z_4$	$(4,4,4)$		16	0
*	$Z_2 \ltimes (Z_2 \times Z_4)$	$(2^3,4)$		16	1
	$D_{3,7,2}$	$(3,3,7)$		21	0
	$SL_2(Z_3)$	$(3,3,6)$		24	0
	S_4	$(3,4,4)$		24	0
*	S_4	$(2^3,3)$		24	1

(cont. on next page)

Table 22. *(continued)*

	Group	Br. Data	Top. Type	Order	Dim.
	$Z_2 \ltimes D_{2,8,5}$	$(2,4,8)$		32	0
	$Z_3 \ltimes (Z_4 \times Z_4)$	$(3,3,4)$		48	0
*	K_{48}	$(2,3,12)$		48	0
*	$D_6 \ltimes (Z_4 \times Z_4)$	$(2,3,8)$		96	0
*	$PSL_2(Z_7)$	$(2,3,7)$		168	0

Notes on Table 22.

(a) The determination of the maximal actions was made using two theorems of Singerman (see below).

(b) Bracketed groups denote actions distinct from hyperelliptic actions of the same groups.

(c) Topological types of cylic group actions with the same branch data are distinguished by their "Nielsen signatures". e.g., the Nielsen signature [1,1,6] for Z_8 with branch data (4,8,8) indicates that the action has a generating vector equivalent to (x,x,x^6), where x is a generator of Z_8.

(d) Presentations of non-standard groups are given below.

$D_{2,8,5}$	$\langle y,z \mid y^2=z^8=e,\ yzy^{-1}=z^5\rangle$
$Z_2 \ltimes D_{2,8,5}$	$\langle x,y,z \mid x^2=y^2=z^8=e,\ yzy^{-1}=z^5,\ xyx^{-1}=yz^4,$ $xzx^{-1}=yz^3\rangle$
$D_{3,7,2}$	$\langle x,y \mid x^3=y^7=e,\ xyx^{-1}=y^2\rangle$
$Z_2 \ltimes (Z_2 \times Z_4)$	$\langle a,b,c \mid a^2=b^2=c^4=[b,c]=[a,c]=1,\ aba^{-1}=bc^2\rangle$
$Z_3 \ltimes (Z_4 \times Z_4)$	$\langle y,z,w \mid y^3=z^4=w^4=[z,w]=1, yzy^{-1}=w,\ ywy^{-1}=(zw)^{-1}\rangle$
$D_6 \ltimes (Z_4 \times Z_4)$	$\langle x,y,z,w \mid x^2=y^3=z^4=w^4=[z,w]=1, xyx^{-1}=y^{-1},$ $xzx^{-1}=w,\ xwx^{-1}=z,\ yzy^{-1}=w,\ ywy^{-1}=(zw)^{-1}\rangle$
K_{48}	$\langle a,b,c \mid a^2=b^3=c^{12}=abc=1\rangle$ (See note (e) below.)

(e) The action of K_{48} was missed by Broughton. It was discovered by Kulkarni ([K6]). The group is isomorphic to the factor group G/B, where $G \simeq SL_2(Z_3)\times Z_4$, and B is the central subgroup of order 2 generated by the element $(-I,u^2)$, with I the identity matrix in $SL_2(Z_3)$ and u a generator for Z_4.

The maximal non-hyperelliptic actions were determined using two crucial theorems of Singerman which list all possible inclusions of finite index between two Fuchsian groups with compact quotient space, such that the corresponding Teichmüller spaces coincide (see ([S2], Theorems 1 and 2). The signatures of these pairs of Fuchsian groups are precisely the pairs of branching data for actions of a group and a subgroup on a compact Riemann surface of genus $g \geq 2$ for which the corresponding subvarieties of the moduli space coincide. In these cases, the action of the subgroup extends automatically to the action of the larger group. We shall call such automatic extensions Singerman-type extensions. (Theorem 5.2, from which Theorem 6.1 follows, is an example of a Singerman-type extension.) Table 23 lists all the Singerman-type extensions involving the nonhyperelliptic actions in genus 3. An action which does not appear in any of these extensions, or which appears as the final member in one of them, is a maximal action.

Table 23. *Singerman-type Extensions in Genus 3*

(a) Z_2 (2;–) $\longrightarrow$ D_4 (2^6) (hyperelliptic).

(b) Z_3 (1;3^2) $\longrightarrow$ D_6 (2^4,3).

(c) D_4 (1;2^2) $\longrightarrow$ $(Z_2)^3$ (2^5) (hyperelliptic).

(d) D_6 (1;3) $\longrightarrow$ D_{12} (2^3,6) (hyperelliptic).

(e) D_8 (1;2) $\longrightarrow$ $Z_2\times D_8$ (2^3,4) (hyperelliptic).

(f) Z_4 (4^4) [1,1,3,3] $\longrightarrow$ D_8 (2^2,4^2) $\longrightarrow$ $Z_2\times D_8$ (2^3,4) (hyperelliptic).

(g) Z_4 (1;2^2) $\longrightarrow$ D_8 (2^5).

(h) Z_7 (7,7,7) [1,1,5] $\longrightarrow$ Z_{14} (2,7,14) (hyperelliptic).

(i) Z_7 (7,7,7) [1,2,4] $\longrightarrow$ $D_{3,7,2}$ (3,3,7) $\longrightarrow$ $PSL_2(Z_7)$ (2,3,7).

(j) Z_8 (4,8,8) [1,1,6] $\longrightarrow$ G_{32} (2,4,8) (hyperelliptic).

(k) Z_8 (4,8,8) [1,2,5] $\longrightarrow$ $D_{2,8,5}$ (2,8,8) $\longrightarrow$ $Z_2\ltimes D_{2,8,5}$ (2,4,8)
$\longrightarrow$ $D_6\ltimes(Z_4\times Z_4)$ (2,3,8).

(l) $Z_4\times Z_4$ (4,4,4) $\longrightarrow$ $Z_3\ltimes(Z_4\times Z_4)$ (3,3,4)
$\longrightarrow$ $D_6\ltimes(Z_4\times Z_4)$ (2,3,8).

(m) Q_8 (1;2) $\longrightarrow$ $Z_2\ltimes(Z_2\times Z_4)$ (2^3,4).

(n) Z_4 (4^4) [1,1,1,1] $\longrightarrow$ $[Z_2\times Z_4]$ (2^2,4^2) $\longrightarrow$ $Z_2\ltimes(Z_2\times Z_4)$ (2^3,4).

(o) A_4 (2^2,3^2) $\longrightarrow$ S_4 (2^3,3).

(p) $SL_2(Z_3)$ (3,3,6) $\longrightarrow$ K_{48} (2,3,12).

(q) S_4 (3,4,4) $\longrightarrow$ $S_4\times Z_2$ (2,4,6) (hyperelliptic).

(r) Z_{12} (3,4,12) $\longrightarrow$ K_{48} (2,3,12).

We now discuss some of the less obvious inclusions among the actions in Tables 21 and 22.

(i) There are three nonhyperelliptic D_8-actions, all included in the (hyperelliptic) action of $Z_2\times D_8$. This is not a contradiction, since $Z_2\times D_8$ contains four subgroups isomorphic to D_8. In the presentation

$$D_8 = \langle t, x, y \mid x^4 = y^2 = t^2 = [t,x] = [t,y] = e,\ yxy^{-1} = e\rangle,$$

the groups $\langle x, y\rangle$, $\langle tx, y\rangle$, $\langle x, ty\rangle$ and $\langle tx, ty\rangle$ are all isomorphic to D_8.

(ii) Both D_6 actions are included in the hyperelliptic action of D_{12}. This is not a contradiction since $D_{12} \simeq Z_2 \times D_6$ contains two subgroups isomorphic to D_6. The argument is similar to the one in (i).

(iii) Both S_4 actions are included in the hyperelliptic action of $S_4 \times Z_2$. This is not a contradiction since $S_4 \times Z_2$ contains two subgroups isomorphic to S_4. Let $\{e, t\} = \langle t\rangle$ be the Z_2 factor in $S_4 \times Z_2$. The set of elements

$$\{(x, e) \mid x \in A_4\} \quad \cup \quad \{(y, t) \mid y \in S_4 - A_4\} \quad \subset \quad S_4 \times Z_2$$

is closed under multiplication (note that $S_4 - A_4$ is just the set of odd permutations in S_4), contains the inverse of each of its elements, and the identity. It is therefore a subgroup which is easily seen to be isomorphic to S_4.

(iv) $Z_2 \ltimes (Z_2 \times Z_4) \subset D_6 \ltimes (Z_4 \times Z_4)$. In the presentations given in Note (c) to Table 22, take $a = wx$, $b = z^2$ and $c = y$.

(v) $S_4 \subset D_6 \ltimes (Z_4 \times Z_4)$. This follows from the fact that S_4 has the semi-direct product structure $D_6 \ltimes (Z_2 \times Z_2)$. (Recall $S_3 \simeq D_6$.)

(vi) $D_8 \subset Z_2 \ltimes (Z_2 \times Z_4)$. In the presentation of Note (c) to Table 22, let $p = b$ and $q = ab$. Then

$$\langle p, q\rangle = \langle p, q \mid p^2 = q^4 = 1,\ pqp^{-1} = q^{-1}\rangle \simeq D_8.$$

(vii) $Q_8 \subset Z_2 \ltimes (Z_2 \times Z_4)$. In the presentation of Note (c) to Table 22, let $i = bc$, $j = ac$, $k = ab$. Then $Q_8 = \langle i, j, k\rangle$. ($i, j, k$ may be interpreted as the usual unit quaternions).

(viii) $S_4 \subset PSL_2(Z_7)$. $PSL_2(Z_7)$ is isomorphic to $GL_3(Z_2)$. The latter group acts transitively on the projective plane $PG(2, 2)$ consisting of the one dimensional subspaces of a three dimensional vector space over the field Z_2. $PG(2, 2)$ contains seven points. $S_4 = GL_2(Z_2) \ltimes (Z_2 \times Z_2) = S_3 \ltimes (Z_2 \times Z_2)$ arises as the stabilizer of a point in $PG(2,2)$.

(ix) $Z_2 \ltimes (Z_2 \times Z_4) \subset K_{48}$. The Sylow 2-subgroup of K_{48} is normal and consists of elements of order 2 and 4, since K_{48} contains exactly 15 elements of those orders, and no others (except the identity) whose order divides 16. The two possibilities for the Sylow 2-subgroup are thus $Z_4 \times Z_4$ and $Z_2 \ltimes (Z_2 \times Z_4)$. Since the former extends automatically to $D_6 \ltimes (Z_4 \times Z_4)$ (see Table 23 (l)), the latter is the only possibility.

Figure 13 shows the complete lattice of maximal actions in genus 3. The left-hand portion of the figure is the lattice of maximal hyperelliptic actions (see Figure 4). Note that the lattice of non-hyperelliptic actions can only link up with the lattice of hyperelliptic actions in the direction of decreasing dimension (downward), since no hyperelliptic action can be included in a nonhyperelliptic action.

Remarks. (1) Of the eight exceptional points in $S_3 \subset M_3$, one is the famous Klein surface, whose full automorphism group is $PSL_2(Z_3)$. This is a Hurwitz group, that is, it has the maximum possible order $(84(g-1))$ for an automorphism group of a surface of genus 3.

(2) Note that the Z_3-maximal variety has empty intersection with H_3. This follows from the fact that the Z_3 action, which has branch data (3^5), fixes more than four points (see Theorem 1.5).

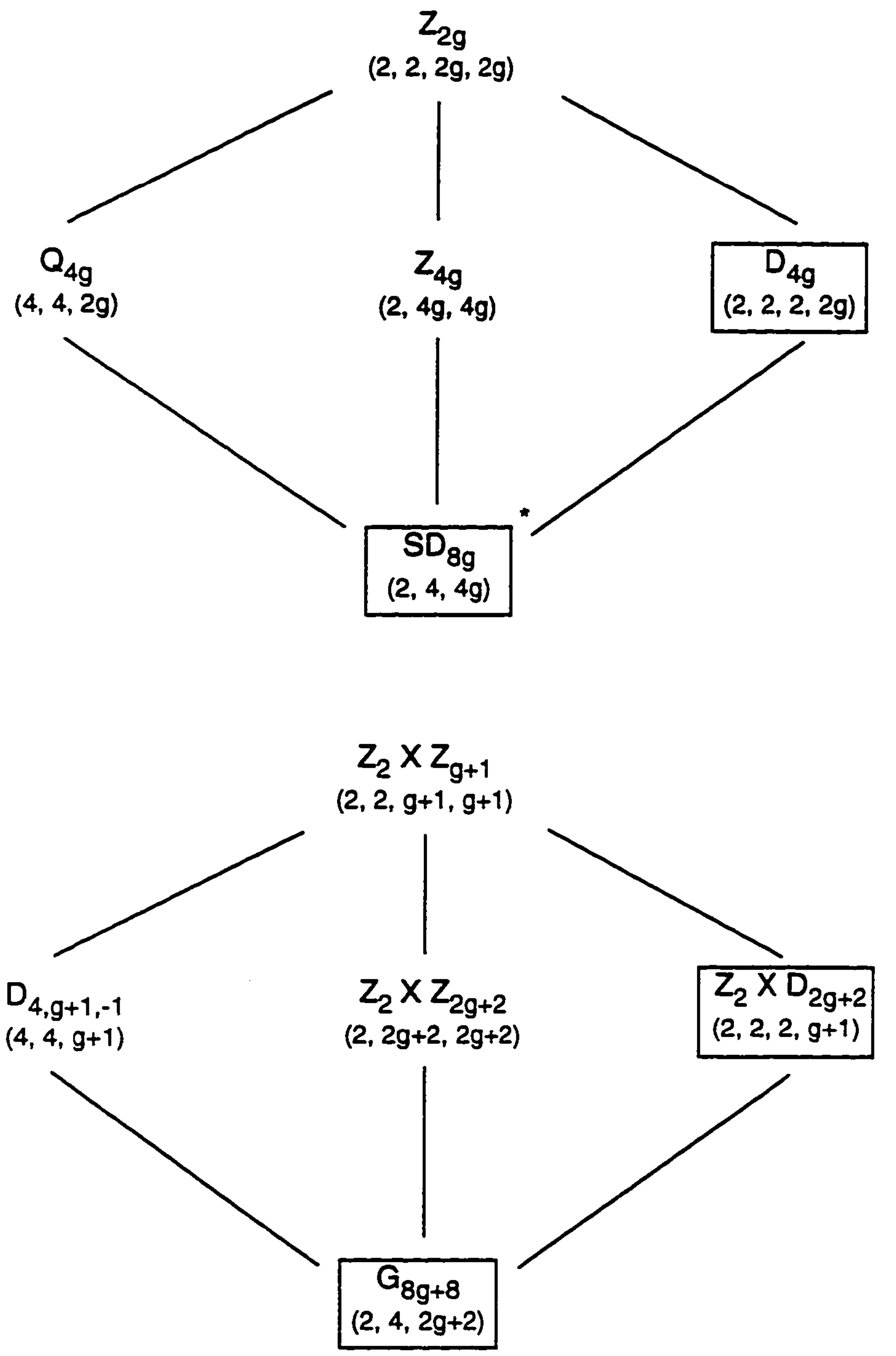
8. Conclusion

I plan to obtain the full lattice of maximal actions in low genera, say, up to genus 10. Knowledge of the lattices of maximal hyperelliptic actions is a big step in this direction. The set of possible branching data are of course the set of solutions of the Riemann-Hurwitz relation for the given genus. The difficulty is that the set of groups which can have a non-hyperelliptic action in a given genus is not known *a priori* as in the hyperelliptic case. Some systematic procedures can be applied to find these groups.

(1) It is easy to determine all the primes p such that a cyclic group of order p acts on a surface of genus g. Such p are bounded above by $2g+1$ (see [H1]). Having determined that Z_p acts, it is then relatively easy to list all possible actions of p-groups. (There are restrictions on the structure of such a group, depending on the highest power of p dividing $g-1$. See [K2].) This information narrows the search for other groups, since their Sylow p-subgroups must be among those in the list. Some of the recent advances in the theory of finite groups – in particular, the interaction of the 2-Sylow subgroup of a given group with the other Sylow p-subgroups – should also prove useful. As the genus and hence the group orders rise, it will be essential to employ a computer algebra system to facilitate some of the group theoretic computations.

(2) In [FK], §V.1.9, a compact Riemann surface is defined to be γ-hyperelliptic ($\gamma \geq 0$) if it is a two-sheeted covering of a compact Riemann surface of genus γ. (Ordinary hyperelliptic surfaces are 0-hyperelliptic by this definition.) By the Riemann-Hurwitz relation (1.1), a γ-hyperelliptic surface of genus g admits a γ-hyperelliptic involution which has $2g+2-4\gamma$ fixed points. It can be shown that any automorphism of a γ-hyperelliptic Riemann surface of genus $g > 4\gamma+1$ which fixes more than $4(\gamma+1)$ points is either the identity or the γ-hyperelliptic involution. It follows that on a γ-hyperelliptic surface of genus $g > 4\gamma+1$, the γ-hyperelliptic involution is unique and central in the full automorphism group of the surface. (The reader will recognize Theorem 1.5 and Corollary 1.5 as special cases ($\gamma = 0$) of these results.) Therefore, the full automorphism group of a γ-hyperelliptic surface must be a central extension by Z_2 of one of the finite automorphism groups of a surface of genus γ. This gives us an inductive way of producing many of the groups acting in higher genera. A further useful fact about γ-hyperelliptic surfaces is: if $g > 4\gamma + 1 + 2r$ ($r > 0$), then a surface of genus g cannot be both γ-hyperelliptic and ($\gamma + r$)-hyperelliptic. This implies, for example (taking $\gamma = 0$ and $r = 1$), that no surface of genus $g > 3$ can be both hyperelliptic and 1-hyperelliptic. Thus the moduli space of 1-hyperelliptic surfaces of genus $g > 3$ is disjoint

from H_g. Further disjointness results in higher genera provide a method of assembling much of the full lattice of maximal actions out of disjoint pieces.

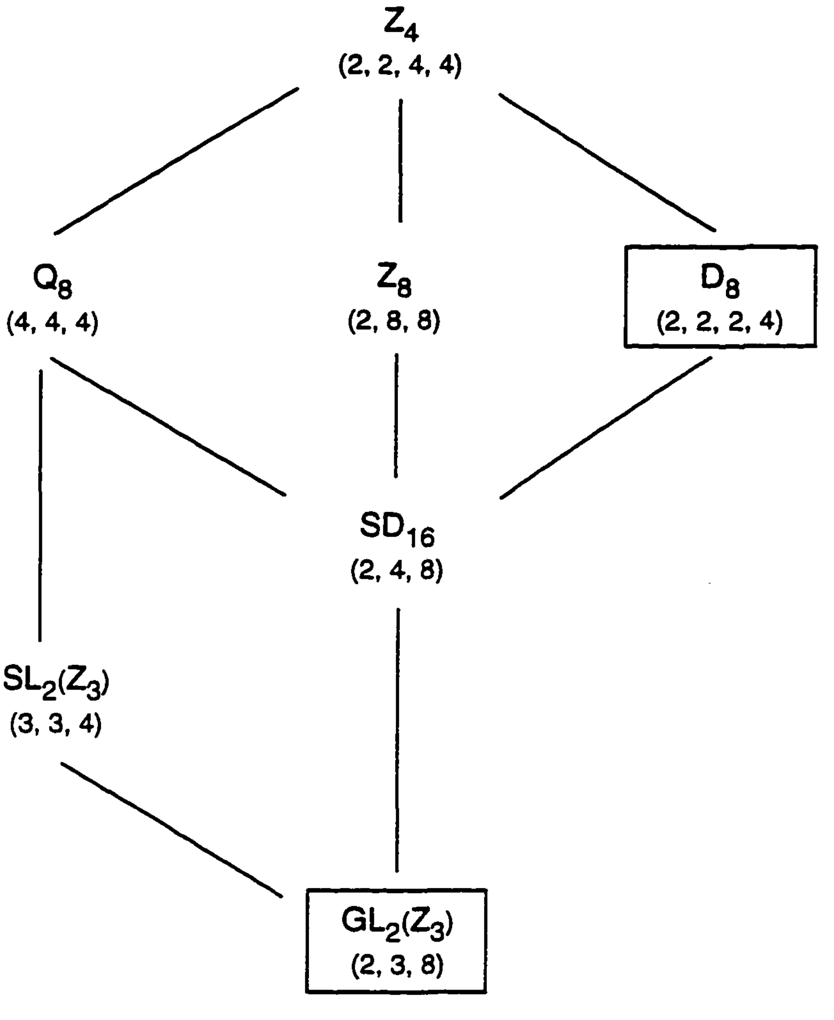


Boxed actions are maximal

* SD_{16} is non-maximal in genus 2 (see Fig. 2)

Portions of the Lattice of Hyperelliptic Actions

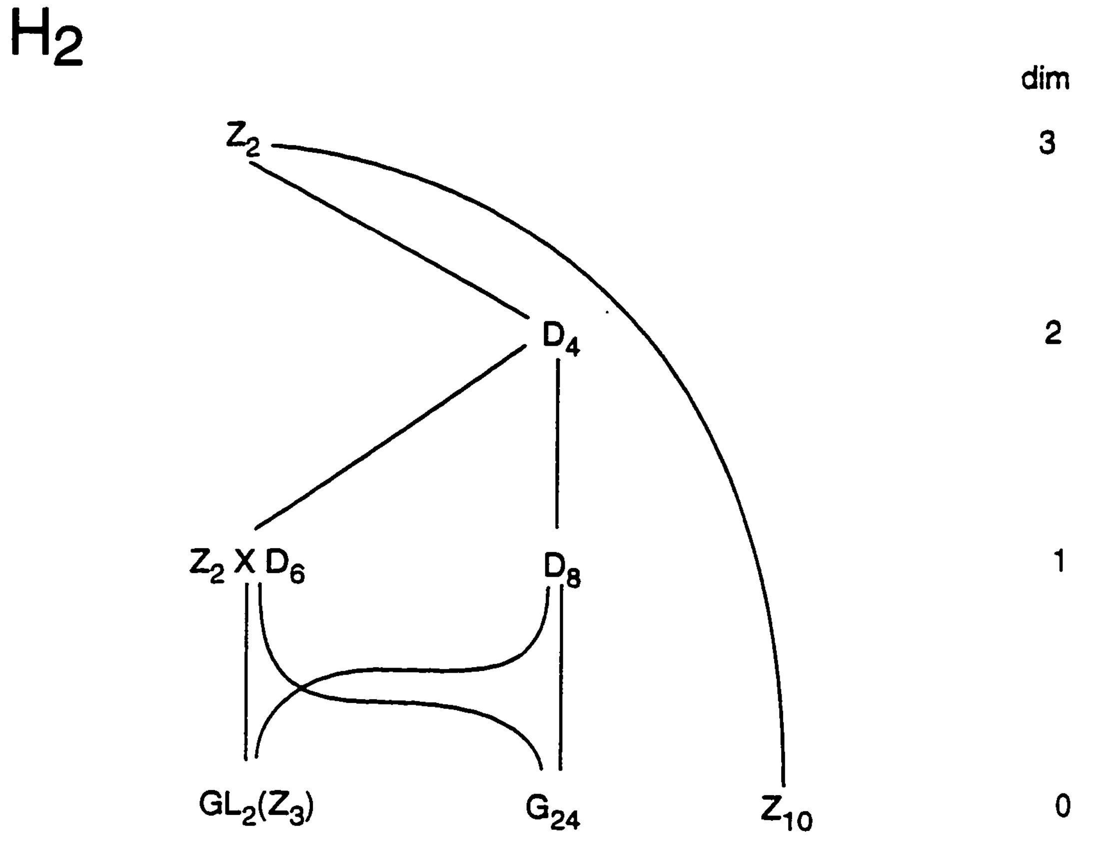
Figure 1.



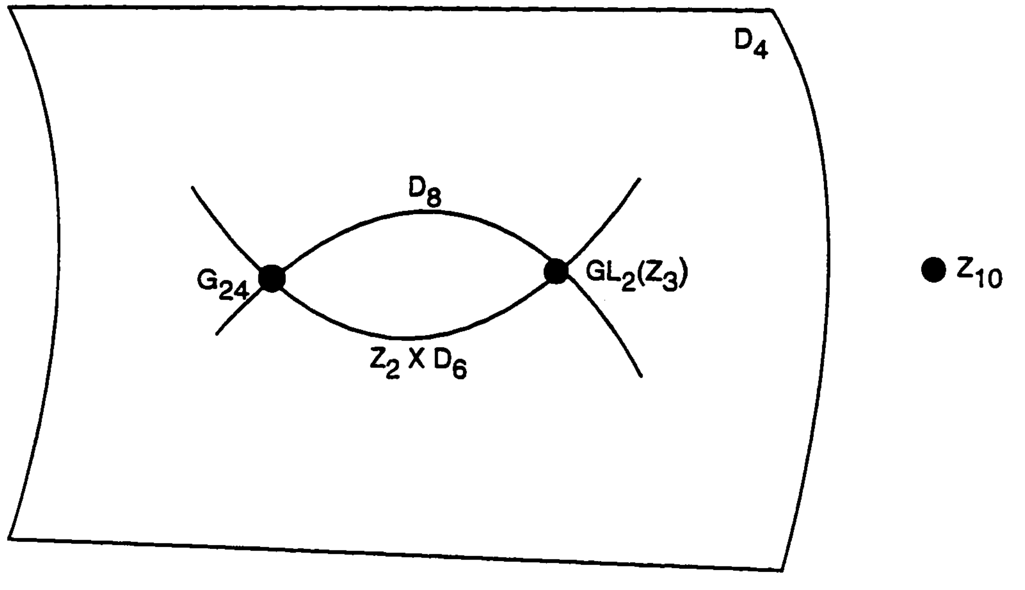
Boxed actions are maximal

A Portion of the Lattice of Actions in Genus 2

Figure 2.

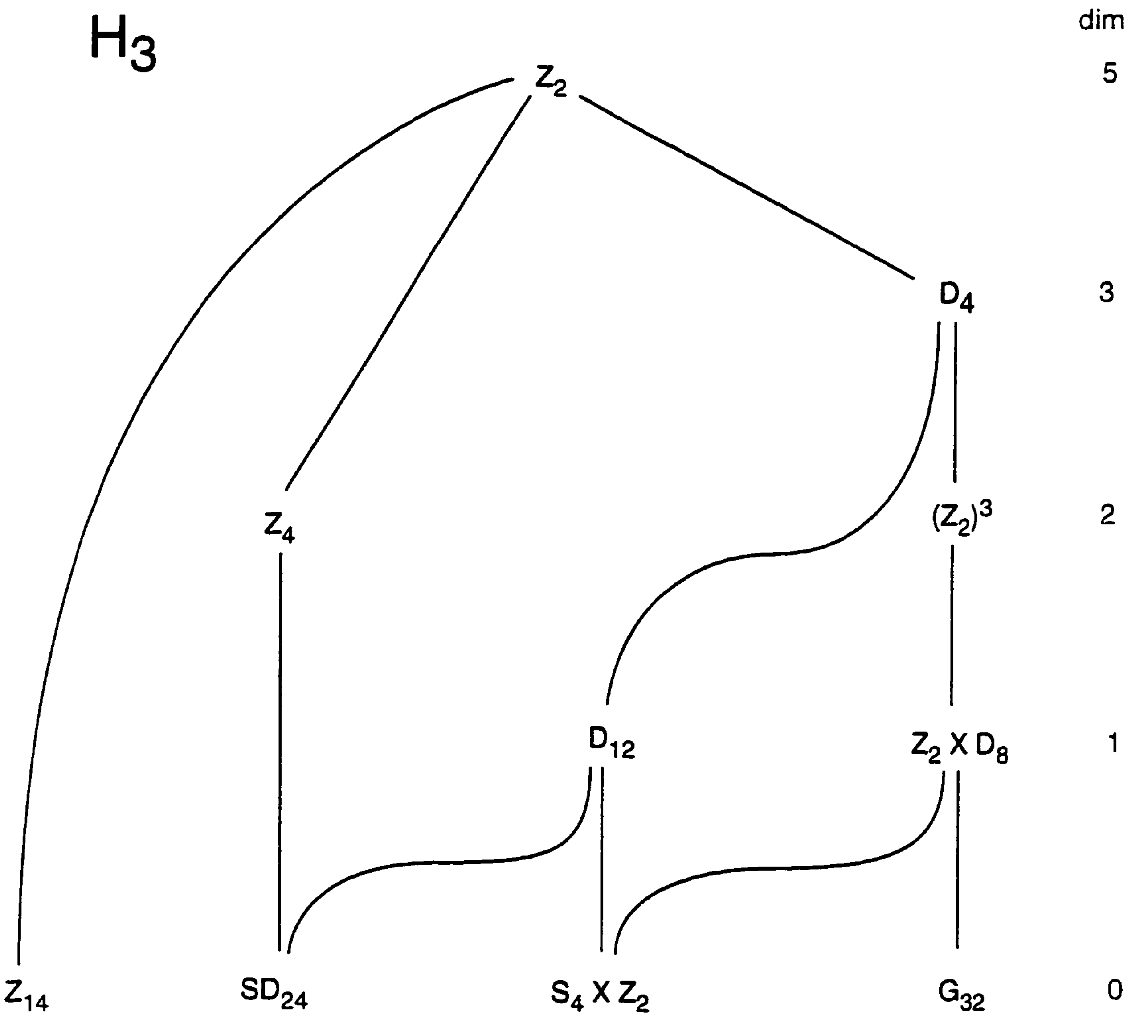


H2
dim
Z2
3
D4
2
Z2 X D6
D8
1
GL2(Z3)
G24
Z10
0

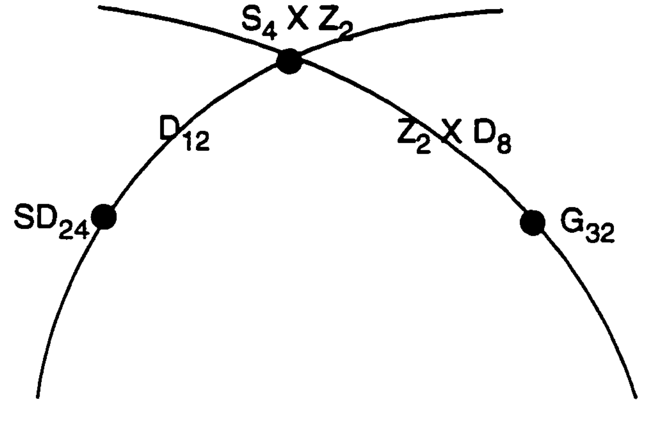


D4
D8
G24
GL2(Z3)
Z2 X D6
Z10

Figure 3.



H3
dim
Z2
5
D4
3
Z4
(Z2)3
2
D12
Z2 X D8
1
Z14
SD24
S4 X Z2
G32
0



S4 X Z2
D12
Z2 X D8
SD24
G32

Figure 4.

H_4

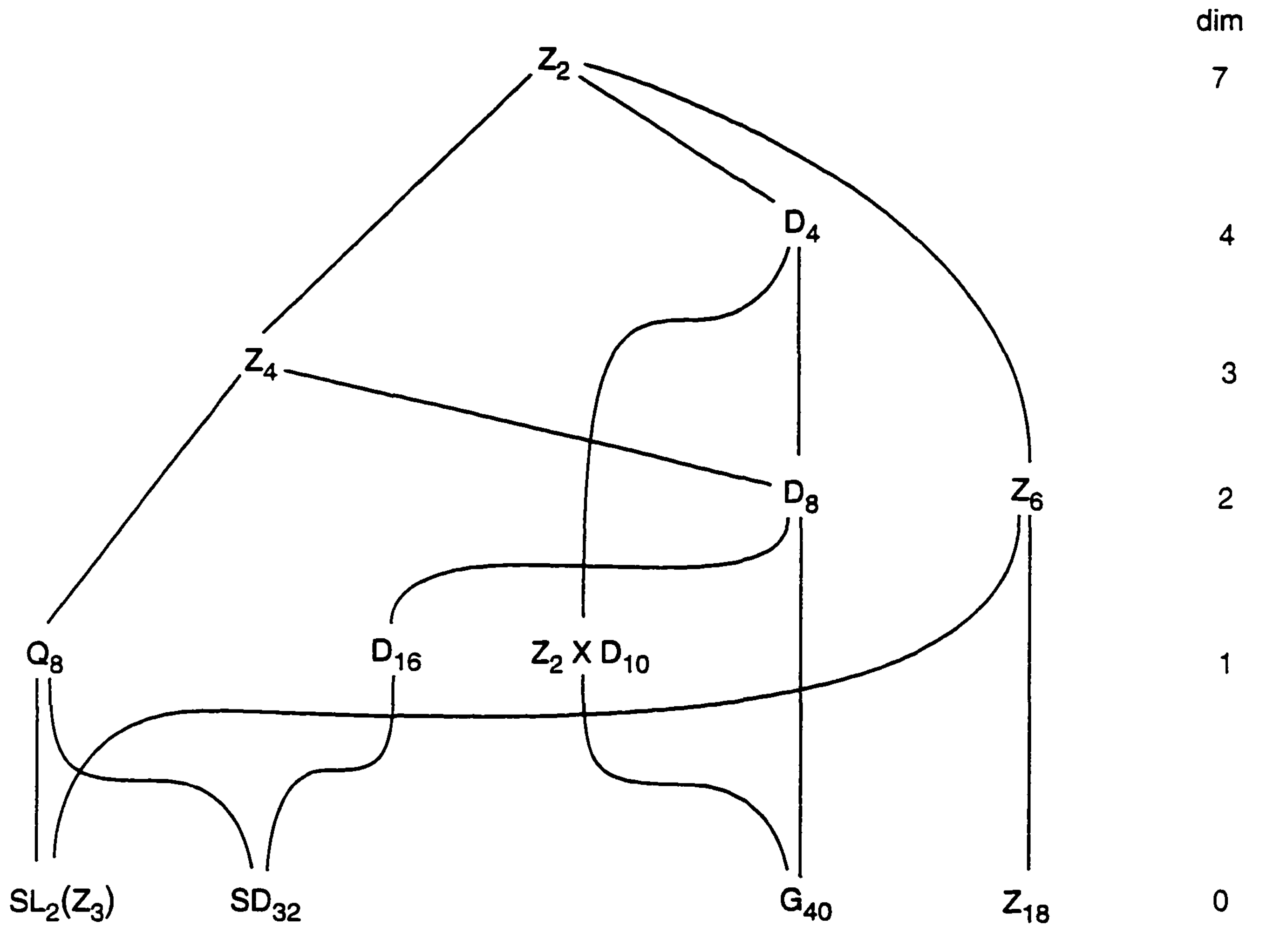


Figure 5.

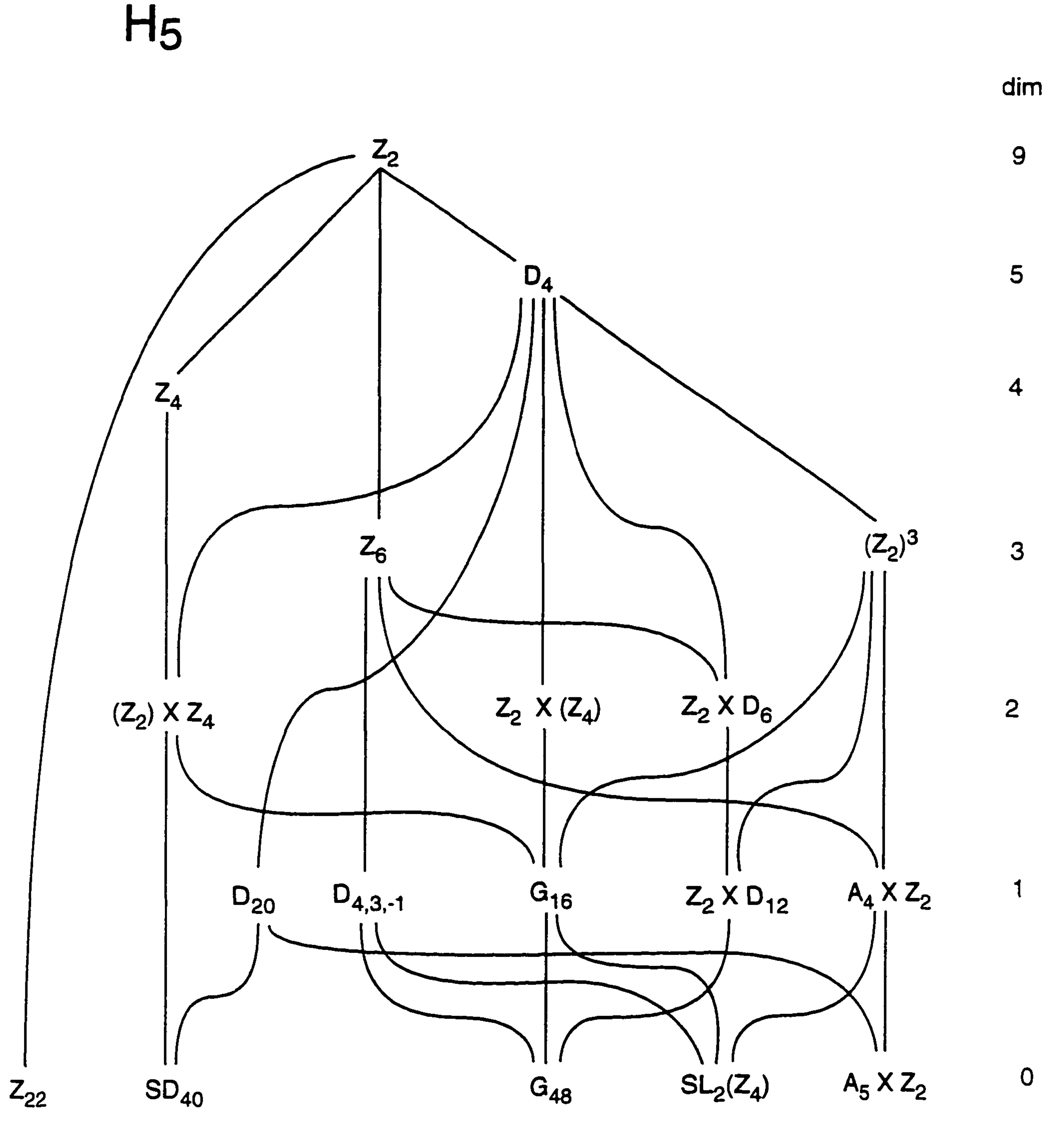

H5
dim
Z2
9
D4
5
Z4
4
Z6
(Z2)3
3
(Z2) X Z4
Z2 X (Z4)
Z2 X D6
2
D20
D4,3,-1
G16
Z2 X D12
A4 X Z2
1
Z22
SD40
G48
SL2(Z4)
A5 X Z2
0

Figure 6.

H_6

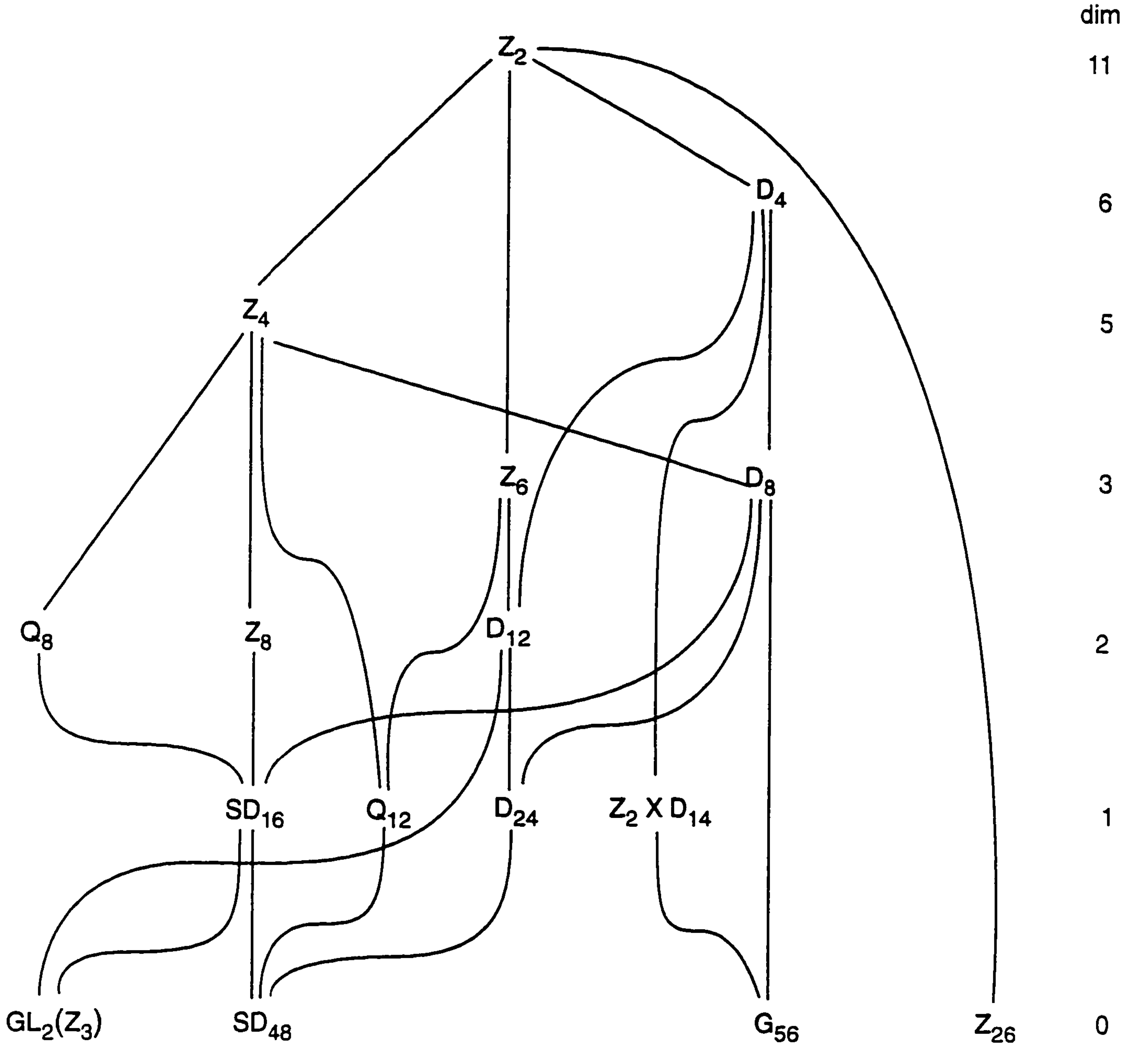

dim
Z_2
11
D_4
6
Z_4
5
Z_6
D_8
3
Q_8
Z_8
D_{12}
2
SD_{16}
Q_{12}
D_{24}
$Z_2 \times D_{14}$
1
$GL_2(Z_3)$
SD_{48}
G_{56}
Z_{26}
0

Figure 7.

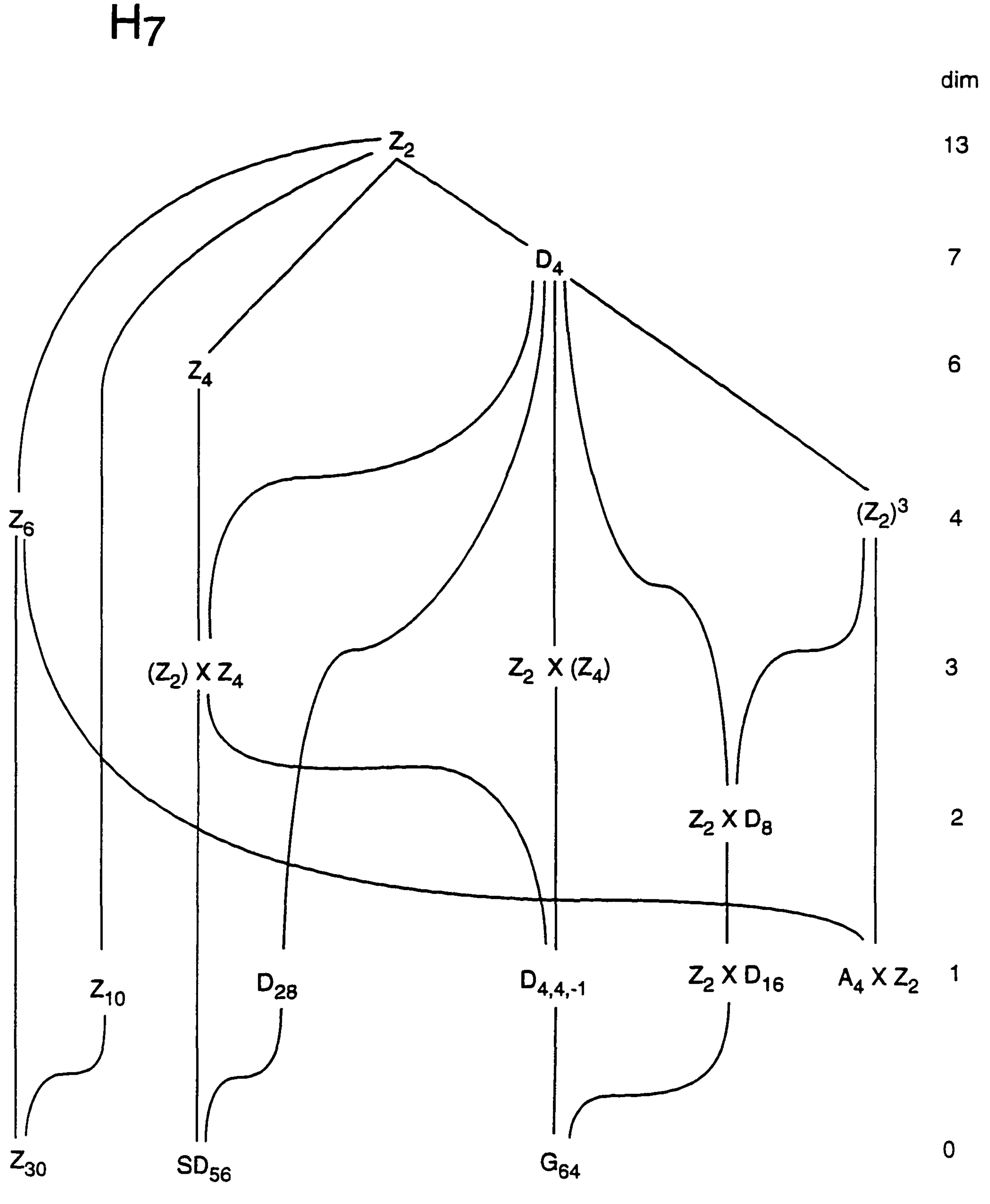

H7
dim
Z2
13
D4
7
Z4
6
Z6
(Z2)3
4
(Z2) X Z4
Z2 X (Z4)
3
Z2 X D8
2
Z10
D28
D4,4,-1
Z2 X D16
A4 X Z2
1
Z30
SD56
G64
0

Figure 8.

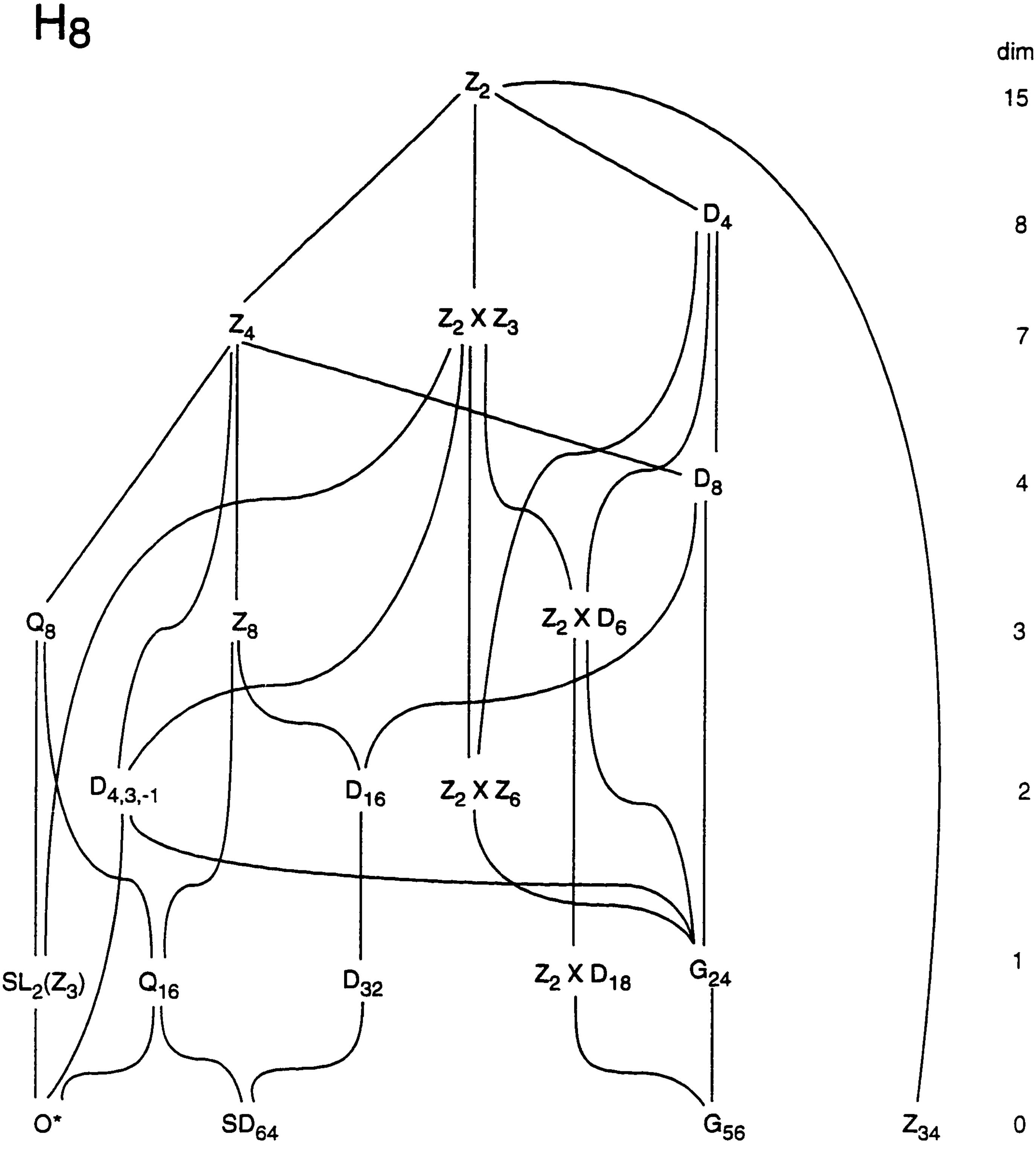

H8
dim
Z2
15
D4
8
Z4
Z2 X Z3
7
D8
4
Q8
Z8
Z2 X D6
3
D4,3,-1
D16
Z2 X Z6
2
SL2(Z3)
Q16
D32
Z2 X D18
G24
1
O*
SD64
G56
Z34
0

Figure 9.

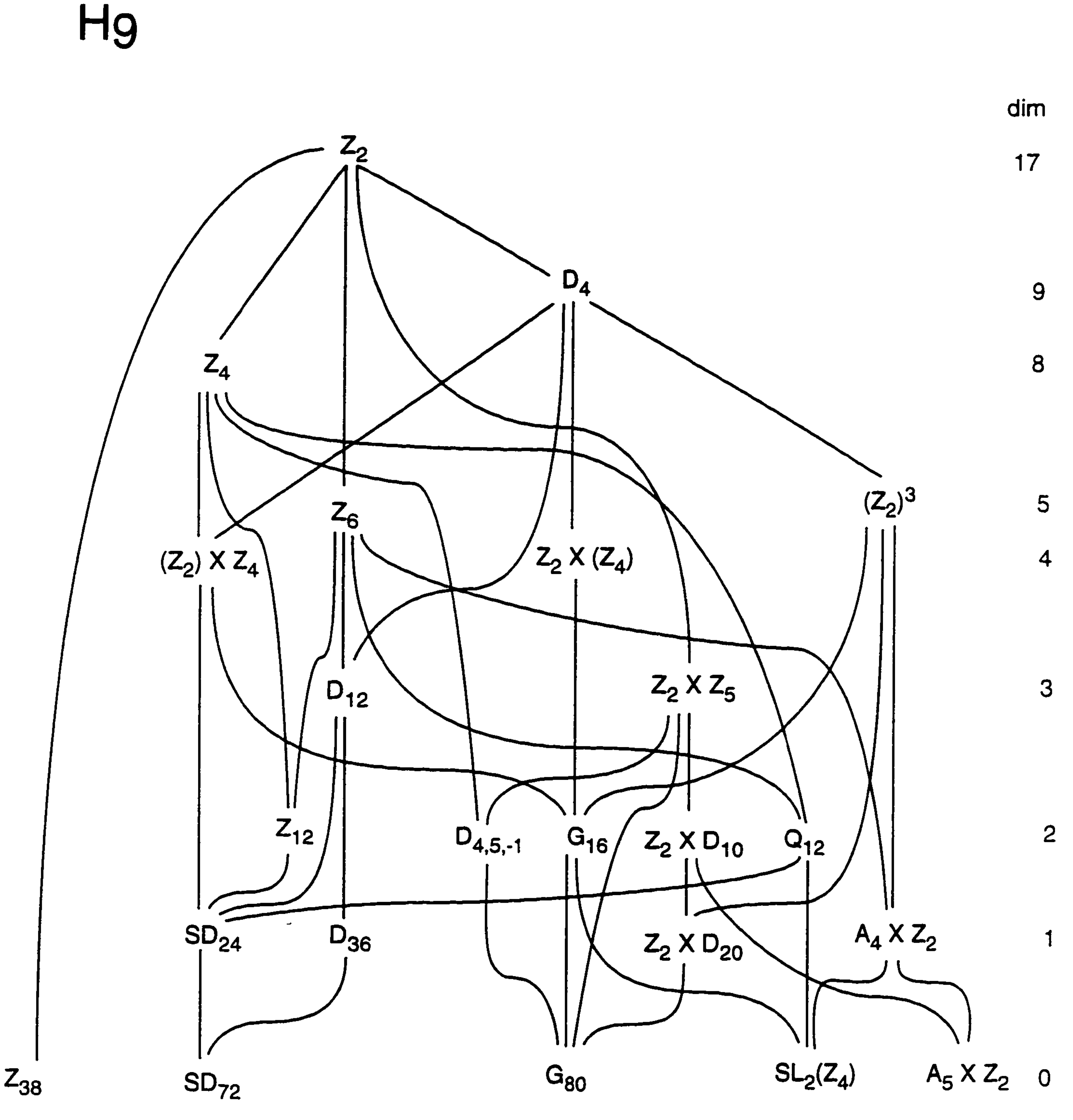
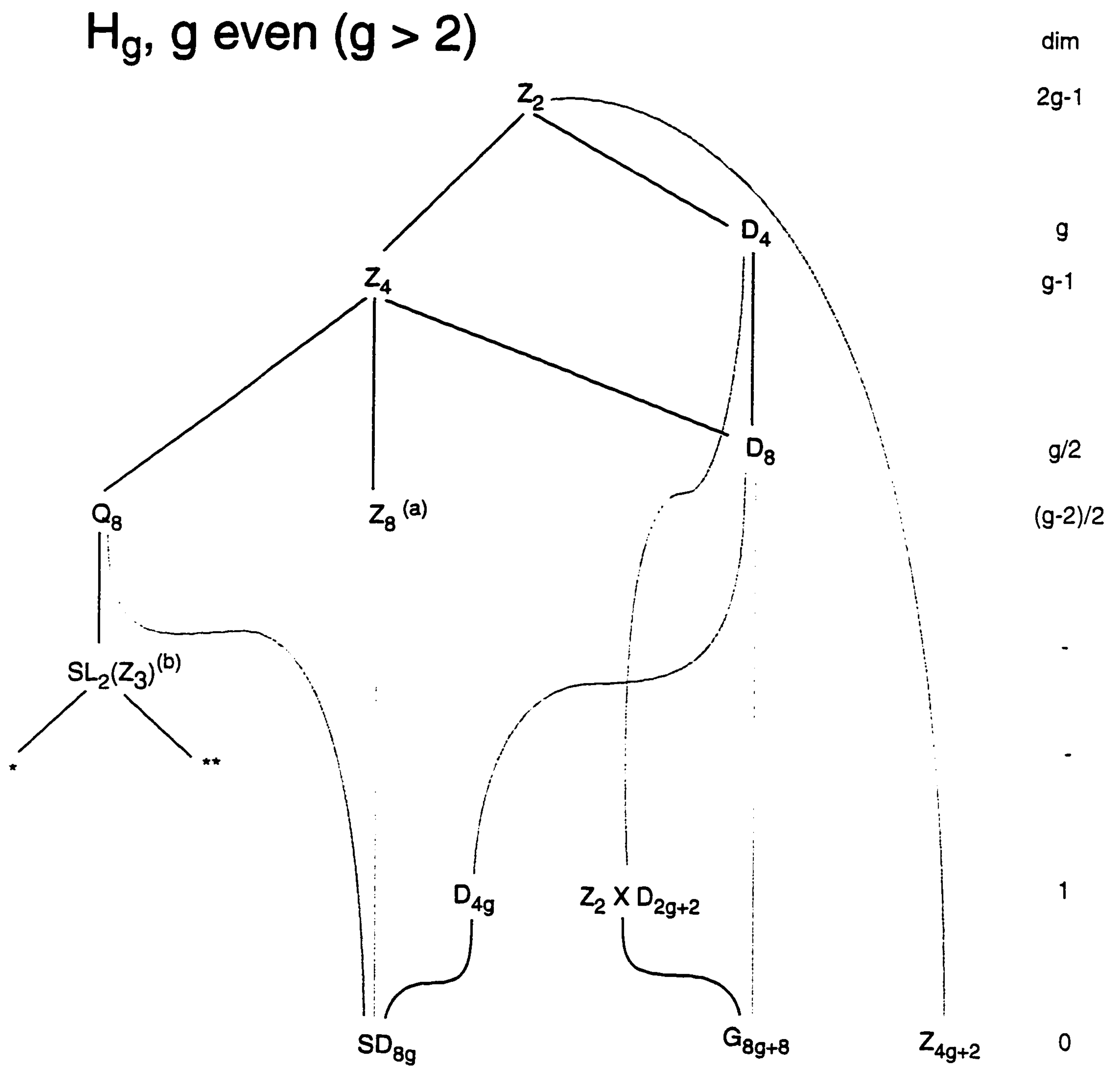

Hg
dim
Z2
17
D4
9
Z4
8
(Z2)3
5
Z6
(Z2) X Z4
Z2 X (Z4)
4
D12
Z2 X Z5
3
Z12
D4,5,-1
G16
Z2 X D10
Q12
2
SD24
D36
Z2 X D20
A4 X Z2
1
Z38
SD72
G80
SL2(Z4)
A5 X Z2
0

Figure 10.



* = $GL_2(Z_3)$ if g = 2, 6 (mod 12); = O* if g = 0, 8 (mod 12)

** = $SL_2(Z_5)$ if g = 0, 14, 20, 24 (mod 30)

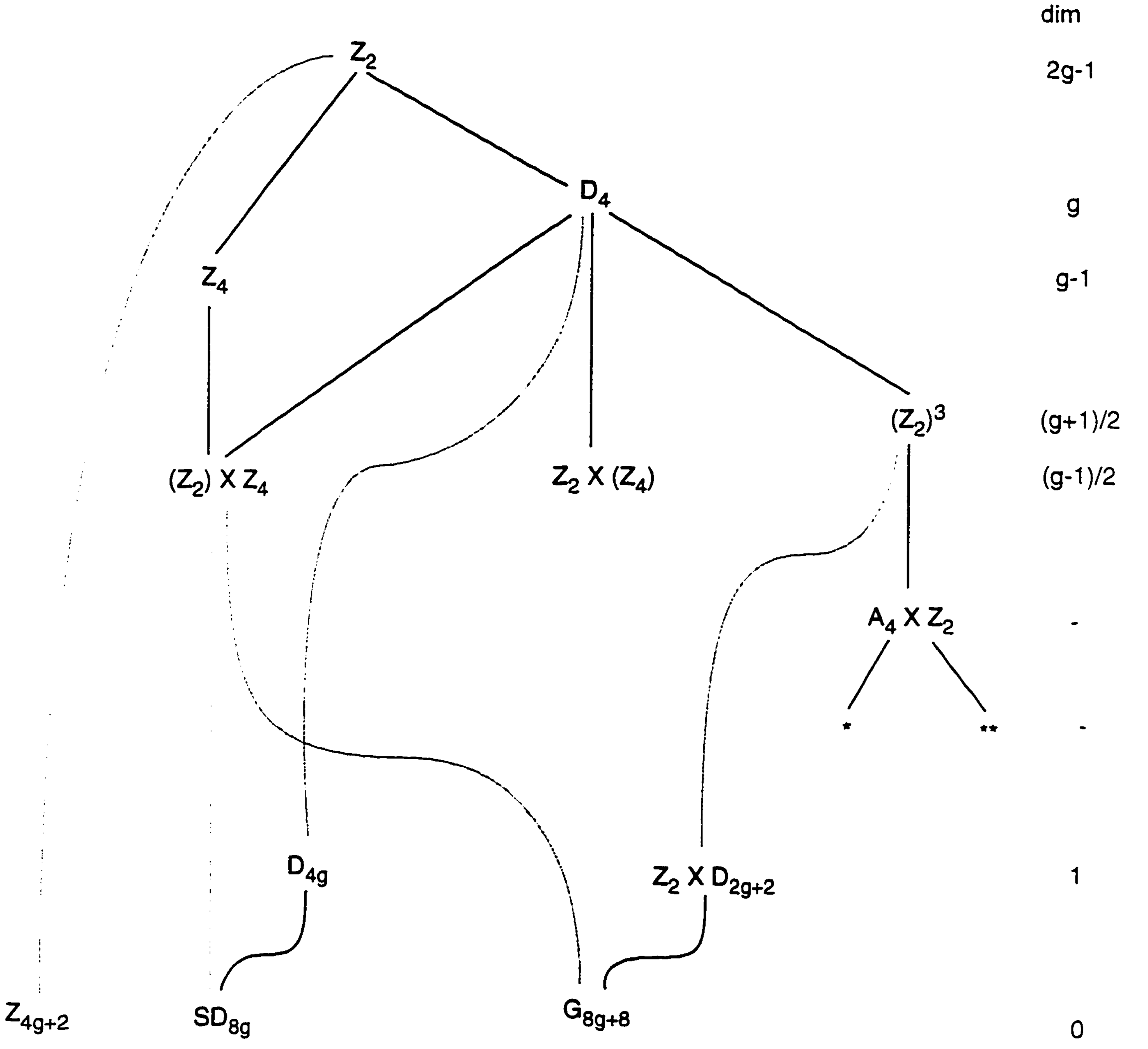
(a) Z_8 is non-maximal in genus 4

(b) $SL_2(Z_3)$ is non-maximal in genus 6

Light lines represent branches of the latice which may need filling in

Figure 11.

H_g, g odd (g > 3)



* = $S_4 \times Z_2$ if g = 3, 11 (mod 12); = $SL_2(Z_4)$ if g = 5, 9 (mod 12)

** = $A_5 \times Z_2$ if g = 5, 9, 11, 15 (mod 30)

Light lines represent branches of the lattice which may need filling in.

Figure 12.

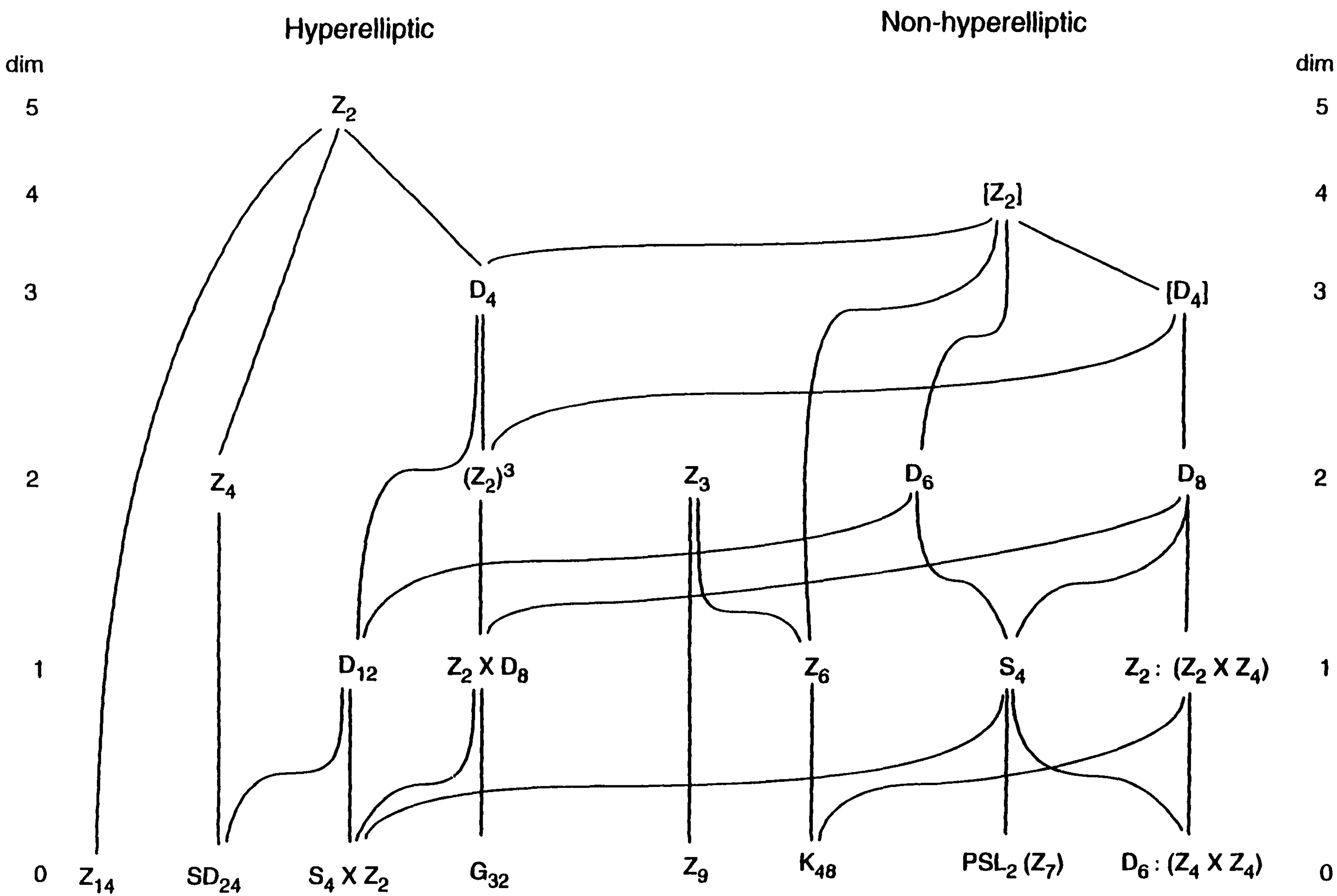


Figure 13. The Lattice of Maximal Actions in Genus 3

Appendix A
Character Tables

(These tables may be found in [FH] and/or [I]) (For the character tables of O^* and $SL_2(Z_4)$, see Tables 18 and 19, §4.)

1. $G = A_4$
   $|G| = 12; \quad \omega = e^{2\pi i/3}$

Class:	1	3_1	3_2	4
$\lvert\mathrm{Cl}(g)\rvert$:	1	4	4	3
χ_1:	1	1	1	1
χ_2:	1	ω	ω^2	1
χ_3:	1	ω^2	ω	1
χ_4:	3	0	0	-1

2. $G = SL_2(Z_3)$
   $|G| = 24; \quad \omega = e^{2\pi i/3}$

Class:	1	2_1	4_1	3_1	3_2	6_1	6_2
$\lvert\mathrm{Cl}(g)\rvert$:	1	1	6	4	4	4	4
χ_1:	1	1	1	1	1	1	1
χ_2:	1	1	1	ω	ω^2	ω	ω^2
χ_3:	1	1	1	ω^2	ω	ω^2	ω
χ_4:	3	3	-1	0	0	0	0
χ_5:	2	-2	0	-1	-1	1	1
χ_6:	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2
χ_7:	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω

3. $G = A_5$
$|G| = 60$; $\alpha_1 = (1+\sqrt{5})/2$; $\alpha_2 = (1-\sqrt{5})/2$

Class:	1	2_1	3_1	5_1	5_2
$\lvert \mathrm{Cl}(g) \rvert$:	1	15	20	12	12
χ_1:	1	1	1	1	1
χ_2:	4	0	1	-1	-1
χ_3:	5	1	-1	0	0
χ_4:	3	-1	0	α_1	α_2
χ_5:	3	-1	0	α_2	α_1

4. $G = SL_2(Z_5)$
$|G| = 120$; $\alpha_1 = (1+\sqrt{5})/2$; $\alpha_2 = (1-\sqrt{5})/2$

Class:	1	2_1	5_1	5_2	10_1	10_2	4_1	3_1	6_1
$\lvert \mathrm{Cl}(g) \rvert$:	1	1	12	12	12	12	30	20	20
χ_1:	1	1	1	1	1	1	1	1	1
χ_2:	5	5	0	0	0	0	1	-1	-1
χ_3:	6	-6	1	1	-1	-1	0	0	0
χ_4:	3	3	α_1	α_2	α_1	α_2	-1	0	0
χ_5:	3	3	α_2	α_1	α_2	α_1	-1	0	0
χ_6:	4	-4	-1	-1	1	1	0	1	-1
χ_7:	4	4	-1	-1	-1	-1	0	1	1
χ_8:	2	-2	$-\alpha_2$	$-\alpha_1$	α_2	α_1	0	-1	1
χ_9:	2	-2	$-\alpha_1$	$-\alpha_2$	α_1	α_2	0	-1	1

5. $G = S_4$
$|G| = 24$

Class:	1	2_1	2_2	3	4
$\|\mathrm{Cl}(g)\|$:	1	6	3	8	6
χ_1:	1	1	1	1	1
χ_2:	1	−1	1	1	−1
χ_3:	2	0	2	−1	0
χ_4:	3	1	−1	0	−1
χ_5:	3	−1	−1	0	1

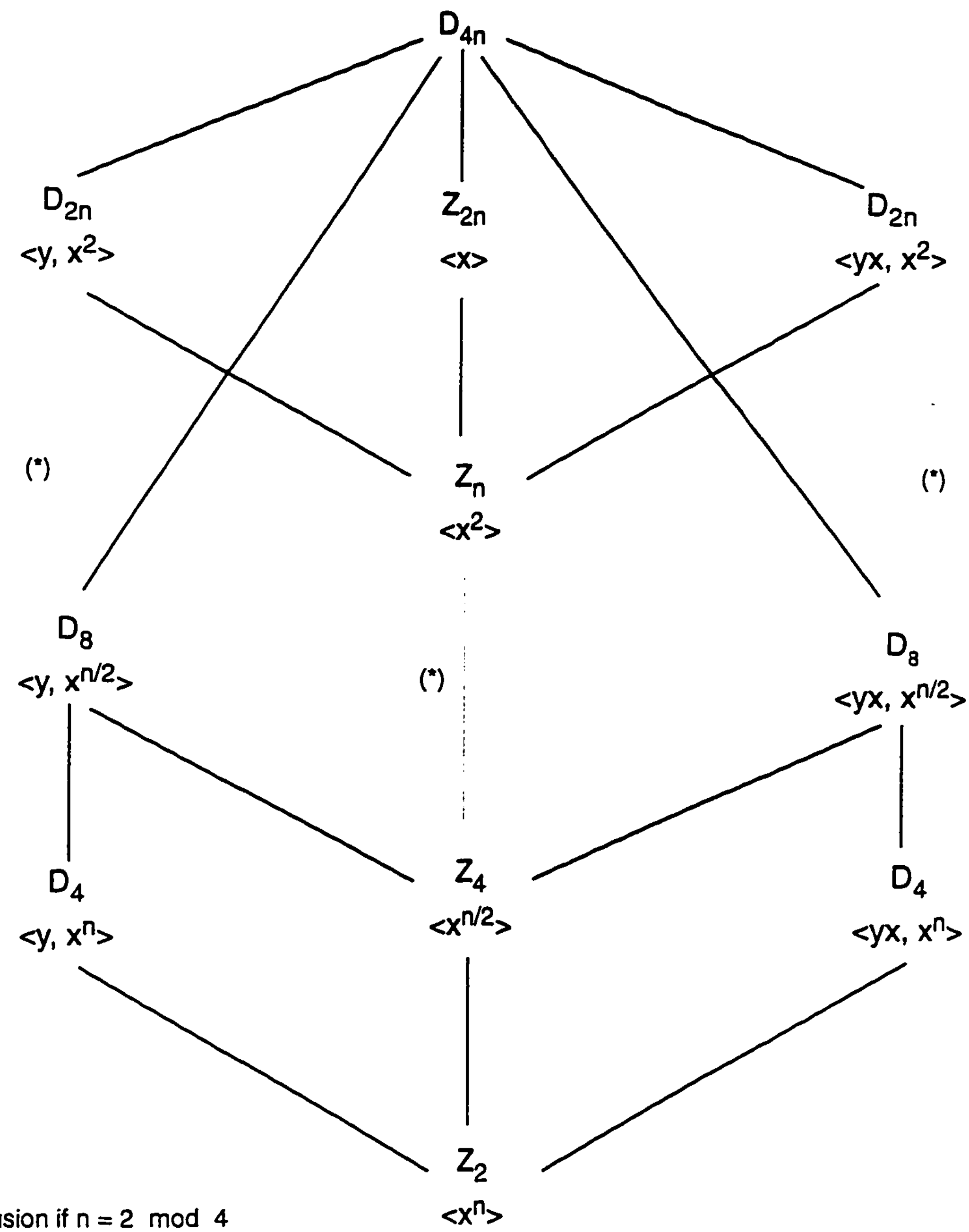
6. $G = GL_2(Z_3)$
$|G| = 48$

Class:	1	2_1	3_1	6_1	4_1	2_2	8_1	8_2
$\|\mathrm{Cl}(g)\|$:	1	1	8	8	6	12	6	6
χ_1:	1	1	1	1	1	1	1	1
χ_2:	1	1	1	1	1	−1	−1	−1
χ_3:	2	2	−1	−1	2	0	0	0
χ_4:	2	−2	−1	1	0	0	$\sqrt{2}$	$-\sqrt{2}$
χ_5:	2	−2	−1	1	0	0	$-\sqrt{2}$	$\sqrt{2}$
χ_6:	3	3	0	0	−1	−1	1	1
χ_7:	3	3	0	0	−1	1	−1	−1
χ_8:	4	−4	1	−1	0	0	0	0

Appendix B
Partial Lattices of Subgroups Containing t

D_{4n}, n even

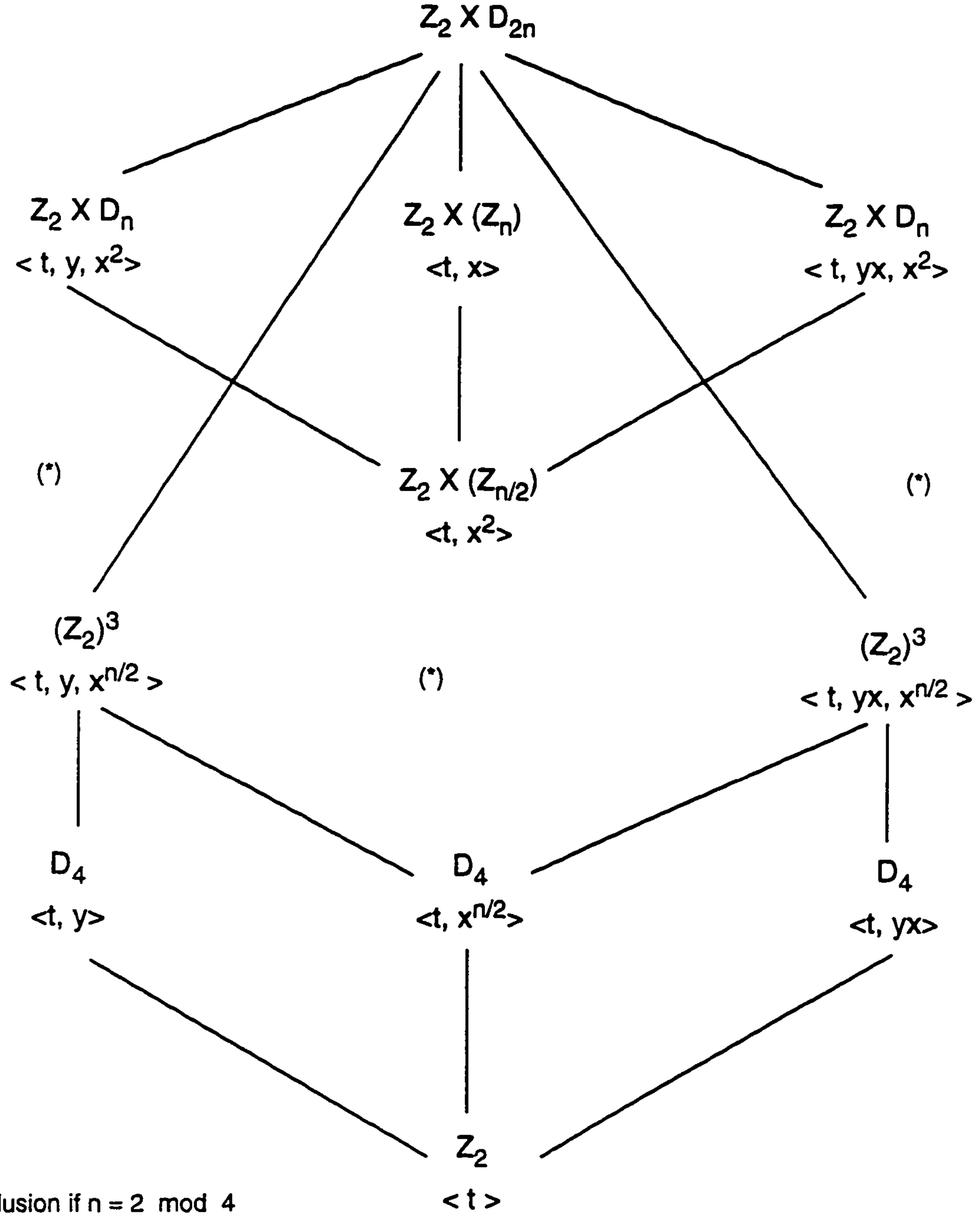
$\langle x, y \mid x^{2n} = y^2 = e,\ yxy^{-1} = x^{-1} \rangle \qquad t = x^n$



(*) No inclusion if n = 2 mod 4

$Z_2 \times D_{2n}$, n even

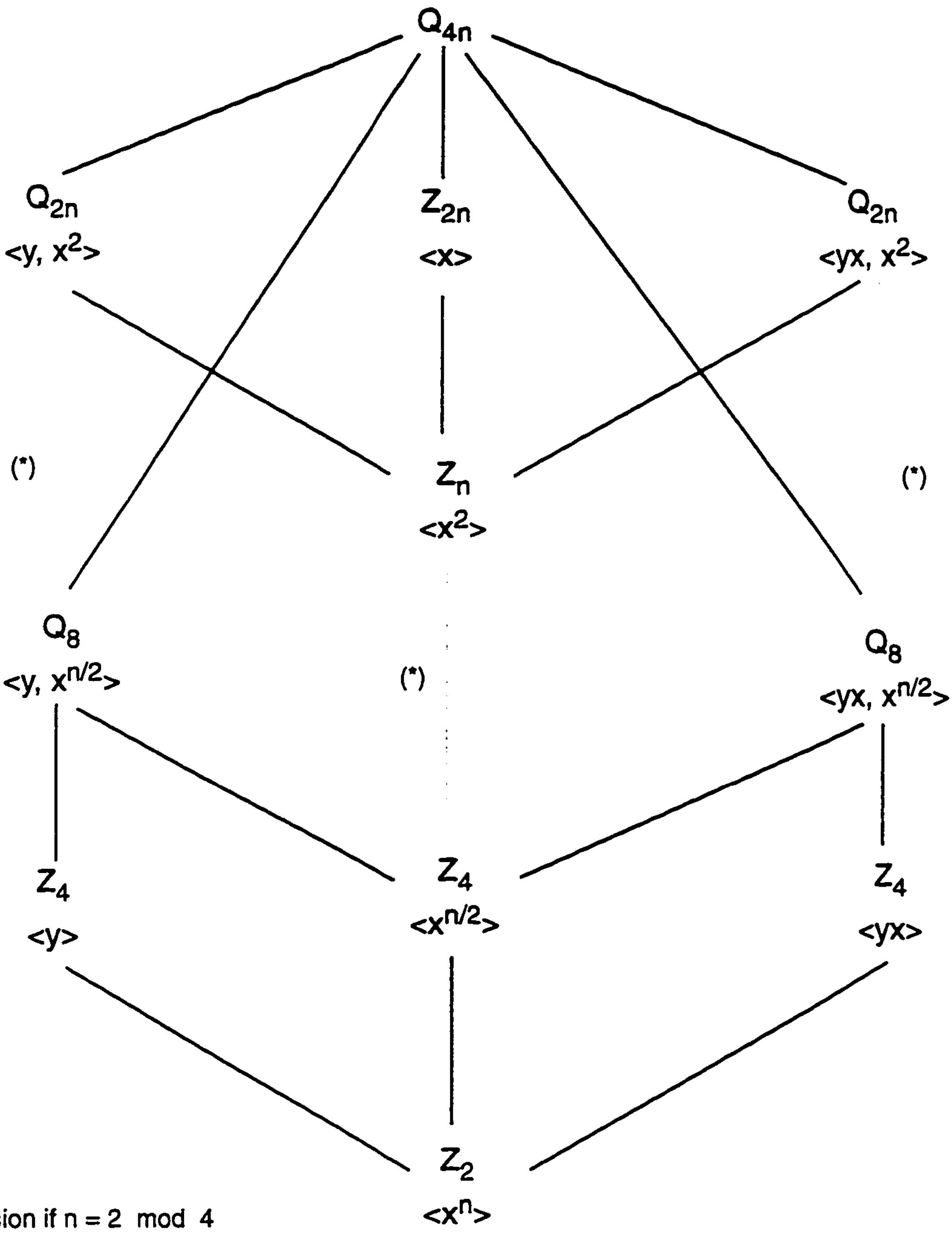
$\langle t, x, y \mid x^n = y^2 = t^2 = e, \quad yxy^{-1} = x^{-1}, \quad [t,x] = [t,y] = e \rangle$



(*) No inclusion if $n = 2 \mod 4$

Q_{4n}, n even (n>2)

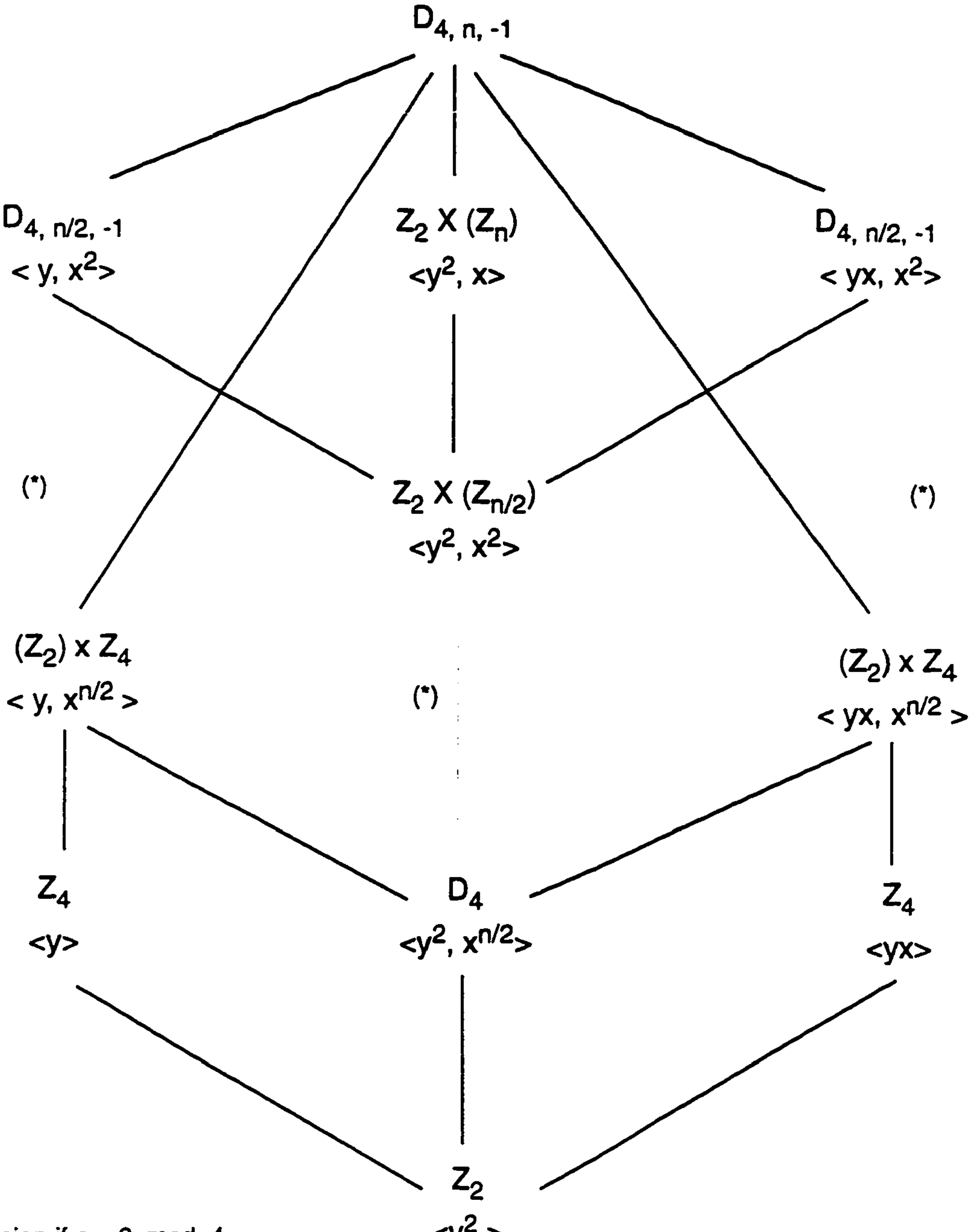
$\langle x, y \mid x^{2n} = e,\ y^2 = x^n,\ yxy^{-1} = x^{-1} \rangle \quad t = x^n = y^2$



(*) No inclusion if n = 2 mod 4

$D_{4,n,-1}$, n even

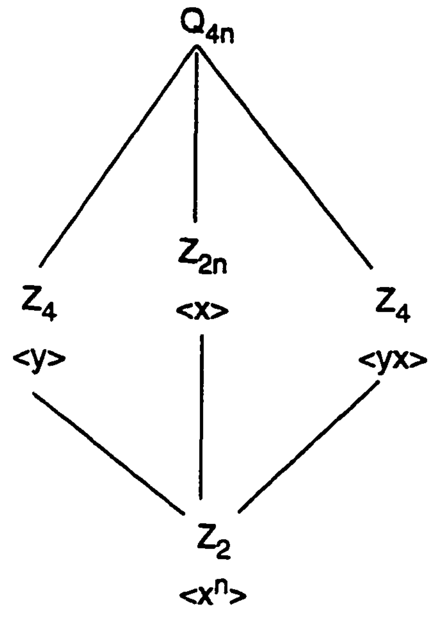
$< x, y \mid x^n = y^4 = e, \quad yxy^{-1} = x^{-1} > \quad t = y^2$



(*) No inclusion if n = 2 mod 4

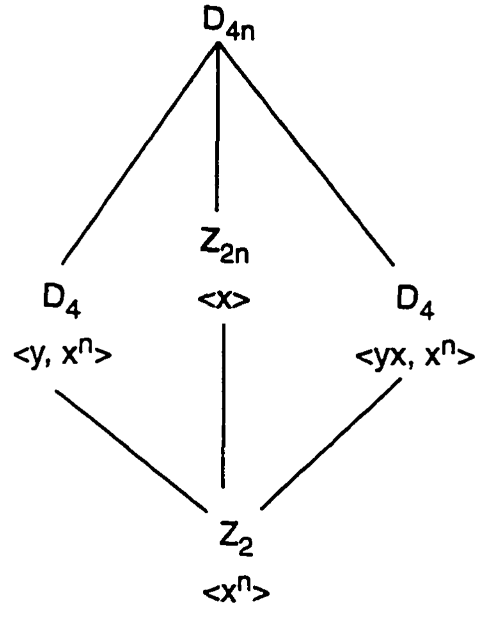
Q_{4n}, n odd (= $D_{4,n,-1}$, n odd)

$\langle x, y \mid x^{2n} = e, y^2 = x^n, \ yxy^{-1} = x^{-1} \rangle \quad t = x^n$



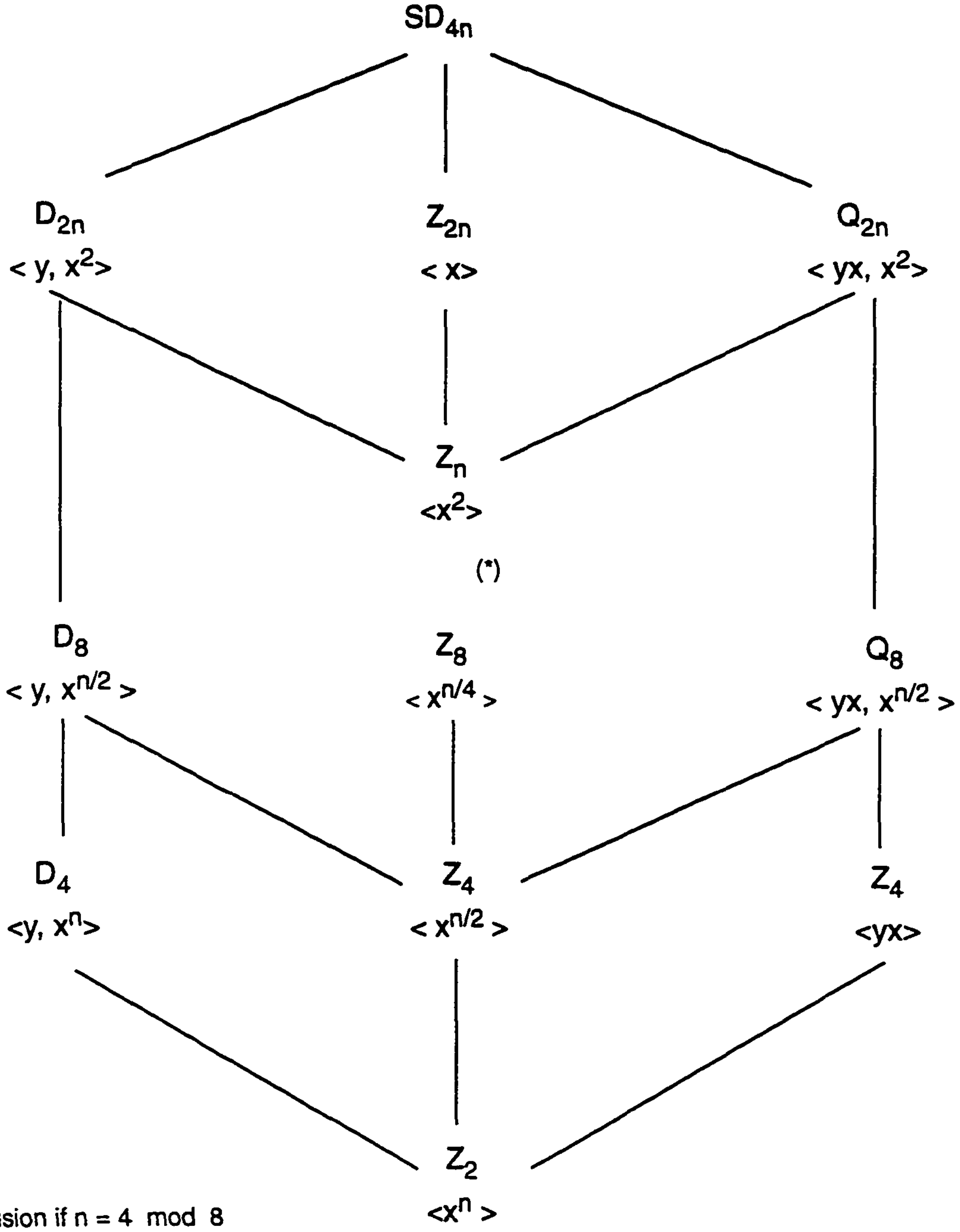
D_{4n}, n odd (= $Z_2 \times D_{2n}$, n odd)

$\langle x, y \mid x^{2n} = y^2 = e, \ yxy^{-1} = x^{-1} \rangle \quad t = x^n$



SD_{4n}, $n = 0$ (4)

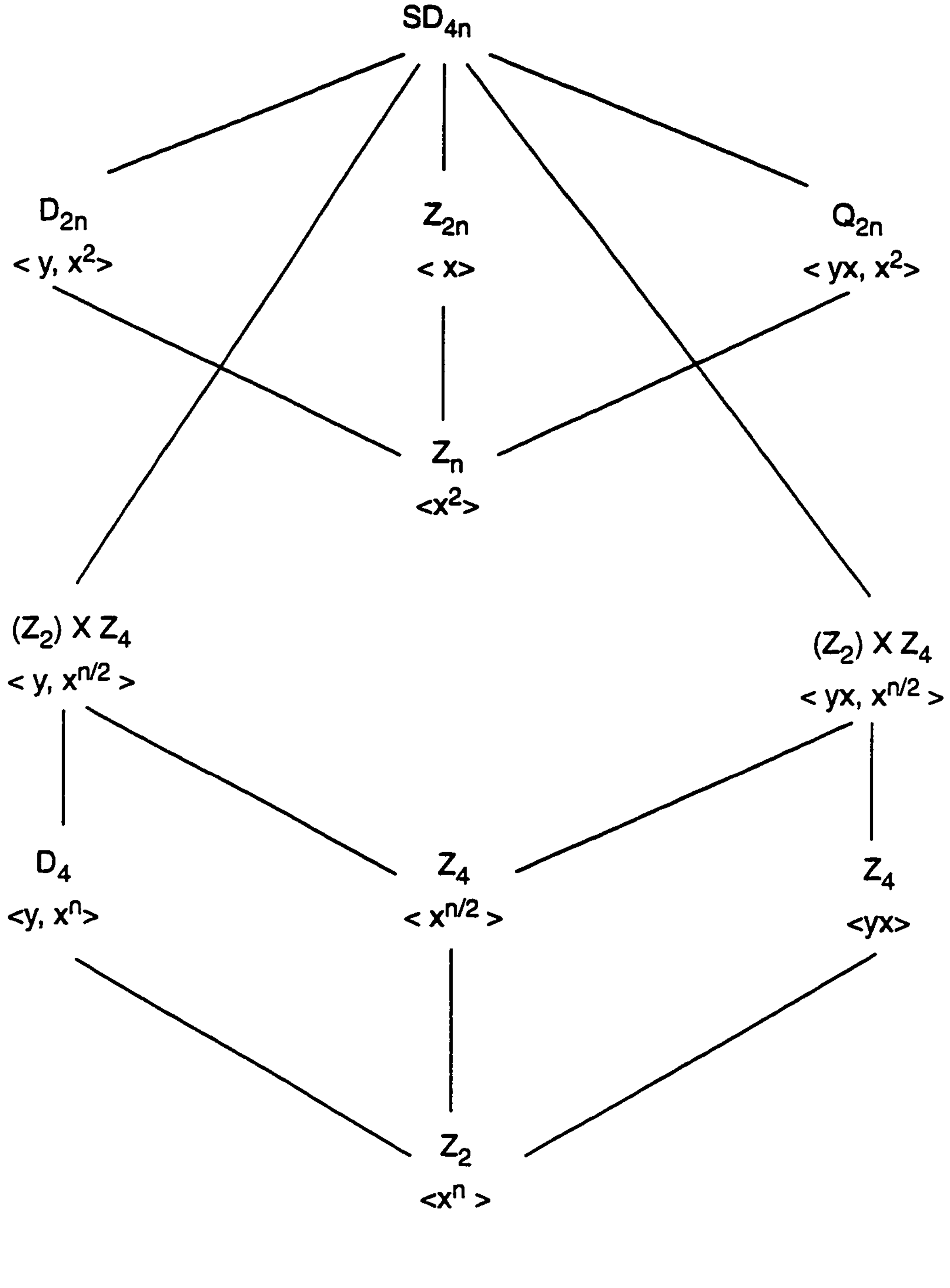
$< x, y \mid x^{2n} = y^2 = e, \quad yxy^{-1} = x^{n-1} > \quad t = x^n$



(*) No inclusion if $n = 4 \mod 8$

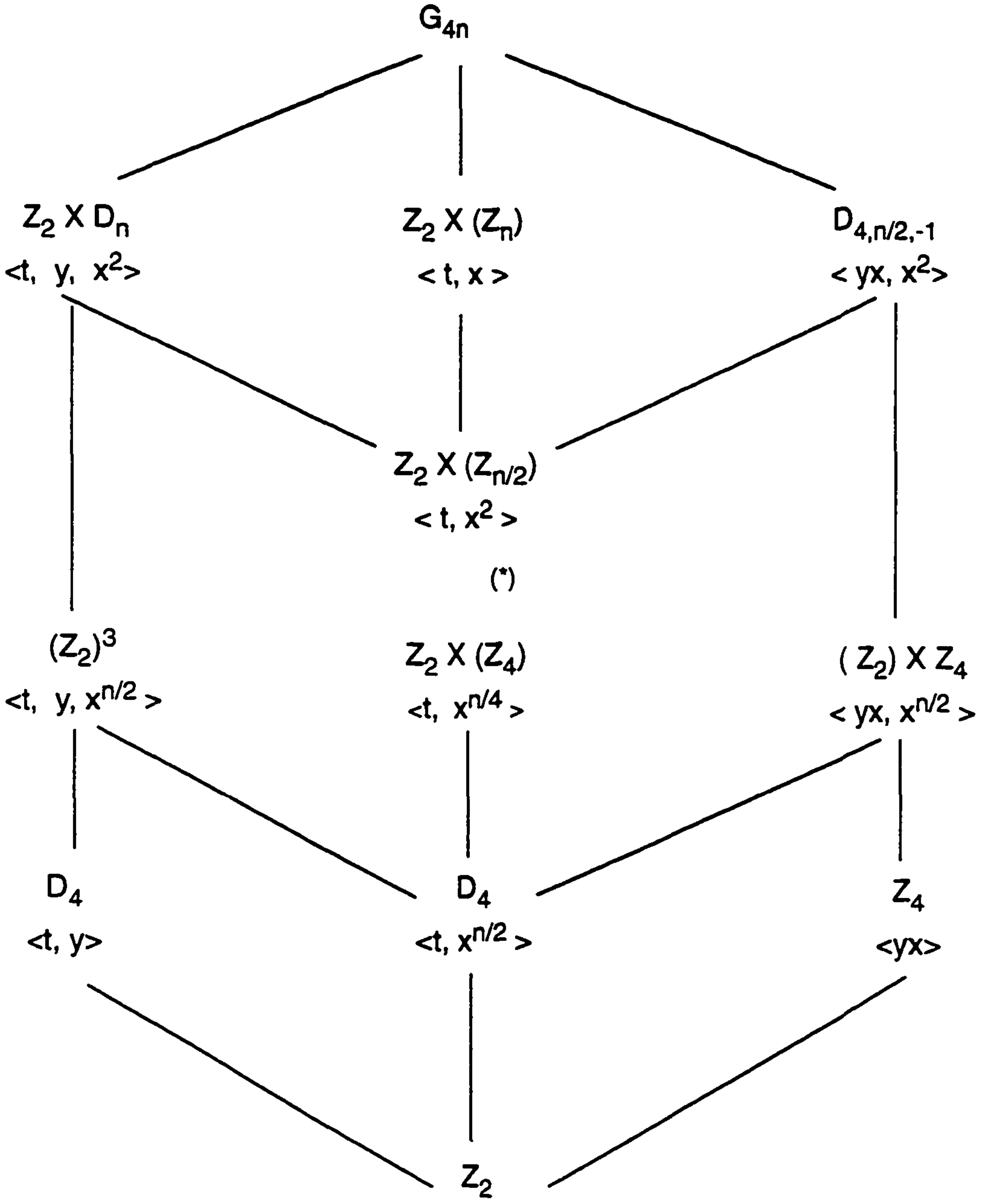
SD_{4n}, $n = 2\ (4)$

$< x, y \mid x^{2n} = y^2 = e, \quad yxy^{-1} = x^{n-1} > \quad t = x^n$



G_{4n}, $n = 0\ (4)$

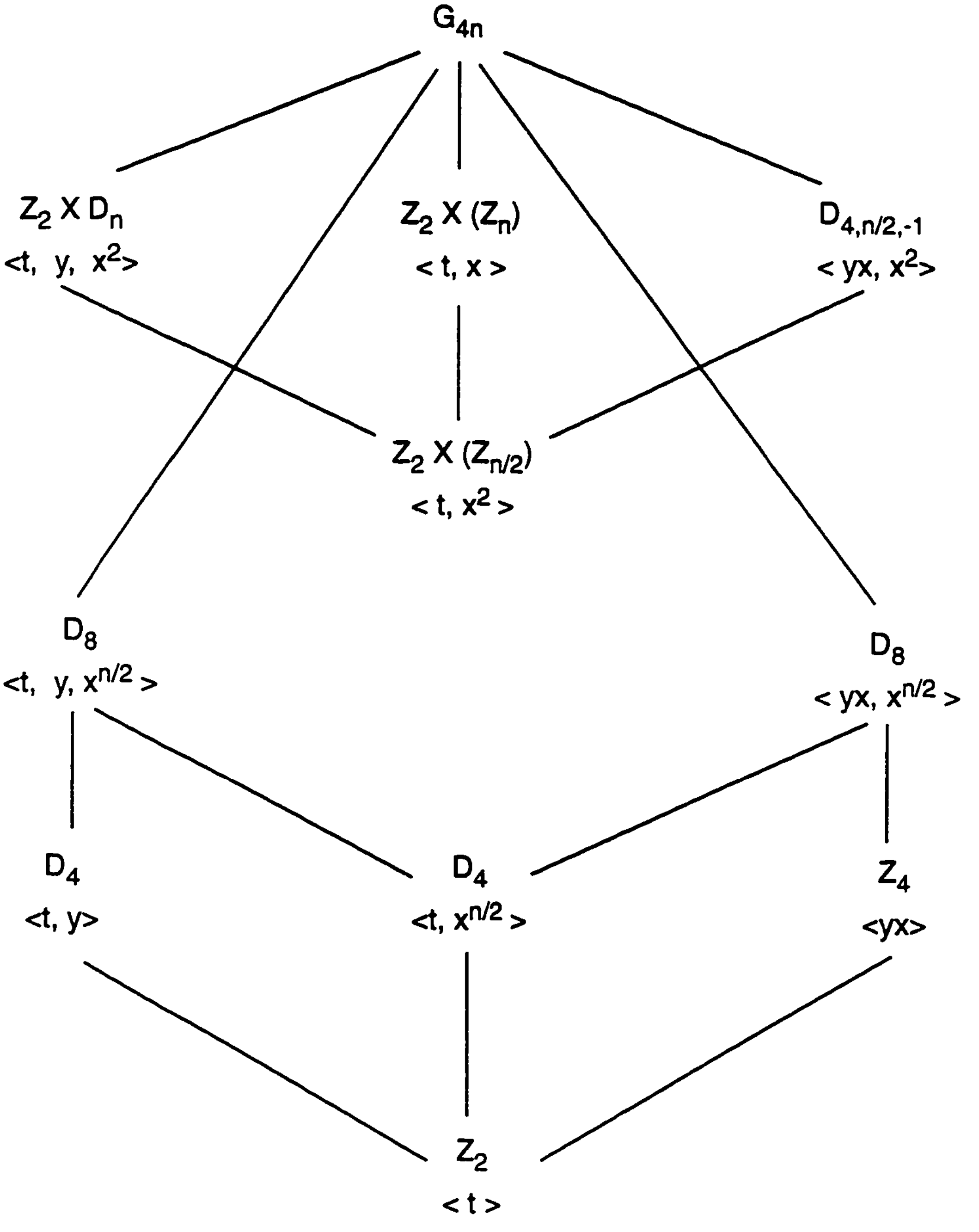
$< x, y, t \mid x^n = y^2 = t^2 = e, \quad yxy^{-1} = tx^{-1}, \quad [t, x] = [t, y] = e >$

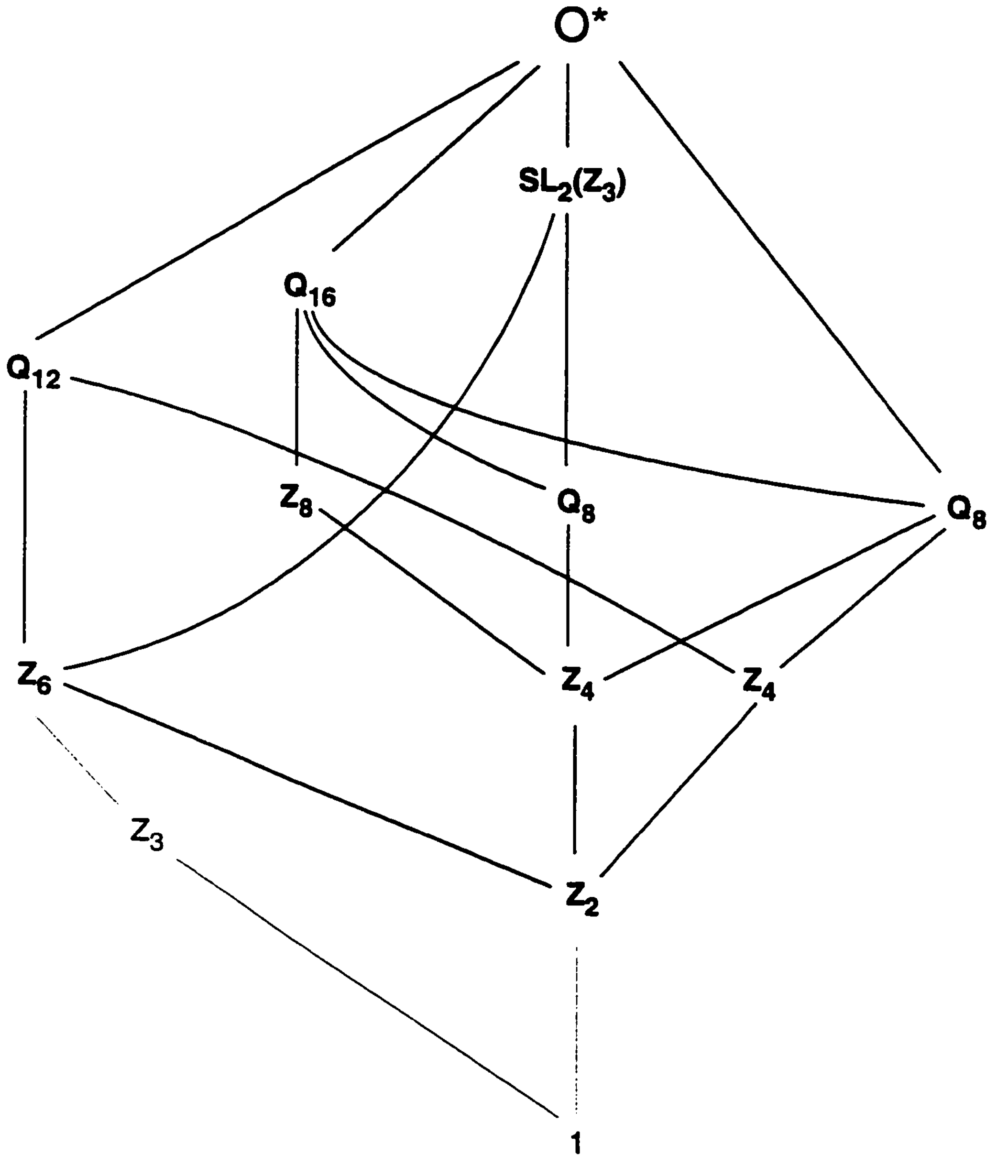


(*) No inclusion if n = 4 mod 8

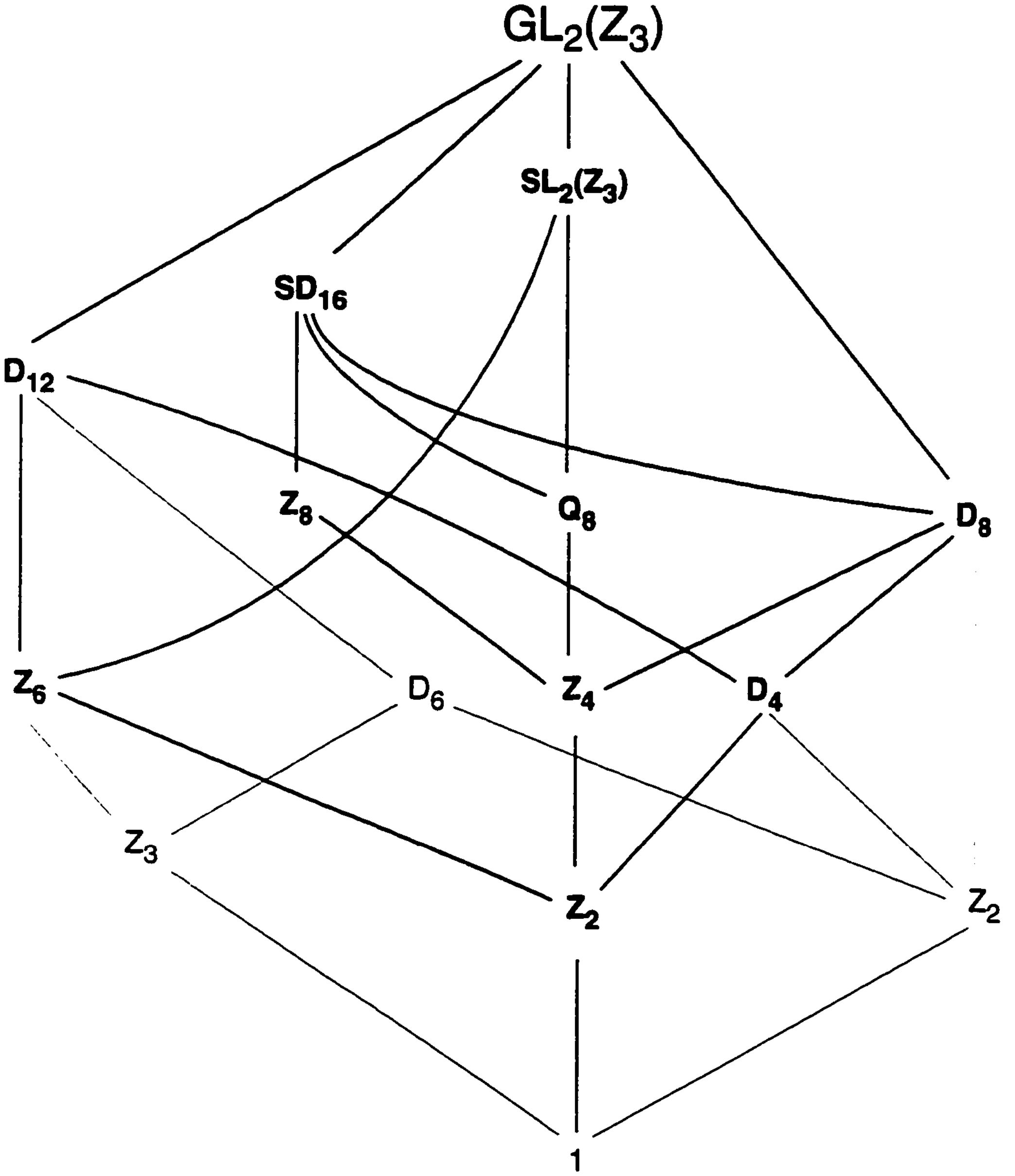
G_{4n}, $n = 2$ (4)

$< x, y, t \mid x^n = y^2 = t^2 = e, \quad yxy^{-1} = tx^{-1}, \quad [t, x] = [t, y] = e >$

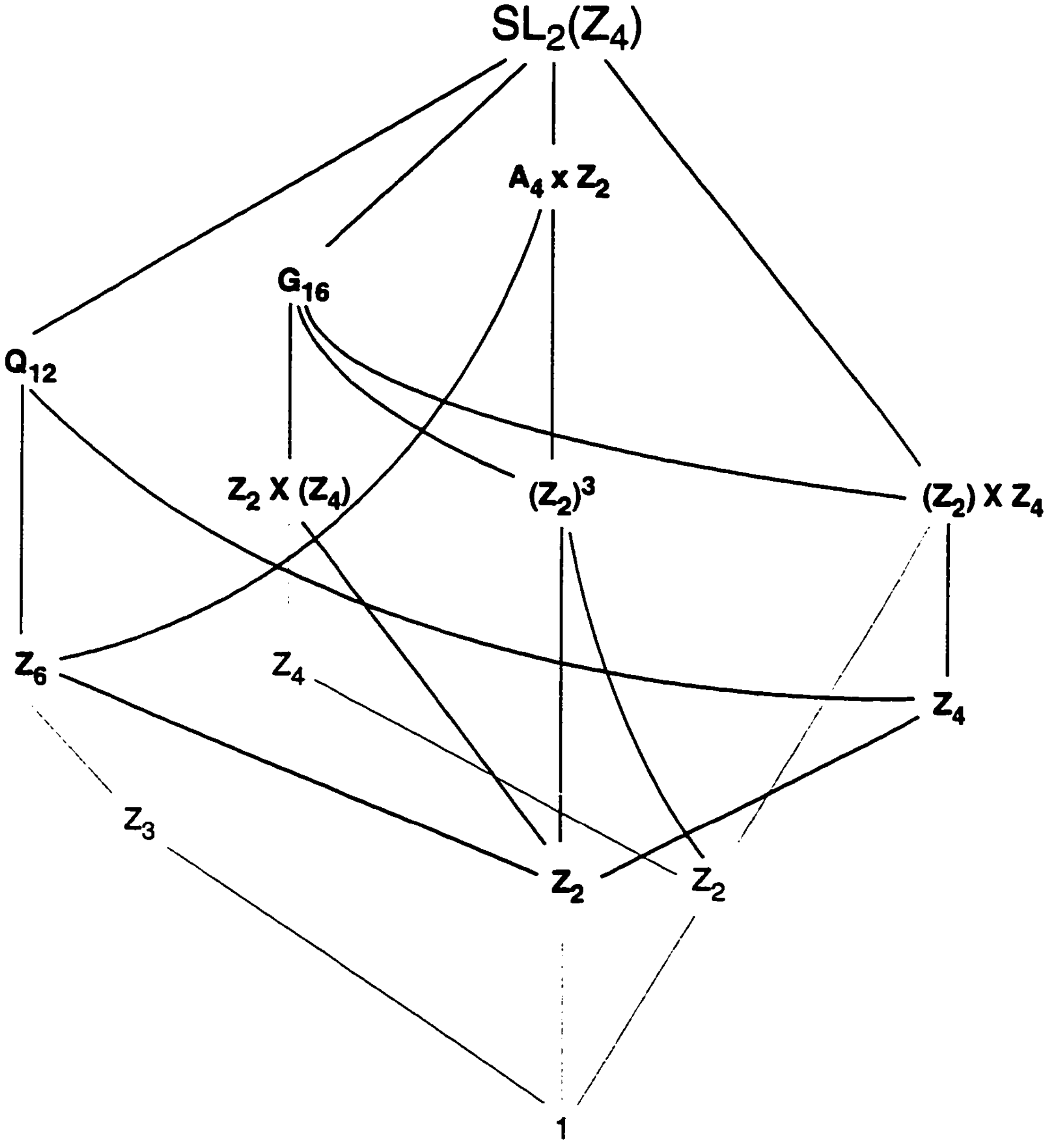




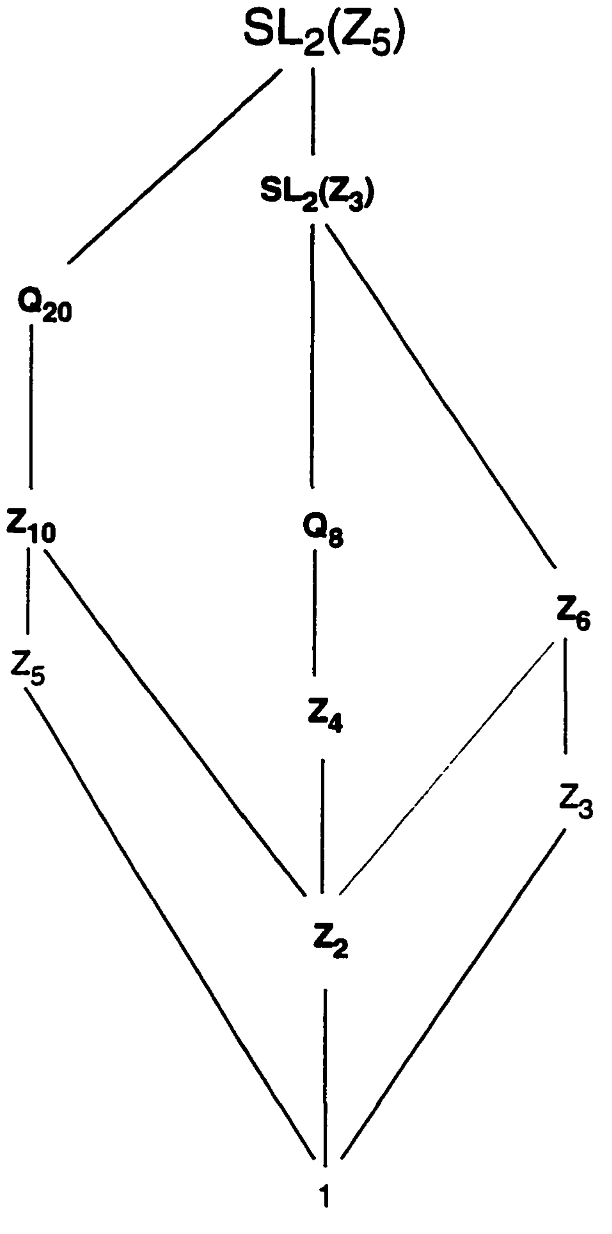
(Subgroups containing t are highlighted in boldface)



(Subgroups containing t are highlighted in boldface)



(Subgroups containing t are highlighted in boldface)



(Subgroups containing t are highlighted in boldface)

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