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SCREENING EFFECTS IN QCD

AND

RENORMALIZATION OF HAMILTONIAN

by

GARNIK G. ALEXANIAN

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1999

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Part I

Screening Effects in QCD

Chapter 1

Introduction

It has been more than twenty years since non-Abelian gauge theory emerged as a primary model for our attempt to describe strong interactions. Due to the numerous successes that it had in predicting and explaining phenomena at high energy ($\sim$ 10 GeV), Quantum Chromodynamics (QCD) is firmly believed to be the "correct" theory describing quarks and gluons. This belief is largely based on the possibility of applying perturbation theory at such regimes, which is expected to produce reliable results that can be tested in the experiment. However, multiple attempts to develop any approach valid for low energy processes have failed so far.

Thus, upto now physical conditions when perturbation theory can be applied to QCD could only be obtained in particle accelerators and can occur naturally only in astrophysical context.

Recently, however, a new possibility for studying the physics of quarks and gluons has appeared. With the development of the relativistic heavy ion colliders (like RHIC being constructed at Brookhaven National Lab) it

is believed to be possible to model some of the processes that take place inside super-dense and hot nuclear matter. This idea is based upon interesting qualitative feature of QCD - the so called "de-confinement phase transition" which is expected on the basis of some lattice simulations as well as theoretical arguments [1, 2]. This transition should happen when ambient temperature is raised higher than certain critical value T_c (which is of the order of the QCD scale $\Lambda_{QCD} \approx 200\ MeV$.) At such a high T quarks and gluons are no longer "locked" inside the hadrons; the plasma of quarks, anti-quarks and gluons is formed.

It is proposed that this new state of matter, the quark-gluon plasma ("QGP" from now on) can perhaps be obtained by colliding two nuclei of heavy ions at relativistic energies. This possibility is very interesting for possible further test of QCD as a model for strong interactions as well as our understanding of physics of early Universe.

There are many questions about wether it is possible to to obtain such a plasma in the laboratory [3]. Will it reach thermal equilibrium? If so, for how long? Perhaps one of the most important questions to ask is what is the signal that this plasma state has been formed, since it will cool and rehadronize on extremely short time-scale [4].

In order to start addressing all these concerns, as a first step it is useful to understand properties of the QGP at equilibrium. The immediate question then is about the possibility of applying perturbation theory to QGP calculations.

There are several well known effects that occur in the usual plasma of electrons and photons at high temperature and density. Effects like Debye

screening, Landau damping and Thomas-Fermi screening are well understood within perturbative analysis of Quantum Electrodynamics (QED) [5]. While successful in describing these phenomena in QED, it turns out that naive application of the standard perturbation theory to high temperature QCD fails. This manifests itself in the fact that several physical quantities that one tries to compute as a power series in coupling constant turn out to be gauge-dependent. For example, this is true for the gluon damping rate in hot QCD. In some cases a new type of infrared divergence appears.

As pointed by Pisarski [6], this happens due to the fact that connection between perturbative expansion in powers of coupling constant and loop-order breaks down at high temperature. Naively higher loop diagrams can contribute to the same order in coupling constant as the lower ones, thus destroying usual power expansion.

This problem has been studied by many authors [6, 7]. By analyzing diagrams that break the loop-power connection one can notice that the "bad" terms come from only a part of the diagram - region of integration at special kinematic regime. One can show that "mixing" comes from one-loop graphs where internal loop momentum is of the order of temperature T ("hard") and all external legs carry the so called "soft" momenta ($\sim gT$). These diagrams have been labeled as Hard Thermal Loops (HTL from now on). In order to be consistent one has to reorganize perturbation series by computing these HTL-diagrams first and then including them as a tree-level action for further computations (we will use $\Gamma(A)$ to denote full HTL action). Then one has to use new propagators and vertices redefined with this new action to perform perturbative computations. In principle, HTL approximation must be used

in any theory considered at sufficiently high temperature. However, it is in the gauge theory and in the non-Abelian gauge theory in particular where it turns out to have great importance. While in scalar field theory only a handful of diagrams have to be included in the HTL action, this quantity is infinite in non-Abelian case. All these new terms are nonlocal and contain all powers of the gauge field. However, the full HTL action is gauge-invariant and the form of $\Gamma(A)$ does not depend on the gauge-fixing condition used to define propagators and vertices [6, 8]. In fact, it can be thought of as a non-Abelian generalization of the Debye mass in QED.

This HTL-effective action has several remarkable properties. First of all, since it represents the correlation between scattering of field quanta off the particles in the thermal bath it is entirely classical. As a matter of fact, it can be derived from the classical kinetic theory without any use of Feynman diagrams [9]. Besides being gauge-invariant it is also closely connected to the eikonal function of the Chern-Simons theory [10] - a feature that will be used in this thesis. This connection provides for several new possibilities to study long-wave length excitations in the QGP state. Using Chern-Simons eikonal form of the Hard Thermal loops one can re-write the whole generating functional as a local expression [12] (using some auxiliary fields, of course) without introducing new degrees of freedom. This may allow a Hamiltonian analysis of the problem, which is very difficult otherwise due to the nonlocality of the $\Gamma(A)$.

Nevertheless, some difficulties remain even *after* the HTL's have been taken into account. The most important one is the so called magnetic mass problem. It arises when one tries to compute the two-loop correction to some

of the physical quantities that have been computed using $\Gamma(A)$. As it turns out, there may exist an infrared divergence even if $\Gamma(A)$ is taken into account - in spite of the inclusion of effective propagators and vertices. The reason for that lies in the fact that only time-dependent components of the gauge field get "screened" by the quadratic terms in $\Gamma(A)$; the spatial part, even though it gets corrected by some momentum-dependent terms from $\Gamma(A)$ still does not have an infrared cut-off for the loop integral for the spacelike momenta; this causes the IR divergence [12].

This problem is very important because one needs to be able to do perturbative calculations at high temperature. There are several ways to approach this difficulty [38]. First, it is possible that there may be a dynamically generated "mass" term for the spatial part of the gluon as well [13]. This is what is called the "magnetic mass". Unlike the $\Gamma(A)$ however, the effective action responsible for this screening does not seem to result from summation of some (infinite) subset of Feynman diagrams for the theory. This is to be expected, though, because arguments for its existence are based on dimensional reduction and low-energy behavior of the 3-dimensional Euclidean Yang-Mills theory, which is essentially nonperturbative. The coupling constant of the effective, 3-d YM theory has dimension of mass $e_{eff}^2 \sim g^2 T$ (g being the dimensionless coupling of the original, 4-d action). Because of this, any possible mass must be just $m = g^2 T \times <some\ number>$. This number is determined purely from the combinatorial coefficients and group-theoretic factors in the loop expansion. Thus, there is no obvious small parameter controlling the mass generation.

Second, one can just try to reorganize the perturbative expansion even

further by adding and subtracting specifically chosen mass terms but treating them formally as different orders in loop expansion. This approach was developed by Nambu and Jona-Lasinio [33] for the meson theory. The consistency conditions for this method lead to the gap equations which determine value of the screening mass. This value should then be included in evaluation of all the physical quantities at the same loop order.

We want to emphasize an important difference between using gap equations to evaluate the value of magnetic screening in a self-consistent resumming of perturbation theory and concluding that some real and positive solution of the gap equation indicates that mass is being generated. The former situation just ensures us that there will be no infrared divergence in loop computations, while the second implies a much stronger statement: that loop expansion for this mass is convergent, with higher loops contributing smaller amounts for some reason.

Upto now, this problem is still unresolved. Several attempts to solve gap equations for the magnetic mass [41, 42, 44, 45, 46, 47, 48] (one of them described in this thesis, [41]) yield different results and raise some questions about the validity of the whole approach. Some hope may come from the lattice simulations [66], which find values that are close but so far do not produce consistent results in different gauges. Another hope is that there are some of the new results from the purely Hamiltonian analysis of the 2+1 YM theory in the continuum [64, 65], which finds the mass of the glueball states to be very close to some of the gap equation results.

In what follows we first discuss the HTL approximation and its connection with the Chern-Simons and WZNW theory (chapter 2). In chapter 3

we consider situation when quarks are kept at very high density and low temperature. In a standard QED plasma this leads to the Thomas-Fermi screening phenomenon. We consider the non-Abelian analog of this effect and show that an effective action similar to that of HTL (only Hard Dense Loops now) appears as a result of summation of the one-loop graphs [34]. We compute the value of the Thomas-Fermi screening mass in this case and show that its value is determined by chemical potential for quarks, rather than the temperature.

In chapter 4 we discuss the problem of the magnetic screening and gap equation scheme for the reorganization of perturbative expansion. We consider possible candidates for the "magnetic mass" term to use with this gap equation and choose one of them using HTL - WZNW connection as a guiding principle [32]. Chapter 5 contains detailed calculation of this mass up to a one-loop order [41]. In chapter 6 we conclude by comparing our results to some of the proposals put forward by other authors and discuss the difficulties and advantages of our approach [40].

Chapter 2

QCD at high temperature

In this chapter we will give a short review of the situation that occurs when a gauge theory is considered at high temperature. In what follows it will be assumed that T is much greater than the T_c and Λ_{QCD} scales. At temperatures this high, all the quarks are essentially deconfined and can be considered (in the first approximation) as free fermions. The gauge degrees of freedom are represented by $N^2 - 1$ gluons.

In order to illustrate some of the specific features of the field theory at high temperature we first start with a simple scalar theory with quartic self-interaction ($\lambda\phi^4$). We assume that theory is massless and has action

$$S = -\frac{1}{2}\int \phi D^2 \phi + \int \lambda\phi^4 \tag{2.1}$$

Even though imaginary time formalism is widely used for treating field theory at finite temperature at equilibrium, we will not use it here. When studying one-loop diagrams the real time form of the propagators will be very useful for us in identifying which diagrams are going to dominate at

high temperature. However, when computing higher order effects and discussing time-dependent processes this form is inadequate - one has to use time-contour approach of [22].

In the real time formalism, the expression for the propagator at finite temperature can be written as

$$G(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \left(\frac{i}{p^2 + i\epsilon} + \frac{2\pi}{e^{\frac{p^0}{T}} - 1} \delta(p^2) \right) \tag{2.2}$$

The one-loop contribution to the effective action Γ is given by the simple tadpole diagram (fig.2.1), and is equal to

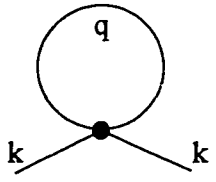


Figure 2.1: $\lambda\phi^4$ tadpole

$$\Gamma_1 = 6\lambda \int G(y,y) d^4y \; \phi^2(x) \tag{2.3}$$

This expression has two parts. The first one is quadratically UV-divergent due to the $1/p^2$ term in the (2.2). It is easy to see that it is temperature-independent. It will be assumed that this kind of divergence is canceled by a standard renormalization procedure and the remaining, temperature dependent part is

$$\Gamma_1 = 6\,\lambda \int \frac{d^3p}{(2\pi)^3}\, dp_0\, \delta(p_0^2 - \vec{p}^2)\, \frac{1}{e^{\frac{p^0}{T}} - 1} \phi^2(x) = \frac{\lambda T^2}{2} \phi^2(x) \tag{2.4}$$

Here we have used

$$\int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6} \tag{2.5}$$

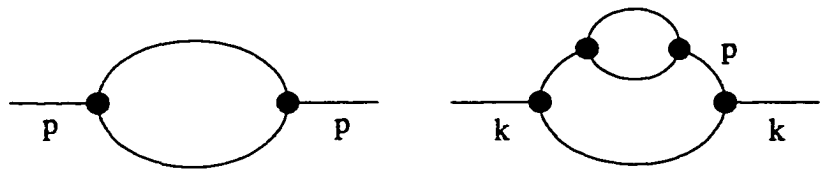


Figure 2.2: $\Pi(p)$ insertion

Expression (2.4) looks just like the mass term in the effective action, with the dynamically generated thermal mass $m \sim \sqrt{\lambda}T$. In order to do the consistent perturbation theory one has to resum all the Π insertions by including mass term (2.4) in the action and defining new propagators. For the $\lambda\phi^4$-field theory this is all there is to resummation - one has to change all the propagators from $1/P^2$ to $1/(P^2+m^2)$, m^2 being the thermal mass defined by (2.4). This not the case in gauge theory. Here the need for resummation can perhaps be better explained by the following argument due to Pisarski [6]. Look at the fig.2.2. It represents elementary vacuum polarization diagram for gluons and its first correction at two-loop level (we will drop all space and color indices for a moment). Due to the Debye screening the small-p expansion of the $\Pi(p)$ is

$$\Pi(p) = g^2 T^2 + g^2 T\,|p| + \cdots \tag{2.6}$$

therefore, at small p regime of integration ($p \sim gT$) ratio of the two expressions is

$$\frac{\Pi(p)}{p^2} \sim \frac{g^2 T^2 + \cdots}{p^2} \sim 1 \tag{2.7}$$

and formally higher-order diagram gives contribution of the same order in coupling constant as the lower one. Since the origin of this problem is in the T^2 contribution to $\Pi(p)$, one therefore needs to identify all of the Feynman graphs that can generate terms proportional to T^2 in the effective action.

Using real-time perturbation theory it is possible to develop a set of special power-counting rules that will allow us to determine which diagrams will be dominant in the high temperature limit.

Quark and gluon propagators at finite temperature can be defined as

$$S(x,y) = \langle \mathcal{T}\, q(x)\bar{q}(y)\rangle_T \tag{2.8}$$

$$G^{ab}_{\mu\nu}(x,y) = \langle \mathcal{T}\, A^a_\mu(x)\bar{A}^b_\nu(y)\rangle_T \tag{2.9}$$

Here $\mathcal{T}$ is the time ordering operator. Index T for the average means that one has to use thermal averages for the creation and annihilation operators. Using Minkowski signature, the quark part can be evaluated as

$$S(x,y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\{\,\Theta(x^0-y^0)[\alpha_p e^{-ip\cdot(x-y)}\gamma\cdot p + \bar{\beta}_p e^{ip'\cdot(x-y)}\gamma\cdot p'] - \Theta(y^0-x^0)[\beta_p e^{-ip\cdot(x-y)}\gamma\cdot p + \bar{\alpha}_p e^{ip'\cdot(x-y)}\gamma\cdot p']\} \tag{2.10}$$

where $p^0 = |\vec{p}|$, $p = (p^0, \vec{p})$, $p' = (p^0, -\vec{p})$, and $\Theta(x)$ is the usual step function. Also

$$\alpha_p = 1 - n_p, \qquad \beta_p = n_p. \tag{2.11}$$

Expressions n_p and $\bar{n}_p$ are quark and anti-quark distribution functions;

$$\langle b^{\dagger\alpha,r}_p b^{\beta,s}_p\rangle = n_p\delta^{rs}\delta^{\alpha\beta} \tag{2.12}$$

$$\langle d^{\dagger\alpha,r}_p d^{\beta,s}_p\rangle = \bar{n}_p\delta^{rs}\delta^{\alpha\beta} \tag{2.13}$$

Here $(b^{\dagger\alpha,r}_p, b^{\alpha,r}_p), (d^{\dagger\alpha,r}_p, d^{\alpha,r}_p)$ are the creation and annihilation operators for quarks and anti-quarks; Greek letters used for spin indices and r, s are color ones. In general, for the plasma of zero chemical potential values of n_p and $\bar{n}_p$ are equal and given by Fermi-Dirac distribution as

$$n_p = \bar{n}_p = \frac{1}{e^{p^0/T}+1} \tag{2.14}$$

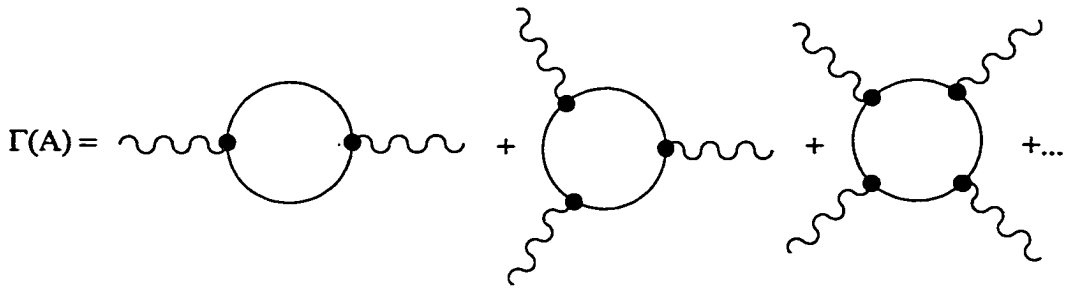


Figure 2.3: Quark Loop contribution to $\Gamma(A)$

It is useful to rewrite quark propagator in momentum space as

$$S(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} \left[i\frac{\gamma\cdot p}{p^2+i\epsilon} - \gamma\cdot p\; 2\pi\; n_p\delta(p^2)\right] \tag{2.15}$$

Similarly, one can evaluate gluon propagator $G(x,y)$ in terms of its creation and annihilation operators $(a_p^{\dagger\alpha,r}, a_p^{\alpha,r})$

$$\begin{aligned} G^{ab}_{\mu\nu}(x,y) &= \delta^{ab}\, g_{\mu\nu} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\Big[\Theta(x^0-y^0)\,\{\alpha_p e^{-ip(x-y)} + \beta_p e^{ip(x-y)}\} \\ &\qquad + \Theta(y^0-x^0)\,\{\beta_p e^{-ip(x-y)} + \alpha_p e^{ip(x-y)}\}\Big] \\ &= \delta^{ab}\, g_{\mu\nu} \int \frac{d^4p}{(2\pi)^4}\left[\frac{i}{p^2+i\epsilon} + 2\pi n_p\delta(p^2)\right] \end{aligned} \tag{2.16}$$

Here $g_{\mu\nu}$ is the metric tensor, a, b are color indices and and $\alpha_p = 1 + n_p$, $\beta_p = n_p$, n_p being the Bose-Einstein distribution function,

$$n_p = \frac{1}{e^{p^0/T}-1} \tag{2.17}$$

The power-counting rules which emerge from the consideration of the integrals in the loop diagrams are:

1. T^3 for each d^3p (loop integration)

2. $1/T$ for each propagator (from $1/(2p^0)$)

3. One power of T for each momentum p_μ in the vertex

4. For more than one propagator in the diagram, factor $\frac{1}{k} \sim \frac{1}{gT}$

5. For each loop that has more than one propagator of the same type a factor of $k/T(\sim g)$.

While items 1-3 are coming from the fact that most of the thermal quanta have momentum of the order of T, the last two perhaps deserve more explanation. Rule number 4 comes from the fact that for a diagram with external momentum k there will be some denominators that involve differences of the "hard" momenta of the propagators $q^0 - p^0 \sim k$. This will occur only if you have more than one such momentum, so one needs more than one propagator in the diagram. Item 5 comes from the fact that there will be difference of the distribution functions multiplying each diagram, $n_q - n_p \sim k/T$. Using these rules one can establish that the dominant diagrams scale as T^2 and consist of all the one loop graphs depicted on fig.2.3, fig.2.4, plus the tadpole graph for gluons, fig.2.5. One can also show that in HTL approximation all diagrams that contain ghosts are subdominant, which is essentially the result of the fact that it is essentially a classical effect, due to the interactions with on-shell particles from the thermal bath. There are several ways to find the explicit expression for the HTL action. One can compute a few first diagrams from fig. 2.3 and 2.4 and find a set of rules that can give generic term in $\Gamma(A)$. In fact, this is how it was first obtained [6, 7]. We will present some elements of this computation in the next chapter where we study the analog of the Thomas-Fermi screening for the QGP. Here we want to describe another,

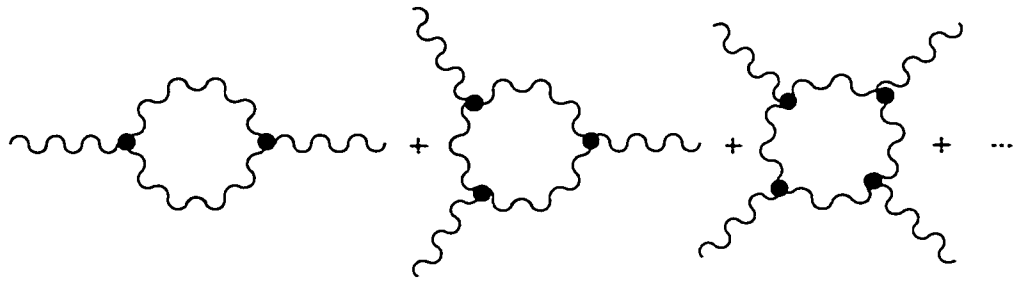


Figure 2.4: Gluon Loop contribution to $\Gamma(A)$

symmetry-based argument for derivation of the HTL action [15] which also illustrates its connection with the Chern-Simons (CS) and WZNW theories.

Let us start by asking the following question: how does $\Gamma(A)$ depend on gauge potentials? It is possible to identify all the tensor structures that are present in $\Gamma(A)$ by analyzing the vertex structure of different diagrams without actually computing them. In order to contribute a T^2-term to $\Gamma(A)$, diagrams from fig.(2.3) and fig.(2.4) that have three or more legs *must* have loop momenta q at each vertex. The only possible tensor structures are $A \cdot q$ and $A^2 q^2$. But $q^2 = 0$ since q is on-shell 4-momentum, so that the only permissible structure is $A \cdot q$. One can rewrite q as $q_\mu = q^0 Q_\mu$, where $Q_\mu = (1, \vec{Q}), \vec{Q}^2 = 1$ defines a 4-dimensional null-vector. Then, any expression of the type $A \cdot q$ becomes $q^0 A \cdot Q$. Therefore, the total contribution to $\Gamma(A)$ from graphs of type (2.3 b,c) and (2.4 b,c) must be some functional of $A \cdot Q$ integrated over all possible directions of the vector $\vec{Q}$, since q_μ is the loop momentum and is integrated,

$$\sim \int W(A \cdot Q) \frac{d\Omega}{4\pi} \tag{2.18}$$

Are there any other tensor structures that can be generated? The answer is yes, since there are three more diagrams that contribute to the to the HTL effective action, namely two vacuum polarization diagrams fig. 2.3 a) and fig. 2.4 a) and the gluon tadpole, fig. 2.5: The tadpole is the only diagram

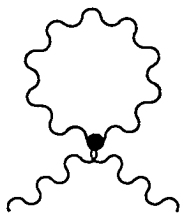


Figure 2.5: Gluon 'tadpole' contribution to $\Gamma(A)$

with the quartic vertex that can contribute in the HTL approximation; its evaluation is very similar to the scalar tadpole on fig.(2.1). We will not need the explicit computation here, the only fact that we need is that this term produces $A_\mu A^\mu$-type expression in the end. The same is true for the other two diagrams - they can produce A^2 type expressions. Using simple angular integration we can rewrite it as

$$A_\mu A^\mu = A_0^2 - \vec{A}^2 = 4A_0^2 - 3\int (A\cdot Q)\frac{d\Omega}{4\pi} \tag{2.19}$$

Notice that the second term on the right-hand side is exactly of the type (2.18); this leaves us with the only other permitted tensor structure - A_0^2.

Since there are no more diagrams that contribute to the HTL, we can say that $\Gamma(A)$ is given by some linear combination of A_0^2 and functional of type (2.18). We will assume the following ansatz for the $\Gamma(A)$ [15]

$$\Gamma = (N + \tfrac{1}{2}N_F)\,\frac{T^2}{12\pi}\,\left[\int d^4x\; 2\pi A_0^a A_0^a + \int d\Omega\; W(A\cdot Q)\right]. \tag{2.20}$$

Here N is number of colors, N_f is equal to number of quark flavors (only light quarks should be taken into account, since for them $T >> m$) and overall coefficient can be fixed by explicitly computing diagrams of figs. 2.5,2.3 a) and 2.4 a). One does not have to evaluate these graphs completely - only coefficient of A_0^2 is needed. It remarkable that the full form of W can be determined from the gauge invariance alone [15]. Requiring that $\Gamma(A)$ do not change under the $SU(N)$ gauge transformation $\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega]$ where ω is a transformation parameter leads to the condition

$$\int \delta W d\Omega = 4\pi \int d^4x \ \dot{A}^a \cdot Q \omega^a \tag{2.21}$$

Taking into account that W must be a functional of $A \cdot Q$ the solution must be such that

$$\delta W = \int d^4x \ \dot{A}^a \cdot Q \omega^a \tag{2.22}$$

Equation (2.22) can be rewritten as

$$\delta W = -\int d^4x (Q \cdot \partial \frac{\delta W}{\delta (A \cdot Q)} + [A \cdot Q, \frac{\delta W}{\delta (A \cdot Q)}])^a \omega^a \tag{2.23}$$

It is convenient to change coordinates from A_0, $A \cdot Q$ to a new pair $u = \frac{1}{2} A \cdot Q$, $v = \frac{1}{2} A \cdot Q'$ where $Q'_\mu = (1, -\vec{Q})$ is another null vector. In a similar way the space coordinates x_μ are replaced by $x_+ = \frac{1}{2} x \cdot Q$, $x_- = \frac{1}{2} x \cdot Q'$ and $\vec{x}^T = \vec{x} - (\vec{x} \cdot \vec{Q}) \vec{Q}$. The volume element d^4x changes to $dx_+ dx_- d^2x^T$. These are the "light-cone" variables. If we define a new function f as

$$f = \frac{\delta W}{\delta (A \cdot Q)} + \frac{1}{2} A \cdot Q \tag{2.24}$$

then the condition (2.23) takes form

$$\frac{\partial f}{\partial u} + [A \cdot Q, f] + \frac{1}{2} \frac{\partial (A \cdot Q)}{\partial v} = 0 \tag{2.25}$$

If we perform the standard Wick rotation with definition of $\frac{1}{2}A\cdot Q = A_+ \to A_z$ and define $a_{\bar{z}} = -f$, then the equation (2.25) (which is originally just gauge-invariance condition) becomes

$$\partial_{\bar{z}} A_z - \partial_z a_{\bar{z}} + [a_{\bar{z}}, A_z] = 0 \tag{2.26}$$

This equation illustrates the connection between Chern-Simons (CS) theory and HTL approximation that we wanted to expose. If one thinks of A_z and $a_{\bar{z}}$ as of two components of the gauge potential of some theory in two dimensions then (2.26) is just a zero field strength condition $F_{z\bar{z}} = 0$. The gauge theory which has this equation as its equation of motion is called the Chern-Simons theory [16, 17]. Using $\partial_z(\frac{1}{\bar{z}-\bar{z}'}) = \pi\delta^{(2)}(z - z')$ and the fact that equation (2.26) does not contain any derivatives with respect to x^T one can solve for $a_{\bar{z}}$ as a power series in A_z as

$$a_{\bar{z}} = \sum(-1)^{n-1} \int \frac{d^2z_1}{\pi} \cdots \frac{d^2z_n}{\pi} \frac{A_z(z_1, \bar{z}_1, x^T)A_z(z_2, \bar{z}_2, x^T) \ldots A_z(z_n, \bar{z}_n, x^T)}{(\bar{z} - \bar{z}_1)(\bar{z}_1 - \bar{z}_2) \ldots (\bar{z}_n - \bar{z})} \tag{2.27}$$

Performing functional integration with respect to A_z we find that

$$W = -4\pi i I(A_z) - \int d^4x \; A_z^a A_z^a \tag{2.28}$$

where I is the eikonal for the CS theory [17, 18]. (with additional integration over the transverse coordinates)

$$I = i\sum \frac{(-1)^n}{n} \int d^2x^T \; \frac{d^2z_1}{\pi} \cdots \frac{d^2z_n}{\pi} \; \frac{\mathrm{tr}(A_z(x_1) \ldots A_z(x_n))}{\bar{z}_{12}\bar{z}_{23} \ldots \bar{z}_{n1}} \tag{2.29}$$

Using properties of the Ω integration and defining $\tilde{I}(A_{\bar{z}}) = (I(A_z))^*$ one can write the final answer as

$$\begin{aligned} \Gamma &= -(N + \tfrac{1}{2}N_F)\frac{T^2}{6\pi} \int d\Omega \left[\int d^4x \; \mathrm{tr}(A_z A_{\bar{z}}) + i\pi I(A_z) + i\pi \tilde{I}(A_{\bar{z}}) \right] \\ &= (N + \tfrac{1}{2}N_F)\frac{T^2}{6} \int d\Omega \; K(A_z, A_{\bar{z}}) \end{aligned} \tag{2.30}$$

Chapter 3

Screening in the dense, cold plasma

The screening of electromagnetic interactions in a plasma of charged particles, the so-called Debye screening, is quite well-known. An analogous effect, the Thomas-Fermi screening, occurs in the case of a degenerate charged fermion system such as the electron gas in metal [35]. Both effects can be easily understood, in a field-theoretic language, by calculating the one loop photon propagator in which the charged particle propagators are at finite temperature and density. (The two effects are in fact quite similar; for the degenerate fermion gas, the excitations generated by the propagating photon are particle-hole pairs which behave like plasma background). The non-Abelian analogs of these screening effects are of considerable interest especially in view of the possibility of obtaining hot and dense quark matter systems in heavy ion collisions [3]. A non-Abelian Thomas-Fermi effect can also be of interest in calculations of the equation of state for quark matter

inside neutron stars.

In this chapter we analyze the non-Abelian Thomas-Fermi screening for degenerate quark matter of finite baryon number. We will use the diagram analysis approach to illustrate both the calculation and its connection to the zero density - high temperature case of Hard Thermal Loops. However, as we have mentioned in the introduction the effective action that emerges in this calculation is essentially classical by nature, so it can be done entirely in the kinetic theory framework with similar results [37].

Since the quark contribution to the two-point function for gluons is similar to that for photons, there should be Thomas-Fermi screening for quark matter. For reasons of non-Abelian gauge invariance, as with Debye screening, there will be higher point contributions, the whole series again summing up to an effective mass term. This term will have the same structure as the effective action for hard thermal loops; the numerical value of the screening mass, however, will be determined by the chemical potential rather than temperature.

Consideration of the one loop two-point function shows that the screening mass $\sim g\mu$ where g is the coupling constant and μ is the chemical potential. It is then clear that a higher loop diagram in which such a term is inserted can give contributions of the same order for the integration range of loop momenta $\lesssim g\mu$. This is exactly as in the hard thermal loop case. One must therefore sum up diagrams with loop momenta $\sim \mu$ and external momenta $\lesssim g\mu$. This effective action must then be used for a self-consistent evaluation of the screening mass. We obtain this effective action in what follows.

Let us start by considering the two-point function for gluons. The con-

served baryon charge is given in terms of the quark field $q(x)$ by $\int q^\dagger q$. With a chemical potential term $\mu \int q^\dagger q$ added to the action, the quark propagator defined as $S(x,y) = \langle Tq(x)\bar{q}(y)\rangle$ given by

$$S(x,y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}[\theta(x^0-y^0)\{\alpha_p\gamma\cdot p e^{-ip(x-y)} + \bar{\beta}_p\gamma\cdot p' e^{ip'(x-y)}\} + \\ - \quad \theta(y^0-x^0)\{\beta_p\gamma\cdot p e^{-ip(x-y)} + \bar{\alpha}_p\gamma\cdot p' e^{ip'(x-y)}\}] \quad (3.1)$$

where $p^0 = |\vec{p}|$, $p = (p^0, \vec{p})$, $p' = (p^0, -\vec{p})$ and $\theta(x)$ is the step function. Also

$$\alpha_p = 1 - n_p, \qquad \beta_p = n_p \quad (3.2)$$

The distribution functions $n_p, \bar{n}_p$ corresponding to quarks and anti-quarks respectively are given by

$$n_p = \frac{1}{e^{(p^0-\mu)/T}+1}, \qquad \bar{n}_p = \frac{1}{e^{(p^0+\mu)/T}+1} \quad (3.3)$$

The one-loop quark graphs are given by the effective action

$$\Gamma = -i\mathrm{Tr}\ \log(1 + S\gamma\cdot A) \quad (3.4)$$

$A_\mu = -it^a g A^a_\mu$ is the gluon vector potential, t^a are hermitian matrices corresponding to the generators of the Lie algebra in the quark representation. In the above expression for Γ a functional trace is implied as well as the trace over the spin and color labels. The two-gluon term in Γ is given by

$$\Gamma^{(2)} = \frac{i}{2}\int d^4x\ d^4y\ \mathrm{Tr}[\gamma\cdot A(x)S(x,y)\gamma\cdot A(y)S(y,x)] \quad (3.5)$$

Using equation (1) and carrying out the time-integrations we get

$$\Gamma^{(2)} = -\frac{1}{2}\int d\mu(k)\frac{d^3q}{(2\pi)^3}\frac{1}{2p^0}\frac{1}{2q^0}\times$$
$$\Big[T(p,q)(\frac{\alpha_p\beta_q}{p^0-q^0-k^0-i\epsilon}-\frac{\alpha_q\beta_p}{p^0-q^0-k^0+i\epsilon})+$$
$$T(p,q')(\frac{\alpha_p\bar{\alpha}_q}{p^0+q^0-k^0-i\epsilon}-\frac{\beta_p\bar{\beta}_p}{p^0+q^0-k^0+i\epsilon})+$$
$$T(p',q)(\frac{\bar{\alpha}_p\alpha_q}{p^0+q^0+k^0-i\epsilon}-\frac{\bar{\beta}_p\beta_q}{p^0+q^0+k^0+i\epsilon})+$$
$$T(p',q')(\frac{\bar{\alpha}_p\bar{\beta}_q}{p^0-q^0+k^0-i\epsilon}-\frac{\bar{\beta}_p\bar{\alpha}_p}{p^0-q^0+k^0+i\epsilon})\Big] \qquad (3.6)$$

where

$$A_\mu(x) = \int\frac{d^4k}{(2\pi)^4}e^{ikx}A_\mu(k) \qquad (3.7)$$

$$d\mu(k) = (2\pi)^4\delta^{(4)}(k+k')\frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} \qquad (3.8)$$

$$T(p,q) = Tr[\gamma\cdot A(k)\ \gamma\cdot p\ \gamma\cdot A(k')\ \gamma\cdot q] \qquad (3.9)$$

and $\vec{p}=\vec{q}+\vec{k}$ in equation(6).

The $i\epsilon's$ can be taken to go to zero at this stage. They were introduced for convergence of time-integrations and contribute to the imaginary part. Here we are interested in screening effects which are described by the real part of the two-point function. (Also for many physical situations, the relevant imaginary part is that of the retarded function which is not directly given by the above time ordered function [11].) Further we are interested in a degenerate gas of quarks; it is therefore appropriate to consider $T \ll \mu$. As $T/\mu \to 0$, the anti-quark occupation numbers $\bar{n}_p \to 0$ (for positive μ). Equation (3.6) then simplifies as

$$\Gamma^{(2)} = -\frac{1}{2}\int d\mu(k)\frac{d^3q}{(2\pi)^3}\frac{1}{2p^0}\frac{1}{2q^0}[T(p,q)\frac{(n_q-n_p)}{p^0-q^0-k^0}+T(p,q')\frac{n_p}{p^0+q^0-k^0}$$

$$- \quad T(p',q)\frac{n_q}{p^0+q^0+k^0}] \tag{3.10}$$

As explained in the introduction, the relevant kinematic regime is $|\vec{p}|, |\vec{q}| \gg |\vec{k}|$, so that $p^0 - q^0 - k^0 \approx -k\cdot Q$, $p^0 + q^0 \pm k^0 \approx 2q^0, Q = (1, \vec{q}/q^0)$. Further,

$$T(p,q) \approx 8{q^0}^2 \mathrm{tr}(A_1 \cdot Q A_2 \cdot Q) \tag{3.11}$$

$$T(p',q) \approx 4{q^0}^2 \mathrm{tr}(A_1 \cdot Q' A_2 \cdot Q + A_1 \cdot Q A_2 \cdot Q' - 2A_1 \cdot A_2) \tag{3.12}$$

$$T(p',q) \approx T(p,q') \tag{3.13}$$

where $A_1 = A(k)$, $A_2 = A(k')$, $Q' = (1, -\vec{q}/q^0)$. Expression (3.10) now simplifies as

$$\Gamma^{(2)} = -\frac{1}{2}\int d\mu(k)\frac{d^3q}{(2\pi)^3}\mathrm{tr}\Big[\frac{dn}{dq^0}\,\frac{A_1\cdot QA_2\cdot Q}{k\cdot Q}2\vec{Q}\cdot\vec{k} - \frac{n}{q^0}(A_1\cdot Q'A_2\cdot Q + A_1\cdot QA_2\cdot Q' - 2A_1\cdot A_2)\Big] \tag{3.14}$$

Using $\int d^3q\,\frac{dn}{dq^0}f(Q) = -\int d^3q\,\frac{2n}{q^0}f(Q)$, and some properties of $\vec{Q}$-integration, $\Gamma^{(2)}$ simplifies to

$$\Gamma^{(2)} = \tag{3.15}$$
$$-\frac{1}{2}\int d\mu(k)\int\frac{d^3q}{(2\pi)^3}\,\frac{n}{2q^0}\,8\,\mathrm{tr}\Big[2A_{1+}A_{2-} - \frac{k\cdot Q'}{k\cdot Q}A_{1+}A_{2+} - \frac{k\cdot Q}{k\cdot Q'}A_{1-}A_{2-}\Big]$$
$$= -\frac{1}{2}\int d\mu(k)\Big(\frac{\mu^2}{4\pi^3}\Big)\int d\Omega\,\mathrm{tr}\Big[2A_{1+}A_{2-} - \frac{k\cdot Q'}{k\cdot Q}A_{1+}A_{2+} - \frac{k\cdot Q}{k\cdot Q'}A_{1-}A_{2-}\Big]$$

where $A_+ = \frac{A\cdot Q}{2}, A_- = \frac{A\cdot Q'}{2}$ and we have taken the limit of $T \to 0$ (small compared to μ). As with hard thermal loops, we can transform this back to coordinate space and write

$$\Gamma^{(2)} = -\frac{\mu^2}{4\pi^3}\int d\Omega\; K^{(2)}(A_+, A_-) \tag{3.16}$$

where

$$K(A_+, A_-) = [\int d^4x \ \mathrm{tr}(A_+A_-) + i\pi I(A_+) + i\pi \tilde{I}(A_-)]$$
$$I(A_+) = i\sum_2^\infty \frac{(-1)^n}{n} \int d^2x^T \frac{d^2z_1}{\pi} \cdots \frac{d^2z_n}{\pi} \frac{tr[A_+(x_1)\cdots A_+(x_n)]}{(\bar{z}_1 - \bar{z}_2)\cdots(\bar{z}_n - \bar{z}_1)} \quad (3.17)$$

$\int d\Omega$ defines integration over the orientations of $\vec{Q}$. $K^{(2)}$ in equation (3.16) denotes the terms in K which are quadratic in $A's$; z and $\bar{z}$ are the Wick-rotated versions of $x\cdot Q'$ and $x\cdot Q$ respectively and x^T is transverse to $\vec{Q}$, i.e $\vec{Q}\cdot\vec{x}^T = 0$. $\tilde{I}(A_-)$ is obtained from $I(A_+)$ by $z \leftrightarrow \bar{z}$. $I(A_+)$, as explained elsewhere, is essentially the eikonal function for Chern-Simons theory. $K(A_+, A_-)$ can be related to the WZW-action for a hermitian matrix $M^\dagger M$ defined in terms of $A_\pm$.

Consider now the three-point function. One has twenty-four terms with denominators involving different combinations of loop momenta and external momenta. The simplification of this expression proceeds in much the same way as for $\Gamma^{(2)}$ - one can neglect terms with denominators that are of the order of μ and leave the terms that involve differences between loop momenta, which are of the order of $k \ll \mu$. After that, the relevant contributions will be

$$\Gamma^{(3)} = \int \frac{d^3q}{(2\pi)^3} 2\Big\{\frac{n_p}{k_2^0 + p^0 - q^0}\frac{1}{r^0 - q^0 - k_3^0} + \frac{n_r}{q^0 - r^0 + k_3^0}\frac{1}{p^0 - r^0 - k_1^0} + \frac{n_p}{p^0 - r^0 - k_1^0}\frac{1}{p^0 - q^0 + k_2^0}\Big\} \ \mathrm{Tr}(A_1\cdot QA_2\cdot QA_3\cdot Q) \quad (3.18)$$

where p, q, r are the loop momenta, k_1, k_2 and k_3 are the momenta of external gluons and $\vec{p} = \vec{q} - \vec{k}_2, \vec{r} = \vec{q} + \vec{k}_3$. After performing the dq^0-integral the final

expression is

$$\Gamma^{(3)} = \frac{\mu^2}{(2\pi)^3}\int d\Omega\Big(\frac{k_2\cdot Q' - k_2\cdot Q}{k_1\cdot Qk_2\cdot Q} + \frac{k_3\cdot Q - k_3\cdot Q'}{k_3\cdot Qk_1\cdot Q}\Big)\ \mathrm{Tr}(A_1\cdot QA_2\cdot QA_3\cdot Q) \tag{3.19}$$

Using definition (3.17) one can show that

$$\Gamma^{(3)} = -\frac{\mu^2}{4\pi^3}\int d\Omega\ K^{(3)}(A_+, A_-) \tag{3.20}$$

The fact that the same coefficient appears in both (12) and (14) is crucial; this guarantees gauge invariance of the full effective action Γ. The non-Abelian gauge-invariant completion of $K^{(2)} + K^{(3)}$ is given by the full K of equation (3.17). The final answer is thus

$$\Gamma = -\frac{\mu^2}{4\pi^3}\int d\Omega\ K(A_+, A_-) \tag{3.21}$$

We now turn to the question of the coefficient of K in equation (3.21). In calculating the screening effects using Γ, we should not change the Lagrangian for the theory, which may be, say, Chromodynamics; we should only rearrange terms in the perturbative expansion and sum certain classes of diagrams. There are then two different but closely related ways of proceeding. We write the action as

$$S = S_0 + \Delta\int d\Omega\ K \tag{3.22}$$

$$S_0 = S_{QCD} - m^2\int d\Omega\ K \tag{3.23}$$

S_0 is used to define propagators and vertices and Δ is treated as being nominally one loop order higher than S_0. In the first case, we use the value as calculated above (or the analogous value for hard thermal loops) for m^2. After calculating higher order corrections, Δ is set to m^2 (so that $S \to S_{QCD}$

). The corrections are in general non-vanishing and this is useful if the corrections are small compared to the lowest order value. The alternative is to keep m^2 as an arbitrary parameter and choose Δ (as a function of m, say $\Delta(m)$) so as to cancel out all the corrections. Upon setting Δ to m^2, we get a gap equation, $\Delta(m) = m^2$, which can be solved for m. In the case of the hard thermal loops, the first approach is satisfactory. The relevant distribution for the momentum-integration is

$$f(q)dq = \frac{q^2 dq}{e^{q/T} + 1} \tag{3.24}$$

The lowest order calculation only evaluates the contribution from the region of $q \gtrsim T$. The probability contained in the region $q \geq \frac{1}{2}T$ for the distribution is approximately 0.99. We can therefore expect that the neglect of the low q-regime is not very significant for the numerical value of the screening mass (within a calculational scheme with re-summations as explained above). For the case of Thomas-Fermi screening, the relevant distribution is

$$f(q)dq \approx \theta(\mu - q)q^2 dq \tag{3.25}$$

The probability contained in the region $q \geq \frac{1}{2}\mu$ is now 0.875. We thus expect that the lowest order value of the coefficient of $-\int d\Omega\ K$ in equation (3.19), viz., $\mu^2/4\pi^3$, is somewhat less accurate than the analogous quantity for hard thermal loops. In this case a self-consistent calculational scheme for including higher order effects might be more appropriate.

Chapter 4

Magnetic mass terms in QCD

In this chapter we discuss the problem of the magnetic screening. We have already mentioned in the introduction that one way to determine the magnetic mass is to use the gap equation method [33], which can be described as follows.

Let us consider a gauge-invariant (arbitrary at first) functional $S_m(A)$ such that when expanded in powers of the gauge field the first term is proportional to $\int A^2\, d^3x$.(In what follows we will consider 3-dimensional Euclidean YM theory, because of the dimensional reduction argument). This will be our trial "mass term". We write,

$$S = S_0 + m^2 S_m(A) - \Delta S_m(A) \tag{4.1}$$

Here S_0 is the original YM action; m and Δ will be determined in the loop expansion by the following rule. Taking $S_0 + m^2 S_m$ as zeroth order term in the loop expansion, let us compute, loop by loop the corrections to the term of the S_m type. If $\gamma^{(n)}$ represents n-th loop correction the total effective

action will be

$$\Gamma = S_0 + m^2 S_m + (\gamma^{(1)} + \gamma^{(2)} + \cdots - \Delta) S_m \tag{4.2}$$

By definition, we want to say that S_m is the dynamically generated mass. Therefore, there should be no loop corrections to it, which implies

$$\gamma^{(1)} + \gamma^{(2)} + \cdots = \Delta \tag{4.3}$$

On the other hand, we would like to keep the original action intact, which means that

$$\Delta = m^2 \tag{4.4}$$

This condition will hold only for certain values of m, since Δ itself is m-dependent due to the condition (4.3). Expression (4.4) is the gap equation we are looking for.

The next question to ask then is: what kind of mass terms should one use? In general one can construct an infinite number of the mass terms for the 3-dimensional Yang-Mills theory. It can be done in the following fashion. The guiding principles are gauge invariance and asymptotic behavior in the ultraviolet limit - as we mentioned earlier any possible mass term should not change the UV limit of the theory.

To the lowest (quadratic) order in gauge field gauge invariance reduces to just transversality of the term,

$$S_2 = M \int A_i^a(x) \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) A_j^a(x) \, d^3x \tag{4.5}$$

Here, as before, k is a 3-dimensional Euclidean vector and A(x) has dimension 1 (as do momenta). The constant in front of the expression has the dimension

of mass and can be "dynamically" generated as a pure number multiplied by the e^2.

As written, this expression is not gauge-invariant with respect to the non-Abelian gauge transformations since they involve the commutator term.. It is necessary therefore to correct this by adding another term which is cubic in A and compensates for the terms that one gets by gauge transforming (4.5). This third-order term will in turn require a 4-th order for gauge invariance. It is easy to see that one can never stop and any invariant mass term will have all orders in the gauge field.

It is easy, in fact, to construct all of these mass terms explicitly. Let us consider expression

$$S_{m_2} = M \ \mathrm{tr} \int F_{ij}(x) \left(\frac{1}{D^2}\right)_{xy} F_{ij}(y) \ d^3x \ d^3y \tag{4.6}$$

It is obviously gauge invariant (since both D and F transform as an adjoint). Dimension of the $1/D^2$ can be determined from the D^2 $(1/D^2)_{xy} = \delta(x-y)$ and equals 1. M therefore has the dimension of mass, as before. Expanding this term in powers of the gauge field we immediately get (4.5) as a first term, and all the corrections hereafter. This expression corresponds to a particular choice of the way to correct the quadratic term to render it gauge-invariant. As we mentioned before, this procedure is not unique. To see this consider another expression,

$$S_{m_3} = M \ \mathrm{tr} \int F_{ij}(x)\left(\frac{1}{D^2}\right)_{xy} F_{jk}(y)\left(\frac{1}{D^2}\right)_{yz} F_{ki}(z) \ d^3x \ d^3y \ d^3z \tag{4.7}$$

This term is gauge invariant as well. Using dimensional analysis developed before one can see that constant M still has dimension of mass. When

(4.7) is expanded in powers of the gauge fields the first term will be cubic in A's and as a consequence of the gauge invariance totally transversal.

Obviously, any linear combination of (4.6) and (4.7) will represent a possible mass term. Now we can generalize this argument to higher orders in the gauge field. In order to do this, notice that when compared to formula (4.6) expression (4.7) has one more integration (brings -3 to the total dimension), one more F (brings +2) and one extra $1/D^2$ (contributes another +1); total of the additional "dimensions" is zero. This is the trick - it tells us that, in fact we can add as many groups of $(1/D^2)Fd^3x_n$ as we want, which makes it possible to write a whole class of mass terms:

$$S_m = M \ \mathrm{tr} \sum_{n=2}^{\infty} C_n \int F_{ij}(x_1) \left(\frac{1}{D^2}\right) F_{jk}(x_2) \cdots \left(\frac{1}{D^2}\right) F_{...}(x_n) \ d^3x_1 \ldots d^3x_n \tag{4.8}$$

Here C_n are pure numbers; different choices of C's correspond to the different mass terms.

Eventhough it is manifestly gauge-invariant, expression (4.8) may not satisfy some of the other conditions that we mentioned as necessary for the "good" magnetic mass term. As one can see this expression is extremely nonlocal, thus increasing the possibility that new degrees of freedom might have been implicitly introduced in the theory. If this is the case they have to decouple at UV regime and must be massive in the IR regime, with mass greater than that of the gluon. Otherwise, it would be possible for the gluon to decay into these new quanta, thus creating complex solutions for the (4.4). We will comment on this issue later, following recent results of several other authors.

Because of these last problems it is still difficult to select any special mass

term for the non-perturbative computation that we want to do. However, one can look at the problem from another point of view. As we have shown in chapter II one can obtain an "electric" screening term by summing an infinite set of Feynman graphs. Yet no similar summation seems to work for the static or "magnetic" part of the gluon. It was suggested in [32] that one can use a Chern-Simons form of the HTL-effective action derived in [10] for a hint. The generalization of the Chern-Simons eikonal to a moving plasma suggests that there is another closely related gauge-invariant mass term which gives screening of static magnetic interactions, in other words, a magnetic mass term [6]. Although nonlocal, the equations of motion for this term can be written in a local way by using auxiliary fields and without introducing additional degrees of freedom. Because of this last property, the ultraviolet behavior of the theory is unchanged. On this basis, it was suggested that this mass term could be used as a gauge-invariant infrared cutoff for loop calculations in QCD at high temperatures. The explicit construction of this term is related to the general parameterization of the null-vectors by use of spinors. It is necessary in order to restore Lorentz invariance of the system at finitel temperature. We will not review it here, but just present the result. Following [32] we can notice that one of the features on the electric mass defined as (2.20) was the pair of (Minkowski) null-vectors $Q_\mu = (1, \vec{Q})$ and $Q'_\mu = (1, -\vec{Q}), |\vec{Q}| = 1$. If one relaxes condition that Q's must be real it is possible to write another pair of null vectors

$$Q_\mu = (0, n) \qquad Q'_\mu = (0, \bar{n}); \tag{4.9}$$

where

$$n_i = (-\cos\theta\cos\varphi - i\sin\varphi, -\cos\theta\sin\varphi + i\cos\varphi, \sin\theta) \tag{4.10}$$

$$\bar{n}_i(\vec{Q}) = n_i(-\vec{Q}) = n_i(\vec{Q})^* \tag{4.11}$$

$$n^2 = \bar{n}^2 = \textstyle\sum_{i=1}^{3} n_i n_i = 0 \tag{4.12}$$

Here angles φ and θ parameterize the unit vector $\vec{Q}$. Now if we just replace all Q_μ expressions in (2.20) by the (4.9), we obtain the magnetic mass term. It will contain only spatial part of the gluon, since, according to (4.9), $Q_0 = 0$.

Chapter 5

Nonperturbative evaluation of the magnetic mass

A consistent implementation of a gauge-invariant infrared cutoff will lead to a gap equation for the value of the magnetic mass. We obtain this equation to one-loop order; as usual, this means a self-consistent summation of one loop self-energy insertions. The magnetic mass to this order is obtained as $\approx 2.38Cg^2T/4\pi$ where g is coupling constant, T is the temperature and C is defined by $C\delta^{ab} = f^{amn}f^{bmn}$, f^{amn} being the structure constants of the gauge group.($C = N$ for an $SU(N)$-gauge theory.) An immediate question is whether the two- and higher loop contributions are smaller than the one-loop terms. Purely based on counting dimensions of momentum integrals and powers of g , we cannot conclude whether higher loop effects are smaller or not. It is really an issue of numerical factors and possible logarithms of g. Now, the magnetic mass, by the standard arguments of dimensional reduction at high temperatures is related to the mass gap of the three-dimensional

QCD. For the mass gap of three-dimensional gauge theories, the one-loop calculations can hardly be adequate. However, for the quark-gluon plasma, perturbative loop expansion, perhaps with re-summations, is expected to be reasonable at high temperatures, at least for a number of processes of interest and our approach is applicable. In any case, qualitatively, it is interesting that a nonzero value can be obtained to this order; computationally, it is useful to have a loop-wise implementation of gauge-invariant magnetic screening, irrespective of the specific numerical value.

In the rest frame of the plasma, the magnetic mass term considered in [32] has the following form

$$\tilde{\Gamma} = -M^2 S_m \tag{5.1}$$

$$S_m = \int d\Omega \; K(A_n, A_{\bar{n}}) \tag{5.2}$$

where $A_n = \frac{1}{2} A_i n_i, A_{\bar{n}} = \frac{1}{2} A_i \bar{n}_i$. n_i is a (complex) three-dimensional null vector of the form

$$n_i = (-\cos\theta\cos\varphi - i\sin\varphi, -\cos\theta\sin\varphi + i\cos\varphi, \sin\theta) \tag{5.3}$$

In Eq.(1), $d\Omega = \sin\theta \; d\theta d\varphi$ and denotes integration over the angles of n_i. $K(A_n, A_{\bar{n}})$ is given by

$$K(A_n, A_{\bar{n}}) = -\frac{1}{\pi} \int d^2x^T \Big[\int d^2z \; \mathrm{Tr}(A_n, A_{\bar{n}}) + i\pi I(A_n) + i\pi I(A_{\bar{n}}) \Big] \tag{5.4}$$

$z = n \cdot \vec{x}, \bar{z} = \bar{n} \cdot \vec{x}$, x^T denotes coordinates transverse to n, i.e., $\vec{x}^T \cdot \vec{n} = 0$. Also

$$I(A_n) = i \sum_2^\infty \frac{(-1)^m}{m} \int \frac{d^2z_1}{\pi} \cdots \frac{d^2z_m}{\pi} \frac{\mathrm{Tr}(A_n(x_1) \ldots A_n(x_m))}{\bar{z}_{12}\bar{z}_{23} \ldots \bar{z}_{m-1m}\bar{z}_{m1}} \tag{5.5}$$

$\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$. The argument of all A's in Eq.(5.5) is the same for the transverse coordinates x^T. The lowest order term in S_m was shown to be

$$S_m = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} A_i^a(-k) A_j^a(k) \Big(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \Big) + \mathcal{O}(A^3) \tag{5.6}$$

This term involves only the transverse potentials, as expected for magnetic screening and on account of gauge invariance. The terms with higher powers of A make S_m invariant under the full non-Abelian gauge transformations.

The general strategy for the inclusion of this term is as follows. We write the action as

$$S = S_0 + M^2 S_m - \Delta\, S_m \tag{5.7}$$

S_0 is the standard quark and gluon part of the action. Below we shall not consider the quark terms since their effects are small and have the same general qualitative features. S_0 will be just the Yang-Mills action. Δ is taken to have a loop expansion, $\Delta = \Delta^{(1)} + \Delta^{(2)} + \ldots$. Calculations can be done in a loop expansion. We require the pole of the propagator to remain at $k_0^2 - \vec{k}^2 = M^2$ (for the transverse potentials) as loop corrections are added. This requires choosing $\Delta^{(1)}$ to cancel the one-loop shift of the pole, $\Delta^{(2)}$ to cancel the two-loop shift of the pole, etc., as is usually done for mass renormalization. Of course, we do not want to change the theory, only rearrange and re-sum various terms. Thus we should impose the condition

$$\Delta = \Delta^{(1)} + \Delta^{(2)} + \ldots = M^2 \tag{5.8}$$

This condition is the gap equation determining M in terms of g^2 and T. This procedure of adding and subtracting a mass term, with a gap equation required for consistency, is very standard, for example, in the Nambu-Jona

Lasinio model. It amounts to the self-consistent summation of self-energy corrections. The difference in the present case is that, for reasons of gauge invariance, the mass term involves an infinite number of interaction vertices as well.

We shall now turn to the explicit one-loop calculations. For the plasma at high temperatures, the effects of magnetic screening will be insignificant for high momentum processes. The regime of interest involves momenta small compared to T. The thermal part of the gluon propagator simplifies as

$$\frac{\delta(k^2 - M^2)}{e^{\omega_k/T} - 1} \approx \frac{T}{2{\omega_k}^2}[\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)] \tag{5.9}$$

$\omega_k = \sqrt{\vec{k}^2 + M^2}$. This is equivalent to using Euclidean three-dimensional propagators, with a coupling constant $e = \sqrt{g^2 T}$; the thermal part of the loop contributions can be done in a three-dimensional theory. This is, of course, the standard dimensional reduction argument. The electrostatic field, with a Debye mass of order gT, will also be neglected for low momentum calculations. The relevant momenta for which this approximation is valid will be of order g^2T; the coupling constant is also self-consistently evaluated at a scale of order g^2T. This is all in keeping with the assumed hierarchy of $\Lambda_{QCD} << g^2T << gT << T$ for the hot quark-gluon plasma.

The action for momenta small compared to T and gT can be written as

$$S = \int d^3x \left[\frac{1}{4e^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{M^2}{e^2}\mathcal{L}_m\right] - \frac{\Delta}{e^2}\int d^3x\, \mathcal{L}_m \tag{5.10}$$

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$. The integral of $\mathcal{L}_m$ is the three-dimensional Euclidean version of Eqs.(5.2,5.4,5.5). There is now only one transverse coordinate x^T. A convenient gauge-fixing term is $\frac{1}{2}\partial \cdot A(1 - M^2\frac{1}{\partial^2})\partial \cdot A$. This

gives the gluon propagator as $\delta^{ab}\delta_{ij}(k^2+M^2)^{-1}$. The sum of all one-loop contributions to the gluon polarization is finite. The functional integral is thus

$$Z = \int [dA]\ \det(-\partial \cdot D) e^{-S} \tag{5.11}$$

$D^{ab}_{\mu} = \partial_{\mu}\delta^{ab} + f^{acb}A^{c}_{\mu}$. The action simplifies as

$$\begin{aligned} S = \int\ \tfrac{1}{2}A^{a}_{i}(-\partial^2 + M^2)A^{a}_{i} \quad &+ \int\ A^{a}_{i}A^{b}_{j}A^{c}_{k}f^{abc}(v_{ijk} + V_{ijk}) \\ &+ \tfrac{1}{4}f^{amn}f^{abc}\int\ A^{m}_{i}A^{n}_{j}A^{b}_{i}A^{c}_{j} + \ldots \end{aligned} \tag{5.12}$$

$$v_{ijk}(k,q,-(k+q)) = \frac{i}{6}[(2k+q)_j\delta_{ik} - (2q+k)_i\delta_{jk} + (q-k)_k\delta_{ij}] \tag{5.13}$$

$$V_{ijk}(k,q,-(k+q)) = -i\,\frac{M^2}{24\pi}\int d\Omega\ \frac{n_i n_j n_k}{k\cdot n}\left[\frac{q\cdot\bar{n}}{q\cdot n} - \frac{(q+k)\cdot\bar{n}}{(q+k)\cdot n}\right] \tag{5.14}$$

This, after angular integration, becomes

$$\begin{aligned} V_{ijk}(k,q,-(k+q)) \ = \ -i\,\frac{M^2}{6}\left[\frac{1}{k^2q^2-(q\cdot k)^2}\right]\Bigg[\left\{\frac{q\cdot k}{k^2} - \frac{q\cdot(q+k)}{(q+k)^2}\right\}k_i k_j k_k \\ + \ \frac{k\cdot(q+k)}{(q+k)^2}(q_i q_j k_k + q_k q_i k_j + q_j q_k k_i) - (q \leftrightarrow k)\Bigg] \end{aligned} \tag{5.15}$$

Expression (5.15) is the contribution from S_m. The four-point vertex involves, in addition to the standard Yang-Mills vertex of Eq.(5.12), a term from S_m of the form $\int d\Omega\ (n_i n_j n_k n_l)$. Since n_i is a null vector we get zero from this term to one loop order. This is one of the advantages of our form of S_m; because n_i is null, Wick contractions at the same point, with a propagator of the form $\delta_{ij}(k^2+M^2)^{-1}$, give zero. Vertices of higher than quartic order do not contribute at one-loop level. The relevant Feynman diagrams are shown in fig. (5.1). The calculations are straightforward although somewhat tedious.

Despite the algebraic complexity of V_{ijk}, there are many cancellations and for the one loop part of the effective action we find

$$\Gamma^{(1-loop)} = \int \frac{d^3k}{(2\pi)^3}\, \tfrac{1}{2} A_i^a(-k)\Pi_{ij}(k)A_j^a(k) + \dots \tag{5.16}$$

$$\Pi_{ij} = \Pi_{ij}^{(1)} + \Pi_{ij}^{(2)} - \frac{\Delta^{(1)}}{e^2}\Big(\delta_{ij} - \frac{k_i k_j}{k^2}\Big) \tag{5.17}$$

$$\begin{aligned}\Pi_{ij}^{(1)} = C\delta_{ij}\Big[-\frac{M}{2\pi} + \int_0^1 d\alpha\, \Big\{\frac{3}{4\pi}\sqrt{k^2(\alpha-\alpha^2)+M^2} - \frac{k}{8\pi}\sqrt{(\alpha-\alpha^2)} \\ - \frac{k^2}{16\pi}\frac{5-2\alpha+2\alpha^2}{\sqrt{k^2(\alpha-\alpha^2)+M^2}}\Big\}\Big] \\ + C\frac{k_i k_j}{16\pi}\int_0^1 d\alpha\, \Big[\frac{3+6(\alpha-\alpha^2)}{\sqrt{k^2(\alpha-\alpha^2)+M^2}} - \frac{2}{k}\sqrt{(\alpha-\alpha^2)}\Big]\end{aligned} \tag{5.18}$$

$$\begin{aligned}\Pi_{ij}^{(2)} = \frac{C}{4\pi}\Big(\delta_{ij} - 3\frac{k_i k_j}{k^2}\Big)\int_0^1 d\alpha\Big[\sqrt{k^2(\alpha-\alpha^2)+M^2} - \sqrt{k^2(\alpha-\alpha^2)+M^2\alpha}\Big] \\ - \frac{C}{8\pi}\Big(\delta_{ij} + \frac{k_i k_j}{k^2}\Big)\int_0^1 d\alpha\Big[2\sqrt{k^2(\alpha-\alpha^2)+M^2\alpha} - \sqrt{k^2(\alpha-\alpha^2)+M^2} \\ - k\sqrt{\alpha-\alpha^2}\Big]\end{aligned} \tag{5.19}$$

$\Pi_{ij}^{(1)}$ is the contribution of the standard Yang-Mills diagrams. $\Pi_{ij}^{(2)}$ involves the new vertex (5.15), by itself and mixed with (5.13). The α-integrals can actually be evaluated in terms of elementary functions but we do not need the explicit form in what follows. The total contribution to Π_{ij} can be written as

$$\Pi_{ij} = \Big(\delta_{ij} - \frac{k_i k_j}{k^2}\Big)\frac{M^2}{e^2}\Big[B + \Pi(K) - \frac{\Delta^{(1)}}{M^2}\Big] \tag{5.20}$$

$$\begin{aligned}\Pi(K) = \frac{Ce^2}{4\pi M}\int_0^1 d\alpha\Big[\frac{9}{2}\sqrt{K(\alpha-\alpha^2)+1-\alpha+\alpha^2} + 1 - 2\sqrt{K(\alpha-\alpha^2)+\alpha^2} \\ - \frac{(K-1)}{4}\frac{(5-2\alpha+2\alpha^2)}{\sqrt{K(\alpha-\alpha^2)+1-\alpha+\alpha^2}} - 5\sqrt{1-\alpha+\alpha^2} \\ - \frac{3}{4}\frac{1}{\sqrt{1-\alpha+\alpha^2}}\Big]\end{aligned} \tag{5.21}$$

$$B = \left(\frac{Ce^2}{4\pi M}\right)\int_0^1 d\alpha\left[5\sqrt{1-\alpha+\alpha^2}+\frac{3}{4}\frac{1}{\sqrt{1-\alpha+\alpha^2}}-3\right]$$
$$\approx \frac{Ce^2}{4\pi M}(2.384) \qquad (5.22)$$

$K = (k^2 + M^2)/M^2$. Notice that Π_{ij} is transverse as expected on grounds of gauge invariance. $\Pi(K)$ vanishes at $K = 0$ and has no other zeros for positive K as can be checked graphically. (For negative K, $\Pi(K)$ is complex and our calculation which uses time-ordered products is not applicable; see ref. [11].) The one-loop corrected inverse propagator has the form $\frac{M^2}{e^2}[K + \Pi(K) + (B - \frac{\Delta^{(1)}}{M^2})]$. Since $\Pi(K) = 0$ at $K = 0$, we see that the pole of the propagator will not be shifted if we choose $B = \frac{\Delta^{(1)}}{M^2}$ or $\Delta^{(1)} \approx \frac{Ce^2M}{4\pi}(2.384)$. The gap equation (5.8) then gives

$$M \approx (2.384)\frac{Ce^2}{4\pi} \approx (2.384)\frac{Cg^2T}{4\pi} \qquad (5.23)$$

With this choice of $\Delta^{(1)}$, the inverse propagator has the form $\frac{M^2}{e^2}[K+\Pi(K)]$. The correction $\Pi(K)$ is significant compared to K, for small K; for large K, it approaches the result for massless Yang-Mills theory; this is a necessary check for any infrared cutoff.

Eventhough Π_{ij} is transverse in agreement with gauge invariance, it depends on the gauge fixing used for the gluon propagator in the loop. The position of the pole and hence the gap equation do not depend on this. One can explicitly check this. The simplest way is as follows. In the effective action, one can have a gauge-dependent Π_{ij}; Generally, the higher point functions are gauge dependent as well. Physical results, such as scattering amplitudes, are independent of the gauge fixing used. Alternatively, one can define a new two-point function, a new three-point function, etc., which are

independent of the gauge-fixing and which lead to the same physical results. One shifts some of the contribution (to the scattering process) from the three-point vertex to the propagator; similar shifts are done for the higher point functions as well. The amount of shift is determined by requiring the physical results to be the same and leads to the pinching procedure [39]. For our gauge choice, the pinching terms arise from the last two diagrams shown in fig.(5.1) (A quark scattering process suffices to identify these terms.)

The total pinching contribution to the two-point function is

$$\tilde{\Pi}(K) = -\frac{Ce^2m}{16\pi}K\int_0^1 d\alpha \left[\frac{5}{\sqrt{K(\alpha-\alpha^2)+1-\alpha+\alpha^2}} - \frac{2\alpha}{\sqrt{K(\alpha-\alpha^2)+\alpha^2}}\right] \tag{5.24}$$

This is to be added to $\Pi(K)$ to obtain the gauge-independent two-point vertex function. From the explicit factor of K in Eq.(5.24), we see that the pole remains at $K = 0$.

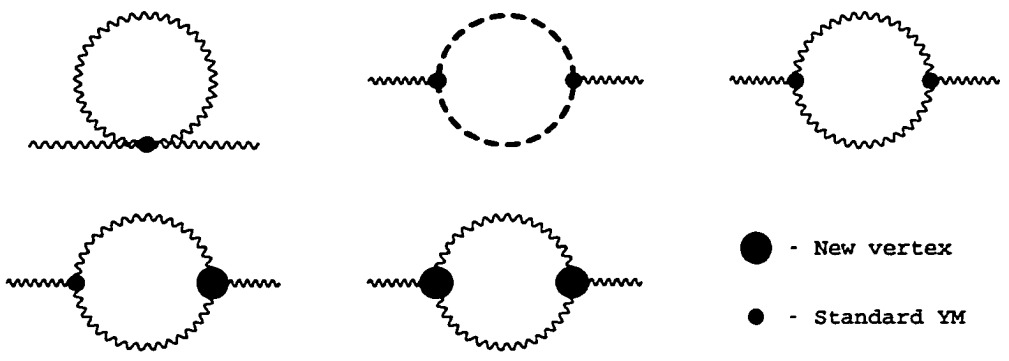
We shall now briefly consider two-loop corrections. These must involve e^4 and since e^2 has the dimension of mass, the two-loop contribution to B has the form

$$B^{(2-loop)} \approx \frac{(Ce^2)^2}{M^2}[\gamma_1 + \gamma_2 \log(T/M)] \tag{5.25}$$

where γ_1, γ_2 are pure numbers. The gap equation now reads

$$x^2 = \frac{2.384}{4\pi}\, x + \gamma_1 + \gamma_2 \log(T/M) \tag{5.26}$$

$M = Ce^2x$. The numerical values of γ_1, γ_2 determine whether the two-loop corrections are small. There are seventy-nine diagrams for Π_{ij} at two-loop level as opposed to five at the one-loop level. Since the vertices given by S_m are fairly complicated we have not completed the two-loop calculation.



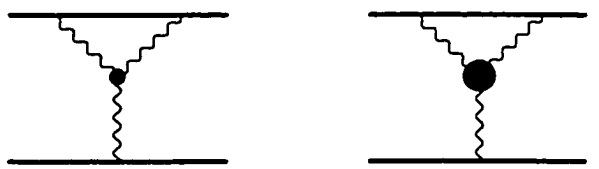


Figure 5.1: Diagrams contributing to $\Pi(K)$ and quark scattering

A preliminary analysis of some of the diagrams suggests that the two-loop effects may be smaller by a factor of three or four. It should be noted that this issue does not impinge on the use of S_m as a gauge-invariant infrared cutoff using the above procedure. It only affects the numerical determination of M in a loop expansion.

Chapter 6

Alternative proposals for the magnetic mass

As we have already mentioned in the beginning of this calculation, there exist, in principle, infinite number of ways to write down gauge-invariant mass terms for the 3-d Yang-Mills theory. The number of possibilities grows even larger if one allows auxiliary fields (like Higgs fields) for the dynamical mass generation. Eventhough we have used the expression "induced" by the form of the electric mass term, it is important to see what happens if some alternative form is used in a similar gap equation framework.

Recently, several authors [42, 44, 45, 46, 47] have investigated a few other such possibilities. The results were mixed. One of the most natural things to try is to use form $F \frac{1}{D^2} F$, discussed earlier in the text. This was used by Jackiw and Pi in [45]. We use term "natural" because a similar term plays role of a mass term in a Schwinger model, and appears in Polyakov's induced

gravity action. Using

$$I_\mu(A) = \mu^2 \int d^3x \text{ tr } F^i \frac{1}{D^2} F^i \tag{6.1}$$

as a mass term, they have obtained the following expression for the vacuum polarization tensor,

$$\begin{aligned}\Pi_{ij}(p) &= N(\delta_{ij} - \hat{p}_i\hat{p}_j) \times \\ &\left(\left(\frac{p^6}{128\pi\mu^5} + \frac{p^4}{32\pi\mu^3} - \frac{3p^2}{32\pi\mu} + \frac{47\mu}{64\pi} - \frac{\mu^3}{16\pi p^2}\right)\frac{2\mu}{p}\tan^{-1}\frac{p}{2\mu}\right. \\ &- \left(\frac{p^6}{32\pi\mu^5} + \frac{p^4}{16\pi\mu^3} - \frac{p^2}{16\pi\mu} + \frac{\mu}{32\pi}\right)\left(\frac{\mu^2}{p^2}+1\right)^2\frac{\mu}{p}\tan^{-1}\frac{p}{\mu} \\ &\left.- \frac{p^2}{32\pi\mu} - \frac{\mu}{4\pi} + \frac{49\mu^3}{96\pi p^2} + \frac{\mu^5}{32\pi p^4} + \frac{p^5}{128\mu^4} + \frac{p^3}{32\mu^2} - \frac{5p}{64}\right)\end{aligned} \tag{6.2}$$

This expression has all the right properties: it is transversal (as required by gauge invariance) and reduces to the pure YM result in the limit $\mu \to 0$ (or $p \to \infty$). Looking at (6.2) one can see that there are threshold singularities at $p^2 = -4\mu^2$, $p^2 = -\mu^2$ and at $p^2 = 0$. While the first one is beyond our region of interest ($p^2 = -4\mu^2$), the second one is "removed" by a prefactor $(\mu^2/p^2 + 1)^2$, the one at $p^2 = 0$ is not - it renders $\Pi(p)|_{p^2=-\mu^2}$ complex when $p^2 < 0$, thus making solutions of the gap equation complex as well. This, of course, indicates an instability of the vacuum of the theory and makes (6.1) unacceptable at least at one-loop level.

This result raises several questions about our approach. As we mentioned before, the dynamically generated mass does not depend on any small parameter in the theory. It means that unless there exist some arguments on why higher loops are less important or there is some purely *numerical* reason why they can be neglected there is no way to tell what order is sufficient

to determine the mass value. In other words, there is a danger that second loop contribution to the gap equation for the mass term proposed by Nair may make it complex as well. The only way to answer this question is to do a two-loop calculation. However, the task seems to be formidable at the moment and may very well be inconclusive if it turns out that the second loop is of the same magnitude as the first one (even if solution remains real).

Nevertheless, there are indirect arguments that indicate some important differences between (6.1) and (5.2). First of all, when considering new vertices introduced by (6.1) one can see that vertices used for "our" mass term are **less** singular for any value of the momenta than the corresponding vertices of (6.1). We have also mentioned before that any "good" candidate for the mass term should not add new degrees of freedom to the theory. The condition that $\Pi(p)$ reduces to the pure YM value $(-7/32p)$ is not enough: it just assures us that in the UV limit any extra degrees of freedom (if any) decouple, and one gets the correct β-function. The way to determine the existence of the extra modes may be described as follows. Let us rewrite an action corresponding to the use of the mass term (6.1) using a multiplet of adjoint auxiliary massless fields Φ^a

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\int F^2 + i\int F^a\Phi^a - \frac{1}{2}\int \Phi^a (D^2)^{ab}\Phi^b \tag{6.3}$$

Shifting Φ^a as

$$\Phi^a \to \Phi^a + \left(\frac{i}{D^2}\right)^{ab} F^b \tag{6.4}$$

one can get the action with the mass term (6.1) with the Φ field entering only as $-\frac{1}{2}\int \Phi^a (D^2)^{ab}\Phi^b$. If one assumes that measure for Φ includes $\det(D)$ (This can be achieved by including a ghost-like term in (6.3)), it can be integrated

out yielding (6.1) as a mass term. However, equation of motion for the field Φ is $\partial^2\Phi^a = F^a$. It has some nontrivial solutions even if $F = 0$, - they are just plane waves. It indicates that nonlocal expression (6.1) may implicitly introduce additional degrees of freedom which, although decoupled at UV-limit, still play some role in the infrared. The region $p^2 < 0$ in Euclidean theory corresponds to the physical values of momenta in the corresponding Minkowski space. The complex value of $\Pi(p)$ may then be just a consequence of the decay of the massive glue states to the massless Φ modes.

Another criticism of our result on the magnetic mass is due to Cornwall [47]. In his paper he looked at the application of the pinch technique to the gap equation method that was also used in [44]. The key idea was to look at the residue of the propagator defined by the pinch technique which we discussed earlier in connection with the position of the pole. The reported value is $Z \approx 150$. This is the main reason for concern, since when computing higher loop contributions such a large value (compared to the "input" value of 1) when raised to higher power (at least 2, since one has more propagators at two-loop order) can make two-loop result much larger than that of one-loop and render mass complex, as in (6.1) case.

We have already mentioned that if one allows some auxiliary fields into the problem, then it is easy to obtain a gauge-invariant mass term by the means of spontaneous symmetry breaking. The non-Abelian Higgs Lagrangian has been considered by Buchmüller and Philipsen in [42, 43] (a similar way to generate mass was suggested by Cornwall in [44]). They considered the action for the scalar coupled to a gauge field as

$$S = \int d^3x \mathrm{Tr}\left(\frac{1}{2}W_{\mu\nu}W_{\mu\nu} + (D_\mu\Phi)^\dagger D_\mu\Phi + \mu^2\Phi^\dagger\Phi + 2\lambda(\Phi^\dagger\Phi)^2\right) \qquad (6.5)$$

This model describes the deconfinement phase transition that is supposed to happen in the standard model at sufficiently high temperature. In the symmetric phase they found mass that has very weak dependence on μ and λ parameters and equals 3/4 of the value obtained in (5.23). This calculation can be done equivalently by use of the action (), where U is the unitary matrix that satisfies $U^\dagger U = 1$.

$$S_{mB\text{-}P} = \mathrm{tr} \int (\partial_\mu \Phi + A_\mu \Phi)^\dagger (\partial_\mu \Phi + A_\mu \Phi) \tag{6.6}$$

Using R_ξ family of gauges (where the $\Pi(p)$ is not transverse) the transverse component in the $\xi = 1$ gauge gives

$$\Pi_{B\text{-}P}^{\text{one-loop}}(p^2) = \frac{Ng^2 m}{8\pi} \left[\left(-\frac{27}{16} \frac{p^2}{m^2} + \frac{9}{4} \right) \frac{2m}{p} \tan^{-1} \frac{p}{2m} - \frac{3}{4} \right] \tag{6.7}$$

Solving the gap equation with this Π yields a real and positive solution

$$m_{B\text{-}P} = \frac{Ng^2}{8\pi} \left[\frac{63}{16} \ln 3 - \frac{3}{4} \right] \tag{6.8}$$

It is very interesting to note that this mass is *exactly* $\frac{3}{4}$ of the mass value that we find in chapter III. There is no explanation of this fact so far. In the end we add that the same result can be obtained by considering the Yang-Mills action with just A^2 mass term [46]. It can be thought of as "unitary gauge" of the model (6.5) and yields an answer identical to (6.8).

Because of all of these problems that seem to appear at one-loop calculations it would be very important to have two-loop results at least for the some of the possibilities considered above. Recently, this question has been investigated by Eberlein [48, 49]. The claim is that it is possible to construct the two-loop gap equation for both the nonlinear σ-model and the SU(2) Higgs lagrangian. It is interesting that at two loops the gap equation will

be quadratic and existence of real positive solutions provides an additional check for the whole concept of using gap equations for the nonperturbative mass evaluation. For the nonlinear σ-model the reported results have weak dependence on the choice of the gauge-fixing parameter ξ, and additional dependence on the scale at which subtraction is performed, μ_{MS} is present. When subtraction scale $\mu_{MS} = 0.3m$, numbers vary from $m = 0.343\ g^2$ at $\xi = 1, \infty$ to $m = 0.35\ g^1$ at $\xi = 10$. Answers tend to become smaller for larger μ_{MS} - taking $\mu = 3\ m$, reported range changes as $m = 0.327\ g^2$ at $\xi = 1, \infty$ to $m = 0.334\ g^1$ at $\xi = 10$.

Similarly, the results of a much more complicated 2-loop calculation of the mass for the full Higgs action (in the symmetric phase) are both μ_{MS} and ξ dependent. In addition, there is another parameter $z = M/m$, where M is the Higgs mass and m is the gauge boson mass. Neglecting the subtraction point dependence (which turns out to be very small numerically) the reported result is $m \approx 0.31\, g^2$ and is almost independent of Higgs mass.

What do these results tell us about the mass generation? Of course, one cannot draw the definite conclusion that value of magnetic screening is as defined by (5.2). Eventhough there seems to be a good agreement between two-loop results, whith the second loop contributing about 20% of the first, the gauge-parameter dependence of the answer is a cause for concern. General arguments that would ensure convergence of the loop expansion are still lacking.

Nevertheless, one can make a series of interesting observations concerning the trial mass term (5.2) proposed in [32] and used for calculations in chapter 3 and 4. Clearly, besides the perhaps empirical connection with the generat-

ing functional of the hard thermal loops, it has some interesting advantages when compared to any of the other proposals. While nonlocal, it still does not seem to introduce any additional degrees of freedom to the theory. The value of the magnetic mass that one obtains by using it with gap equation at one loop level is very close to the two-loop results obtained recently. It is interesting that the one-loop results for the mass terms in Higgs and non-linear σ-model are farther from our term than the two-loop ones. One can hope that this may indicate convergence. Third, the only known lattice result [66] seems to be extremely close as well. And at last, the recent attempt to treat (2+1) Yang-Mills theory in continuum by using Hamiltonian formalism [64, 65] predicts existence of the mass gap of the order of $e^2N/2\pi$, which is again a very close result. Does it all mean that this mass term is somehow special? The above provides some evidence for that; it is our opinion that further investigation of it is required and would be interesting. As a conclusion, we must add that another question that we set to investigate in the beginning of this work seems to have a positive answer: one can set up an infrared finite perturbation theory in non-Abelian gauge theory at high temperature. Gap equation is needed for the self-consistent reorganization of perturbation theory and explicit one-loop calculations yield a real solution for magnetic screening. Several nonperturbative methods may test that in the future [40].

Part II

Renormalization of Hamiltonian

Chapter 7

Introduction

The subject of renormalization for the Hamiltonian operator has been looked into extensively for the past decade [51]. Yet it is hardly being used in any practical computation in field theory, since usually all the intermediate steps are non-covariant and, generally, rather complicated compared to the standard approach. Nevertheless the concept of "integrating out" of the high frequency degrees of freedom has much more physical meaning precisely in the Hamiltonian framework. One way to present it would be to set up the whole procedure as a Born-Oppenheimer approximation used in atomic physics [52]. Another way to look at it would be to introduce a unitary transformation in order to decouple "high" and "low" modes and then look at the low-energy part of the spectrum [53]. In this chapter we want to introduce a renormalization technique in the spirit of the Wilsonian renormalization [54], *i.e.*, integration of fast degrees of freedom, appropriate for the Hamiltonian (Schrödinger) formalism [55]. It is similar in essence to both [52] and [53], but appears to be very different in practice. Basically what we want to see is how

the high frequency degrees of freedom modify the low energy Hamiltonian operator, or equivalently, what Hamiltonian in terms of low frequency modes produces the same low energy physics as the original Hamiltonian. It is also interesting to mention that procedure for renormalization of Hamiltonian operator opens interesting possibilities in interpretation of dynamics of the non-Abelian gauge theories. The kinetic energy operator can be considered as a Laplacian defined on the configuation space of the system. As such it is related to the metric and its renormalization properties can be of some interest with regard to the change of some geometric parameters of this space under RG transformation.

This part of the thesis is organized as follows. In the following chapter we introduce a novel method for the renormalization of the Hamiltonian operator in Quantum Field Theory in the spirit of the Wilson renormalization group (chapter 8). By a series of unitary transformations that successively decouple the high-frequency degrees of freedom and partially diagonalize the high-energy part, we obtain the effective Hamiltonian for the low energy degrees of freedom. We then test this method using simple example from Quantum Mechanics (chapter 9). In the chapter 10 we successfully apply this technique to compute the 2-loop renormalized Hamiltonian in scalar $\lambda\,\phi^4$ theory. In chapter 11 we discuss several ideas on geometry of configuration space in non-Abelian gauge theory and their connection with renormalization method presented in chapter 8. In chapter 12 we compute the 1-loop renormalization of the $SU(N)$ Yang-Mills Hamiltonian. Chapter 13 contains similar analysis of Quantum Electrodynamics. In chapter 14 we discuss the relation between renormalization of the metric and asymptotic freedom. And finally Chapter

15 contains some conclusions.

Chapter 8

Renormalization of Hamiltonian Operator

For the sake of explicitness let us consider a system described by a Hamiltonian $H(\Lambda)$ defined up to the momentum scale (cut-off) Λ; that is, $H(\Lambda)$ produces finite results. Now if μ is a lower momentum scale, we want to find a new Hamiltonian operator $H(\mu)$ that generates the same results for all the physical processes which do not involve momenta greater than μ. The definition of $H(\mu)$ is simple: $H(\mu)$ is the projection of the original Hamiltonian $H(\Lambda)$ onto the low frequency subspace, *i.e.*, the high frequency vacuum,

$$H(\mu) = \mathcal{P}_{low}\, H(\Lambda)\, \mathcal{P}_{low}. \tag{8.1}$$

The claim is that, at least perturbatively, it is possible to decouple the low frequency modes from the high frequency modes and thus give sense to the notion of "low frequency subspace". In essence we will show that after a suitable unitary transformation we can partially diagonalize the Hamiltonian

and construct the vacuum for high momentum modes. Therefore we can rewrite eq. (8.1) as

$$H(\mu) = \langle 0_{high}|U^{\dagger}(\Lambda,\mu)\ H(\Lambda)\ U(\Lambda,\mu)|0_{high}\rangle. \tag{8.2}$$

Now let us explain how to construct the unitary operator $U(\Lambda,\mu)$. As we have explained before, the main goal is to identify the ground state of the high frequency modes and then project the whole Hamiltonian onto this state. So if we can find a unitary transformation that: *i)* separates the low energy modes from the high energy modes and, *ii)* diagonalizes the high energy part of the Hamiltonian (for example in terms of the creation and annihilation operators of the high frequencies), we are done: the ground state will be then the state annihilated by all the high frequency annihilation operators. This task is of course difficult, but in fact we need less than that. Since we only need to identify the vacuum state of the high momenta Hamiltonian and not the whole spectrum, only a partial diagonalization is enough. In fact, only those non-diagonal terms containing purely creation operators or purely annihilation operators have to be removed from the Hamiltonian. Other non-diagonal terms containing both creation and annihilation operators do not change the vacuum state of the theory (once the aforementioned terms have been removed).

We now show how this work in more detail. Suppose that after a unitary transformation we bring a Hamiltonian to the form $H_{diag} + V$ where V contains only terms with at least one creation operator *and* one annihilation operator. That is, V can be written as $V = a_i^{\dagger} M_{ij} a_j$ where M is an arbitrary operator and i, j are generic indices. Then, standard perturbation theory tell

us that the correction to an arbitrary state $|n\rangle$ is given by

$$|\delta n\rangle = \sum_{l\neq n} \frac{\langle l|V|n\rangle}{(E_l - E_n)}|l\rangle + \sum_{l,n\neq m} \frac{\langle l|V|m\rangle\langle m|V|n\rangle}{(E_l - E_m)(E_m - E_n)}|l\rangle + \cdots \tag{8.3}$$

Therefore, if $|n\rangle \equiv |0\rangle$, the vacuum state, it is annihilated by V and there is no correction to it at any order in perturbation theory. The ground state is unaffected by V.

Now we can proceed to find the unitary transformation of eq. (8.2). First we split the original Hamiltonian into four pieces,

$$H = H_1 + H_2 + V_A + V_B. \tag{8.4}$$

Here H_1 contains only the modes with momenta less than μ, H_2 is the *free* part for all the modes with momenta greater than μ and $V_A + V_B$ contains mixing terms and all non-diagonal high-momentum operators; V_A contains the "pure" terms that have only high frequency creation operators or high frequency annihilation operators, but not both, and V_B the "impure" remaining terms (we assume here that V_A and V_B are normal-ordered with respect to the free perturbative vacuum). Then we break the unitary operator U in a product series, $U = U_0\, U_1 \cdots U_n \cdots$ and we compute all the terms successively. The objective of each individual U_n is to partially diagonalize the Hamiltonian, at a given order in λ (a generic coupling of the theory) and μ/Λ. Let us show how to achieve this.

We proceed iteratively by first diagonalizing the Hamiltonian at leading order in λ, up to the desired accuracy in μ/Λ. Consider first a unitary operator U_0 written as $e^{i\Omega_0}$. Now we perform a unitary transformation on equation (8.4), expanding in powers of Ω (in the general case Ω is at least of

order λ so at a given order only a finite number of terms are needed):

$$e^{-i\Omega_0}(H_1+H_2+V_A+V_B)e^{i\Omega_0} = H_1+H_2+V_A+V_B+i\left[H_1,\Omega_0\right]+ \\ +i\left[H_2,\Omega_0\right]+i\left[V_A,\Omega_0\right]+i\left[V_B,\Omega_0\right]\cdots \quad (8.5)$$

As we have explained above, we want to eliminate the "pure" mixing terms V_A, while we generate an expansion in μ/Λ. Thus, we impose the following condition on Ω_0,

$$i\left[H_2,\Omega_0\right]+V_A=0. \quad (8.6)$$

This equation can be solved perturbatively and since commutators with H_2 generate time derivatives of the high frequency fields we have the desired expansion.

Notice that equation (8.6) defines Ω_0 up to the terms that commute with H_2. As it is shown in the appendix A this freedom corresponds to the arbitrariness of the definition of the low energy Hamiltonian up to a unitary transformation. Therefore we will assume some kind of "minimal" scheme - namely that Ω_0, after normal ordering, does not have a part that commutes with H_2 (terms function of $a_k^\dagger a_k$).

After Ω_0 is chosen to cancel V_A in the effective Hamiltonian, a new mixing term of order λ appears from $i\left[H_1,\Omega_0\right]$. However this new term is of a higher order in μ/Λ and will be eliminated by the next unitary transformation $U_1=e^{i\Omega_1}$. Explicitly,

$$e^{-i\Omega_1}e^{-i\Omega_0}(H_1+H_2+V_A+V_B)e^{i\Omega_0}e^{i\Omega_1} = H_1+H_2+V_B+ \\ i\left[H_1,\Omega_0\right]+i\left[H_2,\Omega_1\right]+i\left[H_1,\Omega_1\right]+\text{second order terms}\cdots \quad (8.7)$$

and we now choose Ω_1 so that

$$i\left[H_1,\Omega_0\right]+i\left[H_2,\Omega_1\right]=0. \quad (8.8)$$

Using equations (8.7) and (8.8) we obtain:

$$e^{-i\Omega_1}e^{-i\Omega_0}(H_1+H_2+V_{12})e^{i\Omega_0}e^{i\Omega_1} = H_1+H_2+V_B+\frac{i}{2}[V_A,\Omega_0]+$$
$$i[V_B,\Omega_0]+i[V_B,\Omega_1]+i[H_1,\Omega_1]-\frac{1}{2}[[H_1,\Omega_0],\Omega_0]-$$
$$\frac{1}{2}[[H_1,\Omega_0],\Omega_1]-\frac{1}{2}[[H_1,\Omega_1],\Omega_1]+\text{ higher order terms}\cdots \qquad (8.9)$$

Now it is easy to deduce the logic of the procedure: the next step is to introduce Ω_2 in order to cancel $i[H_1,\Omega_1]$, which is of higher order in μ/Λ, and continue with the same process until we have attained the desired accuracy. Notice that by construction

$$[H_2,\Omega_{n+1}] = -[H_1,\Omega_n] \qquad (8.10)$$

and $H_2 \approx \Lambda, H_1 \approx \mu$, then

$$\Omega_{n+1} \approx \frac{\mu}{\Lambda}\Omega_n, \qquad (8.11)$$

and any new Ω is smaller than the previous one by a factor of μ/Λ.

Of course so far we have only eliminated the high momenta degrees of freedom up to the first order in the coupling constant. Requiring the absence of the λ^2 - order mixing terms will lead to the introduction of a whole new series of unitary transformations, and the same arguments can be applied to them.

As a final remark, notice that the "partial" diagonalization at first order in λ defines a correct low-energy effective Hamiltonian (after projecting onto the high frequency vacuum) which is valid up to the order λ^3. The reason is that since all the relevant terms of order λ have been cancelled, the introduction of the new series of Ω's of order λ^2 will only modify the Hamiltonian at order λ^4.

Chapter 9

A quantum mechanical example

In this chapter we illustrate the ideas of the previous section by studying in some detail a quantum mechanical "toy model" where the main ingredients of our technique can be presented in a simpler context.

Consider the following Hamiltonian of two coupled anharmonic oscillators,

$$H = \frac{\omega_1}{2}\left(P^2 + Q^2\right) + \frac{\omega_2}{2}\left(p^2 + q^2\right) + \lambda\,(Q+q)^4 \tag{9.1}$$

where $\omega_1 << \omega_2$. Now we want to find the effective Hamiltonian for P and Q resulting from "integrating out" the high frequency modes p and q. We choose to compute the effective Hamiltonian up to to order λ^3 and $(\omega_1/\omega_2)^3$.

According to the discussion of the previous section we divide the Hamiltonian in four pieces as in eq. (8.4),

$$H = H_1 + H_2 + V_A + V_B \tag{9.2}$$

where the various terms have the form:

$$H_1 \;=\; \frac{\omega_1}{2}\left(P^2 + Q^2\right) + \lambda\left(\frac{3}{4} + 3\,Q^2 + Q^4\right),$$

$$
\begin{aligned}
H_2 &= \omega_2\, a^\dagger a, \\
V_A &= \lambda\Big\{\frac{3}{2}a^2+\frac{3}{2}a^{\dagger 2}+\frac{1}{4}a^4+\frac{1}{4}a^{\dagger 4}+\sqrt{2}\,Q\left(3a+3a^\dagger+a^3+a^{\dagger 3}\right)+ \\
&\quad 3\,Q^2\left(a^2+a^{\dagger 2}\right)+2^{\frac{3}{2}}Q^3\left(a+a^\dagger\right)\Big\}, \\
V_B &= \lambda\Big\{3a^\dagger a+a^\dagger a^3+6a^{\dagger 2}a^2+a^{\dagger 3}a+3\sqrt{2}\,Q\left(a^\dagger a^2+a^{\dagger 2}a\right)+ \\
&\quad 6\,Q^2\,a^\dagger a\Big\}. \qquad (9.3)
\end{aligned}
$$

In eq. (9.3), a and $a^\dagger$ are the annihilation and the creation operators respectively of the high frequency states, defined through $a \equiv (p+iq)/\sqrt{2}$ and $a^\dagger \equiv (p-iq)/\sqrt{2}$.

Now we are ready to perform the first unitary transformation $U_0 = e^{i\Omega_0}$. As explained in Section 1, Ω_0 is chosen in order to satisfy equation (8.6). Using the expressions given in eq. (9.3) we obtain:

$$
\begin{aligned}
\Omega_0 &= i\lambda\frac{1}{\omega_2}\Big\{\frac{3}{4}(a^{\dagger 2}-a^2)+\sqrt{2}\,Q\left(3(a^\dagger-a)+\frac{1}{3}(a^{\dagger 3}-a^3)\right)+ \\
&\quad \frac{3}{2}\,Q^2\left(a^{\dagger 2}-a^2\right)+2^{\frac{3}{2}}Q^3\left(a^\dagger-a\right)+\frac{1}{16}(a^{\dagger 4}-a^4)\Big\}. \qquad (9.4)
\end{aligned}
$$

After performing the unitary transformation with the above expression for Ω_0, we obtain the leading order approximation of the diagonalization of the whole Hamiltonian: we have eliminated the non-diagonal "pure" terms V_A at the expense of creating new ones in $[H_1, \Omega_0]$. However those new terms are of order of ω_1/ω_2 as can be seen clearly from eqs. (9.3) and (9.4).

The following step is then to cancel this newly created non-diagonal terms with the next unitary transformation $U_1 = e^{i\Omega_1}$ where Ω_1 satisfies equation (8.8). A straightforward calculation gives

$$
\begin{aligned}
\Omega_1 &= i\lambda\frac{\omega_1}{\omega_2^2}\Big\{\frac{3}{4}(a^{\dagger 2}+a^2)+i\sqrt{2}\,P\left(3(a^\dagger+a)+\frac{1}{9}(a^{\dagger 3}+a^3)\right)+ \\
&\quad i\frac{3}{2}\,QP\left(a^{\dagger 2}+a^2\right)-i6\sqrt{2}Q^2P\left(a^\dagger+a\right)\Big\}. \qquad (9.5)
\end{aligned}
$$

Finally, to reach the desired accuracy in $\frac{\omega_1}{\omega_2}$ we need also Ω_2, defined through the relation $[H1, \Omega_1] + [H_2, \Omega_2] = 0$. We find

$$\Omega_2 = -\lambda\frac{\omega_1^2}{\omega_2^3}\left\{12(a^\dagger - a) - i\sqrt{2}\,Q\left(3(a^\dagger - a) - \frac{1}{27}(a^{\dagger 3} - a^3)\right) - i\,\frac{3}{4}\,(P^2 + Q^2)\left(a^{\dagger 2} - a^2\right) + i\,12\sqrt{2}\,QP^2\left(a^\dagger - a\right) i\,6\sqrt{2}\,Q^3\left(a^\dagger - a\right)\right\}. \quad (9.6)$$

Now we are ready to compute the "low energy" effective Hamiltonian. Recalling the discussion of the previous section, we have the following expression for the effective Hamiltonian, up to the order λ^3 and $1/\omega_2^3$:

$$\begin{aligned} H_{eff} = \langle 0|\Big\{ & H_1 + H_2 + V_B + \frac{i}{2}[V_A, \Omega_0] + i[V_B, \Omega_0] + i[V_B, \Omega_1] + i[V_B, \Omega_2] - \\ & \frac{1}{2}[[H_1, \Omega_0], \Omega_0] - \frac{1}{2}[[H_1, \Omega_0], \Omega_1] - \frac{1}{3}[[V_A, \Omega_0], \Omega_0] - \frac{1}{2}[[V_A, \Omega_0], \Omega_1] - \\ & \frac{1}{2}[[V_B, \Omega_0], \Omega_0] - [[V_B, \Omega_0], \Omega_1] - \frac{i}{6}[[[H_1, \Omega_0], \Omega_0], \Omega_0]\Big\}|0\rangle. \end{aligned} \quad (9.7)$$

Here $|0\rangle$ is the vacuum associated to the operators a and $a^\dagger$ leaving the P and Q operators untouched. For the sake of completeness we have written the whole expression derived from the rules described in Section 1. However, some of the terms vanish upon projection.

Finally after a long but straightforward computation we have the desired low frequency Hamiltonian :

$$\begin{aligned} H_{eff} = & \frac{\omega_1}{2}\left(P^2 + Q^2\right) + \lambda\left(\frac{3}{4} + 3\,Q^2 + Q^4\right) + \lambda^2\left(-\frac{21}{8}\frac{1}{\omega_2} + \frac{29}{3}\frac{\omega_1}{\omega_2^2} + \right. \\ & \frac{153}{4}\frac{\omega_1^2}{\omega_2^3} + \left(-31\frac{1}{\omega_2} + 45\frac{\omega_1}{\omega 2^2} + 144\frac{\omega_1^2}{\omega_2^3}\right)Q^2 - \frac{166}{9}\frac{\omega_1^2}{\omega_2^3}\,P^2 + \\ & 162\,i\,\frac{\omega_1^2}{\omega_2^3}\,Q\,P - 81\frac{\omega_1^2}{\omega_2^3}\,Q^2P^2 + 288\,i\,\frac{\omega_1^2}{\omega_2^3}\,Q^3P + \left(36\frac{\omega_1}{\omega_2^2} - \frac{33}{\omega_2}\right)Q^4 + \\ & \left. 72\frac{\omega_1^2}{\omega_2^3}\,Q^4P^2 - \frac{8}{\omega_2}\,Q^6\right) + \lambda^3\left(\frac{333}{16}\frac{1}{\omega_2^2} - 178\frac{\omega_1}{\omega_2^3} + \left(\frac{168}{4}\frac{1}{\omega_2^2} - \right.\right. \end{aligned}$$

$$\frac{3653}{2}\frac{\omega_1}{\omega_2^3}\Big)\,Q^2+\left(\frac{888}{\omega_2^2}-3132\frac{\omega_1}{\omega_2^3}\right)Q^4+\left(\frac{534}{\omega_2^2}-1224\frac{\omega_1}{\omega_2^2}\right)Q^6+$$
$$\frac{96}{\omega_2^2}\,Q^8\Big)+O(\lambda^4,1/\omega_2^4). \tag{9.8}$$

The skeptical reader can verify, using standard Rayleigh-Schrödinger perturbation theory, that the spectrum of the effective Hamiltonian (9.8) is the same as the low energy spectrum (*i.e.*, the part of the spectrum that remains finite if $\omega_2 \to \infty$) of the original Hamiltonian (9.1).

Chapter 10

Renormalization of Scalar Field Theory Hamiltonian

In this chapter we apply the same formalism to the case of a scalar field with the quartic self-interaction, $\lambda\,\phi^4$. We will determine the effective Hamiltonian H up to the two-loop order.

Before proceeding to the loop calculation we have to explain how renormalization is performed in our formalism. As was shown in Section 1, by projecting to the vacuum state for the "high" momentum modes we obtain the effective Hamiltonian H_{eff}. Since our system is supposed to have had an ultraviolet cut-off Λ from the very beginning, H_{eff} will explicitly depend on this UV scale. The renormalization procedure consists of modifying the original Hamiltonian H by introducing renormalization Z-factors that depend on the UV cut-off Λ and some arbitrary "renormalization scale" M. Each of the Z's depends on Λ and M in such a way that the effective Hamiltonian obtained from it does not depend on Λ; in fact it has to look exactly like the

original one except that all Λ's should be replaced by μ - the scale down to which we are integrating out.

Let us start from the "bare" Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}\Pi_\Phi^2(x) + \frac{1}{2}\Phi(x)\left(-\vec{\nabla}^2 + m^2\right)\Phi(x) + \lambda\Phi^4(x)\right), \qquad (10.1)$$

and introduce a Z factor for each composite operator,

$$H = \int \left(\frac{Z_\pi}{2}\Pi_\Phi^2(x) + \frac{Z_\phi}{2}\Phi(x)\left(-\vec{\nabla}^2\right)\Phi(x) + Z_m Z_\phi m^2\Phi^2(x) + \lambda Z_\lambda Z_\phi^2\Phi^4(x)\right) \qquad (10.2)$$

Each of the Z-factors has a perturbative expansion in λ, where we are assuming that λ has been defined by some renormalization prescription at the renormalization scale M (λ is the "renormalized" coupling in the language of the standard renormalization group). Generically

$$Z = 1 + f_1(\Lambda)\lambda + f_2(\Lambda)\lambda^2 + f_3(\Lambda)\lambda^3 + \cdots. \qquad (10.3)$$

The functions f_n will be chosen order-by-order from the requirement that after integration of the modes from μ to Λ all the corrections sum up in such a way that $Z(\Lambda) \to Z(\mu)$. When doing the one-loop corrections one can therefore assume that all the Z's are initially 1 and choose the corresponding f's from the condition that high-cutoff dependence be cancelled after computing the H_{eff} to one loop. For the second loop we use the one-loop Z-corrected Hamiltonian and determine f_n to the next order in λ by the same procedure.

According to our philosophy we have to identify the purely "low" part H_1, the free "high" momentum part H_2 and the interaction terms V_A and V_B. First we split the original field $\Phi(x)$ into its low and high frequency components:

$$\Phi(x) \;=\; \phi(x) + \chi(x),$$

$$\phi(x) = \sum_{k<\mu} \Phi_k e^{ikx}, \quad \chi(x) = \sum_{\mu<k<\Lambda} \Phi_k e^{ikx}. \tag{10.4}$$

By virtue of the equations (10.4), $\int \phi(x)\chi(x) = 0$ and the original Hamiltonian (10.1) can be rewritten as sum on the four pieces $H = H_1 + H_2 + V_A + V_B$:

$$H_1 = \int \left(\frac{1}{2}\pi_\phi^2(x) + \frac{1}{2}\phi(x)\left(-\vec{\nabla}^2 + m^2\right)\phi(x) + \lambda\phi^4(x) \right), \tag{10.5}$$

$$H_2 = \int \left(\frac{1}{2}\pi_\chi^2(x) + \frac{1}{2}\chi(x)\left(-\vec{\nabla}^2 + m^2\right)\chi(x) \right), \tag{10.6}$$

$$V_A + V_B = \lambda \int \left(\chi^4(x) + 4\phi\chi^3 + 6\phi^2\chi^2 + 4\phi^3\chi \right). \tag{10.7}$$

In order to separate V_A and V_B, according to the discussion of Section 1, we introduce the second-quantized form of the high-momentum modes χ,

$$\chi(x) = \sum_{\mu<k<\Lambda} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx} \right) \tag{10.8}$$

Here $\omega_k = \sqrt{k^2 + m^2}$ and can be taken $\omega_k \approx |k|$ for $\mu >> m$. The operators a_k and $a_k^\dagger$ satisfy the standard commutation relations $[a_k, a_k^\dagger] = \delta_{kp}$. Upon substituting this definition in the expressions (10.6) and (10.7) and normal-ordering the result with respect to "high" modes we can identify all the terms that contain only creation or only annihilation operators; these terms form V_A. The rest of the interaction part forms V_B.

$$H_2 = \sum_{\mu<k<\Lambda} \omega_k a_k^\dagger a_k, \tag{10.9}$$

$$V_A = \lambda \sum \left\{ \frac{e^{i(k+p+q+r)x} a_k a_p a_q a_r}{4\sqrt{\omega_k\omega_p\omega_q\omega_r}} + \frac{2\,\phi(x) e^{i(k+p+q)x} a_k a_p a_q}{\sqrt{2\,\omega_k\omega_p\omega_q}} + \right.$$
$$\left. \frac{3\,\phi^2(x)\, e^{i(k+p)x} a_k a_p}{\sqrt{\omega_k\omega_p}} + \frac{4\,\phi^3(x) e^{ikx} a_k}{\sqrt{2\,\omega_k}} + \frac{3}{2}\frac{1}{\omega_p}\,\frac{a_k a_{-k}}{\omega_k} + \text{h. c.} \right\} \tag{10.10}$$

$$V_B = \lambda \sum \left\{ \frac{e^{i(p+q+r-k)x} a_k^\dagger a_p a_q a_r}{\sqrt{\omega_k\omega_p\omega_q\omega_r}} + \frac{3\, e^{i(q+r-k-p)x} a_k^\dagger a_p^\dagger a_q a_r}{\sqrt{\omega_k\omega_p\omega_q\omega_r}} \right.$$
$$\left. + \frac{6\,\phi(x) e^{i(p+q-k)x} a_k^\dagger a_p a_q}{\sqrt{2\,\omega_k\omega_p\omega_q}} + \text{h. c.} \right\} \tag{10.11}$$

Notice that already at this stage we have some contributions to the effective Hamiltonian due to the normal-ordering of the terms of type $\phi^2(x)a_k a_p^\dagger$; this term explicitly depends on the UV cut-off Λ and can be included into H_1,

$$\delta H_1 = 6\lambda \sum_{\mu<k<\Lambda} \frac{1}{2\omega_k} \int \phi^2(x) \approx 3\,\lambda \frac{\Lambda^2-\mu^2}{4\pi^2} \int \phi^2(x), \tag{10.12}$$

where we have made the standard replacement $\sum_k \to \int \frac{d^3k}{(2\pi)^3}$. Equation (10.12) is, of course, the standard "tadpole", one-loop mass renormalization.

In order to pick up the only other one-loop contribution, the coupling constant renormalization, we have to follow the general procedure of Section I and determine Ω_0 for the first unitary transformation. Using equation (8.6) we deduce

$$\Omega_0 = (-i)\lambda \sum \left\{ \frac{4\,\phi^3(x)e^{ikx}a_k}{\sqrt{2\omega_k}\,\omega_k} + \frac{3\,\phi^2(x)e^{i(k+p)x}a_k a_p}{\sqrt{\omega_k\omega_p}(\omega_k+\omega_p)} + \frac{3}{4}\frac{1}{\omega_p}\frac{a_k a_{-k}}{\omega_k^2} + \right.$$
$$\left. \frac{2\phi(x)e^{i(k+p+q)x}a_k a_p a_q}{\sqrt{2\,\omega_k\omega_p\omega_q}(\omega_k+\omega_p+\omega_q)} + \frac{e^{i(k+p+q+r)x}a_k a_p a_q a_r}{4\,\sqrt{\omega_k\omega_p\omega_q\omega_r}(\omega_k+\omega_p+\omega_q+\omega_r)} - \text{h.c.} \right\} \tag{10.13}$$

To find the correction to the Hamiltonian we expand the unitary transformations and project onto the high energy vacuum. The contributions to H_{eff} are exactly the same as in equation (9.7).

Now we have to determine the potentially divergent contributions (if $\Lambda \to \infty$) that will emerge after projection to the vacuum state. In doing so the following naive power-counting rule is useful:

a) Each Ω_n brings $k^{(n+1)}$ (momentum) to the denominator.

b) Each contraction contributes with k^{-1}.

c) For each loop integration we have a contribution of k^3.

Therefore, each term for which the total degree of divergence of a term is greater than or equal to zero, has to be taken into account. Of course, there may be overlapping divergences in computing the proper expressions, but those have to be studied individually, term by term.

By inspection of equation (9.7), is easy to see that at this order there are no contributions to the terms Π^2 and $\Phi(-\vec{\nabla}^2)\Phi$ (*i. e.*, $Z_\pi = Z_\phi = 1$) and the only other one-loop term is the coupling constant renormalization. This contribution comes from the commutator $[V_A, \Omega_0]$ when contracting twice the high energy fields. The rest of the terms are not important at this stage: the commutators $[V_B, \Omega_0]$ and $[H_1, \Omega_0]$ are zero when projected onto the vacuum. After projecting onto the high frequency vacuum we have

$$\begin{aligned} \delta H_1|^{\lambda^2}_{\phi^4} &= -18\lambda^2 \int \phi(x)\phi(y)\, \frac{dp\, dq}{(2\pi)^6}\, \frac{1}{\omega_k \omega_p(\omega_k+\omega_p)}\, e^{i(p-q)(x-y)} \\ &= -\frac{9\lambda^2}{2\pi^2}\, \ln\left(\frac{\Lambda}{\mu}\right) \left[\int \phi^4(x)\, d^3x\right] \end{aligned} \tag{10.14}$$

Expressions (10.12) and (10.14) give the two contributions to H_{eff} at one-loop. From them we can deduce the renormalization factors Z_m and Z_λ:

$$Z_m = 1 - 3\lambda \frac{(\Lambda^2 - M^2)}{2\pi^2}\,, \quad Z_\lambda = 1 + \frac{9\lambda^2}{2\pi^2}\, \ln\left(\frac{\Lambda}{M}\right). \tag{10.15}$$

This finishes the one-loop renormalization procedure. (In refs. [52, 57] there is an extra one-loop counterterm of order λ due to the definition of the "effective" wave functional in the Schrödinger representation). Before going to the next order in coupling, we should notice that there is a two-loop contribution coming from the same term $[V_A, \Omega_0]$ in (9.7), when contracting the high energy fields three times. Its leading divergence is quadratic and it has a logarithmic subleading divergent part that gives rise to the two-loop

wave-function renormalization,

$$\delta H_1|^{\lambda^2}_{\phi^2} = -\frac{12\lambda^2}{(2\pi)^9}\left\{\int d^3r\phi(r)\phi(-r)\right\}\int \frac{d^3p\, d^3q}{\omega_p\omega_q\omega_{r-p-q}(\omega_p+\omega_q+\omega_{r-p-q})}$$
$$= -\frac{3\lambda^2}{\pi^4}(2\ \ln 2-1)\,\Lambda^2\int\frac{1}{2}\phi^2(x)\,d^3x+\frac{3}{8}\frac{\lambda^2}{\pi^4}\ln\left(\frac{\Lambda}{\mu}\right)\int\frac{1}{2}\left(\vec{\nabla}\phi(x)\right)^2 \tag{10.16}$$

Extra contributions to H_{eff} at two-loops come from the terms

$$\delta H = \langle 0|\left(-\frac{1}{2}\left[\left[H_1,\Omega_0\right],\Omega_1\right]+i\left[V_B,\Omega_1\right]-\frac{1}{2}\left[\left[H_1,\Omega_1\right],\Omega_1\right]\right)|0\rangle \tag{10.17}$$

However using the power counting rules described above we can see that the only divergent contribution comes from the first term. Following equation (8.8) we determine the operator Ω_1 as

$$\Omega_1 = (-i)\lambda\sum\left\{\frac{4\left[H_1,\phi^3(x)\right]e^{ikx}a_k}{\sqrt{2\omega_k}\omega_k^2}+\frac{3\ \left[H_1,\phi^2(x)\right]e^{i(k+p)x}a_ka_p}{\sqrt{2\omega_k\omega_p}(\omega_k+\omega_p)^2}+\right.$$
$$\left.\frac{2\left[H_1,\phi(x)\right]e^{i(k+p+q)x}a_ka_pa_q}{\sqrt{2\omega_k\omega_p\omega_q}(\omega_k+\omega_p+\omega_q)^2}+\frac{e^{i(k+p+q+r)x}a_ka_pa_qa_r}{4\sqrt{\omega_k\omega_p\omega_q\omega_r}(\omega_k+\omega_p+\omega_q+\omega_r)^2}-\text{h.c.}\right\} \tag{10.18}$$

and after evaluating a momentum integral similar to the one in eq. (10.16) we finally get

$$\delta H|^{\lambda^2}_{\pi^2} = -\frac{3}{8}\frac{\lambda^2}{\pi^4}\ln\left(\frac{\Lambda}{\mu}\right)\left\{\int\frac{1}{2}\pi_\phi^2\,d^3x\right\} \tag{10.19}$$

It is important to point out that the corrections to $(\vec{\nabla}\phi)^2$, eq. (10.16), and π^2 eq. (10.19), have equal magnitude and opposite sign as it must be, due to the Lorentz covariance of the theory. In fact, since π_ϕ is represented by $\frac{\delta}{\delta\phi}$, then if ϕ^2 gets corrected by Z then π_ϕ^2 should change by $\frac{1}{Z}$, so that the equal time commutator is preserved. Now we can define the two-loop wave-function

renormalization factors Z_ϕ and Z_π as

$$Z_\phi = 1 - \frac{3}{8}\frac{\lambda^2}{\pi^4}\ln\left(\frac{\Lambda}{\mu}\right), \tag{10.20}$$

$$Z_\pi = \frac{1}{Z_\phi} = 1 + \frac{3}{8}\frac{\lambda^2}{\pi^4}\ln\left(\frac{\Lambda}{\mu}\right). \tag{10.21}$$

Finally there is one more two-loop contribution that comes at the order λ^3 and renormalizes the $\phi^4(x)$ term. It comes from the terms

$$\delta H = \langle 0|\left(-\frac{1}{3}\left[\left[V_A,\Omega_0\right],\Omega_0\right] - \frac{1}{2}\left[\left[V_B,\Omega_0\right],\Omega_0\right]\right)|0\rangle \tag{10.22}$$

After a somewhat tedious computation we get

$$\delta H_{\phi^4}^{(2-loop)} = \frac{27}{2}\frac{1}{\pi^4}\lambda^3\left\{\ln\left(\frac{\Lambda}{\mu}\right) + \left[\ln\left(\frac{\Lambda}{\mu}\right)\right]^2\right\}\int \phi^4(x)\, d^3x. \tag{10.23}$$

To compute the entire two-loop correction we have to add to this result the contribution of the one-loop counterterms ,*i.e.*, the one-loop terms with the order λ contributions to the Z-factors. Thus, we can finally deduce the value of the Z_λ factor at two-loops,

$$Z_\lambda = 1 + \frac{9}{2\pi^2}\log(\Lambda/M)\,\lambda^2 + \left(\frac{81}{4\pi^4}(\log(\Lambda/M))^2 - \frac{51}{4\pi^4}\log(\Lambda/M)\right)\lambda^3. \tag{10.24}$$

From equations (10.20) and (10.24) we get the correct two-loop β-function of the theory,

$$\begin{aligned}\beta(\lambda) &= \left.\frac{\partial\lambda}{\partial\log M}\right|_\Lambda = -\left((\partial_\lambda(\lambda\, Z_\lambda)|_{\Lambda,M}\right)^{-1}\lambda\left.\frac{\partial Z_\lambda}{\log M}\right|_{\lambda,\Lambda} \\ &= \frac{9}{2\pi^2}\lambda^2 - \frac{51}{4\pi^4}\lambda^3\,,\end{aligned} \tag{10.25}$$

and the correct two-loop anomalous dimension γ

$$\gamma(\lambda) = \frac{1}{2}\left.\frac{\partial\log Z_\phi}{\partial\log M}\right|_\Lambda = \frac{3}{16\pi^4}\lambda^2. \tag{10.26}$$

In summary, in this chapter we have described a novel perturbative technique of renormalization in the Hamiltonian (Schrödinger) formalism. We have showed that this method successfully gives the two-loop renormalized Hamiltonian for a scalar Field Theory with a quartic potential. In what following chapters we apply this formalism to gauge theory and compare our results with the covariant computations.

Chapter 11

Geometry of the configuration space and mass gap

Looking at the great success that the Standard Model has had since it was introduced more than 20 years ago, it is quite striking that we still lack understanding of the strong interaction part in the low energy regime. In spite of excellent results in the perturbative QCD we are unable to produce any analytical computation of quantities such as the magnetic moment of proton which is known with great accuracy. Some important puzzles, like where does Λ_{QCD} come from, still need to be addressed. Apart from the numerous lattice results (which reinforce our belief that QCD is the right theory for the strong interactions) the situation remains largely the same more than 20 years later. Many attempts to apply a variational approach have not yet produced any effective calculational method for the solution of the problems mentioned before.

It was suggested in [58] that perhaps we need an alternative way to look at

the Yang-Mills theory based on a more geometrical point of view. Namely, one tries to study quantum mechanics of the fields in the space of gauge-inequivalent configurations. The following analogy with quantum mechanics is used. Consider a free particle inside a box of size L. The lowest eigenvalue of the Hamiltonian (which is just a Laplacian) is of the order of $\sim \frac{1}{L^2}$ This is realized by the state of the longest possible wavelength, $\lambda \sim L$. It is clear therefore, that the spectrum of such a system will have a gap due to the fact that L is finite. In other words, the spectrum is going to be discrete as long as distances in the configuration space cannot become arbitrarily large. Feynman's suggestion was to try to adapt this idea to Yang-Mills theory.

In general, in order to determine the distance between field configurations we need to know the metric of the configuration space. The geometry of the configuration space of Non-Abelian gauge theories was considered by many authors ([59, 60, 61]) and, recently, in [62]. The major problem is to extract the metric on a space of the gauge-inequivalent configurations $\mathcal{A}/\mathcal{G}$.

A natural distance between two arbitrary gauge configurations A_1 and A_2 (that is gauge connections *modulo* gauge transformations) is given by [62]

$$||A_1 - A_2||^2 = inf(\int_x Tr((A_1^g(x) - A_2(x))^2)|_g \qquad (11.1)$$

where A_1^g is a gauge-transformed A_1. However the extremum solution of this expression is highly non-local and very difficult to work with. Nevertheless one can try to see some qualitative features of this distance. In [58] it was argued that due to the non-Abelian nature of the gauge group the low-potential-energy configurations are in a space of finite diameter for 2+1 dimensional Yang-Mills theory. According to our very crude analogy with the quantum mechanics it would mean that the kinetic energy operator will have

a discrete spectrum. Of course, these are only qualitative arguments that require a rigorous mathematical formulation if one is to draw any conclusion on the existence and value of the mass gap. Also, as a word of caution we must say that some properties of the configuration space obtained in 2+1 dimensions may not necessarily be the same as in the 3+1 dimensional case (as, for example, the statement above in the case of YM in 3+1 and 1+1 dimensional sigma model [62, 63]).

Another reason to believe that this approach could lead us to a better understanding is the recent progress in Hamiltonian formulation of the 2+1 dimensional Yang-Mills theory. In a series of papers [64, 65] it was shown that by introducing special gauge-invariant variables one can prove that the (properly regularized) volume of the configuration space for the non-Abelian theory is finite, while the corresponding quantity for the Abelian field is infinite. The discreteness of the spectrum of the kinetic term, E^2, has been also shown explicitly and the string tension computed in [65] is in remarkable agreement with the recent Monte-Carlo simulations [66].)

Behind all this discussion a natural question arises: What is the behavior of the configuration space as we integrate out high momenta degrees of freedom? Of course we can not answer completely this question, and even a partial response deserves a profound, probably non-perturbative, analysis. However we still can say something, and answer specific matters as the behavior of the configuration space *metric* under the renormalization group, which encodes much of the properties of the space. Notice that those issues acquire a special significance in Hamiltonian formalism: there the kinetic term is essentially a Laplacian in configuration space and the energy spectrum, at

least in the strong coupling regime, is dominated by it.

The choice of this topic was not fortuitous but we have an idea in mind. In the standard understanding of asymptotic freedom all the significance is put in the interaction potential through the statement that the couplings decrease to zero as the energy at which the theory is tested increases to infinity. However in the Hamiltonian picture we can state the problem in a different way. In the Hamiltonian there is an obvious competition between the kinetic energy and the potential energy. So it is natural to analyze the asymptotic freedom in its "dual" form, *i.e.,* the variation of the kinetic energy (with respect to potential energy) as the renormalization scale changes. In particular we conjectured the following, *in asymptotically free theories the effective distance between configurations* **decreases** *as the momentum is lowered.*

Notice that this question does only makes sense in a Hamiltonian formalism, since it is only in this case that all the geometrical properties of the configuration space can be precisely defined (the Hamiltonian is an operator that acts on the Hilbert space of functionals defined on the configuration space). It can be argued that one can always work in a Lagrangian formalism and at the end construct the respective Hamiltonian by the standard Legendre transformation. However this *cannot* be done if renormalization is involved at any moment. In the Lagrangian formalism, even in the framework of perturbation theory, the process of renormalization generates high time-derivative terms in the Lagrangian, making impossible even the very definition of a Hamiltonian, as the system is governed by high order time-derivatives equations of motion. Even an application of the Ostrogradsky method can in general lead to negative norm states if higher time derivatives

are involved in the Lagrangian. Thus, a prescription for renormalization within the Hamiltonian formalism is indispensable. One must not leave the phase, thereby maintaining the first order time derivative nature of the equations of the system.

Therefore we will use a novel procedure of renormalization of Hamiltonians, already introduced in previous chapters where a successful renormalization of the Hamiltonian of $\lambda\ \phi^4$ was performed. To support our conjecture we will have to do a detailed analysis of two particular examples: Quantum Electrodynamics and Yang-Mills theory in 3+1 dimensions, where we constructed the renormalized Hamiltonian up to one loop. Now let us proceed to explicit calculations.

Chapter 12

Renormalization of non-Abelian gauge theory

12.1 $SU(N)$ Yang Mills theory: Preliminaries

The kinetic energy term for the Hamiltonian of Yang-Mills theory is essentially a Laplace operator on the configuration space. However, due to the gauge invariance, the physical degrees of freedom belong to the space of gauge connections modulo the group of gauge transformations, *i.e*, the space of non-equivalent gauge potentials, and a satisfactory parametrization of this space is needed. In this section we find a perturbatively adequate coordinate system of the configuration space (a similar analysis was done in ref. [71]) and compute the associated metric .

We consider a $SU(N)$ Yang-Mills theory in the temporal gauge, $A_0 = 0$. The canonical variables are the vector potential $A_i^a(x)$ and the electric field

$E^{ai}(x)$. They satisfy canonical commutation relations

$$[A_i^a(x), E^{bj}(y)] = i\delta^{ab}\delta_i^j\delta^{(3)}(x-y) \tag{12.1}$$

which permit the representation of the electric field as

$$E^{ai}(x) = -i\frac{\delta}{\delta A_i^a(x)}. \tag{12.2}$$

The Hilbert space of the theory is supplemented by the Gauss law that enforces a constraint on the wave functionals, essentially only allowing gauge invariant configurations,

$$\mathcal{G}^a(x)\Psi[A] = -iD[A]_i^{ab}\frac{\delta}{\delta A_i^b(x)}\Psi[A] = 0 \tag{12.3}$$

where $D[A]_i^{ab} = \nabla_i\delta^{ab} - ef^{abc}A_i^c(x)$ is the covariant derivative.

The gauge potential can be written also as a Lie algebra valued field $A_i(x) = t^a A_i^a(x)$ where t^a are the hermitian generators of the Lie algebra of $SU(N)$ in the fundamental representation, normalized to $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$.

The Hamiltonian can be written as

$$H = \frac{1}{2}\int d^3x \left(E^{ai}(x)E^{ai}(x) + B^{ai}(x)B^{ai}(x)\right) \tag{12.4}$$

where the magnetic field B_i^a is given as

$$B^{ai}(x) = \epsilon_{ijk}\left(\partial_j A_k^a(x) + \frac{1}{2}ef^{abc}A_j^b(x)A_k^c(x)\right). \tag{12.5}$$

Note that due to the constraint (12.3), not all the degrees of freedom in the Hamiltonian (12.4) are "physical". In order to isolate the physical degrees of freedom we will perform a change of coordinates in such a way that the Gauss law takes its simplest form. We parametrize an arbitrary configuration

in terms of a "gauge fixed" configuration plus gauge transformations. The former will define a coordinate system of the orbit space. We write

$$A_i(x) = g^{-1}(x)\mathcal{A}_i(x)g(x) + ig^{-1}(x)\partial_i g(x) \tag{12.6}$$

where g is a $SU(N)$-valued matrix and $\mathcal{A}_i$ is a configuration satisfying the Coulomb gauge condition

$$\nabla_i \mathcal{A}_i = 0. \tag{12.7}$$

It is well known that the parametrization (12.6) with condition (12.7) is not well defined globally due to the Gribov ambiguity problem. However, we will work in the framework of perturbation theory where the parametrization (12.6-12.7) defines an acceptable isomorphism in the configuration space.

From eq.(12.6) we deduce,

$$\delta A_i = g^{-1}\left(\delta\mathcal{A}_i + [D[\mathcal{A}]_i, i\delta g g^{-1}]\right) g \tag{12.8}$$

where $D[\mathcal{A}_i] = \partial_i - ie\mathcal{A}_i$.

Thus we can decompose the metric of vector valued configurations in "gauge fixed" and "pure gauge" parts

$$\begin{aligned} \delta s^2 &= \int d^d x\ \delta A_i^a(x)\ \delta A_i^a(x) \\ &= \int d^3x\ \left(\delta\mathcal{A}_i^a(x)\ \delta\mathcal{A}_i^a(x) + D[\mathcal{A}]_i^{ab}(i\delta g g^{-1})^b D[\mathcal{A}]_i^{ac}(i\delta g g^{-1})^c\right) \end{aligned} \tag{12.9}$$

The normalization of the wave functionals is then given by

$$\begin{aligned} \langle\Psi_1|\Psi_2\rangle &= \frac{1}{Vol\ G}\int [DA_i^a]\ \Psi_1^*[A]\ \Psi_2[A] \\ &= \int [D\mathcal{A}_i^a]\ \sqrt{G}\ \Psi_1^*[\mathcal{A}]\ \Psi_2[\mathcal{A}] \end{aligned} \tag{12.10}$$

where G is the matrix metric defined by equation (12.9).

Using the Coulomb gauge condition eq.(12.7) we can invert eq.(12.8) and write

$$i\left(\delta g g^{-1}\right)^a = \left(\vec{\nabla}\cdot\vec{D}[\mathcal{A}]\right)^{-1a}_b \vec{\nabla}\cdot(\delta\vec{A}R^{-1})^b \tag{12.11}$$

and

$$\delta\mathcal{A}^a_i = \left(\delta_{ij}\delta^{ab} - D^{ac}_i[\mathcal{A}]\left(\vec{\nabla}\cdot\vec{D}[\mathcal{A}]\right)^{-1c}_a \nabla_j\right)(\delta A_j R^{-1})^b. \tag{12.12}$$

where $R \equiv R(g)$ is the adjoint representation representative of g.

Notice that not all the components of $\mathcal{A}_i$ are independent as they are subject to condition (12.7), so we can parametrize the space of gauge configurations modulo gauge transformations with the $2(N^2-1)$ functions $\mathcal{A}^a_{\hat{\imath}}$, $\hat{\imath} = 1, 2$.

From eqs.(12.11-12.12) we can write the functional derivative as

$$\begin{aligned}\frac{\delta\Psi}{\delta\mathcal{A}^a_i(x)} &= \int R^{-1ad}\left(R^{-1}_y\nabla^x_i(\vec{D}\cdot\vec{\nabla})^{-1}_{(y,x)}\right)^{bd}\frac{\delta\Psi}{i(g^{-1}\delta g)^b(y)} + \\ &\int R^{-1ad}\left(\delta_{ji}\delta^{db}\delta(x,y) + \left(D[\mathcal{A}]^y_j\nabla^x_i(\vec{D}\cdot\vec{\nabla})^{-1}_{(y,x)}\right)^{bd}\right)P_{ji}(y,z)\frac{\delta\Psi}{\delta\mathcal{A}^b_i(z)}\end{aligned} \tag{12.13}$$

where $P_{ij}(x,y) = \delta_{ij}\delta(x,y) - \nabla_i\nabla_j\,\triangle^{-1}(x,y)$ is the projector on the transverse modes.

As previously announced, in these variables the Gauss law has the simple form

$$\frac{\delta\,\Psi[A]}{(i\delta g\; g^{-1})^b} = 0 \tag{12.14}$$

so it is trivially imposed by demanding that the wave functionals be independent of g.

It is useful to define a "transverse" functional derivative as

$$\frac{\delta^T}{\delta\mathcal{A}^a_i(x)} = \int d^3z\; P_{ij}(x,z)\frac{\delta}{\delta\mathcal{A}^a_j(z)} \tag{12.15}$$

restoring rotational invariance at the expense of modifying the canonical commutation relations:

$$\left[\mathcal{A}_i^a(x), \frac{\delta^T}{\delta \mathcal{A}_j^b(y)}\right] = -P_{ij}(x,y)\ \delta^{ab}. \tag{12.16}$$

Now we can write the kinetic energy term in terms of the variables $\mathcal{A}_i^a$ (taking into account eq.(12.15)),

$$\begin{aligned}
\langle \Psi_1 | T | \Psi_2 \rangle &= \frac{1}{Vol\ G} \int [DA_i^a]\ \frac{1}{2} \int d^3x \frac{\delta \Psi_1^*}{\delta A_i^a(x)} \frac{\delta \Psi_2}{\delta A_i^a(x)} \\
&= \int [D\mathcal{A}_i^a]\ \sqrt{\det G}\ \ \frac{1}{2} \int d^3x\ d^3y\ d^3z \times \\
&\left(\delta^{ij}\delta^{ab}\delta(x,y) + \left(D[\mathcal{A}]_j^y \nabla_i^x (\vec{D}\cdot\vec{\nabla})^{-1}_{(y,x)}\right)^{ba}\right) \frac{\delta^T \Psi_1^*}{\delta \mathcal{A}_j^b(y)}\ \times \\
&\left(\delta^{ik}\delta^{ac}\delta(x,z) + \left(D[\mathcal{A}]_k^z \nabla_i^x (\vec{D}\cdot\vec{\nabla})^{-1}_{(z,x)}\right)^{ca}\right) \frac{\delta^T \Psi_2}{\delta \mathcal{A}_k^c(z)}.
\end{aligned} \tag{12.17}$$

That is, the kinetic energy density term takes the form

$$\mathcal{T} = -\frac{1}{2\sqrt{G}} \mathcal{E}^{ia}(x) \sqrt{G}\ G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y)\ \mathcal{E}^{jb}(y) \tag{12.18}$$

where $\mathcal{E}^{ia} = -i\frac{\delta^T}{\delta \mathcal{A}_i^a}$ and $G_{\mathcal{A}/\mathcal{G}}$ is the effective metric of the space of gauge configurations module gauge transformations,

$$\begin{aligned}
G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y) &= \int d^3z\ \left(\delta^{ik}\delta^{ac}\delta(x,z) + \left(D[\mathcal{A}]_i^x \nabla_k^z (\vec{D}\cdot\vec{\nabla})^{-1}_{(x,z)}\right)^{ac}\right) \times \\
&\left(\delta^{jk}\delta^{bc}\delta(y,z) + \left(D[\mathcal{A}]_j^y \nabla_k^z (\vec{D}\cdot\vec{\nabla})^{-1}_{(y,z)}\right)^{bc}\right).
\end{aligned} \tag{12.19}$$

As we mentioned above, the previous analysis was only valid in the framework of perturbation theory where condition (12.7) defines a local system of coordinates on the orbit space. So it is consistent with this approach to compute all the elements of $\mathcal{T}$ (the metric of the space of gauge transformations

G and the metric of the true configuration space $G_{\mathcal{A}/\mathcal{G}}$) in powers of the coupling constant e. In fact, after a straightforward computation, the kinetic energy, up to order e^2, can be shown to be

$$T = \frac{1}{2}\int d^3x d^3y\, G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y)\mathcal{E}^{ai}(x)\mathcal{E}^{bj}(y) - \frac{i}{2}e^2 c_A\, \triangle^{-1}(0)\int d^3x\, \mathcal{A}^a_i\mathcal{E}^{ai}(x) \tag{12.20}$$

where c_A is the Casimir of G in the adjoint representation,

$$G^{(ia)(jb)}_{\mathcal{A}/\mathcal{G}}(x,y) = \delta^{ij}\delta^{ab}\delta(x,y) - e^2 f^{acd}f^{bce}\mathcal{A}^d_i(x)\mathcal{A}^e_j(y)\triangle^{-1}(x,y) + O(e^3) \tag{12.21}$$

and the transverse variables $\mathcal{A}^a_i$ and $\mathcal{E}^a_i$ satisfy the commutation relations (see eq.(12.16):

$$[\mathcal{A}^a_i(x), \mathcal{E}^b_j(y)] = i\delta^{ab}\, P_{ij}(x,y). \tag{12.22}$$

12.2 Renormalization of Yang-Mills Hamiltonian

In this section we will compute the renormalization contribution to the E^2 and B^2 terms in the effective Hamiltonian. In what follows, we will assume the "gauge-fixed" variables from the previous section and will use A and E instead of $\mathcal{A}$ and $\mathcal{E}$. Using expression (12.20) the gauge-fixed Yang-Mills Hamiltonian can be written as follows:

$$H = \frac{1}{2}\int_{x,y}\alpha^{ab}_{ij}(x,y)E^a_i(x)E^b_j(y) + \frac{1}{2}ic_A A^a_i(x)E^a_i(x)G(x,x) + \frac{1}{2}B^a_i(x)B^a_i(x) \tag{12.23}$$

where c_A is N for $SU(N)$ and

$$\alpha_{ij}^{ab}(x,y) = \delta_{ij}\delta^{ab}\delta^3(x-y) + e^2 f^{adc} f^{bde} A_i^c(x) A_j^e(y) G(x,y) + \cdots \quad (12.24)$$

$$G(x,y) = \int \frac{1}{q^2}\, e^{-iq(x-y)} \frac{d^3q}{(2\pi)^3} \quad (12.25)$$

This is the so called "bare" Hamiltonian. For the loop calculations we have to introduce the Z-factors by the following procedure. Using (12.24) let us rewrite expression (12.23) as

$$H = \frac{1}{2} Z_{E^2}(\Lambda) E^2 + \frac{1}{2} Z_{AAEE}(\Lambda) e^2 f^{adc} f^{bde} A_i^c(x) A_j^e(y) G(x,y) E_i^a(x) E_j^b(y)$$
$$+ \frac{1}{2} Z_{B^2}(\Lambda) B^2 + e Z_1(\Lambda) f^{abc} \partial_i A_j^a A_i^b A_j^c + \frac{e^2}{4} Z_4(\Lambda) f^{abc} f^{dec} A_i^a A_j^b A_i^d A_j^e + \quad (12.26)$$

Each of the Z-factors will have the following form

$$Z = 1 + e\, f_1(\Lambda) + e^2\, f_2(\Lambda) + \cdots \quad (12.27)$$

The functions f_n will be chosen order-by-order from the requirement that after integration of the modes from μ to Λ all the corrections sum up in such a way that $f_n(\Lambda) \to f_n(\mu)$ and $Z(\Lambda) \to Z(\mu)$, accordingly. When doing the one-loop corrections one can therefore assume that all the Z's are initially 1 and choose the corresponding f's from the condition that high-cutoff dependence be cancelled after computing the H_{eff} to one loop. Since one-loop wave function renormalization in QCD is of the second order in coupling constant it is easy to see that we need Ω only up to the first order in e. Then there is only one term of V that is relevant:

$$V^{(3)} = e\, f^{abc} \partial_i A_j^a A_i^b A_j^c. \quad (12.28)$$

To compute the renormalization of the quadratic terms in the Hamiltonian we have to assign, according to the general procedure of Section 2, two A's

to be "high" and one "low". Therefore, the relevant part of $V^{(3)}$ looks like [1]

$$V^{(3)} = e\, f^{abc}(2\partial_i A^a_{1j} A^b_{2i} A^c_{2j} + \partial_i A^a_{2j} A^b_{1i} A^c_{2j}) \tag{12.29}$$

We now write A_2 and E_2 in second-quantized form,

$$A^a_{2i}(x) = \sum_k \frac{1}{(2\omega_k)^{\frac{1}{2}}} \left(a^a_{ki} e^{ikx} + a^{a\dagger}_{ki} e^{-ikx}\right) \tag{12.30}$$

$$E^a_{2i}(x) = \sum_k i\left(\frac{\omega_k}{2}\right)^{\frac{1}{2}} \left(a^{a\dagger}_{ki} e^{-ikx} - a^a_{ki} e^{ikx}\right), \tag{12.31}$$

where the creation and annihilation operators satisfy

$$\left[a^a_{ki}, a^{b\dagger}_{pj}\right] = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \delta^{ab} \delta_{kp} \tag{12.32}$$

After normal-ordering, $V^{(3)}$ can have three kind of terms: $a^\dagger_k a^\dagger_p$, $a_k a_p$ and $a^\dagger_k a_p$. As was explained in Section 2, only the first two types will be used to in order to determine Ω, so the final form of $V^{(3)}$ is therefore

$$V^{(3)} = e\, f^{abc} \sum_{p,q} \int d^3x \, \left\{2\left(a^{a\dagger}_{pi} a^{c\dagger}_{qj} e^{-i(p+q)x} + a^a_{pi} a^c_{qj} e^{i(p+q)x}\right) \partial_j A^b_i(x) - \right.$$
$$\left. iq_j \left(a^{a\dagger}_{pi} a^{b\dagger}_{qi} e^{-i(p+q)x} - a^a_{pi} a^b_{qi} e^{i(p+q)x}\right) A^c_j(x)\right\} \frac{1}{2\sqrt{\omega_p\, \omega_q}} \tag{12.33}$$

In order so solve equation (8.9) with the interaction given by (12.29) we use

$$H_2 = \sum_k \omega(k) a^{b\dagger}_{ki} a^b_{ki}, \quad [H_2, a^{a\dagger}_{pi}] = \omega_p a^{a\dagger}_{pi}, \quad [H_2, a^a_{pi}] = -\omega_p a^a_{pi}\,. \tag{12.34}$$

Then,

$$\Omega_0 = i\, e\, f^{abc} \sum_{p,q} \int d^3x \, \left\{2\left(a^{a\dagger}_{pi} a^{c\dagger}_{qj} e^{-i(p+q)x} - a^a_{pi} a^c_{qj} e^{i(p+q)x}\right) \partial_j A^b_i(x) - \right.$$
$$\left. iq_j \left(a^{a\dagger}_{pi} a^{b\dagger}_{qi} e^{-i(p+q)x} + a^a_{pi} a^b_{qi} e^{i(p+q)x}\right) A^c_j(x)\right\} \frac{1}{2\sqrt{\omega_p\, \omega_q}\, (\omega_p + \omega_q)}. \tag{12.35}$$

[1] We use the following convention: fields with subscript 1 indicate "low momenta" fields and fields with subscript 2 indicate "high momenta" fields

According to the equation (8.11), the next Ω will be

$$\Omega_1 = -i\, ef^{abc} \sum_{p,q} \int d^3x \left\{ 2\left(a^{a\dagger}_{pi} a^{c\dagger}_{qj} e^{-i(p+q)x} + a^{a}_{pi} a^{c}_{qj} e^{i(p+q)x} \right) [H_1, \partial_j A^b_i(x)] - iq_j \left(a^{a\dagger}_{pi} a^{b\dagger}_{qi} e^{-i(p+q)x} - a^{a}_{pi} a^{b}_{qi} e^{i(p+q)x} \right) [H_1, A^c_j(x)] \right\} \frac{1}{2\sqrt{\omega_p\, \omega_q}\, (\omega_p + \omega_q)^2} \quad (12.36)$$

To study the renormalization of the metric we have to determine the corrections to the E^2 term in the Hamiltonian. There are only two possibilities. The first one is $-\frac{1}{2}\left[[H_1, \Omega_0], \Omega_1\right]$, since $\Omega_0 \sim A$ and $\Omega_1 \sim E$. The other comes from the normal-ordering of the quadratic part of the α term in the equation (12.24). When computing the double-commutator, there is only one divergent term,

$$\begin{aligned} -\frac{1}{2}\left[[H_1, \Omega_0], \Omega_1\right] \;=\; & e^2 \sum_{k,p} \frac{1}{2} \left[H_1, A^b_i(x)\right] [H_1, A^n_\alpha(y)]\, f^{abc} f^{mnl} k_i q_\alpha \times \\ & \left(\delta^{am}_{kq} \delta^{cl}_{ps} + \delta^{al}_{ks} \delta^{cm}_{pq} \right) P_{j\gamma}(k) P_{j\gamma}(p) \left(e^{-i(k+p)(x-y)} + e^{i(k+p)(x-y)} \right) \times \\ & \frac{1}{4\,(\omega_k + \omega_p)(\omega_q + \omega_s)^2 \sqrt{\omega_k \omega_p \omega_q \omega_s}}. \end{aligned} \quad (12.37)$$

At this point we can say that momenta $(k+p)$ and $-(k+p)$ are essentially the momenta of the "low" fields. Therefore $|k+p| < \mu$. We can now change the summation by the following trick: say $k+p = r$ and $|r| < \mu$. Then $k = r-p$ and the summation goes over k and r. Any possible divergence can only come from the summation over k. Using $f^{adc} f^{bdc} = c_A \delta^{ab}$ the expression (27) can be simplified as follows,

$$\sum_{r,k} \frac{c_A}{4\omega_k \omega_p} \frac{[H_1, A^c_i(r)]\, [H_1, A^c_\alpha(-r)]}{(\omega_k + \omega_p)^3} \left(k_i k_\alpha - k_i p_\alpha\right) P_{j\gamma}(k) P_{j\gamma}(p) \quad (12.38)$$

Noticing that the leading divergence of the expression is logarithmic, which means that we can neglect the difference between k and p (for the divergent

contribution only) and using

$$\sum_k = \int \frac{d^3k}{(2\pi)^3}, \qquad \omega_k = |k|, \qquad k \sim -p \tag{12.39}$$

$$P_{j\gamma}(k)P_{j\gamma}(p) = \left(1 + \frac{(k\cdot p)^2}{k^2p^2}\right) \sim 2, \tag{12.40}$$

we obtain

$$\frac{c_A}{24}\sum_r [H_1, A_i^c(r)]\,[H_1, A_i^c(-r)]\sum_k \frac{1}{\omega_k^3} = -\frac{c_A}{24}\frac{1}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right)\int d^3x\, E^2 \tag{12.41}$$

As we mentioned earlier, there is another term that can correct the E^2 term of the effective Hamiltonian. It is the "Coulomb interaction" term from the kinetic term in (12.24). Looking at the e^2 order correction part of the (12.24,12.28) and choosing the A's to be "high" and the E's to be "low" we have

$$\frac{e^2}{2} f^{adc} f^{bde} \int_{x,y} A_{2i}^c(x) A_{2j}^e(y) G(x,y) E_{1i}^a(x) E_{1j}^b(y) \tag{12.42}$$

As written, this term is not normal-ordered with respect to the high-energy vacuum. Using (12.30),(12.31) and (12.32) we obtain

$$\frac{e^2}{2} f^{adc} f^{bde} \int_x \int_y \sum_k \frac{\delta^{ce} P_{ij}(k)}{2\omega_k} e^{ik(x-y)} G(x,y) E_{1i}^a(x) E_{1j}^b(y) \tag{12.43}$$

Upon using the definition (12.25) of $G(x,y)$ one can see that the leading divergence is logarithmic and that the final expression reads

$$\frac{e^2}{6} c_A \sum_k \frac{1}{\omega_k^3} \int d^3x\, E^2 = \frac{e^2}{6}\frac{c_A}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right)\int d^3x\, E^2. \tag{12.44}$$

The sum of the terms (12.41) and (12.44) gives the total correction to the kinetic energy at one loop:

$$\delta H_{E^2} = \frac{c_A}{8}\frac{e^2}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right)\int d^3x\, E_i^2(x) \tag{12.45}$$

At this point we can say that the Z_{E^2} factor is therefore,

$$Z_{E^2} = 1 - \frac{c_A}{4}\frac{e^2}{2\pi^2} log\left(\frac{\Lambda}{m}\right) + \cdots \tag{12.46}$$

Since the operator E_i is represented by a variational derivative with respect to gauge field, $\frac{\delta}{\delta A_i}$, one would naturally expect that Z_{E^2} should be equal to the $Z_{B^2}^{-1}$. It is therefore an important check on our method to show that it is indeed so. To compute $Z_{B^2} = Z_3$ we need to find out B^2 correction to the effective H. In comparison with the computation of the E^2 correction it is much more involved due to the fact that for most of the terms B^2 comes as a sub-leading divergence. We will not present detailed computation but sketch the main steps and give the final result. B^2 contributions can arise from the following terms: $\frac{i}{2}[V, \Omega_0]$ and normal ordering of the α term from the equation (12.24) again. This term is similar to the (12.42), the only difference being how the "high" and "low" components are assigned:

$$\frac{e^2}{2} f^{adc} f^{bde} A^c_{1i}(x) A^e_{1j}(y) G(x,y) E^a_{2i}(x) E^b_{2j}(y). \tag{12.47}$$

Leading divergence for both terms is quadratic and gives correction of the form A^2. The appearance of this term is related to our choice of the cut-off procedure as a way of regulating the theory; it can be dealt with by introducing a $A^2\Lambda^2$ counter-terms in the bare Hamiltonian and defining appropriate boundary conditions at the ends of the RG flow trajectory [72]. To capture the logarithmic contribution one has to expand the denominators of the Ω_0 up to the second order in the momenta of the "low" fields. Tedious but straightforward computation gives

$$-\frac{i}{2}[V, \Omega_0] = -\frac{27}{120} e^2 c_A \left(\sum_k \frac{1}{k^3}\right) \int d^3x\, B^2. \tag{12.48}$$

The logarithmic divergent part of the normal ordering of the two E's from (12.47) gives

$$\frac{1}{10} e^2 c_A \left(\sum_k \frac{1}{k^3} \right) \int d^3x \, B^2 \tag{12.49}$$

The final result is

$$\delta H_{B^2} = -\frac{c_A}{8} \frac{e^2}{2\pi^2} log \left(\frac{\Lambda}{\mu} \right) \int d^3x \, B^2 \tag{12.50}$$

which makes the corresponding Z factor

$$Z_{B^2} = Z_3 = 1 + \frac{c_A}{4} \frac{e^2}{2\pi^2} log \left(\frac{\Lambda}{m} \right) + \cdots \tag{12.51}$$

This coincides with the value of Z_3 for QCD in the Coulomb gauge obtained in the Lagrangian formalism [73].

12.3 Three-point function renormalization

Let us show briefly the renormalization of the three point function within our formalism. To do that let us recall the general method of Section 2. We will have to be a little more careful in the analysis of the relevant contributions to the renormalized Hamiltonian.

According to the notation of Section 2, we write the Yang-Mills Hamiltonian in terms of the low momentum and high momentum fields as:

$$H = H_1 + H_2 + (V_A^{(3)} + V_B^{(3)} + V^{(4)} + \cdots) \tag{12.52}$$

where H_1 is the part of the Hamiltonian that only contains low momentum fields, H_2 is the part that only contains high momentum fields, and the V's are the "mixing" terms. For convenience we have separated these last

terms according to the vertex number and the high momentum creation-annihilation operators structure

$$V_A^{(3)} = e\, f^{abc} \sum_{p,q} \int d^3x \, \left\{ 2 \left(a^a_{pi} a^c_{qj} e^{i(p+q)x} + a^{a\dagger}_{pi} a^{c\dagger}_{qj} e^{-i(p+q)x} \right) \partial_j A^b_i(x) + \right.$$
$$\left. iq_j \left(a^a_{pi} a^b_{qi} e^{i(p+q)x} - a^{a\dagger}_{pi} a^{b\dagger}_{qi} e^{-i(p+q)x} \right) A^c_j(x) \right\} \frac{1}{2\sqrt{\omega_p\, \omega_q}} , \qquad (12.53)$$

$$V_B^{(3)} = e\, f^{abc} \sum_{p,q} \int d^3x \, \left\{ 2 \left(a^{c\dagger}_{qj} a^a_{pi} e^{i(p-q)x} + a^{a\dagger}_{pi} a^c_{qj} e^{-i(p-q)x} \right) \partial_j A^b_i(x) + \right.$$
$$\left. iq_j \left(a^{a\dagger}_{pi} a^b_{qi} e^{-i(p-q)x} - a^{b\dagger}_{qi} a^{a\dagger}_{pi} e^{i(p-q)x} \right) A^c_j(x) \right\} \frac{1}{2\sqrt{\omega_p\, \omega_q}} , \qquad (12.54)$$

$$V^{(4)} = \frac{e}{2} \sum_{p,q} \int d^3x \left\{ \left(a^a_{pi} a^c_{qi} e^{i(p+q)x} + a^{a\dagger}_{pi} a^{c\dagger}_{qi} e^{-i(p+q)x} \right) f^{abe} f^{cde} A^b_j(x) A^d_j(x) \right.$$
$$\left. + \left(a^a_{pi} a^b_{qj} e^{i(p+q)x} + a^{a\dagger}_{pi} a^{b\dagger}_{qj} e^{-i(p+q)x} \right) \left(f^{abe} f^{cde} + f^{ade} f^{cbe} \right) A^c_i(x) A^d_j(x) \right\} \times$$
$$\frac{1}{2\sqrt{\omega_p\, \omega_q}} . \qquad (12.55)$$

Now we have to analyze which terms contribute to the three point vertices. At this order only Ω at order e is needed, moreover, one can convince oneself that only Ω_0, *i.e,* the first iteration of the unitary transformation leads to divergent contributions. Then eq.(8.5) reads in this case

$$H' = H_1 + H_2 + V_A^{(3)} + V_B^{(3)} + V^{(4)} + \cdots + i[H_1, \Omega_0] + i[H_2, \Omega_0] +$$
$$i[V_A^{(3)}, \Omega_0] + i[V_B^{(3)}, \Omega_0] + i[V^{(4)}, \Omega_0] - \frac{1}{2}[[H_1, \Omega_0], \Omega_0] -$$
$$\frac{1}{2}[[H_2, \Omega_0], \Omega_0] - \frac{1}{2}[[V_A^{(3)}, \Omega_0], \Omega_0] - \frac{1}{2}[[V_B^{(3)}, \Omega_0], \Omega_0] -$$
$$\frac{1}{2}[[V^{(4)}, \Omega_0], \Omega_0] - \frac{i}{6}[[[H_1, \Omega_0], \Omega_0], \Omega_0] - \frac{i}{6}[[[H_2, \Omega_0], \Omega_0], \Omega_0] + \cdots \qquad (12.56)$$

As explained in Section 2, we choose Ω_0 in in such a way that its commutator with H_2 cancels the mixing terms that contain high momentum annihilation operators or high momentum creation operators, but not both.

Also, we note that up to this order, only $V^{(3)}$ is of order e. Then Ω_0 satisfies

$$[H_2, \Omega_0] = iV_A^{(3)} \tag{12.57}$$

(which is precisely the equation that gives eq.(12.35)).

Now it is not difficult to individualize the only terms that contribute to the three point vertex

$$\delta H^{(3)} = \Delta H_1 + i[H_1, \Omega_0]^{(3)} + i[V_A^{(4)}, \Omega_0]^{(3)} - \frac{1}{2}[[V_B^{(3)}, \Omega_0], \Omega_0]^{(3)} \tag{12.58}$$

Here ΔH_1 stands for the normal ordering contribution (tadpole diagram) of order e^3 in the kinetic energy. In fact, at this order, the kinetic energy has a term of the form

$$T_{e^3} = e^3 f^{abc} f^{ade} f^{emn} \int_{xyz} A_i^b(x) G(x,y) A_j^d(y) \partial_j G(y,z) A_k^m(z) E_i^c(x) E_k^n(z) \tag{12.59}$$

which, when properly contracted, generates the contribution:

$$\Delta H_1 = \frac{1}{6} c_A \, e^3 \, log\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c. \tag{12.60}$$

The remaining terms in eq.(12.58), are computed similarly to the ones of Section 2. After a very lengthy, but straightforward computation, we get the following results:

$$\begin{aligned}
[H_1, \Omega_0]^{(3)} &= -i\frac{5}{24} c_A \, e^3 \, \frac{1}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c \, , \\
[V^{(4)}, \Omega_0]^{(3)} &= i\frac{3}{8} c_A \, e^3 \, \frac{1}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c \, , \\
[[V_B^{(3)}, \Omega_0], \Omega_0]^{(3)} &= -\frac{1}{6} c_A \, e^3 \, \frac{1}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c
\end{aligned} \tag{12.61}$$

Finally, adding up all the contributions we have

$$\delta H^{(3)} = \frac{1}{12} c_A \, e^3 \, \frac{1}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right) f^{abc} \int_x \partial_i A_j^a A_i^b A_j^c. \tag{12.62}$$

This result corresponds to a renormalization constant Z_1 equal to:

$$Z_1 = 1 - \frac{1}{12} c_A \, e^2 \, \frac{1}{2\pi^2} log\left(\frac{\Lambda}{m}\right). \tag{12.63}$$

which is the same as the Coulomb gauge result in the Lagrangian approach as well [73].

Chapter 13

Renormalization of QED

In this chapter we will outline a similar computation for Quantum Electrodynamics. The QED Hamiltonian can be written as follows

$$H = \frac{1}{2}\int d^3x \left(E^2 + B^2\right) + \int d^3x \left\{\bar{\psi}\left(i\vec{\gamma}\cdot\vec{\partial} + m\right)\psi + e\bar{\psi}A\cdot\gamma\psi\right\}. \quad (13.1)$$

The imposition of the Gauss law constraint,

$$\partial_i E^i = e\bar{\psi}\gamma^0\psi \quad (13.2)$$

generates an "instantaneous" Coulomb interaction and H takes the form

$$\begin{aligned} H &= \frac{1}{2}\int d^3x \left(E^2 + B^2\right) + \frac{1}{2}\int_x\int_y e^2\,\bar{\psi}(x)\gamma^0\psi(x)G(x,y)\bar{\psi}(y)\gamma^0\psi(y) \\ &\quad + \int d^3x \left\{\bar{\psi}\left(i\vec{\gamma}\cdot\vec{\partial} + m\right)\psi + e\bar{\psi}A\cdot\gamma\psi\right\} \end{aligned} \quad (13.3)$$

where $G(x,y)$ is given by the (12.25). According to the general idea we are supposed to split it in to "high" and "low" energy parts,

$$H_1 = \frac{1}{2}\int d^3x \left(E_1^2 + B_1^2\right) + \int d^3x\,\bar{\psi}_1\left(i\vec{\partial}\cdot\vec{\gamma} + m\right)\psi_1 + e\bar{\psi}_1 A\cdot\gamma\psi_1 \quad (13.4)$$

$$
\begin{aligned}
H_2 &= \sum_k \omega_k b^\dagger_{k\alpha} b_{k\alpha} + \sum_p \omega_p d^\dagger_{p\beta} d_{p\beta} \\
V^{(3)} &= e\int d^3x \left\{\bar\psi_2 A_1\cdot\gamma\psi_2 + e\bar\psi_2 A_2\cdot\gamma\psi_1 + e\bar\psi_1 A_2\cdot\gamma\psi_2\right\} \qquad (13.5)\\
V^{(4)} &= \int_x\int_y e^2\bar\psi_2(x)\gamma^0\psi_1(x)G(x,y)\bar\psi_1(y)\gamma^0\psi_2(y) + \cdots \qquad (13.6)
\end{aligned}
$$

Here dots mean that we show only those terms that play role in one loop effects. Using arguments similar to those in Yang-Mills theory one can see that in order to determine Z factors up to the order e^2 we need Ω only up to the first order. It turns out that only first two iterations in μ/Λ are needed - Ω_0 and Ω_1. In parallel to the previous sections, we determine Z_3 and Z_1 by identifying corrections of the type E^2, B^2 and $\bar\psi A\cdot\gamma\psi$. There is only one commutator that can contribute to the B^2 term: $\frac{i}{2}[V,\Omega_0]$ where one has to take subleading divergence to identify the *log* corrections. E^2 correction is given by two commutators $-\frac{1}{2}[[H_1,\Omega_0],\Omega_1]$ and commutator of the "Coulomb" term with Ω_0 : $i[V^{(4)},\Omega_0]$. The final result is

$$
\begin{aligned}
\delta H_{E^2} &= -\frac{e^2}{6}\frac{1}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right)\int d^3x\, E_1^2 \qquad (13.7)\\
\delta H_{B^2} &= \frac{e^2}{6}\frac{1}{2\pi^2} log\left(\frac{\Lambda}{\mu}\right)\int d^3x\, B_1^2 \qquad (13.8)
\end{aligned}
$$

which leads us to the well-known answer for the Z_3-factor in QED:

$$
Z_{E^2}^{-1} = Z_{B^2} = Z_3 = 1 - \frac{e^2}{3}\frac{1}{2\pi^2} log\left(\frac{\Lambda}{m}\right) \qquad (13.9)
$$

Up to this point all our Z's were identical to those known from the covariant calculations in the Coulomb gauge. However, when computing the other two renormalization constants: Z_1 and Z_2 we find results which are different from the covariant ones. In the case of Z_2 - fermion kinetic term renormalization there are three possible contributions arising from $\frac{i}{2}[V,\Omega_0]$, $-\frac{1}{2}[[H_1,\Omega_0],\Omega_0]$

and normal-ordering of the four-fermion term in the Hamiltonian (13.3). Extracting $\bar{\psi}\vec{\gamma}\cdot\vec{\partial}\psi$-type corrections from each of these terms one can see that they cancel. The similar thing happens in case of Z_1 as well: all the $\bar{\psi}\gamma\cdot A\psi$-type terms cancel at the e^3 order. This makes both Z_1 and Z_2 equal to 1 at one loop. Nevertheless there is no contradiction between our result and the conventional one. In covariant formalism there exists a Ward Identity $Z_1 = Z_2$ which is essential for the maintaining the gauge invariance of the effective action. It is satisfied, presumably, in our case as well. But values of Z_1 and Z_2 are gauge dependent and cancel out from the final expression that defines the beta function for QED, which is determined by Z_3 only. There is an obvious reason why, say Z_2 must be 1 in our case. Since $\bar{\psi}$ and ψ are conjugated variables, in the Hamiltonian formalism one should be represented by the variational derivative with respect to the other. Then, similarly to the E^2 and B^2 terms for the gauge field, they will have inverse renormalization factors which will cancel each other in the final expression for the kinetic term for fermions.

Chapter 14

A geometric interpretation of asymptotic freedom.

Let us recall the Yang-Mills Hamiltonian, after integrating the high momenta modes down to the scale μ (at order e^2), incorporating the correct renormalization factors Z_i.

$$H_{YM} = \frac{1}{2}\frac{1}{Z_3(\mu)}E^2 + \frac{1}{2}Z_3(\mu)(\partial_i A_j^a - \partial_j A_j^a)^2 + Z_1(\mu)\, e_R f^{abc}\partial_i A_j^a A_i^b A_j^c + Z_4(\mu)\,\frac{1}{4}e_R^2 f^{abc}f^{dec}A_i^a A_j^b A_i^d A_j^e + \cdots \tag{14.1}$$

Similar expression for the QED Hamiltonian will be

$$H_{QED} = \frac{1}{2}\frac{1}{Z_3(\mu)}E^2 + \frac{1}{2}Z_3(\mu)B^2 + Z_2(\mu)\bar{\psi}\, i\,\vec{\gamma}\cdot\vec{\partial}\psi + Z_1(\mu)\, e_R\bar{\psi}A\cdot\gamma\psi\cdots \tag{14.2}$$

Here e_R is a fixed quantity at certain scale and all the dependence on the scale is hidden in the Z factors. The definition of the "renormalized" fields through the incorporation of the Z factors was done in analogy with the Lagrangian

covariant approach where the renormalized quantities are included in such a way that the "renormalized" effective action gives finite results when the cut-off is removed. But in the Schrodinger picture this requirement is not necessary as the fields are only coordinates of the configuration space and do not enter explicitly in the computation of correlation functions. With this fact in mind we will alter this requisite and adapt it to our needs.

The usual covariant renormalization program puts the emphasis on the interactions, and the scaling properties of the theory are extracted from the study of the β-functions. However, in a Hamiltonian description the kinetic term plays a significant role since essentially it is nothing but the Laplacian in configuration space. Hence, many of the properties of the QFT can be inferred from the geometrical features of the configuration space (as compactness, boundness, etc.). In particular we are interested in the change of the configuration space metric under the renormalization group flow. We claim that in asymptotically free theories the distance between configurations increases as we move to the UV limit, thus "flattening" the potential energy and consequently fading the interaction. We will support our claim with the analysis of the one-loop Yang-Mills theory.

It is clear then, that in the spirit of our work we want to stress the kinetic term (better, the configuration space metric) over the potential energy and try to understand the asymptotic behavior of the theory through the renormalization flow properties of the distance in the configuration space. For this reason we will rescale the fields in such a way to transfer the renormalization scaling properties to the kinetic term.

Then let us rescale the fields as:

$$A_i^a \to \frac{1}{e_R}\, Z_3(\mu) Z_1(\mu)^{-1}\, A_i^a\,, \qquad E_i^a \to e_R\, Z_3^{-1}(\mu) Z_1(\mu)\, E_i^a \tag{14.3}$$

so the QCD Hamiltonian takes the form:

$$\begin{aligned} H_{QCD} \;=\; & \frac{1}{2} e_R^2 \left(\frac{Z_1^2}{Z_3^3}\right) E_i^{a2} + \frac{1}{e_R^2}\left(\frac{Z_3^3}{Z_1^2}\right)\left\{\frac{1}{2}(\partial_i A_j^a - \partial_j A_j^a)^2 + \right. \\ & \left. + f^{abc}\partial_i A_j^a A_i^b A_j^c + \frac{1}{4} f^{abc} f^{dec} A_i^a A_j^b A_i^d A_j^e \right\} + \cdots \end{aligned} \tag{14.4}$$

where we have used the Slavnov-Taylor identities (adapted for the Hamiltonian formalism) [1]

$$\frac{Z_1}{Z_3} = \frac{Z_4}{Z_1}. \tag{14.5}$$

Note that with this normalization we have "homogenized" the potential term (up to an overall factor) by transferring all the cut-off dependence to the kinetic term. Now we can read from the kinetic term, the cut-off dependence of the (inverse of the) metric. In fact, using the result of Section 3, we can write the cut-off dependent configuration metric as:

$$G_{(ia)(jb)}(x,y;\mu) = \left(1 + e_R^2 \frac{11N}{12}\frac{1}{2\pi^2}\log(\mu/m_R)\right) G^0_{(ia)(jb)}(x,y) + \mathcal{O}(e_R^3) \tag{14.6}$$

where $G^0_{(ia)(jb)}$ is the metric defined in equation (12.19) written in terms of the rescaled fields (14.3) and e_R.

Looking at the QED Hamiltonian and using $Z_1 = 1$ and $Z_2 = 1$ one can see that no rescaling is needed at all, since all the cut-off dependence is

[1] By writing both the Hamiltonian H and Gauss law $\mathcal{G}^a$ in terms of renormalized fields with the explicit Z factors and requiring $[H, \mathcal{G}^a] = 0$ one can get linear relations between Z's that lead to (14.5).

already shifted to the kinetic term,

$$H_{QED} = \frac{1}{2}\left(\frac{1}{Z_3}\right)\ E^2 + \frac{1}{2}Z_3 B^2 + \bar{\psi}\, i\, \vec{\gamma}\cdot\vec{\partial}\psi + e_R\bar{\psi}A\cdot\gamma\psi\cdots \qquad (14.7)$$

Following our procedure the corresponding metric will be

$$G_{(ij)}(x,y;\mu) = \left(1 - e_R^2\,\frac{1}{3}\,\frac{1}{2\pi^2}\,\log(\mu/m_R)\right)\,\delta_{(ij)}\delta(x-y) + \mathcal{O}(e_R^3). \qquad (14.8)$$

At this point it is easy to compare relative behavior of the two metrics under the renormalization flow. Equation (14.6) clearly shows that the distance between configurations decreases as the cut-off is lowered, while the corresponding expression (14.8) increases, sustaining then our claim. Incidentally, it is worthwhile to mention that the combination $\frac{Z_1^2}{Z_3^3}$ that appears in equation (14.4) is precisely the one that defines the β-function of the Yang-Mills theory and, at least in the Lagrangian approach, it has been proved to be independent of the gauge fixing condition. Similarly, QED expression involves only Z_3, since $Z_1 = Z_2$ is 1 to this order, and Z_3 is the only constant that determines beta-function for QED and it is known to be gauge-independent to all loops.

Chapter 15

Summary and Conclusion

There are several issues that are more natural to address in the Hamiltonian picture than in the usual covariant Lagrangian formalism. One of them, which we are interested in, is the relevance of the geometry of the configuration space to the properties of the corresponding Quantum Field Theory. The reason is simple: in the Hamiltonian formalism the kinetic energy term is nothing but a Laplacian operator in the configuration space and its topological and geometrical features determine the nature of its spectrum. Then it is natural to ask what is the behavior of the configuration space as we integrate out high momentum degrees of freedom.

To answer a small part of this question was the aim of this chapter. To be precise we were interested in the following aspect of the problem: the evolution of the distance between field configurations (and more precisely the metric) with the renormalization group in asymptotically free theories. In particular we state the following conjecture: *in asymptotically free theories the effective distance between configurations decreases as high momenta*

degrees of freedom are integrated out.

To support this statement we first developed an original renormalization group technique for Hamiltonian formalism in the framework of perturbation theory. This method resembles the Hamiltonian renormalization approaches of Glazek and Wilson [51] and Wegner [53] and operates by a progressive diagonalization of the Hamiltonian by means of a succession of iterative unitary transformations followed by a projection onto the Hilbert space of the low-momentum degrees of freedom. We have tested this approach by computing renormalized Hamiltonian to two loops in scalar field theory with quartic interaction. Finally we applied the formalism to two conspicuous QFT's: Quantum Electrodynamics and Yang-Mills theory in 3+1 dimensions, where we constructed the renormalized Hamiltonian up to one loop.

Our results were substantially supportive of our conjecture. In the case of Yang-Mills, an asymptotically free theory, the one-loop metric renormalization showed that in fact the distance between configuration increases as the momentum scale increases, and on the contrary for QED, not asymptotically free, the behavior of the metric is the opposite.

We are aware, of course, that our results are not decisive but just consistent with the conjecture. After all we have only studied two examples at one-loop order in perturbation theory. However from the examples considered we can observe a pattern that seems to repeat at any instance: when moving all the weight of the renormalization group onto the configuration space metric, it acquires a renormalization factor which is a function of the same combination of renormalization constants that defines the β-function of the theory, and thus, presumably, inheriting its asymptotic behavior prop-

erties.

Appendix A

In this appendix we show that the ambiguity in the definition of the effective Hamiltonian due to the freedom in the solution of equation (8.6) (and generally (8.10)) is just the standard ambiguity of the Hamiltonian operator, namely the freedom of unitary transformations.

Let us recall that the solution of (8.6), (8.5) or (8.10) is not uniquely defined; if Ω_n is a solution so will be $\Omega_n + O_n$ with O_n satisfying the homogeneous equation $[H_2, O_n] = 0$. Now let us consider two sets of Ω's - say $\Omega_n^{(a)}$ and $\Omega_n^{(b)}$, all of them satisfying the proper equations (8.6) and (8.10) but generated from different type of solutions. Then we have

$$\begin{aligned} H_A &= \ldots e^{-i\Omega_1^{(a)}} e^{-i\Omega_0^{(a)}} H e^{i\Omega_0^{(a)}} e^{i\Omega_1^{(a)}} \ldots \\ H_B &= \ldots e^{-i\Omega_1^{(b)}} e^{-i\Omega_0^{(b)}} H e^{i\Omega_0^{(b)}} e^{i\Omega_1^{(b)}} \ldots \end{aligned} \tag{A.1}$$

Obviously H_A and H_B are unitarily related: $H_A = U^{-1} H_B U$ and consequently they have identical spectrum. What we have to show is that this property remains *after* projecting onto the "high" perturbative vacuum $|0_{high}\rangle$.

Following the prescription given in section 2, we have shown that, up to

given order n in coupling constant λ and a given order m in μ/Λ we can write:

$$\begin{aligned} H_A &= H_1^{(a)}(\text{low}) + H_2^{\text{free}}(\text{high}) + \sum_{k,p>\mu} a_k^\dagger S_{kp}^{(a)} a_p + O\left(\lambda^n, (\mu/\Lambda)^m\right) \\ H_B &= H_1^{(b)}(\text{low}) + H_2^{\text{free}}(\text{high}) + \sum_{k,p>\mu} a_k^\dagger S_{kp}^{(b)} a_p + O\left(\lambda^n, (\mu/\Lambda)^m\right) \end{aligned} \quad \text{(A.2)}$$

where the Hamiltonians $H_1^{(a,b)}$ depend only on the low energy modes, H_2 is the free Hamiltonian for the energy modes and $S^{(a,b)}$ are *arbitrary* operators of low and high frequency modes.

Without losing any generality we can assume that $H_1^{(a)}$) and $H_1^{(b)}$ are diagonal, as they can always be brought to that form with a "low energy modes"-unitary transformation that respects the structure of (A.2).

Now we will show that the low energy Hamiltonians $H_1^{(a)}$ and $H_1^{(b)}$ have the same spectrum and consequently they are unitarily related.

Consider the eigenvalue equations for $H_1^{(a)}$ and $H_1^{(b)}$:

$$H_1^{(a)}\,\psi_\alpha^a = E_\alpha^a\,\psi_\alpha^a\,, \qquad H_1^{(b)}\,\psi_\alpha^b = E_\alpha^b\,\psi_\alpha^b \quad \text{(A.3)}$$

Using standard perturbation theory we can compute the eigenvalues of the whole Hamiltonians, A and B as an expansion in powers of the matrix elements of the interaction terms

$$V^{(a,b)} = \sum_{k,p>\mu} a_k^\dagger S_{kp}^{(a,b)} a_p. \quad \text{(A.4)}$$

We get, for the eigenvalues of the operator H_A,

$$E_{\alpha,n}^{\text{TOT}} = E_\alpha^a + E_n^0 + \langle\alpha,n|V^a|\alpha,n\rangle + \sum_{\gamma,m} \frac{\langle\alpha,n|V^a|\gamma,m\rangle\langle\gamma,m|V^a|\alpha,n\rangle}{E_\gamma^a + E_m^0 - E_\alpha^a - E_n^0} + \cdots \quad \text{(A.5)}$$

and a similar equation is valid for the eigenvalues of the operator H_B. But the low energy spectrum corresponds to those states with $n = 0$, and in this case, due to the particular form of the interaction, all the perturbative contributions vanish and the eigenvalue E^a_α is the *exact* eigenvalue of the whole Hamiltonian (up to the given order in λ and μ/Λ):

$$E^{\mathrm{TOT}}_{\alpha,0} = E^a_\alpha + O\left(\lambda^n, (\mu/\Lambda)^m\right) \tag{A.6}$$

And finally, since both Hamiltonians H_A and H_B have the same spectrum, or since E^{TOT} is the same for both H_A and H_B, we deduce that

$$E^a_\alpha = E^b_\alpha + O\left(\lambda^n, (\mu/\Lambda)^m\right) \ . \tag{A.7}$$

Bibliography

[1] B.Svetitsky and L.Yaffe, *Nucl.Phys.* **B210** (1982) 423 and references therein; R.Pisarski and F.Wilczek, *Phys.Rev.* **D29** (1984) 338.

[2] For a recent review, see B.Petersson, *Nucl.Phys. (Proc.Suppl.)* **B30** (1993) 66.

[3] See, for example, C.P.Singh, *Phys.Rep.* **236** (1993) 147.

[4] Reference 6; K.Rajagopal and F.Wilczek, *Nucl.Phys.* **B379** (1993) 395; *ibid.* **B404** (1993) 577; articles by R.Pisarski and U.Heinz in *Banff/CAP Workshop on Thermal Field Theory*, F.C.Khanna, *et al* (eds.) (World Scientific, Singapore, 1994).

[5] V.P.Silin, *Sov.J.Phys. JETP* **11** (1960) 1136; V.N.Tsytovich, *Sov.Phys. JETP*, **13** (1961) 1249; O.K.Kalashnikov and V.V.Klimov, *Sov.J.Nucl.Phys.*, **31** (1980) 699; V.V.Klimov, *Sov.J.Nucl.Phys.* **33** (1981) 934; *Sov.Phys. JETP* **55** (1982) 199; H.A.Weldon, *Phys.Rev.D* **26** (1982) 1394.

[6] R.Pisarski, *Physica A* **158** (1989) 246; *Phys.Rev.Lett.* **63** (1989) 1129; E.Braaten and R.Pisarski, *Phys.Rev.* **D42** (1990) 2156; *Nucl.Phys.* **B337** (1990) 569; *ibid.* **B339** (1990) 310; *Phys.Rev.* **D45** (1992) 1827.

[7] J. Frenkel and J.C. Taylor, Nucl.Phys. **B334**, 199 (1990); J.C. Taylor and S.M.H. Wong, Nucl.Phys. **B346**, 115 (1990); R. Efraty and V.P. Nair, Phys.Rev.Lett. **68**, 2891 (1992); Phys.Rev. **D47**, 5601 (1993); R. Jackiw and V.P. Nair, Phys.Rev. **D48** (1993) 4991; J.P. Blaizot and E. Iancu, Phys.Rev.Lett. **70**, 3376 (1993); Nucl.Phys. **B417**, 608 (1994)

[8] R.Kobes, G.Kunstatter and A.Rebhan, *Nucl.Phys.* **B355** (1991) 1.

[9] P.F.Kelly *et al*, *Phys.Rev.Lett.* **72** (1994) 3461; *Phys.Rev.* **D50** (1994) 4209.

[10] R.Efraty and V.P.Nair, *Phys.Rev.Lett.* **68** (1992) 2891; *Phys.Rev.* **D47** (1993) 5601.

[11] R.Jackiw and V.P.Nair, *Phys.Rev.* **D48** (1993) 4991.

[12] V.P.Nair, *Phys.Rev.* **D48** (1993) 3432;

[13] A.D. Linde, *Phys.Lett.* **B96** (1980) 289; D. Gross, R. Pisarski and L. Yaffe, *Rev.Mod.Phys.* **53** (1981) 43. *ibid.* **D50** (1994) 4201.

[14] C.Itzykson and J.B.Zuber, *Quantum Field Theory* (McGraw Hill, New York, 1980) pp.594-599.

[15] J.Frenkel and J.C.Taylor, *Nucl.Phys.* **B334** (1990) 199; J.C.Taylor and S.M.H.Wong, *Nucl.Phys.* **B346** (1990) 115.

[16] R.Jackiw and S.Templeton, *Phys.Rev.* **D23** (1981) 2291; J.Schonfeld, *Nucl.Phys.* **B185** (1981) 157; S.Deser, R.Jackiw and S.Templeton, *Phys.Rev.Lett.* **48** (1982) 975; *Ann.Phys.* **140** (1982) 372.

[17] E.Witten, *Commun.Math.Phys.* **121** (1989) 351.

[18] D.Gonzales and A.N.Redlich, *Ann.Phys.(N.Y.)* **169** (1986) 104; B.M.Zupnik, *Phys.Lett.* **B183** (1987) 175; G.V.Dunne, R.Jackiw and C.A.Trugenberger, *Ann.Phys.(N.Y.)* **149** (1989) 197; M.Bos and V.P.Nair, *Phys.Lett.* **B223** (1989) 61; *Int.J.Mod.Phys.* **A5** (1990) 959; S.Elitzur, G.Moore, A.Schwimmer and N.Seiberg, *Nucl.Phys.* **B326** (1989) 108; A.P. Polychronakos, *Ann. Phys.* **203** (1990) 231.

[19] S.P. Novikov, *Usp.Mat.Nauk* **37** (1982) 3; E. Witten, *Commun.Math.Phys.* **92** (1984), 455; V.G.Knizhnik and A.B.Zamolodchikov, *Nucl.Phys.* **B247** (1984) 83.

[20] A.M.Polyakov and P.B.Wiegmann, *Phys.Lett.* **141B** (1984) 223.

[21] See, for example, S.S.Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1964) pp-742-764.

[22] J.Schwinger, *J.Math.Phys.* **2** (1961) 407; P.M.Bakshi and K.Mahanthappa, *J.Math.Phys.* **4** (1963) 1; *ibid.* **4** (1963) 12; L.V.Keldysh, *Sov.Phys. JETP* **20** (1964) 1018.

[23] J.P.Blaizot and E.Iancu, *Phys.Rev.Lett.* **70** (1993) 3376; *Nucl.Phys.* **B 417** (1994) 608.

[24] R.Jackiw, Q.Liu and C.Lucchesi, *Phys.Rev.* **D49** (1994) 6787.

[25] S.Wong, *Nuovo Cim.* **65A** (1970) 689.

[26] See, for example, A.W.Knapp, *Lie Groups, Lie Algebras and Cohomology* (Princeton University Press, Princeton, 1988); A.Pressley and G.Segal, *Loop Groups* (Oxford: Clarendon Press, 1986); J.Sniatycki, *Geometric Quantization and Quantum Mechanics* (Springer-Verlag, New York, 1980).

[27] U.Heinz, *Phys.Rev.Lett.* **51** (1983) 351; *Ann.Phys. (N.Y.)* **161** (1985) 48; *Ann.Phys. (N.Y.)* **168** (1986) 148; H.-Th.Elze and U.Heinz, *Phys.Rep.* **183** (1989) 81.

[28] E.Braaten, in *Hot Summer Daze*, A.Gocksch and R.Pisarski (eds.) (World Scientific, Singapore, 1992).

[29] R.I.Nepomechie, *Phys.Rev.* **D33** (1986) 3670; D.Karabali, Q-H.Park, H.J.Schnitzer and Z.Yang, *Phys.Lett.* **216B** (1989) 307; D.Karabali and H.J.Schnitzer, *Nucl.Phys.* **B329** (1990) 649; K.Gawedzki and A.Kupianen, *Phys.Lett.* **215B** (1988) 119; *Nucl.Phys.* **B320** (1989) 649.

[30] R.Pisarski, *Physica* **A158** (1989) 146; T.Appelquist and R.Pisarski, *Phys. Rev* **D23** (1981) 2305.

[31] J.P.Blaizot and E.Iancu, *Phys.Lett.* **B326** (1994) 138.

[32] V.P. Nair, Phys. Lett. **B352**, 117 (1995);

[33] Y.Nambu and G.Jona-Lasinio, *Phys.Rev.* **122** (1961) 345; *ibid.* **124** (1961) 246.

[34] G. Alexanian and V.P. Nair, Phys.Lett. **B390**, 370 (1997).

[35] See, for example, A.L.Fetter and J.D.Walecka, "Quantum theory of many particle systems", McGraw-Hill, 1971.

[36] R.Jackiw, Q.Liu and C.Lucchesi, *Phys.Rev.* **D49** (1994) 6787; P.F.Kelly *et al*, *Phys. Rev. Lett.* **72** (1994) 3461; *Phys.Rev.* **D50** (1994) 4209; V.P.Nair, *Phys.Rev.* **D48** (1993) 3432; *ibid.* **D50** (1994) 4201; A.K.Rebhan, *Phys.Rev.* **D** (1993) 3967, *Nucl.Phys.* **B 430** (1994) 319.

[37] C. Manuel, *Phys.Rev.* **D53** (1996) 5866.

[38] A. Billoire, G. Lazarides and Q. Shafi, *Phys. Lett.* **103 B** (1981) 450; O.K. Kalashnikov, *JETP Lett.* **39** (1984) 405; T.S. Biro and B. Muller, *Nucl.Phys.* **A 561** (1993) 477; O. Philipsen, in Proceedings of the NATO workshop, Sintra, Portugal, March 1994 (to be published); W. Buchmuller and O. Philipsen, Preprint DESY-94-202 (November 1994); U. Heller, F. Karsch and J. Rank, Phys.Lett. **B355**, 511 (1995); F. Karsch *et al*, Nucl.Phys. **B474**, 217 (1996); F. Karsch, M. Oevers and P. Petreczky, hep-lat/9807035.

[39] J.M. Cornwall, *Phys.Rev.* **D 26** (1982) 1453; J.M. Cornwall and J. Papavassiliou, *Phys.Rev.* **D 40** (1989) 3474.

[40] V. P. Nair, hep-th/9809086

[41] G. Alexanian and V.P. Nair, Phys.Lett. **B352**, 435 (1995).

[42] W. Buchmuller and O. Philipsen, Nucl.Phys. **B443**, 47 (1995).

[43] W. Buchmuller and O. Philipsen, Phys.Lett. **B397** (1997) 112-118

[44] J.M. Cornwall, Phys.Rev. **D10**, 500 (1974); *ibid*, **D26**, 1453 (1982).

[45] R. Jackiw and S.Y. Pi, Phys.Lett. **B368**, 131 (1996).

[46] R. Jackiw and S.Y. Pi, Phys.Lett. **B403**,297 (1997).

[47] J.M. Cornwall, Phys.Rev. **D57**, 3694 (1998).

[48] F. Eberlein, hep-ph/9804460.

[49] F. Eberlein, hep-ph/9811513.

[50] A. Patkos, P. Petreczky and Zs. Szep, Eur.Phys. **C5**, 337 (1998).

[51] S. D. Glazek and K. G. Wilson, *Phys. Rev.* **D48**, 5863(1993); S. D. Glazek and K. G. Wilson, *Phys. Rev.* **D49**, 4214 (1994).

[52] D. Minic and V. P. Nair, *Int. J. Mod. Phys*A**11**, 2749 (1996).

[53] F. Wegner, *Ann. Physik* **3**, 77 (1994); E. L. Gubankova and F. Wegner, *Phys. Rev.* **D58**, 025012 (1998); T. S. Walhout, hep-th/9806097.

[54] K. G. Wilson and J. Kogut, *Phys. Rep.* **12**, 75 (1974); K. G. Wilson, *Rev. Mod. Phys.* **47**, 773 (1975).

[55] S. -Y. Pi and M. Samiullah, *Phys. Rev.* **D36**, 3128 (1987); R. Floreanini and R. Jackiw, *Phys. Rev.***D37**, 2206 (1988).

[56] G. Alexanian and E. Moreno, *"Renormalization of the Hamiltonian and a geometric interpretation of asymptotic freedom"*, hep-th/9809078.

[57] K. Symanzik, *Nucl. Phys* **B 190**, 1 (1980).

[58] R. P. Feynman, *Nucl. Phys.* **B188**, 479 (1981).

[59] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Proc. R. Soc. Lond.* **A 362**, 425 (1978).

[60] I. M. Singer, *Physica Scripta* **24**, 817 (1980);

[61] O. Babelon and C. M. Viallet, *Commun. Math. Phys.* **81** (1981); *Phys. Lett.* **B103**, 45 (1981); P. K. Mitter and C. M. Viallet, *Commun. Math. Phys.* **79**, 457 (1981); *Phys. Lett.* **B85**, 256 (1979);

[62] P. Orland, The Metric on the Space of Yang-Mills Configurations, hep-th/9607134.

[63] M. Kudinov, E. F. Moreno and P. Orland, hep-th/9709031; E. F. Moreno and P. Orland, hep-th/9812176.

[64] D. Karabali and V.P. Nair, *Nucl.Phys.* **B464**, 135 (1996); *Phys.Lett.***B379**, 141 (1996); *Int. J. Mod. Phys* **A12**, 1161 (1997); D. Karabali, C. Kim and V.P. Nair, *Nucl.Phys* **B524**, 661, (1998).

[65] D. Karabali, C. Kim and V.P. Nair, *Phys. Lett.* **B434**, 103, (1998).

[66] M. Teper, *Phys. Lett* **B311** (1993) 223; O. Philipsen, M. Teper and H. Wittig, Nucl.Phys. **B469** 445 (1996); M. Teper, hep-lat/9804008 and references therein.

[67] G. Alexanian and E. F. Moreno, hep-th/9811158, to appear in *Phys.Lett.***B**.

[68] H. Frohlich, *Advances in Physics* **3**, 325 (1954).

[69] B. D. Jones, R. G. Perry and S. D. Glazek, *Phys.Rev.* **D55**, 6561 (1997), hep-th/9605231;
B. D. Jones and R. G. Perry hep-th/9703106.

[70] K. Harada and A. Okazaki, *Phys.Rev.* **D55**, 6198 (1997), KYUSHU-HET-35, hep-th/9610020.

[71] J. L. Gervais and B. Sakita, *Phys.Rev.* **D18**, 453 (1978).

[72] M. D'Attanasio and T. R. Morris, *Phys. Lett.* **B378**, 429 (1996);M. Bonini, M. D'Attanasio and G. Marchesini, *Nucl. Phys.* **B418**, 81 (1994); *Nucl. Phys.* **B429**, 455 (1994).

[73] A. Ali and J. Bernstein *Phys.Rev.* **D12**, 503 (1975).