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by

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Abstract

NON-PERTURBATIVE METHODS IN HAMILTONIAN FORMULATION OF FIELD THEORIES

by

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We discuss two non-perturbative approaches, namely, the WKB approximation and collective field method in the framework of Hamiltonian formulation in field theories. The first part of the thesis consists of the application of a combined WKB and variational method to compact quantum electrodynamics in three space-time dimensions. The ground state wave-functional and energy are derived as also the shift in the energy due to the presence of external static charges. In the second part of the thesis, we formulate the collective field method in canonical Hamiltonian formalism and apply it to O(N) Heisenberg spin-system in two dimensions. The mass-gap and the beta-function are derived for large N. Introduction of the collective field in Lagrangian path integral formalism is also discussed and 1/N corrections computed.

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I. INTRODUCTION AND SUMMARY.

Since the success of gauge theories in explaining weak and electromagnetic interactions[1] and the observation that, at short distances, hadrons behave like a collection of almost free, point-like particles, it is rather widely believed that the theory of strong interactions must be an asymptotically free gauge theory[2]. The most promising candidate seems to be Quantum Chromodynamics (QCD), a gauge model of quarks and gluons based on gauge group SU(3). This is perhaps surprising since no experimentalist has ever seen a quark or a gluon which are the fundamental fields in QCD, nor has any theorist succeeded in calculating any hadron parameter starting from first QCD principles. What persuades the fundamental field theorists, at least, is that, of all renormalisable field theories, only non-abelian gauge theories with a certain maximum number of species of fermions are asymptotically free which seems to be essential for explaining the observed scaling phenomena.

Despite the success of QCD for short distance phenomena, there is still a very difficult question: why is it that we never observed colored objects? In spite of all the efforts the answer, namely, color confinement, is still far from being reliably proved. The difficulty is that the color confinement being a typically large-distance phenomenon, cannot be obtained by direct perturbation theory since QCD is a theory where

coupling constant increases with distance scale (infra-red slavery). There has been many attempts to circumvent this difficulty. The MIT bag model[3] of hadrons, the semiclassical approach by Callan, Dashen and Gross[4] using instanton[5], the lattice gauge theory pioneered by Wilson[6] are a few examples of the various non-perturbative approaches.

The MIT bag model is basically a phenomenological model in which an energy functional, depending on a few parameters, is chosen so as to be consistent with a physical idea that the normal vacuum expels color fields and quarks. When applied to the essentially static problem of low-lying hadron spectroscopy, one obtains a mass-formula which yields a good fit to a surprisingly large number of masses. The disadvantage of this type of treatment is that the starting point is not really a satisfactory quantum theory, and one therefore does not know how to compute quantum corrections or how to deal with non-static problems. Also, not being QCD, the model will not possess asymptotic freedom and will be unable to incorporate scaling in a natural way. Besides, the model has explicit chiral symmetry breaking and thus cannot naturally account for the pion or the successes of PCAC (partial conservation of axial-vector current).

An attempt to rectify these problems was made in the work of Callan, Dashen and Gross which provides a nice qualitative picture of hadrons as a bubble in a dense instanton liquid. Thus it is basically an attempt to "derive"

something like a bag model from QCD which then should have the best of both worlds. However, there are several unresolved problems such as how to control the severe infra-red divergences and most importantly, whether the known instanton field configurations are in principle sufficient to lead to the picture of color confinement.

Another promising attempt seems to be the "lattice gauge theories" in which the ordinary space-time is divided into a four dimensional lattice. An action on the lattice can be defined and quantities like the Green's functions (correlation functions in the euclidean version) can be computed numerically. Essentially the finite lattice spacing provides a cut-off and this is just another way of regularising the divergences of the field theory. At the end of the calculation the original continuum theory is recovered by letting the lattice spacing go to zero. Strong coupling expansion is usually easier in the lattice formulation and confinement has been demonstrated in this limit. However, there are still some crucial questions to be answered. Does this strong coupling confinement survive in the continuum weak coupling limit or does the theory undergo a phase transition at some critical coupling constant ? Is the strong coupling behaviour compatible with the weak coupling asymptotic freedom we get from the continuum perturbation theory ? How does one, in practice, incorporate fermions on the lattice ? All these problems are now being attempted[7] and the main hope seems to lie in the

various numerical calculations being done on computers. But till now only a very few reliable numerical results are available and one has certainly to wait for some time for a concrete overall picture to emerge.

In this thesis, we attempt to develop the canonical Hamiltonian formulation for field theories including gauge theories keeping in mind that a suitable non-perturbative approach is necessary for answering questions like confinement, dynamical mass generation, etc. Specifically, the two non-perturbative approaches we investigate in this thesis are WKB approximation and collective field method.

Ever since the existence of multiple vacua[8] in Yang-Mills theory was discovered, it was conjectured that there might be tunnelling between these different vacua and the true ground state might be a superposition of all these different vacuum states. This tunnelling phenomenon in gauge theories was formulated in terms of the saddle-point approximation of the euclidean functional integration formalism[4,9] of gauge theories. These saddle-points are obtained as the solutions of the classical field equations with euclidean metric. They are extended solutions in euclidean space-time and are known as "instantons"[5], sometimes also referred to as "pseudo-particles". This method of euclidean functional integral is advantageous since many techniques of statistical mechanics can be used and also Lorentz covariance is explicitly maintained. However, it is rather difficult to get the physical

intuitive picture of the tunnelling phenomena, which are familiar to us in non-relativistic quantum mechanics using the language of wave-functions and Schrodinger equation. The main problem to discuss field theoretic tunnelling in this language is that field theory has essentially infinite degrees of freedom and so one has to know how to handle wave-function with infinitely many degrees of freedom (i.e, wave-functional).

Since quantum mechanical tunnelling is most appropriately described by WKB approach, Gervais and Sakita[10] developed the WKB method for constucting wave-functions for many-body systems. The main idea was that the instantons, being classical solutions with imaginary time, are nothing but tunnelling events in the Minkowski space. So the WKB wave-functions are constructed by including quantum fluctuations around the instanton solution.

In the first part of the thesis we follow the procedure for constructing the ground state wave-functional of compact quantum electrodynamics in three space-time dimensions[11]. The model we have in mind is the Georgi-Glashow theory of weak and electromagnetic interactions based on gauge group SO(3) which, in the four dimensional version, has magnetic monopoles discovered by 't Hooft and Polyakov[12]. The static monopole is an instanton in the three dimensional model and Polyakov[13] showed that this model is equivalent to a three dimensional Coulomb gas. The key point is to take the long-range interactions between the pseudo-particles into account

which generates a mass-gap that vanishes non-analytically as the coupling constant tends to zero. The same phenomenon takes place also in the three dimensional abelian lattice gauge model as was demonstrated by Banks, Myerson and Kogut[13a].

That the occurance of multiple classical vacua is implied by the existence of instantons in this model is clear if one goes to a particular canonical gauge (for example, the $A_0=0$ gauge). The different vacua are labelled by an integer n and an instanton (corresponding to a magnetic monopole of charge unity) takes the system from a nth to (n+1)th vacuum when the euclidean time τ goes from $-\infty$ to $+\infty$. So we construct the ground state wave functional in the classically forbidden region by the straight-forward WKB method. The main idea is to use collective coordinates. To explain how it works, let us assume that the Hamiltonian is invariant under a symmetry group G (which may contain gauge transformations, translation, etc) which has parameters denoted by the set $\{\omega^\alpha\}$. Let us assume that the instanton solution, denoted by $\phi_i(\vec{x},x_4=\tau)$ breaks the invariance under G completely. The idea of collective coordinate is to reestablish the symmetry by introducing the parameters ω^α of G as dynamical variables, since we know that the tunnelling will connect the different vacua and the true ground state will be a superposition of all the different vacua. The extra set of dynamical variable ω^α are needed because the degrees of freedom associated with symmetry transformations having no restoring forces can undergo arbitrary

large fluctuations and this cannot be treated by the usual quadratic approximation. However, these set of variables ω^{α} are not sufficient if one has tunnelling from one vacuum to another, since in that case there are other large fluctuations which are not symmetry transformations. To handle this, we introduce the euclidean time τ as a real dynamical variable which describes the essential part of tunnelling. One can then follow the standard WKB procedure to construct the ground state wave functional. However, the existence of massless particles show up in the fact that the wave functional is rather flat in the directions of the infra-red modes and we end up with a result which is the same as would have been obtained by a dilute gas approximation in the euclidean functional integral method where pseudo-particle interactions are neglected.

The reason that the straight-forward WKB method for this model is not quite satisfactory lies in the fact that the long range infra-red modes are not treated properly. To overcome this difficulty, we propose a modified WKB-variational treatment which treats the low-frequency modes correctly. The essence of this method is to use a modified Hamiltonian with constraint. The constraint just reflects the fact that there are instantons and anti-instantons in the theory. When we find the WKB wave-function by quantising around the instanton, the constraint is automatically satisfied by the classical solution. However, we now demand that the constraint be satisfied by the complete field variable including quantum

fluctuations. Thus, although we neglect the pseudoparticle interactions at the classical level, the quantum fluctuations automatically takes this into account. This is not unusal in quantum field theory. A familiar example is the O(3) non-linear sigma model. Here the multi-instanton exact solutions are known. The instantons do not interact classically. However, when one computes the euclidean vacuum-to-vacuum transition amplitude in the two-instanton sector[14], one finds that the instantons interacts through quantum fluctuations. The reason is that in this model the constraint ($\vec{\sigma}^2$=constant) is satisfied by the total $\vec{\sigma}$ field.

Let us point out that the variational procedure we use is not quite the standard variational method in quantum mechanics where one makes an ansatz for a trial wave-function and minimises the expectation value of the Hamiltonian in this trial state. We modify the Hamiltonian itself with constraint; the wave-function is a solution of the Schrodinger equation with this modified Hamiltonian. Since we use a Lagrange multiplier for introducing the constraint, the wave-function now depends on the Lagrange multiplier which we determine at the end by minimising the expectation value of the modified Hamiltonian. We show that the ground state energy obtained this way is the same as the expectation value of the original Hamiltonian in the given state. Our method is equivalent to the ordinary variational procedure in quantum mechanics in the sense that the energy obtained is an upper-bound for the

true ground state energy and agrees completely with Polyakov's result obtained by steepest descent approximation to the euclidean functional integral.

In the second part of the thesis we persue yet another non-perturbative approach which is the collective field method[15]. The motivation is the following. In the WKB method for realistic four dimensional gauge theories like QCD, even if one could control the infra-red behaviour, it is not clear that the instanton field configurations alone will lead to color confinement. In fact, people have tried to include other singular field configurations (for example, merons[16]) in order to derive confinement. Besides, there are other complications due to topology in discussing which are the important relevant regions of field space in non-abelian gauge theory. For example, the phenomenon of Gribov ambiguity[17] implies that the standard gauge fixing is incomplete in the sense that one might inadvertently count equivalent field configurations many times and entirely miss some others. So it seems important to develop an approach which deals only with gauge invariant operators, so that gauge fixing ambiguities do not aries. Collective field is precisely such an approach.

Another motivation for using collective field is that it seems to be a natural framework for discussing 1/N expansion. 't Hooft[18] first demonstrated that the two dimensional U(N) quantum chromodynamics can be realised as a model for mesons in the limit $N\to\infty$. This is achieved by summing all

planar diagrams and has been explored in great details in other models like the N-component vector model[19,19a], various matrix models[20], etc. However, summing planar diagrams for four dimensional QCD is technically rather difficult and the collective field method seems to be a promising alternative to achieve the large N limit.

To illustrate the idea of collective field, let us take a simple system, viz, a collection of N Bose particles in one dimension. The usual way to solve this problem is, of course, to solve the Schrodinger equation directly and select the totally symmetric wave-function. Another alternative is to regard the wave-function as a function of all possible symmetric combination of the coordinates. This way the Bose symmetry is built into the formulation. So the essence is to consider a most general set of commuting operators which are invariant under the symmetry, and explicitly perform a canonical transformation to this new set. Thus, in the example we are discussing, this new set could be the density operator $\rho(q)=\sum_{i=1}^{N}\delta(q-q_i)$ where q_1, q_2, etc are the coordinates. We call this the collective field. The reader should notice that the collective variables are not all independent (originally we had only N degrees of freedom) and the change of variables will necessarily involve a jacobian in the inner product defined in the Hilbert space of states. It is at this point that the large N limit comes in. In this limit, the new set of variables become almost independent and one can define a

field theory in terms of $\rho(q)$ which truly represents the original quantum mechanical system. This is the collective field theory.

So one has to reformulate the Hamiltonian in terms of the new collective field and also find the jacobian of transformation. We have worked out the large N limit of the O(N) Heisenberg spin-system in two dimensions (popularly also known as the non-linear sigma model among field theorists). The spin-spin correlation functions have been obtained in the large N limit. We also obtain the mass-gap and the Callan-Symanzik β-function, and the results agree with the known weak and strong coupling limits.

It is also possible to introduce the collective field in the path-integral formalism where a systematic 1/N expansion becomes apparent. In the Hamiltonian formulation, one has to face the problem that, the collective fields being not all independent for finite N, there might be 1/N correction to the Hamiltonian which are rather difficult to evaluate. In the path-integral method, one introduces the collective field through δ-function condition in the functional integral so one can keep track of all 1/N corrections. Of course, the zeroth order result agrees with that from the Hamiltonian approach but, in addition, a systematic 1/N expansion can be done. We have obtained the leading 1/N correction to the mass gap and the β-function in the O(N) model.

The way the chapters are arranged is as follows. In chapter II, we discuss the WKB method by first working out

a simple quantum mechanical toy model (the double well potential) and then going into compact QED in full details. The ground state wave-functional is derived and the energy shift of the ground state due to presence of external static charges is computed. Since this does not yield the correct result (the existence of a mass-gap), we introduce the modified WKB-variational ansatz for the ground state and derive the correct interaction energy between the external charges. In chapter III, we introduce the collective field theory and work out the Hamiltonian formulation of the O(N) sigma-model. The correlation function, the mass-gap and the beta-function are derived in the large N limit. We then discuss the collective field in path-integral approach and compute 1/N corrections to the relevant quantities. Appendices A and B give certain mathematical details.

II. WKB FORMULATION IN FIELD THEORY.

A. Introduction.

It is known for sometime that non-abelian gauge theories possess a much richer structure than what is apparent from ordinary weak coupling perturbation theory. For example, the existence of multiple vacua in Yang-Mills theory almost inevitably compels us to consider the phenomenon of tunnelling in gauge theories. Unfortunately, the method which is very successful in describing tunnelling in quantum mechanics, that is, the WKB approximation, is rather difficult to generalise to multidimensional problems. In field theory, one has essentially infinite number of degrees of freedom and consequently, the standard practice has been to use the saddle-point approximation in the euclidean functional integral to formulate the problem of tunnelling.

In recent years, however, several authors have constructed the WKB wave-function for many-body systems and some model field theories possessing instanton solutions. The central idea is that the instantons, beimg classical solutions with imaginary time, are nothing but tunnelling events in the Minkowski space. Thus the Wkb wave-function is constructed by including quantum fluctuations around the instanton solution. The phenomenon of tunnelling is much more transparent in this formalism and one gets the same result as is obtained by the euclidean path-integral approach in the dilute gas approximation.

Since this way of describing tunnelling is slightly different from the conventional WKB approach, we shall first work out a quantum mechanical toy model to illustrate the essential point. This method is due to Callan and Coleman[20a]. In the next section, a full-fledged

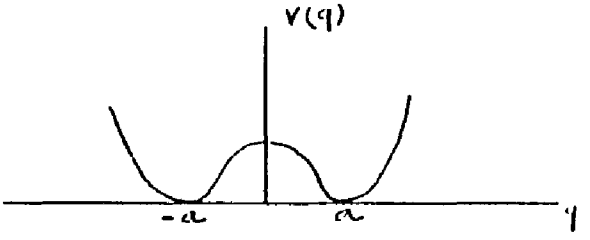
field theoretic model will be discussed in greater details.

B. Double well potential: A toy model.

The model is described by the hamiltonian

$$H = \frac{1}{2} p^2 + V(q) \tag{2.1}$$

where the potential V(q) has the general shape as in figure 1.



V(q) has minima at $q=\pm a$ which are degenerate classical ground states. Near the minima, we assume the potential to be harmonic:

$$V(q) \simeq \frac{1}{2}w^2(q-a)^2 \quad \text{for} \quad q \simeq a \tag{2.2}$$

The conventional way of introducing a parameter g (coupling constant) is as follows. Let us define a function v(x) which is independent of g. Then we shall assume that V(q) can always be written as

$$V(q) = v(gq)/g^2 \tag{2.3}$$

From (2.3) it is obvious that g is related to the height of the barrier. Weak coupling (small g) means a large barrier.

The existence of solutions to euclidean (imaginary time) classical equations of motion can be realised by considering the system with inverted potential (-V(q)). It is easy to see that there will be a solution of zero energy which goes from one classical vacuum to (say, q= -a) at imaginary time $\tau=-\infty$ to the other vacuum (q=a) at $\tau=+\infty$. This is the instanton solution and we denote by $q_{cl}(\tau)$:

$$\partial_\tau^2 q_{cl}(\tau) = \partial V(q_{cl})/\partial q_{cl} \tag{2.4}$$

$$-\frac{1}{2}(\partial_\tau q_{cl}(\tau))^2 + V(q_{cl}(\tau)) = 0 \tag{2.5}$$

$$q_{cl}(\tau=\pm\infty) = \pm a \tag{2.6}$$

The first of these equations is the equation of motion in euclidean time. The second is the condition of zero energy. The third is the initial condition. It is easy to see that equation (2.2) predicts a large τ behaviour as

$$q_{cl}(\tau) \underset{\tau\to\infty}{\simeq} a - Ke^{-W\tau} \tag{2.7}$$

All these can be made a little more transparent by considering specific models satisfying all the necessary conditions (2.2) and (2.3):

Model a: $$V(q) = g^2(q^2-m^2/g^2)^2/4 \tag{2.8}$$

Model b: $$V(q) = m^2(|q|-m/g)^2 \tag{2.9}$$

Both these models satisfy the condition (2.3). The frequency w and the minimum a are given in both cases by

$$w = \sqrt{2}m = \sqrt{2}ag \tag{2.10}$$

the instanton solution satisfying (2.4), (2.5) and (2.6) looks like

Model a: $$q_{cl}(\tau) = m\tanh(w\tau/2)/g \tag{2.11}$$

Model b: $$q_{cl}(\tau) = m(1-e^{-w|\tau|})\epsilon(\tau)/g \tag{2.12}$$

Let us use a rescaled variable

$$Q = gq \tag{2.13}$$

From (2.3) and (2.4) we find the equation of motion for $Q_{cl}(\tau)$:

$$\partial_\tau^2 Q_{cl}(\tau) = \partial v(Q_{cl})/\partial Q_{cl} \tag{2.14}$$

which means that $Q_{cl}(\tau)$ does not depend on g. Thus,

$$a, K \sim O(1/g) \tag{2.15}$$

The Schrodinger equation we have to solve is

$$(-\tfrac{1}{2}\partial_q^2 + V(q))\phi(q) = E\phi(q) \tag{2.16}$$

In terms of variable Q, this looks like

$$\frac{1}{g^2}(-\tfrac{1}{2}g^4\partial_Q^2 + v(Q))\psi(Q) = E\psi(Q) \tag{2.17}$$

where $\psi(Q)=\phi(q)$. So g^2 plays the role of $\hbar$. We can now make the WKB ansatz

$$\begin{aligned} S(Q) &= \ln\psi(Q) = \frac{1}{g^2}S_0 + S_1 + \ldots. \\ E &= \frac{1}{g^2}E_0 + E_1 + \ldots.. \end{aligned} \tag{2.18}$$

The Schrodinger equation becomes

$$\begin{aligned} -\tfrac{1}{2}(S_0')^2 + v(Q) &= E_0 \\ -\tfrac{1}{2}(S_0'') - S_0'S_1' &= E_1, \text{ etc} \end{aligned} \tag{2.19}$$

To solve these equations, we note that the first one is just the Hamilton-Jacobi equation with energy E_0. Since in our case $E_0=0$, we get

$$S_0(Q) = \int^Q dQ'\sqrt{2(v-E_0)} = \pm g^2\int^\tau d\tau'(\partial_\tau q_{cl})^2 \tag{2.20}$$

and S_1 is

$$\begin{aligned} S_1 &= \text{constant} - \tfrac{1}{2}\ln S_0' - E_1\int dQ'(\partial S_0/\partial Q)^{-1} \\ &= \text{constant} - \ln(|\partial_\tau q_{cl}(\tau)|)^{\frac{1}{2}} \mp E_1\tau \end{aligned} \tag{2.21}$$

One can see that our wave-function is written explicitly in terms of τ. In order to construct the ground state wave-function which must be a symmetric function of q, the appropriate choice is

$$\psi_{WKB} = \psi^{+}_{WKB} + \psi^{-}_{WKB} = \frac{A}{\sqrt{|q_{cl,\tau}|}}(e^{\int_0^\tau (q_{cl,\tau})^2 d\tau - E\tau} + e^{-\int_0^\tau (q_{cl,\tau})^2 d\tau + E\tau} \tag{2.22}$$

where E is the energy and A is a constant. Note that τ=0 corresponds to q=0. For large τ, we note that

$$\int_0^\tau (q_{cl,\tau})^2 d\tau' = \tfrac{1}{2}\int_{-\infty}^{\infty} (q_{cl,\tau})^2 d\tau' - \int_\tau^\infty (q_{cl,\tau})^2 d\tau' \tag{2.23}$$

The classical action corresponding to $q_{cl}(\tau)$ is

$$S_{cl} = \int_{-\infty}^{\infty} (\tfrac{1}{2}(q_{cl,\tau})^2 + V(q_{cl}))d\tau' = \int_{-\infty}^{\infty} (q_{cl,\tau})^2 d\tau' \tag{2.24}$$

where we have used equation (2.5). So the first integral in (2.23) is just $\frac{1}{2}S_{cl}$. The second integral can be written as

$$\int_\tau^\infty (q_{cl,\tau})^2 d\tau' = \int_\tau^a dq\sqrt{2V(q)} = \tfrac{1}{2}wx^2 \tag{2.25}$$

where x=a-q is the deviation from the minimum. Now we can use (2.7) to write, for large τ,

$$\psi^{+}_{WKB} = \frac{A}{\sqrt{|q_{cl,\tau}|}} e^{\int_0^\tau (q_{cl,\tau})^2 d\tau' - E\tau}$$

$$\underset{\tau\to\infty}{=} A\exp(\tfrac{1}{2}S_{cl} - \tfrac{1}{2}wx^2 - \tau E + \tfrac{1}{2}\tau w)/\sqrt{wK} \tag{2.26}$$

and, for ψ^{-}_{WKB}

$$\psi^{-}_{WKB} \underset{\tau\to\infty}{\simeq} A\exp(-\tfrac{1}{2}S_{cl}+\tfrac{1}{2}wx^2+\tau E+\tfrac{1}{2}\tau w)/\sqrt{wK} \tag{2.27}$$

In the absence of tunnelling, that is, when the barrier is infinite ($g\to 0$), the actual ground state energy E should be equal to the ground state energy of a simple harmonic oscillator of frequency w. Thus, for weak coupling (small g), we can write

$$E = \tfrac{1}{2}w + \epsilon \qquad \epsilon = \text{small}. \tag{2.28}$$

Then, to lowest order,

$$\begin{aligned} \psi^{+}_{WKB} &\underset{\tau\to\infty}{\simeq} B\exp(\tfrac{1}{2}S_{cl}-\tfrac{1}{2}wx^2)/\sqrt{K} \\ \psi^{-}_{WKB} &\underset{\tau\to\infty}{\simeq} B\sqrt{K}\exp(-\tfrac{1}{2}S_{cl}+\tfrac{1}{2}wx^2)/\sqrt{x} \end{aligned} \tag{2.29}$$

In order to find the ground state energy E, we have to match the WKB wave function with the wave-function we obtain by solving the Schrodinger equation in the harmonic oscillator region, i.e, around q=a. Then we can write

$$(-\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}w^2x^2)\psi = E\psi \tag{2.30}$$

If there were no tunnelling, ψ would be a gaussian peaked around x=0. However, since we have a finite barrier, ψ will also have a small component which increases as we go away from the minimum. Thus we shall have

$$\psi = \psi_0 + \psi_1 \tag{2.31}$$

$$\psi_0 = (w/\pi)^{\frac{1}{4}} e^{-\frac{1}{2}wx^2} \tag{2.31a}$$

$$\psi_1 = \text{small, of order } \epsilon \tag{2.31b}$$

then, the equations (2.30) and (2.28) give

$$(-\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}w^2x^2 - \tfrac{1}{2}w)\psi_0 = 0 \tag{2.32}$$

$$(-\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}w^2x^2 - \tfrac{1}{2}w)\psi_1 = \epsilon\psi_0 \tag{2.33}$$

One should notice that, for $\tau \to \infty$,

$$\psi^{+}_{WKB} \to \psi_0 \tag{2.34}$$

which determines constant B as

$$Be^{\frac{1}{2}S_{cl}}/\sqrt{K} = (w/\pi)^{\frac{1}{4}} \tag{2.35}$$

So, by construction, ψ^{-}_{WKB} should match ψ_1. Going back to equations (2.32) and (2.33), we can determine ϵ as

$$\begin{aligned}\epsilon\psi_0^2 &= \psi_0(-\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}w^2x^2 - \tfrac{1}{2}w)\psi_1 - \psi_1(-\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}w^2x^2 - \tfrac{1}{2}w)\psi_0 \\ &= -\tfrac{1}{2}\partial_x(\psi_0 \overleftrightarrow{\partial}_x \psi_1)\end{aligned} \tag{2.36}$$

Let us integrate both sides from $-x_0$ to x_0, where x_0 is in the overlap region where both WKB and harmonic oscillator wave-function are valid, and, at the same time, x_0 is large enough so that

$$\int_{-\infty}^{x_0}\psi_0^2 dx \simeq \int_{-\infty}^{\infty}\psi_0^2 dx = 1 \tag{2.36a}$$

Then, $$\varepsilon \simeq \int_{-\infty}^{x_0}\partial_x(\psi_0\partial_x\psi_1)dx = -\tfrac{1}{2}(\psi^+_{WKB}\partial_x\psi^-_{WKB})_{x_0} \tag{2.36b}$$

Using (2.29),

$$\begin{aligned}\partial_x\psi^+_{WKB} &\simeq -Bwx\,\exp(\tfrac{1}{2}S_{cl}-\tfrac{1}{2}wx^2)/\sqrt{K}\\ \partial_x\psi^-_{WKB} &\simeq B\sqrt{K}w\,\exp(-\tfrac{1}{2}S_{cl}+\tfrac{1}{2}wx^2)\end{aligned} \tag{2.37}$$

we get

$$\varepsilon \simeq -w\sqrt{(w/\pi)}Ke^{-S_{cl}} \tag{2.38}$$

Thus the energy shift ε has the expected barrier penetration factor $e^{-S_{cl}}$ and is independent of the matching point x_0. It is quite straightforward to show that one gets the same result by doing a functional integral calculation using dilute gas of instantons and anti-instantons.

C.1. Compact Quantum Electrodynamics in (2+1) dimensions[11].

The specific model we will be interested in is compact QED in

(2+1) dimensions. The existence of a mass-gap in this model was first demonstrated by Polyakov using euclidean functional integral technique. His crucial observation was that one has to take into account the long-range interaction between instantons and anti-instantons. Simple dilute gas of pseudoparticles without interactions was not enough; contrary to other confining models without massless particles.

In this section, we shall apply the conventional WKB method (generalisation of the method we have used in the last section) to find the ground state wave-function in the presence of external static electric charge distribution. Using this wave-function we shall compute the shift in the ground state energy in presence of external charges and show that it is the same as obtained in the euclidean path integral approach if one neglects the pseudoparticle interactions. The reason that this conventional WKB approach is not good enough lies in the neglect of long-range interactions. The photons being massless, there are infra-red modes for which the standard WKB method does not apply. To find the correct wave-function for the infra-red modes, we shall use a combined WKB-variational approach.

The main idea is to use a Hamiltonian with contraint. The constraint is just the mathematical statement that there are instantons and anti-instantons in the theory. When we find the WKB wave-function by quantising around the instanton solution, the constraint is automatically satisfied by the classical solution. However, we shall now demand that the constraint be satisfied by the complete field variable including quantum fluctuations. Thus we will be neglecting interaction between the pseudoparticles at the classical level, the quantum fluctuation will include the interaction effect.

C.2. Hamiltonian Formulation.

The model is QED in the Georgi-Glashow theory of weak and electromagnetic interaction based on gauge group SO(3). The Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4}(G^a_{\mu\nu})^2-\tfrac{1}{2}(D_\mu\chi)_a(D_\mu\chi)_a-V(\vec{\chi}) \tag{2.39}$$

where

$$G^a_{\mu\nu} = \partial_\mu A_\nu-\partial_\nu A_\mu+g\varepsilon_{abc}A^b_\mu A^c_\nu \tag{2.40}$$

$$(D_\mu\chi)_a = \partial_\mu\chi_a+g\varepsilon_{abc}A^b_\mu\chi_c \tag{2.41}$$

$$D(\vec{\chi}) = \frac{1}{8}\lambda g^2(\chi_a\chi_a-F^2/g^2)^2 \qquad a,b,c=1,2,3. \tag{2.42}$$

We have defined the potential term $V(\vec{\chi})$ in such a way that

$$V(\vec{\chi}) = \frac{1}{g^2}v(g\vec{\chi}) \tag{2.43}$$

where $v(\vec{\chi})$ is independent of g. Thus g^2 is the conventional semiclassical expansion parameter for fixed λ. In the usual perturbation theory, one expands around the minimum of V as

$$\chi_a = \upsilon_a + \eta_a \tag{2.44}$$

where v_a is the vacuum expectation value of the Higgs field. Choosing $\vec{v}$ in the third direction in the isospin space

$$v_a = \frac{F}{g}\delta_{a3} \tag{2.45}$$

we get a massless photon A_μ^3, heavy charged boson field $W_\mu^\pm = A_\mu^1 \pm iA_\mu^2$ of mass $m_W = F$ and a massive neutral scalar field η_3 of mass $m_\eta = F\sqrt{\lambda}$.

To discuss the canonical Hamiltonian formalism, we work in the $A_0^a = 0$ gauge. Then a point in the field configuration space is denoted by nine real fields A_i^a (a=1,2,3; i=1,2) and χ_a (a=1,2,3). We use compact notation $\vec{\phi}$ for all of them as

$$\vec{\phi}(\vec{x}) = \begin{bmatrix} A_1^a \\ A_2^a \\ \chi_a \end{bmatrix} \tag{2.46}$$

Under a gauge transformation $U(\vec{\lambda}) = \exp(i\vec{\lambda}.\vec{\tau}/2)$ where $\vec{\lambda}$ are the gauge parameters and $\vec{\tau}$ are the Pauli matrices, the original fields A_μ^a and χ_a transform as

$$\begin{aligned} \chi_a\tau_a/2 &\to \chi'_a\tau_a/2 = U(\chi_a\tau_a/2)U^{-1} \\ A_\mu^a\tau_a/2 &\to A_\mu^{a\prime}\tau_a/2 = U(A_\mu^a\tau_a/2)U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \end{aligned} \tag{2.47}$$

Thus the nine component field $\vec{\phi}(\vec{x})$ transforms as

$$\vec{\phi}(x) \to \vec{\phi}(x) = \vec{\phi}_{\{\vec{\lambda}\}}(x) = \vec{\phi}_{(\vec{\lambda})}(x) + \delta\vec{\phi}_{(\vec{\lambda})}(x) \tag{2.48}$$

where $\vec{\phi}_{(\vec{\lambda})}(x)$ is the homogeneous term given by

$$\vec{\phi}_{(\vec{\lambda})}(x) = \begin{bmatrix} \theta(\vec{\lambda}) & 0 & 0 \\ 0 & \theta(\vec{\lambda}) & 0 \\ 0 & 0 & \theta(\vec{\lambda}) \end{bmatrix} \begin{bmatrix} A_1^a \\ A_2^a \\ \chi_a \end{bmatrix} \tag{2.49}$$

with

$$\theta(\vec{\lambda}(x))_{ab} = \delta_{ab}\cos|\vec{\lambda}| + \varepsilon_{abc}\sin|\vec{\lambda}|\frac{\lambda c}{|\lambda|} + 2\sin^2\frac{|\vec{\lambda}|}{2}\lambda_a\lambda_b \tag{2.50}$$

and the inhomogeneous term $\delta\vec{\phi}_{(\vec{\lambda})}(x)$ is given by ($\vec{\lambda}=\hat{n}|\vec{\lambda}|$)

$$\delta\vec{\phi}_{(\vec{\lambda})}(x) = \frac{1}{g}\begin{bmatrix} n_a\partial_1\lambda + \sin\lambda(\partial_1 n_a) + 2\sin^2\frac{\lambda}{2}(\partial_1\hat{n}x\hat{n})_a \\ n_a\partial_2\lambda + \sin\lambda(\partial_2 n_a) + 2\sin^2\frac{\lambda}{2}(\partial_2\hat{n}x\hat{n})_a \\ 0 \end{bmatrix} \tag{2.51}$$

We shall also use a compact notation $\vec{\Pi}(\vec{x})$ for momenta corresponding to $\vec{\phi}(\vec{x})$. Thus

$$\vec{\Pi}(\vec{x}) = -i\frac{\delta}{\delta\vec{\phi}(\vec{x})} = \begin{bmatrix} E_1^a(x) \\ E_2^a(x) \\ \Pi_{\chi_a}(x) \end{bmatrix} \tag{2.52}$$

With these notations, the Hamiltonian in the $A_0^a = 0$ gauge is[21]

$$H = \int d^2x \left\{\tfrac{1}{2}\sum_{\alpha=1}^{9}\Pi_\alpha\Pi_\alpha\right\} + V\{\phi(.)\} \tag{2.53}$$

where the potential energy term is

$$V\{\phi(.)\} = \int d^2x \left\{\tfrac{1}{4}(G_{ij}^a)^2 + \tfrac{1}{2}(D_i\chi)_a(D_i\chi)_a + V(\vec{\chi})\right\} \tag{2.54}$$

Since we are working in the $A_0^a = 0$ gauge, time-independent gauge transformations are still a symmetry of the potential $V\{\phi(.)\}$. The potential V is also invariant under space translation. Thus the local symmetry group, denoted by G, has parameters denoted by ω^α which is actually a set of three arbitrary time-independent functions $\vec{\Lambda}(\vec{x})$ and two space translation parameters X_1 and X_2. So,

$$\{\omega^\alpha\} = \{\ \omega^{i\vec{x}} = \Lambda_i(\vec{x}),\ \omega^{1,2} = X_{1,2}\ \} \tag{2.55}$$

Under G, $\vec{\phi}(\vec{x})$ transforms as

$$\vec{\phi} \to \vec{\phi}_{\{\omega\}}(\vec{x}) = \vec{\phi}_{(\omega)}(\vec{x}) + \delta\vec{\phi}_{(\omega)}(\vec{x}) \tag{2.56}$$

with

$$\vec{\phi}_{(\omega)}(\vec{x}) = \vec{\phi}_{(\vec{\Lambda}(\vec{x}-\vec{X}))}(\vec{x}-\vec{X}) \tag{2.57}$$

$$\vec{\phi}_{\{\omega\}}(\vec{x}) = \vec{\phi}_{\{\vec{\Lambda}(\vec{x}-\vec{X})\}}(\vec{x}-\vec{X}) \tag{2.58}$$

We write down the Schrodinger equation for the wave-functional of the ground state

$$\{\tfrac{1}{2}\int d^2x\ (-i\frac{\delta}{\delta\phi_\alpha(x)})^2 + V\{\phi(.)\}\}\Psi\{\vec{\phi}\} = E\Psi\{\vec{\phi}\} \tag{2.59}$$

To do the WKB formalism, we consider a particular imaginary time classical solution (instanton) denoted by $\vec{\phi}_I(\vec{x},\tau)$. This is nothing but the

monopole solution of 't Hooft and Polyakov[12] transformed to $A_o^a = 0$ gauge. Here τ is the imaginary (euclidean) time x_3. Before actually solving for the WKB wave-functional, we shall require a few important properties of the classical solution which we will discuss in the next section.

C.3. Euclidean Classical Solution.

The static magnetic monopole solution of 't Hooft and Polyakov in (3+1) dimensions is

$$A_i^a = \varepsilon_{iab} x_b \frac{K(x)-1}{gx^2} \tag{2.59}$$

$$x_a = x_a \frac{u(x)}{gx} \tag{2.60}$$

$$x = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \tag{2.61}$$

Boundary conditions for the functions u(x) and K(x) are as follows:

$$u(x) \underset{|x|\to\infty}{\to} F + O(e^{-m_n x}) \tag{2.62}$$

$$K(x) \underset{|x|\to\infty}{\to} 0 + O(e^{-m_w x}) \tag{2.63}$$

$$u(x) \underset{|x|\to\infty}{\to} 0 \tag{2.64}$$

$$K(x) \underset{|x|\to\infty}{\to} 1 \tag{2.65}$$

We now replace $x_3 \to \tau$ so that $x = (x_1,x_2,\tau) = (\vec{r},\tau)$ where $\vec{r}$ is a two dimensional vector. To go to $A_0 = 0$ gauge, we make a gauge transformation

$$A_i^a \to (A_{\{\vec{\rho}\}})^a \tag{2.66}$$

where $\vec{\rho} = \vec{\rho}(\vec{r},\tau)$ is a gauge parameter. We demand

$$(A_{\{\vec{\rho}\}})_3^a = 0 \tag{2.66a}$$

which implies

$$U(\rho)(A_3^a\tau_a/2)U^{-1}(\rho) = \frac{i}{g}(\partial_\tau U(\rho))U^{-1}(\rho) \tag{2.66b}$$

With the ansatz $\vec{\rho} = \rho(\vec{r},\tau)\hat{n}$, $n_a = \epsilon_{3ab}x_b/r$, the equation satisfied by ρ is

$$\partial_\tau \rho(\vec{r},\tau) = \frac{r(1-K(\vec{r},\tau))}{r^2+\tau^2} \tag{2.67}$$

whose solution is

$$\rho(\vec{r},\tau) = \rho(\vec{r},-\infty) + \int_{-\infty}^{\tau} d\tau \, \frac{r(1-K(\sqrt{r^2+\tau^2}))}{r^2+\tau^2} \tag{2.68}$$

Choice of the integration constant $\rho(\vec{r},-\infty)$ reflects the further gauge freedom of time-independent gauge transformations. We shall choose

$$\rho(\vec{r},-\infty) = 0 \tag{2.69}$$

Then we get

$$\rho(\vec{r},\tau) = (\tan^{-1}\frac{\tau}{r} + \frac{\pi}{2}) - \int_{-\infty}^{\tau} d\tau \, \frac{rK}{r^2+\tau^2} \tag{2.70}$$

For $\tau \to -\infty$, $\tan^{-1}\tau/r \to -\pi/2$, and so $\rho \to 0$. For $\tau \to \infty$, $\tan^{-1}\tau/r \to \pi/2$ and

$$\rho(\vec{r}) = \rho(\vec{r},\infty) = \pi - f(\vec{r}) \tag{2.71}$$

$$f(\vec{r}) = \int_{-\infty}^{\infty} d\tau \, \frac{rK(\sqrt{r^2+\tau^2})}{r^2+\tau^2} \tag{2.72}$$

The function $K(x)$ is 1 at $x = 0$ and rapidly goes to zero for $x >> 1/m_W$. Thus,

$$f(\vec{r})_{r>>1/m_W} \overset{=}{} 0 \tag{2.73}$$

For $r \to 0$, there is a contribution to the τ-integral around $\tau \simeq 0$ and one can show that

$$f(\vec{r})_{r<<1/m_W} \overset{\rightarrow}{} \pi - (\text{constant})r \tag{2.74}$$

We shall assume m_W is quite large so that $\rho(r,\infty)$ is equal to π everywhere except at the core of the monopole.

The multiple valuedness of $\rho(r,\infty)$ is evident from the argument of the $\tan^{-1}$ function. If we had chosen

$$\tan^{-1}\tau/r \to (n-\tfrac{1}{2})\pi \text{ for } \tau \to -\infty \tag{2.75}$$

then, for same n,

$$\tan^{-1}\tau/r \to (n+\tfrac{1}{2})\pi \text{ for } \tau \to \infty \tag{2.76}$$

This is simply connected with the multiplicity of the classical ground state of the system. The integer n labels the different vacua and the instanton takes the system from nth to (n+1)th vacuum. We shall from now on restrict ourselves to n=o vacuum. The instanton looks like

$$\vec{\phi}_I(\vec{r},\tau) = \begin{vmatrix} A^a_{1I}(\vec{r},\tau) \\ A^a_{2I}(\vec{r},\tau) \\ x_{aI}(\vec{r},\tau) \end{vmatrix} \tag{2.77}$$

with

$$x_{aI}(\vec{r},\tau) = \frac{u(x)}{g}(\hat{x}_a\cos\rho + \frac{x}{r}(\hat{x}_3\hat{x}_a - \delta_{a3})\sin\rho) \tag{2.78}$$

$$A^a_{iI}\tau^a/2 = \frac{K-1}{gx^2}\{\cos\rho(\tfrac{\vec{\tau}}{2}\wedge\vec{x})_i - n_i(\vec{x}.\tfrac{\vec{\tau}}{2})\sin\rho + 2\sin^2\tfrac{\rho}{2}(\hat{n}.\tfrac{\vec{\tau}}{2})(\delta_{i3}\frac{x^2}{r} - \frac{x_3x_i}{r})\}$$

$$+ \frac{1}{g}\{(\hat{n}.\tfrac{\vec{\tau}}{2})\partial_i\rho + (\partial_i\hat{n}.\tfrac{\vec{\tau}}{2})\sin\rho + 2(\partial_i\hat{n}\wedge\hat{n}).\tfrac{\vec{\tau}}{2}\sin^2\tfrac{\rho}{2}\} \tag{2.79}$$

For $\tau \to -\infty$, $\rho \to 0$ and

$$\vec{\phi}_I(\vec{r},\tau) \to \vec{\phi}_v(\vec{r}) = \begin{bmatrix} 0 \\ 0 \\ -F\delta_{a3}/g \end{bmatrix} \tag{2.80}$$

For $\tau \to \infty$, $\rho \to \rho(\vec{r},\infty)$ and

$$\vec{\phi}_I(\vec{r},\tau) \to \vec{\phi}_{V\{\vec{D}(\vec{r})\}}(\vec{r}) = \begin{vmatrix} -2\partial_1(\tan^{-1}x_2/x_1)\delta_{a3}/g \\ -2\partial_2(\tan^{-1}x_2/x_1)\delta_{a3}/g \\ -F\delta_{a3}/g \end{vmatrix} \qquad (2.81)$$

which is nothing but a vortex at the origin of the (x_1,x_2) plane.

C.4. WKB Wave-function In The Forbidden Region.

We shall build the ground state wave-function in the classically forbidden region by the standard WKB method. The main idea is to use collective coordinates. To explain the motivation, let us consider the instanton solution $\vec{\phi}_I(\vec{r},\tau)$. This breaks the invariance under G (group of time-independent gauge transformations $\vec{\Lambda}(\vec{x})$ and space translations $\vec{X}$) completely. Let us call the set of parameters of G as ($\omega^{i\vec{x}} = \Lambda_i(\vec{x})$; $\omega^{1,2} = X_{1,2}$). The idea of the collective coordinates is to re-establish the symmetry by introducing the parameters ω^α of G as dynamical variables. These are needed because the degrees of freedom associated with the symmetry transformations having no restoring forces can undergo arbitrarily large fluctuations and this cannot be treated by ordinary quadratic approximation. However, these set of variables (ω^α) are not sufficient if one has tunnelling from one vacuum to another, since in that case there are large fluctuations which are not symmetry transformations. To handle these, one introduces the euclidean time τ as a real dynamical variable which describes the essential part of tunnelling. Unlike ω^α, dynamics of τ is non-trivial. Since this is by now standard, the reader is referred to the existing literature for details[10]. We shall now describe the

way this formalism works.

The collective coordinates are introduced by the change of variable

$$\vec{\phi}(\vec{x}) = \vec{\phi}_{I\{\omega\}}(\vec{x},\tau) + \underset{\sim}{\vec{\phi}}_{(\omega)}(\vec{x},\tau) \tag{2.82}$$

$$\underset{\sim}{\vec{\phi}}_{(\omega)}(\vec{x},\tau) = \sum_B \vec{n}_{B(\omega)}(\vec{x},\tau)\eta_B \tag{2.83}$$

$$<\vec{n}_A \mid \vec{n}_B> = \delta_{AB} \tag{2.84}$$

$$\vec{\phi}_{I,\alpha} = \left.\frac{\delta\vec{\phi}_{I\{\omega\}}}{\delta\omega^\alpha}\right|_{\omega=0} \tag{2.85}$$

$$<\vec{n}_B \mid \vec{\phi}_{I,\alpha}> = 0 \tag{2.86}$$

$$<\vec{n}_B \mid \vec{\phi}_{I,\tau}> = 0 \tag{2.87}$$

In the above $\vec{n}_B(\vec{x},\tau)$ are orthogonal to $\vec{\phi}_{I,\tau}$ and $\vec{\phi}_{I,\alpha}$ and together with them form a complete set. We shall denote $\tau = \omega^0$. The index "B" is really a continuous index and we write sum over B for notational simplicity. Another notational abbreviation is the scalar product between two vectors $\vec{u}$ and $\vec{v}$ which is written as

$$<\vec{u} \mid \vec{v}> = \int d^2x\, u^i(\vec{x},\tau)v^i(\vec{x},\tau) \tag{2.88}$$

In usual notation, the orthogonality condition is nothing but the back-ground gauge condition

$$(D_i(A_I)\underset{\sim}{A}_i)_a + g\varepsilon_{abc}\chi^b_I\underset{\sim}{\chi}^c = 0 \tag{2.89}$$

With the above change of variables given by (2.82) - (2.86), the momenta are

$$\pi_i(\vec{x}) = -i \frac{\delta}{\delta \phi_i(x)} = \sum_{\alpha,\beta} \phi^i_{I\{\omega\},\alpha} (I^{-1})_{\alpha\beta} (\tilde{L}_\beta - \underset{\sim}{\phi}^i_{,\beta} \xi^i) + \xi^i_{(\omega)} \qquad (2.90)$$

where

$$\xi^i_{(\omega)} = \sum_{\beta} n^i_{B(\omega)}(\vec{x},\tau)(-i \frac{\delta}{\delta \eta_B}) \qquad (2.91)$$

$$I_{\alpha\beta} = (\phi^i_{I,\alpha} + \underset{\sim}{\phi}^i_{,\alpha}) \cdot \phi^i_{I,\beta} \qquad (2.92)$$

In the equation (2.90), the sum over α,β runs from $\alpha,\beta = 0,1,2,i\vec{x}$. For $\beta=0$,

$$\tilde{L}_\beta = \tilde{L}_o = -i \frac{\partial}{\partial_\tau} \qquad (2.92a)$$

For $\beta \neq 0$, the definition of $\tilde{L}_\beta$ is as follows. Define the group multiplication of G as

$$(\phi_{\{\omega_1\}})_{\{\omega_2\}} = \phi_{\{\Omega(\omega_1,\omega_2)\}} \qquad (2.93)$$

Define two matrices m and n by

$$(m^{-1})^a_b(\xi) = \frac{\partial \Omega^a(\xi,\eta)}{\partial \eta_b}\bigg|_{\eta=0} \qquad (2.94)$$

$$(n^{-1})^a_b(\xi) = \frac{\partial \Omega^a(\eta,\xi)}{\partial \eta_b}\bigg|_{\eta=0} \qquad (2.95)$$

Then, for $\alpha \neq 0$,

$$L_\alpha = (m^{-1})^\beta_\alpha(\omega)(-i\frac{\delta}{\delta\omega_\beta}) \tag{2.96}$$

The reader is referred to the original literature for details. Unlike τ, the dynamics of the other collective coordinates $\omega^\alpha(\alpha\neq 0)$ are rather trivial. The reason is that they are associated with symmetry transformations whose generators commute with the Hamiltonian. In our problem, we shall look for ground state that is translationally invariant. For the gauge transformations, there is a slight complication due to the non-abelian nature of the gauge group. If the gauge group had been abelian,e.g., as in abelian Higgs model, one could have introduced static external electric charge distribution as an eigenvalue condition on the state vector as

$$G^{em}(\vec{x})\Psi_{\rho_{em}} = \rho_{em}(\vec{x})\Psi_{\rho_{em}} \tag{2.97}$$

In our problem, the generators of gauge transformations given by

$$G_a(\vec{x}) = \frac{1}{g} D_i E^a_i(\vec{x}) + \epsilon_{abc}\chi_b(\vec{x})\pi_{\chi_c}(\vec{x}) \tag{2.98}$$

do not commute among themselves. In fact,

$$[G_a(\vec{x}),G_b(\vec{x})] = i\delta(\vec{x}-\vec{y})\epsilon_{abc}G_c(\vec{x}) \tag{2.99}$$

So, although each G_a commute with the Hamiltonian, one cannot simultaneously diagonalise all of them along with H unless each G_a has zero eigenvalue in the state considered. Since we want to introduce external

static electric charges as the eigenvalue of the generator G_{em} of the electromagnetic subgroup, we shall look for the states that are eigenstates of H and G_{em}. In our notation,

$$\Psi^{\rho_{em}}_{WKB}(\tau,\vec{n},\omega^{\alpha}) = \Psi_{WKB}(\tau,\vec{n})D_{\rho_{em}}(\omega) \tag{2.100}$$

where the factor $D_{\rho_{em}}$ is obtained from the condition

$$G_{em}(\vec{x})\Psi^{\rho_{em}}_{WKB} = \rho_{em}(\vec{x})\Psi^{\rho_{em}}_{WKB} \tag{2.101}$$

Operationally, we need D_{ρ} only for $\rho \to \pm\infty$. Then, since $\hat{\chi}^{cl}_{a} \to -F\delta_{a3}/g$, we can identify $-A^{3}_{i}$ as the electromagnetic vector potential (minus sign because $\hat{\chi}^{cl}_{a} \to -\delta_{a3}$) and thus

$$\begin{aligned} D_{\rho_{em}}(\omega) &= \exp(-i\int d^{2}x\rho_{em}(\vec{x})\hat{\chi}^{cl}.\vec{\Lambda}(\vec{x}-\vec{X})) \\ &\to \exp(i\int d^{2}x\rho_{em}(\vec{x})\Lambda_{3}(\vec{x}-\vec{X})) \end{aligned} \tag{2.102}$$

The evaluation of the WKB wave-function is straight-forward. One substitutes the expression (2.90) for $\Pi_{i}(\vec{x})$ into the Hamiltonian H given by (2.53) and expands in power of g. Since this is discussed in detail in the original literature, we shall write down the final result. As in the previous section (see equation 2.22) for the double well potential, the two linearly independent WKB wave-functions of energy E_{i} are

$$\Psi^{\pm}_{WKB}(\tau,\vec{n}) = \frac{\exp(\pm S_{0}(\tau)\mp E_{1}\tau)\exp(-\tfrac{1}{2}<\underset{\sim}{\phi}^{i}|\Omega^{\pm}|\underset{\sim}{\phi}^{i}>)}{(\det g_{\alpha\beta})^{\frac{1}{4}}(\det U^{\pm})^{\frac{1}{2}}} \tag{2.103}$$

where

$$S_0(\tau) = \int_0^\tau d\tau \int d^2x (\phi^i_{I,\tau})^2 \tag{2.104}$$

$$g_{\alpha\beta} = \int d^2x \; \phi^i_{I,\alpha}(\vec{x})\phi^i_{I,\beta}(\vec{x}) \qquad \alpha,\beta=0,1,2,i\vec{x} \tag{2.105}$$

Definitions of the other symbols are as follows. Let us define matrix W

$$< ix|W|jy> \; = \tfrac{1}{2} \left. \frac{\delta^2 V\{\vec{\phi}\}}{\delta\phi^i(\vec{x})\delta\phi^j(\vec{y})} \right|_{\vec{\phi}=\vec{\phi}_I(\tau)} \tag{2.106}$$

for i,j = 1,2,.....,9. Define the evolution operator $\overset{+}{K}{}^-(\tau)$ as

$$\overset{+}{K}{}^-_{,\tau\tau}(\tau) = W(\tau)\overset{+}{K}{}^-(\tau) \tag{2.107}$$

Then the zero modes $\vec{\phi}_{I,\alpha}$ can be written as

$$|\vec{\phi}_{I,\alpha}> \; = \overset{+}{K}{}^-(\tau)|\vec{\chi}_\alpha> \qquad \alpha=0,1,2,i\vec{x} \tag{2.108}$$

Equation (2.108) is to be taken as definition for the set of vectors $|\vec{\chi}_\alpha>$. We define another set of vectors $|\vec{\chi}_M>$ which together with $|\vec{\chi}_\alpha>$ form orthogonal basis. The set $|\vec{\chi}_M>$ are defined by

$$|\vec{v}^{\,+}_{M,\tau\tau}> = \overset{+}{K}{}^-(\tau)|\vec{\chi}_M> \tag{2.109}$$

where $\vec{v}(\tau)$ is the small fluctuation around $\vec{\phi}_I(\tau)$ satisfying

$$|\vec{v}_{,\tau\tau}(\tau)> = W(\tau)|\vec{v}(\tau)> \tag{2.110}$$

The U-matrix of equation (2.103) is given by

$$U_B^{\pm M}(\tau) = n_B(\vec{\tau})\, v_M^{\pm}(\vec{\tau}) \tag{2.111}$$

The $\Omega^{\pm}$ matrix is

$$\Omega^{\pm}_{AB} = \mp \sum_M (U^{\pm -1})_A{}^M (DU^{\pm})_M{}^B \tag{2.112}$$

where

$$D_{AB}(\tau) = \delta_{AB} \frac{\partial}{\partial \tau} + \langle \vec{n}_B | \vec{n}_{A,\tau} \rangle \tag{2.113}$$

C.5. Wave-function around classical vacuum.

We shall find the wave-function around the classical vacuum $\vec{\phi}_V$ given by (2.80). $\vec{\phi}_V$ breaks only the group of gauge transformations, group G_0, whose parameters are $\{\omega_0^\alpha\} = \{\omega_0^{i\vec{x}} = \lambda_i(\vec{x})\}$. Unlike $\vec{\phi}_I$, space translation is not broken by $\vec{\phi}_V$. Analoguous to equations (2.82)-(2.86), we introduce $\{\omega_0\}$ as collective coordinates. The change of variable is

$$\phi^i(\vec{x}) = \phi^i_{V\{\omega_0\}}(\vec{x}) + \underline{\phi}^i{}_{(\omega_0)}(\vec{x}) \tag{2.114}$$

$$\underline{\phi}^i{}_{(\omega_0)}(\vec{x}) = \sum_B n^i_{OB(\omega_0)}(\vec{x}) \eta_{OB} \tag{2.115}$$

$$\langle \vec{n}_{OA} | \vec{n}_{OB} \rangle = \delta_{AB} \tag{2.116}$$

$$\vec{\phi}_{V,\alpha} = \left. \frac{\delta \vec{\phi}_{V\{\omega_0\}}}{\delta \omega_0^\alpha} \right|_{\omega_0 = 0} \tag{2.117}$$

$$\langle \vec{n}_{OB} | \vec{\phi}_{V,\alpha} \rangle = 0 \tag{2.118}$$

As before, $|\vec{n}_{OB}\rangle$ is a set of unit vectors orthogonal to $|\vec{\phi}_{V,\alpha}\rangle$. (2.118) is the usual gauge condition

$$\partial_i A_i^a + g\epsilon_{abc} \chi_V^b \chi^c = 0 \tag{2.119}$$

The Hamiltonian can be derived in the usual way. It is given by

$$H = H_{coulomb} + \frac{1}{2}\int d^2x \, P_i^2(\vec{x}) + \frac{1}{2} \langle \vec{\underline{\phi}} | \tilde{\omega}^2 | \vec{\underline{\phi}} \rangle \tag{2.120}$$

where $\tilde{\omega}^2$ is the quadratic part of the potential energy. As before, the external static electric charge distribution is introduced as the eigenvalue of the generator of electromagnetic subgroup. The wave-function for the ground state is

$$\Psi_0^{\rho_{em}}\{\vec{\underline{\phi}}, \omega_0\} = \Psi_0^{\rho_{em}}\{\vec{\underline{\phi}}\} \, D_{\rho_{em}}\{\omega_0\} \tag{2.121}$$

where

$$\Psi_0^{\rho_{em}}\{\vec{\underline{\phi}}\} = \left\{\frac{\det(\tilde{\omega}_\perp/\pi)}{\det \langle \vec{\phi}_{V,\alpha_i} | \vec{\phi}_{V,\beta_j} \rangle}\right\}^{1/4} \exp\left(-\frac{1}{2}\langle \vec{\underline{\phi}} | \tilde{\omega} | \vec{\underline{\phi}} \rangle\right) \tag{2.122}$$

and

$$D_{\rho_{em}}\{\omega_0\} = \exp\left(i\int \lambda_3(\vec{x}) \rho_{em}(\vec{x}) d^2x\right) \tag{2.123}$$

Since gauge zero-modes of $\tilde{\omega}$ are eliminated by the gauge condition, $\tilde{\omega}_\perp$ of equation (2.122) contains no zero-mode. However, because of the

existence of massless photons, $\tilde{\omega}_1$ contains infra-red modes with almost zero eigenvalue. This renders the wave-function in (2.122) rather unsuitable for the infra-red modes. We shall overcome this difficulty in section (C7) by construvting a modified wave-function.

In equation (2.122), the normalisation of $\Psi_0^{\rho_{em}}\{\vec{\underline{\phi}}\}$ is chosen such that

$$\int D\vec{\underline{\phi}} |\Psi_0^{\rho_{em}}\{\vec{\underline{\phi}}\}|^2 = v_0 = \text{volume of group } G_0 \qquad (2.124)$$

As before, one computes the Hamiltonian H in equation (2.120) corresponding to the change of variable in (2.114). The Schrodinger equation is

$$H\, \Psi_0^{\rho_{em}} = E_0 \Psi_0^{\rho_{em}} \qquad (2.125)$$

When Ψ_0^{ρ} given by (2.122) is substituted in (2.125), $H_{coulomb}$ just gives the coulomb energy of the static charge distribution. Since electric charge density $g\rho_{em}(\vec{x})$ is of order g, the coulomb energy is of order g^2 and we neglect this term. Also, we get

$$E_0 = \frac{1}{2} \mathrm{Tr}\tilde{\omega} \qquad (2.126)$$

In the above discussion of ground state wave-function around the classical vacuum, we have not discussed the fact that, in presence of tunnelling, the correct ground state is not Ψ_0^{ρ} but slightly disturbed. In the next section we shall compute this shift of the wave-function and the energy by matching the appropriate components with the WKB wave-function we found in section (C4).

C.6. The Matching Problem and Shift in Ground State Energy.

In the classical vacuum around $\vec{\phi}_v$, the Schrodinger equation reads

$$H\Psi = \{\frac{1}{2}\int d^2x(-i\frac{\delta}{\delta\phi_i(x)})^2 + \frac{1}{2}<\vec{\phi}|\tilde{\omega}^2|\vec{\phi}>\}\Psi = E\Psi \qquad (2.127)$$

where we have neglected the $H_{coulomb}$ term since it is of order g^2. The $\tilde{\omega}^2$ matrix is given by

$$<i\vec{x}|\tilde{\omega}^2|j\vec{y}> = \frac{\delta^2 V\{\vec{\phi}\}}{\delta\phi_i(x)\delta\phi_j(y)}\Bigg|_{\vec{\phi}=\vec{\phi}_v} \qquad (2.128)$$

It has a very simple form in momentum representation

$$<\vec{k}|\tilde{\omega}^2|\vec{k}> = \delta(\vec{k}-\vec{k}) \left|\begin{array}{ccc|ccc|ccc} m_w^2+k_2^2 & 0 & 0 & -k_1k_2 & 0 & 0 & 0 & im_wk_1 & 0 \\ 0 & m_w^2+k_2^2 & 0 & 0 & -k_1k_2 & 0 & -im_wk_1 & 0 & 0 \\ 0 & 0 & k_2^2 & 0 & 0 & -k_1k_2 & 0 & 0 & 0 \\ \hline -k_1k_2 & 0 & 0 & m_w^2+k_1^2 & 0 & 0 & 0 & im_wk_2 & 0 \\ 0 & -k_1k_2 & 0 & 0 & m_w^2+k_1^2 & 0 & -im_wk_2 & 0 & 0 \\ 0 & 0 & -k_1k_2 & 0 & 0 & k_1^2 & 0 & 0 & 0 \\ \hline 0 & im_wk_1 & 0 & 0 & im_wk_2 & 0 & k_1^2+k_2^2 & 0 & 0 \\ -im_wk_1 & 0 & 0 & -im_wk_2 & 0 & 0 & 0 & k_1^2+k_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_\eta^2+k_1^2+k_2^2 \end{array}\right| \qquad (2.129)$$

This matrix has one eigenmode with eigenvalue $\lambda = k_1^2+k_2^2$, one mode with eigenvalue $\lambda = m_\eta^2+k_1^2+k_2^2$, four modes with eigenvalue $\lambda = m_w^2+k_1^2+k_2^2$, and three with eigenvalue $\lambda = 0$. These correspond to the photon, the massive

neutral scalar, the two massive charged $W^{\pm}$ and the gauge zero modes.

To get the energy shift due to tunnelling, we follow the same procedure as in the case of double well potential. In absence of tunnelling, the wave-function is ψ_0^{em} given by equation (2.122) with eigenvalue E_0 given by (2.126). Thus

$$H\psi_0^{\rho em} = E_0\psi_0^{\rho em} \tag{2.130}$$

In presence of tunnelling, however, the ground state is shifted and the correct wave-function $\psi^{\rho em}$ satisfies

$$H\psi^{\rho em} = E\psi^{\rho em} \tag{2.131}$$

$$E = E_0 + \Delta E \tag{2.132}$$

To lowest order, ΔE is of order $\exp(-S_{cl})$ where S_{cl} is the action (euclidean) of the instanton. To this order, we can expand $\psi^{\rho em}$ as

$$\psi^{\rho em} = \psi_0^{\rho em} + (\Delta E)\psi_1^{\rho em} \tag{2.133}$$

From (2.131), (2.132), and (2.133) we get

$$M\psi_o^{\rho em} = (H-E_o)\psi_o^{\rho em} = 0 \tag{2.134}$$

$$M\psi_1^{\rho em} = \psi_o^{\rho em} \tag{2.135}$$

Since $\Psi_0^{\rho em}$ damps exponentially away from $\vec{\phi}_v$ (neglecting for the moment the fact that there are infra-red modes), one can integrate over a region S in field-configuration space around $\vec{\phi}_v$ such that $\Psi_0^{\rho em}$ is normalised in S. Then,

$$E = \frac{1}{V_0}\int_{\text{vol S}}(D\vec{\phi}^{\dagger})(\Psi_0^{*}\overleftrightarrow{M}(\Delta E)\Psi_1) = -\frac{1}{2V_0}\int_{\text{surface }\Gamma}d\vec{\sigma}(\Psi_0^{*}\overleftrightarrow{\nabla}(\Delta E)\Psi_1) \qquad (2.136)$$

An explanation is necessary at this point. Γ is the surface which encloses the volume S in field-configuration space. $\vec{\nabla}$ is a short-hand notation for the functional derivative with respect to field variable normal to Γ and $d\vec{\sigma}$ is the differential surface element of Γ.

By construction, Γ is the overlap region where we can match $\Psi^{\rho}_{0,1}$ with the WKB wave-functionals. The integral in (2.136) is only on two regions of Γ in the directions of tunnelling where $(\Delta E)\Psi_1$ is not negligible. These are characterised by introducing $\tau^{\pm}$ which label where $\vec{\phi}_I(\tau)$ and $\vec{\phi}_{I\{-\vec{\rho}\}}(\tau)$ intersect Γ. We will explain this shortly. For a detailed discussion, see reference (10).

In the forbidden region, the ground state wave-function is a linear superposition of the two WKB solutions

$$\Psi = C_{-}\Psi^{-}_{WKB}(\tau,\vec{\eta},\omega) + C_{+}\Psi^{+}_{WKB}(\tau,\vec{\eta},\omega) \qquad (2.137)$$

For $\tau \to -\infty$, $\vec{\phi}_I(\tau) \to \vec{\phi}_v$ so that

$$\Psi^{-}_{WKB} \to \frac{\exp(S_{cl}/2)\exp(-<\vec{\phi}|\tilde{\omega}|\vec{\phi}>/2)}{(\det_{\alpha,\beta} <\vec{x}_\alpha|\vec{x}_\beta>)^{1/4}} D\rho\{\omega\} \qquad (2.138)$$

where

$$S_{cl} = \int_{-\infty}^{\infty} d\tau \int d^2x \left(\phi^{i}_{I,\tau}(\vec{x},\tau)\right)^2 \tag{2.139}$$

S_{cl} is nothing but the action of the instanton over euclidean space-time. The vector $|\vec{x}_\alpha>$ are defined in (4.26). Thus, for $\tau \to -\infty$, Ψ^{-}_{WKB} matches with $\Psi^{\rho em}_{0}$ provided

$$D_{\rho_{em}}\{\omega_0\} = D_{\rho_{em}}\{\omega\} \tag{2.140}$$

i.e.,

$$\lambda_3(\vec{x}) = \Lambda_3(\vec{x}-\vec{X}) \tag{2.140a}$$

and

$$C_{-} = e^{-S_{cl}/2} \left\{\frac{\det<\vec{x}_\alpha|\vec{x}_\beta> \det(\tilde{\omega}_\perp/\pi)}{\det_{\alpha_0,\beta_0}<\phi_{\nu,\alpha_0}\phi_{\nu,\beta_0}>}\right\}^{1/4} \tag{2.141}$$

The reader, when comparing this equation with equation (2.26) of the double well problem, should notice that we have changed notation slightly Ψ^{+}_{WKB} now corresponds to Ψ^{-}_{WKB}. In principle, it could be shown that in the limit $\tau \to -\infty$, Ψ^{+}_{WKB} matches with $(\Delta E)\Psi^{\rho}_{1}$. Since we have not computed Ψ^{ρ}_{1} explicitly, we shall determone C_{+} in another way. We could find the WKB solutions for the case where the system tunnells from the vacuum n = -1 given by $\vec{\phi}_{v\{-\vec{\rho}(\vec{r})\}}$ to the vacuum n = 0 given by $\vec{\phi}_{v}$. Then, instead of the change of variables given by equations (2.82)-(2.86), we would use

$$\vec{\phi} = (\vec{\phi}_{I\{\omega\}}(\vec{x},\tau))_{\{-\vec{\rho}(\vec{x})\}} + \sum_{B}(\vec{n}_{B(\omega)}(\vec{x},\tau))_{(-\vec{\rho}(\vec{x}))}\eta_B$$

$$= (\vec{\phi}_{I}(\vec{x}-\vec{X},\tau)_{\{\vec{\Lambda}(\vec{x}-\vec{X})\}})_{\{-\vec{\rho}(\vec{x})\}} + \sum_{B}(\vec{n}_{B}(\vec{x}-\vec{X},\tau)_{(\vec{\Lambda}(\vec{x}-\vec{X}))})_{(-\vec{\rho})}\eta_B \tag{2.142}$$

Now, since for $\tau \to \infty$, $\vec{\phi}_I \to \vec{\phi}^{\,v}_{\{\vec{\rho}(\vec{r})\}}$ we can show that ψ^{+}_{WKB} matches with ψ^{ρ}_{0}. Since the photon is given by $A^{ph}_i = \hat{x}_a A^a_i$, the gauge transformations $\vec{\rho}(\vec{x})$ means that the vector potential A^{ph}_i transforms as

$$(A^{ph}_i)_{\{\vec{\rho}(\vec{x})\}} = A^{ph}_i + \frac{2}{g}\,\partial_i(\tan^{-1}\frac{x_2}{x_1}) = A^{ph}_i + \frac{1}{g}\,\partial_i\sigma(\vec{x}) \qquad (2.143)$$

Thus, for $\tau \to \infty$, ψ^{+}_{WKB} matches with ψ^{ρ}_{0} provided we identify

$$\lambda_3(\vec{x}) = \Lambda_3(\vec{x}-\vec{X}) + \sigma(\vec{x}) - \sigma(\vec{x}-\vec{X}) \qquad (2.144)$$

This gives the constant C_+ as

$$C_+ = C_- \exp(i\int d^2x\,\rho_{em}(\vec{x})\{\sigma(\vec{x})-\sigma(\vec{x}-\vec{X})\}) = C_- \exp(i\eta(\vec{X})) \qquad (2.145)$$

Going back to the evaluation of ΔE, we find that there are two regions in field-space where the integral in (2.136) is non-zero. The first region is characterised by $\tau=\tau^-$ such that $\phi_{I\{\omega\}}(\tau^-)$ lies on the surface Γ and $\psi^{\mp}_{WKB}$ matches with ψ^{ρ}_{0} and $(\Delta E)\psi^{\rho}_{1}$ respectively. Thus one gets

$$\Delta E = -\frac{1}{2v_0}\int D\Lambda D\eta\, dX C_+ C_- \{\frac{\sqrt{\det g}}{g_{00}}(\psi^{-*}\frac{\partial}{\partial\tau}\psi^{+})_{\tau=\tau^-} - \frac{\sqrt{\det g}}{g_{00}}(\psi^{+}\frac{\partial}{\partial\tau}\psi^{-})_{\tau=\tau^+}\}$$

$$= -2\varepsilon\int d^2X\cos\eta(\vec{X}) \qquad (2.146)$$

where $\eta(\vec{X})$ is given by (2.145) and

$$\varepsilon = e^{-S_{cl}}\left(\frac{\det\langle\vec{x}_\alpha|\vec{x}_\beta\rangle\ \det(\tilde{\omega}_\perp/\pi)}{\det\{U^+(\tau^+)U^-(\tau^-)\}\det\{(\Omega^+(\tau^+)+\Omega^-(\tau^-))/2\pi\}}\right) \qquad (2.147)$$

The matrices $\overset{+}{U}^{-}$ and $\overset{+}{\Omega}^{-}$ were defined in section (C4). Equation (2.146) is the same result as can be obtained by doing a dilute-gas calculation in the path-integral formalism where instanton-instanton and instanton-anti-instanton interactions are neglected.

The reason that the standard WKB approximation described above is not quite satisfactory lies in the fact that the long-range infra-red modes were not treated properly. There are two major objections in the above derivation. Firstly, we know that if there are low-frequency modes present, the classical solution $\vec{\phi}_I(\tau)$ approaches $\vec{\phi}_V$ rather slowly (as a power behaviour as opposed to exponential decay for the massive modes). Thus, the approximation made in (2.146), namely that the most rapidly varying term in $\overset{+}{\Psi}^{-}$ is the classical part $\exp(S_0(\tau))$ is not correct for large τ. Secondly, in deriving Ψ_0^{ρ} we have used harmonic approximation which is not valid for infra-red modes. To improve the situation one could in principle go beyond harmonic approximation for the infra-red modes and obtain a better Schrodinger wave-functional. Since this is technically rather difficult, we would obtain a modified wave-functional and ground state energy by a variational approach.

C.7. A Combined WKB-Variational Method.

We have explained in the previous section that to derive the correct interaction energy between the external charges, one has to include long-range pseudoparticle interactions. Polyakov did this explicitly in the functional integral formalism but a simple alternative way is due to Ezawa[22]. He observed that one can derive a simple dual representation in terms of a magnetic potential if one introduces the following constraint

$$\partial_\mu H_\mu = \rho_m \tag{2.148}$$

where ρ_m is the magnetic charge density of the monopoles (instantons) or the anti-monopoles (anti-instantons) and H_μ is the magnetic field given by

$$H_\mu = \frac{1}{2} \epsilon_{\mu\nu\sigma}(\hat{\chi}_a G^a_{\nu\sigma} - \frac{1}{g}\epsilon_{abc}\hat{\chi}_a D_\nu \hat{\chi}_b D_\sigma \hat{\chi}_c) \tag{2.149}$$

Of course, the constraint (2.148) is satisfied by the classical solution. The central idea is to elevate the constraint to the quantum level, i.e., to demand that the constraint be satisfied by the full quantum field.

In the following analysis, we treat only the abelian gauge field (photon) explicitly. We may consider, to avoid unnecessary complications, that all other fields are frozen except for their topological properties. A simple way of achieving this is to assume, for example,

that the Higgs coupling λ and the vector meson mass m_W are very large so that the Higgs field χ_a is frozen at $-(F/g)\delta_{a3}$ and we can neglect quantum fluctuations of χ_a and $W_\mu^{\pm}$.

Since our variational procedure is slightly different from the standard Rayleigh-Ritz variational formalism, let us first explain how the method works. Suppose the Hamiltonian for a system described by a set of dynamical variables (p,q) is given by

$$H = H(p,q) \tag{2.150}$$

We assume that the variables (p,q) satisfy a constraint

$$f(p,q) = 0 \tag{2.151}$$

We construct a modified Hamiltonian

$$\tilde{H} = H(p,q) + \lambda f(p,q) \tag{2.152}$$

where λ is a Lagrange multiplier. Next, we solve the Schrodinger equation

$$\tilde{H}\psi(\lambda) = E(\lambda)\psi(\lambda) \tag{2.153}$$

where $\psi(\lambda)$ and $E(\lambda)$ necessarily depend on the parameter λ. From (2.153), $E(\lambda)$ can be written as

$$E(\lambda) = \langle\psi(\lambda)|\tilde{H}|\psi(\lambda)\rangle/\langle\psi(\lambda)|\psi(\lambda)\rangle \tag{2.154}$$

Now, we determine the optimum value of λ by minimising $E(\lambda)$ with respect to λ:

$$\left.\frac{\partial E(\lambda)}{\partial \lambda}\right|_{\lambda_0} = 0 \tag{2.155}$$

The advantage of this procedure is that (2.155) implies that the constraint is automatically satisfied in the state $\psi(\lambda_0)$. That is, one can prove, from equation (2.153), (2.154) and (2.155) that

$$\frac{\langle\psi(\lambda_0)|f(p,q)|\psi(\lambda_0)\rangle}{\langle\psi(\lambda_0)|\psi(\lambda_0)\rangle} = 0 \tag{2.156}$$

This means that the expectation value of the modified Hamiltonian $\tilde{H}$ in the state $\psi(\lambda_0)$ is the same as that of the original Hamiltonian H in state $\psi(\lambda_0)$:

$$E(\lambda_0) = \frac{\langle\psi(\lambda_0)|\tilde{H}|\psi(\lambda_0)\rangle}{\langle\psi(\lambda_0)|\psi(\lambda_0)\rangle} = \frac{\langle\psi(\lambda_0)|H|\psi(\lambda_0)\rangle}{\langle\psi(\lambda_0)|\psi(\lambda_0)\rangle} \tag{2.157}$$

To prove that $E(\lambda_0)$ is indeed an upper bound for the true ground state energy E_0, one can proceed as follows. Suppose we had used $\psi(\lambda)$ as a variational trial wave-function in the standard Rayleigh-Ritz variational method. Then we would get an upper bound $\varepsilon(\lambda_1)$ where

$$\varepsilon(\lambda) = \frac{\langle\psi(\lambda)|H|\psi(\lambda)\rangle}{\langle\psi(\lambda)|\psi(\lambda)\rangle} \tag{2.158}$$

$$\left.\frac{\partial \varepsilon(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_1} = 0 \tag{2.158a}$$

$$E_0 \leq \varepsilon(\lambda_1) \tag{2.158b}$$

From (2.157), we have

$$E(\lambda_0) = \varepsilon(\lambda_0) \tag{2.159}$$

But since $\lambda=\lambda_1$ is the point where ε is a minimum, we get from (2.158) and (2.159)

$$E_0 \leq \varepsilon(\lambda_1) \leq E(\lambda_0) \tag{2.160}$$

Having set up the formalism, we now go back to our problem. In the (2+1) dimensional Minkowski space, the modified Hamiltonian is

$$\tilde{H} = H - \int d^2x B(\vec{x})\{\varepsilon_{ij}\partial_i E_j - \frac{i}{2}\partial_\tau \varepsilon_{ij}F_{ij} + i\rho_m\} \tag{2.161}$$

In the above $B(\vec{x})$ is a Lagrange multiplier field assumed, for calculational simplicity, to be τ-independent. Since we are looking for a "correction" to the wave-function and the ground state energy shift given by (2.147), we shall assume that the magnitude of the B field is small. This can only be justified at the end when we obtain the value of B variationally to be of order g. Also, in the above, we are looking for modification in the photon part of the wave-functional which, in our case, can be identified with $-A_i^3$ away from the core of the instanton. Thus in the above expression for $\tilde{H}$, E_i and F_{ij} refer to photon.

H in equation (2.161) is the Hamiltonian we have been using, i.e., given by (2.53) with $\vec{\pi}(\vec{x})$ substituted from equation (2.90). The modified Schrodinger equation is now

$$\tilde{H}\tilde{\psi} = \tilde{E}\tilde{\psi} \tag{2.162}$$

Let us first evaluate $\tilde{\psi}^{-}_{WKB}$. A straightforward computation gives

$$\tilde{\psi}^{-}_{WKB} = \psi^{-}_{WKB} \exp(-i\int d^2x B(\vec{x}\,)\varepsilon_{ij}\partial_i \underset{\sim}{A}_j) \tag{2.163}$$

where $\underset{\sim}{A}_j$ is the quantum fluctuation. To prove this, one substitutes (2.163) in (2.162) and computes both sides of the equation keeping terms upto order B. Since the constraint is satisfied by the classical solution, one has contribution only from the quantum part. Of course, the energy E is no longer equal to $E_1(=Tr\omega/2)$. This shift will be calculated when matching is done.

In the allowed region classical vacuum $\vec{\phi}_v$, the modified ground state wave-functional is similarly given by

$$\psi_o^{\rho em} \to \tilde{\psi}_o^{\rho em} = \psi_o^{\rho em}\{\underset{\sim}{\phi}\}\exp(-i\int d^2x B\varepsilon_{ij}\partial_i \underset{\sim}{A}_j) \tag{2.164}$$

This changes the ground state energy E_o as

$$E_o \to \tilde{E}_o = E_o + \frac{1}{2}\int d^2x(\partial_i B)^2 \tag{2.165}$$

Finally, we come to the matching problem. As before, for $\tau \to -\infty$,

$$\tilde{C}_- \tilde{\psi}^{-}_{WKB} \underset{\tau\to-\infty}{\to} \psi_o^{\rho em}\{\underset{\sim}{\phi}\} \tag{2.166}$$

This means that the constant C_- remains unchanged. For C_+, however, there is an additional phase factor involved. To show that, one observes

$$\psi^{+}_{WKB} \to \tilde{\psi}^{+}_{WKB} = \psi^{+}_{WKB} \exp(-i\int d^2x B\varepsilon_{ij}\partial_i A_j - 2i\int_0^{\tau} B(\vec{x})\rho_m d^2x d\tau) \quad (2.167)$$

The reason for this additional phase factor is that ψ^{+} can be obtained from ψ^{-} by simply changing the classical solution from an instanton to an anti-instanton which satisfies the constraint (2.148) with the change $\rho_m \to -\rho_m$. The reader can convince himself that (2.167) is indeed correct by directly substituting (2.167) into (2.162).

The modified constant $\tilde{C}_{+}$ can now be evaluated as

$$C_{+} \to \tilde{C}_{+} = C_{+} \exp(2i\int_0^{\infty} d^2x d\tau B(\vec{x})\rho_m) \quad (2.168)$$

Thus, finally, including the energy shift given by (2.165), the total energy shift of the ground state is

$$\Delta E(\rho_{em}) = \int d^2x \left(\frac{1}{2}(\partial_i B)^2 - 2\varepsilon \cos\{\eta(\vec{x}) + \frac{4\pi B(\vec{x})}{g}\}\right) \quad (2.169)$$

where we have used the fact that, by construction, instanton is located at $\vec{x}=\vec{X}$ and $\tau=0$ so that

$$\int_{-\infty}^{\infty} d\tau \int d^2x B(\vec{x})\rho_m(\vec{x},\vec{X},\tau) = \frac{4\pi}{g} B(\vec{X}) \quad (2.170)$$

Equation (2.169) is the same result obtained by euclidean calculation including pseudo-particle interaction. To compare with Polyakov's result let us change variables

$$J(\vec{x}) = -\eta(\vec{x}), \quad \chi(\vec{x}) = -\frac{4\pi}{g} B(\vec{x}) - \eta \quad (2.171)$$

Then, equation (2.169) becomes

$$\Delta E(\rho_{em}) = \frac{16\pi^2}{g^2}\int d^2x\{ \frac{1}{2}(\nabla(\chi-J))^2-M^2\cos\chi \} \qquad (2.172)$$

where

$$M^2 = \frac{32\pi^2\varepsilon}{g^2} \qquad (2.173)$$

The reader may compare this with equation (5.20) of Polyakov's work[13]. Now we can minimise $\Delta E(\rho_{em})$ by varying $\chi(\vec{x})$ which leads to

$$\nabla^2\chi = \nabla^2 J + M^2\sin\chi \qquad (2.174)$$

For ρ_{em} corresponding to two arbitrary static external charges $\pm e$ at $(\pm R,0)$,

$$\rho_{em} = e\delta(y)\{\delta(x-R)-\delta(x+R)\}/g \qquad (2.175)$$

and equation (2.145) gives

$$\nabla^2 J = -\nabla^2\eta = \frac{4\pi e}{g}\,\delta'(y)\theta_R(x) \qquad (2.176)$$

$$\theta_R(x) = 1 \quad \text{if} \quad -R < x < R$$
$$= 0 \quad \text{otherwise}$$

For integral charge e=g, the only solution of (2.174) with a discontinuity of 4π across y-axis is

$$\chi = 2\pi \quad (y>0), \quad \text{and} = -2\pi \quad (y<0) \qquad (2.177)$$

which does not lead to confinement[23]. However, for half-integer charge e=g/2, there is a non-trivial solution

$$\chi = 4 \tan^{-1}(e^{-My}) \qquad y>0$$
$$= -4 \tan^{-1}(e^{My}) \qquad y<0 \qquad (2.178)$$

which leads to a linear potential between the charges.

C.8. Conclusion.

In this chapter we have constructed an explicit wave-function for the ground state of compact QED using a combined variational and WKB approach. By imposing the constraint at the quantum level, we have, in effect, taken the long-range pseudo-particle interaction into account and this, in turn, implies the existence of a mass-gap in the theory. In the case of scale-invariant theories like non-abelian pure gauge theory in four dimensions, the same problem, namely, long-range pseudo-particle interaction arises. As expected, this gives rise to severe infra-red divergences. In the euclidean path-integral formulation, one has no other choice but to put a cut-off for instanton sizes and hope that physical quantities would be independent of the cut-off. In absence of a reliable calculation, it is our opinion that such a procedure is quite ambiguous and we hope that a variational treatment similar to the one we proposed for the ground state will avoid these infra-red difficulties.

In the next chapter, we are going to try another approach to pure gauge theory by the method of collective field. The reason is that

even if one could control the infra-red behaviour in the WKB approach, it is not clear whether the known instanton field configurations are sufficient to lead to color confinement. Besides, there are rather complicated topological problems involved in the discussion of field configuration of non-abelian gauge theories. For example, it has been demonstrated that one cannot fix gauge uniquely by a single non-singular gauge condition such as the coulomb gauge. This is the so-called Gribov ambiguity. Simply stated, if one uses the standard coulomb gauge formalism, one would count the same field configuration many times and also might miss some other configurations entirely. In other words, we need an approach which treats gauge-invariance correctly. Collective field theory is such an approach. It is formulated completely in terms of gauge invariant operators and thus the problem of gauge fixing ambiguities does not arise.

III. COLLECTIVE FIELD THEORY.

A. Introduction.

In this chapter we shall explore the quantum collective field method which was developed by Jevicki and Sakita. To illustrate the idea, it is nice to review a simple many-body problem, namely, that of the ground state of a collection of N Bose particles. A typical Hamiltonian would look like

$$H = \frac{1}{2}\sum_{i=1}^{N}\hat{p}_i^2 + \frac{1}{2}\sum_{i\neq j} v(\hat{x}_i,\hat{x}_j) \tag{3.1}$$

Here $v(x_i,x_j)$ is the two-particle interaction energy. One could also include a general one-particle common potential, but for our purpose this is general enough. The reader can get a detailed discussion in the original literature[15].

The particles being bosons, the symmetry of the problem demands that the ground state is symmetric under pair-exchange. So the idea is to write the ground state wave-function as a functional of the density operator which is the most general operator invariant under the symmetry of the problem,

$$\text{density operator} = \hat{\rho}(x) = \sum_{i=1}^{N}\delta(x-\hat{x}_i) \tag{3.2}$$

Notice that $\rho(x)$ satisfies a constraint

$$\int \rho(x)dx = N \tag{3.3}$$

The essence of the quantum collective field method is to consider a most general (usually overcomplete) set of commuting operators (such as $\rho(x)$) and explicitly perform a change of variables to this new set (collective field). Thus, we shall assume that the wave-function is a functional of $\rho(x)$:

$$\psi(x_1,x_2,.....,x_N) = \Phi\{\rho(.)\} \tag{3.4}$$

Next step is to express the Hamiltonian in terms of $\rho(x)$ and the conjugate momenta $\hat{\pi}(x)$:

$$\hat{\pi}(x) = -i \frac{\delta}{\delta\rho(x)} \tag{3.5}$$

The kinetic energy can be written as

$$\frac{1}{2} \int dx\ \omega(x;\{\rho\})(i\hat{\pi}(x)) + \frac{1}{2} \int dx \int dy\ \Omega(x,y;\{\rho\})\hat{\pi}(x)\hat{\pi}(y) \tag{3.6}$$

where we have denoted

$$- \sum_i \nabla_i^2\rho(x) = \omega(x;\{\rho\})$$

$$\sum_i \nabla_i\rho(x)\nabla_i\rho(y) = \Omega(x,y;\{\rho\}) \tag{3.7}$$

Now, at this point the relevance of the large N limit becomes apparent. That is, in this limit the new variables become almost independent and one can define a genuine field theory which approaches the original

quantum mechanical system. This is the collective field theory. What has been done is simply a change of variable from the original N degrees of freedom to the new set of infinitely many variables.

Going back to our example, we see that the scalar product of any two states is

$$\langle 1|2\rangle = \int \prod_{i=1}^{N} dx_i \psi_1^*(\vec{x})\psi_2(\vec{x})$$

$$= \int (D\rho(x))\Phi_1^*\{\rho\}\Phi_2\{\rho\}\int \prod_{i=1}^{N} dx_i \prod_x \delta(\rho(x)-\sum_i \delta(x-x_i)) \qquad (3.8)$$

The second integral in this expression defines the jacobian $J\{\rho\}$ and then the rescaled wave-functions

$$\Psi\{\rho\} = J\{\rho\}^{1/2}\Phi\{\rho\} \qquad (3.9)$$

have a simple scalar product

$$(\Psi_1,\Psi_2) = \int (D\rho(x))\Psi_1^*\{\rho\}\Psi_2\{\rho\} \qquad (3.10)$$

Now, in order to make the kinetic energy term given by (3.6) hermitean, we consider the similarity transformation (3.9). The effect of this transformation on operators $\hat{\rho}(x)$ and $\hat{\pi}(x)$ is:

$$\hat{\rho}(x) \to \hat{\rho}(x)$$

$$\hat{\pi}(x) \to \hat{\pi}(x) - iC(x;\{\rho\}) \qquad (3.11)$$

where

$$C(x;\{\rho\}) = -\frac{1}{2}\frac{\delta \ln J\{\rho\}}{\delta \rho(x)} \tag{3.12}$$

Then the kinetic energy term becomes

$$\frac{1}{2}i\int dx\omega(x;\{\rho\})(\pi(x)-iC(x;\{\rho\}))+\frac{1}{2}\int dxdy\Omega(x,y;\{\rho\})(\pi-iC)_x(\pi-iC)_y \tag{3.13}$$

The important point to note now is that, instead of evaluating the jacobian based on the definition (3.8), we can determine it more efficiently by demanding hermiticity of Hamiltonian after the similarity transformation. This gives an equation for C:

$$\omega(x;\{\rho\})+\int dy\,\frac{\delta\Omega(x,y;\{\rho\})}{\delta\rho(y)}-2\int dy\Omega(x,y;\{\rho\})C(y;\{\rho\}) = 0 \tag{3.14}$$

This equation can be formally solved as

$$C = \frac{1}{2}\,\Omega^{-1}(\omega+i(\hat{\pi}\Omega)) \tag{3.15}$$

where $\Omega^{-1}(x,y)$ is the inverse of Ω.

Now we can write the complete hermitean Hamiltonian:

$$H = \frac{1}{2}\int dx\int dy\ \pi(x)\Omega(x,y;\{\rho\})\pi(y) + V\{\rho\} + \Delta V\{\rho\} \tag{3.16}$$

where $V\{\rho\}$ is the original potential expressed in terms of ρ and the extra piece $\Delta V\{\rho\}$ is given by

$$\Delta V\{\rho\} = \frac{1}{8}\int dx\int dy\{\omega(x;\{\rho\})+\int dz\ \frac{\delta\Omega(x,z;\{\rho\})}{\delta\rho(z)}\}\Omega^{-1}(x,y;\{\rho\})\{\omega(y;\{\rho\})+$$

$$\int dz'\ \frac{\delta\Omega(y,z';\{\rho\})}{\delta\rho(z')}\}-\frac{1}{4}\int dx\ \frac{\delta\omega(x;\{\rho\})}{\delta\rho(x)}-\frac{1}{4}\int dx\int dy\ \frac{\delta^2\Omega(x,y;\{\rho\})}{\delta\rho(x)\delta\rho(y)} \qquad (3.17)$$

In case of gauge theory, we have gauge invariance instead of Bose symmetry of the above problem. The most general gauge invariant operators one can think of in this case are the path-ordered phase factors

$$W\{\Gamma\} = \text{tr P exp}\{i\oint_{\Gamma}\vec{A}(\vec{x}).d\vec{x}\} \qquad (3.18)$$

and consequently one assumes the ground state wave-functional as

$$\psi\{A\} = \Phi\{W\} \qquad (3.19)$$

Assuming at this point that the motivation is apparent enough, we shall treat a somewhat simpler problem which involves a global (rather than a local) symmetry; namely, the Heisenberg spin model in two dimensions (popularly known as O(N) sigma model in field theory). This model has the virtue of all the important charcateristics of a full-fledged four dimensional gauge theory (that is, asymptotic freedom, dynamical mass generation, etc). The interested reader is reffered to the original literature for the discussion of a non-abelian gauge model[15].

B. Collective Field Theory and Large N Limit of O(N) Model.

Ever since 't Hooft's work on the two dimensional U(N) quantum

chromodynamics as a toy model for mesons, 1/N expansion has been explored in great details in various N-component vector models, Gross-Neveu model and others. The central idea is the observation that the large N limit can be obtained by summing planar diagrams. However, it is technically rather difficult to sum the planar diagrams for realistic theories like Yang-Mills theory in four dimensions.

Collective field theory provides a very useful set-up to obtain the large N limit of the above mentioned theories. In this chapter we shall analyse the O(N) Heisenberg spin system using the collective field approach. The main idea is to construct an effective theory in terms of a collective field such that the large N limit can be obtained by a stationary point method.

In this section we introduce the collective field in the Hamiltonian formalism to derive an effective Hamiltonian. The advantage is that the large N limit is just the semi-classical expansion of the effective Hamiltonian. We shall derive the mass-gap and the β-function for $N\to\infty$. In the next section, we introduce the same collective field in the Lagrangian path integral formalism. The computation of 1/N corrections to physical quantities is rather straightforward using Feynman diagrams.

The Hamiltonian for the O(N) Heisenberg spin system in d-dimensions is given by

$$H = -K \sum_{\vec{m},\vec{\mu}} \vec{\sigma}(\vec{m}).\vec{\sigma}(\vec{m}+\vec{\mu}) \tag{3.20}$$

with

$$\vec{\sigma}(m).\vec{\sigma}(m) = 1 \tag{3.21}$$

where $\vec{\sigma}(\vec{m})$ is the N-dimensional classical spin vector of unit length at the lattice site $\vec{m}$. $\vec{\mu}$ denotes the unit lattice vector. In the continuum limit (lattice spacing $a \to 0$), the hamiltonian $\mathcal{H}$ reduces to the euclidean action for the O(N) non-linear sigma-model:

$$\mathcal{H} \to S_E = \frac{K}{2} \int d^d x \sum_{i=1}^{d} (\partial_i \vec{\sigma}(\vec{x}))^2 (a)^{2-d} \tag{3.22}$$

apart from an unimportant constant. If we now perform a "Wick rotation"

$$x_i \to x_i \ (i=1,2,\ldots,d-1), \quad x_d \to it \tag{3.23}$$

we get the Minkowski action

$$S_M = \frac{1}{2g} \int dt \, L \tag{3.24}$$

where

$$L = \int d^{d-1} x (\partial_\mu \vec{\sigma})^2 \tag{3.25}$$

$$Ka^{2-d} = 1/g \tag{3.26}$$

So instead of discussing the original Hamiltonian $\mathcal{H}$, we can discuss an equivalent physical system described by the Lagrangian L of equation (3.25). The advantage is that the Lagrangian L is defined in (d-1) dimensional space. In order to calculate a physical quantity such as spin-spin correlation function in the original d-dimensional system, we can calculate the same quantity at "equal time" using the (d-1) dimensional Lagrangian L and use the property of isotropy to obtain the spin-spin

correlation at "unequal time".

It has been shown that the quantum (d-1) dimensional Hamiltonian corresponding to the Lagrangian $L^{(24)}$ is

$$H = \frac{g}{2a} \sum_{\vec{m}} \vec{J}^2(\vec{m}) - \frac{1}{ag} \sum_{m,\mu} \vec{\sigma}(\vec{m}).\vec{\sigma}(\vec{m}+\vec{\mu}) \tag{3.27}$$

where $\vec{m}$ and $\vec{\mu}$ now refer to the (d-1) dimensional "spatial lattice". $\vec{J}$ is the angular momentum generator for O(N).

There are several interesting properties of the Hamiltonian H given by (3.27). For strong coupling ($g \to \infty$), or, high temperature, the dominant part is

$$H \to \frac{g}{2a} \sum_{\vec{m}} \vec{J}^2(\vec{m}) \tag{3.28}$$

which has "local O(N)" invariance. For weak coupling ($g \to 0$), or, low temperature,

$$H \to - \frac{1}{ag} \sum_{m,\mu} \vec{\sigma}(\vec{m}).\vec{\sigma}(\vec{m}+\vec{\mu}) \tag{3.29}$$

and the invariance is that of "global O(N)",i.e., when all the spins are rotated simultaneously by equal amounts. This gives us a hint that so far as the ground state is concerned (for finite g), we should look for a collective field which is singlet under global O(N), so that the ground state wave-function ψ_0 depends only on the collective field. Such a field is

$$q_{\vec{m},\vec{n}} = \vec{\sigma}(\vec{m}).\vec{\sigma}(\vec{n}) \tag{3.30}$$

From now on, for simplicity, we shall work in dimension d=2, although generalisation to any higher dimension is quite simple. In terms of the q-variables, the constraint condition (3.21) becomes

$$q_{mm} = 1 \tag{3.31}$$

We shall also assume periodic boundary condition for the original $\vec{\sigma}$ variable

$$\vec{\sigma}(M+1) = \vec{\sigma}(1) \tag{3.32}$$

where M is the total number of lattice sites.

The basic feature of the collective field variables is that they are not all independent. What we have done is that instead of the original M(N-1) independent degrees of freedom of the $\vec{\sigma}$-variable, we have chosen M(M-1)/2 degrees of freedom for q. Thus, although q-variables are not all independent for finite N, they become independent in the limit $N\to\infty$ and $M\to\infty$. Assuming the ground state of the system to be non-degenerate we can take it to be a function of q only:

$$\psi_0 = \psi_0(q) \tag{3.33}$$

Using the chain rule of differentiation, the Hamiltonian H of equation (3.27) can be written as

$$H = -\frac{g}{2a}\sum_m T(m) - \frac{1}{ag}\sum_m q_{m,m+1} \tag{3.34}$$

where

$$T(m) = -4 \sum_{k}{}' \sum_{k'}{}' \Omega(m,k,k') p_{mk} p_{mk'} - 2i(N-1) \sum_{k}{}' q_{mk} p_{mk}$$

$$\Omega(m,k,k') = q_{kk'} - q_{mk} q_{mk'} \tag{3.35}$$

$$p_{mk} = -i\partial/\partial q_{mk}$$

and the sum $\sum_{K}{}'$ means that the term $m=k$ has been omitted. Although the Hamiltonian H of equation (3.34) is adequate for calculating the large N limit, it is usually more convenient to use the original Hamiltonian $\mathcal{H}$ of equation (3.20) to compute $1/N$ corrections to physical quantities like the ground state energy. However, the collective field variable q_{mn} has a natural connection with $1/N$ expansion as we shall see in the next section.

Going back to the Hamiltonian H of equation (3.34), we see that it is not hermitean in the ordinary sense. This is because the scalar product in the Hilbert space of states is defined with a jacobian $|J|$ as can be seen as follows:

$$\begin{aligned} \langle 1|2\rangle &= \int \prod_{m} d\vec{\sigma}(m) \phi_1^*(\vec{\sigma}) \phi_2(\vec{\sigma}) \\ &= \int \prod_{m} d\vec{\sigma}(m) \prod_{m<k} dq_{mk}\, \delta(q_{mk} - \vec{\sigma}(m).\vec{\sigma}(k)) \psi_1^*(q) \psi_2(q) \\ &= \int \prod_{m<k} dq_{mk}\, \psi_1^*(q) \psi_2(q) |J| \end{aligned} \tag{3.36}$$

where

$$\psi(q) = \phi(\vec{\sigma}), \quad |J| = \int d\vec{\sigma}\, \delta(q_{mk} - \vec{\sigma}(m).\vec{\sigma}(k)) \tag{3.37}$$

We can eliminate the jacobian by a similarity transformation

$$\psi(q) \to \Psi(q) = \sqrt{|J|}\ \psi(q) \tag{3.38}$$

and correspondingly all operators undergo a transformation

$$\hat{O} \to \hat{O}' = |J|^{1/2}\ \hat{O}\ |J|^{-1/2} \tag{3.39}$$

Since $|J|$ is a function of q, we get

$$\begin{aligned} q_{mk} &\to q_{mk} \\ p_{mk} &\to p_{mk} - iC_{mk} \\ C_{mk} &= \partial(\ln |J|^{1/2})/\partial q_{mk} \end{aligned} \tag{3.40}$$

Under these transformations the scalar product becomes

$$\langle 1 | 2 \rangle \to \int \prod_{m<k} dq_{mk}\ \Psi_1^*(q)\,\Psi_2(q) \tag{3.41}$$

and consequently the Hamiltonian should become hermitean. So we demand

$$H(p,q) \to H(p-iC,q) = \{ H(p-iC,q)\}^{\dagger} \tag{3.42}$$

and get an equation for C as

$$\sum_k{}' C_{mk}\Omega(m,k,k') = (N-2M-1)q_{mk'}/4 \qquad (m \neq k') \tag{3.43}$$

For m=k', the left hand side is zero and so the complete set of equation is

$$\sum_{\text{all } k} C_{mk}\Omega(m,k,k') = (N-2M-1)(q_{mk'}-\delta_{mk'})/4 \qquad (3.44)$$

This can easily be solved as

$$C_{mk} = (\text{constant})\delta_{mk} - (N-2M-1)(q^{-1})_{mk}/4 \qquad (3.45)$$

where q^{-1} is the inverse matrix of q. The coefficient of δ_{mk} is arbitrary and it drops out of the new effective Hamiltonian, which is given by

$$H_{eff} = \frac{2g_o}{a} \sum_m \sum_{k,k'}'' \frac{1}{N} p_{mk}\Omega(m,k,k')p_{mk'} + V_{eff}(q) \qquad (3.46)$$

where the effective potential is

$$V_{eff}(q) = NV_o(q) + \Delta V$$

$$V_o(q) = -\frac{1}{ag_o}\sum_m q_{m,m+1} + \frac{g_o}{8a}\sum_m (q^{-1})_{mm} \qquad (3.47)$$

$$g_o = gN$$

and ΔV is an extra piece containing O(1) and O(1/N) terms

$$\Delta V = \frac{g_o}{8a}\{-2(2M+1) + \frac{1}{N}(2M+1)^2\}\,\mathrm{Tr}(q^{-1}) \qquad (3.48)$$

We shall seek the large N limit of the system keeping g_o finite. By a

rescaling of p_{mk} as

$$p_{mk} \to \tilde{p}_{mk} = p_{mk}/N \tag{3.49}$$

we get

$$H_{eff} = N\left(\frac{2g_o}{a}\sum_m \sum_{k,k'} \tilde{p}_{mk}\Omega(m,k,k')\tilde{p}_{mk'} + V_o\right) + \Delta V \tag{3.50}$$

because of equation (3.49) we have

$$[p,q] = -i/N \tag{3.51}$$

From (3.51) it is clear that 1/N plays the role of $\hbar$ so that the 1/N expansion is nothing but the semi-classical expansion of the Hamiltonian system described (3.50). In this systematic 1/N expansion one would get divergent higher order terms, i.e., terms proportional to M, M^2, etc. These divergent terms should be cancelled by the divergent terms of ΔV in (3.50)(see appendix A). ΔV therefore plays the role of counterterms.

The ground state in the limit $N\to\infty$ can be obtained from the classical solution of the Hamiltonian system of (3.50). We look for static solutions only. The equation is simply obtained by minimising $V_o(q)$ subject to constraint (3.31):

$$-\frac{g_o}{4a}(q^{-2})_{mn} - \frac{1}{ag_o}(\delta_{m+1,n}+\delta_{n+1,m}) + \lambda(m)\delta_{mn} = 0 \tag{3.52}$$

The Lagrange multiplier $\lambda(m)$ is to be determined from (3.31).

The solution of (3.52), which we denote by q^o_{mn}, can be interpre-

ted as usual as the vacuum expectation value of the corresponding operator in the limit $N\to\infty$:

$$q^0_{mn} = \langle\vec{\sigma}(m).\vec{\sigma}(n)\rangle_{vac} \tag{3.53}$$

From this interpretation and the translational invariance of the vacuum, one would expect that q^0_{mn} depends only on the difference $(m-n)$ and $\lambda(m)$ independent of m. With this ansatz we obtain the following solution of (3.52) after a straightforward calculation:

$$q^0_{mn} = \frac{1}{M}\sum_{j=1}^{M}\frac{\exp(2\pi i(m-n)j/M)}{(4a\lambda/g_0 - 8\cos(2\pi j/M)/g_0^2)^{1/2}} \tag{3.54}$$

In the infinite volume limit ($M\to\infty$), q^0_{mn} is given by

$$q^0_{mn} = \int_0^{2\pi}(d\theta/2\pi)\exp\{i(m-n)\theta\}(4a\lambda/g_0-8\cos\theta/g_0^2)^{-1/2} \tag{3.55}$$

The Lagrange multiplier λ is determined from the following equation which is a result of the constraint:

$$\begin{aligned} K(k) &= \tfrac{\pi}{2}(4a\lambda/g_0 + 8/g_0^2)^{1/2} \\ k^2 &= (16/g_0^2)(4a\lambda/g_0 + 8/g_0^2)^{-1} \end{aligned} \tag{3.56}$$

where $K(k)$ is the complete elliptic integral of the first kind[25].

Let us briefly indicate what the solution looks like from equation (3.56) for weak coupling ($g_0 << 1$) and strong coupling ($g_0 >> 1$).

For $g_0 \ll 1$,

$$\lambda \simeq \frac{g_0}{4a}\left(\frac{8}{g_0^2} + \frac{256}{g_0^2} e^{-4\pi/g_0}\right) \tag{3.57}$$

and for $g_0 \gg 1$,

$$\lambda \simeq g_0/4a \tag{3.58}$$

The mass-gap μ (defined as the inverse of correlation length)

$$\mu = -\lim_{|m-n|\to\infty}\left(\frac{1}{a|m-n|}\ln q^0_{mn}\right) \tag{3.59}$$

is related to λ by

$$\mu^2 = \lambda g_0/a - 2/a^2 \tag{3.60}$$

The β-function of the theory is defined by

$$\beta(g) = a\frac{dg}{da} \tag{3.61}$$

We can calculate $\beta(g)$ by demanding that the mass-gap μ is a physical quantity independent of a:

$$\frac{d\mu}{da} = 0 \tag{3.62}$$

From (3.57) and (3.58) we get

$$\beta(g) \simeq g^2N/2\pi \quad \text{for} \quad gN \ll 1 \tag{3.63}$$

and

$$\beta(g) \simeq g \quad \text{for} \quad gN >> 1 \tag{3.64}$$

In the intermediate coupling region, one can solve (3.56) numerically. However, an excellent approximation can be obtained by noticing that the main contribution to the integration in (3.55) comes from small θ. We expand $\cos\theta$ keeping upto quadratic term and do the integration in (3.55). We quote only the final result

$$\mu = \frac{2\pi}{a}\left(\frac{\alpha}{\alpha^2-1}\right)$$

$$\frac{\beta(g)}{g} = \frac{gN}{2\pi}\left(\frac{\alpha^2-1}{\alpha^2+1}\right) \tag{3.65}$$

$$\alpha = \exp(2\pi/g_0)$$

Equation (3.65) has the correct limits given by (3.63) and (3.64). The expression for μ, however, differs by a factor of $(2\pi/8)$ from the weak coupling limit. This difference is due to the approximation we made in doing the integration. The β-function also agrees with the results found in low-temperature expansion[26] and high-temperature expansion[24,27].

In concluding this section a few comments are in order. All the results obtained here can easily be generalised to a d-dimensional model. For example, the classical solution given by equation (3.55) can be generalised to d=3 from which it can be deduced that, unlike the two-dimensional model which has critical coupling constant $g_c=0$, the three dimensional model has a non-zero critical coupling constant. The simplification we have achieved by using an effective field q is that all the

information about the ground state of the system (for large N) is contained in the potential energy term, so that the large N approximation reduces to the ordinary semi-classical approximation ($\hbar \to 0$). However, to obtain a systematic 1/N expansion, the effective Hamiltonian H_{eff} may not be very suitable since all the q_{mn} variables are not independent. To overcome this difficulty, we shall introduce the same collective field in Lagrangian path integral formulation in the next section.

C. Collective Field in Path Integral Formulation.

The vacuum-to-vacuum transition amplitude for the system described by the Lagrangian of (3.25) is

$$Z = \int \prod_m \{d\vec{\sigma}(m)\delta(\vec{\sigma}^2(m)-1)\}\exp\{iS(\vec{\sigma})\} \tag{3.66}$$

with

$$S(\vec{\sigma}) = \int dt\{ \frac{a}{2g} \sum_m \dot{\vec{\sigma}}(m).\dot{\vec{\sigma}}(m) + \frac{1}{ag} \sum_m \vec{\sigma}(m).\vec{\sigma}(m+1)\} \tag{3.67}$$

We introduce the collective field q_{mn} in Z as

$$Z = \int \prod_m d\vec{\sigma}(m)\delta(\vec{\sigma}^2-1) \prod_{m,n} \{dq_{mn}\delta(q_{mn}-\vec{\sigma}(m).\vec{\sigma}(n))\}\exp\{iS(\vec{\sigma})\} \tag{3.68}$$

Since we want to calculate $\langle q_{mn} \rangle$, we introduce a source for the $\vec{\sigma}$-field as

$$Z(\vec{J}) = \int D\vec{\sigma}Dq\delta(q_{mn}-\vec{\sigma}(m).\vec{\sigma}(n))\delta(q_{mm}-1)\exp\{i(S(\vec{\sigma})+\int dt \sum_m \vec{\sigma}(m,t).\vec{J}(m,t))\}$$

$$= \int D\vec{\sigma}DqD\gamma\delta(q_{mn}-1)\exp(i\{\int dt(\frac{a}{2g}\dot{\vec{\sigma}}^2 + \frac{1}{ag}q_{m,m+1} + \vec{J}.\vec{\sigma} + \frac{1}{2}\gamma_{mn}(\vec{\sigma}.\vec{\sigma}-q_{mn}))\})$$

in which the $\vec{\sigma}$-integration can be done as

$$Z(\vec{J}) = \int Dq D\gamma \delta(q_{mm}-1)\exp\{-\frac{N}{2}\mathrm{tr}\ln O + i\int dt(\frac{1}{ag}q_{m,m+1} - \frac{1}{2}\gamma_{mn}q_{mn})$$

$$- \frac{i}{2}\int dtdt' J_\alpha(m,t)O^{-1}_{mn}(t,t')J_\alpha(n,t')\} \qquad (3.69)$$

where

$$O_{mn}(t,t') = (-\frac{a}{g}\partial_t^2\delta_{mn} + \gamma_{mn}(t))\delta(t-t') \qquad (3.70)$$

$$\int dt' \sum_n O_{mn}(t,t')O^{-1}_{nk}(t',t'') = \delta_{mk}\delta(t-t'') \qquad (3.71)$$

Define a new variable

$$\beta_{mn} = g\gamma_{mn} \qquad (3.72)$$

and write $Z(\vec{J})$ as

$$Z(\vec{J}) = (\text{constant})\int DqD\beta D\lambda \exp\{iNS(q,\beta,\lambda) - \frac{i}{2}\int dtdt'\vec{J}O^{-1}\vec{J}\} \qquad (3.73)$$

where the action $S(q,\beta,\lambda)$ is given by

$$S(q,\beta,\lambda) = \frac{i}{2}\mathrm{tr}\ln(-a\partial_t^2\delta_{mn}+\beta_{mn}) + \int dt(\frac{1}{ag_o}q_{m,m+1} - \frac{1}{2g_o}\beta_{mn}q_{mn}) - \frac{1}{2}\int dt\lambda_m(q_{mm}-1) \qquad (3.74)$$

From now on we shall drop all $\vec{J}$-independent multiplicative constant appearing in front of $Z(\vec{J})$. The required vacuum expectation value of q_{mn} is

$$\langle q_{mn}(t)\rangle_{vac} = \langle\sigma(m,t).\sigma(n,t)\rangle_{vac} = \frac{1}{i^2}\frac{\delta^2 \ln Z(\vec{J})}{\delta J_\alpha(m,t)\delta J_\alpha(n,t)}\Big|_{\vec{J}=0}$$

or,

$$<q_{mn}(t)>_{vac} = iN<O^{-1}_{mn}(t,t)>_{q,\beta,\lambda} \tag{3.75}$$

where $<(...)>_{q,\beta,\lambda}$ means

$$<(...)>_{q,\beta,\lambda} = \frac{\int DqD\beta D\lambda(...)\exp\{iNS(q,\beta,\lambda)\}}{\int DqD\beta D\lambda\exp\{iNS(q,\beta,\lambda)\}} \tag{3.76}$$

Since $g_0=gN$ is fixed when large N limit is taken, the action $S(q,\beta,\lambda)$ does not depend on N. because of the explicit presence of N in the exponent, equation (3.76) is very suitable for a diagramatic expansion in powers of 1/N. The clue is to find the saddle point $(q^0_{mn},\beta^0_{mn},\lambda^0_m)$ of the action $S(q,\beta,\lambda)$ as

$$\frac{\delta S(q,\beta,\lambda)}{\delta(q_{mn},\beta_{mn},\lambda_m)}\Big|_{q^0,\beta^0,\lambda^0} = 0 \tag{3.77}$$

If we now expand $S(q,\beta,\lambda)$ around the saddle point given by (3.77) as

$$S(q,\beta,\lambda) = S(q^0+\tilde{q},\beta^0+\tilde{\beta}^0+\ ,\lambda^0+\tilde{\lambda}) \tag{3.78}$$

the trln term becomes

$$\mathrm{trln}(-a\partial_t^2\delta_{mn}+\beta^0_{mn}+\tilde{\beta}_{mn}) = \mathrm{tr}\ \ln G^{-1} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{tr}(G\tilde{\beta})^n \tag{3.79}$$

where G is given by

$$(-a\partial_t^2\delta_{mn} + \beta^0_{mn})G_{nk}(t,t') = \delta_{mk}\delta(t-t') \tag{3.80}$$

The saddle points are given by (assuming static solutions)

$$q^o_{mm} = 1$$

$$\frac{1}{2ag_o}(\delta_{m,n+1}+\delta_{n,m+1}) - \frac{1}{2g_o}\beta^o_{mn} - \frac{1}{2}\lambda^o_m\delta_{mn} = 0 \qquad (3.81)$$

$$-\frac{1}{2g_o}q^o_{mn}(t) + \frac{i}{2}G_{mn}(t,t) = 0$$

As before we demand that q^o_{mn} and β^o_{mn} are functions of (m-n) only and λ^o_m independent of m. Then the solutions are given by

$$\lambda^o_m = \lambda$$

$$q^o_{mn} = \frac{1}{M}\sum_{j=1}^{M}\exp\{2\pi i(m-n)j/M\}q(j)$$

$$\beta^o_{mn} = \frac{1}{M}\sum_{j=1}^{M}\exp\{2\pi i(m-n)j/M\}\beta(j) \qquad (3.82)$$

$$q(j) = \{4a\lambda/g_o - 8\cos(2\pi j/M)/g_o^2\}^{-1/2}$$

$$\beta(j) = -g_o\lambda + \frac{2}{a}\cos(2\pi j/M)$$

and

$$G_{mn}(t,t') = \frac{1}{M}\sum_{j=1}^{M}e^{\{2\pi i(m-n)j/M\}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G(\omega,j)e^{i\omega(t-t')}$$

$$G(\omega,j) = \{a\omega^2 + \beta(j) + i\varepsilon\}^{-1},\ \varepsilon \to 0^+ \qquad (3.83)$$

Thus we get the same result as in equation (3.54).

To obtain a systematic 1/N expansion, we go back to $Z(\vec{J})$ which, after the shift in fields, becomes

$$Z(\vec{J}) = e^{iNS_{cl}} \int D\tilde{q} D\tilde{\beta} D\tilde{\lambda} \exp\{iN(\frac{i}{2}\mathrm{tr}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}(G\tilde{\beta})^n - \frac{1}{2}\int dt \tilde{q}_{mm}\tilde{\lambda}_m$$

$$- \frac{1}{2g_0}\int dt \tilde{\beta}_{mn}\tilde{q}_{mn}) - \frac{i}{2}\int dt dt' \vec{J}O^{-1}\vec{J}\} \qquad (3.84)$$

where S_{cl} is the classical value of the action $S(q,\beta,\lambda)$. So far as a general Green's function of the original $\vec{\sigma}$-field is concerned, we can integrate out the $\tilde{\lambda}$ and the $\tilde{q}$ fields and finally get

$$Z(\vec{J}) = \int D\tilde{\beta}_m \exp\{-\frac{N}{2}\mathrm{tr}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}(G\tilde{\beta})^n - \frac{i}{2}\int dt dt' \vec{J}(t)O^{-1}(t,t')\vec{J}(t')\} \qquad (3.85)$$

where $\tilde{\beta}_m$ is just the diagonal part of $\tilde{\beta}_{mn}$, i.e.,

$$\tilde{\beta}_m = \tilde{\beta}_{mm} \qquad (3.86)$$

Then $\langle q_{mn}\rangle$ is given by

$$\langle q_{mn}(t)\rangle = iN\langle O^{-1}_{mn}(t,t)\rangle_{\tilde{\beta}} = iN\langle O^{-1}_{mn}(t,t)\rangle_{\phi} \qquad (3.87)$$

where ϕ and $\langle(\ldots)\rangle_{\phi}$ are

$$\phi_m = \sqrt{N}\tilde{\beta}_m \qquad (3.88)$$

$$\langle(\ldots)\rangle_{\phi} = \frac{\int D\phi(\ldots)\exp S(\phi)}{\int D\phi \exp S(\phi)} \qquad (3.89)$$

with

$$S(\phi) = -\frac{1}{2}N^{1-n/2}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}\,tr(G\phi)^n \tag{3.90}$$

Equation (3.89) leads to a natural 1/N expansion in terms of Feynman diagrams. From (3.90) it is easy to see that the ϕ-propagator is of order N^0, the ϕ^3 vertex is of order $N^{-1/2}$, ϕ^4 vertex is of order N^{-1} and so on. In general, a ϕ^k vertex is of order $N^{1-k/2}$. The Feynman rules are

mt ———— nt' = ϕ-propagator = $-2D^{-1}_{mn}(t,t')$

k_1t_1 —<— k_2t_2, k_3t_3 = ϕ^3 vertex = $\frac{-1}{6\sqrt{N}}G_{k_1k_2}(t_1,t_2)G_{k_2k_3}(t_2,t_3)G_{k_3k_1}(t_3,t_1)$

and similar rules for other vertices. We have not shown them explicitly because we do not need them to calculate $\langle q_{mn}\rangle$ to order 1/N. The matrix $D_{mn}(t,t')$ is defined by

$$D_{mn}(t,t') = G_{mn}(t,t')G_{nm}(t',t) \tag{3.91}$$

Expanding O^{-1} in (3.87) as

$$\begin{aligned} O^{-1} &= \{\tfrac{1}{g}(-a\partial_t^2\delta_{mn} + \beta^0_{mn} + \tfrac{1}{\sqrt{N}}\phi_m\delta_{mn})\}^{-1} \\ &= g\{G^{-1} + \tfrac{1}{\sqrt{N}}\phi\}^{-1} \\ &= g\{G - \tfrac{1}{\sqrt{N}}G\phi G + \tfrac{1}{N}G\phi G\phi G + O(N^{-3/2})\} \end{aligned} \tag{3.92}$$

we get, upto order $1/N$,

$$\langle q_{mn}(t)\rangle = ig_0\{G_{mn}(t,t) - \frac{1}{\sqrt{N}}\int dt' G_{mk}(t,t')\langle\phi_k(t')\rangle G_{kn}(t',t)$$

$$+ \frac{1}{N}\int dt'dt'' G_{mk}(t,t')G_{kl}(t',t'')G_{ln}(t'',t)\langle\phi_k(t')\phi_l(t'')\rangle \quad (3.93)$$

The tadpole $\langle\phi_k(t)\rangle$ can be calculated (upto order $N^{-1/2}$) as

$\langle\phi_k(t)\rangle$ = k,t ——○

$$= -\frac{2}{\sqrt{N}}\frac{1}{M^2}\sum_{j_1,j_2}\int\frac{d\omega_1}{(2\pi)}\frac{d\omega_2}{(2\pi)}\{D^{-1}(0,0)D^{-1}(\omega_1-\omega_2,j_1-j_2)\times$$

$$G^2(\omega_1,j_1)G(\omega_2,j_2)\} \quad (3.94)$$

and finally

$$\langle q_{mn}(t)\rangle = ig_0(\ a + b + c\ )$$

$$a = \frac{1}{M}\sum_j \exp\{2\pi i(m-n)j/M\}\int\frac{d\omega}{2\pi}\,G(\omega,j)$$

$$b = -\frac{\langle\phi\rangle}{\sqrt{N}M}\sum_j \exp\{2\pi i(m-n)j/M\}\int\frac{d\omega}{2\pi}\,G^2(\omega,j)$$

$$c = -\frac{2}{NM^2}\sum_{j_1,j_2}\int\frac{d\omega_1}{(2\pi)}\frac{d\omega_2}{(2\pi)}\exp\{2\pi i(m-n)j_1/M\}G^2(\omega_1,j_1)\times$$

$$G(\omega_1-\omega_2,j_1-j_2)D^{-1}(\omega_2,j_2) \quad (3.95)$$

In order to simplify the calculation, we shall use the same approximation

as we did in equation (3.65), i.e.,

$$G_{mn}(t,t') = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{\exp\{i\omega(t-t')+i\theta(m-n)\}}{a\{\omega^2-\mu^2-(4/a^2)\sin^2(\theta/2)+i\varepsilon\}}$$

$$\simeq \int\frac{d^2k}{(2\pi)^2}\frac{\exp(ik.x)}{k^2-\mu^2+i\varepsilon} \qquad (3.96)$$

where

$$k^2 = k_\mu k_\mu = \omega^2-k_1^2$$

$$\theta = 2\pi j/M = ak_1$$

$$\sin^2(\theta/2) \simeq \theta^2/4$$

$$a^2\mu^2 = a\lambda g_0-2 \qquad (3.97)$$

We trust this approximation only for weak coupling. Performing a "Wick rotation",

$$\frac{1}{g_0}\langle q_{mn}(t)\rangle = \int\frac{d^2k}{(2\pi)^2}\,\frac{e^{ik_1a(m-n)}}{k^2+\mu^2+\Sigma(k)}$$

$$\Sigma(k) = \frac{2}{11}\{\int\frac{d^2q}{(2\pi)^2}2G(k-q)D^{-1}(q)-\int\frac{d^2qd^2p}{(2\pi)^2}D^{-1}(0)D^{-1}(q)G\ (p)G(p-q)\}$$

$$G(p) = (p^2+\mu^2)^{-1} = (p_1^2+p_2^2+\mu^2)^{-1}$$

$$D(p) = \int\frac{d^2q}{(2\pi)^2}2G(q)G(q-p) = \frac{\ln\{(\sqrt{p^2+4\mu^2}+\sqrt{p^2})/(\sqrt{p^2+4\mu^2}-\sqrt{p^2})\}}{2\pi\sqrt{p^2(p^2+4\mu^2)}} \qquad (3.98)$$

See appendix B for derivation of the last formula in (3.98). Equation

(3.98) was first derived by Abe and Hikami using a slightly different approach[28]. Our approach should be advantageous if one wishes to compute general Green's functions involving $\vec{\sigma}$-field.

To calulate mass-correction to order 1/N, we expand $\Sigma(p)$ around $p^2=-\mu^2$ as

$$\Sigma(p) = \Sigma_1 + (p^2+\mu^2)\Sigma_2 + \Sigma_3(p)$$

$$\Sigma_1 = \Sigma(p^2=-\mu^2) \tag{3.99}$$

$$\Sigma_2 = \frac{\partial\Sigma(p)}{\partial p^2}\bigg|_{p^2=-\mu^2}$$

The wave-function renormalisation constant Z_2 for the $\vec{\sigma}$-field is then

$$Z_2 = 1 - \Sigma_2 \tag{3.100}$$

and the mass correction $\delta\mu^2$ is

$$\delta\mu^2 = \Sigma_1 = \Sigma(p^2=-\mu^2) \tag{3.101}$$

Since the expression for $\Sigma(p)$ is divergent, we put an euclidean cut-off Λ and get

$$\delta\mu^2 = -\frac{4\mu^2}{N}\int_0^{\lambda} f(x)dx$$

$$\lambda = \Lambda^2/4\mu^2$$

$$f(x) = \frac{1}{2(x+1)} + \frac{\sqrt{x/(x+1)}-1}{\ln\{\frac{(\sqrt{(x+1)}+\sqrt{x})}{(\sqrt{(x+1)}-\sqrt{x})}\}} \tag{3.102}$$

We shall define the β-function as

$$\beta(g) = -\Lambda \frac{dg}{d\Lambda} \tag{3.103}$$

and evaluate it from the condition

$$\frac{d}{d\Lambda}(\mu^2+\delta\mu^2) = 0 \tag{3.104}$$

The reader should note that the β-function will, in general, be different from the β-function we used before. However, for weak coupling, they should be same since g=0 is the critical coupling constant. To order 1/N, we find (see appendix B)

$$\beta(g) = -\frac{g_o^2(1-\alpha)}{2\pi N\alpha}\{1 - \frac{\alpha-1}{N} f(\lambda)\}$$

$$\alpha = \exp(4\pi/g_o) \tag{3.105}$$

which, for weak coupling, reduces to the known result[25]

$$\beta(g) \simeq \frac{g^2(N-2)}{2\pi} + \frac{g^3N}{4\pi^2} \tag{3.106}$$

IV. CONCLUSION.

It is generally believed that to solve the important problems of QCD like color confinement and the structure of hadrons, it is essential to go beyond perturbation theory and develop suitable non-perturbative formalisms. At the same time, this must also be compatible with the asymptotic freedom one observes at short distances. In this thesis, we have attempted to develop two such non-perturbative formulations; the modified WKB approximation and the collective field approach.

The modified WKB-variational approach seems to be suitable for describing theories which have tunnelling and in the model we discussed (the compact quantum electrodynamics in three space-time dimensions) there is a generation of mass-gap. In the language of euclidean functional integration, this is due to long-range pseudo-particle interactions which means that different tunnelling events are somewhat strongly correlated. This is reproduced in our formalism by treating the long-range infra-red modes correctly through a variational approach. The same thing, in principle, can be done for SU(3) pure Yang-Mills theory which has instantons. However, we are not sure at this point whether the instantons are sufficient to lead to confinement or whether one has to include more complicated field configurations. Possibly a suitable variational ansatz can overcome this difficulty, but we have not found it yet.

Another approach we have discussed is the collective field method which has already been successfully applied to a large class of models like the vector models, matrix models, U(N) gauge theory and others. The problem of gauge invariance is avoided since all collective

field variables are manifestly singlets. This seems to be a promising method in the sense that large N limit is achieved rather naturally and the technical problem of summing planar diagrams is reduced to solving a classical equation. As we have shown the collective field can also be introduced into the functional integral formulation and 1/N corrections can be computed. Application of this method to various other models are presently under study.

Appendix A. Cancellation of Divergence in O(N) Model.

In this appendix we shall prove the cancellation of leading divergences in the ground state energy of H_{eff} in equation (3.46) by the counterterm ΔV given by (3.48). The Hamiltonian is

$$H_{eff} = \frac{2g}{a} \sum p_{mk} \Omega(m,k,k') p_{mk'} + NV_o(q) + \Delta V \tag{A.1}$$

We have shown that the large N limit is achieved by minimising $V_o(q)$ subject to the constraint (3.31). Let us call this solution q^o. Then we expand q as

$$q_{mn} = q^o_{mn} + Q_{mn} \tag{A.2}$$

in H_{eff} and keep upto quadratic part. Classically the ground state energy is given by $NV_o(q^o)+ V(q^o)$. We want to find the quantum correction as the sum of all zero-point energies. To achieve this, we write the quadratic part of $NV_o(q)$ as

$$NV_o(q)\Big|_{quadratic} = \frac{gN^2}{8a} tr(q^{o^{-1}} Q q^{o^{-1}} Q q^{o^{-1}}) \tag{A.3}$$

The kinetic energy (upto quadratic in fluctuation) is

$$K.E\Big|_{quadratic} = \frac{2g}{a} \sum p_m(k) \Omega^o_m(k,k') p_m(k') \tag{A.4}$$

$$p_m(k) = p_{mk}, \quad \Omega^o_m(k,k') = \Omega(m,k,k')\Big|_{q=q^o} \tag{A.5}$$

To find the quantum zero-point energy of the quadratic part, we define a transformation of Q as

$$Q'_m(k) = \sum_{k'} \alpha_m(k,k')Q_m(k') \tag{A.6}$$

Then, $p_m(k)$ transforms as

$$p_m(k) = -i\frac{\partial}{\partial Q_m(k)} = \sum_{k'} \alpha_m(k',k)\left(-i\frac{\partial}{\partial Q'_m(k')}\right) = (\alpha_m^T p'_m)(k) \tag{A.7}$$

where are using an obvious matrix notation (T stands for transpose). So we are treating the indices m and k in Q_{mk} as belonging to two different types of labels (for example, one may think of "m" as an internal symmetry index and "k" as a lattice index). The transformation (A.6) is defined for each m. Our purpose is to choose the matrix $\alpha_m(k,k')$ such that the kinetic energy term (A.4) becomes diagonal, i.e., we assume

$$p_m^T \Omega_m^0 p_m = p_m'^T(\alpha_m \Omega_m^0 \alpha_m^T)p'_m = p_m'^T p_m \tag{A.8}$$

which means, for each m,

$$\alpha_m \Omega_m^0 \alpha_m^T = 1 \tag{A.9}$$

To find such α_m, let us write Ω^0 as

$$\Omega_m^0(k,k') = q^0(k,k') - q^0(m,k)q^0(m,k') = q^0(k,k') - B_m(k,k') \tag{A.10}$$

where

$$B_m(k,k') = q^0(m,k)q^0(m,k') \tag{A.11}$$

and define β and γ_m as

$$q^o(k,k') = (\beta^T\beta)(k,k')$$

$$\alpha_m^T = \beta^{-1}\gamma_m \tag{A.12}$$

Then condition (A.9) yields

$$\gamma_m^T(\beta^{-1})^T(q^o-B_m)\beta^{-1}\gamma_m = 1 \tag{A.13}$$

or,

$$(\gamma_m^T)^{-1}\gamma_m^{-1} = 1 - (\beta^{-1})^T B_m\beta^{-1} \tag{A.14}$$

Solution for γ_m is

$$\gamma_m^{-1} = 1 - (\beta^T)^{-1}B_m\beta^{-1} \tag{A.15}$$

So we can write the quadratic part of H_{eff} as

$$H_{eff}\Big|_{quad} = \frac{2g}{a}\sum p_m'^T p_m' + \frac{gN^2}{8a}\sum Q_m'^T D_{mm''}^T D_{m''m'}Q_{m'}' \tag{A.16}$$

where we have defined

$$D_{mm'}(k,k') = \sum_p W_{mm'}(k,p)\alpha_{m'}^{-1}(p,k')$$

$$W_{mm'}(k,k') = q^{o^{-1}}(m,m')(\beta^T)^{-1}(k,k') \tag{A.17}$$

We can now split matrix D as

$$D = D^{(o)} + D^{(1)}$$

$$D^{(o)}_{mm'}(k,k') = \sum_{p} W_{mm'}(k,p)\beta^{T}(p,k') \tag{A.18}$$

$$D^{(1)}_{mm'}(k,k') = -\sum_{p,q} W_{mm'}(k,p)B_{m'}(p,q)\beta^{-1}(q,k')$$

Notice that D is a double matrix (in two different indices) and it can be shown that

$$\text{tr } D^{(o)} = M \text{ tr } q^{o^{-1}}, \quad \text{tr } D^{(1)} = -\text{tr } q^{o^{-1}} \tag{A.19}$$

where M is the total number of lattice points. Thus the zero-point energy is

$$E = \frac{1}{2}\left(\frac{4g}{a} \cdot \frac{gN^2}{4a}\right)^{1/2} \text{tr}D = \frac{gN}{2a}(M-1)\text{tr}(q^{o^{-1}}) \tag{A.20}$$

Recall that the leading divergence in ΔV is

$$\Delta V(q^o) = -\frac{gN}{2a}\left(M+\frac{1}{2}\right) \text{tr}(q^{o^{-1}}) + O(1/N) \tag{A.21}$$

which exactly cancels the divergence in (A.20) proportional to M.

Appendix B. Numerical Details for Computing 1/N Corrections.

We first derive the expression for D(p) given by

$$D(p) = \int\frac{d^2k}{4\pi^2}\,(k^2+m^2)^{-1}\{(k-p)^2+m^2\}^{-1}$$

$$= \int\frac{d^2k}{4\pi^2}\int_0^\infty dx\int_0^\infty dy\,\exp\{-x(k^2+m^2)-y(k^2+p^2-2p.k+m^2)\}$$

$$= \int dx\int dy\,\exp\{-m^2(x+y)-\frac{xyp^2}{x+y}\int\frac{d^2k}{4\pi^2}e^{-(x+y)k^2}$$

$$= \int\frac{dxdy}{4\pi(x+y)}\exp\{-m^2(x+y)-\frac{xyp^2}{x+y}\} \qquad (B.1)$$

Changing variable to $2x=u+v$, $2y=u-v$, we get

$$D(p) = \int_0^\infty\frac{du}{2}\int_{-u}^{u}\frac{dv}{4\pi u}\,\exp\{-(m^2+\frac{p^2}{4})u+\frac{v^2p^2}{4u}\}$$

$$= \frac{1}{4\pi(m^2+p^2/4)}\sum_{n=0}^{\infty}\frac{x^n}{(2n+1)}$$

$$= \frac{F(1,1/2,3/2,x)}{(p^2+4m^2)} \qquad (B.2)$$

where F is the confluent hypergeometric function and x is given by

$$x = p^2/(p^2+4m^2) \qquad (B.3)$$

Actually, in a general d-dimensional space,

$$D(p) = \Gamma(2-\frac{d}{2})2^{-d}\pi^{-d/2}(m^2+\frac{p^2}{4})^{-2+d/2}F(2-\frac{d}{2},\frac{1}{2},\frac{3}{2},x) \qquad (B.4)$$

Using the formula

$$\sum_{n=0}^{\infty} \frac{x^n}{2n+1} = \frac{1}{2\sqrt{x}} \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}} \right) \tag{B.5}$$

we get the expression for D(p) in two dimension as

$$D(p) = \frac{1}{2\pi\sqrt{(p^2(p^2+4m^2))}} \ln\left(\frac{\sqrt{(p^2+4m^2)}+\sqrt{p^2}}{\sqrt{(p^2+4m^2)}-\sqrt{p^2}} \right) \tag{B.6}$$

To get β-function, we first note that

$$\mu^2+\delta\mu^2 = \mu^2\left(1- \frac{4}{N_0}\int_0^{\infty} f(x)dx\right) \tag{B.7}$$

where f(x) and are given by (3.102). The relation between μ^2 and cut-off Λ is

$$\Lambda^2 = \mu^2(\alpha-1)$$

$$\alpha = \exp(4\pi/g_0) \tag{B.8}$$

Now it is straightforward to derive equation (3.105) for β-function from equations (3.103), (3.104) and (B.8). It is also easy to show that the asymptotic expansion for f(x):

$$f(x) \underset{x\to\infty}{\to} \frac{1}{2x} - \frac{1}{2x\ln(4x)} + O(1/x^2)$$

$$f(x) \underset{x\to 0}{\to} -\frac{1}{2\sqrt{x}} + 1 + O(\sqrt{x}) \tag{B.9}$$

For weak coupling limit,

$$\lambda = \frac{\alpha-1}{4} \simeq \frac{\alpha}{4} >> 1$$

$$f(\lambda) \simeq \frac{2}{\alpha} \qquad (B.10)$$

and we get (3.106).

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