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STOCHASTIC QUANTIZATION AND THE LARGE N REDUCTION OF QUANTUM FIELD THEORIES

by

JORGE ALFARO

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

STOCHASTIC QUANTIZATION AND THE LARGE N REDUCTION OF QUANTUM FIELD THEORIES

by

JORGE ALFARO

Adviser: Professor B. Sakita

We discuss the Reduction in the large N limit of field theories using the stochastic quantization of Parisi and Wu. The quenched momentum prescription is derived for both globally and locally U(N) symmetric theories in the continuum. Using our approach, the Gross-Kitazawa constraint is not needed to reduce the U(N) gauge theory. Furthermore, the stochastic quantization is extended to include U(N) variables and applied to derive the quenched momentum prescription of U(N) lattice gauge theory. Finally, we use the stochastic regularization to compute the one loop large N, mass renormalization in the four dimensional NxN Hermitian matrix model.

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Table of contents.

I. INTRODUCTION AND SUMMARY.

The Yang-Mills theory of three colored quarks and gluons, Quantum Chromodynamics (QCD), is today widely accepted as the theory of strong interactions.

This is mainly due to the successful predictions of the theory in experiments testing the short distance properties of hadrons(1). In this region the hadrons behave as a set of point-like, almost free particles; QCD incorporates this behavior of hadrons and makes rigorous predictions about it, because of the property of non-abelian gauge theories known as asymptotic freedom(2). In an asymptotically free theory, the effective coupling constant (running coupling constant) goes to zero for large momenta (small distance); and therefore the strength of the interaction falls off and a perturbative expansion in the effective coupling constant is justified.

It happens, however, that most of the common properties of nuclear matter, e.g., masses, lifetimes and decay ratios of elementary particles are of a non-perturbative nature. Moreover, one of the most fundamental problems of quark physics, the confinement of quarks inside colorless hadrons, is out of the reach of weak coupling perturbation theory.

Considerable progress in the understanding of QCD has come from the use of computer simulations to study the lattice version of the theory[3,4]. In this approach, the theory is regulated by defining it on a lattice. The lattice spacing acts as a gauge invariant regulator that must be taken to zero to recover the continuum theory. Although the theory loses manifest Poincare invariance, it possesses explicit gauge invariance. Moreover, since for a non-zero lattice spacing the correlation functions are finite, they can be conveniently evaluated by computer simulation (Monte Carlo methods).

If we wish to have analytic results about QCD, in regions where the usual perturbation expansion is not applicable, some other approximation to the model should be devised; an approximation independent of the value of e (in fact e is not a free parameter in QCD, since through the renormalization group it fixes the scale of masses in the theory[32]). This seems hopeless, since QCD is a one coupling constant theory.

Nevertheless, there is a hidden parameter in QCD discovered by 'tHooft[5]. He suggested generalizing QCD, the gauge theory based on SU(3) group, to a gauge theory based in U(N) group. The theory simplifies for N large and the hope is that this is a good approximation for N=3.

To see this, consider U(N) Gauge theory which has one coupling constant e. This theory can be quantized in the usual way, fixing a gauge and adding the necessary Fadeev-Popov ghosts. However, for our purpose, we just need to count powers of e and N in Feynman's diagrams.

By counting powers of e and N in Feynman graphs, it is found that planar graphs have factors of $(e^2N)^a$ $a=0,1,2....$ All others Feynman graphs have factors of $(e^2N)^a/N^b$, $b=2,3,...$

The large N limit is defined by making N large, but e^2N fixed. In this way planar graphs dominate, and all non-planar graphs are neglected.

In some simple cases, e.g. Two Dimensional Quantum Chromodynamics, the large N limit can be computed analytically and the results are encouraging: a simple integral equation gives the masses of quark-anti-quark bound states (mesons)(5). Moreover, the elastic scattering of mesons is mediated by the interchange of other mesons and not of unphysical quarks(6). This agrees with what is expected according to Regge phenomenology(31). In four dimensions, no one has succeded in summing the planar series, but general properties of the approximation can be deduced(7)

In studying the large N limit of U(N) lattice gauge theory, Eguchi and Kawai(8) discovered that the Schwinger-Dyson equations of this model coincide with the Schwinger-Dyson equations of a reduced model (Eguchi-Kawai model). The reduced model consists of a one site lattice over which we define a vector valued U(N) matrix. Although they claimed that this remarkable coincidence holds for arbitrary values of the coupling constant, it was found by

Bhanot, Heller and Neuberger[9] that the coincidence is not true for weak coupling. For small coupling constant, a $(U(1))^d$ symmetry of the Eguchi Kawai model, which is necessary to prove the equivalence of the two sets of Schwinger-Dyson equations, is spontaneously broken (d is the dimension of space time). The spontaneous breakdown of the symmetry is due to quantum fluctuations around the minima of the reduced action, which dominate the weak coupling limit of the theory. To restore the symmetry, the authors of reference (9) propose to quench the minima of the reduced action.

In a short paper by Parisi[10], the quenching prescription was generalized to global symmetries as well.

Subsequently, Gross and Kitazawa[11], and Das and Wadia[12] (see also Migdal[13]) proved the quenched momentum prescription, both in the continuum and in the lattice, to all orders in perturbation theory. The novel feature incorporated by these authors into the reduction of U(N) gauge theory is the introduction of additional constraints. Without these constraints, the quenched model would reduce to the Eguchi-Kawai model.

The purpose of this thesis is to report on a study of the reduction of degrees of freedom in the large N limit of field theories using the stochastic quantization of Parisi and Wu[14].

In Section III.B of this work, we use the stochastic quantization method of Parisi and Wu to rederive the quenched momentum prescription for the large N Hermitian matrix model. The basic idea of our derivation of the quenched momentum prescription is quite simple. In the stochastic quantization method, the correlation functions of Euclidean field theory are obtained by i)solving the Langevin equation in the presence of a random noise η, ii) taking the random average with respect to η iii) letting the fictitious time go to infinity. We first note, that in the large N limit, it is possible to replace the random white noise appearing in the Langevin equation of stochastic quantization by a special reduced form without losing the Gaussian distribution property of the original white noise. Therefore the whole content of the original theory is reproduced in this limit by solving the Langevin equation with this particular form of reduced random noise. The solution to the equation is obtained from the solution of a reduced Langevin equation, accordingly a reduced model.

We shall consider in section III.C. the locally symmetric gauge theory, which is most intriguing in the large N reduction, since, as it has been shown in references (11) and (12), new complicated constraints have to be imposed in order to reduce the theory consistently. (See reference (15) for a different approach to this problem).

In the usual method of quantization (path integrals) constraints are needed whenever we reduce a theory with zero mass parameters. Without some constraint the reduced propagator of the field is singular, due to the contribution coming from its diagonal components. Formally these components contribute only non-leading terms in the large N expansion. However their effect is already divergent for finite N, if massless particles exist in the model, making it impossible to neglect them by being non-leading terms. So we are forced to eliminate them by introducing a constraint by hand. Clearly, one of the roles of the constraint is as a regulator of the otherwise singular reduced theory.

In the stochastic quantization the fictitious time provides us with a natural regulator. As long as t is finite we can neglect non-leading N terms safely.

However, there is another aspect of the reduction which merits separate attention. If we start with either a globally or locally U(N) symmetric theory the Green's functions will satisfy a set of Ward's identities reflecting the symmetry of the underlying dynamics. In general, these identities are not satisfied by the reduced Green's functions, except when they are originally invariant under the symmetry group, in which case they are obviously satisfied. In section III.C.2 we shall remark on how such a constraint could be used to remove this defect in the reduced theory.

The point mentioned in the previous paragraph does not introduce any complication for the reduction of the matrix model, since there all correlations we are interested in, being invariant under the global U(N) group, satisfy the Ward's identities trivially.

It is in the reduction of the gauge theory that problems appear, for the usual quantization procedure utilizes Green's functions which, being globally invariant, are not locally invariant under U(N). For them the Ward's identities implied by the local U(N) invariance are not true in the reduced theory.

However, if we agree to work with gauge invariant correlation functions only (strictly speaking, these are the ones we can calculate in the stochastic quantization without gauge fixing), then the reduced Langevin equation, <u>without extra constraints</u> will reproduce the full $U(\infty)$ gauge theory.

The arguments introduced above are somewhat formal in nature because of the need to renormalize the theory. As is known, the reduced Langevin equation will reproduce the Green's functions of the original theory only if those have been previously regulated to make them at most logarithmically divergent, so that shifting of momentum integration variables is permitted. In the case of the gauge theory the method of higher covariant derivatives could be used to regulate the model. But we want to argue that, since the reduced Langevin equation of the gauge theory is gauge covariant for fixed random momenta P_μ and since we agree to use gauge invariant correlation functions only (for them the Ward identities of the original theory are satisfied), then we may avoid a divergence higher than logarithmic in loop momentum integrals and, for this very reason the Λ cutoff defining the reduced random noise is an adequate regulator. Instead of proving the last statement we offer a perturbative calculation supporting this claim. (Section III.C.3).

Nevertheless, in exchange for avoiding the constraint to reduce the gauge theory, we have to reformulate the whole content of the model employing gauge invariant Green's functions only.

Continuing our work we analyze the reduction of the U(N) lattice gauge theory. First we find the Langevin equation appropiate for stochastic quantization over the U(N) group. Following this, the same method of reduction developed in Section III is used to obtain the reduced model, which coincides with the quenched Eguchi- Kawai model introduced by Das and Wadia[12].

Finally, we use the recently invented technique of stochastic regularization[16] to clarify the formal problems, refered to above, introduced by the divergences of quantum field theories.

II. STOCHASTIC QUANTIZATION.

A. Introduction.

The most widely used methods of quantization, the path integral and Hamiltonian formalism, lead to complications when they are applied to Gauge Theories[17]. In these models, the local gauge invariance of the action implies the non-existence of the gauge field propagator and renders the perturbation theory invalid.

The problem created by the local gauge invariance is avoided in both methods by introducing a gauge condition. The gauge condition would select just one point from each orbit of the gauge group. Furthermore, and in order to get the correct path integral measure, Fadeev-Popov ghosts should be added to the action. In this way an overall infinite constant is factored out (the volume of the local gauge group), permitting the definition of a sensible perturbation theory.

Although the gauge fixing does not introduce new difficulties for Abelian gauge theories or perturbative calculations in the non-Abelian case, it is expected to fail for large non-Abelian field configurations. For such sufficiently large configurations the orbits of the gauge group will intersect the hyperplane defined by the gauge condition several times, thus making it impossible to fix the gauge completely (Gribov ambiguity)[18]. The Gribov ambiguity will manifest itself in regions of a vanishing Fadeev-Popov determinant, in non-perturbative calculations. In the neighborhood of these regions, the standard quantization procedures will not be applicable.

Below, we shall discuss a different method of quantization, the so-called stochastic quantization method of Parisi and Wu[14]. The main advantage of this method is that we do not need to fix a gauge to calculate the relevant (gauge invariant) Green's functions in a Gauge Theory.

The stochastic quantization of Parisi and Wu emerges as an analogy between equilibrium Statistical Mechanics and Euclidean Quantum Field Theory. In both cases, we need to compute correlation functions (vacuum expectation values) in an ensemble with a weight factor given by the Boltzmann distribution e^{-S}, S being the Euclidean action in Quantum Field Theory or the Hamiltonian of the Statistical System.

The idea is to introduce an extra parameter (fictitious time) t and postulate adequate stochastic evolution equations in t. These equations have to be such that for $t\to\infty$ the stochastic averages must be determined by the canonical distribution (Boltzmann's distribution). In a sense, the new parameter t measures the approach to equilibrium of the stochastic system.

There is some freedom to select the stochastic equations with the right behavior for large t. For the moment, we select the simplest possibility and postpone the discussion of the general case to section II.C.

B. Rules of Stochastic Quantization

Let us now turn to the mathematical formulation of the method; we shall restrict our discussion to bosonic variables for the sake of simplicity[28].

The physical content of a Quantum Field Theory, defined by the Euclidean classical action S, is describable using Green's functions (Correlation functions).

$$\langle \phi_{l_1}(x_1) \ldots \phi_{l_n}(x_n) \rangle = \frac{\int D\phi \; \phi_{l_1}(x_1) \ldots \phi(x_{l_n}) \, e^{-S}}{\int D\phi \; e^{-S}} \tag{2.1}$$

$\phi_l(x)$ is a boson field defined on the space time point x; l denotes any set of internal degrees of freedom and Lorentz's indices as well.

We shall say that $\eta_l(x,t)$ is a random variable with Gaussian distribution if the following properties are satisfied:

$$\langle \eta_l(x,t) \, \eta_{l'}(x',t') \rangle_\eta = 2 \, \delta_{ll'} \, \delta(x-x') \, \delta(t-t') \tag{2.2}$$

$$\langle \eta_{\ell_1}(x_1,t_1)\,\eta_{\ell_2}(x_2,t_2)\ldots\eta_{\ell_{2n}}(x_{2n},t_{2n})\rangle_\eta =$$

$$= \sum_{\substack{\text{possible} \\ \text{combinations}}} \prod_{\text{pair}} \langle \eta_{\ell_i}(x_i,t_i), \eta_{\ell_j}(x_j,t_j)\rangle_\eta \tag{2.3}$$

We shall refer to (2.3) as "Wick's decomposition property". Equations (2.2) and (2.3) may also be considered the definition of $\langle\ \rangle_\eta$. A concrete representation of the Gaussian average is provided by the path integral formula:

$$\langle \eta_{\ell_1}(x_1,t_1)\ldots\eta_{\ell_n}(x_n,t_n)\rangle_\eta =$$

$$= \frac{\int \mathcal{D}\eta \;\; \eta_{\ell_1}(x_1,t_1)\ldots\eta_{\ell_n}(x_n,t_n)\; e^{-1/4\int dx\,dt\,\sum_\ell \eta_\ell(x,t)^2}}{\int \mathcal{D}\eta \;\; e^{-1/4\int dx\,dt\,\sum_\ell \eta_\ell(x,t)^2}} \tag{2.4}$$

Below we shall list the steps to calculate Green's function (2.1) by means of the stochastic quantization. We shall provide a general proof of equivalence between stochastic quantization and the path integral method of quantization in the next section.

Firstly, we solve the following stochastic evolution equation (Langevin equation)

$$\frac{\partial}{\partial t}\phi_\ell(x,t) = -\frac{\delta S}{\delta \phi_\ell(x,t)} + \eta_\ell(x,t) \tag{2.5a}$$

$$\phi_\ell(x,0) = \phi_\ell^0(x) \tag{2.5b}$$

Next, we calculate the average of,

$$\langle \phi_{\ell_1}(x_1,t),\ \phi_{\ell_2}(x_2,t) \ldots \phi_{\ell_n}(x_n,t) \rangle_\eta \tag{2.6}$$

where $\phi_\ell(x,t)$ is the solution of the Langevin equation (2.5). Notice that (2.6) contains the variables evaluated at equal fictitious times only.

Finally, the large t limit of equation (2.6) gives the Euclidean Green's function (2.1). That is:

$$\langle \phi_{\ell_1}(x_1,t) \ldots \phi_{\ell_n}(x_n,t) \rangle_\eta \xrightarrow[t\to\infty]{} \langle \phi_{\ell_1}(x_1) \ldots \phi_{\ell_n}(x_n) \rangle \tag{2.7}$$

In most of the cases we shall encounter in this work, the choice of initial condition ($\phi_\ell^0(x)$) will be irrelevant, since for large t, the system is not sensitive to it (See equation (2.29)). From now on we shall take $\phi_\ell^0(x)=0$.

The solution of (2.5) is a function of x,t and a functional of η_1, but in order to simplify the notation we have not written the η-dependence of ϕ_l (x,t) explicity. It must, however, always be kept in mind.

Now, we present a simple proof of the equality (2.4) which not only makes clear the limitations of the method but also helps to generalize the method to other theories such as the U(N) lattice gauge theory. (See Section IV)

C. Fokker-Planck Equation[19,20]

We denote the probability distribution at t by $P(\phi, t)$, and define it by

$$\langle F(\phi_l(x,t))\rangle_\eta = \int D\phi \; F(\phi_l(x))\, P[\phi_l(x), t] \tag{2.8}$$

where $F(\phi(x))$ is an arbitrary functional of $\phi(x)$ and P is normalized as $\int D\phi P(\phi, t) = 1$. We then take a time derivative of (2.8). The left hand side is then given by

$$\sum_l \int dx \left\langle \dot{\phi}(x,t) \frac{\delta F[\phi(x,t)]}{\delta \phi_l(x,t)} \right\rangle_\eta =$$

$$= \sum_l \int dx \left[\left\langle -\frac{\delta S}{\delta\phi_l(x,t)} \frac{\delta F[\phi(x,t)]}{\delta \phi_l(x,t)} \right\rangle_\eta + \left\langle \eta_l(x,t) \frac{\delta F[\phi(x,t)]}{\delta \phi_l(x,t)} \right\rangle_\eta \right] \tag{2.9}$$

where we have used the Langevin equation (2.5) for $\dot{\phi}_l(x,t)$. Using (2.8) we can write the first term of this expression as

$$\sum_l \int dx \int \mathcal{D}\phi \left(-\frac{\delta S}{\delta\phi_l(x)}\frac{\delta F}{\delta\phi_l(x)}\right) P[\phi,t] =$$

$$= \int \mathcal{D}\phi\ \ F[\phi] \sum_l \int dx \frac{\delta}{\delta\phi_l(x)}\left(\frac{\delta S}{\delta\phi_l(x)} P[\phi,t]\right)$$

(2.10)

Using the definition of η-average (2.4), we write

$$\langle \eta_l(x,t) \frac{\delta F}{\delta\phi_l(x,t)}\rangle_\eta = 2 \langle \frac{\delta}{\delta\eta_l(x,t)} \frac{\delta F}{\delta\phi_l(x,t)} \rangle_\eta =$$

$$= 2 \int dx' \sum_{l'} \langle \frac{\delta\phi_{l'}(x',t)}{\delta\eta_l(x,t)} \quad \frac{\delta^2 F}{\delta\phi_l(x,t)\,\delta\phi_{l'}(x',t')} \rangle_\eta =$$

$$= \langle \frac{\delta^2 F}{\delta\phi_l^2(x,t)} \rangle_\eta = \int \mathcal{D}\phi\ F \ \frac{\delta^2}{\delta\phi_l^2(x)} P[\phi,t]$$

(2.11)

In this calculation we have used:

$$\frac{\delta}{\delta \eta_{\ell}(x,t)} \phi_{\ell'}(x,t) = 1/2 \, \delta(x-x') \, \delta_{\ell\ell'}$$

(2.12)

to be proved later. Combining (2.8), (2.9), (2.10)and (2.11) we obtain the equation for $P[\phi,t]$, the Fokker-Planck equation:

$$\frac{\partial}{\partial t} P[\phi,t] = \sum_{\ell} \int dx \, \frac{\delta}{\delta \phi_{\ell}(x)} \left(\frac{\delta}{\delta \phi_{\ell}(x)} + \frac{\delta S}{\delta \phi_{\ell}(x)} \right) P[\phi,t]$$

(2.13)

The initial condition to be used with (2.13) is obtained from the initial condition of the Langevin equation (2.5b) and the definition of $P(\phi;t)$ (2.8):

$$P[\phi,0] = \prod_{x,\ell} \delta[\phi_{\ell}(x) - \phi_{\ell}^{0}(x)]$$

(2.14)

In order to prove (2.12), we construct a differential equation for $\frac{\delta \phi_{\ell}(x,t)}{\delta \eta_{\ell'}(x',t')}$ by taking a functional derivative of the Langevin equation (2.5a) with respect to $\eta_{\ell'}(x',t')$:

$$\frac{\partial}{\partial t} \frac{\delta \phi_{\ell}(x,t)}{\delta \eta_{\ell'}(x',t')} =$$

$$= - \sum_{\ell''} \int dx'' \frac{\delta^2 S}{\delta\phi_\ell(x,t)\,\delta\phi_{\ell''}(x'',t)} \frac{\delta}{\delta\eta_{\ell'}(x',t')} \phi_{\ell''}(x'',t) + \delta_{\ell\ell'}\,\delta(x-x')\,\delta(t-t') \tag{2.15}$$

The solution of this equation is unique since

$$\frac{\delta}{\delta\eta_{\ell'}(x',t')} \phi_\ell(x,t) = 0 \qquad \text{for } t'>t \tag{2.16}$$

We obtain

$$\frac{\delta}{\delta\eta_{\ell'}(x',t')} \phi_\ell(x,t) =$$

$$= \Theta(t-t')\Big[\ \delta_{\ell\ell'}\,\delta(x-x') - \int_{t'}^{t} dt_1 \frac{\delta^2 S}{\delta\phi_\ell(x,t_1)\,\delta\phi_{\ell'}(x',t_1)} +$$

$$+ \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \sum_{\ell_2} \int dx_2 \frac{\delta^2 S}{\delta\phi_\ell(x,t_1)\,\delta\phi_{\ell_2}(x_2,t_1)} \frac{\delta^2 S}{\delta\phi_{\ell_2}(x_2,t_2)\,\delta\phi_{\ell'}(x',t_2)} + \cdots \Big] \tag{2.17}$$

from which (2.12) follows since $\theta(0) = 1/2$.

<u>Proof of Equality (2.7)</u>: First notice that the Fokker-Planck equation can be written as

$$-\tfrac{1}{2}\frac{\partial}{\partial t}\Psi[\phi,t] = \hat{H}_{F.P.}\Psi[\phi,t]$$

(2.18)

where Ψ is related to the probability distribution at time t by

$$P[\phi,t] = e^{-\frac{1}{2}S}\Psi[\phi,t]$$

(2.19)

and

$$\hat{H}_{F.P.} = \tfrac{1}{2}\int dx \sum_{\ell} \hat{Q}^{\dagger}_{\ell}(x)\,\hat{Q}_{\ell}(x)$$

(2.20)

with

$$\hat{Q}_{\ell}(x) = -i\frac{\delta}{\delta\phi_{\ell}(x)} - i/2\,\frac{\delta S}{\delta\phi_{\ell}(x)}$$

(2.21)

where $\hat{Q}^{\dagger}_{\ell}(x)$ is the Hermite adjoint of $\hat{Q}_{\ell}(x)$ in the Hilbert space such that the measure of scalar products is given by

$$(\Psi_1, \Psi_2) = \int D\phi\; \Psi_1[\phi]^{*}\,\Psi_2[\phi]$$

(2.22)

Let E be an eigenvalue of $\hat{H}_{F.P.}$ and $X_E(\Phi)$ the corresponding eigenvector

$$\hat{H}_{F.P.} \chi_E(\phi) = E\,\chi_E(\phi)$$

(2.23)

We conclude from the expression of $\hat{H}_{F.P.}$ given by (2.20) that the eigenvalues are non-negative,

$$E \geqslant 0$$

(2.24)

and the eigenfunction of zero eigenvalue should satisfy

$$\hat{Q}_\ell(x)\,\chi_0[\phi] = 0$$

(2.25)

for all ℓ and x. The solution of (2.25) is given by

$$\chi_0[\phi] \propto e^{-\frac{1}{2}S}$$

(2.26)

Using the eigenfunctions one obtains the following general solution of Fokker-Planck equation:

$$\Psi[\phi,t] = \sum_E e^{-2Et}\,\chi_E[\phi]\,C_E$$

(2.27)

Therefore, if the spectrum of $\hat{H}_{F.P.}$ has a "mass gap" (i.e. no continuum spectrum above zero) we obtain

$$\Psi[\phi,\infty] \propto \chi_0[\phi] \tag{2.28}$$

accordingly

$$P[\phi,t] \underset{t\to\infty}{\longrightarrow} \frac{e^{-S}}{\int D\phi\, e^{-S}} \tag{2.29}$$

which is independent of the initial condition. Inserting (2.4) into (2.8) we obtain (2.7).

REMARKS: If $\hat{H}_{FP}$ does not have a mass gap, then (2.29) is not true as it stands and the answer will depend very much on the particular entity we are averaging over P. In general the evolution of the system at large times will depend on the initial condition. This is true even for the free particle (Appendix I). $\hat{H}_{FP}$ does not have a mass gap if the ground state wave function $\chi_0[\phi]$ is not normalizable; the most important example of this is the gauge theory. For this theory the norm of χ_0 in the Hilbert space defined by (2.22) is not a finite number due precisely to the contribution to the norm coming from integration over the orbits of the local gauge group.

Notice that $P[\phi,t]$ will still approach the distribution (2.29) for large t, if the evolution equation is

$$\frac{\partial}{\partial t}P=\int dx\,dy\sum_{\ell,\ell'}M_{\ell\ell'}(x,y)\frac{\delta}{\delta\phi_\ell(x)}\left[\frac{\delta}{\delta\phi_{\ell'}(y)}+\frac{\delta S}{\delta\phi_{\ell'}(y)}\right]P[\phi,t] \tag{2.30}$$

$M_{ll'}(x,y)$ is a positive operator satisfying

$$M_{\ell\ell'}(x,y)=M_{\ell'\ell}(y,x) \tag{2.31}$$

In this case the Langevin equation is

$$\frac{\partial}{\partial t}\phi_\ell(x,t)=-\int dy\sum_{\ell'}M_{\ell\ell'}(x,y)\frac{\delta S}{\delta\phi_{\ell'}(y,t)}+\eta_\ell(x,t) \tag{2.32}$$

with

$$\langle\eta_\ell(x,t)\,\eta_{\ell'}(x',t')\rangle_\eta=2M_{\ell\ell'}(x,x')\,\delta(x-x')\,\delta(t-t') \tag{2.33}$$

Consequently, we have at our disposal a rather large set of stochastic equations with the time evolution necessary to implement the program of Stochastic Quantization. That is, that for large t, $P[\phi,t]$ should approach the distribution (2.29).

In order to see the truth of our last statement we may repeat the steps done to prove (2.7) starting now from the Langevin equation (2.32). However we prefer an alternative method.

Notice that $M_{ll'}(x,y)$ is a positive operator if there exists an operator $A_{ll'}(x,y)$ such that,

$$\int dz \sum_{l''} A_{ll''}(x,z)\, A_{l''l'}(z,y) = M_{ll'}(x,y) \tag{2.34}$$

where $A_{ll'}(x,y)$ has real, non-zero eigenvalues a defined by,

$$\int dy \sum_{l'} A_{ll'}(x,y) B_{l'}(y) = a\, B_l(x) \tag{2.35}$$

Introduce the change of variables,

$$\bar{\Phi}_l(x) = \int dy \sum_{l'} A^{-1}_{ll'}(x,y)\ \ \phi_{l'}(y) \tag{2.36}$$

which is well defined since all eigenvalues of A (x,y) are non-zero. We get,

$$\frac{\partial}{\partial t} P[\bar{\Phi},t] = \int dx \sum_{l'} \frac{\delta}{\delta \bar{\Phi}_{l'}(x)} \left[\frac{\delta}{\delta \bar{\Phi}_{l'}(x)} + \frac{\delta S}{\delta \bar{\Phi}_{l'}(x)} \right] P[\bar{\Phi},t] \tag{2.37}$$

Therefore,

$$P[\bar{\Phi},t] \xrightarrow[t\to\infty]{} \frac{e^{-S[\bar{\Phi}]}}{\int D\bar{\Phi}\, e^{-S[\bar{\Phi}]}} \tag{2.38}$$

The Langevin equation for $\bar{\Phi}$ is (2.5a),

$$\frac{\partial}{\partial t}\bar{\Phi}_\ell(x,t) = -\frac{\delta S[\bar{\Phi}]}{\delta \bar{\Phi}_\ell(x,t)} + \bar{\eta}_\ell(x,t)$$

$$\langle \bar{\eta}_\ell(x,t)\bar{\eta}_{\ell'}(x',t')\rangle_{\bar{\eta}} = 2\ \delta(x-x')\ \delta(t-t')$$

(2.39)

In terms of ϕ_ℓ we get,

$$\frac{\partial}{\partial t}\phi_\ell(x,t) = -\int dy \sum_{\ell'} M_{\ell\ell'}(x,y)\frac{\delta S}{\delta \phi_{\ell'}(y,t)} + \int dy \sum_{\ell'} A_{\ell\ell'}(x,y)\bar{\eta}_{\ell'}(y,t)$$

(2.40)

which coincide with (2.32) and (2.33) upon the identification,

$$\eta_\ell(x,t) = \int dy \sum_{\ell'} A_{\ell\ell'}(x,y)\ \bar{\eta}_{\ell'}(y,t)$$

(2.41)

The proof is made complete by the following statement,

$$\langle F(\phi[\eta,t])\rangle_\eta = \langle F(\phi[A\bar{\eta},t])\rangle_{\bar{\eta}}$$

(2.42)

Therefore,

$$\langle F(\phi[\eta,t])\rangle_\eta \underset{t\to\infty}{\longrightarrow} \frac{\int \mathcal{D}\bar{\phi}\, F(A\bar{\phi})\, e^{-S[\bar{\phi}]}}{\int \mathcal{D}\bar{\phi}\, e^{-S[\bar{\phi}]}}$$

$$= \frac{\int \mathcal{D}\phi\, F[\phi]\, e^{-S[\phi]}}{\int \mathcal{D}\phi\, e^{-S[\phi]}} \tag{2.43}$$

D. Perturbative Solution of the Langevin Equation[14].

As an illustration, we apply the method to obtain a perturbative expansion of Green's function (2.1) in the model defined by the following Euclidean action,

$$S[\phi] = \int d^4x \left\{ \tfrac{1}{2}(\partial_\mu \phi(x))^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\} \tag{2.44}$$

$\phi(x)$ is a real Bose field defined on the Euclidean space time point x.

According to equation (2.5), we shall solve the following Langevin equation,

$$\frac{\partial}{\partial t}\phi(x,t) = (\Box - m^2)\,\phi(x,t) - \frac{\lambda}{3!}\phi(x,t)^3 + \eta(x,t) \tag{2.45a}$$

$$\phi(x,0) = 0$$

(2.45b)

To get a perturbative expansion we rewrite system (2.45) in the form of an integral equation.

To do this, we introduce the Green's function $G(x,t)$,

$$G(x,t) = \Theta(t) \int \frac{d^d k}{(2\pi)^d} e^{ikx} e^{-(k^2+m^2)t}$$

(2.46)

which satisfies the following equation and boundary condition:

$$\left(\frac{\partial}{\partial t} - \Box + m^2\right) G(x,t) = \delta(x)\,\delta(t)$$

(2.47a)

$$G(x,t) = 0 \qquad\qquad t < 0$$

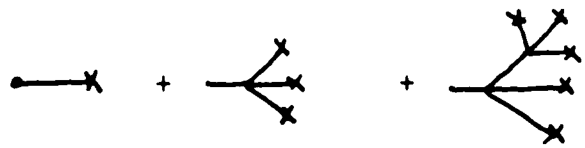
(2.47b)

We can easily verify that the solution of (2.45) also solves the integral equation,

$$\phi(x,t) = \int_0^\infty d\tau \int d^d y \;\; G((x-y);t-\tau)\left[-\frac{\lambda}{3!}\phi^3(y,\tau)+\eta(y,\tau)\right]$$

(2.48)

In general, the solution of equation (2.48) can be found by iteration and is diagramatically expressed as,

$\phi(x,t) =$ 

(2.49)

Here $\phi(x,t)$ is made of all possible tree graphs arranged in a t-sequence (t decreases toward the right side of the page), constructed from four point vertices. All final lines are crossed, except for the starting point which carries the label (x,t).

In the diagram a cross represents η, the line -- depicts G(x,t) defined in equation (2.46). Moreover, integration over x,t of vertices and crosses is assumed.

As an example of the technique, we calculate the Two Point function, defined by

$$\Delta(x,y) = \langle \phi(x)\,\phi(y) \rangle \tag{2.50}$$

If $\lambda = 0$ the solution of (2.48) is

$$\phi(x,t) = \int d\tau\, d^d z\; G(x-z;t-\tau)\,\eta(z,\tau)$$

$$\phi(x,t) = \int d\tau\, d^d z\; G(x-z;t-\tau)\,\eta(z,\tau) \tag{2.51}$$

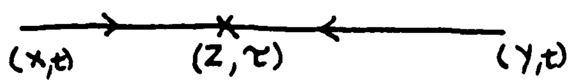
Therefore,

$$\langle \phi(x,t)\ \phi(y,t)\rangle_\eta = 2\int d\tau\ d^dz\, G(x-z;t-\tau)\ G(y-z,t-\tau)$$

(2.52)

We have already computed the η -average using equation (2.2)

The quantity (2.52) is represented graphically by:



(2.53)

The arrow points toward the past in fictitious time. Notice that a cross reverses the direction of fictitious time, and introduces a factor of 2.

If we integrate over τ, we get,

$$\langle \phi(x,t)\ \phi(y,t)\rangle_\eta = \int \frac{d^dK}{(2\pi)^d}\, \frac{e^{iK(x-y)}}{K^2+m^2}\left(1-e^{-2(K^2+m^2)t}\right)$$

$$\xrightarrow[t\to\infty]{} \int \frac{d^dK}{(2\pi)^d}\, \frac{e^{iK(x-y)}}{K^2+m^2}$$

(2.54)

which is the well-known Euclidean Feynman propagator.

In the next order in λ , $O(\lambda)$, there are two graphs contributing to the Two Points function (2.50), each having a combinational factor 3 which cancels the corresponding 3 in the definition of the vertex.

(a) (b)

$$(2.55)$$

The explicit evaluation of the η-average using equations (2.2), (2.3) and (2.49) gives

$$(a) = 2\lambda \int \frac{d^dK}{(2\pi)^d}\, e^{iK(x-y)} \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1\, e^{-2(K^2+m^2)(t-\tau_1)}$$

$$\int \frac{d^dq}{(2\pi)^d} \int_0^{\tau_2} d\tau_3\, e^{-2(q^2+m^2)(\tau_2-\tau_3)} \tag{2.56}$$

It is easy to see that (a) = (b). Therefore,

$$\lim_{t\to\infty} (a+b) = \frac{\lambda}{2} \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2+m^2} \int \frac{d^dK}{(2\pi)^d} \frac{e^{iK(x-y)}}{(K^2+m^2)^2} \tag{2.57}$$

which also agrees with the standard result.

E. Stochastic Quantization of Gauge Theories

The dynamics of the U(N) gauge theory is defined by the Euclidean action,

$$S(A) = 1/4 \int d^4x \; \mathrm{tr}\, F_{\mu\nu}^2(x)$$

(2.58)

$A_\mu(x)$ is a vector-valued matrix field belonging to the adjoint representation of U(N), and

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ie[A_\mu(x), A_\nu(x)]$$

(2.59)

where e is a coupling constant.

The action S(A) is invariant under the local gauge transformation,

$$A_\mu(x) \rightarrow U(x) A_\mu(x) U(x)^\dagger - i/e \; U(x) \partial_\mu U(x)^\dagger$$

(2.60)

The Langevin equation of this theory is given by

$$\frac{\partial}{\partial t} A_\mu^{ij}(x,t) = \partial_\nu F_{\nu\mu}^{ij}(x,t) - ie[A_\nu(x,t), F_{\nu\mu}(x,t)]^{ij} + \eta_\mu^{ij}(x,t)$$

(2.61a)

$$A_\mu^{ij}(x,0) = 0$$

(2.61b)

where $\eta^{ij}(x,t)$ is a random source with Gaussian distribution and

$$\langle \eta^{ij}_{\mu}(x,t)\, \eta^{k\ell}_{\nu}(x',t')\rangle_\eta = 2\, \delta_{\mu\nu}\, \delta_{i\ell}\, \delta_{jk}\, \delta(x-x')\, \delta(t-t')$$

(2.62)

There is an important new element in the stochastic quantization of gauge theories of which we need to be aware. To understand what it is, let us calculate the Two Point function as in section II.D. We obtain to $O(e^0)$

$$\langle A^{ij}_{\mu}(x,t)\, A^{k\ell}_{\nu}(y,t)\rangle_\eta =$$

$$= \int \frac{d^d k}{(2\pi)^d}\, e^{ik(x-y)} \left\{ \frac{1-e^{-2k^2 t}}{k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + 2\,\frac{k_\mu k_\nu}{k^2}\, t \right\} \delta_{i\ell}\, \delta_{jk}$$

(2.63)

This last expression becomes singular for $t \to \infty$.

On the other hand if we calculate the following gauge invariant object we shall get

$$\langle \mathrm{tr}\, F_{\mu\nu}(x,t)\, F_{\alpha\beta}(x,t)\rangle_\eta =$$

$$= N^2 \int \frac{d^d k}{(2\pi)^d} \left[k_\nu k_\alpha \delta_{\mu\beta} - k_\nu k_\beta \delta_{\mu\alpha} - k_\mu k_\alpha \delta_{\nu\beta} + k_\mu k_\beta \delta_{\nu\alpha} \right]$$

$$\frac{1-e^{-2k^2 t}}{k^2}$$

(2.64)

This correlation function has a well defined equilibrium value.

Hence gauge invariant correlation functions have a finite value for large t, whereas gauge variant correlation functions diverge for large t. This situation persists at every order in the coupling constant (Parisi and Wu)[14]

As we mentioned on page (20) this happens because the Fokker-Planck Hamiltonian corresponding to the action (2.58) does not have a normalizable ground state wave function and a "mass gap" due to the symmetry (2.60); hence some expectation values will diverge for large t. (See the example of the free particle in Appendix I)

In order to understand why gauge invariant correlation functions have a uniform expansion in the coupling constant e, we study below the approach to equilibrium of gauge invariant expectations values in the gauge theory. We shall follow the discussion of Zwanziger and Baulieu[22] We are going to see that, for gauge invariant quantities, it is possible to find a normalizable ground state of a modified Fokker-Planck Hamiltonian, and then for large t the probability distribution will approach the normalizable ground state. Moreover, we shall show that the new ground state coincides with the Fadeev-Popov ansatz for some important gauges. A similar argument can be developed using directly the Langevin equation and we shall discuss it in Section III.C.3.

To begin, we consider a gauge invariant functional of A, F(A). Then, by definition

$$\langle F(A(x,t))\rangle_\eta = \int \mathcal{D}A\, F(A)\, P(A,t) \tag{2.65}$$

where P(A,t) satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t} P(A,t) = -\hat{L}\, P(A,t)$$

$$\hat{L} = -\int dx \sum_{a,b,\mu} \frac{\delta}{\delta A_\mu^{ba}(x)} \left(\frac{\delta}{\delta A_\mu^{ab}(x)} + \frac{\delta S(A)}{\delta A_\mu^{ab}(x)} \right) \tag{2.66}$$

S(A) is the classical action of gauge theory defined by equation (2.58). The initial condition satisfied by P(A,t) is:

$$P[A,0] = \delta(A) \tag{2.67}$$

The formal solution of this system is given by

$$P[A,t] = e^{-\hat{L}t} P[A,0] \tag{2.68}$$

Therefore

$$\langle F(A(x,t))\rangle_\eta = \int \mathcal{D}A \; F(A) \; e^{-\hat{L}t} \, P[A,0]$$

$$= \left[\left(e^{-\hat{L}^\dagger t} \right) F(A) \right]_{A=0}$$

(2.69)

where $\hat{L}^\dagger$ is the Hermitian adjoint operator of $\hat{L}$

$$\hat{L}^\dagger = - \int dx \sum_{a,b,\mu} \left(\frac{\delta}{\delta A_\mu^{ab}(x)} - \frac{\delta S(A)}{\delta A_\mu^{ab}(x)} \right) \frac{\delta}{\delta A_\mu^{ba}(x)}$$

(2.70)

Since F(A) is gauge invariant, it satisfies $\hat{G}(A)\, F(A)=0$; where $\hat{G}(A)$ is the generator of local gauge transformations defined by:

$$G^\alpha(A) = \sum_{a,b,\mu} \left\{ ie\,[I^\alpha, A_\mu(x)]^{ab} \frac{\delta}{\delta A_\mu^{ab}(x)} - (I^\alpha)^{ab} \partial_\mu \frac{\delta}{\delta A_\mu^{ab}(x)} \right\}$$

(2.71)

the I^α are generators of the Lie-Algebra of the group U(N).

We see that adding an arbitrary linear combination of $\hat{G}^\alpha$ to $\hat{L}^\dagger$ does not change $\langle F(A)\rangle_\eta$. So, for a gauge invariant F(A), the approach to equilibrium is governed by the v-dependent operator

$$\hat{L}^{\dagger}_{v} = \hat{L}^{\dagger} + \int d^4x \sum_{\alpha} v^{\alpha}(x,A)\, \hat{G}^{\alpha}(x,A) \tag{2.72}$$

where the v's are arbitrary functions of x and A.

This change in the Fokker-Planck evolution operator is equivalent to changing the related Langevin equation to

$$\frac{\partial}{\partial t} A_{\mu}(x,t) = D_{\nu} F_{\nu\mu}(x,t) + D_{\mu} v(x,A) + \eta_{\mu}(x,t) \tag{2.73}$$

For an interpretation of v , see our discussion of gauge fixing in the Langevin equation at the beginning of the section III.C.3.

We have defined D_{μ} (covariant derivative) as usual by

$$D_{\mu} = \partial_{\mu} - ie[A_{\mu},\ ] \tag{2.74}$$

The Fokker-Planck equation with v=0 does not have a mass gap, so the demonstration of equivalence of stochastic quantization and path integral quantization is incomplete. Moreover, the remarkable cancelation of divergences for large t in gauge invariant correlation functions, encountered order by order in the perturbative expansion, requires an explanation. The explanation comes from the fact that by selecting v appropiately a mass gap of the Fokker-Planck equation is generated, giving a normalizable ground state. We develop this idea below.

The v-dependent evolution operator is

$$\hat{L}_v = \hat{L} - \int d^d x \sum_{a,b,\mu} \frac{\delta}{\delta A_\mu^{ab}(x)} \left(D_\mu v \right)^{ab} \tag{2.75}$$

$$V = \sum_\alpha V^\alpha I^\alpha \tag{2.76}$$

$\hat{L}_v$ may be written in the alternative form:

$$\hat{L}_v = \hat{L} - \int d^d x \sum_{a,b,\mu} \left\{ \partial_\mu \frac{\delta}{\delta A_\mu^{ab}(x)} V^{ab} - ie \left[A_\mu(x), \frac{\delta V}{\delta A_\mu^{ab}(x)} \right]^{ab} \right\} \tag{2.77}$$

The equilibrium distribution satisfies

$$\hat{L}_v \chi_v = 0 \tag{2.78}$$

Because a discussion of solutions of(2.78) is rather longwinded and would take us away from our purpose we just state the results and refer the reader to the literature for a proof[22].

One solution of equation (2.70) is

$$\chi_v = P_{FP}(A) =$$

$$= N \int \mathcal{D}c\, \mathcal{D}\bar{c}\ \exp - [S(A) + \int d^dx\ \mathrm{tr}[\frac{(\partial\cdot A)^2}{2\alpha} + \partial_\mu \bar{C} D_\mu C]]$$

(2.79)

$$V(x,A) = -\frac{N}{P_{FP}} \int d^dy\ \frac{\delta}{\delta A_\nu^{ba}(y)} \int \mathcal{D}C\, \mathcal{D}\bar{C}\ C(x)\, \partial_\nu \bar{C}^{ba}(y)$$

$$\exp - [S(A) + \int d^dx\ \mathrm{tr}[\frac{(\partial\cdot A)^2}{2\alpha} + \partial_\mu \bar{C} D_\mu C]]$$

(2.80)

Thus, for large t gauge invariant expectation values calculated using the solution of the Langevin equation (2.61) will be given by distribution (2.79) which is the standard Fadeev-Popov ansatz.

F. Schwinger-Dyson Equations

We have previously defined the stochastic quantization method and showed that it agrees with the standard Fadeev-Popov ansatz. In this section, we shall study the connection discovered by Marchesini(23), between the Schwinger-Dyson equations of a given model and the equilibrium condition of the same model when described using the stochastic quantization.

The Schwinger-Dyson equations are an infinite set of relations among the Green's functions of a theory, reflecting the maximum symmetry of path integral measure. These equations are supposed to determine the functions uniquely and in this sense are the most general set of independent relations among the Green's functions.

Usually the following method is employed to derive the Schwinger-Dyson equations.

Let $F(\phi)$ be any functional of the field and $S(\phi)$ the Euclidean action defining the model. Then,

$$\langle F(\phi)\rangle = \int \mathfrak{D}\phi\, F(\phi)\, e^{-S[\phi]} \tag{2.81}$$

for $\mathfrak{D}\phi$ is invariant under,

$$\phi_l(x) \longrightarrow \phi_l(x) + \delta\phi_l(x) \tag{2.82}$$

with $\delta\phi_1(x)$ independent of $\phi_1(x)$.

We have,

$$\langle F(\phi)\rangle = \int \mathcal{D}\phi \; F(\phi+\delta\phi)\, e^{-S[\phi+\delta\phi]}$$

$$= \langle F(\phi)\rangle + \int dx \sum_{\ell} \left\langle \frac{\delta F}{\delta\phi_\ell(x)} - F\frac{\delta S}{\delta\phi_\ell(x)} \right\rangle \delta\phi_\ell(x)$$

$$+ \ldots$$

(2.83)

Since $\delta\phi_\ell(x)$ is an arbitrary functional of x, independent of $\phi_\ell(x)$, we must have,

$$\left\langle \frac{\delta F}{\delta\phi_\ell(x)} - F\frac{\delta S}{\delta\phi_\ell(x)} \right\rangle = 0$$

(2.84)

These are the Schwinger-Dyson equations for this model.

Now we consider a similar object in the stochastic quantization,

$$\langle f_\ell(\phi)\rangle_\eta = \int \mathcal{D}\phi\, f_\ell(\phi)\; P[\phi,t]$$

(2.85)

where $P[\phi,t]$ satisfies the Fokker-Planck equation (2.13)

If $\langle f(\phi)\rangle_\eta$ approaches equilibrium for large t, we have,

$$\frac{\partial}{\partial t}\langle f_\ell(\phi)\rangle_\eta \xrightarrow[t\to\infty]{} 0$$

(2.86)

Using the Fokker-Planck equation (2.13), we can express this last condition in the following way,

$$\frac{\partial}{\partial t}\langle f_\ell(\phi)\rangle_\eta = \int \mathcal{D}\phi \; f_\ell(\phi)\,(-\hat{L})\; P[\phi,t]$$
$$= -\int \mathcal{D}\phi \; [\hat{L}^+ f_\ell(\phi)]\; P[\phi,t] \tag{2.87}$$

Therefore

$$0 = -\langle \hat{L}^+ f_\ell(\phi)\rangle \tag{2.88}$$

Using the explicit form of L defined by equation (2.13) we have

$$\int dx \sum_{\ell'} \left\langle \frac{\delta^2}{\delta\phi^2_{\ell'}(x)} f_\ell(\phi) - \frac{\delta S}{\delta\phi_{\ell'}(x)}\frac{\delta f_\ell}{\delta\phi_{\ell'}(x)}\right\rangle = 0 \tag{2.89}$$

Now, if we identify,

$$\frac{\delta f_\ell}{\delta\phi_{\ell'}(x)} = \delta(x-y)\;\delta_{\ell\ell'}\;F[\phi] \tag{2.90}$$

then the equilibrium condition reads,

$$\left\langle \frac{\delta F}{\delta\phi_\ell(y)} - \frac{\delta S}{\delta\phi_\ell(y)}F\right\rangle = 0 \tag{2.91}$$

which coincides with the Schwinger-Dyson equation (2.84).

Remarks: Equilibrium condition (2.86) implies the Schwinger-Dyson equation (2.84) as we just showed. However, the Schwinger-Dyson equations do not imply that the Green's functions will reach equilibrium for large t. In fact, the Schwinger-Dyson equations require that,

$$\frac{\partial}{\partial t}\langle f(\phi)\rangle_\eta \xrightarrow[t\to\infty]{} 0 \tag{2.92}$$

It could happen that $\langle f(\phi)\rangle_\eta$ diverges as t^ϵ for large t with $\epsilon<1$.

III. DERIVATION OF QUENCHED MOMENTUM PRESCRIPTION BY MEANS OF STOCHASTIC QUANTIZATION.

A. Introduction.

The object of this section is to report on a study of the reduction of degrees of freedom in the large N limit using the stochastic quantization, already revised on a previous section.

We emphasize the following two aspects of the stochastic method for our purpose: First the Langevin equation can be viewed as a classical equation of motion for a system with an external source function. Second, the random source $\eta(x,t)$ can be quite arbitrary in its x,t dependence in so far as the random average has the Gaussian distribution property given by (2.2) and (2.3). These aspects are exploited to reduce the number of degrees of freedom for large N.

Incidentally, we want to mention that in Appendix II we refer to a related application of the stochastic quantization to the large N limit.

B. Large N reduction of an NxN Hermitian Matrix Model(24,26)

We illustrate our method with an example of an

Hermitian matrix model defined by the action:

$$S[\phi]=\int d^4x \; \mathrm{tr}\left\{ \tfrac{1}{2}(\partial_\mu \phi(x))^2 + \tfrac{1}{2}m^2\phi(x)^2 + \frac{g}{4N}\phi(x)^4 \right\}$$

$$S[\phi]=\int d^4x \; \mathrm{tr}\left\{ \tfrac{1}{2}(\partial_\mu \phi(x))^2 + \tfrac{1}{2}m^2\phi(x)^2 + \frac{g}{4N}\phi(x)^4 \right\}$$

(3.1)

where ϕ is an Hermitian NxN matrix field. This model has a global U(N) symmetry and a reflection symmetry:

$$\phi(x) \to U\,\phi(x)\,U^\dagger \;, \quad U \in U(N)$$

$$\phi(x) \to -\phi(x)$$

(3.2)

Accordingly, we consider a set of invariant Green's functions defined by

$$\langle \mathrm{tr}\,(\phi(x_1)\,\phi(x_2)\ldots\phi(x_n))\rangle \quad , n \text{ even}$$

(3.3)

The Langevin equation of this model is given by

$$\frac{\partial}{\partial t}\phi_{ij}(x,t) = (\Box - m^2)\,\phi_{ij}(x,t) - g/N\,(\phi^3(x,t))_{ij} + \eta_{ij}(x,t)$$

(3.4)

where $\eta_{ij}(x,t)$ is an Hermitian NxN random source with Gaussian distribution. Therefore in accordance with (2.7)

$$\langle \mathrm{tr}\,(\phi(x_1)\ldots\phi(x_n))\rangle = \lim_{t\to\infty}\langle \phi(x_1,t)\ldots\phi(x_n,t)\rangle_\eta \qquad (3.5)$$

Solving the Langevin equation by iteration and inserting the solution into (3.5) and (3.3), we obtain a formal power series expansion of Green's function:

$$
\begin{aligned}
&\langle \mathrm{tr}\,(\phi(x_1)\ldots\phi(x_n))\rangle \\
&= \lim_{t\to\infty} \sum_{m=0}^{\infty} (g/N)^m \int\ldots\int dy_1\,dt_1\ldots dy_{2m+n}\,dt_{2m+n} \\
&\quad K_m(x_1,\ldots x_n, t;\, y_1 t_1, \ldots, y_{2m+n}\, t_{2m+n}) \quad \times \\
&\quad \times \langle \mathrm{tr}(\eta(y_1,t_1)\ldots\eta(y_{2m+n},t_{2m+n}))\rangle_\eta
\end{aligned}
\tag{3.6}
$$

where the K_m's are scalar functions. A very nice point of this expression is that all the U(N) indices appear through η, so that one can study the large N limit by examining only the random average. Of course, as we showed in the last section, expression (3.6) generates the standard Feynman-Dyson expansion if one inserts the Wick decomposition property (2.3).

Next we prove the following proposition.

Proposition: Let $\eta_{ij}(t)$ (i, j = 1,... ,N) be a random source with Gaussian distribution and p_i^α (i = 1,...,N; α =1,... ,d) be a random number with uniform distribution in the hypercube $[-\Lambda/2, \Lambda/2]^d$. As long as one restricts oneself to invariant expectation values of the form

$$\langle tr(\eta(x_1,t_1)\ldots\eta(x_{2m},t_{2m}))\rangle_\eta$$

in the large N limit

$$\eta_{ij}(x,t) = \left(\frac{\Delta}{2\pi}\right)^{d/2} e^{i(p_i-p_j)x}\,\bar{\eta}_{ij}(t) \tag{3.7}$$

serves as variables with Gaussian distribution.

The proof reduces to show that (3.7) gives the Wick decomposition property. Let us first examine the case of m = 1.

$$\sum_{ij} \langle \eta_{ij}(x,t)\,\eta_{ji}(x',t')\rangle$$

$$= \sum_{ij}\left(\frac{\Delta}{2\pi}\right)^d \int \prod_{\alpha,k} \frac{dp_k^\alpha}{\Lambda}\, e^{i(p_i-p_j)(x-x')} \langle \bar{\eta}_{ij}(t)\,\bar{\eta}_{ji}(t')\rangle_{\bar{\eta}}$$

$$= \left(\frac{\Delta}{2\pi}\right)^d \Big\{ 2(N^2-N)\,\delta(t-t')\int \prod_{\alpha,k} \frac{dp_k^\alpha}{\Lambda}\, e^{i(p_i-p_j)(x-x')} + 2N\,\delta(t-t')\int \prod_{\alpha,k} \frac{dp_k^\alpha}{\Lambda} \Big\} \tag{3.8}$$

In leading order in N we can neglect the last term. We then obtain

$$\langle \mathrm{tr}\left(\eta(x,t)\,\eta(x',t')\right)\rangle_\eta \sim 2N^2\,\delta(x-x')\,\delta(t-t') \tag{3.9}$$

which is the desired result. Neglection of the second term is justified only when N is so large that the following inequality is satisfied:

$$\left(\frac{\Delta}{2\pi}\right)^d L^d \ll N \tag{3.10}$$

Here L^d is the space-time volume. This shows the degree of large N about which we have been speaking.

Next let us examine the m=2 case.

$$\sum_{ijkl} \langle \eta_{ij}(x_1,t_1)\,\eta_{jk}(x_2,t_2)\,\eta_{kl}(x_3,t_3)\,\eta_{li}(x_4,t_4)\rangle =$$

$$= \left(\frac{\Delta}{2\pi}\right)^{2d} \sum_{ijk\ell} \int \prod \frac{dp}{\Lambda} e^{i[(p_i-p_j)x_1+(p_j-p_k)x_2+(p_k-p_\ell)x_3]} \times$$

$$\times \; e^{i(p_\ell-p_i)x_4} \langle \bar{\eta}_{ij}(t_1)\, \bar{\eta}_{jk}(t_2)\, \bar{\eta}_{k\ell}(t_3)\, \bar{\eta}_{\ell i}(t_4) \rangle_{\bar{\eta}}$$

(3.11)

Since $\bar{\eta}$ has Gaussian distribution, we have

$$\langle \bar{\eta}_{ij}(t_1)\, \bar{\eta}_{jk}(t_2)\, \bar{\eta}_{k\ell}(t_3)\, \bar{\eta}_{\ell i}(t_4) \rangle_{\bar{\eta}}$$

$$= 2^2 \delta_{i\ell}\, \delta_{jk}\, \delta_{ji}\, \delta_{k\ell}\, \delta(t_1-t_2)\, \delta(t_2-t_4)$$

$$+ 2^2 \delta_{ik}\, \delta(t_1-t_2)\, \delta(t_3-t_4)$$

$$+ 2^2 \delta_{j\ell}\, \delta(t_1-t_4)\, \delta(t_2-t_3)$$

(3.12)

Thus

$$\langle tr(\eta(x_1,t_1)\ldots\eta(x_4,t_4))\rangle$$

$$= \left(\frac{\Delta}{2\pi}\right)^{2d} \sum_{ijk\ell} \int \prod \frac{dp}{\Lambda} e^{i(p_i-p_j)(x_1-x_2)} e^{i(p_i-p_\ell)(x_3-x_4)}$$

$$2\; \delta_{ik}\, \delta(t_1-t_2)\, 2\, \delta(t_3-t_4) + \ldots$$

$=$

$$= 4\ \delta(t_1-t_2)\ \delta(t_3-t_4)\left(\frac{\Lambda}{2\pi}\right)^{2d}\ \times$$

$$\times\left\{ (N^3-3N^2+N)\int_{\substack{i\neq j\\ i\neq l\\ l\neq j}} \prod \frac{dp}{\Lambda}\ e^{i(p_i-p_j)(x_1-x_2)+i(p_i-p_l)(x_3-x_4)}\right.$$

$$+ N^2\int \prod \frac{dp}{\Lambda}\ e^{i(p_i-p_l)(x_3-x_4)} + N^2\int \prod \frac{dp}{\Lambda} e^{i(p_i-p_j)(x_1-x_2)}$$

$$\left. + N(N-1)\int \prod \frac{dp}{\Lambda}\ e^{i(p_i-p_j)(x_1-x_2+x_3-x_4)}\ \right\} + \ldots$$

$$\approx\ 4\ \delta(t_1-t_2)\ \delta(t_3-t_4)\ N^3\left[\ \delta(x_1-x_2)\ \delta(x_3-x_4) + \frac{1}{N}\left(\frac{\Lambda}{2\pi}\right)^d \times \right.$$
$$\left. \times\ \delta(x_1-x_2+x_3-x_4)\right]$$
$$+\ \ldots \tag{3.13}$$

We neglect the second term in the square bracket in the large N limit using (3.10), and obtain

$$\langle\, \mathrm{tr}\,(\eta(x_1,t_1)\,\eta(x_2,t_2)\,\eta(x_3,t_3)\,\eta(x_4,t_4))\rangle$$
$$\approx \mathrm{tr}\ \Big\{\ \langle\eta(x_1,t_1)\,\eta(x_2,t_2)\rangle\ \langle\eta(x_3,t_3)\,\eta(x_4,t_4)\rangle$$
$$+\langle\eta(x_2,t_2)\ \eta(x_3,t_3)\rangle \langle\ \eta(x_4,t_4)\ \eta(x_1,t_1)\rangle\ \Big\} \tag{3.14}$$

It is clear now how the argument continues. An important point to notice is that if N were large but finite η's given by (3.7) cease to be Gaussian for $m \gtrsim N$ because we do not have enough p's to produce independent δ-functions.

The Langevin equation (3.4) with the random source given by (3.7) has a solution given by

$$\phi_{ij}(x,t) = e^{i(p_i - p_j)x}\, \bar{\phi}_{ij}(t) \tag{3.15}$$

where $\bar{\phi}_{ij}(t)$ is a solution of the reduced Langevin equation

$$\frac{\partial}{\partial t} \bar{\phi}_{ij}(t) = -\left[(p_i - p_j)^2 + m^2\right] \bar{\phi}_{ij}(t) - g/N\, \bar{\phi}^3_{ij}(t) + \left(\frac{\Delta}{2\pi}\right)^{d/2} \bar{\eta}_{ij}(t) \tag{3.16}$$

Inserting (3.15) into (3.5) and using the proposition, we can show in a straightforward manner the large N equivalence between the model defined by (3.1) and the reduced model with quenching prescription:

$$\langle tr\,(\phi(x_1)\ldots\phi(x_n))\rangle = \lim_{t\to\infty} \langle tr(\phi(x_1,t)\ldots\phi(x_n,t))\rangle_\eta$$

$$= \lim_{t\to\infty} \int \prod \frac{dp}{\Lambda} \sum_{ijk\ldots} e^{i(p_i - p_j)x_1} \cdots \langle \bar{\phi}_{ij}(t)\, \phi_{jk}(t) \ldots \rangle_{\bar{\eta}} \tag{3.17}$$

$$= \int \prod \frac{dp}{\Lambda} \sum_{ijk\ldots} e^{i(p_i - p_j)x_1} \cdots \langle \bar{\Phi}_{ij}\, \bar{\Phi}_{jk} \cdots \rangle_{\bar{S}} \tag{3.18}$$

where $\langle \ldots \rangle_{\bar{S}}$ is defined by

$$\langle \cdots \rangle_{\bar{S}} \equiv \frac{\int \mathcal{D}\bar{\phi}\, (\ldots)\, e^{-\bar{S}(\bar{\phi})}}{\int \mathcal{D}\bar{\phi}\ e^{-\bar{S}(\bar{\phi})}} \tag{3.19}$$

$$\bar{S} = (2\pi/\Lambda)^4 \left\{ \sum_{ij} 1/2 \left[(p_i - p_j)^2 + m^2 \right] \bar{\phi}_{ij} \bar{\phi}_{ji} + g/N\, \mathrm{tr}\, \bar{\phi}^4 \right\} \tag{3.20}$$

This is the Gross-Kitazawa[11] version of quenched momentum prescription. If we derive the Langevin equation associated with this action using (2.5) we obtain (3.16) in which t is replaced by $(\Lambda/2\pi)^4 t$. Since $t\to\infty$ is taken to derive the equality (3.5) this scale difference does not affect the conclusion.

Because decomposition (3.7) is true in leading N order only, in rigour, equation (3.17) should read,

$$\ell.o.\ \langle \mathrm{tr}\,(\phi(x_1)\ldots\phi(x_n))\rangle$$

$$= \lim_{t\to\infty}\ \ell.o.\int_{-\Lambda/2}^{\Lambda/2} \prod \frac{dp}{\Lambda}\ \mathrm{tr}\langle e^{iPx_1}\bar{\Phi}(t)\bar{e}^{iPx_1}\ldots e^{iPx_n}\bar{\Phi}(t)\bar{e}^{iPx_n}\rangle_\eta \tag{3.21}$$

where l.o. stands for "leading order in large N". An important point to notice in this expression is that l.o. is calculated before $\lim_{t\to\infty}$, otherwise the reduction (3.7) and (3.15) cannot be used to solve the original Langevin equation (3.4).

For the massive case, by interchaging these limits, we are able to write equation (3.18), which is the path integral version of the quenched momentum prescription. We wish to argue that this is not correct for the massless theory because

$$\lim_{t\to\infty} \mathrm{tr}\langle e^{iPx_1}\bar{\Phi}(t)\bar{e}^{iPx_1}\ldots e^{iPx_n}\bar{\Phi}(t)\bar{e}^{iPx_n}\rangle_\eta$$

does not exist (at least in perturbation theory). This fact will be exhibited explicity in the following simple calculations for the case of $g=0$.

As we have seen, the perturbation series of the Langevin equation is obtained by iterating the following integral equation:

$$\bar{\Phi}_{ij}(t) = \int_0^t d\tau\, e^{-(p_i-p_j)^2(t-\tau)} \left[(\Lambda/2\pi)^{d/2} \bar{\eta}_{ij}(\tau) - g/N\, \bar{\Phi}^3(\tau)_{ij} \right]$$

(3.22)

Let us first calculate l.o. $tr\langle\phi^2(x)\rangle_S$ for the case $g=0$

$$l.o.\, tr\langle\phi^2(x)\rangle_S = \lim_{t\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \int \left[\frac{dp}{\Lambda}\right] \sum_{ij} \langle \bar{\Phi}_{ij}(t)\, \bar{\Phi}_{ji}(t)\rangle_{\eta}$$

(3.23)

Inserting the solution (3.22) we obtain

$$\sum_{ij} \langle \bar{\Phi}_{ij}(t)\, \bar{\Phi}_{ji}(t)\rangle_{\eta} = \left(\frac{\Lambda}{2\pi}\right)^d \sum_{ij} \frac{1-e^{-2(p_i-p_j)^2 t}}{(p_i-p_j)^2}$$

(3.24)

Thus, if we take first the $t\to\infty$ limit we obtain,

$$\lim_{t\to\infty} \sum_{ij} \langle \bar{\Phi}_{ij}(t)\, \bar{\Phi}_{ji}(t)\rangle_{\eta} = \lim_{t\to\infty} \left(\frac{\Lambda}{2\pi}\right)^d 2tN$$

(3.25)

namely the large t divergence.

On the other hand, if we take the large N limit first as it should be according to (3.21) we obtain,

$$\lim_{t\to\infty} \lim_{N\to\infty} \left[\frac{1}{N^2} \sum_{i\neq j} \int \left[\frac{dp}{\Lambda}\right] \left(\frac{\Lambda}{2\pi}\right)^d \frac{1-e^{-2(p_i-p_j)^2 t}}{(p_i-p_j)^2} + \left(\frac{\Lambda}{2\pi}\right)^d \frac{2t}{N} \right]$$

(3.26)

which is free from large t divergences.

Generalizing the previous calculation, we readily see that equation (3.21) is free from large t-divergences order by order in perturbation theory.

We want to emphasize that no constraint on the dynamical variables is needed to obtain the correct large N limit as long as we adhere to equation (3.21). It is only when we define and describe the large N reduction by ordinary path integrals that a constraint is required to regularize the diagonal part of the field. In the stochastic quantization the role of the regulator is played by fictitious time t.

C.1. U(N) Gauge Theory

Keeping the discussion of the previous chapter in mind we proceed to the large N reduction of the U(N) gauge theory.

The theory is defined by the Euclidean action.

$$S[A] = 1/4 \int d^d x \; \mathrm{tr}\, F_{\mu\nu}^2(x) \tag{3.27}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ie\,[\,A_\mu(x), A_\nu(x)] \tag{3.28}$$

$A_\nu(x)$ is an NxN Hermitian matrix field. The corresponding Langevin equation is given by

$$\dot{A}_\mu(x,t) = D_\nu\, F_{\nu\mu}(x,t) + \eta_\mu(x,t) \tag{3.29}$$

I have chosen the initial condition

$$A_\mu(x,0) = 0 \tag{3.30}$$

In (3.29) D_ν is the covariant derivative defined by

$$D_\nu = \partial_\nu - ie[\,A_\nu,\ \ ] \tag{3.31}$$

and $\eta_\mu(x,t)$ is a Gaussian random source with second momenta

$$\langle \eta_\mu^{ij}(x,t)\,\eta_\nu^{j'i'}(x',t')\rangle_\eta = 2\ \delta_{\mu\nu}\,\delta_{ii'}\,\delta_{jj'}\,\delta(x-x')\,\delta(t-t') \tag{3.32}$$

In gauge theories we are interested in correlation functions of gauge invariant operators only, i.e., operators that do not change under the following gauge transformation,

$$A_\mu(x) \rightarrow U(x)\,A_\mu(x)\,U(x)^\dagger + \frac{i}{e}\,U(x)\,\partial_\mu U(x)^\dagger, \quad U(x) \in U(N) \tag{3.33}$$

In particular gauge invariant operators are globally invariant under

$$A_\mu(x) \to U\, A_\mu(x)\, U^\dagger \tag{3.34}$$

where U is an x-independent U(N) matrix.

Thus all the arguments concerning the Hermitian matrix model apply to this case. So we use the following reduction for $\eta_\mu(x,t)$

$$\eta_\mu(x,t) = \left(\frac{\Lambda}{2\pi}\right)^{d/2} e^{iPx}\, \bar{\eta}_\mu(t)\, e^{-iPx} \tag{3.35}$$

and ansatz for $A_\mu(x,t)$

$$A_\mu(x,t) = e^{iPx}\, \bar{A}_\mu(t)\, e^{-iPx} \tag{3.36}$$

Then, we find the following Langevin equation for A (t)

$$\dot{\bar{A}}_\mu(t) = \frac{1}{e}\left[P_\nu - e\bar{A}_\nu(t), \left[P_\nu - e\bar{A}_\nu(t), P_\mu - e\bar{A}_\mu(t)\right]\right] + \left(\frac{\Lambda}{2\pi}\right)^{d/2} \bar{\eta}_\mu(t) \tag{3.37a}$$

$$\bar{A}_{\mu}(0) = 0$$

(3.37b)

Accordingly, we must have

$$l.o. \langle F(A) \rangle_S = \lim_{t \to \infty} l.o. \int [\tfrac{dp}{N}] \langle F(e^{iPx} \bar{A} e^{-iPx}) \rangle_{\bar{\eta}}$$

(3.38)

for any gauge invariant F(A).

We see that (3.38) is free from large t divergences.

In fact, these divergences arise from two sources. Firstly, there are those due to the non-gauge invariant part of the propagator, but they cancel from expectation values of gauge invariant quantities. In the second place, there are those would-be large-t divergences due to the diagonal part of the propagator but they are of non-leading order in N (for finite t) so they disappear when we calculate l.o..

At this point the following remark is relevant. Equation (3.38) may not be true for gauge variant functions, for reasons we give in the following section.

C.2. Ward Identities

Here we shall review the Ward Identities of both globally and locally U(N) symmetric theories in the context of stochastic quantization. Special emphasis will be laid on the invariance of the Gaussian property of the random noise under U(N) transformations, since this property corresponds to the invariance of the path integral integration measure, a feature we need for the derivation of Ward identities in the usual quantization procedure. We shall show that ansatz (3.7) does not preserve the U(N)-invariance of the Gaussian property of the original random noise, so none of the Ward identities of the original model will be satisfied in the reduced theory, unless they are trivially satisfied, i.e., when the Green's functions are invariant by themselves.

To make the discussion clearer we shall use the following model,

$$S[\phi] = \int d^dx \; \mathrm{tr}\left\{ \tfrac{1}{2} \left(\partial_\mu \phi(x) \right)^2 + \tfrac{1}{2} m^2 \phi(x)^2 + g/N \, \phi(x)^4 \right\} \tag{3.39}$$

$\phi(x)$ is an NxN Hermitian matrix field defined on Euclidean space-time point x.

The Langevin equation is,

$$\frac{\partial}{\partial t}\phi(x,t) = \Box\phi(x,t) - m^2\phi(x,t) - g/N\,\phi(x,t)^3 + \eta(x,t)$$

(3.40)

Let $F(\phi)$ be any functional of ϕ. We now derive the Ward identity satisfied by $F(\phi)$.

The first step is to notice that $\tilde{\phi} = U\phi U^{\dagger}$ satisfies the Langevin equation with $\tilde{\eta} = U\eta U^{\dagger}$ as a random source if ϕ satisfies equation (3.40). So,

$$\phi[U\eta U^{\dagger}] = U\phi[\eta]U^{\dagger}$$

The equivalence of the two stochastic descriptions is completed by observing that $\tilde{\eta}$ is again a random source with the same Gaussian distribution as η, so that $\langle\ \rangle_{\eta} = \langle\ \rangle_{\tilde{\eta}}$. In fact,

$$\langle\tilde{\eta}^{ij}(x,t)\,\tilde{\eta}^{k\ell}(x',t')\rangle = 2\sum_{\bar{\ell}\bar{j}} U^{i\bar{\ell}}\,U^{\dagger\bar{\ell}\ell}\,U^{k\bar{j}}\,U^{\dagger\bar{j}j}\,\delta(x-x')\,\delta(t-t')$$

$$= 2\,\delta_{i\ell}\,\delta_{jk}\,\delta(x-x')\,\delta(t-t')$$

(3.41)

Notice that (3.41) holds even if $U=U(x)$ as in the gauge theory.

Thus,

$$\langle F(\tilde{\phi})\rangle_{\eta} = \langle F(\phi)\rangle_{\eta}$$

(3.42)

It follows that,

$$\int d^dx \; tr < \frac{\delta F}{\delta \phi(x,t)} [I^\alpha, \phi(x,t)] >_\eta \tag{3.43}$$

for all α. I^α are generators of U(N) algebra.

There is one instance where (3.43) is trivially satisfied. It happens when $F(\tilde{\phi}) = F(\phi)$. In this case, property (3.41) is not needed to prove (3.43).

To reduce the Langevin equation (3.40) we write,

$$\eta(x,t) = \left(\frac{\Delta}{2\pi}\right)^{d/2} e^{iPx} \tilde{\eta}(t) e^{-iPx} \tag{3.44}$$

It is easy to see that (3.44) does not preserve property (3.41). In fact,

$$\sum_{\substack{jk \\ no}} < U_{ij} \eta_{jk}(x,t) U^\dagger_{k\ell} U_{mn} \eta_{no}(x',t') U^\dagger_{op} > =$$

$$= 2\delta(t-t') \left\{ \sum_{j \neq k} U_{ij} U^\dagger_{jp} U_{mk} U^\dagger_{k\ell} \delta(x-x') + \left(\frac{\Delta}{2\pi}\right)^d \sum_j U_{ij} U^\dagger_{jp} U_{mj} U^\dagger_{j\ell} \right\} \tag{3.45}$$

To no surprise, equation (3.43) is not satisfied for general F. It is only satisfied for U(N) invariant F as $tr[\phi(x_1) \ldots \phi(x_N)]$, but these are all we are interested in for this model.

The same problem appears in the reduction of the gauge theory.

Strictly speaking, we cannot calculate non-gauge invariant quantities using stochastic quantization of the gauge theory, because secular terms appear in their perturbative expansion.

However, we could first fix a gauge and then evaluate a non-gauge invariant quantity such as

$$\langle \mathrm{tr}\, A_\mu(x,t)\, A_\nu(y,t)\rangle_\eta$$

It should be clear by now that the reduced model will violate the Ward identity for this object, since the reduced random noise does not respect the invariance of the Gaussian property under local U(N) gauge transformations,

$$\tilde{\eta}(x,t) = U(x)\,\eta(x,t)\,U(x)^\dagger$$
$$U(x) \in U(N) \tag{3.46}$$

In the remaining part of this section we say a few words about the need of imposing a constraint in the reduced model in order to preserve the Ward identities of the full theory.

Consider, for instance, system (3.39) for $g=0$ and F given by

$$F = \phi^{ab}(y)\,\phi^{cd}(z) \tag{3.47}$$

The Ward identity (3.43) is:

$$\begin{aligned} T^{\alpha}_{cd;ab}(z,y) &\equiv \langle \phi^{cd}(z)\,[I^{\alpha},\phi(y)]^{ab}\rangle \\ &\quad + \langle \phi^{ab}(y)\,[I^{\alpha},\phi(z)]^{cd}\rangle \\ &= 0 \end{aligned} \tag{3.48}$$

It is straightforward to verify that the reduced model ansatz (3.15) satisfies equation (3.48) for any c≠d and a≠b, but for a≠b;c=d we have,

$$\begin{aligned} T^{\alpha}_{cc;ab}(z,y) &= \frac{1}{m^2}\left(\frac{\Delta}{2\pi}\right)^d \Big[(I^{\alpha})^{ab}\,\delta_{bc} - (I^{\alpha})^{ab}\,\delta_{ac}\Big] \\ &\quad + \Big[(I^{\alpha})^{cb}\,\delta_{ad} - (I^{\alpha})^{ad}\,\delta_{bc}\Big]\,\Delta(y-z) \\ &\neq 0 \end{aligned}$$

$$\Delta(y-z) \equiv \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(y-z)}}{p^2+m^2} \tag{3.49}$$

We can inmediately see the constraint $\tilde{\phi}^{aa}=0$ enforces the Ward identity at least in this simple case. The introduction of the constraint modifies the Langevin equation. The new equation is:

$$\frac{\partial}{\partial t}\tilde{\Phi}^{ab} = -(1-\delta_{ab})\left\{ [P_\mu,[P_\mu,\Phi]]^{ab} + g/N(\tilde{\Phi}^3)^{ab} - \tilde{\eta}^{ab}\left(\frac{\Delta}{2\pi}\right)^{d/2} \right\} \tag{3.50}$$

For the gauge theory, we expect that a modified Langevin equation similar to (3.50) should exist for the variable $\bar{A}_\mu(t)$. For one equation of this sort, see reference (29)

However, we wish to repeat that no such constraint is needed if we agree to calculate gauge invariant correlation functions only, rendering (3.38) valid as it is.

Still, we have to understand the problem of renormalization ,i.e., the handling of divergences appearing in loop integrals.

Since the reduced model is gauge invariant under

$$\bar{A}_\mu(t) \rightarrow U\,\bar{A}_\mu(t)\,U^\dagger - \frac{1}{e}U[P_\mu, U^\dagger] \tag{3.51}$$

for fixed P_μ ,then, we argue that all the divergences of loop integrals of gauge invariant Green's functions are logarithmic. A general proof of this proposition lies outside the scope of this thesis, but we express its plausibility by an explicit calculation in the next section.

C.3. Perturbative Calculation

We have chosen to evaluate the simplest gauge invariant correlation function.

$$\Gamma \equiv \langle \mathrm{tr}\, F_{\mu\nu}(x)^2 \rangle \tag{3.52}$$

According to (3.38)

$$\Gamma = \lim_{t\to\infty} \ell.\mathrm{o.} \int \frac{[dp]}{[\Lambda]} \langle \mathrm{tr}[P_\mu - e\bar{A}_\mu(t), P_\nu - e\bar{A}_\nu(t)]^2 \rangle_{\bar{\eta}} \tag{3.53}$$

$\bar{A}_\mu$ is the solution to the equation,

$$\begin{aligned} \frac{\partial}{\partial t} \bar{A}_\mu(t) = \frac{1}{e\kappa} [P_\nu - e\bar{A}_\nu(t), [P_\nu - e\bar{A}_\nu(t), P_\mu - e\bar{A}_\mu(t)]] \\ + \left(\frac{\Delta}{2\pi}\right)^{d/2} \bar{\eta}_\mu(t) \end{aligned} \tag{3.54}$$

$$\bar{A}_\mu(0) = \bar{A}_\mu^c \tag{3.55}$$

We write κ, $\bar{A}_\mu^c$ explicitly for the following reasons. We are interested in the renormalization properties of Γ, and therefore we want to consider the divergences appearing in the loop integrals and neglect the spurious ones due to the locality of Γ ($\langle \mathrm{tr}\, F_{\mu\nu}^2(x) \rangle$) is divergent even at the tree level). A convenient way to do this is to use the WKB expansion, i.e., an expansion in powers of $\hbar$

However before we show how this is done, let us introduce a simplification in the calculation.

If we were to use equation (3.54) to evaluate Γ we would find that the longitudinal part of the propagator does not reach equilibrium for large t. However, this longitudinal part does not contribute to gauge invariant expectation values[14].

At this point, we prefer to fix a gauge in order to eliminate the longitudinal part of the propagator from the very beginning. The argument we develop below complements the discussion of gauge fixing using the Fokker-Planck formalism of Section II.E.

A natural way to fix the gauge follows from the observation that the quantities we are interested in, as Γ, have a larger invariance than has the Langevin equation. Equation (3.54) is covariant under the following t-independent gauge transformation

$$\bar{A}_\mu(t) \rightarrow U\bar{A}_\mu(t)U^\dagger - 1/e\, U[P_\mu, U^\dagger] \tag{3.56}$$

whereas the group of invariance of Γ contains t-dependent U's as well.

$$\bar{A}_\mu(t) \rightarrow U(t)\bar{A}_\mu(t)U(t)^\dagger - 1/e\, U(t)[P_\mu, U(t)^\dagger] \tag{3.57}$$

This fact implies the existence of a whole set of Langevin equations, obtained from (3.54) by using a transformation (3.57), and we see that all of them give the same answer for Γ.

It is convenient to parametrize this set by an arbitrary function v, so that the v-Langevin equation of the set is,

$$\frac{\partial}{\partial t} \bar{A}_\mu(t) = \frac{1}{e\hbar}[P_\nu - e\bar{A}_\nu(t), [P_\nu - e\bar{A}_\nu(t), P_\mu - e\bar{A}_\mu(t)]]$$

$$+ i[P_\mu - e\bar{A}_\mu(t), v] + (\Lambda/2\pi)^{d/2}\bar{\eta}_\mu(t)$$

$$\bar{A}_\mu(0) \approx \bar{A}_\mu^c$$

(3.58)

Observe that the last equation coincides with (2.73).

Fixing the gauge, amounts to specifying the function v.

We have chosen to work with, (Landau background field gauge)

$$v = \frac{i}{\hbar}[P_\mu - e\bar{A}_\mu^c, \bar{A}_\mu(t) - \bar{A}_\mu^c]$$

(3.59)

To obtain an expansion in powers of $\hbar$ we write,

$$\bar{A}_\mu(t) = \bar{A}_\mu^c + \hbar^{1/2} B_\mu(t)$$

$$\frac{\partial}{\partial t} B_\mu(t) = \hbar^{-1/2} \Big\{ [P_\nu - e\bar{A}_\nu^c - e\hbar^{1/2} B_\nu(t), [$$

$$[P_\nu - e\bar{A}_\nu^c - e\hbar^{1/2} B_\nu(t), P_\mu - e\bar{A}_\mu^c - e\hbar^{1/2} B_\mu(t)]]$$

$$-[P_\mu - e\bar{A}_\mu^c - e\hbar^{1/2} B_\mu(t), [P_\nu - e\bar{A}_\nu^c, \hbar^{1/2} B_\nu(t)]] \Big\}$$

$$+ \left(\frac{\Lambda}{2\pi}\right)^{d/2} \bar{\eta}_\mu(t)$$

$$B_\mu(0) = 0 \tag{3.60}$$

$\bar{A}_\mu^c$ must satisfy the classical equation of motion,

$$[P_\nu - e\bar{A}_\nu^c, [P_\nu - e\bar{A}_\nu^c, P_\mu - e\bar{A}_\mu^c]] \tag{3.61}$$

and we have rescaled the fictituous time

$$t \rightarrow \hbar^{-1} t \tag{3.62}$$

To order $\hbar^{1/2}$, the effective Langevin equation is,

$$\dot{B}_\mu(t) = -[P_\nu - A_\nu, [P_\nu - A_\nu, B_\mu(t)]] + 2[F_{\nu\mu}, B_\nu(t)]$$

$$+ \left(\frac{\Lambda}{2\pi}\right)^{d/2} \bar{\eta}_\mu(t)$$

$$B_\mu(0) = 0$$

$$A_\nu \equiv e\bar{A}_\nu^c \tag{3.63}$$

and $F_{\mu\nu}$ is defined as,

$$F_{\mu\nu} = [P_\mu - A_\mu, P_\nu - A_\nu] \tag{3.64}$$

To get a perturbative expansion in powers of A_μ, we rewrite (3.63) in integral form,

$$B_\mu^{ab}(t) = \int_0^\infty d\tau\ \theta(t-\tau)\ e^{-P_{ab}^2(t-\tau)} \{ [P_\alpha, [A_\alpha, B_\mu(\tau)]]^{ab}$$

$$+ [A_\alpha, [P_\alpha, B_\mu(\tau)]]^{ab} - [A_\alpha, [A_\alpha, B_\mu(\tau)]]^{ab}$$

$$+ 2[F_{\alpha\mu}, B_\alpha(\tau)]^{ab} + \left(\frac{\Lambda}{2\pi}\right)^{d/2} \eta(\tau)^{ab} \} \tag{3.65}$$

It must be kept in mind that we have to sum over repeated Greek indices and that

$$P_\mu^{ab} \equiv P_\mu^a - P_\mu^b \tag{3.66}$$

where,

$$P_\mu^{ab} \equiv P_\mu^a \, \delta_{ab} \tag{3.67}$$

Solving (3.65) by iteration, we get an expansion in powers of A_μ.

According to equation (3.53), Γ can be written as follows,

$$\Gamma = \lim_{t\to\infty} \ell.o. \int [\tfrac{dp}{N}] \, tr\langle [P_\mu - A_\mu - e\hbar^{1/2} B_\mu, P_\nu - A_\nu - e\hbar^{1/2} B_\nu]^2 \rangle_{\bar\eta} \tag{3.68}$$

$$\approx \lim_{t\to\infty} \ell.o. \int [\tfrac{dp}{N}] \Big\{ tr\, F_{\mu\nu}^2 + 2e^2\hbar \, tr\, F_{\mu\nu} \langle [B_\mu, B_\nu] \rangle_{\bar\eta} + 2e^2\hbar \, [\tilde\nabla_1 - \tilde\nabla_2] \Big\} \tag{3.69}$$

The new quantities $\tilde\nabla_{1,2}$ are,

$$\tilde\nabla_1 = \langle tr\, [P_\mu - A_\mu, B_\nu]^2 \rangle_{\bar\eta} \tag{3.70}$$

$$\tilde\nabla_2 = \langle tr\, [P_\mu - A_\mu, B_\nu][P_\nu - A_\nu, B_\mu] \rangle_{\bar\eta} \tag{3.71}$$

It is sufficient to calculate $\langle B_\mu \; B_\nu \rangle_{\bar\eta}$ up to $O(A^2)$.

The relevant graphs are shown in Fig.2 on page 71

The answers are,

$$\lim_{t\to\infty} \text{l.o.} \int [\tfrac{d\rho}{N}] \, \mathrm{tr}\, F_{\mu\nu} \langle [B_\mu, B_\nu] \rangle_\eta =$$

$$= -4 \sum_{a,b} \int \int \frac{d^4p}{(2\pi)^4} \frac{F^{ab}_{\mu\nu} F^{ba}_{\mu\nu}}{(p-p_a)^2 (p-p_b)^2} \tag{3.72}$$

$$\lim_{t\to\infty} \text{l.o.} \int [\tfrac{d\rho}{N}] \, \tilde{V}_1 = 8 \sum_{a,b} \int \int \frac{d^4p}{(2\pi)^4} \frac{F^{ab}_{\mu\nu} F^{ba}_{\mu\nu}}{(p-p_a)^2 (p-p_b)^2} \tag{3.73}$$

$$\lim_{t\to\infty} \text{l.o.} \int [\tfrac{d\rho}{N}] \, \tilde{V}_2 =$$

$$= 8 \sum_{a,b} \int \int \frac{d^4p}{(2\pi)^4} \frac{(p-p_a)_\mu (p-p_a)_\nu F^{ab}_{\epsilon\nu} F^{ba}_{\epsilon\mu}}{[(p-p_a)^2]^2 (p-p_b)^2}$$

$$- \tfrac{1}{2} \sum_{a,b} \int \int \frac{d^4p}{(2\pi)^4} \frac{F^{ab}_{\mu\nu} F^{ba}_{\mu\nu}}{(p-p_a)^2 (p-p_b)^2} \tag{3.74}$$

We have normalized Γ so that it vanishes for zero background field.

After making one substraction we obtain

$$\Gamma_R(A)_{a^2} = \int \frac{d^4K}{(2\pi)^4} \, \mathrm{tr}\, F^2_{\mu\nu}(K) \left[1 - \frac{5e^2N\hbar}{(4\pi)^2} \ln \frac{K^2}{a^2} + \frac{2\, e^2\hbar N}{(4\pi)^2} \right] \tag{3.75}$$

In the calculation of $\tilde{V}_1$ and $\tilde{V}_2$, the would-be quadratic divergences in P_μ cancel before integrating over the random momenta, and so do the linear terms in A_μ. The remaining terms are at most logarithmically divergent in P_μ, as expected. (See discussion at the end of page 61).

As one instance of this cancelation we write explicitly the contribution to $\tilde{V}_1$ of graphs (1) (5) and (6) of Fig 2.

$$2 \sum_{\substack{cd \\ \bar{c}\bar{d}}} (A_\mu^{\bar{d}c} A_\mu^{d\bar{c}} - \delta_{d\bar{c}} (A_\mu^2)^{\bar{d}c}) \langle B_\nu^{cd} B_\nu^{\bar{c}\bar{d}} \rangle =$$

$$= 8 \sum_{c \neq d} [A_\alpha^{cc} A_\alpha^{dd} - (A_\alpha^2)^{cc}] / p_{cd}^2$$

From graph (1)

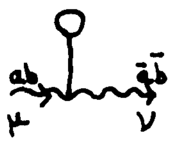
$$2 \sum_{\substack{cd \\ \bar{c}\bar{d}}} p_\mu^c p_\mu^{dc} \delta_{\bar{d}c} \delta_{d\bar{c}} \langle B_\nu^{cd} B_\nu^{\bar{c}\bar{d}} \rangle =$$

$$= 8 \sum_{c \neq d} [(A_\alpha^2)^{cc} - A_\alpha^{cc} A_\alpha^{dd}] / p_{cd}^2$$

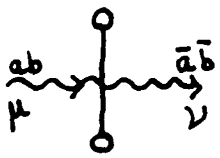
From graphs (5,6)

They cancel each other.

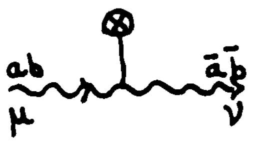
Fig. 1 Feynman rules derived from equation (3.65)



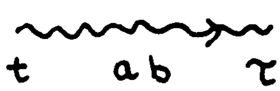
$$\delta_{\mu\nu}\left\{A_\alpha^{a\bar a}\,\delta_{b\bar b}\,(p_a+p_{\bar a}-2p_b)_\alpha+A_\alpha^{\bar b b}\,\delta_{a\bar a}(p_b+p_{\bar b}-2p_a)_\alpha\right\}$$



$$-\delta_{\mu\nu}\left\{(A_\alpha^2)^{a\bar a}\,\delta_{b\bar b}+(A_\alpha^2)^{\bar b b}\,\delta_{a\bar a}-2A_\alpha^{a\bar a}A_\alpha^{b\bar b}\right\}$$



$$2\left\{F_{\nu\mu}^{a\bar a}\,\delta_{b\bar b}-\delta_{a\bar a}\,F_{\nu\mu}^{\bar b b}\right\}$$



$$\theta(t-\tau)\,e^{-p_{ab}^2(t-\tau)}$$

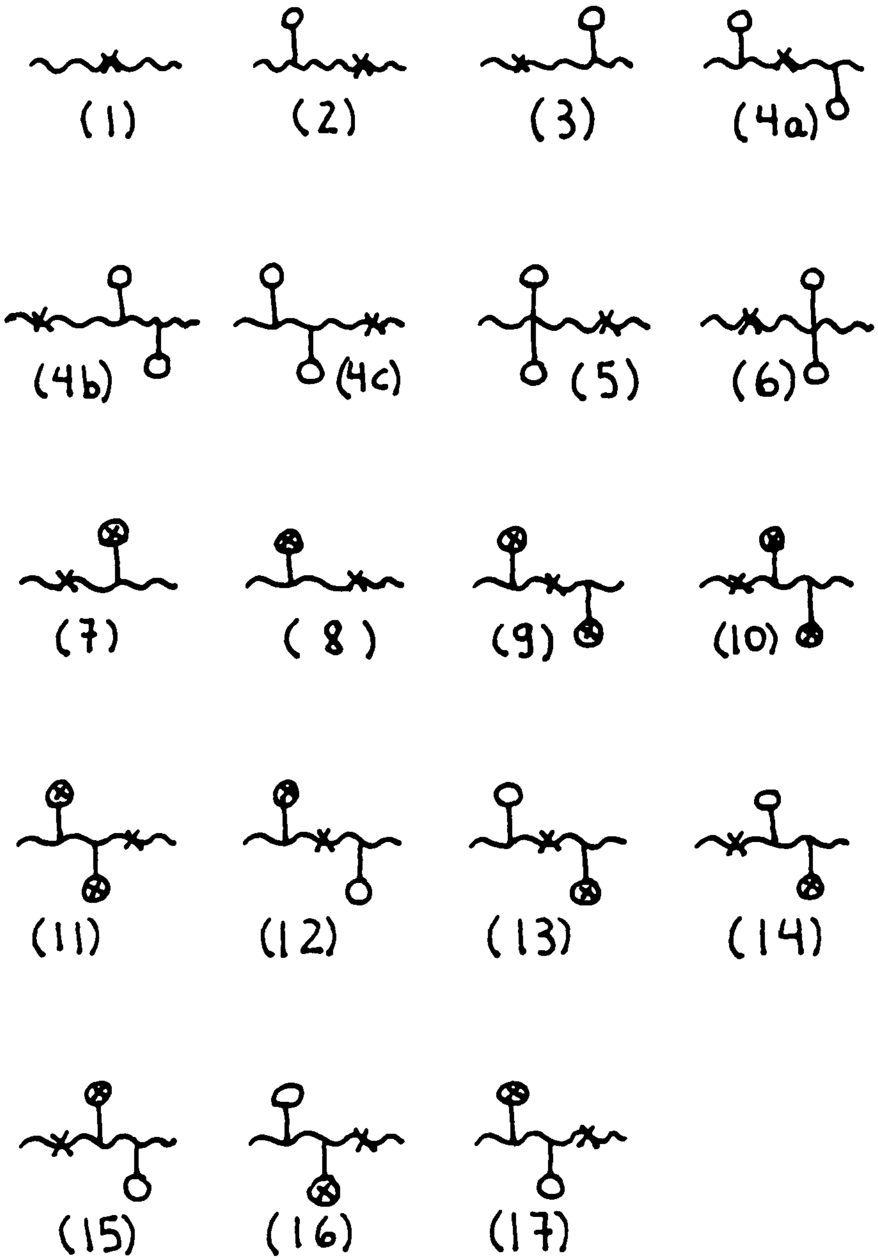
The arrow points toward the past in fictitious time t. To complete the rules we need the following.

a) To find a cross means to reverse the direction of t, transpose the indices of the next line and multiply by

$$2\;(\Lambda/2\pi)^d$$

b) Integrate over t's of each vertex and cross, from zero to infinity.

Fig. 2: Graphs contributing to $\langle B_\mu B_\nu \rangle_q$ to $O(A^2)$.



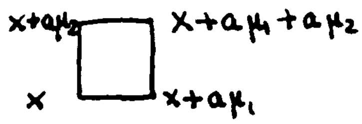
IV. LATTICE GAUGE THEORIES.

A. Introduction.

We outline the lattice formulation of gauge theories.[3]

The main advantage of defining a gauge theory on a lattice is that the lattice spacing acts as a gauge invariant regulator. Furthermore, since this formulation is free from infinities at all stages, it is specially suited for doing numerical calculations. These numerical calculations are the only available information supporting the confining property of non-Abelian gauge theories[4].

We shall introduce a hypercubical lattice of side a. Any minimal square on this lattice will be called a fundamental plaquette. For instance a fundamental plaquette centered on x may be:



We define an action on the fundamental plaquette by

$$S_{\Box} = \mathrm{tr}\left\{ U_{x,x+a\mu_1} U_{x+a\mu_1,x+a\mu_1+a\mu_2} U^{\dagger}_{x+a\mu_2,x+a\mu_1+a\mu_2} U^{\dagger}_{x,x+a\mu_2} + h.c. \right\} \tag{4.1}$$

The U variables are defined on the links of the fundamental plaquette and they belong to the adjoint representation of G (the group of symmetries). Notice also that the plaquette has an orientation such that

$$U^{+}_{x,x+a\mu} = U_{x+a\mu,x} \tag{4.2}$$

Clearly $S_{\square}$ is invariant under the local G-gauge transformation:

$$U_{x,x+a\mu} \rightarrow V(x)\, U_{x,x+a\mu}\, V(x+a\mu)^{-1} \quad , \; V \in G \tag{4.3}$$

Wilson's action defining the gauge theory on the lattice is given by:

$$S_W = \sum_{\text{plaquettes}} S_{\square} \tag{4.4}$$

where S_W is obviously invariant under transformation (4.3)

The quantum mechanics of the theory is described by the partition function.

$$Z = \int \prod [dU] \; e^{1/g^2 \, S_W} \tag{4.5}$$

$[dU]$ is the Haar-measure of the group G and g is a coupling constant.

The correspondence of the lattice quantization and the standard continuum quantization is obtained by the identification:

$$U_{x,x+a\mu} = e^{i\frac{a}{2} A_\mu(x)} \tag{4.6}$$

$A_\mu(x)$ belongs to the Lie Algebra of G.

It is easy to check that

$$\begin{aligned} S_w \xrightarrow[a\to 0]{} & \text{constant} - \frac{a^4}{4} \sum_x \operatorname{tr} F_{\mu\nu}^2(x) \\ & = \text{constant} - 1/4 \int d^4x \operatorname{tr} F_{\mu\nu}^2(x) \end{aligned} \tag{4.7}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i\,[A_\mu(x), A_\nu(x)] \tag{4.8}$$

That is, we get the correct classical limit.

In a lattice gauge theory only gauge invariant correlation functions must be considered since, according to Elitzur[30], gauge variant correlation functions are identically zero.

A useful set of gauge invariant correlation functions is provided by the Wilson loop that we shall define below.

On the lattice we consider a closed path C.

The Wilson loop is defined by

$$W[C] = \mathrm{tr} \langle \prod_{i \in C} U_i \rangle \tag{4.9}$$

where i denotes a link belonging to the curve C.

It is clear that W(C) is invariant under (4.3)

It is possible to express the property of confinement of the pure gauge theory in terms of the Wilson's loop.

We have two possibilities:

i) $W[C] \underset{\text{large } C}{\sim} e^{-\sigma A}$ (confinement criteria) (4.10)

ii) $W[C] \underset{\text{large } C}{\sim} e^{-\sigma' P}$ (deconfining criteria) (4.11)

A is the minimal area enclosed by C; P is the length of C; and σ is called the string tension.

It is not difficult to show that in the strong coupling limit[27] ($g \to \infty$)

$$W[C] \sim e^{-A/a^2 \ln g^2} \tag{4.12}$$

So the lattice gauge theory confines in this limit.

This is true even for Quantum Electrodynamics where electrons and positrons are found free. So the connection with the usual concept of confinement is not straightforward.

To clarify this last point we resort to the following Renormalization Group argument

When $a \to 0$, the string tension is divergent (for fixed g).

On the other hand, σ is an observable quantity that fixes the scale of the glue-ball masses, for example. The answer is provided by the renormalization theory: the coupling constant g has to be a function of the cut off such that physical quantities are finite for $a \to 0$. Moreover, the arbitrariness in defining a finite part of the divergent Green's functions actually implies the stronger condition:

$$a \frac{d\sigma}{da} = \left(a \left(\frac{\partial\sigma}{\partial a}\right)_g + a \frac{\partial g}{\partial a} \left(\frac{\partial\sigma}{\partial g}\right)_a \right) = 0 \tag{4.13}$$

or

$$\left(-a \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial g}\right)\sigma = 0 \tag{4.14}$$

where β is defined by $\beta = -a(\partial g/\partial a)$ (4.15)

On dimensional grounds we must have:

$$\sigma(g,a) = f(g)/a^2 \tag{4.16}$$

Then

$$\sigma(g) = -1/2\left(\frac{\partial\sigma}{\partial g}\right)\beta(g) \tag{4.17}$$

Since $\beta = \beta(g)$ is defined for strong and weak couplings, this provides a way of relating both limits of the theory.

For strong couplings, from equation (4.12) we have:

$$\beta(g) = g \ln g^2 \tag{4.18}$$

When $g \to 0$ we have the usual asymptotic freedom result[2], for a SU(3) color group,

$$\beta(g)/g = -\beta_0 g^2 - \beta_1 g^4$$

$$\beta_0 = \frac{11}{16\pi^2} \quad ; \quad \beta_1 = \frac{102}{(16\pi^2)^2} \tag{4.19}$$

There are two possibilities:

i) If the strong coupling B(g) given by equation (4.18) can be extrapolated all the way down to g=0 where it matches equation (4.19) without ever finding a zero of B(g); then

the theory will confine since $\sigma(g)$ will never be zero.
ii) If in between $g=0$ and $g=+\infty$ a zero of $B(g)$ exists σ will vanish there and the theory will have a "phase transition" to a deconfining phase.

To determine which of the aforementioned scenarios is realized in the Yang-Mills theory has proven to be very difficult, although recently considerable progress in this direction has been made using numerical simulations in computers[4].

After these brief remarks, we proceed to derive the Langevin equation for lattice gauge theories. Using this Langevin equation and the methods introduced in section III, we shall obtain the reduced form of the theory, i.e., the quenched Eguchi-Kawai model[12].

B Langevin Equation for Lattice Gauge Theories[20]

We can extend the arguments of section III into lattice gauge theories, in which the dynamical variables are the link variables. Let us consider U(N) lattice theory as an example and let $U(x)$ be the link variable on the link $(x,x+\mu)$, which we assume belongs to the fundamental representation of U(N).

In order to use our methods of reduction in the case of U(N) lattice gauge theory we must find a Langevin equation for $U_\mu(x,t)$ such that

$$\lim_{t\to\infty} \langle F(U_\mu(x,t))\rangle_\eta = \frac{\int \prod_{x,\mu} [dU_\mu(x)]\, F[U_\mu(x)]\, e^{S_W(U)}}{\int \prod_{x,\mu} [dU_\mu(x)]\, e^{S_W(U)}} \tag{4.20}$$

S_W (U) is defined in (4.4), F is an arbitrary function of U (x,t), $[dU_\mu(x)]$ is the Haar measure of U(N) and $\langle\ \rangle_\eta$ is given by (2.4), properly modified for x integer.

To find the Langevin equation whose solution $U_\mu(x,t)$ satisfies equation (4.20), we first write an Hermitian and non-negative Fokker-Planck Hamiltonian $\hat{H}_W$ such that

$$\hat{H}_W \; e^{1/2 S_W} = 0 \tag{4.21}$$

H_W is Hermitian in the Hilbert space defined by the scalar product,

$$\langle \Psi(U), \Phi(U)\rangle = \int \prod_{x,\mu} [dU_\mu(x)]\, \Psi^*[U]\, \Phi(U) \tag{4.22}$$

To construct H_W an analogy with (2.20) we write

$$\hat{H}_W = 1/2 \sum_{x,\mu,\alpha} \hat{Q}_{\alpha\mu}(x)^\dagger\, \hat{Q}_{\alpha\mu}(x) \tag{4.23}$$

where we defined

$$\hat{Q}_{\alpha\mu} = \hat{E}^{\alpha}_{\mu}(x) - \frac{1}{2}\left[\hat{E}^{\alpha}_{\mu}(x)\, S_w\right] \tag{4.24}$$

and $\hat{Q}^{\dagger}_{\alpha\mu}$ is the Hermitian adjoint of $\hat{Q}_{\alpha\mu}$ in the Hilbert space with scalar product (4.22).

$\hat{E}^{\alpha}_{\mu}(x)$ will be specified below. But first notice that (4.23) ensures the Hermiticity and non-negativity of H_w.

To guaranty (4.21) we shall choose $\hat{E}^{\alpha}_{\mu}(x)$ such that

$$\hat{E}^{\alpha}_{\mu}(x)\, e^{\frac{1}{2} S_w(U)} = \frac{1}{2}\left[\hat{E}^{\alpha}_{\mu}(x)\, S_w\right] e^{\frac{1}{2} S_w(U)} \tag{4.25}$$

Moreover, $\hat{E}^{\alpha}_{\mu}(x)$ must satisfy the formula of integration by parts needed in (2.10). That is

$$\int \prod_{x,\mu} \left[dU_{\mu}(x)\right] \left\{\left[\hat{E}^{\alpha}_{\mu}(x)\, F(U)\right] G(U) + F(U)\left[\hat{E}^{\alpha}_{\mu}(x) G(U)\right]\right\} = 0 \tag{4.26}$$

for any F(U) and G(U).

Formula (4.26) is the key to find $\hat{E}^{\alpha}_{\mu}(x)$. In fact (4.26) is nothing else than the expression of the invariance of the integration measure under the group of symmetry having generators $\hat{E}^{\alpha}_{\mu}(x)$.

It is known that the Haar measure is invariant under the transformation;

$$U_\mu(x) \rightarrow V_\mu(x)\, U_\mu(x) \qquad\qquad V_\mu(x) \in U(N)$$

$$\approx U_\mu(x) + i I^\alpha U_\mu(x)\, \Theta^\alpha_\mu(x) \tag{4.27}$$

for small $\Theta^{\alpha}_{\mu}(x)$. In this equation I^α are generators of the algebra of U(N) satisfying,

$$\mathrm{tr}\left(I^\alpha I^\beta\right) = \delta^{\alpha\beta} \qquad\qquad \alpha, \beta = 1, \ldots N^2$$

$$\sum_\alpha (I^\alpha)_{ij} (I^\alpha)_{kl} = \delta_{il}\, \delta_{jk} \qquad\qquad i, j = 1, \ldots N \tag{4.28}$$

We define $\hat{E}^{\alpha}_{\mu}(x)$ by:

$$F\left(U_\mu(x) + i I^\alpha U_\mu(x) \Theta^\alpha_\mu(x)\right) \approx$$

$$\approx F\left(U_\mu(x)\right) + i \sum_{\alpha\mu x} \Theta^\alpha_\mu(x) \hat{E}^\alpha_\mu(x) F(U) \tag{4.29}$$

The invariance of the Haar measure under (4.27) implies (4.26).

A realization of the operator $\hat{E}^{\alpha}_{\mu}(x)$ is readily obtained from (4.29) by expanding the left hand side of (4.29) in a Taylor series in $\Theta^{\alpha}_{\mu}(x)$.

We get

$$\hat{E}^\alpha_\mu(x)\, F(U) = \sum_{ab} \frac{\partial F}{\partial U_\mu(x)^{ab}} \left(I^\alpha U_\mu(x)\right)^{ab} \tag{4.30}$$

We see that this realization of $\hat{E}^{\alpha}_{\mu}(x)$ guaranties (4.21) From (4.30) follows that,

$$[\hat{E}^{\alpha}_{\mu}(x) F(U)] G(U) = [\hat{E}^{\alpha}_{\mu}(x) F(U)] G(U) \tag{4.31}$$

for all G(U).

From now on we assume the action of the operator $\hat{E}^{\alpha}_{\mu}(x)$ on functions of U is defined by (4.30).

Then, in analogy with equation (2.13), we write

$$\frac{\partial}{\partial t} P(U,t) = -\sum_{\alpha\mu x} \hat{E}^{\alpha}_{\mu}(x) [\hat{E}^{\alpha}_{\mu}(x) + (\hat{E}^{\alpha}_{\mu}(x) S_W)] P(U,t) \tag{4.32}$$

Therefore,

$$\frac{\partial}{\partial t} \langle F(U) \rangle_{\eta} = \int \prod_{x\mu} [dU_{\mu}(x)] F(U) \frac{\partial}{\partial t} P(U,t) =$$

$$= \langle \sum_{\alpha\mu x} [- \hat{E}^{\alpha}_{\mu}(x) \hat{E}^{\alpha}_{\mu}(x) F(U) + (\hat{E}^{\alpha}_{\mu}(x) S_W) \hat{E}^{\alpha}_{\mu}(x) F(U)] \rangle_{\eta} \tag{4.33}$$

To obtain (4.33) equations (4.32) and (4.26) are used.

The t-evolution of U(x,t) must respect the unitary property of U(x,t). Therefore

$$U_\mu(x,t+\Delta t) = e^{i\,\theta^\alpha_\mu(x,\Delta t)I^\alpha}\,U_\mu(x,t)$$

$$\approx U_\mu(x,t) + i\,\theta^\alpha_\mu(x,\Delta t)\,I^\alpha\,U_\mu(x,t) \tag{4.34}$$

$\theta^\alpha_\mu(x,\Delta t)$ vanishes for $\Delta t=0$. So that, according to (4.29)

$$\frac{\partial}{\partial t}F(U) = i\sum_{\alpha,\mu,x}\lim_{\Delta t\to 0}\frac{\theta^\alpha_\mu(x,\Delta t)}{\Delta t}\,\hat{E}^\alpha_\mu(x)F(U) \tag{4.35}$$

Choose $F(U) = U_\nu(x,t)$. We get

$$\frac{\partial}{\partial t}U_\nu(x,t) = i\sum_{\alpha,\mu,y}\lim_{\Delta t\to 0}\frac{\theta^\alpha_\mu(y,\Delta t)}{\Delta t}\,\delta_{\mu\nu}\,\delta(x,y)\,U_\mu(x,t)$$

$$= i\sum_{\alpha}\lim_{\Delta t\to 0}\frac{\theta^\alpha_\nu(x,\Delta t)}{\Delta t}\,I^\alpha\,U_\nu(x,t) \tag{4.36}$$

Comparing (4.35) and (4.36) we get,

$$\frac{\partial}{\partial t}F(U) = \sum_{\alpha\mu x}\mathrm{tr}\left(\frac{\partial U_\mu(x,t)}{\partial t}\,U_\mu(x,t)^\dagger I^\alpha\right)\hat{E}^\alpha_\mu(x)\,F(U) \tag{4.37}$$

Therefore

$$\langle \sum_{\alpha\mu x} \mathrm{tr}\left(\frac{\partial}{\partial t} U_\mu(x,t)\, U_\mu(x,t)^\dagger I^\alpha \right) \hat{E}^\alpha_\mu(x)\, F(U) \rangle_\eta =$$

$$= \sum_{\alpha\mu x} \langle [-\hat{E}^\alpha_\mu(x)\, \hat{E}^\alpha_\mu(x)\, F(U) + (\hat{E}^\alpha_\mu(x)\, S_w)\, \hat{E}^\alpha_\mu(x) F(U)] \rangle_\eta \tag{4.38}$$

The last equation is satisfied if we choose

$$\mathrm{tr}\left(\frac{\partial}{\partial t} U_\mu(x,t)\, U_\mu(x,t)^\dagger I^\alpha \right) = (\hat{E}^\alpha_\mu(x)\, S_w) + i\, \eta^\alpha_\mu(x,t) \tag{4.39}$$

where $\eta^\alpha_\mu(x,t)$ is a random variable having Gaussian distribution (2.4). To see that (4.39) implies (4.38) we observe that, from equation (2.4),

$$\langle \eta^\alpha_\mu(x,t)\, \hat{E}^\alpha_\mu(x) F(U) \rangle_\eta = 2 \langle \frac{\delta}{\delta \eta^\alpha_\mu(x,t)} (\hat{E}^\alpha_\mu(x)\, F(U)) \rangle_\eta \tag{4.40}$$

But

$$2 \langle \frac{\delta}{\delta \eta^\alpha_\mu(x,t)} (\hat{E}^\alpha_\mu(x)\, F(U)) \rangle_\eta =$$

$$= 2 \sum_{y\nu\beta} \langle \mathrm{tr}\left(\frac{\delta U_\nu(y,t)}{\delta \eta^\alpha_\mu(x,t)}\, U_\nu(y,t)^\dagger I^\beta \right) \hat{E}^\beta_\nu(y)\, \hat{E}^\alpha_\mu(x)\, F(U) \rangle_\eta \tag{4.41}$$

To obtain (4.41) we have used (4.37).

It is easy to show that (See the derivation of equation (2.12)):

$$\frac{\delta U_\nu(y,t)}{\delta \eta^\alpha_\mu(x,t)} = \frac{i}{2}\,\delta(x,y)\,\delta_{\mu\nu}\,I^\alpha\,U_\nu(x,t) \tag{4.42}$$

Now the proof that (4.39) implies (4.38) is complete.

We prefer to write (4.39) in a more recognizable form,

$$\frac{\partial}{\partial t} U_\mu(x,t) = \sum_\alpha I^\alpha U_\mu(x,t)[\hat{E}^\alpha_\mu(x,t)\;S_W] + i\;\eta_\mu(x,t)\,U_\mu(x,t) \tag{4.43}$$

where

$$\eta_\mu(x,t) = \sum_\alpha I^\alpha\;\eta^\alpha_\mu(x,t) \tag{4.44}$$

(4.43) is the Langevin equation for the lattice gauge theory.

It is now a simple matter to use the method of large N reduction of III to obtain the quenched Eguchi-Kawai model.

C. Quenched Eguchi-Kawai model[20].

In this section we use the appropiate Langevin equation (Equation (4.43)) to study the large N reduction for the U(N) lattice gauge theory.

The reduced model that we find is the quenched Eguchi-Kawai model proposed by Das and Wadia[12].

We first write explicity the Langevin equation (4.43) for Wilson action S_W

$$S_W = \beta \sum_x \sum_{\mu \neq \nu} \mathrm{tr}\, U_\mu(x)\, U_\nu(x+\mu)\, U_\mu(x+\nu)^\dagger\, U_\nu(x)^\dagger \tag{4.45}$$

where we define $B = 1/g^2$ (4.46)

Now since the constraint

$$U_\mu(x)^\dagger\, U_\mu(x) = 1 \tag{4.47}$$

must hold; we have

$$\frac{\partial\, U_\mu(x)^{\dagger\, ab}}{\partial\, U_\nu(y)^{cd}} = -\,\delta_{\mu\nu}\,\delta_{xy}\, U_\mu(x)^{\dagger\, ac}\, U_\mu(x)^{\dagger\, db} \tag{4.48}$$

Replacing $\hat{E}^a_\mu(x,t)S_W$ given by (4.30) into (4.43) we find

$$\frac{\partial\, U_\mu^{ik}(x,t)}{\partial\, t} = \sum_{ab} U_\mu(x,t)_{ib}\, \frac{\partial S_W}{\partial U_\mu(x)^{ab}}\, U_\mu(x,t)_{ak} + i[\eta_\mu(x,t)\, U_\mu(x,t)]_{ik} \tag{4.49}$$

and

$$\frac{\partial S_W}{\partial U_\mu(x)^{ab}} = \beta \sum_{\nu \neq \mu} \Big\{ U_\nu(x+\mu)\, U_\mu(x+\nu)^\dagger\, U_\nu(x)^\dagger$$

$$+ U_\nu(x-\nu+\mu)^\dagger\, U_\mu(x-\nu)^\dagger\, U_\nu(x-\nu)$$

$$- U_\mu(x)^\dagger\, U_\nu(x-\nu)^\dagger\, U_\mu(x-\nu)\, U_\nu(x+\mu-\nu)\, U_\mu(x)^\dagger$$

$$- U_\mu(x)^\dagger\, U_\nu(x)\, U_\mu(x+\nu)\, U_\nu(x+\mu)^\dagger\, U_\mu(x)^\dagger \Big\}^{ba} \tag{4.50}$$

We remind the reader that $\eta_\mu(x,t)$ is a random source with Gaussian distribution; that is: the η's obey Wick's decomposition property (2.3) and

$$\langle \eta_\mu^{ij}(x,t)\, \eta_\nu^{\ell m}(x',t') \rangle_\eta = 2\, \delta_{\mu\nu}\, \delta_{im}\, \delta_{j\ell}\, \delta_{xx'}\, \delta(t-t') \tag{4.51}$$

The Kronecker delta in x has replaced the Dirac delta appearing in (2.2) because now x is an integer number.

We have the following proposition.

Proposition: Let $\eta_{ij}^{\mu}(t)$ (i,j = 1,,.. N) be a random source with Gaussian distribution and p_i^α (i=1, N; α =1..,d) be a random number with uniform distribution in the interval

$(-\pi, \pi)$. As long as one restrict oneself to invariant expectation values of the form $\langle tr(\eta^{\mu_1}(x_1,t_1)\ldots\eta^{\mu_{2m}}(x_{2m},t_{2m}))\rangle_\eta$ in the large N limit

$$\eta^{\mu}_{ij}(x,t) = e^{i(p_i - p_j)x}\,\bar{\eta}^{\mu}_{ij}(t) \tag{4.52}$$

serves as variables with Gaussian distribution.

<u>Proof</u>: The demonstration follows exactly as the proof that (3.7) is a Gaussian source in the large N limit, so we do not repeat it here. However, we want to offer an example calculating

$$\begin{aligned}
\langle tr\,\eta^{\mu_1}(x_1,t_1)\,\eta^{\mu_2}(x_2,t_2)\rangle &= \sum_{ij}\langle \eta^{\mu_1}_{ij}(x_1,t_1)\,\eta^{\mu_2}_{ji}(x_2,t_2)\rangle = \\
&= \sum_{ij}\int_{-\pi}^{\pi}\prod_{K}\frac{d^d p_K}{(2\pi)^d}\, e^{i(p_i-p_j)(x_1-x_2)} \langle \bar{\eta}^{\mu_2}_{ji}(t_2)\,\bar{\eta}^{\mu_1}_{ij}(t_1)\rangle_{\bar{\eta}} \\
&= 2\,\delta_{\mu_1\mu_2}\,\delta(t_1-t_2)\sum_{ij}\int_{-\pi}^{\pi}\prod_{K}\frac{d^d p_K}{(2\pi)^d}\, e^{i(p_i-p_j)(x_1-x_2)} \\
&= 2\,\delta_{\mu_1\mu_2}\,\delta(t_1-t_2)\left\{(N^2-N)\,\delta_{x_1x_2} + N\right\} \\
&\approx 2N^2\,\delta_{\mu_1\mu_2}\,\delta(t_1-t_2)\,\delta_{x_1x_2}
\end{aligned} \tag{4.53}$$

which is the right answer. From this example we see that the approximation makes sense if

$$N \gg \sum_x = \text{number of lattice sites} \tag{4.54}$$

We can now reduce (4.49) using η (x t) given by (4.52) and the ansatz

$$U_\mu(x,t) = e^{iPx}\, \overline{U}_\mu(t)\, e^{-iPx} \tag{4.55}$$

where

$$P_\mu^{ij} = p_\mu^i \delta_{ij} \tag{4.56}$$

We get

$$\begin{aligned}\dot{\overline{U}}_\mu = \beta \overline{U}_\mu \sum_{\nu\neq\mu} \{ & e^{iP\cdot\mu}\overline{U}_\nu e^{-iP\cdot\mu} e^{iP\cdot\nu} \overline{U}_\mu^\dagger e^{-iP\cdot\nu} \overline{U}_\nu^\dagger \\ & + e^{iP(\mu-\nu)}\overline{U}_\nu^\dagger e^{-iP\cdot\mu} \overline{U}_\mu^\dagger \overline{U}_\nu e^{iP\cdot\nu} \\ & - \overline{U}_\mu^\dagger e^{-iP\cdot\nu} \overline{U}_\nu^\dagger \overline{U}_\mu e^{iP\cdot\mu} \overline{U}_\nu e^{-iP\cdot(\mu-\nu)} \overline{U}_\mu^\dagger \\ & - \overline{U}_\mu^\dagger \overline{U}_\nu e^{-iP\cdot\nu} \overline{U}_\mu e^{-iP\cdot(\nu-\mu)} \overline{U}_\nu^\dagger e^{-iP(\mu-\nu)} \overline{U}_\mu^\dagger \} \overline{U}_\mu \\ & + i\, \overline{\eta}_\mu \overline{U}_\mu \end{aligned} \tag{4.57}$$

The reduced Langevin equation (4.57) formally implies a reduced action S_R (see the discussion of equation (3.21)):

$$S_R = \beta \sum_{\nu\neq\mu} \mathrm{tr}[(\overline{U}_\mu e^{iP\cdot\mu})(\overline{U}_\nu e^{iP\cdot\nu})(\overline{U}_\mu e^{iP\cdot\mu})^\dagger (\overline{U}_\nu e^{iP\cdot\nu})^\dagger] \tag{4.58}$$

The reduced action S_R is precisely the one found by Das and Wadia.[12]

V STOCHASTIC REGULARIZATION AND LARGE N REDUCTION.

Until now our treatment of the reduction in the large N limit of non-gauge fields theories has been rather formal. This is so because of the divergences appearing in higher orders of perturbation theory.

Recently, a new regularization method has become available, the Stochastic Regularization(16). It has the merit of preserving all the symmetries of the action, but at the expense of non-locality. We shall see that Stochastic Regularization combines efficiently with our methods of reduction.

But, what is Stochastic Regularization?

To answer this question we shall make a quick review of the stochastic quantization.

The main ingredient of stochastic quantization is to write a Langevin equation

$$\frac{\partial}{\partial t}\phi_\ell(x,t) = -\frac{\delta S}{\delta \phi_\ell(x,t)} + \eta_\ell(x,t) \tag{5.1}$$

$\eta_\ell(x,t)$ is a Gaussian random source satisfying:

$$\langle \eta_\ell(x,t)\,\eta_{\ell'}(x',t')\rangle_\eta = 2\,\delta_{\ell\ell'}\,\delta(x-x')\,\delta(t-t') \tag{5.2}$$

The Stochastic Regularization replaces $\delta(t-t')$ by a family of functions $B_A(t-t')$. This family will converge (as a distribution) to $\delta(t-t')$ when $A\to\infty$.

In general we shall have:[16]

$$B_A(t-t') \underset{t\to t'}{\sim} (t-t')^n, \quad n=0,1,2.. \tag{5.3}$$

Choosing n sufficiently large, it is possible to regularize the divergence of renormalizable theories. Moreover, since the Stochastic Regularization does not touch the x-dependent part of the η correlation, it will preserve all the symmetries of the action S. (See section III.C.2).

Now we shall proceed to apply the Stochastic Regularization to the large N reduction.

The simple case and the only one we shall examine in this section is the massive NxN Hermitian matrix model introduced in (3.1),

$$\frac{\partial}{\partial t}\phi(x,t) = (\Box - m^2)\,\phi(x,t) - g/N\;\phi(x,t)^3 + \eta(x,t) \tag{5.4a}$$

$$\phi(x,0) = 0 \tag{5.4b}$$

Introducing the reduced source

$$\eta(x,t) = \left(\frac{\Lambda}{2\pi}\right)^{d/2} e^{iPx}\, \bar{\eta}(t)\, e^{-iPx}$$

(5.5)

and the ansatz:

$$\phi(x,t) = e^{iPx}\bar{\phi}(t)e^{-iPx}$$

(5.6)

we obtain the reduced Langevin Equation

$$\frac{\partial}{\partial t}\bar{\Phi}^{ij}(t) = -(p_{ij}^2 + m^2)\,\bar{\Phi}^{ij}(t) - g/_N\,\bar{\Phi}^3(t)_{ij} + \left(\frac{\Lambda}{2\pi}\right)^{d/2}\bar{\eta}^{ij}(t)$$

$$\bar{\Phi}^{ij}(0) = 0$$

$$p_{ij} \equiv p_i - p_j$$

(5.7)

The perturbation expansion is obtained as in section II.D by rewriting (5.7) as an integral equation,

$$\bar{\Phi}^{ij}(t) = \int d\tau\; G_t(p_{ij},\tau)\left\{-g/_N\,\Phi^3(\tau)_{ij} + \left(\frac{\Lambda}{2\pi}\right)^{d/2}\bar{\eta}^{ij}(\tau)\right\}$$

$$G_t(p_{ij},\tau) = \Theta(t-\tau)\, e^{-(p_{ij}^2+m^2)(t-\tau)}$$

(5.8)

As an example we calculate the one loop mass renormalization in this theory. For that, we shall examine:

$$\ell.o.\langle \mathrm{tr}\,\phi(x)\phi(y)\rangle \underset{m\neq 0}{=} \ell.o.\lim_{t\to\infty}\int\left[\frac{dp}{\Lambda}\right]e^{ip_{ij}(x-y)}\langle\bar{\phi}^{ij}(t)\,\bar{\phi}^{ji}(t)\rangle_{\bar{\eta}} \tag{5.9}$$

where the last expresion follows from equation (3.17).

The one loop contribution to $\langle\bar{\phi}^{ij}(t)\;\bar{\phi}^{ji}(t)\rangle_{\bar{\eta}}$ is given by the 2(a), where (a) is defined as follows

$$\begin{aligned}(a) = \left(\frac{\Lambda}{2\pi}\right)^{2d}\int & d\tau_1\,d\tau_2\,d\tau_3\,d\tau_4\,d\tau_5\;\theta(t-\tau_1)\,\theta(\tau_1-\tau_2)\\ & \theta(\tau_1-\tau_3)\,\theta(\tau_1-\tau_4)\;\theta(t-\tau_5)\\ & e^{-(p_{ik}^2+m^2)(\tau_1-\tau_2)}\quad e^{-(p_{k\ell}^2+m^2)(\tau_1-\tau_3)}\\ & e^{-(p_{ij}^2+m^2)(t-\tau_1)}\quad e^{-(p_{\ell j}^2+m^2)(\tau_1-\tau_4)}\\ & e^{-(p_{ij}^2+m^2)(t-\tau_5)}\quad\times\\ \times\langle\bar{\eta}_{ik}(\tau_2)\;\bar{\eta}_{k\ell}(\tau_3)\;\bar{\eta}_{\ell j}(\tau_4)\;\bar{\eta}_{ji}(\tau_5)\rangle_{\bar{\eta}}\end{aligned} \tag{5.10}$$

The leading large N contribution of $\langle \quad \rangle_{\bar\eta}$ will be

$$\langle \bar\eta_{ik}(\tau_2)\, \bar\eta_{k\ell}(\tau_3)\rangle_{\bar\eta}\ \langle \bar\eta_{\ell j}(\tau_4)\, \bar\eta_{ji}(\tau_5)\rangle_{\bar\eta} \,+$$

$$+\, \langle \bar\eta_{ik}(\tau_2)\, \eta_{ji}(\tau_5)\rangle_{\bar\eta}\ \langle \bar\eta_{k\ell}(\tau_3)\, \bar\eta_{\ell j}(\tau_4)\rangle_{\bar\eta} \tag{5.11}$$

Therefore,

$$(a) = 8\left(\frac{\Lambda}{2\pi}\right)^{2d} \int d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4\, d\tau_5\ \Theta(t-\tau_1)\, \Theta(\tau_1-\tau_2)$$

$$\Theta(\tau_1-\tau_3)\, \Theta'(\tau_1-\tau_4)\, \Theta(t-\tau_5) \sum_{ijk} e^{-(p_{ik}^2+m^2)(2\tau_1-\tau_2-\tau_3)} \times$$

$$\times\, e^{-(p_{ij}^2+m^2)(2t-\tau_4-\tau_5)}\, B_A(\tau_2-\tau_3)\, B_A(\tau_4-\tau_5) \tag{5.12}$$

Introducing

$$B_A(\tau) = \tfrac{1}{2} A^2 |\tau|\, e^{-A|\tau|} \tag{5.13}$$

We have

$$B_A(\tau) \xrightarrow[A\to\infty]{} \delta(\tau)$$

(5.14)

Then

$$(a) \underset{\text{large } t}{\sim} \left(\frac{\Lambda}{2\pi}\right)^{2d} (A^2)^2 \sum_{ijk} \frac{1}{(p_{ik}^2+m^2+A)^2\,(p_{ik}^2+m^2)} \times$$

$$\times \frac{1}{(p_{ij}^2+m^2)^2\,(p_{ij}^2+m^2+A)^2}$$

(5.15)

Therefore, the one loop contribution to the propagator is

$$\text{l.o.} \quad \langle T_r \phi(x)\phi(y)\rangle = 2\,g/N \sum_{ijk} \int \frac{d^4p_i}{(2\pi)^4}\,\frac{d^4p_k}{(2\pi)^4}\,\frac{d^4p_j}{\Lambda^4}$$

$$\frac{(A^2)^2}{(p_{ij}^2+m^2)^2(p_{ij}^2+m^2+A)^2}\; e^{i\,p_{ij}(x-y)}$$

$$\frac{1}{(p_{ik}^2+m^2+A)^2(p_{ik}^2+m^2)}$$

(5.16)

Now

$$\int \frac{d^4 p_k}{(2\pi)^4} \frac{1}{(p_{ik}^2 + m^2 + A)^2} \frac{1}{p_{ik}^2 + m^2}$$

is independent of i because the integral is convergent and the shifting of variable $p_k \to p_k + p_i$ is permitted. Without regularization the integral is .

$$\int \frac{d^4 p_k}{(2\pi)^4} \frac{1}{p_{ik}^2 + m^2}$$

Here shifting of integration variables is forbidden. Thus, the one loop contribution to l.o. $\langle tr\, \phi(x) \phi(y) \rangle$ is

l.o. $\langle tr\, \phi(x) \phi(y) \rangle$ =

$$= 2 g N^2 (A^2)^2 \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{(p^2+m^2)^2 (p^2+m^2+A)^2} \times$$

$$\times \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2+A)^2} \frac{1}{q^2+m^2} \qquad (5.17)$$

So that the one particle irreducible graph is

$$\Gamma_2 = g A / 8\pi^2 \qquad (5.18)$$

This is the one loop contribution to the mass renormalization of $\Phi(x)$.

The Stochastic Regularization can also be used to regulate the U(N) Gauge Theory which has been considered in section III, but we believe that the Λ cutoff, introduced in (3.7) is sufficient to regulate this theory consistently.

VI. CONCLUSION

In this thesis we have applied the stochastic quantization of Parisi and Wu to the study of the large N Reduction In this context, we want to emphasize the following aspects.

We have been able to derive the quenched momentum prescription in a straightforward elegant and universal way; our derivation can be applied to globally and locally symmetric theories without any modification. Moreover, it has been possible to obtain an idea of how large N should be to validate the approximation, i.e., N ought to be larger than the number of space time points (Equation (4.54)).

We avoid the constraints that several authors have found necessary to introduce in the reduction of the U(N) gauge theory In addition to this, we have attempted to clarify this aspect of the Reduction by observing that the Ward Identities of the original theory are not satisfied by the reduced Green's functions, except when these last functions are originally invariant.

The stochastic regularization technique combines efficiently with the stochastic form of the Reduction. This point may be important for the future non-perturbative work and for computer simulations.

We have derived the Langevin equation for the lattice gauge theory and use it to obtain the quenched Eguchi-Kawai model.

The previous considerations lead us to believe that the stochastic quantization is a powerful method to study the large N limit of Quantum Field theories and deserves further consideration.

Appendix I Fokker-Planck Equation for the Free Particle

Here we discuss in detail the eigenfunctions of Fokker-Planck operator for the case of S=0. To make the problem simpler we assume a system of finite degree of freedom, and we denote the variables by $b^i (i = 1, . . N)$. The corresponding Fokker-Plank Hamiltonian is given by

$$\hat{H}_{FP} = -1/2 \sum_i \frac{\partial^2}{\partial b_i^2} \tag{I.1}$$

The solution of the Fokker-Plank equation with the initial condition $P(\vec{b};0) = \delta(\vec{b}-\vec{b}_0)$ is given by

$$P(\vec{b},t) = \Psi(\vec{b},t) = \int \frac{d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{b}} e^{-k^2 t} e^{-i\vec{k}\cdot\vec{b}_0} \tag{I.2}$$

We calculate the expectation value defined by

$$\langle F(\vec{b})\rangle_t \equiv \int d\vec{b}\; F(\vec{b})\, P(\vec{b},t) \tag{I.3}$$

for three cases: i) F =const.; ii) $F = b^i b^j$; and iii) $F = N b^i b^j - \vec{b}^2 \delta_{ij}$

i) $\langle F\rangle_t = F$ (independent of t and b_0)

(I.4a)

ii) $\langle b^i b^j \rangle_t = 2\,\delta_{ij}\,t + b_0^i b_0^j$ (divergent for large t)

(I.4b)

iii) $\langle N b^i b^j - \vec{b}^2 \delta_{ij} \rangle_t = N b_0^i b_0^j - \delta_{ij} \vec{b}_0^2$

(independent of t)

(I.4c)

In this analysis we first note that the Fokker-Planck Hamiltonian (I.1) has a continuous spectrum above zero, and it is invariant by translations and O(N) rotations. Although the denominator of (2.29) is infinite in this case, we can interpret it as a distribution as the previous calculation indicates. Here the fictitious time t serves a role as the regulator. We expect from this simple analysis that even when $\hat{H}_{FP}$ has a continuous spectrum in general we can give (2.29) a meaning of distribution such that it gives a definite result provided the expectation values are restricted to a certain class (case (i) and (iii) in the previous example).

Appendix II Factorization of The Correlation Functions of Invariant Operators in the Large N Limit.

We consider the generalization of expression (3.6) appropiate to calculate correlations between two U(N) invariant functionals of ϕ . We shall have:

$$\langle tr[\phi(x_1)\ldots\phi(x_n)]\, tr[\phi(z_1)\ldots\phi(z_{n'})]\rangle$$

$$= \lim_{t\to\infty}\sum_{m,k=0}^{\infty}(g/N)^{m+k}\int\ldots\int dy_1\, dt_1\ldots dy_{2m+n}\, dt_{2m+n}\, dy_1'\, dt_1'\ldots dy'_{2k+n'}\ldots$$

$$K_m(x_1\ldots x_n, t; y_1 t_1 \ldots y_{2m+n}\, t_{2m+n})\, K_k(z_1\ldots z_{n'}, t;\, y_1' t_1' \ldots y'_{2k+n'}\, t'_{2k+n'})$$

$$\times\langle tr[\eta(y_1,t_1)\ldots\eta(y_{2m+n},t_{2m+n})]\, tr[\eta(y_1',t_1')\ldots\eta(y'_{2k+n'},t'_{2k+n'})]\rangle_\eta \qquad \text{(II.1)}$$

We see that all N-dependence is contained in the η-average. Furthermore since η is a Gaussian random variable, it is easy to verify that

$$\langle tr[\eta(y_1,t_1)\ldots\eta(y_{2m+n},t_{2m+n})]\, tr[\eta(y_1',t_1')\ldots\eta(y'_{2k+n'},t'_{2k+n'})]\rangle_\eta$$

$$\approx\langle tr[\eta(y_1,t_1)\ldots\eta(y_{2m+n},t_{2m+n})]\rangle_\eta\langle tr[\eta(y_1',t_1')\ldots\eta(y'_{2k+n'},t'_{2k+n'})]\rangle_\eta$$

$$+\ O\left(\frac{1}{N^2}\right) \qquad \text{(II.2)}$$

Replacing II.2 into II.1 we conclude that,

$$\langle \mathrm{tr}[\phi(x_1)\ldots\phi(x_n)]\,\mathrm{tr}[\phi(z_1)\ldots\phi(z_{n'})]\rangle$$

$$\approx \langle \mathrm{tr}[\phi(x_1)\ldots\phi(x_n)]\rangle\langle \mathrm{tr}[\phi(z_1)\ldots\phi(z_{n'})]\rangle +$$

$$+ \; O(1/N^2)$$

(II.3)

This is the factorization property of invariant operators at large N. The stochastic quantization has permitted a quick demonstration of this important proposition.

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