# The Bousfield-Kan Spectral Sequence for Morava K-Theory

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# Abstract


## The Bousfield-Kan Spectral Sequence for Morava K-Theory

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Given a homology theory $E$, we can construct the Bousfield-Kan spectral sequence. Even though one can set this spectral sequence with great generality, the $E_2$-term turns out to be an Ext group in some non-abelian category. In practical terms this description limits our ability to make computations. If we require $E$ to be a Landweber exact homology theory and with some mild assumptions on the space $X$, then we can relate the $E_2$-term to an Ext group in an abelian category, which in turn can be calculated as the homology of some subcomplex of the stable cobar complex. Although Morava's K theories do not satisfy this property, there is a spectral sequence converging to the $E_2$-term of the BK spectral sequence. The input to this spectral sequence can be calculated again as the homology of some unstable cobar complex. In the case of $K(1)$ and for any space $X$ such that $K(1)_*(X)$ is cofree as a coalgebra, this spectral sequence collapses and we get a complete description of the $E_2$-term of the BK spectral sequence for $X$. As observed by N. Kuhn, this turns out to be isomorphic to the stable $E_2$-term. Using this we determine all the differentials and thus prove convergence of the spectral sequence to the unstable $K(1)$-completion of the odd spheres.

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# Contents

# Chapter 1

# Preliminaries

## 1.1 Introduction

This chapter will deal with background material. Most of the material of the category spectra can be found in [Ada74] and in [Swi75].

**Notation 1.1.1.** *The category of pointed topological spaces will be denoted by $\mathcal{T}$ and the CW category by $\mathcal{T}_{CW}$. When we talk about spaces we mean topological spaces with base-point and maps will mean continous base-preserving maps. The set of (base-point preserving) clases of homotopic maps between $X$ and $Y$ will be denoted by $[X, Y]$. The $n^{th}$ complex projective space and the infinite projective plane will be denoted by $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$ respectively.*

## 1.2 Spectra

**Definition 1.2.1.** *A spectrum $E$ is a collection of spaces $\{\underline{E}_n\}_{n\in\mathbb{Z}}$ and maps $\epsilon_n : \Sigma\underline{E}_n \to \underline{E}_{n+1}$. If the adjoint maps $\tilde{\epsilon} : \underline{E}_n \to \Omega\underline{E}_{n+1}$ are weak homotopy equivalences then $E$ is called an $\Omega$-spectrum*

Any spectrum $E$ is homotopy equivalent to an $\Omega$-spectrum $F$ by defining $\underline{F}_n = \varinjlim_k \Omega^k \underline{E}_{n+k}$.

Let $\mathcal{S}$ be the category of spectra. We have adjoint functors $\Sigma^\infty : \mathcal{T} \to \mathcal{S}$ and $\Omega^\infty : \mathcal{S} \to \mathcal{T}$. Let $S$ denote the sphere spectrum $\Sigma^\infty(S^0)$.

**Definition 1.2.2.** *A ring spectrum $E$ is a spectrum with maps $m : E\wedge E \to E$ and $u : S \to E$ called the unit map such that the following diagrams commute*

$$\begin{array}{ccc} E\wedge E\wedge E & \xrightarrow{m\wedge id} & E\wedge E \\ {\scriptstyle id\wedge m}\downarrow & & \downarrow{\scriptstyle m} \\ E\wedge E & \xrightarrow[m]{} & E \end{array}$$

$$\begin{array}{ccccc} S\wedge E & \xrightarrow{u\wedge id} & E\wedge E & \xleftarrow{id\wedge u} & E\wedge S \\ \simeq\downarrow & & {\scriptstyle m}\downarrow & & \simeq\downarrow \\ E & = & E & = & E \end{array}$$

If we have a multiplicative spectrum $E$ then $E^n(X) = [X, \underline{E}_n]$ defines a cohomology theory with products over $\mathcal{T}$. There is also a corresponding (reduced) homology theory defined by $E_n(X) = \varinjlim_k \pi_{n+k}(X \wedge \underline{E}_k)$. The coeficients of the theory $E$ are defined by $E_*(*)$ and denoted by $E_*$. We have $E^k(*) = E_{-k}(*)$.

If we have a multiplicative cohomology theory $E^*(-)$, we can always find a multiplicative $\Omega$-spectrum $\{\underline{E}_n\}_{n\in\mathbb{Z}}$ such that $E^n(X) \cong [X, \underline{E}_n]$ for $X \in \mathcal{T}_{CW}$. See §9 of [Swi75].

**Definition 1.2.3.** *A module spectrum $F$ over $E$ is a spectrum with a map $p : E \wedge F \to F$ making the following diagrams commute*

$$\begin{array}{ccc} E\wedge E\wedge F & \xrightarrow{m\wedge id} & E\wedge F \\ {\scriptstyle id\wedge p}\downarrow & & {\scriptstyle p}\downarrow \\ E\wedge F & \xrightarrow[p]{} & E \end{array}$$

$$\begin{array}{ccc} S\wedge F & \xrightarrow{u\wedge id} & E\wedge F \\ \simeq\downarrow & & {\scriptstyle p}\downarrow \\ F & = & F \end{array}$$

## 1.3 Hopf Rings

We denote by $\mathcal{CO}$ the category of (graded) coalgebras over $A$.

**Definition 1.3.1.** *A Hopf ring $C$ is a ring object in the category of (graded) coalgebras, e.i. a Hopf algebra with multiplication map $* : C \otimes C \to C$ and an extra map $\circ : C_i \otimes C_j \to C_{i+j}$ satisfying the following commutative diagrams*

$$\begin{array}{ccc} C \otimes C \otimes C & \xrightarrow{id \otimes \circ} & C \otimes C \\ {\scriptstyle \circ \otimes id}\downarrow & & \downarrow{\scriptstyle \circ} \\ C \otimes C & \xrightarrow{\circ} & C \end{array}$$

$$\begin{array}{ccc} C \otimes C & \xrightarrow{\circ} & C \\ {\scriptstyle \tau}\downarrow & & {\scriptstyle \chi}\downarrow \\ C \otimes C & \xrightarrow{\circ} & C \end{array}$$

$$\begin{array}{ccc} A \otimes C & \xrightarrow{\cong} & C \\ {\scriptstyle \epsilon \otimes id}\downarrow & & {\scriptstyle id}\downarrow \\ C \otimes C & \xrightarrow{\circ} & C \end{array}$$

$$\begin{array}{ccc} A \otimes C & \longrightarrow & A \\ {\scriptstyle \epsilon \otimes id}\downarrow & & \downarrow \\ C \otimes C & \xrightarrow{\circ} & C \end{array}$$

$$\begin{array}{ccccc} C \otimes C \otimes C & \xrightarrow{\psi \otimes id \otimes id} & C \otimes C \otimes C \otimes C & \xrightarrow{id \otimes \tau \otimes id} & C \otimes C \otimes C \otimes C \\ {\scriptstyle id \otimes *}\downarrow & & & & {\scriptstyle \circ \otimes \circ}\downarrow \\ C \otimes C & \xrightarrow{\circ} & C & \xleftarrow{*} & C \otimes C \end{array}$$

*where $\tau$ is the twisting map and $\chi$ is the inverse map with respect to the $*$ product.*

Suppose that $E_*(-)$ is a commutative homology theory and $G^*(-)$ be a commutative cohomology theory with products. Let $\{\underline{G}_k\}_{k \in \mathbb{Z}}$ be the representing $\Omega$-spectrum. Define $E_*(G_*) = \{E_*(\underline{G}_k)\}_{k \in \mathbb{Z}}$. If $E_*(\underline{G}_k)$ is a free $E_*$-module for all $k$, then we have a Künneth isomorphism making $E_*(G_*)$ a Hopf Ring. The H-space structure of each $\underline{G}_k$ induces the $*$ product and the ring product of $G$ induces the circle product $\circ : E_*(G_m) \otimes E_*(G_n) \to E_*(G_{m+n})$

Let $x \in G^n$. Then $x$ is represented by an element $x : * \to \underline{G}_n$. This defines a map $x_* : E_* \to E_*(\underline{G}_n)$. Define $[x] \in E_0(\underline{G}_n)$ to be $x_*(1)$. With this we can reinterpret the diagrams of definition 1.3.1

**Lemma 1.3.2.** *Let $a \in E_i(G_m), b \in E_j(G_n)$ and $c \in E_k(G_p)$*

1. $a * b = (-1)^{ij} b * a$
2. $[0_m] * a = a$ *where* $[0_m] = (0_m)_*(1)$ *and* $0_m \in G^m(*)$
3. $[0_k] \circ a = 0$ *for* $k > 0$
4. $[1] \circ a = a$
5. $a \circ b = (-1)^{ij}[-1]^{mn} \circ b \circ a$
6. $[n] \circ b = \sum b' * b'' \cdots b^{(n)}$
7. $a \circ (b * c) = \sum (-1)^{|a''||b|}(a' \circ b) * (a'' \circ c)$ *where* $\psi(a) = \sum a' \otimes a''$

## 1.4 Orientable Theories

We want to describe some Hopf rings. But first we need to talk about orientable theories. Recall that $\mathbb{C}P^1 \simeq S^2$ and there is a map $i : S^2 \to \mathbb{C}P^\infty$.

**Definition 1.4.1.** *A cohomology theory $E^*(-)$ is complex orientable if there exist $x \in E^*(\mathbb{C}P^\infty)$ such that $E^*(S^2)$ is a free $E_*$-module generated by $i^*(x)$*

We assume that the generator lives in $E^2(\mathbb{C}P^\infty)$. This can always be achieved and has some advantages for our exposition of Hopf rings.

**Theorem 1.4.2.** *If $E$ is orientable then*

1. $E^*(\mathbb{C}P^\infty) = E^*[[x]]$

2. $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*(\mathbb{C}P^\infty)\widehat{\otimes}E^*(\mathbb{C}P^\infty)$

3. $E_*(\mathbb{C}P^\infty)$ *is a free* $E_*$*-module generated by* $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ *where*

   $< x^i, \beta_j >= \delta_{i,j}$

4. *The coproduct on* $E_*(\mathbb{C}P^\infty)$ *is given by* $\delta(\beta_n) = \sum_{k=0}^{n} \beta_{n-k} \otimes \beta_k$

The simbol $\widehat{\otimes}$ means that infinite sums are allowed. Complex orientable theories include $BP, MU, KU, E(n), K(n)$ and $H$. The theory $KO$ is not complex orientable.

## 1.5 Some Hopf Rings

In this section we describe some Hopf rings that will be used latter on.

If $G$ is orientable we can take the generator $x$ of $E^2(\mathbb{C}P^\infty)$ and represent it as a map $x : \mathbb{C}P^\infty \to \underline{G}_2$. With this we can define elements $b_i \in E_{2i}(G_2)$ by $b_i = x_*(\beta_i)$. Let $e_1 \in E_1(\underline{G}_1)$ be the element such that $e_1 \circ e_1 = b_1$.

### 1.5.1 $BP_*(\underline{\mathbf{BP}}_*)$

By [RW77] the spaces $BP_*(\underline{BP}_k)$ are free $BP_*$-modules so, $BP_*(\underline{BP}_*)$ is a Hopf ring. Recall for for each prime $p$, we have $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ with $|v_i| = 2(p^i - 1)$.

Let $I = (i_1, i_2, \ldots)$ and $J = (j_0, j_1, \ldots)$ be sequences of non-negative integers almost all zero. Let also $v^I = [v_1^{i_1} v_2^{i_2} \cdots], b^J = b_{(0)}^{j_0} \circ b_{(1)}^{j_1} \circ \cdots$ where $b_{(n)} = b_{p^n}$ and $\Delta_k$ be the sequence with 1 in the $k^{th}$ place and zero elsewhere.

**Definition 1.5.1.** *The element* $v^I \circ b^J \circ e_1^s$ *with* $s = 0, 1$ *is said to be allowable if*

$$J = p\Delta_{k_1} + p^2\Delta_{k_2} + \cdots + p^n\Delta_{k_n} + J'$$

*where* $k_1 \leq k_2 \leq \cdots \leq k_n$ *and* $J'$ *is non-negative implies* $i_n = 0$

**Theorem 1.5.2.**

$$BP_*(\underline{BP}_*) \cong BP_*[v^I \circ b^J] \otimes \Lambda(v^I \circ b^J \circ e_1)$$

*where* $v^I \circ b^J \circ e_1^s$ *is allowable*

The proof of this fact is in [RW77].

### 1.5.2 $K(n)_*(\underline{\mathbf{K(n)}}_*)$

For $p$ an odd prime and $n$ a positive integer we have the Morava $K$-theories $K(n)$. $K(1)$ is just one of the $2(p-1)$ summands of complex $K$-theory with mod $p$ coefficients. Since $K(n)_* = \mathbb{Z}_p[v_1, v_1^{-1}]$, we have a Künneth theorem and so $K(n)_*(\underline{\mathrm{K(n)}}_*)$ is a Hopf ring.

**Lemma 1.5.3.** *We have elements* $e_1 \in K(n)_1(\underline{K(n)}_1)$, $a_i \in K(n)_{2i}(\underline{K(n)}_1)$ *and* $b_i \in K(n)_{2i}(\underline{K(n)}_2)$ *such that*

1. $\psi(a_n) = \sum_{i=0}^{n} a_{n-i} \otimes a_i$ *and* $\psi(b_n) = \sum_{i=0}^{n} b_{n-i} \otimes b_i$
2. $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$
3. $b_{(i)}^{*p} = 0$
4. $a_{(i)}^{*p} = 0$ *for* $i < n-1$
5. $a_{(n-1)}^{*p} = v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{\circ(p^n-1)} \circ [v_n]$
6. $v_n e_1 = b_{(0)}^{\circ(p^n-1)} \circ e_1 \circ [v_n]$

*7.* $b_{(n)}^{p^n} \circ [v_n] = v_n^{p^k} b_{(k)}$

*where* $a_{(i)} = a_{p^i}$ *and* $b_{(i)} = b_{p^i}$

With these elements we can describe $K(n)_*(\underline{K(n)}_*)$. Let $I = (i_0, i_1, \ldots, i_{n-1})$ be a collection of $0 \leq i_k \leq 1$, $I(1)$ be the sequence with all ones and $J$ a sequence of non-negative integers with each $j_k < p^n$ and almost all zero. Let also $\rho(I) = min\{k | i_{n-k} = 0\}$ for $I \neq I(1)$. from [Wil84] we have the following.

**Theorem 1.5.4.**

$$K(n)_*(\underline{K(n)}_*) = \bigotimes_{j_0 < p^n - 1} \Lambda(a^I b^J \circ e_1 \circ [v_n^s]) \bigotimes_{j_0 < p^n - 1} P(a^{I(1)} b^J \circ [v_n^s])$$

$$\bigotimes_{I \neq I(1),\ if\ i_0 = 1\ then\ j_0 < p^n - 1} TP_{\rho(I)}(a^I b^J \circ [v_n^s])$$

The proof of this theorem is [Wil84] and in [BKW99]. Indeed, in [Wil84] a description of the Hopf ring $E_*(\underline{K(n)}_*)$ is given whenever $E$ is a module spectrum over $BP$ such that $\mu(I_n) = 0$ where $I_n = (p, v_1, \ldots, v_{n-1}) \subset BP_*$ and $\mu : BP \to E$ is the module spectrum map. These include $K(m), k(n)$ for $m \geq n$, and $H\mathbb{Z}_p$.

# Chapter 2

# The Bousfield-Kan Spectral Sequence

## 2.1 Introduction

In this chapter we give some background material and construct the Bousfield-Kan spectral sequence. We also give a description of the $E_2$-term based on cosimplicial and as the homology of an object related to the stable cobar complex.

**Notation 2.1.1.** *We assume $E$ is a multiplicative $\Omega$-spectrum with unit. We write $\mathcal{HO}$ for the associated homotopy category, $\mathcal{M}$ the category of free $E_*$-modules and $\mathcal{A}$ the category of $E_*$-modules. Finally, we denote $E_*(E)$ by $\Gamma$.*

## 2.2 The Bousfield-Kan Spectral Sequence

We begin by recalling form [Ada74] the construction of the stable Adams spectral sequence based on $E$. We start by constructing the fiber of the map $X \simeq X \wedge S \xrightarrow{1\wedge i} X \wedge E$ where $i$ is the unit map. We call this map $D(X)$. We continue inductively this process taking in the $n^{th}$ stage the fiber of the map $D^n(X) \simeq D^n(X) \wedge S \to D^n(X) \wedge E$ and calling it $D^{n+1}(X)$.

We get a tower under $X$

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ D^2(X) & \longrightarrow & E \wedge D^2(X) \\ \downarrow & & \\ D(X) & \longrightarrow & E \wedge D(X) \\ \downarrow & & \\ X & \longrightarrow & E \wedge X \end{array}$$

We apply $\pi_*(-)$ and define $E_1^{s,t} = \pi_{t-s}(D^s(E(X)))$ for $t - s \geq 0$ and zero otherwise. Observe that $\pi_*(E \wedge D^n(X)) = E_*(D^n(X))$. Under suitable hypothesis on $X$ and $E$, this spectral sequence converges to the $E$-localization of $X$.

We would like to do an analogue construction for the unstable case. The problem is finding an unstable analogue to $E \wedge D^n(X)$.

Let $X$ be a space. We define a functor from $E : \mathcal{HO} \to \mathcal{HO}$ as follows. Let $X$ be a space then

$$E(X) = \Omega^\infty(E \wedge \Sigma^\infty X)$$

This construction, for $E_*(X)$ a free $E_*$-module, basically gives a copy of the $n^{th}$ space in the $\Omega$-spectrum representing $E$ for each generator of $E_n(X)$. For $n \geq 0$ we have

$$\pi_n(E(X)) \cong E_n(X)$$

Where the right is reduced $E$ homology.

The composition $\eta : X \to \Omega^\infty\Sigma^\infty X \to E(X)$ defines a map called the Herewicz map. We can get the fiber of this map and get the fiber sequence $D(X) \to X \xrightarrow{\eta} E(X)$. Iterating this procedure we get a tower under $X$

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ D^2(X) & \longrightarrow & D^2(E(X)) \\ \downarrow & & \\ D(X) & \longrightarrow & D(E(X)) \\ \downarrow & & \\ X & \longrightarrow & E(X) \end{array}$$

define $E_1^{s,t} = \pi_{t-s}(D^s(E(X)))$ for $t - s \geq 0$ and zero otherwise. We call this spectral sequence the Bousfield-Kan spectral sequence. Next, we give a description of the $E_2$-term.

## 2.3 A Cosimplicial Description of the $E_2$-term

The following definitions will allow us to describe the $E_2$-term as the cohomotopy of some cosimplicial group.

**Definition 2.3.1.** *A cosimplicial object* $\mathbf{X}$ *over a category* $\mathcal{C}$ *is a collection of objects* $X_i \in \mathcal{C}, n \geq 0$ *such that for each* $0 \leq n$ *there are maps* $d^i : X_n \to X_{n+1}$ *and* $s^i : X_{n+1} \to X_n$ *with* $0 \leq i \leq n$ *satisfying the following identities*

$$\begin{aligned} d^j d^i &= d^i d^{j-1} & i < j \\ s^j d^i &= d^i s^{j-1} & i < j \\ &= id & i = j, j+1 \\ &= d^{i-1} s^j & i > j+1 \\ s^j s^i &= s^{j-1} s^j & i > j \end{aligned}$$

In our case the category $\mathcal{C}$ will be either the category $\mathcal{A}$, $\mathcal{T}$ or $\mathcal{HO}$.

Given a cosimplicial object $\mathbf{X}$ over $\mathcal{HO}$ we can apply $\pi_*(-)$ to get a cosimplicial object over the category of abelian groups. This defines a cochain complex $ch(\pi_*(\mathbf{X}))$ with $\delta^n =$

$\sum_{i=0}^{n}(-1)^i d^i$. The homology of this cosimplicial group is called the cohomotopy of $\pi_*(\mathbf{X})$ and denoted by $\pi^*\pi_*\mathbf{X}$.

Our objective is to turn the functor $E : \mathcal{HO} \to \mathcal{HO}$ into a cosimplicial object over $\mathcal{HO}$. We use the following definition.

**Definition 2.3.2.** *A triple $(G, \mu, \eta)$ over the category $\mathcal{C}$ is a functor $G : \mathcal{C} \to \mathcal{C}$ and natural transformations $\mu : G^2 \to G$ and $\eta : 1 \to G$ such that we have the following commutative diagrams:*

$$\begin{array}{ccccc} G & \xrightarrow{G\eta} & G^2 & \xleftarrow{\eta G} & G \\ \| & & \downarrow{\scriptstyle \mu} & & \| \\ G & = & G & = & G \end{array}$$

$$\begin{array}{ccc} G^3 & \xrightarrow{G\mu} & G^2 \\ {\scriptstyle \mu G}\downarrow & & \downarrow{\scriptstyle G\mu} \\ G^2 & \xrightarrow{\mu} & G \end{array}$$

Using the triple $(G, \mu, \eta)$ we can construct a functor $\mathbf{G}$ from $\mathcal{C}$ to the category of cosimplicial objects over $\mathcal{C}$ as follows: let $X \in \mathcal{C}$ and define $\mathbf{G}(X)_n = G^{n+1}(X)$ and the maps $d^i = G^i\eta G^{n-i} : G^n(X) \to G^{n+1}(X)$ and $s^i = G^i\mu G^{n-i} : G^{n+2}(X) \to G^{n+1}(X)$ with $0 \leq i \leq n$.

The natural transformation $\mu : E^2 \to E$, induced by the multiplicative structure of $E$, together with the Heurewitz map $\eta$ makes the functor $(E, \mu, \eta)$ a triple in the category $\mathcal{HO}$. This in turn gives us a functor $\mathbf{E}$ from $\mathcal{HO}$ into cosimplicial objects over $\mathcal{HO}$.

**Theorem 2.3.3.** *Let $X \in \mathcal{T}$. Then*

$$E_2^{s,t}(X) \cong \pi^s\pi_t\mathbf{E}(X)$$

The proof of this theorem can be found in [BK72].

## 2.4 The category $\mathcal{M}(G)$ and and an alternate description of the $E_2$-term

There is an alternate description of the $E_2$-term using the dual concept of triple. The advantage of this description is that it will enable us to describe the $E_2$-term as the homology of a subcomplex of the stable complex.

**Definition 2.4.1.** *A cotriple $(G, \delta, \epsilon)$ in a category $\mathcal{C}$ is a functor $G : \mathcal{C} \to \mathcal{C}$ and natural transformations $\delta : G \to G^2$ and $\epsilon : G \to 1$ such that the following diagrams commute*

$$\begin{array}{ccccc} G & \xleftarrow{G\epsilon} & G^2 & \xrightarrow{\epsilon_G} & G \\ \| & & \uparrow{\scriptstyle\delta} & & \| \\ G & = & G & = & G \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{G\delta} & G^2 \\ {\scriptstyle\delta_G}\downarrow & & \downarrow{\scriptstyle\delta} \\ G^2 & \xrightarrow{\delta} & G^3 \end{array}$$

*Given a cotriple $G$, a $G$-coalgebra is an object $C \in \mathcal{C}$ and a map $\psi : C \to G(C)$ such that the following diagrams commute:*

$$\begin{array}{ccc} C & \xrightarrow{\psi} & G(C) \\ \| & & \downarrow{\scriptstyle\epsilon} \\ C & = & C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\psi} & G(C) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle G\psi} \\ G(C) & \xrightarrow{\delta} & G^2(C) \end{array}$$

*A map $f : C \to D$ is a $G$-coalgebra map if the following diagram commutes*

$$\begin{array}{ccc} C & \xrightarrow{\psi} & G(C) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(f)} \\ D & \xrightarrow{\psi} & G(D) \end{array}$$

We denote the category of $G$-coalgebras over $\mathcal{C}$ as $\mathcal{C}(G)$. If $C \in \mathcal{C}$, then $G(C)$ is a $G$-coalgebra with $\psi = \delta$. Given a $G$-coalgebra $(Z, \psi)$, we can define a triple by setting

$$\mu = G(\epsilon) : G^2(Z) \to G(Z)$$

$$\eta = \psi : Z \to G(Z)$$

We define a cotriple $(G, \delta, \epsilon)$ over $\mathcal{M}$. But first we impose the following restrictions on $E$

**Hypothesis 2.4.2.** *Let $E$ is an $\Omega$-spectrum represented by $\{\underline{E}_k\}$, We assume*

1. *$E$ is a multiplicative, associative, homotopy commutative, CW spectrum with unit*
2. *$E_*(\underline{E}_k)$ is a free $E_*$-module for all $k$*

Let $M \in \mathcal{M}$ and let $F$ be the spectrum such that $\pi_*(F) = M$. Define $G(M) = E_*(\Omega^\infty F)$. By [BCM78] we know that this defines a triple over $\mathcal{M}$. With this we have the category of $G$-coalgebras over $\mathcal{M}$, or $\mathcal{M}(G)$. For $M \in \mathcal{M}(G)$ there is a resolution

$$G(M) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} G^2(M) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} \cdots$$

We call this the $G$-resolution of $M$. Aplying $Hom_{\mathcal{M}(G)}(E_*(S^t), M)$ and taking the homology of this complex gives $Ext^s_{\mathcal{M}(G)}(E_*(S^t), M)$.We write $Ext^{s,t}_{\mathcal{M}(G)}(M)$, for $Ext^s_{\mathcal{M}(G)}(E_*(S^t), M)$.

**Theorem 2.4.3.** *Let $E$ satisfy hypothesis 2.4.2. Let also $X$ be a simply connected space such that $E_*(X) \in \mathcal{M}$. Then*

$$E_2^{s,t} = Ext^{s,t}_{\mathcal{M}(G)}(E_*(X)) \qquad t > s \geq 0$$

This is proven in [BT00]. They impose an additional condition on the spectrum $E$(the primitives of $E_*(\underline{E}_k)$ inject into $E_*(E)$). But this condition, by [Kar98], is not really necessary for the previous theorem. This condition will be used in the next section to prove still another characterization of the $E_2$-term.

In practice, the previous characterizations of the $E_2$-term are of little use. The problem is that it does not provide an explicit way to produce elements. The next description will give us an explicit way of calculating. But first we need to know about the derived functors of the primitives.

## 2.5 Derived Functors of the Primitives

Let $A$ be a commutative ring with unit, $\mathcal{A}$ be the category of graded modules over $A$ and $\mathcal{CO}$ the category of colagebras over $A$. There is a functor $S : \mathcal{A} \to \mathcal{CO}$ that assigns, for any $M \in \mathcal{A}$, the cofree, cocommutative, coassociative coalgebra generated by $M$. This defines a functor of a cotriple $(S, \delta, \epsilon)$ in $\mathcal{A}$.

For each $C \in \mathcal{CO}$, we can define $P(C)$ as the set of primitive elements of $C$. This defines a functor to abelian groups.

**Definition 2.5.1.** *Let $C \in \mathcal{CO}$. Then*

$$R^i P(C) = H^i(chP(\tilde{\mathbf{S}}(C)))$$

Here are some of the basic properties of this functor

**Theorem 2.5.2.** *Let $C, D \in \mathcal{CO}$*

1. $R^0 P(C) = P(C)$
2. $R^i P(C \otimes D) = R^i P(C) \oplus R^i P(D)$
3. *If $C$ is cofree then $R^i P(C) = 0$ for $i > 0$*

Property 3 will be especially important for the next section.

## 2.6 The Category $\mathcal{A}(U)$ and the Composite Functor Spectral Sequence

Before going on, one more condition will be imposed on $E$

**Hypothesis 2.6.1.** *Let $PE_*(\underline{E}_k)$ be the module of primitives of $E_*(\underline{E}_k)$ and let $\sigma$ be the stabilization map. Then the following composition is an injection*

$$PE_*(\underline{E}_k) \to E_*(\underline{E}_k) \xrightarrow{\sigma} E_*(E)$$

Let $U(M) = PG(M)$. hypothesis 2.6.1 ensures that $U$ is the functor of a cotriple on $\mathcal{M}$ and is a subcotriple of $(G, \delta, \epsilon)$. We can extend the functor such that $U$ is the functor of a cotriple over the category of(not necessarily free) modules $\mathcal{A}$. Let $M \in \mathcal{A}$ and let $F_0, F_1 \in \mathcal{M}$ be such that

$$F_0 \xrightarrow{a} F_1 \to M \to 0$$

is exact. We define $U(M) = Coker(a)$.

**Theorem 2.6.2.** *Suppose $E_*(\underline{E}_n)$ is cofree for all $n$. There is a spectral sequence*

$$E_2^{i,j,t} = Ext^i_{\mathcal{A}(U)}(E_*(S^t), R^jP(E_*(X))) \Longrightarrow Ext^{i+j}_{\mathcal{M}(G)}(E^i_*(S^t), E_*(X))$$

*If $E_*(X)$ is also cofree, then the spectral sequence colapses and we have*

$$Ext^i_{\mathcal{A}(U)}(E_*(S^t), PE_*(X)) \cong Ext^{i+j}_{\mathcal{M}(G)}(E^i_*(S^t), E_*(X))$$

The poof of this theorem can be found on [BCM78]. This theorem was used to study the BK spectral sequence based on $BP$(see [Wil84]). Unfortunately, for many other homology theories, this hypothesis is not satisfied. There is an alternative proof of this theorem for a class of theories related to $BP$.

## 2.7 The $E_2$-Term for Landweber Exact Homology Theories

Recall that for a prime $p$, $BP$ is a homology theory with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. If $F$ is a spectrum, let $F_* = \pi_*(F)$.

**Definition 2.7.1.** *Let $F$ be a $BP$-module spectrum. Then we say $F_*(-)$ is an Landweber exact homology theory if multiplication by $v_n \in BP_*$ acts injectively on $F_*/F_*(v_0, v_1, \ldots, v_{n-1})$*

**Theorem 2.7.2.** *Suppose that $F_*(-)$ is a Landweber exact homology theory. Then for any $X$*

$$E_*(X) \cong E_* \otimes_{BP_*} BP_*(X)$$

For this class of theories we have the following theorem.

**Theorem 2.7.3.** *Let $E$ be a Landweber exact ring spectrum satisfying hypothesis 2.4.2 and 2.6.1, let $X$ be an H-space such that $E_*(X)$ is generated by elements of odd degree and $E_*(X) \cong \Lambda(M)$ as coalgebras where $M \in \mathcal{M}$. Then*

$$Ext^s_{\mathcal{A}(U)}(E_*(S^t), M) \cong Ext^s_{\mathcal{M}(G)}(E_*(S^t), \Lambda(M))$$

*Where $\Lambda(M)$ be the exterior algebra with $P(\Lambda(M)) = M$.*

## 2.8 The Calculation of $Ext^{s,t}_{\mathcal{A}(U)}(M)$

The advantage of working in the category $\mathcal{A}(U)$ is that we can calculate the $E_2$-term explicitly from some subcomplex of the stable cobar complex.

Theorem 2.7.2 implies that any for any Landweber exact homology theory, there is an isomorphism

$$\Gamma = E_*(E) \cong E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_*$$

where $BP_*(BP) = \mathbb{Z}_{(p)}[t_1, t_2, \ldots]$ and $p$ is a fixed prime. By [BCM78] we can substitute the $t_i$ by $h_i = c(t_i)$, where $c$ is the canonical anti-isomorphism. Let $J = (j_1, j_2, \ldots)$ where most of the $0 \leq j_i$ are zero. Define $h^J = h_1^{j_1} h_2^{j_2} \ldots \in \Gamma$.

**Theorem 2.8.1.** *Let $E$ satisfy hypothesis 2.4.2 and 2.6.1 and $M \in \mathcal{M}$ then*

1. *Let $l(J) = \sum j_i$. Then*

$$U(M) = Span_{E_*}\{h^J \otimes m \in \Gamma \otimes M \mid 2l(J) < |m|\}$$

2. *Suppose that $M$ is an unstable $\Gamma$-comodule, with coaction $\Psi : M \to U(M)$. Then the differential is given by*

$$\begin{aligned} d([\gamma_1|\ldots|\gamma_n]m) =& [1|\gamma_1|\ldots|\gamma_n]m \\ &+ \sum_{j=1}^{n} (-1)^j [\gamma_1|\ldots|\gamma_j'|\gamma_j''|\ldots|\gamma_n] \\ &+ (-1)^{n+1} \sum [\gamma_1|\ldots|\gamma'_n]m' \end{aligned}$$

*where $\Psi(\gamma_j) = \sum \gamma' \otimes \gamma''$ and $\Psi(m) = \sum \gamma' \otimes m'$.*

The proof of this theorem is given in [BCM78].

# Chapter 3

# Convergence Issues

## 3.1 Introduction

Up to this point we have not discussed convergence of the spectral sequence. We begin with the definition of $E$-localization and some of its properties.

## 3.2 The Unstable $E$-Localization

**Definition 3.2.1.** *Let $X$ be a space. The (unstable) $E$-localization of $X$ is a space $X_E$ and a map $\xi_E : X \longrightarrow X_E$ such that $E_*(X) \xrightarrow{\xi_*} E_*(X_E)$ is an isomorphism and for any space $Y$ and any map $f : X \longrightarrow Y$ that induces an isomorphism on $E$-homology, there is a unique map $g : X_E \longrightarrow Y$ such that $g \circ \xi_E = f$. We say that a space $X$ is $E$-local if there exist $Y \in \mathcal{T}$ such that $X \simeq Y_E$.*

In [Bou75] it is proven the existence of (unstable) $E$-localizations. It is also proven that if $E$ is a connected spectrum and $X$ is simply connected then the $E$-localization turns out to be the same as the $HG$-localization where $HG$ is just ordinary homology with coefficients in $G = \mathbb{Z}[J^{-1}], \oplus_{p\in J}\mathbb{Z}_p$ and $J$ is just a set of primes. If $E$ fails to be connective then the localization is a stranger object. For example, in [BT00] is proven that $\pi_{2n-2}(S^{2n+1}_{E(1)}) = \mathbb{Q}/\mathbb{Z}_{(p)}$.

We would like the $E$-localization to be the target of our spectral sequence.

## 3.3 The Completion

The cosimplicial standard simplex, denoted by $\mathbf{\Delta}$, consist, in codimension $n$, of the standard $n$-simplex $\Delta[n]$ with the usual coface and codegeneracy maps.

Recall that if $\mathbf{X}$ and $\mathbf{Y}$ are cosimplicial spaces, then the function space $Hom(\mathbf{X}, \mathbf{Y})$ is defined to be the space in which $Hom(\mathbf{X}, \mathbf{Y})_n$ consist of maps $\Delta[n] \times \mathbf{X} \to \mathbf{Y}$ and with face and degeneracy maps defined by the compositions

$$\Delta[n-1] \times \mathbf{X} \overset{d^i \times id}{\rightarrow} \Delta[n] \times \mathbf{X} \rightarrow \mathbf{Y}$$

$$\Delta[n+1] \times \mathbf{X} \overset{s^i \times id}{\rightarrow} \Delta[n] \times \mathbf{X} \rightarrow \mathbf{Y}$$

**Definition 3.3.1.** *Lex $\mathbf{X}$ be a cosimplicial space. The total space $Tot_\infty(\mathbf{X})$ of $\mathbf{X}$ is defined to be the function space*

$$Hom(\mathbf{\Delta}, \mathbf{X})$$

If we let $\Delta^{[s]}$ denote the $s$-skeleton of $\mathbf{\Delta}$, *i.e.* the subsimplicial space generated by simplices of dimension less or equal to $n$, and define $Tot_s(\mathbf{X})$ to be $Hom(\Delta^{[s]}, \mathbf{X})$, then we can express the completion as

$$\varprojlim Tot_s(\mathbf{X})$$

**Definition 3.3.2.** *Let $(E, \mu, \eta)$ be a triple over $\mathcal{T}$. Then the (unstable) $E$-completion of a space $X$ is defined by*

$$Tot_\infty \mathbf{E}(X)$$

*And is denoted by $E^\wedge X$.*

The $E$-completion of $X$ can also be seen as the inverse limit of a tower of fibrations consisting of $Tot_s(\mathbf{E}(X))$

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ F_2 & \longrightarrow & X_2 \\ & & \downarrow \\ F_1 & \longrightarrow & X_1 \\ & & \downarrow \\ F_0 & = & X_0 \end{array}$$

If we let $E_1^{s,t} = \pi_{t-s} F_s$, this also defines a spectral sequence. Unfortunately, our spectral sequence comes from a tower over $X$ and not from a tower under $X$. The most we can say is that, given a tower over $X$, one can get a tower under $\Omega X$ by the taking the fibers of the maps $X_s \to X_{s-1}$.

## 3.4 $\mathbb{S}$-Algebras and Convergence

**Definition 3.4.1.** *We say $E$ is an $\mathbb{S}$-algebra if $E$ is a strictly commutative, associative, ring spectrum with unit.*

Spectra like $MU$, $BP$, $K$, and $E(1)$ are known to be $\mathbb{S}$-algebras. It is also known that $K(n)$ is not an $\mathbb{S}$-algebra for any $n$.

**Lemma 3.4.2.** *Suppose $E$ is an $\mathbb{S}$-algebra. Then $E(X)$ is the functor of a triple over $\mathcal{T}$ for any $X \in \mathcal{T}$*

This enable us to define the $E$-completion and moreover, we have the following theorem

**Theorem 3.4.3.** *If $E$ is an $\mathbb{S}$-algebra and $X$ a simply connected space, then the spectral sequence induced by the tower over $X$ is equivalent to the spectral sequence induced by the tower under $E^\wedge X$*

The proof of this fact is in [BT00].

We would like to apply this results to spectra like $K(n)$. The problem is that these are not $\mathbb{S}$-algebras. Fortunately, this can be fixed by modifying the definitions.

## 3.5 The General Case

**Definition 3.5.1.** *A functor $T : \mathcal{C} \to \mathcal{C}$ is called an augmented functor if there is a natural transformation $\chi : 1 \to T$. It is denoted by $(T, \chi)$.*

For any spectrum with unit $E$ we can define an augmented functor $\mathbf{T}_E : \mathcal{HO} \to \mathcal{HO}$ where $T_E(X) = E(X)$ and $\chi = \eta$.

If we have an augmented functor $T$ and a space $X$, we can define a restricted cosimplicial space $\widehat{\mathbf{T}}(X)$ by having $\widehat{\mathbf{T}}(X)_k = T^{k+1}(X)$ and with maps $d^i : \widehat{\mathbf{T}}(X)_{k-1} \to \widehat{\mathbf{T}}(X)_k$ defined by $d^i = T^i \chi T^{k-i}$ for $0 \le i \le k-1$. This is just a cosimplicial space where we forget about the codegeneracies.

**Definition 3.5.2.** *Let $E$ be a ring spectrum with unit (not necessarily a $\mathbb{S}$-Algebra). Then the $E$-completion of $X$, denoted by $E^\wedge X$, is defined as*

$$\underleftarrow{holim} \widehat{\mathbf{T}}_E(X)$$

If $E$ is an $\mathbb{S}$-algebra then this definition agrees with the definition of $E$-completion given in definition 3.3.2.

**Definition 3.5.3.** *A modified cosimplicial space* $\mathbf{X}$ *is a restricted cosimplicial space with codegeneracies satisfying the following identities*

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && i < j \\ s^j d^i &\simeq d^i s^{j-1} && i < j \\ &\simeq id && i = j, j+1 \\ &\simeq d^{i-1} s^j && i > j+1 \\ s^j s^i &\simeq s^{j-1} s^j && i > j \end{aligned}$$

For any ring spectrum with unit $E$, the cosimplicial space over $\mathcal{HO}$, $\mathbf{E}(X)$, can be considered a modified cosimplicial space over $\mathcal{T}$ which we still denote by $\mathbf{E}(X)$.

**Definition 3.5.4.** *For $X \in \mathcal{T}$ and $E$ a ring spectrum with unit, define the Bousfield-Kan spectral sequence of $X$ as the Bousfield-Kan spectral sequence for $\widehat{\mathbf{T}}_E(X)$*

**Theorem 3.5.5.** *Let $E_2^{s,t}$ be the $E_2$-term of the spectral sequence defined in definition 3.5.4. Then $E_2^{s,t} = \pi^s \pi_t \mathbf{E}(X)$.*

See [BH] for the proof.

## 3.6 Proving Convergence

None of these facts guarantee convergence of the spectral sequence. It only says that if the spectral sequence converges, then it converges to the completion. One of the only tools that we have for dealing with convergence comes from chapter IX of [BK70].

**Theorem 3.6.1.** *Suppose we have a tower as in section 3.3 and, for $i \geq 1$, $X = holim_{\leftarrow} X_n$. Then if*

$$\lim_{\substack{\leftarrow \\ s}}{}^1 E_r^{s,s+i} = 0 = \lim_{\substack{\leftarrow \\ s}}{}^1 E_r^{s,s+i+1}$$

*then the spectral sequence converges completely to $\pi_*(X)$*

See chapter IX §5 of [BK70] for the definition of completely convergent. This is automatically satisfied if the spectral sequence has a horizontal vanishing line.

Even supposing that theorem 3.6.1 applies, we do not know that the completion is the $E$-localization of $X$. The most one can say is that there is a map $j : X_E \to E^\wedge X$. In most of the cases one checks that one has a vanishig line, compute the completion and then compate the result with the localization. For example, in [BT00], Bendersky and Thompson compute the $E_2$-term of the BK spectral sequence based on $E(1)$ for the odd spheres. By dimensional arguments, all differentials are trivial. They conclude that there is a homotopy equivalence between the conective cover of the $E(1)$-completion and the $E(1)$-localization.

In the case in which $E$ is a connective spectrum with unit, $X$ is simply connected and there is a Thom map, *i.e.* a unit preserving map $\iota : E \to H\mathbb{Z}$ where $H\mathbb{Z}$ is the integral Eilenberg-Mac Lane spectrum, then we have the following theorem proved in [BCM78]

**Theorem 3.6.2.** *Let $E$ be a connective spectrum with unit and with a Thom map $\tau : E \to H\mathbb{Z}$ where $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum and suppose $X$ is a simply connected space. Then the Bousfield-Kan spectral sequence converges completely to $X_E$.*

A similar argument but using the concept of $p$-local Thom map is used to prove that the spectral sequence converges for $BP$.

# Chapter 4

# The Bousfield-Kan Spectral Sequence for Morava $K$-Theory

## 4.1 Introduction

The results of the previous chapters enable us to define the Bousfield-Kan spectral sequence for $K(n)$. But since Morava's $K$-theories are not Landweber exact, theorem 2.7.3 is not valid. Still, the description of $K(n)_*(K(n))$ is close enough to that of $E_*(E)$ when $E$ a Landweber exact theory to conjecture that an alternate description of the $E_2$-term can be found(See [Yag78]). We begin by studing the functor $S$ defined in 2.5.

**Notation 4.1.1.** *We assume $p$ is an odd prime and $\{\underline{E}_n\}_{n\in\mathbb{N}}$ is a multiplicative $\Omega$-spectrum. Let also $X$ be a space such that $E_*(X)$ is a free $E_*$-module. Let $G$ be the cotriple associated to $E$ and $U = PG$. We will write $G(X)$ for $G(E_*(X))$ and $U(X)$ for $U(E_*(X))$. Also, $\mathbb{Z}_p^\wedge$ will denote the $p$-adic integers and $A(n_1, n_2, \ldots, n_k)$ will denote free $A$-module generated by elements in dimensions $n_1, n_2, \ldots, n_k$.*

## 4.2 The Q-Primitives Functor

Let $\mathcal{M}$ be the category of free $A$-modules and let $\mathcal{HP}$ be the subcategory of $\mathcal{M}$ consisting of Hopf algebras. Recall the functor $S$ from [Bou70]. We would like to give a Hopf algebra structure to $S(H)$ whenever $H \in \mathcal{HP}$ in such a way as to make all maps be in $\mathcal{HP}$.

We have a functor $S : \mathcal{HP} \to \mathcal{HP}$ defined as follows: let $H \in \mathcal{HP}$ and consider it as an element in $\mathcal{M}$. Let D(H) be the (continuos) dual of $H$. Let $F$ be the free algebra generated by $D(H)$. Then there is an $A$-module map $i : D(H) \to F$. Let $q : F \to Q(F)$ be the natural map to the indecomposables. We can choose a section $j : Q(F) \to D(H)$ such the $q \circ i = j$. This induces an isomorphism of modules. Define the coalgebra structure on $x \in Q(F)$ by $\psi(x) = i \otimes i(\psi(i^{-1}(x)))$. For $x$ decomposable, let $x = x_1 \ldots x_n \in F$ with $x_i \in Q(F), 0 \leq i \leq n$. Then $\psi(x) = \psi(x_1) \ldots \psi(x_n)$.

Since D(H) is an algebra we have a unique algebra map $j : F \to D(H)$ such that $j \circ i = id$.

**Claim 4.2.1.** *$j$ is a Hopf algebra map*

*Proof.* Since $j$ is an algebra map we only have to prove is that is a coalgebra map. By definition we have the following commutative diagram

$$\begin{array}{ccc} Q(F) & \xrightarrow{j} & H \\ \psi \downarrow & & \psi \downarrow \\ Q(F) \otimes Q(F) & \xrightarrow{j \otimes j} & H \otimes H \end{array}$$

where the horizontal maps are isomorphisms with inverse $i, i \otimes i$ respectively. Let $x = x_1 \ldots x_n \in F$ with $x_i \in Q(F), 0 \leq i \leq n$. Then

$$j \otimes j(\psi(x)) = j \otimes j(\psi(x_1 \ldots x_n)) = j \otimes j(\psi(x_1) \ldots \psi(x_n))) =$$

$$j \otimes j(\psi(x_1)) \ldots j \otimes j(\psi(x_n)) = \psi(j(x_1)) \ldots \psi(j(x_n)) = \psi(j(x_1 \ldots x_n)) = \psi(j(x))$$

□

Then we can take $S(H)$ to be the (continous) dual of $F$. This defines a cotriple $(S, \delta, \epsilon)$ where $\delta = S(j^*)$ and $\epsilon = i^*$.

**Remark 4.2.2.** *By the previous argument we can deduce that $\delta$ is a Hopf algebra map and $\epsilon$ is a algebra map.*

If we consider this resolution as a resolution in the category of coalgebras then this is the acyclic resolution of $H$ (as a coalgebra) by models of [Bou70]. The advantage is that we can work with indecomposable primitives. We say a Hopf algebra is cofree if its cofree as a coalgebra.

Let $K \in \mathcal{HP}$. We have a functor from Hopf algebras over $A$ to $A$-modules

$$P^Q(K) = Im[P(K) \to Q(K)]$$

**Remark 4.2.3.** *Let $H$ is a Hopf algebra over a ring with characteristic zero. If $H$ is free as an algebra then $P(H) = P^Q(H)$*

**Definition 4.2.4.** *Let $K$ be a Hopf algebra. Then the derived functors of $P^Q$ are defined*

$$R^q P^Q(K) = H^q(chP^Q(\tilde{\mathbf{S}}(K)), \partial)$$

*Where $\tilde{\mathbf{S}}(K)$ is the unaugmented resolution induced by the cotriple $(S, \delta, \epsilon)$.*

The natural map $q : P(K) \to P^Q(K)$ induces a natural transformation $q_* : R^i P(K) \to R^i P^Q(K)$.

**Lemma 4.2.5.** *The map $q_*$ onto for all $i$*

*Proof.* It is enough to prove that if $x \in S^n(H)$ is indecomposable then $d^i(x)$ is also indecomposable. Suppose that x is indecomposable and $d^i(x) = ab$. Then by remark 4.2.2 and by the properties of cotriples we know that $x = s^i d^i(x) = s^i(ab) = s^i(a)s^i(b)$.

□

**Corollary 4.2.6.** *Let $K$ be Hopf algebra such that $R^iP(K) = 0$ for $i > n$. Then $R^iP^Q(K) = 0$ for $i > n$.*

*Proof.* This follows from the previous lemma. □

We will say a Hopf algebra K is $Q$-nice if $R^iP^Q(K) = 0$ for $i > 1$. By the previous lemma if $K$ is nice (in the sense of [Bou70]), then is $Q$-nice.

Thanks to the next lemma, we will see that $R^iP^Q(-)$ inherits the nice properties of $R^iP(-)$.

**Lemma 4.2.7.** *Let $\{H_i\}_{i\in\mathbb{N}}$ be a collection of objects from $\mathcal{HP}$. Then*

1. $R^0P^Q(H) = P^Q(H)$
2. $\bigoplus_{j\in\mathbb{N}} R^iP^Q(H_j) \to R^iP^Q(\bigotimes_{j\in\mathbb{N}} H_j)$ *is surjective*

*Proof.* We have the following commutative diagram

$$\begin{array}{ccc} P(H) & \xrightarrow{\cong} & R^0P(H) \\ \downarrow & & \downarrow{\scriptstyle q^*} \\ P^Q(H) & \xrightarrow{\epsilon} & R^0P^Q(H) \end{array}$$

where the columns are surjective. This yields the surjectivity of $\epsilon$. Suppose $x \in P^Q(H)$. Then, by the same argument of lemma 4.2.2 we can see that $\epsilon(x)$ is also indecomposable and thus non-zero.

To prove part 2 we begin with the case of two Hopf algebras. Consider, as in proposition 3.3 of [Bou70], the cosimplicial resolutions $\mathbf{X}$ and $\mathbf{Y}$ for $H_1$ and $H_2$ respectively. Then by the Eilenberg-Zilber theorem, $\mathbf{X} \otimes \mathbf{Y}$ is a cosimplicial resolution for $H_1 \otimes H_2$ and since $P^Q(\tilde{\mathbf{X}} \otimes \tilde{\mathbf{Y}}) \cong P^Q(\tilde{\mathbf{X}}) \oplus P^Q(\tilde{\mathbf{Y}})$ the result follows. For the general case let $\{K_i\}_{i\in\mathbb{N}}$ be a directed set in $\mathcal{HP}$ with $K_n = \bigotimes_{j=1}^n H_j$ and maps

$$e_n : K_n = \bigotimes_{j=1}^{n} H_j \overset{id\otimes\cdots\otimes id\otimes u}{\longrightarrow} K_{n+1} = \bigotimes_{j=1}^{n+1} H_j$$

where $u : A \to H_{n+1}$ is the unit map.

since $\varinjlim K_i = \bigotimes_{j\in\mathbb{N}} H_j$, $R^i P^Q(A) = 0$. From the commutative diagram

$$\begin{array}{ccc}
\varinjlim_n R^i P(H_n) & \xrightarrow{\cong} & R^i P(\varinjlim_n H_n) \\
\downarrow & & \downarrow \\
\varinjlim_n R^i P^Q(H_n) & \longrightarrow & R^i P^Q(\varinjlim_n H_n) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

we deduce that there is an onto map $\varinjlim_n R^i P^Q(H_n) \to R^i P^Q(\varinjlim_n H_n)$ and so we have

$$\bigoplus_{j\in\mathbb{N}} R^m P^Q(H_j) = \varinjlim_n \bigoplus_{j=1}^{n} R^m P^Q(H_j) =$$

$$\varinjlim_n R^m P^Q(\bigotimes_{j=1}^{n} H_j) \overset{onto}{\to} R^m P^Q(\varinjlim_n K_n) = R^m P^Q(\bigotimes_{j\in\mathbb{N}} H_j)$$

□

**Definition 4.2.8.** *Let $E$ be a multiplicative homology theory and $\{\underline{E}_n\}_{n\in\mathbb{Z}}$ its corresponding spectrum. Let also $E_*(\underline{E}_n)$ be a Hopf algebra for all $n$. We say $E$ is $Q$-nice if $E_*(\underline{E}_n)$ is $Q$-nice for all $n$.*

We would like to apply the $P^Q$ derived functors to the $G$-resolution of $X$. But, $d^0$ is not a Hopf algebra map. Fortunately, since for any $X$, we have $G(X) \cong \bigotimes_{n\in\Sigma} E_*(\underline{E}_n)$ it is enough to require:

**Hypothesis 4.2.9.** *The map $d^0 : G^n(X) \to G^{n+1}(X)$ takes decomposables to decomposables for any $n \geq 0$*

From now on we assume that the homology theory $E$ satisfies hypothesis 4.2.9.

**Definition 4.2.10.** *Let $R^s_q P^Q_E(X)$ be the homology of the following cochain complex*

$$R^q P^Q(G(X)) \underset{\rightarrow}{\overset{\rightarrow}{}} R^q P^Q(G^2(X)) \begin{matrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{matrix} \cdots$$

*(The derived functors of the Q-primitives applied to the G-resolution of M)*

## 4.3 The (generalized) Composite Functor Spectral Sequence

We begin by imposing the following hypothesis on the homology theroy $E$.

**Hypothesis 4.3.1.** *The composition $P^Q(E_*(\underline{E}_n)) \rightarrow E_*(\underline{E}_n) \xrightarrow{\sigma} E_*(E)$ is injective*

As with the functor $U = PG$, hypothesis 4.3.1 implies that the endofunctor $U_Q = P^Q G$ is the functor of a cotriple in $\mathcal{M}$. This in turn allows us to construct the category of $U_Q$-coalgebras.

**Theorem 4.3.2.** *There is a spectral sequence*

$$\overline{E}_2^{m,n,t} = Ext^{m,t}_{U_Q}(R^n_0 P^Q_E(X)) \Rightarrow E_2^{n+m,t}(X)$$

*(converging to the $E_2$-term of the Bousfield-Kan SS based on E-theory) with differential $d_r : \overline{E}_r^{m,n,t} \longrightarrow \overline{E}_r^{m+r,n-r+1,t}$*

*Proof.* Fix a $t \geq 0$. We form,for $n, m \geq 0$ the double complex concentrated at degree $t$

$$D^{m,n,t} = D^{m,n}(X)_t = U^n_Q P^Q G^{m+1}(X)_t$$

For each fixed $n$ we have

$$\begin{array}{ccccccc} \cdots \longrightarrow & D^{q-1,n,t} & \longrightarrow & D^{q,n,t} & \longrightarrow \cdots \\ & \| & & \| & \\ \cdots \longrightarrow & U_Q^n P^Q G^q(X)_t & \longrightarrow & U_Q^n P^Q G^{q+1}(X)_t & \longrightarrow \cdots \end{array}$$

Where the maps are induced by the $G$-resolution of $X$. Next we fix $m$. We have

$$\begin{array}{ccccccc} \cdots \longrightarrow & D^{m,q-1,t} & \longrightarrow & D^{m,q,t} & \longrightarrow \cdots \\ & \| & & \| & \\ \cdots \longrightarrow & U_Q^{q-1} P^Q G^{m+1}(X)_t & \longrightarrow & U_Q^q P^Q G^{m+1}(X)_t & \longrightarrow \cdots \\ & \| & & \| & \\ \cdots \longrightarrow & Hom_{E_*}^t(U_Q^{q-1} P^Q G^m(X)) & \longrightarrow & Hom_{E_*}^t(U_Q^q P^Q G^m(X)) & \longrightarrow \cdots \\ & \| & & \| & \\ \cdots \longrightarrow & Hom_{U_Q}^t(U_Q^q P^Q G^m(X)) & \longrightarrow & Hom_{U_Q}^t(U_Q^{q+1} P^Q G^m(X)) & \longrightarrow \cdots \end{array}$$

Where $Hom_{U_Q}^t(-) = Hom_{U_Q}(E_*(S^t), -)$ and the bottom row is just the $P^Q$-complex for $P^Q G^m(X)$. Define $\overline{E_1}^{m,n,t} = H^{m+n}({}_iF^m(Tot(\mathbf{D}))/{}_iF^{m+1}(Tot(\mathbf{D}))$ where $i = 1, 2$ depending on the filtering of the complex. Fixing $m$ and taking homology we get

$$\overline{E_1}^{m,n,t} = Ext_{U_Q}^{m,t}(U_Q(G^m(X))) = \begin{cases} G^m(X)_t & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Taking homology again we have $E_2^{m,t}(X)$

Fixing $n$ and taking homology we find

$$\overline{E_1}^{m,n,t} = U_Q^m R_0^n P_E^Q(X)_t$$

taking homology again we find the following

$$\overline{E_2}^{m,n,t} = Ext_{U_Q}^{m,t}(R_0^n P_E^Q(X))$$

□

This tells us that knowledge of the functors $R_0^m P_E^Q(X)$ is enough to give us the $E_2$-term of the BKSS based on $E$. Now we need information about these objects. We have the following theorem that gives us a way to approach these objects.

**Theorem 4.3.3.** *There is a spectral sequence*

$$E_2^{i,j} = R_i^j P_E^Q(X) \Rightarrow R^{i+j} P^Q(E_*(X))$$

*with differential* $d_r : E_r^{i,j} \to E_r^{i+r,j-r+1}$

*Proof.* Consider the following double complex

$$D^{i,j} = P^Q S^{j+1} G^{i+1}(X)$$

Fixing $i$ first we have

$$\begin{array}{ccccccc} \cdots \longrightarrow & D^{i,j-1} & \longrightarrow & D^{i,j} & \longrightarrow \cdots \\ & \| & & \| & \\ \cdots \longrightarrow & P^Q S^j G^{i+1}(X) & \longrightarrow & P^Q S^{j+1} G^{i+1}(X) & \longrightarrow \cdots \end{array}$$

this is just the functor $P^Q$ applied to the $S$-resolution for $G^{i+1}(X)$.

Fixing $j$ now we have

$$\begin{array}{ccccccc} \cdots \longrightarrow & D^{i-1,j} & \longrightarrow & D^{i,j} & \longrightarrow \cdots \\ & \| & & \| & \\ \cdots \longrightarrow & P^Q S^{j+1} G^{i}(X) & \longrightarrow & P^Q S^{j+1} G^{i+1}(X) & \longrightarrow \cdots \end{array}$$

this map is induced by the $G$-resolution of $M$. Define the $E_1$-term the same as the previous theorem. Fixing $i$ again and taking homology, we have

$$E_1^{i,j} = \begin{cases} R^{i+1} P^Q(G^{j+1}(X)), & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

taking homology again we have $R_i^j P_E^Q(X)$. Lets fix $j$ now. We get

$$E_1^{i,j} = \begin{cases} P^Q S^{j+1}(E_*(X)), & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

homology again gives $R^j P^Q(E_*(X))$.

□

**Corollary 4.3.4.** *Suppose $E$ is $Q$-nice. Then there is a long exact sequence*

$$\cdots \to R_0^k P_E^Q(X) \to R^k P_E^Q(E_*(X)) \to R_1^{k-1} P_E^Q(X) \to R_0^{k+1} P_E^Q(X) \to \cdots$$

*Proof.* Since $E_*(\underline{E}_k)$ is nice, and since $G^r(X)$ is just a tensor product of these Hopf algebras then, for $k > 1$, $R_k^j P_E^Q(X) = 0$. We get a spectral sequence with just two rows. We turn this into a long exact sequence. The only non-trivial differential must be $d_2 : E_2^{i,i} \to E_2^{i-2,i-1}$. So the only things that survive to $E_\infty$ are the following:

$$E_3^{i,1} = Ker(d_2 : E_2^{i,1} \to E_2^{i+2,0})$$

$$E_3^{i,0} = Coker(d_2 : E_2{}^{i,1} \to E_2{}^{i+2,0})$$

Since $E_\infty^{i,j} = 0$ if $j > 1$ then it follows that $R^{i+1}P^Q(M) = F^{0,i+1} = F^{k,i-k}$ for $0 < k < i+1$. We also have exact sequences

$$0 \to F^{i+1,0} \to F^{i,1} \to E_\infty^{i,1} \to 0$$

$$0 \to E_\infty^{i,1} \to E_2^{i,1} \xrightarrow{d_2} E_2^{i+2,0} \to E_\infty^{i+2,0} \to 0$$

Combining these exact sequences with the fact that $F^{i+1,0} = E_{\infty}^{i+1,0}$ gives the result.

□

**Remark 4.3.5.** *Since by [BH] we know that if E is Landweber exact then $E_*(\underline{E}_n)$ is a free algebra for all n, then by remark 4.2.3 we know that, for all n, $P(E_*(\underline{E}_n)) = P^Q(E_*(\underline{E}_n))$. So this new approach will not give anything new for Landweber exact theories. Also, all of the previous theorems still holds if one substitutes the functor $P^Q$ by P.*

We want to apply the previous results to $K(n)$. But first we need to make sure that hypothesis 4.2.9 is satisfied. We use the following lemma to accomplish that.

**Lemma 4.3.6.** *Let $\sigma : E_*(\underline{E}_m), \to E_*(E)$ be the stabilization map. Suppose $Ker(\sigma)$ is the set of decomposable elements. Then hypothesis 4.2.9 is satisfied.*

*Proof.* Let $I_n = QG^n(M)$ and $I_{n+1} = QG^{n+1}(M)$ for $n \geq 0$. Recall that $d^0 = \eta_*$. Since the stabilization map $\sigma$ commutes with differentials we have

$$\sigma(d^0(xy)) = d^0(\sigma(xy)) = 0$$

and by hypothesis we have $d^0(xy) \in I_{n+1}^2$.

□

**Lemma 4.3.7.** *For $E = K(n)$ hypothesis 4.3.1 is satisfied and $Ker(\sigma)$ is the set of decomposable elements.*

*Proof.* By [Wil84], $\Gamma_{n,m} = K(n)_*(\underline{K(n)}_m)$ has generators $a^I \circ b_{(0)}^{j_0} \circ b^J \circ e_1^{\epsilon}$ with $\epsilon = 0, 1, i_k = 0, 1, 0 \leq j_k < p^n, j_0 < p^n - 1$. These elements stabilize to $\tau^I b^J$. So $\sigma$ is injective on the indecomposables.

□

**Lemma 4.3.8.** *$K(n)$ is $Q$-nice*

*Proof.* By [Wil84] we know that $K(n)_*(\underline{K(n)}_k)$ is a tensor product of exterior, truncated and polynomial Hopf algebras. The exterior algebra only contributes zero derived functors, the (primitively generated) polynomial algebra is a tensor product of coalgebras of the type $T(x_{2n})$(see [BCR82]) and this has primitive dimension one. So the only question is the truncated algebra. But by Theorem 2.1 of [Wil84] this is just a divided power algebra mod $p$ and this is cofree.

□

**Remark 4.3.9.** *Hypothesis 4.3.1 is not satisfied by $E(1)$ mod $p$. The element $v_1 b_{(0)}^{p-1} \circ b_{(1)}^n \circ e_1 - b_{(1)}^n \circ e_1 \circ [v_1] \in E(1)_*(\underline{E(1)}_{2n+1})$ is non-zero unstably but it is in $Ker(\sigma)$. This does not happen in $K(1)$ because we have the extra relation $v_1 b_{(0)}^{p-1} \circ b_{(1)}^n \circ e_1 = b_{(1)}^n \circ e_1 \circ [v_1]$.*

## 4.4 Calculation of $U_Q(M)$ for Morava $K$-Theories

All these facts will enable us to compute the $E_2$-term using $Ext_{U_Q}$. But for that knowledge of $U_Q(M)$ and the differentials is needed.

From [Yag78] we know that

$$K(n)_*(K(n)) = \Lambda[\tau_0, \dots, \tau_{n-1}] \otimes K(n)_* \otimes_{BP_*} BP_*(BP)/(v_n t_i^{p^n} - v_n^{p^i} t_i)$$

As in [BCM78] we use the basis consisting of $h_n$ instead of $t_n$. Since the $\tau_k$ comes from the dual of the Steenrod algebra, this implies that the canonical anti-isomorphism is given by $\tau_k + \sum_{i=0}^{k} t_{k-i}^{p^i} c(\tau_i) = 0$ or $\tau_k = c(\tau_k)$ mod decomposables. Let $\beta_k = c(\tau_k)$. Then the exterior part of $\Gamma_n$ is generated by $\beta_i$ with $0 \leq i \leq n-1$.

**Notation 4.4.1.**

$$<_i = \begin{cases} < & i = 0 \\ \leq & i = 1 \end{cases}$$

**Theorem 4.4.2.** *Let $\Gamma_n = K(n)_*(K(n))$ and $M$ a $K(n)_*$-module. Then*

1. $U_Q(M) = Span_{K(n)_*}\{\beta^I h^J \otimes m \in \Gamma_n \otimes M \mid l(I) + 2l(J) <_{i_0} |m|\}$

2. *$U_Q(M)$ injects into the stable cobar complex*

3. *Suppose that $M$ is an unstable $\Gamma$-comodule, with coaction $\Psi : M \to U_Q(M)$. Then the differential is given by*

$$\begin{aligned} d([\gamma_1|\dots|\gamma_n]m) =& [1|\gamma_1|\dots|\gamma_n]m \\ &+ \sum_{j=1}^{n} (-1)^j [\gamma_1|\dots|\gamma_j'|\gamma_j''|\dots|\gamma_n] \\ &+ (-1)^{n+1} \sum [\gamma_1|\dots|\gamma\prime_n]m' \end{aligned}$$

*where $\Psi(\gamma_j) = \sum \gamma' \otimes \gamma''$ and $\Psi(m) = \sum \gamma' \otimes m'$.*

*Proof.* The primitive elements in $K(n)_*(\underline{K(n)}_m)$ are

1. Exterior: $[v_n^k] \circ a^I \circ b_0^{j_0} \circ b^J \circ e_1$ with $i_0 = 1$ or $j_0 = 1$

2. Truncated: $[v_n^k] \circ a^I \circ b_0^{j_0} \circ b^J$ with $I \neq I(1)$ and $i_0 = 1$ or $j_0 = 1$

3. Polynomial: $([v_n^k] \circ a^{I(1)} \circ b_0^{j_0} \circ b^J)^{*kp}, k \geq 1$

The first and second cases are also indecomposable. In the third case the only one that is indecomposable is when $k = 1$. All of the cases suspends to $\beta^I h^J \otimes v_n^k i_m$. We have for all cases

$$l(I) + 2j_0 + 2l(J) - kq_n = m \qquad \text{or}$$

$$l(I) + 2l(J) \leq l(I) + 2j_0 + 2l(I) = m + kq_n = |v_n^k i_m|$$

Where $q_n = |v_n|$. The inequality on the left is strict if $j_0 = 1$ and is an equality if $i_0 = j_0 = 1$. This proves 1.

Part 2 follows immediately from 1. Part 3 follows from the fact that we know the differential in the stable cobar. Since $U_Q(M)$ injects, the result follows. □

At last we see why we have chosen the derived functors of the $Q$-primitives instead of just the usual derived functors of the primitives, these do not inject into the stable object(($[v^k] \circ a_{(0)} \circ b_{(0)}^{j_0} \circ b^I)^{*kp}$ is killed by $\sigma$ for $k > 0$). We only need information about the higher derived functors of $P^Q$.

**Theorem 4.4.3.** *Suppose that $R^q P^Q(E_*(X)) = 0$ for $q \geq n$*

1. $R_0^0 P^Q(X) \cong P^Q(E_*(X))$
2. $R_0^q P^Q(X) \cong R_1^{q+2} P^Q(X)$ *for* $q \geq n$
3. $R_0^1 P^Q(X)$ *injects into* $R^1 P^Q(E_*(X))$ *and if* $E_*(X)$ *is cofree then* $R_0^1 P^Q(X) \cong 0$

*Proof.* Using corollary 4.3.4 and since $R^0 P^Q(E_*(X)) = P^Q(E_*(X))$ we have the following

$$R_1^{-2} P^Q(X) \to R_0^0 P^Q(X) \to P^Q(E_*(X)) \to R_1^{-1} P^Q(X)$$

since the first and the last terms are zero, we have the isomorphism.

Using corollary 4.3.4 again and for $q \geq n$

$$\cdots \to R^{q+1}P^Q(E_*(X)) \to R_1^q P^Q(X) \to R_0^{q+2}P^Q(X) \to R^{q+2}P^Q(E_*(X)) \to \cdots$$

where the first and the last term are zero. This gives 2.

To obtain the third relation we have

$$0 = R_1^{-1}P^Q(X) \to R_0^1 P^Q(X) \to R^1 P^Q(E_*(X))$$

this gives 3.

□

## 4.5 Applications to $E_2(S^{2n+1}; K(1))$

We apply all the previous results to $K(1)$. In this case we can say much more.

**Corollary 4.5.1.** *Supose that $R^i P^Q(K(1)_*(X)) = 0$ for $i \geq n$. Then for $i \geq n+1$*

$$R_0^i P^Q(X) = 0$$

*Proof.* Suppose $i > n+1$ and $R_0^i P^Q(X) \neq 0$ and we have a generator $x$. Since we can get new generators from $x$ by multiplying by $v_1^k$, we consider $R^1 P^Q(G^{i+2}(X)) \otimes \mathbb{Z}_p$. By the universal coefficients theorem and Lemma 4.3.8 we know that the only part contributing first derived functors is the polynomial part. This is of the form $T(x_{2n})$ of [BCR82]. The generators are of the form $x = a_0 \circ b^J$ of dimension $2(1 + \sum j_i p^i)$ with $j_0 < p-1$. Since $R^1 P(T(x_{2n})) = \mathbb{Z}_p(2np, 2np^2, \ldots)$ then, by corollary 4.2.5, $R^1 P^Q(T(x_{2n})) = \mathbb{Z}_p(2np^{i_1}, 2np^{i_2}, \ldots)$. So we have generators of degree $2p^{i_k}(1 + \sum j_i p^i)$. Now lets look at generators for

$$R^0P^Q(G^{i+1}(X)) \otimes \mathbb{Z}_p = P^Q(G^{i+1}(X)) \otimes \mathbb{Z}_p$$

This has generators $a_{(0)} \circ b^K \circ e_1^\epsilon$ and $b_{(0)} \circ b^K \circ e_1^\epsilon$ of degrees $2(1 + \sum_{k \geq 0} k_i p^i) + \epsilon$ with $k_0 < p - 1$ in both cases. By (1) of 4.4.3 the dimension of the generators have to agree. So $\epsilon = 0$. But the generators of the first case was divisible by $p^{i_k}$ and none of the other two are.

□

In fact this result implies the following

**Theorem 4.5.2.** *Let $X$ be an H-space and suppose $K(1)_*(X)$ is cofree. Then*

$$E_2^{s,t}(X) \cong Ext_{U_Q}^{s,t}(P^Q(K(1)_*(X)))$$

*Proof.* Since $K(1)_*(X)$ is cofree then, by 4.5.1, we know that we have only zero derived functors of $P^Q$ and so the spectral sequence of theorem 4.3.2 collapses and the result follows.

□

**Remark 4.5.3.** *By [Kuh89], there are no unstable $K(n)_*(K(n))$-comodules. This implies that whenever we have a collapsing to the zero line of the spectral sequence of theorem 4.3.2, then there is an isomorphism $E_2(X) \cong E_2(\Sigma^\infty X)$ where the object on the right is the stable Adams spectral sequence. In particular, the $E_2$-terms of the stable spectral sequence for the sphere and the unstable spectral sequence for the odd sphere agree. This does not happen with the even sphere. It can be shown that $R^1P^Q(S^{2n})$ is a $K(1)_*$-module with a generator in dimension $4n$. This leads to a long exact sequence*

$$\cdots \to Ext_{U_Q}^{s,t}(S^{2n}) \to Ext_{\mathcal{M}(G)}^{s,t}(S^{2n}) \to Ext_{U_Q}^{s-1,t}(S^{4n}) \to \cdots$$

*where the last map on the left has bidegree (2,0).*

With all these facts we can completely determine $E_2$-term of the spectral sequence for odd spheres.

**Lemma 4.5.4.** *Suppose $\gamma = [\gamma_1|\dots|\gamma_n]m \in U^n_Q(M)$ with $\gamma_i$ and $m$ (as an element in the comodule) primitive. Then $\gamma \in Ker(d^n)$*

*Proof.* We have the following

$$d([\gamma_1|\dots|\gamma_n]m) = [1|\dots|\gamma]m$$

$$+\sum_{i=1}^{n}(-1)^i[\gamma_1|\dots|\gamma_{i-1}|1|\gamma_i|\dots|\gamma_n]m$$

$$+\sum_{j=1}^{n}(-1)^i[\gamma_1|\dots|\gamma_j|1|\gamma_{j+1}|\dots|\gamma_n]m$$

$$+(-1)^{n+1}[\gamma_1|\dots|\gamma_n|1]m$$

Reindexing the first sum gives $\sum_{j=0}^{n}(-1)^{j+1}[\gamma_1|\dots|\gamma_j|1|\gamma_{j+1}|\dots|\gamma_n]m$ and adding this to the second sum gives $-[1|\dots|\gamma_n]m + (-1)^n[\gamma_1|\dots|\gamma_n]m$ and this is cancel by the first and the last term.

□

Lets get some elements. Since $c(\tau_0) = \tau_0$ we do not use the $\beta$ notation. The only elements in $\Gamma = K(1)_*(K(1))$ which are primitive are $h_1^{pk}, k > 0$, and $\tau_0$. But, by [Yag78], in $\Gamma$ we have $h_1 v_1^p = h_1^p v_1$, so we only have to consider $h_1$ and $\tau_0$ . So we see that in $M = K(1)_*(S^{2n+1})$, $n > 0$, are the ones that have combinations of $\tau_0$ and $h_1$.

**Claim 4.5.5.** *Let $\gamma = [\gamma_1|\dots|\gamma_n]i_{2n+1} \in U^n_Q(M)$ with $\gamma_j \in \{\tau_0, h_1\}$. Then*

1. $[\tau_0|\dots|\tau_0|h_1]i_{2n+1} + (-1)^k[\tau_0|\dots|\tau_0|\overset{k}{h_0}|\tau_0|\dots|\tau_0]i_{2n+1} \in Im(d^{n-1})$

2. *If $\gamma \neq [\tau_0|\dots|\tau_0]i_{2n+1}$, $[\tau_0|\dots|\tau_0|h_0]i_{2n+1}$ then $\gamma \in Im(d_{n-1})$.*

*Proof.* For 1, we have

$$d([\tau_0|\dots|\overset{k}{\tau_0 h_1}|\dots|\tau_0]i_{2n+1})$$

$$= (-1)^k([\tau_0|\dots|\overset{k}{h_1}|\tau_0|\dots|\tau_0]i_{2n+1} + [\tau_0|\dots|\overset{k+1}{h_1}|\tau_0|\dots|\tau_0]i_{2n+1})$$

For the general case, take a sum of the elements in the left of the equation where $\tau_0 h_1$ starts in the last place and moves to the $k$ place.

For 2, by 1, it is enough to prove that $[\tau_0|\dots|\tau_0|h_1|\dots|h_1|h_1]i_{2n+1} \in Im(d_{n-1})$. We have

$$d(\frac{(-1)^n}{2}[\tau_0|\dots|\tau_0|h_1|\dots|h_1^2]i_{2n+1}) = [\tau_0|\dots|\tau_0|h_1|\dots|h_1]i_{2n+1}$$

□

From now on the element $[\gamma_1|\dots|\gamma_n]i_{2n+1}$ will be represented in homology by $\gamma_1\dots\gamma_n i_{2n+1}$ or, if it is clear on which sphere we are working on, $i_{2n+1}=1$ and $\gamma_1\dots\gamma_n$, with the convention that $deg(\gamma_1\dots\gamma_n) = 2n+1+\sum deg(\gamma_i)$. Since for $K(1)$ the right action and the left actions commute we immediately have

$$E_1^{0,t}(S^{2n+1}) = \begin{cases} \mathbb{Z}_p, & \text{if } t = 2n+1+kq,\ k\in\mathbb{Z} \text{ generated by } v_1^k \\ 0 & \text{otherwise} \end{cases}$$

The element $\tau_0$ generates a tower over in filtration 1 we also have $v_1^k h_1$ and $\tau_0$ generates a tower over it.

## 4.6 Composition Parings in the Spectral Sequence

By [Bou], there is also composition paring in the spectral sequence for $t-s\geq 1$ and $r\geq 2$:

$$E_r^{s,m+t}(X)\otimes E_r^{s',t'}(S^m) \xrightarrow{\circ} E_r^{s+s',t+t'}(X)$$

Given the natural map $i : X \to X^{\wedge}_E$, where $X^{\wedge}_E$ is the completion, there is a commutative diagram

$$\begin{array}{ccc} \pi_{t+m}X \otimes \pi_{t'}S^m & \xrightarrow{*} & \pi_{t+t'}X \\ {\scriptstyle i_*\otimes i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ \pi_{t+m}X^{\wedge}_E \otimes \pi_{t'}(S^m)^{\wedge}_E & \xrightarrow{\circ} & \pi_{t+t'}X^{\wedge}_E \end{array}$$

**Lemma 4.6.1.** *The composition corresponds(up to sign) to the Yonneda product in the category $U_Q$.*

$$Ext^s_{U_Q}(K(1)_*(S^{t+2m+1}), K(1)_*(S^{2n+1})) \otimes Ext^{s'}_{U_Q}(K(1)_*(S^{t'}), K(1)_*(S^{2m+1})) \to$$

$$Ext^s_{U_Q}(K(1)_*(S^{t+2m+1}), K(1)_*(S^{2n+1})) \otimes Ext^{s'}_{U_Q}(K(1)_*(S^{t+t'}), K(1)_*(S^{t+2m+1})) \to$$

$$Ext^{s+s'}_{U_Q}(K(1)_*(S^{t+t'}), K(1)_*(S^{2n+1}))$$

*Proof.* This follows from [Bou]. □

We use this result to study compositions by $\tau_0$.

**Claim 4.6.2.** *For $k > 0$ $\tau_0^k \neq 0$*

*Proof.* Let $f \in F^{1,2n+2}$ represent $\tau_0$. Suppose $f^k$ maps to $\tau_0^k$. Then by naturality we have

$$\begin{array}{ccc} F^{s,2n+1+s} \otimes F^{1,2n2+2} & \longrightarrow & E_2^{s,2n+1+s} \otimes E_2^{1,2n+2} \\ \downarrow & & \downarrow \\ F^{s+1,2n+s+1} & \longrightarrow & E_2^{s+1,2n+s+1} \end{array}$$

This gives $f^k * f \mapsto f^{k+1} \mapsto \tau_0^{k+1}$.

□

As in the stable case, we have a tower representing multiplication by $p$.

**Lemma 4.6.3.** *Suppose that the spectral sequences for $X$ and $S^{2n+1}$ converge. Suppose also that $x \in E_2$ survives to $E_\infty$ and represents $\alpha \in \pi_*(X)$. Then $a_{(0)} \circ x$ represents $p\alpha \in \pi_*(X)$.*

*Proof.* By naturality is enough to prove that $\tau_0 * 1 (= \tau_0 * i_{2n+1})$ is just $p$. The element $1 (= i_{2n+1})$ represents the generator of $\pi_{2n+1}(S^{2n+1})$. We have the following diagram

$$\begin{array}{ccccccccc}
 & & & & \pi_{2n+1}(\hat{S}^{2n+1}) & & \mathbb{Z}_p & & \\
 & & & & \| & & \| & & \\
 & & 0 & \longrightarrow F^{1,2n+2} \longrightarrow & F^{0,2n+1} & \longrightarrow & E_\infty^{0,2n+1} & \longrightarrow & 0 \\
 & & & \| & & & & & \\
0 & \longrightarrow & F^{2,2n+1} & \longrightarrow F^{1,2n+2} \longrightarrow & E_\infty^{1,2n+2} & \longrightarrow & 0 & & \\
 & & & & \| & & & & \\
 & & & & \mathbb{Z}_p & & & &
\end{array}$$

let $g \in F^{0,2n+1}$ correspond to the preimage of $i_{2n+1} \in E_\infty^{0,2n+1}$. Then $\tau_0 * 1 = a_{(0)} \in E_\infty^{1,2n+2}$ corresponds to $f$. Since $f$ push forward twice is zero in the first line we have that $f = pg$.

□

Form this follows that we have infinite towers, for $k \in \mathbb{Z}$, in dimensions $t-s = 2n+1+kq$, generated by $v_1^k \tau_0^s$ and towers in dimension $t-s = 2n+kq$ generated by $v_1^k \tau_0^s h_1$. The only thing missing is knowledge about the differentials.

## 4.7 Convergence of the Stable Adams Spectral Sequence

The next result will give us the missing piece. We define $\nu(k)$ as $k = ap^{\nu(k)}$ with $p \nmid a$.

**Theorem 4.7.1.** *The stable Adams Spectral Sequence based on $K(1)$ of the sphere converges for $t-s>0$ and $v_1^k$ supports a $d_{\nu(k)+2}$ differential*

*Proof.* Since the only place in which the $E_2$-term has classes is in dimensions $t-s = kq-1, kq$ with $k \in \mathbb{Z}$, it is enough to worry about those dimensions. By [HRM77] we know that the $E(1)$ S.S. for the sphere converges and $\pi_*(S^\wedge_{E(1)})$ has a $\mathbb{Z}_{p^{\nu(k)+1}}$ generated by $\alpha_k = d_1(v_1^k)/p^{\nu(k)+1}$ in dimension $kq-1$. We have a map of ring spectra $j : E(1) \to K(1)$

which induces a map between the spectral sequence of these spectra and sends $\alpha_k$ to $v_1^{k-1}h_1$. Since $E_r^{s,t} = 0$ for any $t-s = kq-2$ the element $v_1^{k-1}h_1 \in \varprojlim_s \pi_{kq-1}(\overline{K(1)}^s)$ and since there is an onto map to $\pi_{kq-1}(S^{\wedge}_{K(1)})$ we have $v_1^{k-1}h_1 \in \pi_{kq-1}(S^{\wedge}_{K(1)})$. this implies that $1 \leq ord(v_1^{k-1}h_1) \leq p^{\nu(k)+1}$. Since multiplication by $\tau_0$ represents multiplication by $p$ then the tower over $v_1^{k-1}h_1$ has to be killed at filtration less or equal than $\nu(k)+2$. So $v_1^k$ supports a $d_r$ differential where $r \leq \nu(k)+2$.

We prove $\nu(k)+2 = r$ by induction on $d_n$. For $k=1$, we have, by the previous paragraph, that $d_2(v_1) = \tau_0 h_1$ and using the derivation rule we have $d_2(v_1^k\tau_0^w) = kv_1^{k-1}\tau_0^{w+1}h_1$. Suppose now that $d_m(v_1^{kp^{m-2}}\tau_0^w) = kv_1^{kp^{m-2}-1}\tau_0^{w+m-1}h_1$ for $m < n$. By induction hypothesis we know that $d_{n-1}(v_1^{p^{n-2}}) = pv_1^{p^{n-2}-1}\tau_0^{n-2}h_1 = 0$. So the smallest differential in which $v_1^{p^{n-2}}$ is non-zero is $d_n = d_{\nu(n-2)+2}$. The general result follows from using the derivation rule. □

**Corollary 4.7.2.** *The Bousfield-Kan spectral sequence based on $K(1)$ for the sphere converges completely and the completion in the sense of [BH] is given by*

$$\pi_m((S^{2n+1})^{\wedge}) = \begin{cases} \mathbb{Z}_{p^{\nu(k)+1}} & m = 2n+kq, k \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_p^{\wedge} & m = 2n+1, 2n \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* The differentials on $v_1^k$ for $k \geq 0$ can be deduced by the stable differentials. For $k \leq 0$, [Bou] says that $d_r$ is a derivation for $r > 1$. We have the following formula

$$0 = d_r(1) = d_r(v_1^k v_1^{-k}) = d_r(v_1^k)v_1^{-k} + v_1^k d_r(v_1^{-k})$$

or $d_r(v_1^{-k}) = -v_1^{-2k}d_r(v_1^k)$. Since we have a vanishing line then by [BK70] we know that the spectral sequence converges completely.

□

## 4.8 Some Remarks on the General $K(n)$ Spectral Sequence

Although we have not been able to prove that $R_0^i P^Q(X)$ vanish when the groups $R^i P^Q(K(n)_*(X))$ vanish for $n > 1$, we have the following theorem.

**Theorem 4.8.1.** *The zero line of the spectral sequence of Theorem 4.3.2 for $K(n)$ injects into the stable Adams spectral sequence.*

*Proof.* Since $d_r$ has degree $(r, -r+1, 0)$ then $\bar{E}_{r+1}^{m,0,t} = \bar{E}_r^{m,0,t}/Im(d_r)$ for all $r > 1$. So we have a sequence of groups

$$\bar{E}_2^{m,0,t} \to \bar{E}_3^{m,0,t} \to \cdots \to \bar{E}_\infty^{m,0,t} = \bar{E}_\infty^{m,0,t} = F^{m,0}/F^{m+1,-1}$$

Where $F^{m,n} = Im[H^{m+n}({}_2F^m(Tot(D)) \to H^{m+n}(Tot(D))]$ and ${}_2F(Tot(D))$ is the filtration of the double complex $D$ of the proof of theorem 4.3.2. Since ${}_2F^m(Tot(D))_n = 0$ if $m > n$ then $F^{m+1,-1} = 0$ and we have that $\bar{E}_2^{m,0,t} = F^{m,0}$ and we have a map

$$Ext_{U_Q}^{m,t}(P^Q K(n)_*(X)) = \bar{E}_2^{m,0,t} \to \bar{E}_\infty^{m,0,t} = F^{m,0} \to F^{0,m} = E_2^{m,t}(X)$$

Since this construction commutes with stabilization we have a commutative diagram

$$\begin{array}{ccccc} Ext_{U_Q}^{m,t}(P^Q K(n)_*(X)) & \longrightarrow & Ext_{\mathcal{M}(G)}^{m,t}(K(n)_*(X)) & = & E_2^{m,t}(X) \\ \cong\downarrow & & \sigma\downarrow & & \sigma\downarrow \\ Ext_{\Gamma_n}^{m,t'}(K(n)_*(X)) & \xrightarrow{\cong} & Ext_{\Gamma_n}^{m,t'}(K(n)_*(\Sigma^\infty X)) & = & E_2^{m,t'}(\Sigma^\infty X) \end{array}$$

where $\Gamma_n = K(n)_*(K(n))$. □

## 4.9 The $K(1)$-completion of $S^{2n+1}$ and its Relation to the Work of Farjoun

In [Far00], Faorjoun defines a tower $\{Y_n(X)\}$ under $X$ as follows. Let $E(X)$ be the functor defined on section 2.2 and let $Y_1(X) = E(X)$. Define $Y_n(X) = fb[Y_{n-1}(X) \to E(Y_{n-1}(X)/X)]$ where $fb$ means the homotopy fiber. The null homotopy from $X$ into the cofiber gives a map $X \to Y_n(X)$. Let $Y_\infty(X) = \varprojlim Y_n(X)$. He puts forward the following two questions.

**Question 4.9.1.** *Let $X$ be an H-space of finite type. Is the natural map $X_E \to Y_\infty$ a homotopy equivalence.*

**Question 4.9.2.** *When does the natural map of towers $\{Y_n(X)\}_{n\in\mathbb{N}} \to \{D^n(X)\}_{n\in\mathbb{N}}$ has a left inverse.*

Although we can not prove 4.9.1 or get necessary conditions to get 4.9.2, we can deduce the following from our work.

**Lemma 4.9.3.** *Let $E = K(1)$. Then 4.9.1 and 4.9.2 can not be true at the same time for $X = S^{2n+1}$.*

*Proof.* Suppose 4.9.1 and 4.9.2 are true. Then we have an injective map $\pi_*(S^{2n+1}_{K(1)}) \to \pi_*(K(1)^\wedge(S^{2n+1}))$. The $K(1)$-localization of the odd spheres was calculated in [Bou99]. We have

$$\pi_i(S^{2n+1}_{K(1)}) = \begin{cases} \mathbb{Z}_{p^{\nu(k)+1}} & i = 2n + qk, 2n-1+kq \quad k \in \mathbb{Z} - \{0\} \\ \mathbb{Z}_p^\wedge & i = 2n, 2n-1 \\ \mathbb{Z}_p^\wedge \oplus \mathbb{Z}_p^\wedge & i = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Comparing this with the result of corollary 4.7.2 we see that the map can not be injective.

□

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