Inverse limits of models of set theory and the large cardinal hierarchy near a high-jump cardinal

by

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Abstract

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This dissertation consists of two chapters, each of which investigates a topic in set theory, more specifically in the research area of forcing and large cardinals. The two chapters are independent of each other.

The first chapter analyzes the existence, structure, and preservation by forcing of inverse limits of inverse-directed systems in the category of elementary embeddings and models of set theory. Although direct limits of directed systems in this category are pervasive in the set-theoretic literature, the inverse limits in this same category have seen less study. I have made progress towards characterizing the existence and structure of these inverse limits. Some of the most important results are as follows. An inverse limit exists if and only if a natural source exists. If the inverse limit exists, then it is given either by the entire thread class or by a rank-initial segment of the thread class. Given sufficient large cardinal hypotheses, it is consistent that there are systems with no inverse limit, systems with inverse limit given

by the entire thread class, and systems with inverse limit given by a proper subset of the thread class. Inverse limits are preserved by forcing in both directions under fairly general assumptions but not in all cases. Prikry forcing and iterated Prikry forcing are important techniques for constructing some of the examples in this chapter.

The second chapter analyzes the hierarchy of the large cardinals between a supercompact cardinal and an almost-huge cardinal, including in particular high-jump cardinals. I organize the large cardinals in this region by consistency strength and implicational strength. I also prove some results relating high-jump cardinals to forcing. A high-jump cardinal is the critical point of an elementary embedding $j : V \to M$ such that M is closed under sequences of length $\sup\{\, j(f)(\kappa) \mid f : \kappa \to \kappa \,\}$. Two of the most important results in the chapter are as follows. A Vopěnka cardinal is equivalent to a Woodin-for-supercompactness cardinal. There are no excessively hypercompact cardinals.

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Next, I would like to thank the many students and professors at CUNY and elsewhere who have been my friends and mentors, a few of whom I will name here. The set theory scene at CUNY is strong and vibrant, and there is a great sense of community at the Friday seminars. Brent Cody and Victoria Gitman deserve special thanks for being two of my most significant mentors among the students and young faculty. George Leibman wrote the LaTeX document class for CUNY doctoral dissertations, saving me much time and effort. Kaethe Minden has been a close friend and has helped me through some difficult times. Thanks also goes to Everett Piper, who enthusiastically read drafts of my dissertation and offered suggestions and ideas.

I would especially like to thank Menachem Magidor, who helped me to prove a major theorem about inverse limits when we met in Toronto in the autumn of 2012. Professor Magidor was generous with his time and ideas, and he helped me to solve a problem that I had been stuck on for the better part of a year. It was an honor for me to collaborate with him.

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I would also like to acknowledge the work of set theorists and mathematicians over the entire course of human history. One of my favorite things about conducting research in mathematics is the feeling that I am contributing my small part to a major human endeavor and that I am collaborating with the giants of the past.

Finally, I would like to thank the people who helped to make the typesetting of this dissertation possible, including the following people. Donald Knuth and Leslie Lamport invented TeX and LaTeX respectively and made them available as free software. Many people worked towards the production of TeXstudio, the feature-rich open-source LaTeX editor that I used to write my dissertation. I used the Commutative Diagrams in TeX package, by Paul Taylor, for typesetting of commutative diagrams.

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Chapter 1

Inverse limits of elementary embeddings between models of set theory

1.1 Introduction

Set theorists often consider direct limits of directed systems of elementary embeddings. For instance, these systems occur frequently in the study of iterated ultrapowers in inner model theory and in other areas of set theory. However, inverse limits of inverse-directed systems of elementary embeddings are rarely studied. I was motivated to study inverse limits when inverse-directed systems appeared in section 5 of my joint paper, [HKP12]. Unlike in the case of direct limits, not every inverse-directed system of elementary embeddings between models of set theory has an inverse limit.

I will work in a category-theoretic framework: the objects are model-theoretic structures satisfying a certain theory, and the morphisms are elementary embeddings. The focus is on models of ZFC or of a substantial fragment of ZFC, but a few of the results apply to much more general structures as well.

Let $\mathcal{C}$ be any category in which the morphisms are elementary embeddings. Then it is well-known that every directed system over $\mathcal{C}$ must have a direct limit and that this direct limit is given by the class of coherent threads through the directed system, along with the inclusion maps.[1]

In the case of inverse limits over $\mathcal{C}$, the situation is more complicated. There may be inverse-directed systems with no inverse limit, and for systems that do have an inverse limit, this inverse limit may be given by a proper subset of the class of coherent threads. In particular, given sufficient large cardinal axioms, all of the above situations can occur in the category of ZFC models and elementary embeddings.

Another interesting contrast between direct and inverse limits in the category of elementary embeddings and transitive models of set theory is as follows. Direct limits always exist, but they are not necessarily well-founded. Indeed, much work in the study of direct limits (for example, in the study of iterated ultrapowers) goes into showing that particular direct limits are well-founded. On the other hand, while inverse limits may not always exist, if they exist, then they are necessarily well-founded. (I will prove this fact in lemma 7.)

An outline of some of the most important results of the chapter follows. All of the systems discussed in this outline are inverse-directed systems of elementary embeddings and models of ZFC.

- The inverse limit of a system exists if and only if there is some natural source mapping into the system in such a way that the appropriate commutative laws hold. That is

[1] I will prove this well-known fact about direct limits in lemma 11. The fact is also true in certain other categories, though not in all categories.

to say, with no further hypotheses, there exists such a natural source satisfying the universal property for inverse limits. (theorem 19, section 1.3)

- If every model of a system satisfies $V = HOD$, then the inverse limit of the system exists and is given canonically by the entire thread class. (theorem 23, section 1.5)

- If there exists a measurable cardinal, then it is consistent that there is a system of transitive models with no inverse limit, and this system has order type ω^*.[2] (theorem 28, section 1.6)

- If there exists a 1-extendible cardinal, then it is consistent that there is a system of transitive models with an inverse limit, but this inverse limit is not given by the entire thread class of the system, but rather by a proper subset of the thread class. Furthermore, this system has order type ω^*. (theorem 29, section 1.7)

- Under fairly general conditions (but not in all cases), inverse limits are preserved by forcing in both directions. (theorems 37 and 39, section 1.8)

The definitions of a system of structures, a direct limit, and an inverse limit follow. These definitions should not be surprising, but different authors use these terms with different meanings, especially in the case of the inverse limit, so I will define precisely the sense in which I will use them.

Definition 1. Let $\mathcal{C}$ be a category. Let I be an indexing set equipped with a partial order, $\leq$. **A system over the category** $\mathcal{C}$ (also called a diagram in $\mathcal{C}$) indexed by I is a

[2]The symbol ω^* denotes the reverse order of ω.

collection of objects and morphisms of $\mathcal{C}$ of the form $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$, satisfying the following requirements.[3] For all pairs of indices of the form $\alpha \leq \beta \in I$, the morphism $j_{\alpha\beta}$ maps from $M^{(\alpha)}$ to $M^{(\beta)}$. These morphisms commute: that is to say, if $\alpha \leq \beta \leq \gamma$ are elements of I, then $j_{\beta\gamma} \circ j_{\alpha\beta} = j_{\alpha\gamma}$. Furthermore, for all indices $\alpha \in I$, the morphism $j_{\alpha\alpha}$ is the identity morphism.

The index set I is **directed** if and only if for all indices $\alpha, \beta \in I$, there exists an index $\gamma \in I$ such that $\alpha, \beta \leq \gamma$. In this case, I also say that the system is a directed system. The index set is **inverse-directed** if and only if for all indices $\alpha, \beta \in I$ there exists an index $\gamma \in I$ such that $\gamma \leq \alpha, \beta$. In this case, I also say that the system is an inverse-directed system.

Note that I index the $\mathcal{C}$-objects by superscripts rather than subscripts. The motivation for this convention is as follows. Given a transitive model, $(M^{(\alpha)}, \in)$, of ZF and an ordinal $\beta \in M^{(\alpha)}$, I will use the expression $M^{(\alpha)}_\beta$ to denote the set $M^{(\alpha)} \cap V_\beta$. More generally, this notation extends to all ZF-models $(M^{(\alpha)}, \in_\alpha)$, even ill-founded models: given such a model and an $M^{(\alpha)}$-ordinal β, the expression $M^{(\alpha)}_\beta$ denotes the subset of elements of $M^{(\alpha)}$ that $M^{(\alpha)}$ thinks have $\in_\alpha$-rank less than β.

A partial order I can be viewed as a category whose objects are given by the elements of I, with exactly one morphism from α to β whenever $\alpha \leq \beta$ and no other morphisms. It follows that in an abstract sense, a system over $\mathcal{C}$ is simply a functor from I to $\mathcal{C}$. More generally, one could take I to be a any category, not just a partial order, but I will restrict

[3] I will more commonly denote such a system by explicity describing the category $\mathcal{C}$, for instance, *a system of elementary embeddings and models of* ZFC.

my attention to the case where I is a partial order.

I will next define the direct limit of a directed system and the inverse limit of an inverse-directed system. One could use these same definitions to define direct and inverse limits of arbitrary systems, but I will restrict my attention to direct limits of directed systems and inverse limits of inverse-directed systems. Immediately after I give these definitions, I will show that if a direct or inverse limit exists, then it is unique up to unique canonical isomorphisms, so that it makes sense to refer to *the* direct limit or *the* inverse limit.

In the category-theoretic literature, there are a few alternative equivalent notations for discussing direct limits and inverse limits. In the general categorical context, a direct limit is more commonly called a colimit, and an inverse limit is more commonly called a limit. However, the terms *direct limit* and *inverse limit* are common when discussing directed systems and inverse-directed systems. An inverse limit is a natural source satisfying a particular universal property, and a direct limit is a natural sink satisfying a particular universal property. Natural sources and natural sinks (the notation used by Herrlich and Strecker in [HS73]) are also commonly called cones, (in particular by Mac Lane in [ML10]). I personally think that *natural source* and *natural sink* sound more descriptive, so I use that terminology. In the definitions below, I indicate all of the alternative notations given in [HS73] and in [ML10].

Definition 2. Let $\langle N^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be a directed system over a category $\mathcal{C}$. Let $N^{(\infty)}$ be an object of $\mathcal{C}$, and for each index $\alpha \in I$, let $j_{\alpha\infty} : N^{(\alpha)} \to N^{(\infty)}$ be a $\mathcal{C}$-morphism such that the commutative rule $j_{\beta\infty} \circ j_{\alpha\beta} = j_{\alpha\infty}$ holds whenever $\alpha \leq \beta$. Then the pair $(N^{(\infty)}, \langle j_{\alpha\infty}\rangle_{\alpha\in I})$ is a **natural sink** (cone to the base $N^{(\infty)}$, cocone) for the system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$. This natural sink is a **direct limit** (colimit, inductive limit) of the system if and only if the following universal property is satisfied. Whenever $(N^{(\infty')}, \langle j_{\alpha\infty'}\rangle_{\alpha\in I})$ is a natural sink for the same system, then there is a unique $\mathcal{C}$-morphism $j_{\infty\infty'} : N^{(\infty)} \to N^{(\infty')}$ such that for all indices $\alpha \in I$, the commutative rule $j_{\infty\infty'} \circ j_{\alpha\infty} = j_{\alpha\infty'}$ holds.

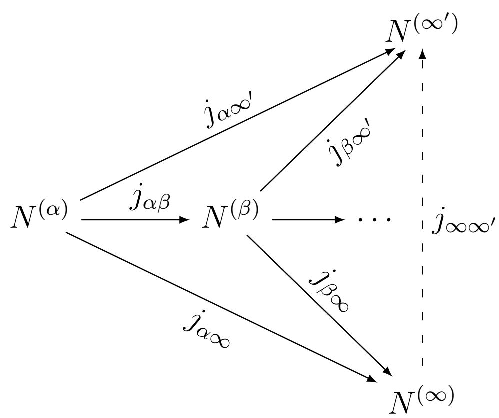


Figure 1.1: Definition of the direct limit

The definition of the inverse limit is obtained by reversing the arrows in the definition of the direct limit.

Definition 3. Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system over a category $\mathcal{C}$. Let $M^{(\infty)}$ be an object of $\mathcal{C}$, and for each index $\alpha \in I$, let $j_{\infty\alpha} : M^{(\infty)} \to M^{(\alpha)}$ be a $\mathcal{C}$-morphism such that the commutative rule $j_{\alpha\beta} \circ j_{\infty\alpha} = j_{\infty\beta}$ holds whenever $\alpha \leq \beta$. Then the pair $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ is a **natural source** (cone from the base $M^{(\infty)}$, cone) for the system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$. This natural source is an **inverse limit** (limit, projective limit) of the system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ if and only if the following universal property is satisfied. Whenever $(M^{(\infty')}, \langle j_{\infty'\alpha}\rangle_{\alpha\in I})$ is a natural source for the same system, there is a unique $\mathcal{C}$-morphism

$j_{\infty'\infty} : M^{(\infty')} \to M^{(\infty)}$ such that for all indices $\alpha \in I$, the commutative rule $j_{\infty\alpha} \circ j_{\infty'\infty} = j_{\infty'\alpha}$ holds.

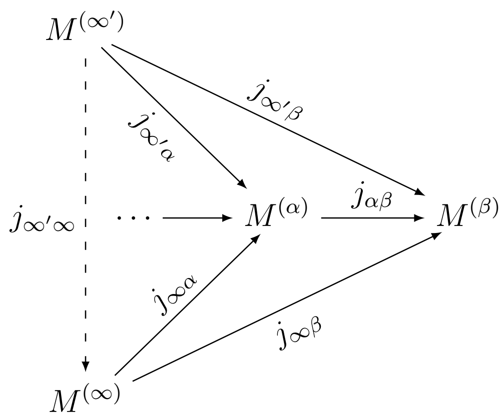


Figure 1.2: Definition of the inverse limit

Lemma 4 (see [HS73, pp. 135-136]). *Let $\mathcal{C}$ be any category. Then the direct and inverse limits of systems over the category $\mathcal{C}$ are unique up to unique canonical isomorphisms. These isomorphisms are given as follows.*

1. *Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be a directed system over the category $\mathcal{C}$, and let the natural sinks $(M^{(\infty)}, \langle j_{\alpha\infty}\rangle_{\alpha\in I})$ and $(M^{(\infty')}, \langle j_{\alpha\infty'}\rangle_{\alpha\in I})$ be two direct limits of this system. Then the maps $j_{\infty\infty'}$ and $j_{\infty'\infty}$ given by the universal property for direct limits are isomorphisms.*

2. *Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system over the category $\mathcal{C}$, and let the natural sources $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ and $(M^{(\infty')}, \langle j_{\infty'\alpha}\rangle_{\alpha\in I})$ be two inverse limits of this system. Then the maps $j_{\infty\infty'}$ and $j_{\infty'\infty}$ given by the universal property for inverse limits are isomorphisms.*

Proof. I present the proof for direct limits first. To show that $j_{\infty\infty'}$ and $j_{\infty'\infty}$ are isomorphisms, I must show that $j_{\infty'\infty} \circ j_{\infty\infty'} = \mathrm{id}_{M^{(\infty)}}$ and $j_{\infty\infty'} \circ j_{\infty'\infty} = \mathrm{id}_{M^{(\infty')}}$. (This is the category-theoretic definition of an isomorphism.) Note that both of the morphisms $\mathrm{id}_{M^{(\infty)}}$ and $j_{\infty'\infty} \circ j_{\infty\infty'}$ witness that the universal property is satisfied for $(M^{(\infty)}, \langle j_{\alpha\infty} \rangle_{\alpha \in I})$ with respect to itself. By definition, a morphism witnessing the universal property must be unique, so $j_{\infty'\infty} \circ j_{\infty\infty'} = \mathrm{id}_{M^{(\infty)}}$. Essentially the same argument shows that $j_{\infty\infty'} \circ j_{\infty'\infty} = \mathrm{id}_{M^{(\infty')}}$, so the proof is complete. Note that the isomorphism $j_{\infty\infty'}$ is unique in the sense that it is the only isomorphism from $M^{(\infty)}$ to $M^{(\infty')}$ that commutes appropriately with the maps $j_{\infty\alpha}$ and $j_{\infty'\alpha}$.

The proof for inverse limits follows the same line of reasoning as the proof for direct limits, so I will not write it out. □

The simplest nontrivial examples of inverse-directed systems are systems of order type ω^*, the reverse of the order type of ω. That is to say, the index set I is the set ω ordered by the relation $\leq_I$, which is given by $n \leq_I m \iff n \geq m$.

$$\cdots \xrightarrow{j_{3,2}} M^{(2)} \xrightarrow{j_{2,1}} M^{(1)} \xrightarrow{j_{1,0}} M^{(0)}$$

Similarly, the simplest nontrivial directed systems are the systems of order type ω.

$$M^{(0)} \xrightarrow{j_{0,1}} M^{(1)} \xrightarrow{j_{1,2}} M^{(2)} \xrightarrow{j_{2,3}} \cdots$$

Given a model M of ZFC and an elementary embedding $j : M \to M$, one can generate both an inverse-directed system of order type ω^* and a directed system of order type ω and in which each object is the model M and each morphism is the map j.

$$\cdots \xrightarrow{j} M \xrightarrow{j} M \xrightarrow{j} M$$

$$M \xrightarrow{j} M \xrightarrow{j} M \xrightarrow{j} \cdots$$

I will wrap up the introduction by discussing the dependencies between the results in this chapter. Section 1.2 discusses some metamathematical preliminaries. Readers wishing to skip this section may safely assume that the entire chapter takes place under the background theory of Kelley-Morse. Section 1.3 defines the thread class and proves some lemmas of fundamental importance for the rest of the chapter. Theorem 19 of section 1.4 proves the most fundamental theorem of the entire chapter about the existence of inverse limits: the inverse limit of a system exists if and only if the system has a natural source. The remainder of section 1.4, as well as the rest of the sections of the chapter, may be read in any order, except that section 1.6, in which a system with no inverse limit is constructed, must be read before section 1.7, in which a system with an inverse limit which is a proper subset of the thread class is constructed, as section 1.7 builds on a construction from section 1.6. Section 1.8 also makes a brief reference to the construction from section 1.7, but the main ideas from section 1.8 can be understood independently of section 1.7.

1.2 Metamathematical preliminaries

Part of this section is adapted from section 1 of [HKP12].

Discussing elementary embeddings between proper class models of set theory leads to some metamathematical challenges. The usual second-order set theory is the theory NGBC of von Neumann, Gödel and Bernays, which is a conservative extension of ZFC. However, the NGBC theory does not have a class satisfaction predicate, which can lead to some difficulties. Alternatively, one can use the stronger second-order set theory KM of Kelley and Morse.

The metamathematical problems arise in part because statements M *is a model of* ZFC and $j : M \to N$ *is an elementary embedding* cannot necessarily be expressed in a first-order way in NGBC. Another problem is that NGBC does not allow for quantification over proper classes in the comprehension axiom. However, in case M and N are transitive models of ZF containing all of the ordinals, there is a partial solution, as follows.

First of all, given a transitive proper class M, one can express that M is a model of ZF in a first-order way in NGBC, using the following lemma.

Lemma 5 ([Jec03, Theorem 13.9])**.** *A transitive proper class M is a model of* ZF *if and only if it is closed under the finitely many Gödel operations and is almost universal, i.e. for every subset $X \subseteq M$, there is a set $Y \in M$ such that $X \subseteq Y$. These properties of M are expressible as a single first-order assertion using class parameter M.*

Next, given two transitive class models of ZF, each containing all of the ordinals, one can use lemma 6 below to express, in a first-order way in NGBC, that there is an elementary embedding between them. This lemma uses the following definition. A map $j : M \to N$ is **cofinal** if and only if for every set $y \in N$ there is a set $x \in M$ such that $y \in j(x)$.

Lemma 6 (Gaifman, [Gai74])**.** *Let M and N be transitive classes such that M satisfies* ZF*. If $j : M \to N$ is Δ_0-elementary and cofinal, then j is fully elementary.*

Formally, the conclusion of lemma 6 should be regarded as a lemma scheme, stating that for each metatheoretical natural number n, the map j is Σ_n-elementary. Before [Gai74], this lemma was already known in the case that both M and N are known to satisfy ZF. Gaifman proved it in the case where only M is known to satisfy ZF.

In case the model M contains all the ordinals, then from a metamathematical perspective, every elementary embedding $j : M \to N$ must be cofinal. Thus, I will adopt the convention that if M and N are transitive proper class models of ZF, by an elementary embedding $j : M \to N$ I mean a Δ_0-elementary cofinal embedding. However, in the case where M is a set and N is a proper class, I do not adopt this convention, because it is conceivable from a metamathematical perspective that there would be an elementary embedding that is not cofinal. This makes sense because given an inaccessible cardinal κ, it follows from the Reflection Theorem that there are many smaller ordinals α such that $V_\alpha \prec V_\kappa$. In this case, the inclusion map from V_α to V_κ is an elementary embedding that is not cofinal.

With lemmas 5 and 6 in mind, I next consider how to formalize the theory of direct and inverse limits of systems of elementary embeddings and proper class models of ZF, under the background theory GBC. (In the case of systems of set models, there are no metamathematical difficulties.)

Such a directed system or inverse-directed system is a collection of models of set theory and elementary embeddings, but this collection can be encoded by a single proper class. Furthermore, the following lemma will show that the inverse limit of such a system is well-founded.

Lemma 7. *Suppose that an inverse-directed system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ of well-founded models of* ZF *and elementary embeddings has an inverse limit, $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$. Then this inverse limit must be well-founded.*

Proof. Let $\alpha \in I$ be an arbitrary index. Then any infinite descending $\in$-sequence in the inverse limit model $M^{(\infty)}$ gives rise to an infinite descending $\in$-sequence in $M^{(\alpha)}$, which contradicts the fact that $M^{(\alpha)}$ is well-founded. □

Since the inverse limit model is well-founded, it follows that by applying the Mostowski Collapse, one can assume without loss of generality that this model is transitive, and one can express that $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ is an inverse limit in a first-order way using lemma 6.[4]

However, one might also want to discuss systems of proper class models for which the inverse limit is a set, or even systems in which the models are ill-founded proper classes. In these case, lemmas 5 and 6 may not suffice. Instead, there are several alternative options, depending on the context.

One option is to work in the theory KM. The theory KM allows for class satisfaction predicates, so this solves all of the difficulties. Another option is to work in NGBC but reformulate the analysis so that all of the models in question are actually set models. This is actually related to the KM approach, because given an inaccessible cardinal κ, the model $V_{\kappa+1}$, interpreted by taking the proper classes to be precisely the sets of $\in$-rank κ, satisfies the theory KM.

[4]This analysis does not suffer from circular reasoning, because only the Σ_0-elementarity of $j_{N\alpha}$ was required to establish that N was well-founded, and this Σ_0-elementarity is first-order expressible.

Another option is to work in NGBC but restrict the analysis to Σ_n elementarity for a given n or to work with schemes of formulas stating that a given map is Σ_n elementary for all n. This approach is valid because there are class satisfaction predicates for Σ_n formulas. While this approach will work in many situations, it requires greater care than the approaches of working with set models or working in KM. For instance, to express that a given system does not have an inverse limit, it would be necessary to specify a natural number n such that the system does not have a Σ_n-elementary inverse limit. This is because the negation of a scheme is not in general expressible even as a scheme.

Taking into account these benefits and drawbacks, Kelley-Morse set theory seems to be the most natural environment for studying inverse limits of models of set theory and elementary embeddings. This is because this topic frequently requires the consideration of elementary embeddings from a set to a proper class. For this reason, this chapter should be formally considered to be written in the background theory KM. However, the reader can apply the analysis from this section to determine the extent to which the results can be formalized in NGBC.

In particular, any theorem which deals only with embeddings from sets to sets and from transitive proper classes to transitive proper classes can be formalized in NGBC. Furthermore, theorems which show that an inverse limit cannot exist by showing that a map cannot be elementary for a particular formula of bounded complexity can also be formalized in NGBC.

1.3 The thread class

In this section, I define the thread class and the corresponding projection maps of a system of elementary embeddings and models of set theory, and I analyze how the thread class is related to the direct or inverse limit of the system. More generally, these definitions make sense in any category in which the objects are model-theoretic structures, or even more generally in any concrete category. However, I will restrict my definitions to categories of model-theoretic structures and elementary embeddings, as this is my main topic of study.

Definitions 8 and 9 define the thread class of an inverse-directed system and of a directed system. These two definitions are similar but different. It is possible that a system is both directed and inverse-directed, but the sense of the thread class that I wish to discuss in such a case will be clear from the context.

Definition 8. Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an *inverse-directed* system over a category $\mathcal{C}$ such that every object of $\mathcal{C}$ is a model-theoretic structure with some particular signature $\mathcal{L}_{\mathcal{C}}$. Define the **thread class**, T, of the system as follows. An element of T, also called a **thread** through the system, is an I-tuple of the form $\langle x_\alpha\rangle_{\alpha\in I}$, such that $x_\alpha \in M^{(\alpha)}$ for each index α and such that whenever $\alpha \leq \beta \in I$, it follows that $j_{\alpha\beta}(x_\alpha) = x_\beta$.

Associated with T are the **projection maps**. For each index $\beta \in I$, there is a corresponding projection map $\pi_\beta : T \to M^{(\beta)}$, which maps a thread of the form $\langle x_\alpha\rangle_{\alpha\in I}$ to x_β.

In case the objects are transitive models of set theory, the thread class is endowed with a relation $\in_T \subseteq T \times T$, defined by

$$\langle x_\alpha \rangle_{\alpha \in I} \in_T \langle y_\alpha \rangle_{\alpha \in I} \iff (\exists \beta \in I) x_\beta \in y_\beta \iff (\forall \beta \in I) x_\beta \in y_\beta.$$

More generally, one can define relations, functions, and constants on the thread class in a way analogous to the definition of $\in_T$, regardless of the language $\mathcal{L}_\mathcal{C}$.

Definition 9. Let $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ be a *directed* system over a category $\mathcal{C}$ such that every object of $\mathcal{C}$ is a model-theoretic structure with some particular signature $\mathcal{L}_\mathcal{C}$. The threads and thread class of this directed system are defined in the same way as for an inverse-directed system (as described in definition 8), with one important exception. The threads are of the form $\langle x_\alpha \rangle_{\alpha \in J}$, where J is a subset of I. The subset $J \subseteq I$ must be maximal in the sense that for every index $\alpha \in I$, if there is some index $\beta \in J$ and some set $y \in M^{(\alpha)}$ such that either $j_{\alpha\beta}(y) = x_\alpha$ or $j_{\beta\alpha}(x_\beta) = y$, then $\alpha \in J$.

The associated **inclusion maps** $\langle i_\alpha \rangle_{\alpha \in I}$ are given as follows. The map $i_\alpha : M^{(\alpha)} \to T$ takes the set $x \in M^{(\alpha)}$ to the unique thread $\langle x_\alpha \rangle_{\alpha \in J} \in T$ such that $x_\alpha = x$.

In case the objects are transitive models of set theory, then the relation $\in_T$ is given by

$$\langle x_\alpha \rangle_{\alpha \in J} \in_T \langle y_\alpha \rangle_{\alpha \in J'} \iff (\exists \alpha \in J \cap J') x_\alpha \in y_\alpha \iff (\forall \alpha \in J \cap J') x_\alpha \in y_\alpha.$$

More generally, one can define relations, functions, and constants on the thread class in a way analogous to the definition of $\in_T$, regardless of the language $\mathcal{L}_\mathcal{C}$.

In both the directed and the inverse-directed cases, given an index $\beta \in I$ and an element $x \in M^{(\beta)}$, I will say that the set x **lies on the thread** $\langle x_\alpha \rangle$ if and only if $x_\beta = x$. In the directed case, every element $x \in M_\beta$ lies on a thread — this is why the inclusion maps are total. But in the inverse-directed case, in general, not every element $x \in M^{(\beta)}$ lies on a thread. Rather, the element $x \in M^{(\beta)}$ lies on a thread if and only if it belongs to the image of the βth projection map, that is to say, $x \in \pi_\beta$ " T. Thus, the projection maps $\langle \pi_\alpha \rangle_{\alpha \in I}$ are injective but are not necessarily surjective. Those sets that do not lie on threads are precisely the ones that are not in the image of a projection map. It is useful to note that if the set x is definable in $M^{(\beta)}$ using parameters that lie on threads, then x must also lie on a thread.

In an inverse-directed system, if the maps $j_{\alpha\beta}$ are injective, I claim that an element $x \in M^{(\beta)}$ lies on a thread if and only if x is in the image of the map $j_{\alpha\beta}$ for every index $\alpha < \beta$. Clearly, if x lies on a thread, then this criterion is satisfied. For the converse, for every index γ, define x_γ as follows. Pick $\xi \leq \gamma, \beta$, and let $x_\gamma = j_{\xi\gamma}(j_{\xi\beta}^{-1}(x))$. This definition is independent of the choice of ξ, since the system is inverse-directed. The resulting tuple $\langle x_\alpha \rangle_{\alpha \in I}$ is a thread, and $x_\beta = x$.

For a directed system in the category of models of set theory and elementary embeddings (or more generally in any category whose objects are model-theoretic structures and whose morphisms are elementary embeddings, and even in some more general situations), it is well-known that the direct limit always exists, and it is given by the thread class and the inclusion maps. I sketch the proof of this fact in lemma 11.

The case for inverse-directed systems is a bit more complicated. Given an inverse-directed system of elementary embeddings and models of ZFC, the inverse limit may or may not exist.

If it does exist, then it will be given in a canonical way by either the whole thread class or by a particular type of subset of the thread class, along with the projection maps. That the inverse limit must have this form is proven in theorem 19. In section 1.5, I give some examples of situations where the inverse limit exists and is given by the entire thread class. Later, in theorems 28 and theorem 29, I will give examples where the inverse limit does not exist and where it is given by a proper subset of the thread class.

Given any concrete category $\mathcal{C}$ (i.e. a category in which the objects have underlying sets), the **forgetful functor** for $\mathcal{C}$ is a functor from $\mathcal{C}$ to the category of sets and set-maps, given by taking the objects and morphisms of $\mathcal{C}$ to the underlying sets and set-maps.

With this definition in mind, the following lemma gives an alternative characterization of the thread class and projection maps. This lemma is essentially equivalent, modulo the choice of notation, to the theorem given on page 110 of [ML10] along with the dual of that theorem.

Lemma 10. *Let $\mathcal{C}$ be a concrete category. Let F be the forgetful functor from $\mathcal{C}$ to the category of sets and set-maps. Then the direct limit of the image of a directed system under F is given by the thread class, and the inverse limit of the image of an inverse-directed system under F is also given by the thread class. To be more precise, the following conclusions hold.*

1. *Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system over $\mathcal{C}$. Let T be the thread class of the system, and let $\langle \pi_\alpha\rangle_{\alpha\in I}$ be its projection maps. Then the pair $(T, \langle \pi_\alpha\rangle_{\alpha\in I})$ is the inverse limit of the image of $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ under the forgetful functor F.*

2. *Let* $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ *be a directed system over* $\mathcal{C}$. *Let* T *be the thread class of the system, and let* $\langle i_\alpha\rangle_{\alpha\in I}$ *be its inclusion maps. Then the pair* $(T, \langle i_\alpha\rangle_{\alpha\in I})$ *is the direct limit of the image of* $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ *under the forgetful functor* F.

Proof. For this proof, I work in the category of sets and set maps, that is, assume that the forgetful functor has already been applied.

I begin with the inverse-directed case. It is clear that the pair $(T, \langle \pi_\alpha\rangle_{\alpha\in I})$ is a natural source for the system, so it suffices to verify the universal property. Suppose that $(M^{(\infty')}, \langle j_{\infty'\alpha}\rangle_{\alpha\in I})$ is also a natural source for the system. To verify the universal property, it suffices to specify a unique function $j_{\infty'\infty}$ from $M^{(\infty')}$ to T that witnesses the universal property. Such a function must necessarily be given by taking an element $x \in M^{(\infty')}$ to the corresponding thread. That is to say, $j_{\infty'\infty}(x) = \langle j_{\infty'\alpha}(x)\rangle_{\alpha\in I}$. The function $j_{\infty'\infty}$ is the unique function witnessing that the universal property is satisfied. Thus, the pair $(T, \langle \pi_\alpha\rangle_{\alpha\in I})$ is the inverse limit of the system.

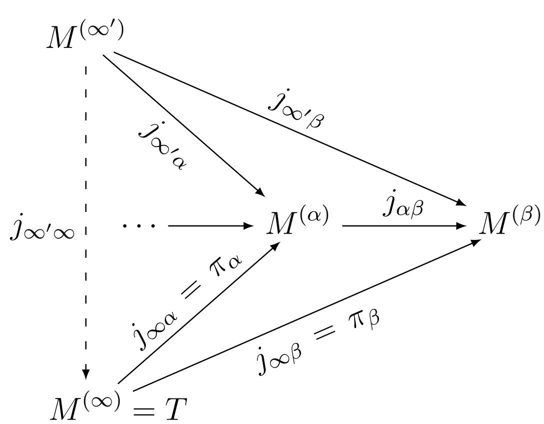


Figure 1.3: Inverse limit in the category of sets

Next, the proof for the directed case is similar. It is clear that $(T, \langle i_\alpha \rangle_{\alpha \in I})$ is a natural sink for the system. Suppose that $(N^{(\infty')}, \langle j_{\alpha\infty'} \rangle_{\alpha \in I})$ is also a natural sink for the system. It suffices to produce a unique function $j_{\infty\infty'} : T \to N^{(\infty')}$ witnessing that the universal property is satisfied. The function $j_{\infty\infty'}$ must map a thread t in accordance with its components. Therefore, the function $j_{\infty\infty'}$ is defined as follows. Given a thread $t \in T$, let $\xi \in I$ be an index such that t is in the image of the inclusion map i_ξ. Such an index must exist, and the definition of $j_{\infty\infty'}$ will not depend on which index ξ is chosen. Define $j_{\infty\infty'}(t) = j_{\xi\infty'} \circ \pi_\xi$. The function $j_{\infty\infty'}$ is indeed the unique function witnessing that the universal property is satisfied, so $(T, \langle i_\alpha \rangle_{\alpha \in I})$ is indeed the direct limit of the system $\langle N^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$.

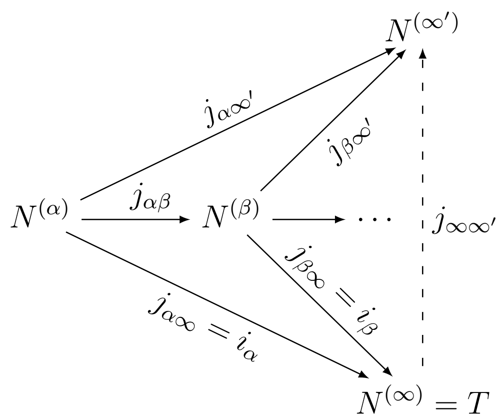


Figure 1.4: Direct limit in the category of sets

□

Next, I sketch the proof of the well-known fact that the direct limit of a system of elementary embeddings and model-theoretic structures is always given by the thread class. This lemma also applies in certain other categories, but I will restrict my attention to categories where the morphisms are elementary embeddings. The proof of essentially the same lemma is also sketched in [Jec03, lemma 12.2].

Lemma 11. *Let $\mathcal{C}$ be a category in which the objects are model-theoretic structures and the morphisms are elementary embeddings. Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be a directed system over this category with thread class T and inclusion maps $\langle i_\alpha\rangle_{\alpha\in I}$. Then $(T, \langle i_\alpha\rangle_{\alpha\in I})$ is the direct limit of this system.*

Proof sketch. The first point is to show that for any index $\alpha \in I$, the inclusion map $i_\alpha : M^{(\alpha)} \to (T, \in_T)$ is an elementary embedding, and so $\langle T, \langle i_\alpha\rangle_{\alpha\in I}\rangle$ is a natural sink. This can be verified by the Tarski-Vaught criterion.

To complete the proof, one must also show that the universal property is satisfied. This is done in almost the same way as in lemma 10. Given a natural sink $(M^{(\infty')}, \langle j_{\alpha\infty'}\rangle_{\alpha\in I})$, specify the map $j_{\infty\infty'} : T \to M^{(\infty')}$ as in lemma 10, mapping a thread in accordance with its components. There is one additional wrinkle: it is necessary to show that the map $j_{\infty\infty'} : T \to M^{(\infty')}$ is an elementary embedding. The important fact to use in verifying this elementarity is that every element of T is in the image of some inclusion map i_α. □

It is natural to ask whether the analogue to lemma 11 holds.

Question 12. *Suppose $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ is an inverse-directed system of elementary embeddings and models of set theory. Must this system have an inverse limit?*

Question 13. *Suppose $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ is an inverse-directed system of elementary embeddings and models of set theory. If the inverse limit of this system exists, must it be given by the thread class of the system?*

I will answer both of these questions in the negative in sections 1.6 and 1.7. Sometimes, the inverse limit of a system of elementary embeddings and models of ZFC does not exist at all. If it does exist, then it may be given by the entire thread class or by a proper subset of the thread class. To begin analyzing inverse limits, I define the thread form of a natural source below. I will then show in theorems 16 and 17 that the thread form of an inverse limit is a canonical form for the inverse limit.

Definition 14. Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system of elementary embeddings and models of set theory with thread class T and projection maps $\langle\pi_\alpha\rangle_{\alpha\in I}$. Let $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ be a natural source for this system. Let S be the subclass of the thread class consisting of those threads arising from elements of $M^{(\infty)}$ via the maps $j_{\infty\alpha}$. That is to say, $S = \{\, \langle j_{\infty\alpha}(x)\rangle_{\alpha\in I} \mid x \in M^{(\infty)}\,\}$. Then the structure

$$\big((S, \in_T\restriction S\times S), \langle\pi_\alpha \restriction S\rangle\big)_{\alpha\in I}$$

is the **thread form** of the natural source $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$. If, additionally, the natural source $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ is an inverse limit of the system, then $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ is the **thread form of the inverse limit** of the system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$.

More generally, suppose that $\langle X^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ is an inverse-directed system over a category $\mathcal{C}$ of model-theoretic structures and elementary embeddings. The thread form of any natural source for this system can be described analogously with the case for models of set theory, using the appropriate relations, functions, and constants of the thread class.

I will usually abbreviate the notation for the thread form of the inverse limit, writing $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ rather than $\big((S, \in_T\restriction S\times S), \langle\pi_\alpha \restriction S\rangle\big)_{\alpha\in I}$.

The following basic lemma will be helpful.

Lemma 15. *Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system of elementary embeddings and models of set theory with thread class T and projection maps $\langle\pi_\alpha\rangle_{\alpha\in I}$. Let $S\subseteq T$, and suppose that there exists some index $\xi\in I$ such that the restricted projection map $\pi_\xi\restriction S : S\to M^{(\xi)}$ is elementary. Then for every index $\alpha\in I$, the restricted projection map $\pi_\alpha\restriction S : S\to M^{(\alpha)}$ is elementary. More generally, if $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ is a natural source for the system in the category of functions and models of set theory (i.e. for the image of the system under the forgetful functor to this category), and one of the maps $j_{\infty\alpha}$ is elementary, then they are all elementary.*

Proof. Let $\alpha\in I$ be any index. Since the system is inverse-directed, there is an index $\gamma\in I$ such that $\gamma\leq\alpha,\xi$. The commutative rule $\pi_\xi\restriction S = j_{\gamma\xi}\circ\pi_\gamma\restriction S$ holds, and the maps $\pi_\xi\restriction S$ and $j_{\gamma\xi}$ are elementary by hypothesis. It follows that $\pi_\gamma\restriction S$ is also elementary. Since $\pi_\alpha\restriction S = \pi_\gamma\restriction S\circ j_{\gamma\alpha}$, it follows that $\pi_\alpha\restriction S$ is also elementary, as desired.

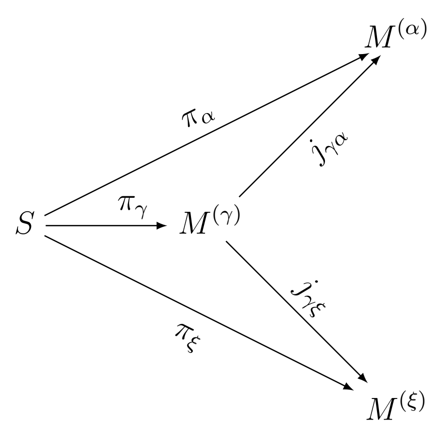


Figure 1.5: If one projection is elementary, then they all are elementary

The more general claim follows the same line of reasoning. □

The analogue of lemma 15 applies to other concrete categories (i.e. categories whose objects have underlying sets), if the following hypothesis holds. If $f : Y \to Z$ and $h : X \to Z$ are morphisms and $g : X \to Y$ is any function such that $h = f \circ g$, then g is also a morphism.

Next, theorem 16 indicates the relationship between a natural source and its thread form.

Theorem 16. *Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system of model-theoretic structures and elementary embeddings. Suppose that $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ is a natural source for this system. Let $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ be the thread form of this natural source. Let $f : M^{(\infty)} \to S$ be the function sending an element of $M^{(\infty)}$ to the corresponding thread. To be precise, this function is given by $f(x) = \langle j_{\infty\alpha}(x)\rangle_{\alpha\in I}$.*

Then the map f is an isomorphism, and for each index $\alpha \in I$, the identity $\pi_\alpha \circ f = j_{\infty\alpha}$ is true. Furthermore, if $(M^{(\infty')}, \langle j_{\infty'\alpha}\rangle_{\alpha\in I})$ is also a natural source for the system and f' and S' are defined analogously to f and S, then $S' \subseteq S$ if and only if there exists an elementary embedding $j_{\infty'\infty} : M^{(\infty')} \to M^{(\infty)}$ such that the commutative rule $j_{\infty'\alpha} = j_{\infty\alpha} \circ j_{\infty'\infty}$ holds for each index α. In this case, the first diagram shown on the next page commutes. The second diagram shown on the next page provides an example of the situation for a system of order type ω^*, illustrating the way in which a thread (suggested by arrows of the form $\Longrightarrow$) is derived from a natural source.

Moreover, the natural source $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ is an inverse limit of the system if and only if its thread form, $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ is an inverse limit of the system.

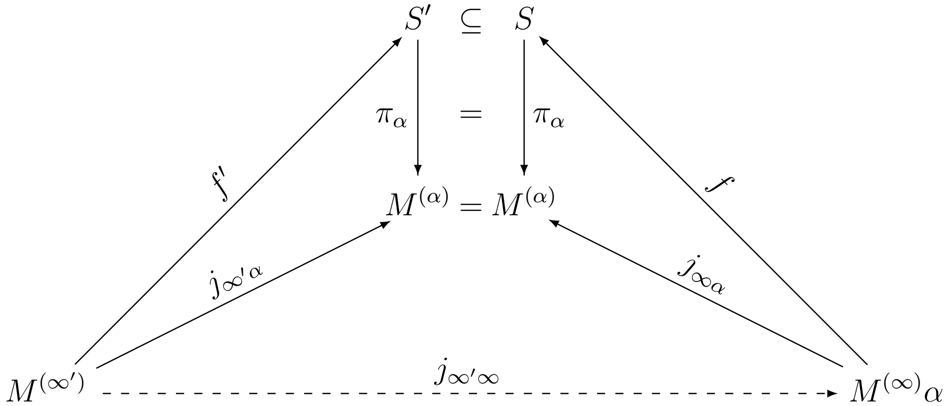


Figure 1.6: Natural sources and their thread forms, part 1

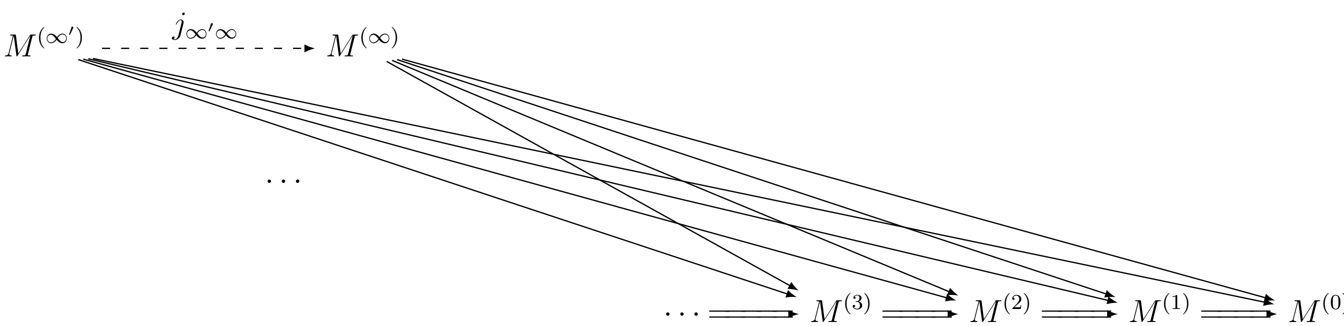


Figure 1.7: Natural sources and their thread forms, part 2

Proof. It is immediate that the function f is an isomorphism from $M^{(\infty)}$ onto S. Furthermore, by definition of f, it follows that $\pi_\alpha \circ f = j_{\infty\alpha}$ for every index $\alpha \in I$. Since f is an isomorphism, the restricted projection map $\pi_\alpha \restriction S : S \to M^{(\alpha)}$ is an elementary embedding, and so $(S, \langle \pi_\alpha \rangle_{\alpha \in I})$ is indeed a natural source.

Suppose that $(M^{(\infty')}, \langle j_{\infty'\alpha} \rangle_{\alpha \in I})$ is a natural source for the same system. The function f' and the class of threads S' are defined analogously to f and S. Thus, the natural source $(S', \langle \pi_\alpha \rangle_{\alpha \in I})$ is the thread form of $(M^{(\infty')}, \langle j_{\infty'\alpha} \rangle_{\alpha \in I})$, and $\pi_\alpha \circ f' = j_{\infty'\alpha}$ for every index α.

Suppose there exists an elementary embedding $j_{\infty'\infty} : M^{(\infty')} \to M^{(\infty)}$ such that for each index α, the commutative rule $j_{\infty'\alpha} = j_{\infty\alpha} \circ j_{\infty'\infty}$ holds. These commutative rules show that each thread coming out of $M^{(\infty')}$ corresponds to a thread coming out of $M^{(\infty)}$, and so $S' \subseteq S$. Formally speaking, one can chase through the definitions of f and f' to verify that $S' \subseteq S$ as follows.

$$S' = f' \text{ " } M^{(\infty')} = \{ (\langle j_{\infty'\alpha}(x)\rangle_{\alpha\in I}) \mid x \in M^{(\infty')} \} = \{ (\langle j_{\infty\alpha} \circ j_{\infty'\infty}(x)\rangle_{\alpha\in I}) \mid x \in M^{(\infty')} \}$$

$$= f \text{ " } (j_{\infty'\infty} \text{ " } M^{(\infty')}) \subseteq f \text{ " } M^{(\infty)} = S.$$

Conversely, suppose $S' \subseteq S$. Then the image of $j_{\infty'\alpha}$ is contained in the image of $j_{\infty\alpha}$ for each index α. Thus, it is possible to define $j_{\infty'\infty} = j_{\infty'\alpha} \circ j_{\infty\alpha}^{-1}$. It follows immediately that the map $j_{\infty'\infty}$ is elementary and satisfies the commutative rule $j_{\infty'\alpha} = j_{\infty\alpha} \circ j_{\infty'\infty}$ for each index α.

Finally, it is easy to verify, using the commutative diagram above, that the natural source $(M^{(\infty)}, \langle j_{\infty\alpha}\rangle_{\alpha\in I})$ is an inverse limit of the system if and only if its thread form $(S, \langle \pi_\alpha\rangle_{\alpha\in I})$ is an inverse limit.

□

Theorem 17 applies theorem 16 to reframe the definition of the inverse limit in terms of threads. The universal property translates into a maximality condition. In particular, theorem 17 shows that the thread form of the inverse limit is unique, if it exists (not just unique up to canonical isomorphism, but literally unique).

Theorem 17. *Let* $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ *be an inverse-directed system of model-theoretic structures and elementary embeddings with thread class* T *and projection maps* $\langle\pi_\alpha\rangle_{\alpha\in I}$*. Let* S *be a subclass of the thread class. Then the following are equivalent.*

1. *The pair* $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ *is the thread form of the inverse limit of the system.*

2. *For some index (or equivalently, for all indices)* $\xi \in I$*, the restricted projection map* $\pi_\xi \restriction S : S \to M^{(\xi)}$ *is an elementary embedding, and furthermore, the subclass* $S \subseteq T$ *is the largest subclass of* T *with this property. That is to say, given a subclass* $R \subseteq T$ *such that* $\pi_\xi \restriction R : R \to M^{(\xi)}$ *is an elementary embedding, then* $R \subseteq S$*, and in particular,* $R \prec S$*.*

Moreover, if the full projection map $\pi_\xi : T \to M^{(\xi)}$ *is an elementary embedding, then* $(T, \langle\pi_\alpha\rangle_{\alpha\in I})$ *is the thread form of the inverse limit of the system.*

Proof. I begin by proving (1) $\implies$ (2). Suppose that $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ is the inverse limit of the system. It is immediate that the restricted projection maps $\pi_\alpha \restriction S$ are all elementary. Fix some index $\xi \in I$, and let R be a subclass of T such that the restricted projection map $\pi_\xi \restriction R$ is elementary. Then all of the projection maps $\pi_\alpha \restriction R$ are elementary by lemma 15. It follows that there must be an elementary embedding $j : R \to S$ witnessing that the universal property for inverse limits is satisfied. Since this map j must satisfy the appropriate commutative rules, it must be the case that j is the inclusion map, and so S satisfies statement 2.

Next, I prove the converse, namely (2) $\implies$ (1). Suppose that $S \subseteq T$ satisfies statement 2 with respect to some index $\xi \in I$. It is immediate from the definitions that the projection maps $\langle \pi_\alpha \rangle_{\alpha \in I}$ commute with the maps $\langle j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$. To show that $(S, \langle \pi_\alpha \rangle_{\alpha \in I})$ is a natural source, I must also show that all of the restricted projection maps $\pi_\alpha \restriction S$ are elementary, but this follows immediately from lemma 15.

Finally, I verify the universal property. Towards this end, I will set $(M^{(\infty)}, \langle j_{\infty\alpha} \rangle_{\alpha \in I}) = (S, \langle \pi_\alpha \rangle_{\alpha \in I})$ for the sake of notational convenience. Let $(M^{(\infty')}, \langle j_{\infty'\alpha} \rangle_{\alpha \in I})$ be a natural source for the system. Refer to the first diagram accompanying theorem 16. The isomorphism f is given by the identity map. Let $S' = f' \text{"} M^{(\infty)}$, that is to say, $(S', \langle \pi_\alpha \rangle_{\alpha \in I})$ is the thread form of the natural source $(M^{(\infty')}, \langle j_{\infty'\alpha} \rangle_{\alpha \in I})$. By theorem 16, there is an elementary embedding $j_{\infty'\infty} : M^{(\infty')} \to M^{(\infty)}$ establishing the universal property if and only if $S' \subseteq S$. But hypothesis 2 ensures that $S \subseteq S'$ is indeed the case. It follows that $(S, \langle \pi_\alpha \rangle_{\alpha \in I})$ is indeed the thread form of the inverse limit. □

The following corollary summarizes an important point.

Corollary 18. *The inverse limit of a system $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ of elementary embeddings and models of set theory exists if and only if the system has an inverse limit in thread form, $(S, \langle \pi_\alpha \rangle_{\alpha \in I})$. This inverse limit $(S, \langle \pi_\alpha \rangle_{\alpha \in I})$ is equal to the thread form of every inverse limit of the system.*

Proof. The proof follows immediately from theorems 16 and 17. □

1.4 The inverse limit exists if and only if a natural source exists

The main idea of this section is to prove, in theorem 19, that an inverse-directed system of elementary embeddings and models of ZFC has an inverse limit if and only if it has a natural source. Furthermore, this theorem shows that the inverse limit, if it exists, must be given by a rank-initial segment of the thread class.

Theorem 19. *Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system of elementary embeddings and models of* ZFC *with thread class T and projection maps $\langle\pi_\alpha\rangle_{\alpha\in I}$. Then the following are equivalent.*

1. *The inverse limit of the system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ exists.*
2. *There exists a natural source for the system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$.*
3. *There exists a thread-form natural source for the system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$. That is to say, there exist a subclass $R\subseteq T$ such that for some index (or equivalently, for all indices) $\xi\in I$, the restricted projection $\pi_\xi\restriction R : R\to M^{(\xi)}$ is an elementary embedding.*

Assuming the inverse limit exists, let $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ be its thread form. Let $\langle\in_\alpha\rangle_{\alpha\in I}$ be the set membership relations of the models $\langle M^{(\alpha)}\rangle_{\alpha\in I}$. Given the further hypotheses that I is atomless and that $\in_\alpha=\in$ for every index $\alpha\in I$, it follows that either $S=T$ or else there is an ordinal δ such that $S={}^I(V_\delta)\cap T$. More generally speaking, without this further hypothesis, for every index $\alpha\in I$, the model π_α " S is an $\in_\alpha$-rank-initial segment of the class π_α " T.

Proof. The implication (1) $\implies$ (2) is immediate. The implication (2) $\implies$ (3) is easily proved: given a natural source for the system, its thread form (described formally in definition 14) is a thread-form natural source for the system.

Next, I prove the crux of the theorem, namely the implication, (3) $\implies$ (1). Let S be the union of all subclasses $R \subseteq T$ such that $\pi_\xi \restriction R$ is an elementary embedding. My goal is to show that $(\pi_\xi \text{ " } S) \prec M^{(\xi)}$. If these subclasses R were ordered linearly by inclusion, then this goal would be achieved immediately by the elementary chain lemma. So to achieve the goal, I will "fatten up" the classes $R^{(\xi)}$ so that the resulting fattened up classes are linearly ordered by inclusion.

Fix a subclass $R \subseteq T$ and an index $\xi \in I$ such that the restricted projection $\pi_\xi \restriction R : R \to M_\xi$ is an elementary embedding. Let $R^{(\xi)} = \pi_\xi \text{ " } R$, and let $T^{(\xi)} = \pi_\xi \text{ " } T$.

Let the closure (i.e. the fattening up) $\mathrm{cl}(R^{(\xi)})$ denote the minimal $\in_\xi$-rank-initial segment of $T^{(\xi)}$ containing $R^{(\xi)}$. That is to say, $\mathrm{cl}(R^{(\xi)})$ is the subclass of $T^{(\xi)}$ consisting of those sets $x \in T^{(\xi)}$ such that for some set $y \in R^{(\xi)}$, the model $M^{(\xi)}$ believes that the $\in_\xi$-rank of x is less than or equal to the $\in_\xi$-rank of y.

I claim that $\mathrm{cl}(R^{(\xi)}) \prec M^{(\xi)}$. I will verify this claim using the Tarski-Vaught criterion. Let φ be any formula, and suppose that

$$M^{(\xi)} \models \exists x \varphi(x, b_0, \ldots, b_n)$$

for some parameters $b_0 \ldots b_n \in \mathrm{cl}(R^{(\xi)})$. From the definition of $\mathrm{cl}(R^{(\xi)})$, it follows that there is an $M^{(\xi)}$-ordinal $\gamma \in R^{(\xi)}$ such that $b_0 \ldots b_n \in M^{(\xi)}_\gamma$. Since $R^{(\xi)} \prec M^{(\xi)}$, there exists a Skolem function $f^\gamma_\varphi \in R^{(\xi)}$ for φ whose domain consists of all n-tuples of elements of $M^{(\xi)}_\gamma$.

In particular,

$$M^{(\xi)} \models \varphi(f^{\gamma}_{\varphi}(b_0, \ldots, b_n), b_0, \ldots, b_n).$$

The function f^{γ}_{ϕ} and the parameters $b_0 \ldots b_n$ are all elements of $T^{(\xi)}$. That is to say, they all lie on threads. It follows that the witness $f^{\gamma}_{\varphi}(b_0, \ldots, b_n)$ also lies on a thread. Furthermore, since the function f^{γ}_{φ} is an element of $R^{(\xi)}$, it follows that there is an $M^{(\xi)}$-ordinal $\eta \in R^{(\xi)}$ such that for all sets $c_0, \ldots c_n \in M^{(\xi)}$, the $\in_{\xi}$-rank of $f^{\gamma}_{\varphi}(c_0, \ldots, c_n)$ (as calculated in the model $M^{(\xi)}$) is less than η. Putting everything together, I conclude that $f^{\gamma}_{\varphi}(b_0, \ldots, b_n) \in \mathrm{cl}(R^{(\xi)})$, and so the Tarski-Vaught criterion is verified, and $\mathrm{cl}(R^{(\xi)}) \prec M^{(\xi)}$.

Recall that I defined $S \subseteq T$ as the union of all subclasses $R \subseteq T$ such that $\pi_{\xi} \restriction R$ is an elementary embedding. For such a subclass $R \subseteq T$, continue to define $R^{(\xi)}$ and $\mathrm{cl}(R^{(\xi)})$ as before. Furthermore, let $\mathrm{cl}(R)$ denote that subset of T such that $\pi_{\xi} \restriction \mathrm{cl}(R) = \mathrm{cl}(R^{(\xi)})$, and let $S^{(\xi)} = \pi_{\xi}\,''\,S$.

In order to show that $(S, \langle \pi_{\alpha} \rangle_{\alpha \in I})$ is the inverse limit of the system $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$, it suffices, by theorem 17, to show that the restricted projection map $\pi_{\xi} \restriction S$ is an elementary embedding. This is equivalent to showing that $S^{(\xi)} \prec M^{(\xi)}$. But this follows immediately from the elementary chain lemma, since the class $S^{(\xi)}$ is the union of the elementary chain formed by the fattened up classes $\mathrm{cl}(R^{(\xi)})$. This completes the proof of the implication (3) $\implies$ (1).

Next, I will show that for every index α, the class $S^{(\alpha)} := \pi_{\alpha}\,''\,S$ is a rank-initial segment of $M^{(\alpha)}$. The case $\alpha = \xi$ is immediate, because $S^{(\xi)} = \mathrm{cl}(S^{(\xi)})$. More generally, since $\pi_{\alpha} \restriction S$ is elementary for every index $\alpha \in I$ (by lemma 15), the same reasoning that shows

$S^{(\xi)} = \mathrm{cl}(S^{(\xi)})$ also shows that $S^{(\alpha)} = \mathrm{cl}(S^{(\alpha)})$, and so $S^{(\alpha)}$ is indeed an $\in_\alpha$-rank-initial segment of $T^{(\alpha)}$.

In particular, if the models $M^{(\alpha)}$ have the same set membership relation as V, then for every index $\xi \in I$, either $S^{(\xi)} = T^{(\xi)}$ or else there is an ordinal δ such that $S^{(\xi)} = T^{(\xi)} \cap V_\delta$. Furthermore, if the partial order I is atomless, then for any thread $t \in T$ and for any index α, the $\in$-rank of $\pi_\alpha(t)$ must be the same as the $\in$-rank of $\pi_\xi(t)$, as otherwise the foundation axiom would be contradicted. This completes the proof. □

Next, I present some lemmas about the structure of the thread class. Lemmas 20 and 21 characterize the thread-form natural sources of a system. In particular, lemma 20 characterizes those situations in which the inverse limit is the entire thread class.

Lemma 20. *Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system of elementary embeddings and models of* ZFC *with thread class T and projection maps $\langle\pi_\alpha\rangle_{\alpha\in I}$. Let $\in_T$ be the set membership relation of the thread class. Then $(T, \langle\pi_\alpha\rangle_{\alpha\in I})$ is the inverse limit of the system if and only if for every thread $t \in T$ consisting of nonempty sets at every coordinate, there is a thread $s \in T$ such that $s \in_T t$.*

Proof. For the forwards direction, if $(T, \langle\pi_\alpha\rangle_{\alpha\in I})$ is the inverse limit of the system, then the structure $(T, \in_T)$ satisfies ZFC and in particular satisfies the axiom of extensionality, and so every thread of nonempty sets has a member with respect to the relation $\in_T$, and the desired result follows.

For the converse, let $\xi \in I$ be an arbitrary index. In light of theorem 17, it suffices to show that the projection map $\pi_\xi : T \to M_\xi$ is elementary. In other words, taking $T^{(\xi)} = \pi_\xi \text{ " } T$, it suffices to show that $T^{(\xi)} \prec M^{(\xi)}$. Recall from the comments following definition 14 that $T^{(\xi)}$ consists of precisely those elements of $M^{(\xi)}$ that lie on threads.

To prove the desired elementarity relation, I will verify the Tarski-Vaught criterion. It suffices to consider formulas with a single parameter, because if the parameters $b_0 \ldots b_n$ all lie on threads, then so does the tuple $(b_0, \ldots b_n)$. Suppose that for some formula φ and for some parameter $b \in T^{(\xi)}$,

$$M^{(\xi)} \models (\exists x)\varphi(x, b).$$

Since the set b lies on a thread, so does the set of witnesses of minimal rank,

$$w = \{\, x \in M^{(\xi)} \mid M^{(\xi)} \models \varphi(x, b) \text{ and } x \text{ is of minimal } \in\text{-rank} \,\},$$

and this set is nonempty in $M^{(\xi)}$. Let $t \in T$ be the thread on which this set lies. It follows from the hypothesis of the lemma that there is a thread $s \in T$ such that $s \in_T t$. It follows from the definition of $\in_T$ that $\pi_\xi(s) \in w$, and so $M^{(\xi)} \models \varphi(x, \pi_\xi(s))$. □

Lemma 21 applies the ideas of lemma 20 to rank-initial subsets of the thread class. For the sake of clarity, I will state it in the context where the set membership relations of the models $M^{(\alpha)}$ are the same as the $\in$ relation of V, but it can be generalized to accommodate ill-founded models as well.

Lemma 21. *Let $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ be an inverse-directed system of elementary embeddings and transitive models of set theory with thread class T and projection maps π. Let δ be a limit ordinal. Let $\xi \in I$ be an index. Let R be a subset of T such that $\pi_\xi \text{ "}\, R = V_\delta \cap (\pi_\xi \text{ "}\, T)$.*

Then the restricted projection map $\pi_\xi \upharpoonright R : R \to M^{(\xi)}$ is an elementary embedding if and only if both of the following two conditions hold.

- *For every thread $t \in R$ consisting of nonempty sets at every coordinate, there is a thread $s \in R$ such that $s \in_T t$.*
- $M_\delta^{(\xi)} \prec M^{(\xi)}$

Proof. For the forwards direction, suppose that the restricted projection $\pi_\xi \upharpoonright R$ is elementary. Then the structure $(R, \in_T)$ satisfies ZFC and in particular the axiom of extensionality, and so every thread of nonempty sets in R has a $\in_T$ member.

To see that $M_\delta^{(\xi)} \prec M^{(\xi)}$, I will verify the Tarski-Vaught criterion. It suffices to consider formulas with a single parameter, because the model $M_\delta^{(\xi)}$ is closed under pairing, since δ is a limit ordinal. Let φ be a formula, let $b \in M_\delta^{(\xi)}$ be a parameter, and suppose that $M^{(\xi)} \models (\exists x)\varphi(x, b)$. Let $R^{(\xi)} = \pi_\xi$ " R. Since $R^{(\xi)} \prec M^{(\xi)}$, it follows that for every ordinal $\gamma < \delta$, there is a Skolem function $f_\varphi^\gamma \in R^{(\xi)}$ with domain $M_\gamma^{(\xi)}$ providing witnesses for the satisfaction of φ in the model $M^{(\xi)}$. In particular, if $b \in V_\gamma$, then $M^{(\xi)} \models \varphi(f_\varphi^\gamma(b), b)$. Furthermore, since the Skolem function f_φ^γ is an element of $R^{(\xi)} \subseteq M_\delta^{(\xi)}$, the model $R^{(\xi)}$ can compute the supremum of the $\in$-ranks of the outputs of this function, and this supremum is less than δ. It follows that $f_\varphi^\gamma(b) \in M_\delta^{(\xi)}$, and the Tarski-Vaught criterion is satisfied.

To prove the converse direction of the lemma, suppose that the two conditions of the lemma hold. It suffices to verify that $R^{(\xi)} \prec M^{(\xi)}$. Again, I verify the Tarski-Vaught criterion. The set $R^{(\xi)}$ is closed under pairing, so it suffices to consider formulas with one parameter. Suppose that $M^{(\xi)} \models (\exists x)\varphi(x,b)$. Since the parameter b lies on a thread, the set w of witnesses x of minimal rank satisfying the formula $\varphi(x,b)$ in $M^{(\xi)}$ also lies on a thread. Furthermore, this minimal $\in$-rank must be less than δ, because $M^{(\xi)}_\delta \prec M^{(\xi)}$. It follows that $t \in R$. By hypothesis, there is a thread $s \in R$ such that $s \in_T t$. Thus, $\pi_\xi(s) \in w$ is a witness, and the proof is complete. $\square$

Lemma 22 shows that in simple cases, if thread of nonempty sets has an $\in_T$-member, as in the hypothesis of lemma 20, then every thread is constant. This lemma draws upon an idea from theorem 23 of [HKP12]. I state it as a lemma about models of set theory, but really the models only need to satisfy a weak fragment of set theory in order for the lemma to work.

Lemma 22. *Let* $\ldots M^{(2)} \subseteq M^{(1)} \subseteq M^{(0)}$ *be models of set theory. Let* $j : M^{(1)} \to M^{(0)}$ *be an elementary embedding, such that for each natural number* n*, the restricted function* $j \restriction M^{(n+1)}$ *maps elementarily from* $M^{(n+1)}$ *to* $M^{(n)}$ *These embeddings give rise to an inverse-directed system of the following form.*

$$\cdots \xrightarrow{\ j\ } M^{(2)} \xrightarrow{\ j\ } M^{(1)} \xrightarrow{\ j\ } M^{(0)}$$

Let T *denote the thread class of this system. Suppose that every thread of nonempty sets has an* $\in_T$*-member. Then it follows that every element* $\langle t_n\rangle_{n\in\omega}$ *of* T *is a constant thread. That is to say,* $t_n = t_0$ *for every natural number* n.

Proof. Suppose that the hypotheses are satisfied, but there exists a nonconstant thread, $\langle x_n\rangle_{n\in\omega} \in T$. Without loss of generality, assume $\langle x_n\rangle_{n\in\omega}$ is such that x_0 is of minimal $\in$-rank among all such threads. It must be the case that $\text{rank}_\in(x_0)$ is a fixed point of the embedding j, as otherwise $\langle \text{rank}_\in(x_n) \mid n < \omega \rangle$ would be an infinite descending sequence of ordinals. Let $d_n = x_n \triangle x_{n+1}$. Then $\langle d_n \rangle_{n \in \omega}$ is a thread, because the models $M^{(n)}$ are transitive and $M^{(n+1)} \subseteq M^{(n)}$ for all n. The thread $\langle x_n \rangle_{n \in \omega}$ does not equal the thread of empty sets, so by hypothesis, there is some thread $\langle y_n \rangle_{n \in \omega}$ such that $\langle y_n \rangle_{n \in \omega} \in_T \langle x_n \rangle_{n \in \omega}$. It follows that $\text{rank}_\in(y_0) < \text{rank}_\in(x_0)$. However, I will show that $\langle y_n \rangle_{n \in \omega}$ is not constant, contradicting the minimality of $\text{rank}_\in(x_0)$. Either $y_0 \in x_0 - x_1$ and $y_1 \in x_1 - x_2$, or else $y_0 \in x_1 - x_0$, in which case $y_1 \in x_2 - x_1$. In either of these two cases, $y_0 \neq y_1$, and so the proof is complete. □

1.5 Situations where the inverse limit exists and is given by the entire thread class

In this section, I prove some results that give examples of inverse-directed systems for which the inverse limit exists and is given by the entire thread class. I begin with the following simple application of lemma 20.

Theorem 23. *Let $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ be an inverse-directed system of elementary embeddings and models of* ZFC *with thread class T and projection maps $\langle \pi_\alpha \rangle_{\alpha \in I}$. Suppose that for some index $\alpha \in I$ (or equivalently, for all such indices), the model $M^{(\alpha)}$ satisfies $V = \text{HOD}$. Then the inverse limit of this system exists and has thread form $(T, \langle \pi_\alpha \rangle_{\alpha \in I})$.*

Proof. By theorem 19, it suffices to show that every nonempty thread in T has an $\in_T$-member. Let $\langle x_\alpha \rangle_{\alpha \in I} \in T$ be a thread. For each index $\alpha \in I$, let y_α be the $HOD^{M^{(\alpha)}}$-least element of x_α. Then $\langle y_\alpha \rangle_{\alpha \in I}$ is a thread, and $\langle y_\alpha \rangle_{\alpha \in I} \in_T \langle x_\alpha \rangle_{\alpha \in I}$. □

The following lemma will be useful for producing examples of inverse-directed systems in the present section and also in sections 1.6 and 1.7.

Lemma 24. *Suppose that in some model $N^{(0)}$ of set theory, the measure $\mu = \mu_0$ is a normal measure on a cardinal κ. Let $j_\mu : N^{(0)} \to N^{(1)}$ be the ultrapower embedding generated by μ. Let $N^{(\omega)}$ be the ωth model in the system of iterated ultrapowers generated by μ.*

$$N^{(0)} \xrightarrow{j_{\mu_0}} N^{(1)} \xrightarrow{j_{\mu_1}} N^{(2)} \xrightarrow{j_{\mu_2}} \cdots \longrightarrow N^{(\omega)}$$

Then $j_\mu(\mu_\omega) = \mu_\omega$, and the restricted embedding $j_\mu \restriction N^{(\omega)} : N^{(\omega)} \to N^{(\omega)}$ is elementary.

Proof. Let φ be a formula, and let $x \in N^{(\omega)}$ be a parameter. By the elementarity of the embedding j, it follows that the model $N^{(0)}$ believes that $\varphi(x)$ is true in the model given by the ωth iterate of the measure μ if and only if the model $N^{(1)}$ believes that the formula $\varphi(j_\mu(x))$ is true in the model given by the ωth iterate of the measure $j_\mu(\mu)$. However, the ωth iterate of μ as calculated in $N^{(0)}$ is the same model as the ωth iterate of $j_\mu(\mu)$ as calculated in $N^{(1)}$; namely, both of these models are equal to $N^{(\omega)}$, and the same reasoning shows that $j_\mu(\mu_\omega) = \mu_\omega$. □

More generally, lemma 24 applies to systems of iterated ultrapowers generated by other types of normal measures as well, for instance, supercompactness measures.

Theorem 25. *In the model $N^{(0)}$, let μ be a normal measure on a cardinal κ. Let the system $\langle N^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \le \beta \le \omega}$ be a system of iterated ultrapowers of length ω generated by μ, as in the diagram accompanying lemma 24. Then there exists an inverse-directed system of order type*

ω^* *of the form*

$$\cdots \xrightarrow{j_\mu} N^{(\omega)} \xrightarrow{j_\mu} N^{(\omega)} \xrightarrow{j_\mu} N^{(\omega)}.$$

The inverse limit of this system exists and has form $(N^{(0)}, \langle j_{0\omega} \rangle_{n<\omega})$*, that is to say, the inverse limit is given by the model* $N^{(0)}$ *along with the embedding* $j_{0\omega}$ *at every coordinate.*

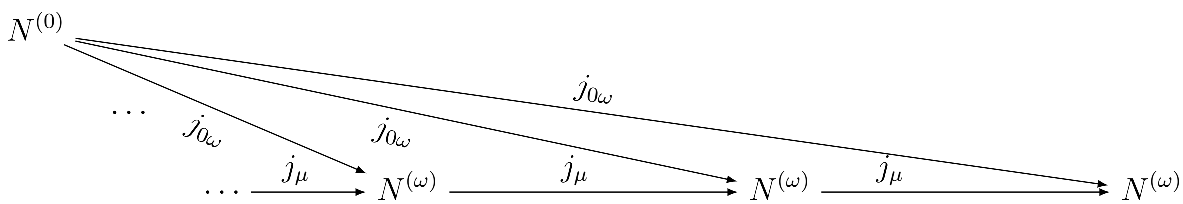


Figure 1.8: A system formed by restricting j_μ to $N^{(\omega)}$

Furthermore, the thread form of this inverse limit is given by the entire thread class, which consists entirely of constant threads.

Proof. Before proving the theorem, I will address a notational ambiguity. For the rest of the proof, the signifiers $j_{\alpha\beta}$ representing elementary embeddings should be taken as *de re* (i.e. absolute) signifiers. For instance, if I say that $N^{(2)} \models j_{2,5}(x) = y$, then the signifier $j_{2,5}$ refers to $j_{2,5}$ *de re*, not to $N^{(2)}$'s *de dicto* version of $j_{2,5}$, which would be denoted *de re* as $j_{4,7}$.

The inverse-directed system indicated in the theorem statement is well-defined due to lemma 24. In light of theorems 16 and 17 and lemma 22, it suffices to show that the pair $(N^{(0)}, \langle j_{0\omega} \rangle_{n<\omega})$ is a natural source for the system and that its thread form is given by the entire thread class.

For each natural number n, let $j_n = j_{n+1,n}$. Let T denote the thread class of the system, and let $\langle y_n \rangle_{n\in\omega} \in T$. Since $\langle y_\alpha \rangle_{\alpha\in I}$ is a thread, it follows by definition that $j_0(y_{n+1}) = y_n$

for all $n \in \omega$. I will first show that in fact for all natural numbers $m, n \in \omega$, the identity $j_m(y_{n+1}) = y_n$ holds. The proof is by induction on m. The base case has just been verified.

For the inductive step, suppose that for all n, the identity $j_m(y_{n+1}) = y_n$ holds. Then

$$j_{m+1}(y_{n+1}) = j_m(j_m)(j_m(y_{n+2})) = j_m(j_m(y_{n+2})) = y_n.$$

The first equality follows from the definition of j_m and the inductive assumption, the second from the commutative square for elementary embeddings, and the third again from the inductive assumption. I conclude that $j_m(y_{n+1}) = y_n$ for all natural numbers m and n. By repeatedly applying this fact, it also follows that $j_{0m}(y_m) = y_0$.

Next, I show that the thread $\langle y_n \rangle_{n \in \omega}$ is actually constant, that is, $y_n = y_0$ for all n. Furthermore, I will show that there is a set $x \in M^{(0)}$ such that each y_n is equal to $j_{0\omega}(x)$. First, since $y_0 \in N^{(\omega)}$, it must be "born" somewhere, that is to say, $y_0 = j_{m\omega}(x)$ for some particular natural number m and set x. In other words, the model $N^{(m)}$ believes that there is some set x such that y_0 is the image of x under the ωth iterate of μ_m.[5] By applying the inverse of the map j_{0m} to this statement and taking into account the fact from above that $j_{0m}(y_m) = y_0$, it follows that $N^{(0)}$ believes that there is some x such that y_m is the image of x under the ωth iterate of μ_0. In other words, there is some particular set x such that $N^{(0)} \models y_m = j_{0,\omega}(x)$. By applying the embedding j_0 to this statement, it follows that $N_1 \models j_0(y_m) = j_{1\omega}(j_0(x))$. But since $j_{1\omega}(j_0(x)) = j_{0\omega}(x) = y_m$, it follows that $j_0(y_m) = y_m = j_{0\omega}(x)$, and therefore the thread $\langle y_n \rangle_{n \in \omega}$ is constant as claimed.

This same argument shows that for *any* set $x \in N^{(0)}$, the identity $j_0 \circ j_{0,\omega}(x) = j_{0,\omega}(x)$

[5]Here, μ_m is a *de re* variable name, as described in the comments preceding the theorem.

holds. It follows that $(N^{(0)}, \langle j_{0\omega} \rangle_{n<\omega})$ is a natural source and that its thread form is given by the entire thread class. This completes the proof. $\square$

Theorem 25 shows that theorem 23 cannot be extended to a biconditional, since the existence of a measurable cardinal does not imply $V = \mathrm{HOD}$.

1.6 A situation where the inverse limit does not exist

In this section, I answer question 12 in the negative by exhibiting an inverse-directed system of models of set theory and elementary embeddings with no inverse limit. I first introduce a useful lemma for proving the nonexistence of inverse limits. I then use this lemma to produce an example of a system with no inverse limit, using a Prikry forcing construction.

Lemma 26. *Let M be a transitive model of* ZFC*, and suppose that there is an inverse-directed system of elementary embeddings of order type ω^*, as follows.*

$$\cdots \xrightarrow{j_2} M \xrightarrow{j_1} M \xrightarrow{j_0} M$$

Suppose that there are ordinals κ and λ such that for every embedding j_m, the critical point of j_m is κ, and the least ordinal fixed point of j_m above κ is λ. Suppose that $\mathrm{cof}(\lambda)^M < \kappa$*. If the system has an inverse limit with thread form $(S, \langle \pi_n \rangle_{n\in\omega})$, then the constant thread given by λ is not an not an element of the thread form of the inverse limit. In other words, $\lambda \notin \pi_0 \,''\, S$. In particular, if λ is definable in M, then the system has no inverse limit.*

Proof. Suppose to the contrary that $(S, \langle \pi_n \rangle_{n\in\omega})$ is the thread form of the inverse limit of the system and that $\lambda \in S^{(0)}$, where $S^{(0)} = \pi_0 \,''\, S$. By elementarity, the model $S^{(0)}$ believes that

λ has cofinality $\mathrm{cof}(\lambda)^M$, and so in $S^{(0)}$ there is a sequence of ordinals, $\sigma : \mathrm{cof}(\lambda)^M \to \lambda$, such that the sequence σ is cofinal in λ. Let α_0 denote the least element in σ above κ. Let ξ be the ordinal such that $\alpha_0 = \sigma(\xi)$. Since $\xi < \kappa$ and σ lies on a thread, it follows that α_0 lies on a thread, $\langle \alpha_n \rangle_{n\in\omega}$. The thread $\langle \alpha_n \rangle_{n\in\omega}$ must be eventually constant as $n \to \omega$, as otherwise the foundation axiom is contradicted. However, $\kappa < \alpha_0 < \lambda$, and λ is the least fixed point of each j_{mn} above κ, so the thread $\langle \alpha_n \rangle_{n\in\omega}$ cannot be eventually constant as $n \to \omega$. This is a contradiction, so it follows that $\lambda \notin \pi_0$ " S. In particular, if λ is definable in M, then λ is in the image of any elementary embedding mapping into M, so the restricted projection map $\pi_0 \restriction S$ cannot be elementary, and the inverse limit does not exist. $\square$

A quick corollary for the case of models of $V = \mathrm{HOD}$ follows.

Corollary 27. *Suppose M is a transitive model of $ZFC + V = HOD$, and let $j : M \to M$ be an elementary embedding with critical point κ. Then if j has a fixed point above κ, the least fixed point of j above κ, if it exists, is regular in M.*

Proof. Let λ be the least fixed point of j above κ. By theorem 23, the inverse-directed system

$$\cdots \xrightarrow{\ j\ } M \xrightarrow{\ j\ } M \xrightarrow{\ j\ } M$$

of order type ω^* has an inverse limit, and the thread form of this inverse limit is given by the entire thread class. In particular, the thread with constant value λ is an element of the inverse limit. It follows from lemma 26 that the cofinality of λ in M must be at least κ. Furthermore, this cofinality must lie on a thread, since λ lies on a thread. Every thread of ordinals must be constant, as otherwise the foundation axiom is contradicted. Therefore, the cardinal λ is regular in the model M. $\square$

Next, I will review some ideas related to Prikry forcing. Prikry forcing was invented by Karel Prikry in his dissertation, [Pri68]. For a modern account, see [Jec03, pp. 400 - 404]. For an extended account of Prikry forcing and of many related forcing notions, including iterated Prikry forcing, Magidor forcing, and Radin forcing, see [Git10]. Prikry forcing adds a cofinal ω-sequence to a measurable cardinal κ, while preserving cardinals and adding no new sets to V_κ.

The Prikry forcing, $\mathbb{P}$, is defined with respect to a particular normal measure, μ, on the measurable cardinal κ. The conditions are ordered pairs of the form (s, B). The **stem**, s, is a finite sequence of ordinals below κ, and B is a set in μ such that the least element of B is above the greatest element of S. The partial order is defined by end-extending the stems and shrinking the measure-one sets in such a way that any new ordinals added to the stem are members of the larger measure-one set. Given two conditions $p \leq q$, the condition p is a **direct extension** of the condition q if and only if $p \leq q$ and p and q have the same stems. Prikry forcing satisfies the **Prikry property**, that is to say, given any condition p and any formula φ in the forcing language, there is a direct extension of p deciding φ.

The **Mathias criterion** states that an ω-sequence $s \subseteq \kappa$ is $\mathbb{P}$-generic if and only if every set in μ contains all but finitely many elements of the sequence s ([Mat73], see also [Jec03, theorem 21.14]).

Let μ be a normal measure on κ in a transitive model N_0. Consider the corresponding system of iterated ultrapowers, $\langle N^{(\xi)}, \kappa_\xi, \mu_\xi \rangle_{\xi \in \mathrm{ORD}}$. An application of the Mathias criterion shows that the sequence $\langle \kappa_n \rangle_{n<\omega}$ is $N^{(\omega)}$-generic for the Prikry forcing defined in N_ω with respect to κ_ω and μ_ω (see [Jec03, theorem 21.15]).

This genericity result plays a key role in the proof of theorem 28 below. Lemma 26 also plays a key role.

Theorem 28. *Let $\kappa = \kappa_0$ be an ordinal, and let μ_0 be a subset of $P(\kappa)$ such that*

$$L[\mu_0] \models \mu_0 \text{ is a normal measure on } \kappa.$$

Let $\langle L[\mu_\xi], \kappa_\xi, \mu_\xi \rangle_{\xi \le \omega}$ be the iteration in $L[\mu_0]$ of ultrapowers by the measure μ_0.

$$L[\mu_0] \xrightarrow{j_{\mu_0}} L[\mu_1] \xrightarrow{j_{\mu_1}} L[\mu_2] \xrightarrow{j_{\mu_2}} \cdots \longrightarrow L[\mu_\omega]$$

Then there is an inverse-directed system of order type ω^* of the form

$$\cdots \xrightarrow{j_{\mu_0}} L[\mu_\omega] \xrightarrow{j_{\mu_0}} L[\mu_\omega] \xrightarrow{j_{\mu_0}} L[\mu_\omega].$$

This system lifts over Prikry forcing to produce a system of the form

$$\cdots \xrightarrow{j} L[\mu_\omega][s] \xrightarrow{j} L[\mu_\omega][s] \xrightarrow{j} L[\mu_\omega][s],$$

where $s = \langle \kappa_n \rangle_{n \in \omega}$ is the critical sequence of the system of iterated ultrapowers, and this lifted system has no inverse limit.

Proof. Let $\mathbb{P}$ be the Prikry forcing with respect to κ_ω and μ_ω, as defined in $L[\mu_\omega]$. Consider the sequence $t = \langle \kappa_{n+1} \rangle_{n \in \omega}$. The result cited just before the theorem shows that the sequence

$s = \langle \kappa_n \rangle_{n \in \omega}$ is $\mathbb{P}$-generic over $L[\mu_\omega]$. It follows by the Mathias criterion that the sequence t is also $\mathbb{P}$-generic over $L[\mu_\omega]$. Furthermore, since every transitive model of ZFC containing s contains t and vice-versa, it follows that the Prikry sequence s and t give rise to the same forcing extension $L[\mu_\omega][s] = L[\mu_\omega][t]$.[6]

In the model $L[\mu_\omega]$, the cardinal κ_ω is the least ordinal that is measurable in an inner model. This can be formulated as a first-order definition: the ordinal κ_ω is the least ordinal α such that there is a subset $U \subseteq \mathcal{P}(\alpha)$ so that α is measurable in $L[U]$. To see that κ_ω is indeed the least such ordinal, suppose there were a smaller such ordinal, δ, and a set $U \subseteq P(\delta)$ such that U were a normal measure on δ in $L[U]$ and $U \in L[\mu_\omega]$. Then $L[\mu_\omega]$ would be an iterated ultrapower of $L[U]$ (see [Jec03, theorem 19.14]), but $L[U]$ would also be a subclass of $L[\mu_\omega]$, which is a contradiction.

Since $\mathbb{P}$ is Prikry forcing with respect to κ_ω, it follows that $L[\mu_\omega][s] \cap V_{\kappa_\omega} = L[\mu_\omega] \cap V_{\kappa_\omega}$. It further follows that κ_ω is definable in $L[\mu_\omega][s]$ as the least ordinal which is measurable in an inner model.

Let $j_{\mu_0} : L[\mu_0] \to L[\mu_1]$ be the ultrapower embedding generated by the measure μ_0. From lemma 24, it follows that the restricted map $j_{\mu_0} \upharpoonright L[\mu_\omega] : L[\mu_\omega] \to L[\mu_\omega]$ is an elementary embedding, that $j_{\mu_0}(\mu_\omega) = \mu_\omega$ and that $j_{\mu_0}(\mathbb{P}) = \mathbb{P}$. Furthermore, $j_{\mu_0}(\kappa_n) = \kappa_{n+1}$, for all natural numbers n. It follows that the lifting criterion $j_{\mu_0}{}'' G_s = G_t$ is satisfied, where G_s and G_t generic filters generated by the Prikry sequences s and t respectively. Therefore, the embedding $j_{\mu_0} \upharpoonright L[\mu_\omega]$ lifts over $\mathbb{P}$ to an elementary embedding $j : L[\mu_\omega][s] \to L[\mu_\omega][s]$.

[6] By the Mathias criterion, the measure μ_ω is definable from s, so $L[\mu_\omega][s] = L[s]$, but I use the notation $L[\mu_\omega][s]$ to emphasize that this model is a Prikry forcing extension of $L[\mu_\omega]$

The cardinal κ_ω is the least ordinal fixed point of the lifted embedding j above its critical point κ_0. Furthermore, as previously noted, κ_ω is definable in $L[\mu_\omega][s]$ without parameters as the least ordinal that is measurable in an inner model. Therefore, if the inverse system of order type ω^* generated by j has an inverse limit with thread form $(S, \langle \pi_\alpha \rangle_{\alpha \in I})$, then κ_ω must be an element of π_0 " S. It follows from lemma 26 that this system has no inverse limit. □

In the proof of theorem 28, the reason for working in $L[\mu_\omega]$ instead of in V was to ensure that the ordinal κ_ω was definable without parameters. More generally, this theorem could be carried out in other models, so long as κ_ω is definable without parameters or with parameters fixed by j_{μ_0}. However, if the technique from the theorem is carried out in such a way that κ_ω is not definable, then the resulting system may have an inverse limit. This is the topic of the next section.

Note that before the Prikry forcing, the inverse system generated by $j_\mu : L[\mu_\omega] \to L[\mu_\omega]$ did have an inverse limit. This fact can be proven by a direct application of either theorem 23 or theorem 25.

1.7 A situation where the inverse limit exists and is given by a proper subset of the thread class

In this section, I answer question 13 in the negative, showing that it is possible to have an inverse-directed system of elementary embeddings and models of set theory for which the inverse limit exists but is given by a proper subset of the thread class. My technique is to build on the Prikry forcing construction from the previous section.

One might think that doing the Prikry forcing construction from the proof of theorem 28 in any model of set theory with a measurable cardinal (rather than just in $L[\mu]$) would produce an inverse-directed system with no inverse limit. But it turns out that such a construction with an inverse limit is possible, as I will show in the present section.

Theorem 29. *Suppose that the cardinal κ is 1-extendible. Then there is an inverse-directed system of elementary embeddings and transitive set models of set theory of order type ω^* such that the inverse limit of this system exists, but this inverse limit is given by a proper subset of the thread class of the system.*

The remainder of this section will be devoted to proving theorem 29. This proof is rather lengthy and requires several lemmas.

Let μ be a normal measure on a cardinal $\kappa = \kappa_0$. Consider the directed system of iterated ultrapower embeddings generated by μ, beginning in $V = N^{(0)}$. (More generally, $N^{(0)}$ could be any model of set theory in which μ is a normal measure.)

$$N^{(0)} \xrightarrow{j_{\mu_0}} N^{(1)} \xrightarrow{j_{\mu_1}} N^{(2)} \xrightarrow{j_{\mu_2}} \cdots \longrightarrow N^{(\omega)}$$

According to lemma 24, the restricted embedding $j_\mu \restriction N^{(\omega)} : N^{(\omega)} \to N^{(\omega)}$ is elementary, and so there is an inverse-directed system as follows.

$$\cdots \xrightarrow{j_\mu} N^{(\omega)} \xrightarrow{j_\mu} N^{(\omega)} \xrightarrow{j_\mu} N^{(\omega)}.$$

I would like to use the techniques from the previous section but to ensure that the cardinal $\kappa_\omega = j_{0,\omega}(\kappa)$ is not definable without parameters in the model $N^{(\omega)}[s]$ (i.e. the forcing extension of $N^{(\omega)}$ by Prikry forcing with respect to κ_ω). One particularly strong way of achieving this goal is if the elementarity relation $N^{(\omega)}_{\kappa_\omega} \prec N^{(\omega)}[s]$ is satisfied. This idea can be modified slightly by chopping off the universe at an inaccessible cardinal θ above κ_ω and seeking to satisfy the elementarity relation $N^{(\omega)}_{\kappa_\omega} \prec N^{(\omega)}[s]_\theta$. This chopping off simplifies matters from a metamathematical perspective. It turns out that this elementarity relation is sufficient to achieve the desired example; this is essentially the content of lemma 31. After proving lemma 31, the remainder of the section will be devoted to satisfying the hypothesis of lemma 31.

As a preliminary, I present a simple lemma about Prikry forcing, lemma 30. The conclusion of lemma 30 is the same as the consequence usually derived from the almost-homogeneity of a forcing notion. It is known as folklore that one-step Prikry forcing, as defined on page 41, admits no nontrivial automorphisms.[7] However, in [BK79, lemma 2.2], it is shown that there is an almost-homogeneous partial order that is forcing-equivalent to Prikry forcing.[8] Lemma 2.2 of [BK79] can be seen as an alternative proof of my lemma 30.

Lemma 30. *Let $\mathbb{P}$ be a notion of Prikry forcing. Let φ be a formula in the forcing language using only check-names (i.e. canonical names for elements of the ground model) as parameters. Suppose that some condition $(s, B) \in \mathbb{P}$ forces φ. Then every condition in $\mathbb{P}$ forces φ.*

Proof. Suppose that some condition $(s, B) \in \mathbb{P}$ forces φ. To prove the lemma, it suffices to

[7] Arthur Apter told me in personal communication that E.L. Bull related the rigidity of Prikry forcing to Apter many years ago.

[8] Lemma 2.2 of [BK79] actually proves a more general fact about iterated Prikry forcing; the result stated above is the special case of that lemma for a one-step iteration.

show that if $(s', B') \in \mathbb{P}$ is another condition, then (s', B') does not force $\neg\varphi$. Let g be a V-generic Prikry sequence for $\mathbb{P}$ such that g is compatible with the condition $(s, B \cap B')$. Let g' be another Prikry sequence obtained by modifying finitely many coordinates of g, so that g' is compatible with the condition $(s', B \cap B')$. It follows that g' is also compatible with the weaker condition (s', B'). Since only finitely many coordinates of the Prikry sequence were modified, it follows that $V[g] = V[g']$. Furthermore, if the check-name $\check{x}$ is a parameter in the formula φ, then its interpretations by g and g' are the same. It follows that the truth value of the formula φ^g in the model $V[g]$ is the same as the truth value of the formula $\varphi^{g'}$ in the model $V[g']$. Since g is compatible with (s, B), and $(s, B) \Vdash \varphi$, it follows that $V[g'] \models \varphi^{g'}$. Therefore, it cannot be the case that (s', B') forces $\neg\varphi$, and the proof is complete. □

In particular, it follows from lemma 30 that if $\mathbb{P}$ is a notion of Prikry forcing with respect to some cardinal κ and there is a generic Prikry sequence g for this forcing such that $V_\kappa \prec V[g]_\theta$, then this fact is true in every forcing extension. The fact is true because the elementarity relation $\check{V}_\kappa \prec \check{V}[\dot{g}]_{\check{\theta}}$ can be expressed in the forcing language using only the parameters $\check{V}_\kappa$ and $\check{\theta}$. The name $\dot{g}$ for the generic filter is not needed to express this elementarity relation, because $V[g]_\theta$ is definable in $V[g]$ as the collection of all sets of $\in$-rank less than θ.

Lemma 31. *Let the measure μ_0 be a normal measure on the cardinal κ_0. Let $V = N^{(0)}$, and let $\langle N^{(\alpha)}, \mu_\alpha, \kappa_\alpha, j_{\alpha\beta}\rangle_{\alpha,\beta\le\omega}$ be the system of iterated ultrapowers generated by μ_0.*

$$V = N^{(0)} \xrightarrow{j_0} N^{(1)} \xrightarrow{j_1} N^{(2)} \xrightarrow{j_2} \cdots \longrightarrow N^{(\omega)}$$

Let θ be an inaccessible cardinal above κ_0. Let the partial order $\mathbb{P}$ be the Prikry forcing with respect to μ_0, and suppose that there is a V-generic Prikry sequence g for $\mathbb{P}$ such that

$$V_{\kappa_0} \prec V[g]_\theta.$$

Let s denote the critical sequence $\langle \kappa_n \rangle_{n\in\omega}$. It follows from the Mathias criterion (as discussed in section 1.6) that the sequence s is Prikry-generic for the forcing $j_{0\omega}(\mathbb{P})$. Thus, there is an inverse-directed system of elementary embeddings and models of ZFC *of the form shown in the diagram below.*

$$\cdots \xrightarrow{\ j\ } N^{(\omega)}[s]_\theta \xrightarrow{\ j\ } N^{(\omega)}[s]_\theta \xrightarrow{\ j\ } N^{(\omega)}[s]_\theta$$

This system has an inverse limit, but the thread form of this inverse limit is not given by the entire thread class. In particular, the inverse limit is given by the model V_{κ_0} along with the inclusion map i at every coordinate.

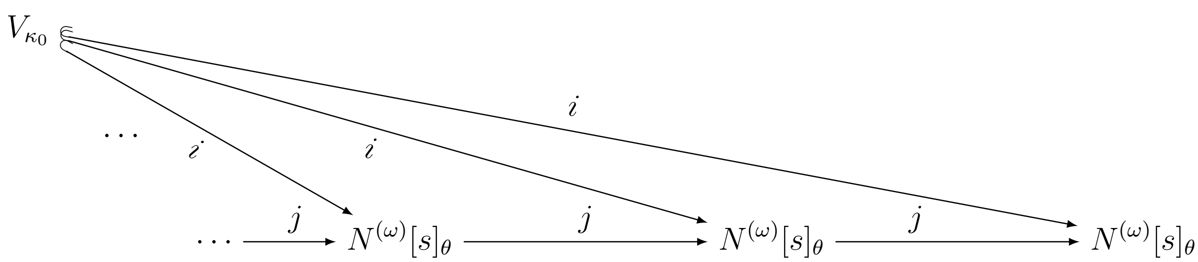


Figure 1.9: A system with a small inverse limit

Proof. First of all, by lemma 30, it follows that $\mathbb{1} \Vdash_{\mathbb{P}} \check{V}_{\kappa_0} \prec \check{V}[\dot{g}]_\theta$, where $\mathbb{1}$ is the trivial condition of $\mathbb{P}$. (As discussed in the comments immediately following lemma 31, an equivalent formula of the forcing language can be expressed using only check-names.)

Next, I define the map j shown in the diagram and show that it is elementary. This map j is derived from the ultrapower embedding j_0 generated by the normal measure μ_0. By lemma

24, the restriction $j_0 \upharpoonright N^{(\omega)} : N^{(\omega)} \to N^{(\omega)}$ is an elementary embedding. Furthermore, the cardinal θ is fixed by the ultrapower embedding j_0, because θ is an inaccessible cardinal above the size of the measure generating the embedding j_0. It follows that the further restriction $j_0 \upharpoonright N_\theta^{(\omega)} : N_\theta^{(\omega)} \to N_\theta^{(\omega)}$ is an elementary embedding. Finally, the same reasoning used in theorem 28 of section 1.6 shows that the embedding $j_0 \upharpoonright N_\theta^{(\omega)}$ lifts over the forcing $\mathbb{P}$, so the lifted map $j : N^{(\omega)}[s]_\theta \to N^{(\omega)}[s]_\theta$ is indeed an elementary embedding.

Next, I show that $V_{\kappa_0} \prec N^{(\omega)}[s]_\theta$. In order to demonstrate this elementarity relation, I will demonstrate that

$$V_{\kappa_0} \prec N_{\kappa_\omega}^{(\omega)} \prec N^{(\omega)}[s]_\theta.$$

The elementarity relation $V_{\kappa_0} \prec N_{\kappa_\omega}^{(\omega)}$ holds because $j_{0\omega}(V_{\kappa_0}) = N_{\kappa_\omega}^{(\omega)}$, and the critical point of $j_{0\omega}$ is κ_0. Next, note that $j_{0\omega}(\theta) = \theta$, because the map $j_{0\omega}$ is a direct limit of ultrapowers and the ultrapowers involved are bounded in size below the inaccessible cardinal θ. By applying the embedding $j_{0\omega}$ to the hypothesis $\mathbb{1} \Vdash_{\mathbb{P}} V_{\check{\kappa}_0} \prec V[g]_{\check{\theta}}$, it follows that $N_{\kappa_\omega}^{(\omega)} \prec N^{(\omega)}[s]_\theta$.

Let T and $\langle \pi_n \rangle_{n\in\omega}$ be the thread class and projection maps of the inverse-directed system in question. The embedding j fixes the model V_{κ_0} pointwise, so it follows that $(V_{\kappa_0}, \langle i \rangle_{n\in\omega})$ is a natural source, as illustrated in the diagram accompanying the statement of the lemma. It follows that the inverse-directed system in question has an inverse limit by theorem 19. Let $(S, \langle \pi_n \rangle)$ be the thread form of this inverse limit. From lemma 26, it follows that the thread with constant value κ_ω is not an element of S (i.e. $\kappa_\omega \notin \pi_0 \text{ " } S$), so the inverse limit S is a proper subset of the thread class T. Furthermore, by theorem 19, the set S must be a rank-initial subset of the thread class T, and by theorem 17, it must be the case that $R \subseteq S$, where R is the thread form of the natural source $(V_{\kappa_0}, \langle i \rangle_{n\in\omega})$. Since κ_ω is the least ordinal fixed point of j above κ_0, it follows that $(R, \langle \pi_\alpha \rangle_{\alpha \in I})$ is the thread form of the inverse limit. In other words, the inverse limit is given by the natural source $(N^{(0)}_{\kappa_0}, \langle i \rangle_{n\in\omega})$. □

In order to achieve the hypothesis $V_{\kappa_0} \prec V[g]_\theta$ of lemma 31, it is necessary that Prikry sequences have been shot through many cardinals above κ_0. In lemma 34 below, this goal will be achieved using iterated Prikry forcing. Before presenting lemma 34, I present some background material on Prikry forcing and iterated Prikry forcing, beginning with the following basic lemma.

Iterated Prikry forcing was invented by Magidor in [Mag76]. In describing this iteration, I refer to iterated Prikry forcing with *Magidor support* in order to distinguish it from other versions of iterated Prikry forcing invented later by Gitik and described in Gitik's Handbook of Set Theory article, [Git10]. This Handbook of Set Theory article is a comprehensive reference for iterated Prikry forcing and for related forcing notions. The handbook article also provides an exposition of the Magidor iteration of Prikry forcing (beginning on page 1424). I will define the Magidor iteration according to Magidor's original definition from [Mag76]; I believe that this definition yields a forcing notion equivalent to the Magidor iteration as described in Gitik's handbook article, but I do not rely on this equivalence.

Definition 32. Let A be a set of measurable cardinals, and for each measurable cardinal $\eta \in A$, let μ_η be a normal measure on η that does not concentrate on measurable cardinals. The **iterated Prikry forcing with Magidor support**, $\mathbb{P}$, with respect to these cardinals and normal measures, $\langle \eta, \mu_\eta \rangle_{\eta \in A}$, is defined as follows. For each cardinal $\eta \in A$, assume recursively that $\tilde{\mu}_\eta$ is a $\mathbb{P}_{<\eta}$-name for a normal measure on η extending μ_η. (Later, I will explain why such a name exists.) The forcing conditions are defined recursively as sequences of the form $\langle \check{s}_\eta, \dot{B}_\eta \rangle_{\eta \in A}$, such that the stem $\check{s}_\eta$ is a canonical name for finite sequence of ordinals below η from the ground model and such that every condition in $\mathbb{P}_{<\eta}$ forces that $\dot{B}_\eta$ is a name for a measure-one set with respect to the measure $\tilde{\mu}_\eta$ and that every member of $\check{s}_\eta$ is below every member of $\dot{B}_\eta$. The support is finite on the stems and full on the measure-one sets. That is to say, for all but finitely many values of η, the stem $\check{s}_\eta$ is empty.

The partial order is defined as would be expected for an iteration of Prikry forcing. That is to say, $p \leq q$ if and only if for every cardinal $\eta \in A$, every condition in $\mathbb{P}_{<\eta}$ forces that the ηth coordinate of p extends the ηth coordinate of q in the sense of one-step Prikry forcing on η. Direct extensions are formed by only shrinking the measure-one sets. That is to say, the condition p is a **direct extension** of the condition q if and only if $p \leq q$ and furthermore, the conditions p and q have the same stems.

It will be convenient in some proofs (and especially in the proof of lemmas 33 and 34), to think of the initial portion $\mathbb{P}_{<\eta}$ of $\mathbb{P}$ as literally a subset of $\mathbb{P}$, namely $\mathbb{P}_{<\eta} = \mathbb{P} \cap V_\eta$. Thus, every $\mathbb{P}_{<\eta}$-name will also be a $\mathbb{P}$-name. This can be achieved by reframing the definitions so that the conditions are *partial* sequences with domain A. In case $(s_\eta, \dot{B}_\eta)$ is undefined, then this is taken as equivalent to the ηth coordinate being defined as the trivial condition, $(\emptyset, \check{\eta})$.

As in the case of one-step Prikry forcing, the Prikry property is satisfied: for every condition $p \in \mathbb{P}$ and for every formula φ in the forcing language, there is a direct extension of p deciding φ. This is proven in [Mag76, lemma 2.1]. It follows that for any cardinal $\kappa \in A$, the forcing $\mathbb{P}_{>\kappa}$ beyond stage κ does not add any new subsets to V_κ, because the suborder of $\mathbb{P}_{>\kappa}$ defined by the direct extension relation is $\leq\kappa$-closed.

In order for the forcing to be well-defined, it must be shown recursively that for each measurable cardinal η such that $\eta \in A$, there is a $\mathbb{P}_{<\eta}$-name $\tilde{\mu}_\eta$ for a normal measure on η extending μ_η. This fact is proven in [Mag76, theorem 2.5]. In case η is not a limit point of the set A of measurable cardinals, then it follows from the Levy-Solovay theorem of [LS67] that the name $\tilde{\mu}_\eta$ is a name for the normal measure generated by μ_η. In case η is a limit point of A, then the name $\tilde{\mu}_\eta$ is given as follows, by stating which forcing conditions force a given name for a subset of η to be an element of $\tilde{\mu}_\eta$. Let $j : V \to M$ be the ultrapower embedding generated by the measure μ_η. Let $\dot{B}$ be a $\mathbb{P}_{<\eta}$-name for a subset of η, and let $p \in \mathbb{P}_{<\eta}$ be a forcing condition. Then $p \Vdash_{\mathbb{P}} \dot{B} \in \tilde{\mu}_\eta$ if and only if there is a condition $q \in j(\mathbb{P})$ such that p is an initial segment of q, the condition q is a direct extension of the condition $j(p)$, and $q \Vdash_{j(\mathbb{P}_{<\eta})} \check{\kappa} \in j(\dot{B})$. (Requiring simply that q has the same stems as $j(p)$ would give rise to an equivalent definition, since such a q can always be extended to a direct extension of $j(p)$.)

One convenient feature of iterated Prikry forcing with Magidor support is that it functions somewhat like a product forcing in that the order of the forcing can sometimes be rearranged. The details are given in lemma 33 below, which I proved in collaboration with Magidor, who has kindly permitted me to include the proof in my dissertation. This lemma was previously known as folklore to Magidor and to Arthur Apter, among others.

Before presenting lemma 33, I review some ideas about equivalences between forcing notions. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions. Let f be an isomorphism (i.e. a $\leq$-preserving bijection) from a dense subset of $\mathbb{Q}$ to a dense subset of $\mathbb{P}$. The isomorphism f induces an isomorphism from the Boolean algebra of the forcing $\mathbb{Q}$ to the Boolean algebra of the forcing $\mathbb{P}$. This isomorphism of Boolean algebras in turn induces an isomorphism between the classes of $\mathbb{Q}$-names and $\mathbb{P}$-names and between the corresponding forcing languages. I will denote all of these isomorphisms by f. These isomorphisms preserve structure in the following ways. (The proof is by standard inductive arguments.) If $G \subseteq \mathbb{Q}$ is V-generic, then $f(G) \subseteq \mathbb{P}$ is also V-generic, and $\dot{x}^G = f(\dot{x})^{f(G)}$ for every $\mathbb{Q}$-name $\dot{x}$. For every condition $q \in \mathbb{Q}$ and for every formula φ in the forcing language of $\mathbb{Q}$, if $q \Vdash_{\mathbb{Q}} \varphi$, then $f(q) \Vdash_{\mathbb{Q}} f(\varphi)$.

Lemma 33 (joint with Magidor)**.** *Let A be a set of measurable cardinals. For each cardinal $\eta \in A$, let μ_η be a normal measure on η such that μ_η does not concentrate on measurable cardinals. Let $\kappa \in A$. Let $\mathbb{R} = \mathbb{R}_{<\kappa} * \dot{\mathbb{R}}_\kappa * \dot{\mathbb{R}}_{\text{tail}}$ be iterated Prikry forcing with Magidor support with respect to these measurable cardinals and normal measures, $\langle \eta, \mu_\eta \rangle_{\kappa \in A}$.*

Then there is a forcing notion $\mathbb{Q}$ such that $\mathbb{Q}$ is forcing-equivalent to $\mathbb{R}$, and $\mathbb{Q}$ is a rearrangement of $\mathbb{R}$ such that the κth stage comes last. That is to say, the forcing notion $\mathbb{Q}$ can be factored as $\mathbb{Q} = \mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{\text{tail}} * \dot{\mathbb{Q}}_\kappa$, where $\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{\text{tail}}$ is the iterated Prikry forcing with Magidor support with respect to the measurable cardinals and normal measures $\langle \eta, \mu_\eta \rangle_{\eta \in A - \{\kappa\}}$ and $\dot{\mathbb{Q}}_\kappa$ is a $(\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{\text{tail}})$-name for the one-step Prikry forcing on κ with respect to the normal measure extending μ_κ and named by the name $\tilde{\mu}_\kappa$, as defined above.

Proof. We will prove the following three statements by induction on cardinals $\eta \in A$ such that $\kappa < \eta$. For statement (1), the same proof will also work in the case $\eta = \sup(A)$, thereby proving the lemma.

1. There is an isomorphism, f_η, from a dense subset of $\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)} * \dot{\mathbb{Q}}_\kappa$ to a dense subset of $\mathbb{R}_{<\kappa} * \dot{\mathbb{R}}_\kappa * \dot{\mathbb{R}}_{(\kappa,\eta)}$. This isomorphism is given by rearranging the order of the coordinates, that is to say,[9]

$$f_\eta(q_{<\kappa} * \dot{q}_{(\kappa,\eta)} * \dot{q}_\kappa) = q_{<\kappa} * \dot{q}_\kappa * \dot{q}_{(\kappa,\eta)}.$$

Furthermore, the isomorphism f_η preserves direct extensions.

2. Let $\tilde{\mu}_\eta$ be the particular $(\mathbb{R}_{<\kappa} * \dot{\mathbb{R}}_\kappa * \dot{\mathbb{R}}_{(\kappa,\eta)})$-name for a normal measure on η extending μ_η defined just before the statement of this lemma. Let $\tilde{\mu}_\eta^-$ be the $(\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)})$ name for a measure on η extending μ_η, defined in the same way. Then

$$\Vdash_{\mathbb{R}_{<\eta}} f_\eta(\tilde{\mu}_\eta^-) \subseteq \tilde{\mu}_\eta.$$

3. Whenever $\dot{B}$ is a $\mathbb{R}_{<\eta}$-name for an element of $\tilde{\mu}_\eta$, there is a $(\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)})$-name, $\dot{C}$, for an element of $\tilde{\mu}_\eta^-$ such that

$$\Vdash_{\mathbb{R}_{<\eta}} f_\eta(\dot{C}) \subseteq \dot{B}.$$

Assume inductively that for all cardinals $\eta' \in A$ such that $\eta' < \eta$, the statements above are proven.

To prove statement (1), we begin by defining the domain of the isomorphism f_η. Suppose that $\dot{B}$ is a $(\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)})$-name for an element of $\tilde{\mu}_\kappa$. The tail forcing $\dot{\mathbb{Q}}_{(\kappa,\eta)}$ adds no new

[9]The notation $q_{<\kappa} * \dot{q}_{(\kappa,\eta)} * \dot{q}_\kappa$ delineates the three sections of the forcing condition $q \in \mathbb{Q}$. That is to say, the condition $q_{<\kappa} \in \mathbb{Q}_{<\kappa}$ has support less than κ, the condition (technically, $\mathbb{Q}_{<\kappa}$-name for a condition) $\dot{q}_{(\kappa,\eta)} \in \dot{\mathbb{Q}}_{(\kappa,\eta)}$ has support on the open interval (κ, η), and the (name for a) condition $\dot{q}_\kappa \in \dot{Q}_\kappa$ has support at κ. The concatenation $q_{<\kappa} * \dot{q}_{(\kappa,\eta)} * \dot{q}_\kappa$ is an element of $\mathbb{Q}$.

subsets to κ. It follows that for every condition $p \in (\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)})$, there is an extension $q \leq p$ and a $\mathbb{Q}_{<\kappa}$-name $\dot{C}$ for a subset of κ such that

$$q_{<\kappa} \Vdash_{\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)}} \dot{B} = \dot{C}.$$

Therefore, the subset of $\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)} * \dot{\mathbb{Q}}_\kappa$ consisting of those conditions in which the name for a measure-one subset of κ is a $\mathbb{Q}_{<\kappa}$-name is a dense subset of the forcing notion $\mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)} * \dot{\mathbb{Q}}_\kappa$. This dense subset is the domain of the function f_η. The function f_η, given by reordering the coordinates, is well-defined on this domain. It follows from inductive assumptions (2) and (3) that the image of f_η is dense in $\mathbb{R}_{<\kappa} * \dot{\mathbb{R}}_\kappa * \dot{\mathbb{R}}_{(\kappa,\eta)}$. Finally, it follows immediately from the definitions that the map f_η is an isomorphism that also preserves direct extensions.

Next, we prove statement (2). In case the cardinal η is not a limit point of the set A of measurable cardinals, then it follows from the Levy-Solovay theorem that the measures named by $\tilde{\mu}_\eta$ and $\tilde{\mu}_\eta^-$ are both generated by the measure μ_η. In this case, statement (2) follows immediately. The interesting case occurs when the cardinal η *is* a limit point of A. The rest of the proof of statement (2) will address this case.

Let $\mathbb{Q}^- = \mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)}$. It suffices to show that if $\dot{B}$ is a $\mathbb{Q}^-$-name for a subset of η and $q \in \mathbb{Q}^-$ is a forcing condition in the domain of f such that $q \Vdash_{\mathbb{Q}^-} \dot{B} \in \tilde{\mu}_\eta^-$, then $f(q) \Vdash_{\mathbb{R}_{<\eta}} f(\dot{B}) \in \tilde{\mu}_\eta$. Suppose that

$$q \Vdash_{\mathbb{Q}^-} \dot{B} \in \tilde{\mu}_\eta^-.$$

Let $j : V \to M$ be the ultrapower embedding generated by the normal measure μ_η. Let $f = f_\eta$. By the definition of $\tilde{\mu}_\eta^-$, it follows that in the partial order $j(\mathbb{Q}^-)$, there is a direct extension p of $j(q)$ such that q is an initial segment of p and

$$p \Vdash_{j(\mathbb{Q}^-)} \check{\eta} \in j(\dot{B}).$$

By the elementarity of j and by inductive assumption (1), the function $j(f)$ is a direct-extension-preserving isomorphism from the forcing notion $j(\mathbb{Q}^-)*j(\dot{\mathbb{Q}}_\kappa)$ to the forcing notion $j(\mathbb{R}_{<\eta})$. The domain of the function $j(f)$ is defined analogously to that of the function f. Since the conditions p and $j(q)$ both have trivial κth coordinates, it follows that they are both in the domain of $j(f)$. Since $j(f)$ is a direct-extension-preserving isomorphism, it follows that $j(f)(p)$ is a direct extension of $j(f)(j(q))$, and

$$j(f)(p) \Vdash_{j(\mathbb{R}_{<\eta})} j(f)(\check{\eta}) \in j(f)(j(\dot{B})).$$

Since $j(f)$ is a forcing equivalence, it follows that $j(f)(\check{\eta}) = \check{\eta}$.[10] Furthermore, it follows from the elementarity of j that $j(f)(j(q)) = j(f(q))$ and that $j(f)(j(\dot{B})) = j(f(\dot{B}))$. Thus, I conclude that $j(f)(p)$ is a direct extension of $j(f(q))$ and

$$j(f)(p) \Vdash_{j(\mathbb{R}_{<\eta})} \check{\kappa} \in j(f(\dot{B})).$$

By the definition of $\tilde{\mu}_\eta$, this nearly suffices to show that

$$f(q) \Vdash_{\mathbb{R}_{<\eta}} f(\dot{B}) \in \tilde{\mu}_\eta,$$

which would complete the proof. It remains to be shown that $f(q)$ is an initial segment of $j(f)(p)$. This follows immediately from the definition of f and the elementarity of j. However, we will write out the details below.

[10] These two instances of the symbol $\check{\eta}$ denote the canonical names for the cardinal η with respect to two different forcing notions.

Considering q as an element of $\mathbb{Q}^- * \dot{\mathbb{Q}}_\kappa$, write $q = q_{<\kappa} * \dot{q}_{(\kappa,\eta)} * \dot{q}_\kappa$. Then it follows from the definition of f that $f(q) = q_{<\kappa} * \dot{q}_\kappa * \dot{q}_{(\kappa,\eta)}$. Furthermore, since $q \in \mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)}$, it follows that $\dot{q}_\kappa$ is given by the trivial condition $\check{\mathbb{1}}_\kappa = (\check{\emptyset}, \check{\kappa})$. Keeping in mind that the critical point of j is η, it follows that the condition p, considered as a member of $j(\mathbb{Q}^-) * j(\dot{\mathbb{Q}}_\kappa)$, has the form $p = p_{<\kappa} * \dot{p}_{(\kappa,\ j(\eta))} * \dot{p}_\kappa$. Since p was actually defined as a member of $j(\mathbb{Q}^-)$, it follows that p_κ is trivial. Furthermore, the condition p was defined such that q (considered as an element of $\mathbb{Q}^-$) is an initial segment of p. It follows that $p = q_{<\kappa} * \dot{q}_{(\kappa,\eta)} * \dot{p}_{[\eta,j(\eta))} * \check{\mathbb{1}}_\kappa$. Since j is an elementary embedding, it follows that the isomorphism $j(f)$ is defined analogously to f, by rearranging the coordinates, and so $j(f)(p) = q_{<\kappa} * \check{\mathbb{1}}_\kappa * \dot{q}_{(\kappa,\eta)} * \dot{p}_{[\eta,j(\eta))}$. Thus, $f(q)$ is indeed an initial segment of $j(f)(p)$. This completes the proof of statement (2).

Finally, the proof of statement (3) is similar to the proof of the Levy-Solovay theorem, as follows. Let $G^- = G_{<\kappa} * G_{(\kappa,\eta)}$ be V-generic for $\mathbb{Q}^- = \mathbb{Q}_{<\kappa} * \dot{\mathbb{Q}}_{(\kappa,\eta)}$. Work in the forcing extension $V[G^-]$. Let $\mathbb{Q}_\kappa$ and μ^-_η be the interpretations of the $\mathbb{Q}^-$-names $\dot{\mathbb{Q}}_\kappa$ and $\tilde{\mu}^-_\eta$ by the generic filter G^-. The $(\mathbb{Q}^- * \dot{\mathbb{Q}}_\kappa)$-names $f^{-1}_\eta(\dot{B})$ and $f^{-1}_\eta(\tilde{\mu}_\eta)$ have partial interpretations by the generic filter G^-. These partial interpretations are $\mathbb{Q}_\kappa$-names, which I will denote by $\dot{B}^{G^-}$ and $\tilde{\mu}^{G^-}_\eta$. To prove statement (3), it suffices to show that there is a set $C \in \mu^-_\eta$ such that $\Vdash_{Q_\kappa} \check{C} \subseteq \dot{B}$.

For each condition $q \in \mathbb{Q}_\kappa$, let C_q be the set of ordinals given by

$$C_q = \{\, \alpha < \kappa \mid q \Vdash_{\mathbb{Q}_\kappa} \check{\alpha} \in \dot{B}^{G^-} \,\}.$$

It must be the case that

$$\Vdash_{\mathbb{Q}_\kappa} \left(\bigcup_{q \in \check{\mathbb{Q}}_\kappa} \check{C}_q \right) \supseteq \dot{B}^{G^-} \in \tilde{\mu}_\eta^{G^-}.$$

Since $\kappa < \eta$, this is a small union relative to η, and so there is some particular condition $q \in \mathbb{Q}_\kappa$ such that $\Vdash_{\mathbb{Q}_\kappa} \check{C}_q \in \tilde{\mu}_\eta^{G^-}$. By inductive hypothesis (2), it follows that $C_q \in \mu_\eta^-$. This completes the proofs of statement (3) and of lemma 33. □

I am finally ready to present lemma 34, which shows, together with lemma 33, that the hypothesis of lemma 31 can be satisfied.

Lemma 34 (joint with Magidor). *Suppose that the cardinal κ is 1-extendible, as witnessed by an elementary embedding $j : V_{\kappa+1} \to V_{\theta+1}$. Let A be the set of measurable cardinals below θ, and for each cardinal $\eta \in A$, let μ_η be a normal measure on η such that μ_η does not concentrate on a set of measurable cardinals. Furthermore, assume that $\langle \eta, \mu_\eta \rangle_{\eta \in A} = j\big(\langle \eta, \mu_\eta \rangle_{\eta \in A \cap \kappa}\big)$. Let $\mathbb{P}$ be the iterated Prikry forcing with Magidor support with respect to the measurable cardinals and normal measures $\langle \eta, \mu_\eta \rangle_{\eta \in A}$. Then there is a V-generic filter $G \subseteq \mathbb{P}$ such that $V[G]_\kappa \prec V[G]_\theta$.*

Proof. Let $G_{<\kappa}$ be V-generic for $\mathbb{P}_{<\kappa}$. Working in $V[G_{<\kappa}]$, we will define a master condition for the tail forcing, $\mathbb{P}_{[\kappa,\theta)}$. This master condition will be defined so that the elementarity relation $V[G]_\kappa \prec V[G]_\theta$ is satisfied.

The master condition $q \in \mathbb{P}_{[\kappa,\theta)}$ will be a condition with empty stems, that is to say, the condition q has the form $q = \langle (\emptyset, \dot{B}_\eta) \rangle_{\eta \in A \cap [\kappa,\theta)}$. To define q, we will specify the names $\dot{B}_\eta$ of measure-one sets. For each cardinal $\eta \in A \cap [\kappa, \theta)$, if $\eta \neq \kappa$, then let the name $\dot{B}_\eta$ be a name for the intersection, over all conditions $p \in \mathbb{P}_{<\kappa}$, of the measure-one set named in the ηth coordinate of $j(p)$. Since this is a small intersection relative to η, it follows that $\dot{B}_\eta$ is also a name for a measure-one set in $\tilde{\mu}_\eta$, so q is indeed a forcing condition in $\mathbb{P}_{[\kappa,\theta)}$. The name $\dot{B}_\kappa$ cannot be defined in the same way, because the intersection would not be small relative to κ. Instead, let $\dot{B}_\kappa = \check{\kappa}$. (Later in the proof, we will make special accommodations to handle this κth coordinate.) Let $G_{[\kappa,\theta)} \subseteq \mathbb{P}_{[\kappa,\theta)}$ be a $V[G_{<\kappa}]$-generic filter containing the master condition q. Let $G = G_{<\kappa} * G_{[\kappa,\theta)}$.

We now verify the elementarity relation $V[G]_\kappa \prec V[G]_\theta$. Let φ be a formula with one free variable, and let $x \in V[G]_\kappa$ be a parameter. It suffices to show that if $V[G]_\kappa \models \varphi(x)$ then $V[G]_\theta \models \varphi(x)$. Furthermore, as I noted shortly after the definition of the Magidor iteration, the tail forcing $\dot{\mathbb{P}}_{[\kappa,\theta)}$ adds no new sets of $\in$-rank less than κ since the direct extension relation of this tail forcing is $\leq\kappa$-closed. Thus, $V[G]_\kappa = V[G_{<\kappa}]_\kappa$, so it suffices to show that if $V[G_{<\kappa}]_\kappa \models \varphi(x)$ then $V[G]_\theta \models \varphi(x)$. The set x is an element of V_η for some measurable cardinal $\eta < \kappa$, since the measurable cardinals are unbounded below κ. Therefore, the set x is added before the ηth stage of forcing, and so x has a $\mathbb{P}_{<\eta}$-name $\dot{x}$ such that $\dot{x} \in V_\eta$. Since $\mathbb{P}_{<\eta} \subseteq \mathbb{P}_{<\kappa} \subseteq \mathbb{P}$, it follows that $\dot{x}$ is also a $\mathbb{P}_{<\kappa}$ name for x and a $\mathbb{P}$-name for x.

Fix a formula φ in the forcing language of $\mathbb{P}_{<\kappa}$ with one free variable. Suppose that $V[G_{<\kappa}]_\kappa \models \varphi(x)$. Then there is a forcing condition $p \in G_{<\kappa}$ such that

$$p \Vdash_{\mathbb{P}_{<\kappa}} \check{V}[\dot{G}_{<\kappa}]_{\check{\kappa}} \models \varphi(\dot{x}). \tag{1.1}$$

I claim that the forcing relation

$$\{ (p, \tau) \in \mathbb{P}_{<\kappa} \times V_\kappa \mid p \Vdash_{\mathbb{P}_{<\kappa}} \check{V}[\dot{G}_{<\kappa}]_{\check{\kappa}} \models \varphi(\tau) \} \tag{1.2}$$

is definable in the model $V_{\kappa+1}$. The ordered pair (p, τ) must be encoded using a flat pairing function so that it is an element of $V_{\kappa+1}$. Towards proving this claim, first note that $\mathbb{P}_{<\kappa} \subseteq V_{\kappa+1}$, the partial order $(\mathbb{P}_{<\kappa}, <)$ is definable in the model $V_{\kappa+1}$ from the parameter $\langle \eta, \mu_\eta \rangle_{\eta \in A}$, and this definition is absolute between $V_{\kappa+1}$ and V. The related standard forcing relation for $\mathbb{P}_{<\kappa}$, namely

$$\{ (p, \tau) \in \mathbb{P}_{<\kappa} \times V \mid p \Vdash_{\mathbb{P}_{<\kappa}} \varphi(\tau) \}, \tag{1.3}$$

is defined in V by recursion on complexity. The definition of the standard forcing relation (1.3) is given in reference texts on forcing, for instance in [Kun80, pp. 195-196]. Since every element of $V[G_{<\kappa}]_\kappa$ has a name in V_κ, it follows that the definition of the forcing relation (1.2) is given by the relativization to names $\tau \in V_\kappa$ of the definition of the standard forcing relation (1.3). That is to say, where the definition of the standard forcing relation (1.3) quantifies over all $\mathbb{P}_{<\kappa}$-names in defining the case where the formula φ begins with a quantifier, the definition of the forcing relation (1.2) quantifies only over $\mathbb{P}_{<\kappa}$-names $\tau \in V_\kappa$. This relativized definition is absolute between $V_{\kappa+1}$ and V, so the claim is proven: the forcing relation (1.2) is definable in the model $V_{\kappa+1}$.

Note that the formula defining the partial order $(\mathbb{P}_{<\kappa}, <)$ in $V_{\kappa+1}$ using the parameter $\big(\langle \eta, \mu_\eta \rangle_{\eta \in A \cap \kappa}\big)$ also defines the partial order $(\mathbb{P}, <)$ in V_θ using the parameter $\langle \eta, \mu_\eta \rangle_{\eta \in A} = j\big(\langle \eta, \mu_\eta \rangle_{\eta \in A \cap \kappa}\big)$. Keeping in mind that $j(\dot{x}) = \dot{x}$, it follows, by applying the embedding j to formula (1.1), that

$$j(p) \Vdash_{\mathbb{P}} \check{V}[\dot{G}]_{\check{\theta}} \models \varphi(\dot{x}).$$

Next, to accommodate the way in which the master condition was defined at the κth coordinate, let $r \in \mathbb{P}$ be the forcing condition which is equal to $j(p)$ at every coordinate other than κ but has the trivial condition at the coordinate κ. By lemma 33, the forcing $\mathbb{P}$ is forcing equivalent to an iterated Prikry forcing in which the one-step Prikry forcing at κ comes last. The set x has a check-name with respect to this final one-step Prikry forcing, and the model $V[G]_\theta$ is definable in $V[G]$ from only the parameter θ (without the parameter G) so it follows from lemma 30 that

$$r \Vdash_{\mathbb{P}} \check{V}[\dot{G}]_{\check{\theta}} \models \varphi(\dot{x}).$$

The initial part of the condition r, namely $r \restriction \kappa$, is equal to p, so $r \restriction \kappa \in G_{<\kappa}$. The iterated Prikry forcing with Magidor support is defined with finite support on the stems, so the condition $j(p)$ has empty stems at and above the κth coordinate. Thus, the master condition q ensures that $r \in G$, and the proof is complete. □

The iterated Prikry forcing described in lemma 34 preserves the measurability of θ, as was discussed earlier in this section. It follows in particular that the inaccessibility of the cardinal θ is preserved. Furthermore, the forcing described in lemma 34 can be rearranged, using lemma 33, so that the final stage is one-step Prikry forcing at κ. Therefore, the hypothesis of lemma 31 is satisfied, and an application of lemma 31 completes the proof of the main theorem of this section, theorem 29.

I suspect that the large cardinal hypothesis of a 1-extendible cardinal, used to prove theorem 29, can be weakened. The full strength of this hypothesis was only used to prove lemma 34, and I conjecture that this lemma, or a suitable modification of it, can be proven from a much weaker hypothesis.

1.8 Interaction of inverse limits with forcing

In this section, I consider the extent to which the existence or nonexistence of inverse limits is preserved by forcing with a thread of forcing notions over an inverse-directed system of elementary embeddings and models of set theory. I will show in theorems 37 and 39 that under fairly general assumptions (though not in all cases), inverse limits are preserved in both the upwards and downwards directions. In order to prove these theorems, some preliminary considerations are required.

The following theorem of Laver about the definability of ground models will be very useful in analyzing these forcing extensions of inverse-directed systems.

Theorem 35 ([Lav07, theorem 3], see also [Rei07, section 2])**.** *Let $V[G]$ be a forcing extension of V by the poset $\mathbb{P}$. Then V is definable in $V[G]$ using the parameter $V_{\delta+1}$, where $\delta = (|\mathbb{P}|^+)^V = (|\mathbb{P}|^+)^{V[G]}$.*

Next, I will discuss some notational and metamathematical issues. Let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ be an inverse-directed system of elementary embeddings and transitive models of set theory with thread class T and inverse limit $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ in thread form. It will be useful at times to work in the inverse limit model, that is, the model $(S, \in_T)$, which will sometimes be denoted as simply S. The elements of this model are threads, $\langle x_\alpha \rangle_{\alpha \in I}$. I will sometimes use the abbreviated notation $\langle x_\alpha \rangle$ in place of $\langle x_\alpha \rangle_{\alpha \in I}$, as sometimes this abbreviated notation will make the writing easier to follow.

I will also use descriptions such as *a thread of ordinals* or *a thread of forcing notions.* For instance, a thread of ordinals is a thread $\langle \gamma_\alpha \rangle_{\alpha \in I}$ such that each γ_α is an ordinal. If such a thread $\langle \gamma_\alpha \rangle_{\alpha \in I}$ is a member of S, then from the perspective of the model $(S, \in_T)$, the set $\langle \gamma_\alpha \rangle_{\alpha \in I}$ is actually an ordinal, whereas from the perspective of V it is a thread of ordinals.

Going back and forth between the perspectives of $(S, \in_T)$ and V leads to some ambiguity, because if a set were a thread of ordinals from the perspective of the model $(S, \in_T)$, then from the perspective of V, that set would be a thread of threads of ordinals. However, I will not have occasion to actually talk about threads of threads, so the notation *a thread of ordinals* should be taken to refer to a thread from the perspective of V and an ordinal from the perspective of S.

Next, I present a lifting criterion for systems of elementary embeddings.

Theorem 36. *Let $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ be an inverse-directed system of elementary embeddings and models of* ZFC. *Let $\langle \mathbb{P}_\alpha \rangle_{\alpha \in I}$ be a thread of forcing notions. Suppose that there is a tuple of generic filters $\langle G_\alpha \rangle_{\alpha \in I}$ such that for each index $\alpha \in I$, the filter G_α is $M^{(\alpha)}$-generic for $\mathbb{P}_\alpha$. Then the following two statements are equivalent.*

1. *There is a lifted system $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ such that each map $j^*_{\alpha\beta}$ is a lift of the map $j_{\alpha\beta}$ and such that the tuple of generic filters $\langle G_\alpha \rangle_{\alpha \in I}$ is a thread of the lifted system.*

2. *The following* ***global lifting criterion*** *is satisfied: for all pairs of indices $\alpha \leq \beta$ in I, the lifting criterion $j_{\alpha\beta}\,''\,G_\alpha \subseteq G_\beta$ is satisfied.*

Proof. Suppose that statement 1 holds. Then statement 2 follows immediately from the elementarity of the embeddings $j_{\alpha\beta}$. Conversely, suppose that statement 2 holds. It follows from the lifting criterion for a single embedding that each embedding $j_{\alpha\beta}$ lifts to an embedding $j^*_{\alpha\beta}$. The criterion $j_{\alpha\beta}\,''\,G_\alpha \subseteq G_\beta$ ensures that these embeddings are defined in such a way that the lifted system, $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$, commutes. □

Note that in the examples from sections 1.6 and 1.7 using Prikry forcing, even though the system lifted over Prikry forcing, the global lifting criterion was not satisfied. The generic filters did not form a thread in the lifted system.

I wish to analyze forcing extensions of the inverse limit model $(S, \in_T)$. This analysis involves some metamathematical subtleties, which I will now discuss. Suppose that $\langle \mathbb{P}_\alpha\rangle_{\alpha\in I}$ is a thread of forcing notions and that for each index $\alpha \in I$, the filter G_α is $M^{(\alpha)}$-generic for $\mathbb{P}_\alpha$. Suppose further that the global lifting criterion of theorem 36 is satisfied. Then for any particular pair of indices $\alpha \leq \beta$, the elementary embedding $j_{\alpha\beta}$ lifts over the forcing $\mathbb{P}_\alpha$ to an elementary embedding $j^*_{\alpha\beta} : M^{(\alpha)}[G_\alpha] \to M^{(\beta)}[G_\beta]$ given by $j^*_{\alpha\beta}(\dot{x}_\alpha^{G_\alpha}) = \dot{x_\beta}^{G_\beta}$ in such a way that the generics form a thread, $\langle G_\alpha\rangle_{\alpha\in I}$, in the lifted system.

$$\begin{array}{ccc} M^{(\alpha)}[G_\alpha] & \xrightarrow{\;j^*_{\alpha\beta}\;} & M^{(\beta)}[G_\beta] \\ \cup\!\!\mid & & \cup\!\!\mid \\ M^{(\alpha)} & \xrightarrow{\;j_{\alpha\beta}\;} & M^{(\beta)} \end{array}$$

Figure 1.10: Lifting the embedding $j_{\alpha\beta}$

Suppose that the inverse-directed system $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ has an inverse limit, with thread form $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$ and that $\langle\mathbb{P}_\alpha\rangle \in S$. It seems reasonable to say that the thread of generics $\langle G_\alpha\rangle$ is $\langle P_\alpha\rangle$-generic over the model S and to construct the forcing extension $S[\langle G_\alpha\rangle]$. But it is worthwhile to think a bit about what this means, since the models S makes use of a nonstandard set-membership relation, $\in_T$.

In the context of the usual $\in$ relation, a filter G is generic over a transitive model M for a partial order $\mathbb{P}$ if and only if G meets every dense subset of $\mathbb{P}$ which is an element of M. Consider next a model $(N, \in_N)$ of set theory, not necessarily transitive, and suppose that the model N believes that $\mathbb{P}$ is a partial order. Let $\tilde{\mathbb{P}}$ be the set of N-elements of $\mathbb{P}$. That is to say, $\{\, \tilde{\mathbb{P}} = x \in N \mid x \in_N \mathbb{P}\,\}$. Let $\tilde{G} \subseteq \tilde{\mathbb{P}}$ be an N-generic filter for $\tilde{\mathbb{P}}$, in the following sense. If the model N believes that D is a dense subset of P, and if $\tilde{D} = \{\, x \in N \mid x \in_N D\,\}$, then $\tilde{D} \cap \tilde{G} \neq \emptyset$. Then one can construct a set G and a forcing extension $N[G]$ of N, whose set-membership relation $\in_{N[G]}$ extends the set-membership relation $\in_N$, and such that $\tilde{G} = \{\, x \in N[G] \mid x \in_{N[G]} G\,\}$. To be precise, the elements of $N[G]$ are equivalence classes $[\sigma]_{\tilde{G}}$ of $\mathbb{P}$-names, under the equivalence relation $\equiv_{\tilde{G}}$ given by

$$\sigma \equiv_{\tilde{G}} \tau \iff (\exists p \in \tilde{G})\; N \models \big(p \Vdash \sigma = \tau\big),$$

and the relation $\in_{N[G]}$ is defined by

$$[\sigma]_{\tilde{G}} \in_{N[G]} [\tau]_{\tilde{G}} \iff N \models \big(p \Vdash \sigma = \tau\big).$$

The element $G \in N[G]$ is given by the equivalence class $[\dot{G}]_{\tilde{G}}$ of the canonical name $\dot{G}$ for the generic filter. The ground model N is seen to be isomorphic to a submodel of $N[G]$ by identifying an element $x \in N$ with the equivalence class $[\check{x}]_{\tilde{G}}$ of its canonical name, and the relation $\in_{N[G]}$ extends the set-membership relation of this isomorphic copy of N. For any name τ, from the perspective of the model $N[G]$, the set $[\tau]_{\tilde{G}}$ is the interpretation of the name $\tau = [\check{\tau}]_{\tilde{G}}$ by the generic filter G. Accordingly, I will write τ^G rather than $[\tau]_{\tilde{G}}$ from now on.

I now narrow the analysis to the case where the model N is given by the inverse limit model, S, of an inverse-directed system, $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$. Suppose that the global lifting criterion of theorem 36 is satisfied. The partial order $\mathbb{P}$ is given by the thread of partial orders $\langle \mathbb{P}_\alpha\rangle$. Let

$$\tilde{G} = \{\, \langle p_\alpha\rangle \in S \mid (\forall \alpha \in I)\ p_\alpha \in G_\alpha \,\}.$$

In order to show that the forcing construction described above works, I must verify that whenever S believes that $D = \langle D_\alpha\rangle$ is a dense subset of $\mathbb{P}$, then $\tilde{\mathbb{P}} \cap \tilde{D}$ is nonempty. Towards this end, let $A = \langle A_\alpha\rangle$ be a thread of antichains such that the model S believes that $A \subseteq D$ is a maximal antichain in $\mathbb{P}$. For each index α, the generic filter G_α meets the antichain A_α in the model $M^{(\alpha)}$ at exactly one point, $p_\alpha \in M^{(\alpha)}$. The point p_α is definable in the model $M^{(\alpha)}[G_\alpha]$ from the parameters G_α and A_α, both of which lie on threads in the lifted system, since the global lifting criterion is satisfied. It follows that p_α also lies on a thread, $\langle p_\alpha\rangle$. Since $\langle \mathbb{P}_\alpha\rangle \in S$ and S is a rank-initial segment of the thread class, it follows that $\langle p_\alpha\rangle \in S$, and so the construction works, and it makes sense to speak of the forcing extension, $S[G]$.

Each element $x \in S[G]$ is given by the interpretation $\langle \dot{x}_\alpha\rangle^G$ of some thread of names, $\langle \dot{x}_\alpha\rangle$. I will identify this element $\langle \dot{x}_\alpha\rangle^G \in S[G]$ with the thread $\langle (\dot{x}_\alpha)^{G_\alpha}\rangle_{\alpha\in I}$ of the lifted system, so that the forcing extension $S[G]$ can be considered as a subclass of the thread class of the lifted system. It is simple to check that this identification is an isomorphism from the model $S[G]$ to a subclass of the thread class of the lifted system. In particular, the generic filter G is identified with the thread $\langle(\dot{G}_\alpha)^{G_\alpha}\rangle = \langle G_\alpha\rangle$ of the lifted system, and the set-membership relation $\in_{S[\langle G_\alpha\rangle]}$ is identified with the set-membership relation of the thread class of the lifted system. From this point forward, I will denote this forcing extension of S by $S[\langle G_\alpha\rangle]$ rather than $S[G]$.

Taking the above considerations into account, I will prove in theorem 37 that forcing preserves inverse limits in the upwards direction so long as the global lifting criterion of theorem 36 is satisfied.

Theorem 37. *Roughly speaking, if the global lifting criterion holds, then the inverse limit of the lifted system is the lift of the inverse limit of the ground system.*

To be precise, let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ (the ground system) be an inverse-directed system of elementary embeddings and transitive models of ZFC *with thread class T and projection maps $\langle\pi_\alpha\rangle_{\alpha\in I}$. Suppose that this system has an inverse limit, with thread form $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$. Suppose that $\langle\mathbb{P}_\alpha\rangle \in S$ is a thread of forcing notions and that for each index $\alpha \in I$, the filter G_α is $M^{(\alpha)}$-generic for $\mathbb{P}_\alpha$. Suppose further that the global lifting criterion of theorem 36 holds, that is to say, $j_{\alpha\beta}\,"\,G_\alpha \subseteq G_\beta$ for every embedding $j_{\alpha\beta}$. Let $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ denote the lifted system, and let T_G and $\langle\pi^G_\alpha\rangle_{\alpha\in I}$ denote the thread class and projection maps of this lifted system. Then the following conclusions hold.*

- *The projection maps $\langle\pi_\alpha\rangle_{\alpha\in I}$ lift to elementary embeddings $\pi^*_\alpha : S[\langle G_\alpha\rangle_{\alpha\in I}] \to M_\alpha[G_\alpha]$,* *and the lifted embeddings π^*_α and $j^*_{\alpha\beta}$ commute in the same way as the corresponding ground maps.*

- *The pair $(S[\langle G_\alpha\rangle], \langle \pi^*_\alpha\rangle_{\alpha\in I})$ is a natural source for the lifted system, and the lifted projection maps π^*_α are the restrictions of the projection maps of the lifted system, $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$, to $S[\langle G_\alpha\rangle]$.*

- *The lifted inverse-directed system, $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$, has an inverse limit, and the thread form of this inverse limit is $(S[\langle G_\alpha\rangle], \langle \pi^*_\alpha\rangle_{\alpha\in I})$.*

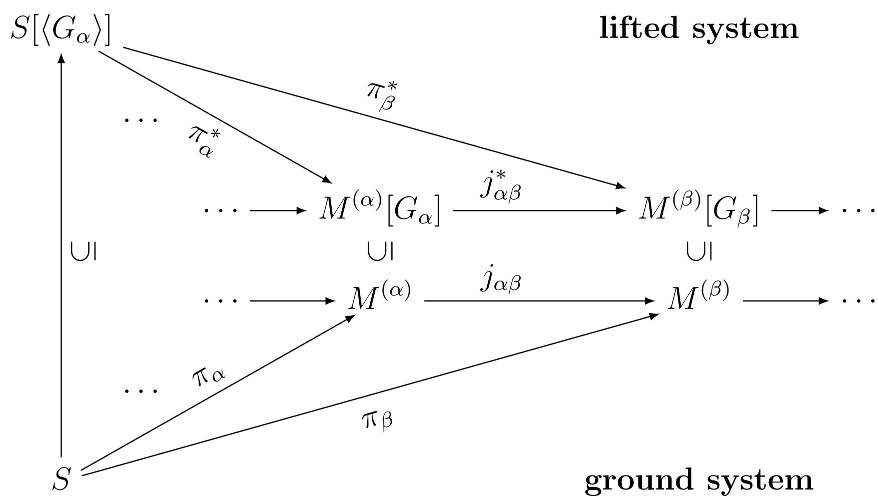


Figure 1.11: Inverse limits of the ground system and the lifted system

Proof. Let $\xi \in I$ be an index. The lifting criterion π_ξ " $\langle G_\alpha\rangle \subseteq G_\xi$ is satisfied. Therefore, the projection map π_ξ lifts to an elementary embedding $\pi^*_\xi : S[\langle G_\alpha\rangle] \to M_\xi[G_\xi]$. Each element x of $S[\langle G_\alpha\rangle]$ has a $\langle \mathbb{P}_\alpha\rangle$-name, $\langle \dot{x}_\alpha\rangle \in S$. The lifted projection maps must be given by $\pi^*_\xi\big(\langle \dot{x}_\alpha\rangle^{\langle G_\alpha\rangle}\big) = \pi_\xi\big(\langle \dot{x}_\alpha\rangle\big)^{G_\xi} = \dot{x}_\xi^{G_\xi}$ and $j^*_{\alpha\beta}(\dot{x}_\alpha^{G_\alpha}) = j_{\alpha\beta}(\dot{x}_\alpha)^{G_\beta}$. Therefore, the lifted elementary embeddings commute in the same fashion as the ground elementary embeddings, according to the rule $j^*_{\alpha\beta} \circ \pi^*_\alpha = \pi^*_\beta$. Since I have already (immediately before the statement of the theorem) identified $\langle \dot{x}_\alpha \rangle^{\langle G_\alpha \rangle}$ with $\langle \dot{x}_\alpha^{G_\alpha} \rangle$, it follows immediately that $(S[\langle G_\alpha \rangle], \langle \pi^*_\alpha \rangle_{\alpha \in I})$ is a thread-form natural source for the system. In particular, $\pi^*_\xi = \pi^G_\xi \upharpoonright S[\langle G_\alpha \rangle]$ for each index ξ.

I have just shown that for any index $\xi \in I$, the restricted projection map $\pi^G_\xi \upharpoonright S[\langle G_\alpha \rangle] : S[\langle G_\alpha \rangle] \to M_\xi[G_\xi]$ is elementary, so it follows from theorem 17 that the inverse limit of the lifted system $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ exists and has thread form $(R^*, \langle \pi^*_\alpha \rangle_{\alpha \in I})$ for some class of threads R^* such that $S[\langle G_\alpha \rangle] \subseteq R^* \subseteq T_G$.

To complete the proof, it suffices to demonstrate the inclusion $R^* \subseteq S[\langle G_\alpha \rangle]$, so that the equality $R^* = S[\langle G_\alpha \rangle]$ will be verified. Towards this end, I will apply theorem 35 on the definability of the ground model. By theorem 35, there is a formula φ such that in each model $M^{(\alpha)}[G_\alpha]$, the formula $\varphi(x, M^{(\alpha)}_{\delta_\alpha+1})$ defines the ground model $M^{(\alpha)}$, where $\delta_\alpha = (|P_\alpha|^+)^{M^{(\alpha)}}$.[11] By hypothesis, the thread $\langle \mathbb{P}_\alpha \rangle$ is an element of S, so this thread is also an element of R^*, since $S \subseteq R^*$. It follows that the thread $\langle M^{(\alpha)}_{\delta_\alpha+1} \rangle$ is also an element of R^*.

In the ZFC-model $(R^*, \in_{T_G})$, the formula $\varphi(x, \langle M^{(\alpha)}_{\delta_\alpha+1} \rangle)$ defines some ground model R such that $R^* = R[\langle G_\alpha \rangle]$. Let $\xi \in I$ be an index. The projection map $\pi^G_\xi \upharpoonright R[\langle G_\alpha \rangle]$ maps elementarily into $M^{(\xi)}[\langle G_\alpha \rangle]$. Restricting the domain of this elementary embedding to the definable subclass $R \subseteq R[\langle G_\alpha \rangle]$ and restricting the codomain using the same formula produces another elementary embedding $\pi^G_\xi \upharpoonright R : R \to M^{(\xi)}$. However, since $(S, \langle \pi_\alpha \rangle_{\alpha \in I})$

[11] If the partial order I is atomless, then all of the δ_α are the same, as otherwise there would be an infinite descending sequence of ordinals.

is the thread form of the inverse limit of the ground system, $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\le\beta\in I}$, it follows by theorem 17 that $R \subseteq S$. Therefore, $R^* = R[\langle G_\alpha\rangle] \subseteq S[\langle G_\alpha\rangle]$, so the proof is complete. □

If the hypothesis of theorem 37 is weakened to simply require that the system lifts, but not necessarily in such a way that the generics form a thread, then the Prikry forcing example from theorem 28 shows that the inverse limit of the lifted system may not exist.

Question 38. *Suppose that the hypotheses of theorem 37 all hold, except that the thread of forcing notions $\langle P_\alpha\rangle_{\alpha\in I}$ is not an element of the inverse limit model S. Must the lifted system have an inverse limit?*

A trivial example pertaining to question 38 is as follows. Work in the inverse-directed system given in the proof of theorem 29. This system, along with its inverse limit, $(V_{\kappa_0}, \langle i\rangle_{n\in\omega})$, is shown in the following diagram. I will review the structure of this system briefly here, refer to lemma 31 for the full details.

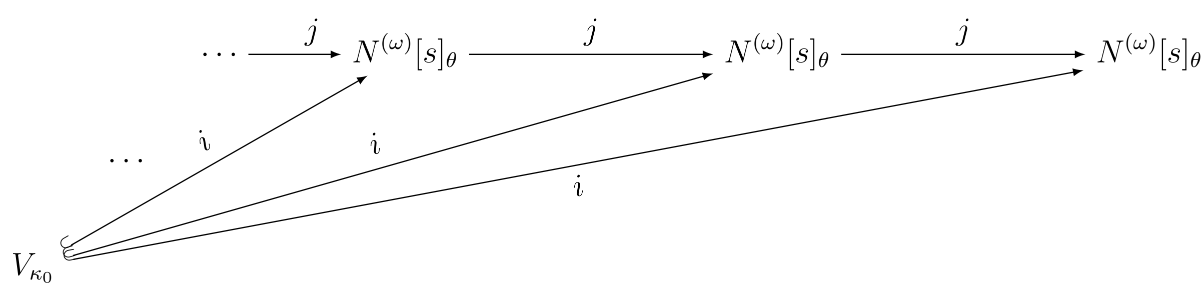


Figure 1.12: A system with a small inverse limit

The inverse limit model, V_{κ_0}, corresponds to a proper subclass of the thread class. The elementary embedding i is the inclusion map, and κ_ω is a fixed point of j above κ_0.

Let $\mathbb{P}$ be the trivial forcing over $N^{(\omega)}[s]$ with a single condition, κ_ω. Then the forcing $\mathbb{P}$ lies on a thread, but this thread is not an element of V_{κ_0}. Of course, since the forcing is trivial, the lifted system is identical to the ground system and therefore has an inverse limit.

Next, in theorem 39, I show that inverse limits are also preserved over forcing in the downwards direction.

Theorem 39. *Roughly speaking, if the global lifting criterion holds, then the ground of the inverse limit is the inverse limit of the grounds.*

To be precise, let $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ (the ground system) be an inverse-directed system of elementary embeddings and transitive ZFC*-models with thread class T and projection maps $\langle\pi_\alpha\rangle_{\alpha\in I}$. Let $\langle\mathbb{P}_\alpha\rangle_{\alpha\in I}\in T$ be a thread of forcing notions, and suppose that the global lifting criterion of theorem 36 holds. Let $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$ denote the corresponding lifted system, and let T_G and $\langle\pi^G_\alpha\rangle_{\alpha\in I}$ denote its thread class and projection maps.*

Suppose the lifted system, $\langle M_\alpha[G_\alpha], j^*_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$, has an inverse limit, with thread form $(S^*, \langle\pi^G_\alpha\rangle_{\alpha\in I})$. Furthermore, suppose that the thread of forcing notions $\langle\mathbb{P}_\alpha\rangle_{\alpha\in I}$ is an element of S^*. Then the ground system, $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$, has an inverse limit with thread form $(S, \langle\pi_\alpha\rangle_{\alpha\in I})$, such that $S^* = S[\langle G_\alpha\rangle]$.

Proof. The diagram illustrating theorem 37 also illustrates this theorem.

For each index $\alpha\in I$, let $\delta_\alpha = |\mathbb{P}_\alpha|^+$ as calculated in the model $M^{(\alpha)}$. By theorem 35, there is a formula φ such that the formula $\varphi(x, M^{(\alpha)}_{\delta_\alpha+1})$ defines the ground model $M^{(\alpha)}$ in the forcing extension $M^{(\alpha)}[G_\alpha]$.

By hypothesis, $\langle P_\alpha\rangle\in T$, and so $\langle M^{(\alpha)}_{\delta_\alpha+1}\rangle\in T$ as well. Since $T\subseteq T_G$, it follows that $\langle M^{(\alpha)}_{\delta_\alpha+1}\rangle\in T_G$. By theorem 19, the domain, S^*, of the inverse limit is either all of T_G or else has the form $T_G\cap({}^I V_\eta)$ for some ordinal η. Since $(S^*, \in_{T_G})$ is a model of ZFC and $\langle P_\alpha\rangle\in S^*$

by hypothesis, it follows that $\langle M^{(\alpha)}_{\delta_\alpha+1}\rangle \in S^*$.

Therefore, the formula $\varphi(x, \langle M^{(\alpha)}_{\delta_\alpha+1}\rangle)$ defines a ground model, S, in the model $(S^*, \in_{T_G})$, such that S consists of threads from the ground system, that is, $S \subseteq T$. Let $\xi \in I$ be an index. By hypothesis, the restricted projection map $\pi^G_\xi \restriction S^* : S^* \to M^{(\xi)}[G_\xi]$ is an elementary embedding. Using the formula $\varphi(x, \langle M^{(\alpha)}_{\delta_\alpha+1}\rangle)$ to further restrict the domain of this embedding to a definable class, and restricting the codomain to the corresponding class defined by $\varphi(x, M^{(\xi)}_{\delta_\xi+1})$ yields an elementary embedding $\pi \restriction S : S \to M^{(\xi)}$.

Since the map $\pi \restriction S : S \to M^{(\xi)}$ is elementary, the ground system, $\langle M^{(\alpha)}, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in I}$, has a natural source, so it follows from theorem 19 that the inverse limit of the ground system exists and has thread form $(S', \langle\pi_\alpha\rangle_{\alpha\in I})$ for some class of threads S' and that $S^* = S'[\langle G_\alpha\rangle]$. Furthermore, the proof of theorem 37 shows that S and S' are defined in S^* by the same formula with the same parameter, so it follows that $S = S'$. □

1.9 Inverse limits of rank-into-rank embeddings

Given cardinals κ and λ, the large cardinal axiom $I_3(\kappa, \lambda)$ states that there exists a nontrivial elementary embedding $j : V_\lambda \to V_\lambda$ with critical point κ. The large cardinal axiom $I_1(\kappa, \lambda)$ states that for some ordinal λ, there exists a nontrivial elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ. (An $I_1(\kappa, \lambda)$ embedding may also be viewed as a Σ^1_ω-elementary embedding from V_λ to V_λ.)

By the Kunen inconsistency, there cannot be an elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$. Therefore, if j is an $I_1(\kappa, \lambda)$ or $I_3(\kappa, \lambda)$ embedding, then λ must be the least ordinal fixed point of j above κ. In particular, $\lambda = \sup_{n<\omega} j^n(\kappa)$. It follows by the elementary chain lemma that $V_\kappa \prec V_\lambda$.

In this section, I completely characterize inverse limits of systems of order type ω^* of I_1 embeddings and of I_3 embeddings. Laver has already analyzed the inverse limits of these systems in [Lav97]. Laver does not explicitly define the inverse limit in terms of a universal property. Instead, he analyzes an inverse-directed system and shows that what he calls the *inverse limit map* is an elementary embedding. This inverse limit map is equivalent to the map that I call the projection map $\pi_0 : T \to M^{(0)}$. Thus, by theorem 17, if the inverse limit map is elementary, then the inverse limit exists, and the domain of its thread form is the entire thread class.

Most of this section will be a summary of Laver's results, translated into my notation. However, I will also prove some original results. The results of the present section are summarized by the following theorem.

Theorem 40. *Every inverse-directed system of I_3 embeddings of order type ω^*, as in the following diagram, has an inverse limit in the category of elementary embeddings and models of* ZFC*.*

$$\cdots \xrightarrow{j_2} V_\lambda \xrightarrow{j_1} V_\lambda \xrightarrow{j_0} V_\lambda$$

Every inverse-directed system of I_1 embeddings of order type ω^* as in the following diagram has an inverse limit in the category of elementary embeddings and $\in$-structures if and only if the critical points of the embeddings j_m do not attain their $\liminf$ *infinitely many times as*

$m \to \omega$.

$$\cdots \xrightarrow{j_2} V_{\lambda+1} \xrightarrow{j_1} V_{\lambda+1} \xrightarrow{j_0} V_{\lambda+1}$$

In case any of the above systems has an inverse limit, then the thread form of the inverse limit is given by the entire thread class, which is isomorphic to V_κ, where κ is the lim inf *of the critical points of the embeddings j_m.*

I will present the proof of theorem 40 by breaking it down into parts. By replacing the embeddings $j_m = j_{m+1,m}$ with finite compositions of such embeddings if necessary, I will assume without loss of generality that the critical points κ_m of the embeddings j_m are nondecreasing as m tends to ω. In particular, $\liminf_{m\in\omega} \kappa_m = \sup_{m\in\omega} \kappa_m$. The κ_m are eventually constant as $m \to \omega$ if and only if the critical points of the original embeddings (before the replacement with finite compositions) attained their lim inf infinitely many times.

I begin with the analysis of systems of I_3 embeddings, in the case where the critical point is eventually constant. Laver calls this case the trivial case.

Proposition 41 ([Lav97, p.86]). *Let*

$$\cdots \xrightarrow{j_2} V_{\lambda} \xrightarrow{j_1} V_{\lambda} \xrightarrow{j_0} V_{\lambda}$$

be an inverse-directed system of I_3 embeddings of order type ω^*. Let T be the thread class of this system. Suppose that as n tends to ω, the critical points of the j_n eventually have constant value κ. Then the system has an inverse limit. The Mostowski collapse of T is V_κ, and the inverse limit of the system is given by $(V_\kappa, \langle i\rangle_{n\in\omega})$, where i is the inclusion map.

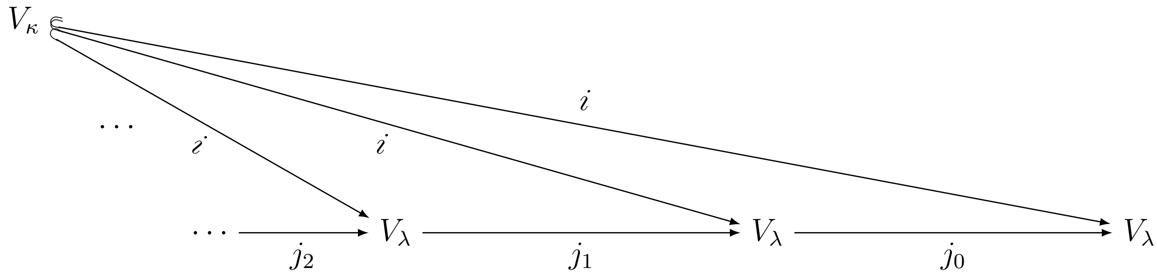


Figure 1.13: A system of I_3 embeddings with constant critical point

Proof. Without loss of generality, every embedding j_m has critical point κ. Since κ is the critical point of an $I_3(\kappa, \lambda)$ embedding, it follows that $V_\kappa \prec V_\lambda$. Furthermore, no embedding j_m has any ordinal fixed points above κ. It follows that the threads are all constant, with one constant thread for each member of V_κ, and the Mostowski collapse of T is V_κ. Since $V_\kappa \prec V_\lambda$, an application of theorem 17 completes the proof. □

Next, I present the analysis of systems of I_3 embeddings where the critical points are not eventually constant.

Proposition 42 ([Lav97, lemma 3.1]). *Let $\langle V_\lambda, j_{mn} \rangle_{n \leq m}$ be an inverse-directed system of order type ω^* of I_3 embeddings with thread class T and projection maps $\langle \pi_n \rangle_{n \in \omega}$.*

$$\cdots \xrightarrow{j_2} V_\lambda \xrightarrow{j_1} V_\lambda \xrightarrow{j_0} V_\lambda$$

For each natural number m, let $j_m = j_{m+1,m}$, and let κ_m be the critical point of j_m. Suppose that κ_m is increasing as m tends to ω, let $\kappa = \sup_{m \in \omega} \kappa_m$, and assume that $\kappa_m < \kappa$ for all m.

Then the following conclusions hold.

1. *The Mostowski collapse of T is V_κ. Let $f : V_\kappa \to T$ be the inverse of the Mostowski collapse isomorphism, and for each natural number n, let $\hat{\pi}_n = \pi_n \circ f$. Then $\hat{\pi}_0$ " κ is unbounded in λ.*

2. *The inverse limit of this system exists and is given by $(V_\kappa, \langle \hat{\pi}_n \rangle_{n\in\omega})$.*

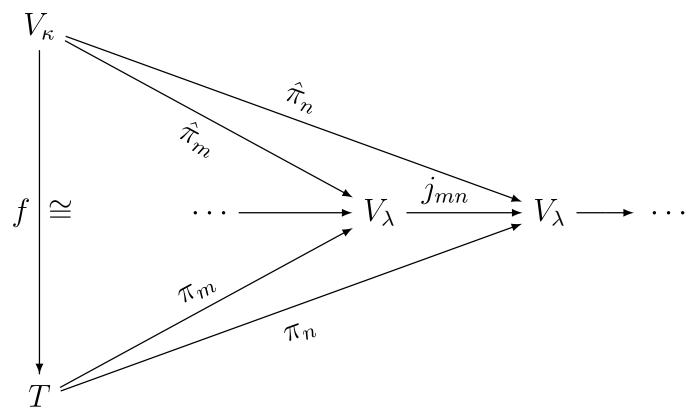


Figure 1.14: A system of I_3 embeddings with increasing critical points

Proof. I prove statement (1) first. Since κ is the supremum of the critical points, it follows that for every set $x \in V_\kappa$, there is a thread with almost constant value x. Conversely, every thread eventually has a constant value in V_κ as $m \to \omega$. This is because no embedding j_m has an ordinal fixed point above κ, so every thread must eventually be below the critical point of all further embeddings as $m \to \omega$. Therefore, the Mostowski collapse of T is V_κ.

Suppose towards a contradiction that $\sup \hat{\pi}_0$ " $\kappa = \delta_0 < \lambda$. For each natural number m, let $\delta_m = \sup \pi_m$ " κ. Then $\lambda > \delta_0 \geq \delta_1 \geq \delta_2 \geq \ldots$. Pick n such that δ_n is minimal. Note that $\delta_n \geq \kappa > \kappa_i$ for each natural number i. Let η be an ordinal such that $\eta < \delta_n$ but $j_n(\eta) \geq \delta$. Suppose that $\hat{\pi}_{n+1}(\kappa_i) > \eta$ for some natural number i. Then $\hat{\pi}_n(\kappa_i) > \delta_n$, contradicting the definition of δ_n. From this contradiction, it follows that $\hat{\pi}_{n+1}(\kappa_i) \leq \eta$. Since i was arbitrary, it follows that $\delta_{n+1} \leq \mu < \delta_n$. This contradicts the minimality of δ_n, so statement (1) is proven.

To prove statement (2), note that κ_m and $\hat{\pi}_0(\kappa_m)$ are critical points of a I_3 embeddings on V_λ. Therefore, V_{κ_m} and $V_{\pi_0(\kappa_m)}$ are both elementary substructures of V_λ. It follows by the elementary chain lemma that $\hat{\pi}_0 : V_\kappa \to V_\lambda$ is elementary. By theorem 17, it follows that the inverse limit of the system $\langle V_\lambda, j_{mn} \rangle_{n \leq m}$ exists and is given by $(V_\kappa, \langle \hat{\pi}_n \rangle_{n \in \omega})$. This completes the proof of proposition 42. □

Next, I consider I_1 embeddings. Laver addresses these embeddings with the following theorem. Note that a Σ^1_ω embedding from V_λ to V_λ is equivalent to an elementary embedding from $V_{\lambda+1}$ to $V_{\lambda+1}$.

Theorem 43 ([Lav97, theorem 3.3]). *Let $\langle V_\lambda, j_{mn} \rangle_{n \leq m}$ be an inverse-directed system of order type ω^* in which the maps j_{mn} are all Σ^1_α embeddings for some particular ordinal $\alpha \leq \omega$. For each natural number m, let $j_m = j_{m+1,m}$.*

$$\cdots \xrightarrow{j_2} V_{\lambda+1} \xrightarrow{j_1} V_{\lambda+1} \xrightarrow{j_0} V_{\lambda+1}$$

Suppose that the critical points κ_m of the embeddings j_m do not attain their $\liminf$ *infinitely many times as $m \to \infty$. Then this system has an inverse limit in the category of Σ^1_α-elementary embeddings and models of* ZFC, *and the domain of the thread form of the inverse limit is the entire thread class.*

For the full proof of theorem 43, I refer the reader to Laver's paper. The proof is by induction on α. The base case is given by proposition 42.

Laver does not address systems of I_1 embeddings for which the critical point is eventually constant. He considers them to be part of the trivial case, which he addresses for I_3 embeddings (proposition 41) but not for I_1 embeddings. It turns out that such systems lack an inverse limit. I address this situation in proposition 44 below.

Proposition 44. *Let $\langle V_{\lambda+1}, j_{mn}\rangle_{n\leq m}$ be an inverse-directed system of I_1 embeddings of order type ω^*. Suppose that the critical point of $j_m = j_{m+1,m}$ is eventually constant as m tends to ω. Then the system has no inverse limit.*

$$\cdots \xrightarrow{j_2} V_{\lambda+1} \xrightarrow{j_1} V_{\lambda+1} \xrightarrow{j_0} V_{\lambda+1}$$

Proof. The cardinal λ is definable in V_λ as the greatest ordinal. Therefore, if the inverse limit exists and has thread form $(S, \langle\pi_n\rangle_{n\in\omega})$, then λ must be an element of $\pi_0 " S$. However, the cofinality of λ is ω. Furthermore, the model $V_{\lambda+1}$ satisfies enough of ZFC to carry out the proof of lemma 26. Therefore, since $\text{cof}(\lambda) = \omega$, the inverse limit does not exist. This completes the proof of proposition 44. □

Putting together the above results completely characterizes inverse limits of systems of I_1 and I_3 embeddings of order type ω^*. In particular, theorem 40 has been proven.

I conclude the chapter by proving that the thread class of the system described in proposition 44 is isomorphic to $V_{\kappa+1}$, provided that every embedding j_m is the same.

Theorem 45. *Let $j : V_{\lambda+1} \to V_{\lambda+1}$ be an I_1 embedding with critical point κ. Consider the corresponding system of elementary embeddings and models of set theory.*

$$\cdots \xrightarrow{j} V_{\lambda+1} \xrightarrow{j} V_{\lambda+1} \xrightarrow{j} V_{\lambda+1} \xrightarrow{j} V_{\lambda+1}.$$

The thread class, T, of this system is isomorphic to $V_{\kappa+1}$, which is not elementarily equivalent to $V_{\lambda+1}$

Proof. First, I show that every thread of nonempty sets has an $\in_T$-member. Let $\langle A_n \rangle_{n\in\omega}$ be a thread of nonempty sets, and let α_0 be the minimal $\in$-rank of a member of A_0. Then α_0 lies on a thread, since A_0 lies on a thread. Furthermore, $\alpha_0 < \lambda$, so it follows that $\alpha_0 < \kappa$, since j has no fixed points between κ and λ. Let a_0 be any member of A_0 of $\in$-rank α_0. Then a_0 is fixed by j, and so a_0 lies on a thread. This shows that every thread of nonempty sets has an $\in_T$-member as claimed. It follows that every thread is constant by lemma 22.

Therefore, the thread class, T, is isomorphic to the set of fixed points of j. So it suffices to show that the Mostowski collapse of this set of fixed points is equal to $V_{\kappa+1}$. Every fixed point that is an element of V_κ gets mapped to itself by the Mostowski collapse. The only remaining fixed points have $\in$-rank λ. Such a fixed point B must be mapped to $B \cap V_\kappa$ by the Mostowski collapse. To show that the Mostowski collapse maps surjectively onto $V_{\kappa+1}$, suppose that $B_0 \subseteq V_\kappa$ has rank κ. For each natural number n, let $B_{n+1} = j(B_n)$, and let $B_\omega = \cup_n B_n$. Then $j(B_\omega) = \cup_n j(B_n) = \cup_n B_{n+1} = B_\omega$. The final equality follows because $B_0 \subseteq B_1$. Therefore, B_ω is a fixed point of j, and the Mostowski collapse maps B_ω to B_0.

Since κ is inaccessible, $V_{\kappa+1}$ is a model of Kelley-Morse set theory, interpreting the sets of maximum rank as proper classes. However, under this same interpretation, $V_{\lambda+1}$ does not even satisfy the axiom of replacement — there is a cofinal ω-sequence in λ, and ω is a set, but λ is a proper class. It follow that the thread class is not elementarily equivalent to $V_{\lambda+1}$ □

1.10 Inverse limits of systems constructed in nonstandard models

Consider an inverse-directed system constructed in some ω-nonstandard model, N, of set theory, such that the order-type of the system is given by the reverse of the ω of N. Restricting this system to the standard ω^* produces a new inverse-directed system. The inverse limit of this new system always exists. A sketch of the proof is given by Victoria Gitman in unpublished notes. Such systems arise when studying computably saturated models of set theory, as in [GH10]. These systems are constructed with no large cardinal assumptions, using computably saturated models of ZFC. Thus, no large cardinal assumption is required to produce an inverse-directed system of models of ZFC with an inverse limit.

Below, in theorem 46, I prove a more general result using the same ideas. The special case mentioned above occurs when I is the nonstandard ω of N and I^+ is the standard ω.

Before presenting theorem 46, some technical prerequisites are needed. Let $(N, \in_N)$ be a possibly nonstandard model of set theory. From the perspective of N, let $\langle \tilde{M}^{(\alpha)}, \tilde{j}_{\alpha\beta} \rangle_{\alpha \leq \beta \in_N \tilde{I}}$ be an inverse-directed system of elementary embeddings and model-theoretic structures. Then it is possible to construct an inverse-directed system $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ in V which corresponds to the system in N via relativization.

To be precise, this system is defined as follows. For the sake of simplicity, I will consider systems where the model-theoretic structure $\tilde{M}^{(\alpha)}$ hs a single binary relation, $\tilde{E}_\alpha$, from the perspective of N. The analysis can be extended to other sorts of structures, although things become more complicated if there are functions or relations of nonstandard arity.

The partial order I is given by $\alpha \in I \iff \alpha \in_N \tilde{I}$, and $\alpha \leq \beta \iff N \models \alpha \leq \beta$. The domain of the model $M^{(\alpha)}$ is given by $\{\, \alpha \mid \alpha \in_N M^{(\alpha)} \,\}$. Given $x, y \in M^{(\alpha)}$, then $xE_\alpha y \iff N \models x\tilde{E}_\alpha y$. The map $j_{\alpha\beta}$ is given by $j(x) = y \iff N \models \tilde{j}_{\alpha\beta}(x) = y$. The map $j_{\alpha\beta}$ is indeed an elementary embedding, since the models $M^{(\alpha)}$ and $\tilde{M}^{(\alpha)}$ agree on the truth values of corresponding standard formulas.

Theorem 46. *Let $(N, \in_N)$ be a model of set theory. From the perspective of the model N, let $\langle \tilde{M}_\alpha, \tilde{j}_{\alpha\beta} \rangle_{\alpha \leq \beta \in_N \tilde{I}}$ be an inverse-directed system of elementary embeddings and model-theoretic structures. Let $\langle M^{(\alpha)}, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in I}$ be the corresponding inverse-directed system in V, as described above. Suppose that $I = I^- \sqcup I^+$ is a disjoint partition of I into an initial segment and a final segment such that this partition is not known to N. Formally speaking, this means that $I^+ = \{\, \alpha \in I \mid (\forall \beta \in I^-) \beta < \alpha \,\}$, and there is no element $\tilde{I}^+ \in N$ such that $\alpha \in I^+ \iff \alpha \in_N \tilde{I}^+$ for all indices α. Suppose further that I^- is directed and I^+ is inverse-directed. Let*

$$\mathfrak{M}^+ = \langle M_\alpha, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in_N I^+} \quad \text{and} \quad \mathfrak{M}^- = \langle M_\alpha, j_{\alpha\beta} \rangle_{\alpha \leq \beta \in_N I^-}.$$

Let T^- be the thread class of the directed system $\mathfrak{M}^-$, and let T^+ be the thread class of the inverse-directed system $\mathfrak{M}^+$. Then the inverse limit of $\mathfrak{M}^+$ exists, and its thread form is $(T^+, \langle \pi_\alpha \rangle_{\alpha \in I^+})$. Furthermore, T^+ is isomorphic to T^-, and the direct limit of $\mathfrak{M}^-$ is given by $(T^-, \langle i_\alpha \rangle_{\alpha \in I^-})$.

Proof. The direct limit of $\mathfrak{M}^-$ is given by $(T^-, \langle i_\alpha \rangle_{\alpha \in I^-})$, by lemma 11.

The following diagram will illustrate the proof.

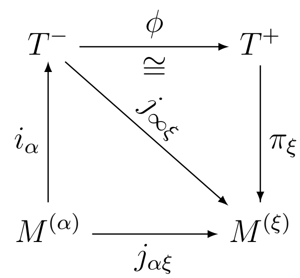


Figure 1.15: System constructed in a nonstandard model

Fix an index ξ such that $\xi \in I^+$. For any index α such that $\alpha \in I^-$, the map $j_{\alpha\xi}$ is an elementary embedding from $M^{(\alpha)}$ to $M^{(\xi)}$, and these maps commute in the appropriate fashion. It follows from the universal property for direct limits that there is a map $j_{\infty\xi} : T^- \to M^{(\xi)}$ such that $j_{\alpha\xi} = j_{\infty\xi} \circ i_\alpha$ for every index $\alpha \in I^-$.

Let $\langle \pi_\alpha \rangle_{\alpha \in I^+}$ be the projection maps of the inverse-directed system $\mathfrak{M}^+$, and let $\langle i_\alpha \rangle_{\alpha \in I^-}$ be the inclusion maps of the directed system $\mathfrak{M}^-$. There is a map $\phi : T^- \to T^+$ such that the map $j_{\infty\xi}$ factors as $j_{\infty\xi} = \pi_\xi \circ \phi$. In particular, this map ϕ is given as follows. Let $x \in T^-$. Then t is born in some model $M^{(\alpha)}$, that is to say, $x = i_\alpha(x_\alpha)$ for some x_α. Let $\phi(x) = \langle j_{\alpha\gamma}(x_\alpha) \rangle_{\gamma \in I^+}$. Since the system commutes, the definition of ϕ does not depend on the choice of α, and so ϕ is well-defined.

Next, I will show that $\phi : (T^-, \in_{T^-}) \to (T^+, \in_{T^+})$ is an isomorphism. Once I have shown this, then it follows that the projection map $\pi_\xi : T^+ \to M^{(\xi)}$ is elementary, since $j_{\infty\xi}$ is elementary and $j_{\infty\xi} = \pi_\xi \circ \phi$. Therefore, it will follow from theorem 17 that the inverse limit of $\mathfrak{M}^+$ is indeed given by $(T^+, \langle \pi_\alpha \rangle_{\alpha \in I^+})$.

That the map ϕ is preserves atomic formulas follows immediately from the definitions. Since the maps $j_{\alpha\gamma}$ are injective, the map ϕ is also injective. It remains to be shown that ϕ

is surjective. Towards this end, I will fix a thread $\langle x_\alpha \rangle_{\alpha \in I+} \in T^+$ and show that $\langle x_\alpha \rangle_{\alpha \in I+}$ is in the image of ϕ. Fix $\gamma \in I^+$. Let $J \supseteq I^+$ be the largest subset of I to which the thread $\langle x_\alpha \rangle_{\alpha \in I^+}$ extends. That is to say, $J = \{\, \beta \in I \mid (\exists x_\beta \in M_\beta) j_{\beta\gamma}(x_\beta) = x_\gamma \,\}$. The set J corresponds to a set definable in N and therefore must be a proper superset of I^+. Let $\beta \in J - I^+$, and let $x_\beta \in M^{(\beta)}$ such that $j_{\beta\gamma}(x_\beta) = x_\gamma$. Then $\langle x_\gamma \rangle_{\gamma \in I+} = \phi(x_\beta)$, so the map ϕ is indeed surjective. □

As previously mentioned, an important application of theorem 46 is when N is ω-nonstandard, I is α^* (that is, the reverse order of α) for some ordinal α of N, and I^+ is ω^*. Such an example can be constructed, by starting with an inverse-directed system in V and then taking the ultrapower of V by an ultrafilter on ω. In this way, it is possible to construct inverse limits of nonstandard analogues of systems for which no inverse limit exists. For instance, start with the system $\langle L[\mu_\omega][s], j_0 \rangle_{n<\omega}$ produced in theorem 28 by Prikry forcing. This system has no inverse limit. Let U be an ultrafilter on ω, and let $h : V \to V^\omega/U$ be the corresponding ultrapower. Apply theorem 46, to the system $h\Big(\langle L[\mu_\omega][s], j_0 \rangle_{n<\omega}\Big)$, taking $I^+ = \omega$. This yields a related nonstandard system with an inverse limit.

1.11 Ideas for further research

I conclude the chapter with some ideas for further research.

In sections 1.6 and 1.7, I produced an example of a system with no inverse limit from the assumption of a measurable cardinal, and I produced an example of a system with a small inverse limit from the assumption of a 1-extendible cardinal. But as section 1.10 shows, it is possible to produce a system with an inverse limit from no large cardinal hypothesis at all.

Question 47. *What is the large cardinal strength, if any, of an inverse-directed system of elementary embeddings and models of* ZFC *with no inverse limit? With an inverse limit that is given in thread form by a proper subset of the thread class? Is this consistency strength affected by requirements on the order type of the system or by the requirement that the models be transitive?*

With respect to question 47, I conjecture that it is possible, without large cardinal hypotheses, to produce a system with no inverse limit or with a small inverse limit, but that the easiest ways of doing this will either use a system with a fat order type (i.e. with no linear cofinal sequence) or else will use ill-founded models. To produce a system of order type ω^* consisting of transitive models with no inverse limit or with a small inverse limit would be more difficult and would perhaps require large cardinal hypotheses.

All of the examples that I give of systems with no inverse limit or with a small inverse limit make use of lemma 26. This observation gives rise to the following question.

Question 48. *Does there exist an inverse-directed system of transitive models of* ZFC *and elementary embeddings of the form*

$$\cdots \xrightarrow{j} M \xrightarrow{j} M \xrightarrow{j} M \xrightarrow{j} M$$

for which the inverse limit either does not exist or is not given in thread form by the entire thread class, and in which every fixed point of j above the critical point of j is regular in M?

If question 48 is answered in the negative, then this would suggest that large cardinal hypotheses may be required to obtain an inverse-directed system of order type ω^* consisting of transitive models with no inverse limit. This is because large cardinals are required for the Prikry-type forcings that change cofinalities without collapsing cardinals.

One way of constructing example systems to answer questions 47 and 48 might be to work with models generated by indiscernibles. However, one should keep in mind that in light of theorem 23, the models used in such a construction must not satisfy $V = HOD$.

A major area of inquiry for the chapter has been to determine necessary and sufficient conditions for the existence of inverse limits in the category of elementary embeddings and models of set theory. I have provided one necessary and sufficient condition in theorem 19: an inverse limit exists if and only if a natural source exists. However, this condition is closely related to the definition of an inverse limit. It would be interesting to produce a necessary and sufficient condition that is not, *prima facie*, related to the definition of the inverse limit. I formalize this idea in the following soft question.

Question 49. *Can a necessary and sufficient condition for the existence of inverse limits of elementary embeddings between models of set theory be given in such a way that this condition is not,* prima facie*, related to the definition of inverse limits, and such that this condition gives mathematical insight into the properties of inverse limits?*

Finally, more work can be done in analyzing the interactions between inverse limits and forcing. In that vein, I will close the chapter by restating the following question from section 1.8.

Question 50 (restatement of question 38)**.** *Suppose that the global lifting criterion holds, so that an inverse-directed system lifts over forcing in such a way that the generic filters form a thread in the lifted system, but that the thread of forcing notions* $\langle \mathbb{P}_\alpha \rangle_{\alpha \in I}$ *is not an element of the inverse limit model* S *of the ground system. Must the lifted system have an inverse limit?*

Chapter 2

Large cardinals between supercompact and almost-huge, including high-jump cardinals

2.1 Introduction

The purpose of this chapter is to examine the consistency and implicational strengths of several large cardinals falling between supercompact and almost-huge cardinals. Many of these cardinals are variants of the high-jump cardinals, which are described in definition 52. I will also investigate superstrong cardinals, which are weaker than supercompact cardinals but are closely related to high-jump cardinals. Many of the cardinals that I will discuss have been used by Apter, Hamkins, and Sargsyan to prove several results about universal indestructibility in [AH99], [AS07], [Apt08], and [Apt12].

Perhaps the most interesting result in this chapter is the main result of section 2.6, that a Woodin-for-supercompactness cardinal is equivalent to a Vopěnka cardinal. Another noteworthy result is that there are no excessively hypercompact cardinals, which is proven in section 2.8.

Recall that an almost-huge cardinal κ is characterized by an elementary embedding $j : V \to M$ with critical point κ such that M is closed under $<j(\kappa)$-sequences in V.[1] Although this definition is second-order, it has an equivalent first-order definition in terms of ultrafilters. However, the second-order definition should be considered as a fully formal definition in the context of NGBC set theory.

Many of the large cardinals that I will discuss here are natural weakenings of the definition of an almost-huge cardinal, formed by reducing the level of closure of the target model. Indeed, in my study of these cardinals, a key methodology is to define new large cardinals by weakening or otherwise modifying existing large cardinal definitions.

Often, the weaker large cardinals will still be sufficient for proving many of the same results as the stronger large cardinals. Eventually, by repeatedly weakening definitions, one hopes to obtain an equiconsistency, as is done in [AS10]. However, in this chapter, I focus on the large cardinals themselves rather than their applications.

The chart at the end of the introduction summarizes the relationships between the large cardinals discussed in this chapter. Most of the remaining sections of the chapter will be dedicated to proving these relationships. The arrows on the chart represent relationships between the cardinals, as indicated in the key. A solid arrow from A to B indicates a direct implication: every cardinal with property A has property B. A dotted arrow means that a cardinal of type A is *strictly* stronger in consistency than a cardinal of type B. That is to say, if there is a cardinal of type A, then it is consistent with ZFC that there is a cardinal of

[1] When I speak of an elementary embedding in this chapter, I always intend to denote an elementary embedding with a critical point between transitive proper class models of ZFC, unless otherwise stated.

type B. A double arrow indicates that both of these relationships hold.

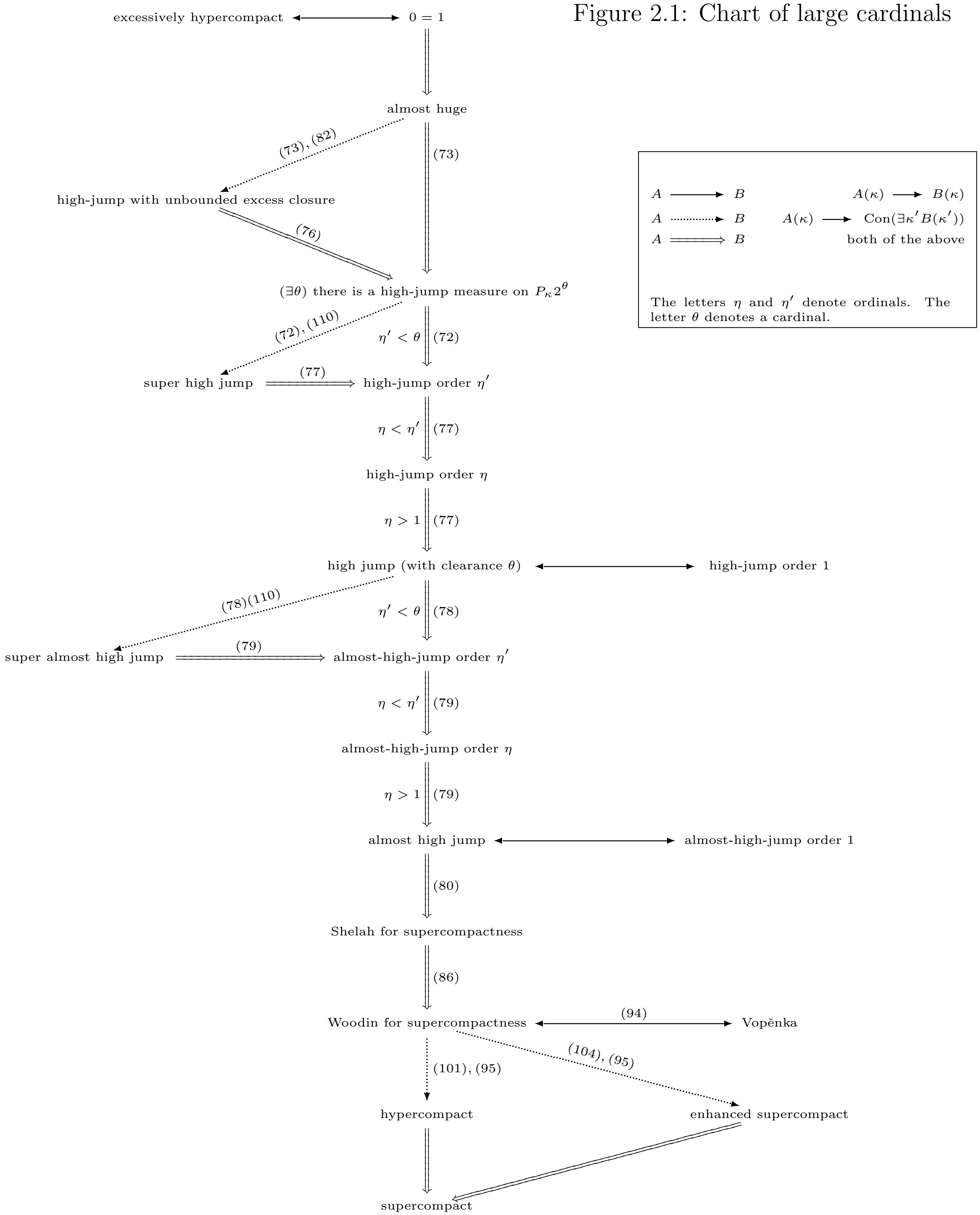
The arrows are labeled with theorem numbers referring to the theorems, propositions, and corollaries in which the corresponding results are proven. Dashed arrows are labeled with two numbers: one for a theorem demonstrating the consistency implication and one for a theorem demonstrating the failure of the direct implication.

The organization of the remainder of the chapter is as follows. The sections after section 2.2 can mostly be read out of order. I have noted the most important dependencies between the sections below.

In section 2.2, I define the clearance of an elementary embedding and use this property to define the high-jump cardinals. I also define and analyze the related notions of almost-high-jump cardinals, Shelah-for-supercompactness cardinals, and high-jump functions. In section 2.3, which depends on section 2.2, I further examine some of the definitions given in section 2.2 and give an analysis of how they might be modified. In section 2.4, which depends on section 2.2, I analyze properties of the clearance of an embedding and prove theorems tying together the ideas of the clearance of an embedding, the almost-high-jump cardinals, and the superstrong cardinals. The next few sections are arranged mostly by decreasing strength of the large cardinal notions studied. In section 2.5, which depends on section 2.2 and on lemma 66, I define and analyze several large cardinals above a Vopěnka cardinal and below an almost-huge cardinal. In section 2.6, I define the Vopěnka and Woodin-for-supercompactness cardinals and prove that they are equivalent. In section 2.7, I define and analyze universal high-jump functions. In section 2.8, I define the hypercompact cardinals and the excessively hypercompact cardinals, and I show that the existence of an excessively hypercompact cardinal is inconsistent with ZFC. In section 2.9, I define the enhanced supercompact cardinals and analyze their place in the large cardinal hierarchy. In section 2.10, which depends on section 2.2, I consider the relationship between high-jump cardinals and forcing. In section 2.11, which depends on section 2.2 and on lemma 66, I develop analogues of Laver functions for high-jump cardinals and related cardinals. In section 2.12, I review open problems and directions for further research.

Throughout the chapter, I use the label *theorem* to denote very important results. The results labeled as *propositions* vary in their mathematical depth. Some of them might more appropriately be considered as examples or observations.

Figure 2.1: Chart of large cardinals



2.2 High-jump cardinals, almost-high-jump cardinals, and Shelah-for-supercompactness cardinals

In this section, I will define high-jump cardinals, almost-high-jump cardinals, and Shelah-for-supercompactness cardinals.[2] I will also give characterizations for these large cardinals in terms of ultrafilters and prove a lemma about factor embeddings that will be very useful for the rest of the chapter.

The clearance of an elementary embedding, defined below in definition 51, is a key concept for defining several large cardinals. The motivation for defining the clearance is for use as a weaker substitute for $j(\kappa)$ in large cardinal definitions.

Definition 51. Let $j : M \to N$ be an elementary embedding with critical point κ. The **clearance** of j denotes the ordinal

$$\sup\{\, j(f)(\kappa) \mid f : \kappa \to \kappa \,\}.$$

The notation *clearance* is borrowed from the sport of pole vaulting, where the clearance is the height of the bar that the pole vaulter must clear. A high-jump embedding is like a pole vaulter: for a cardinal to be high jump, the closure of the embedding must successfully clear the clearance, as is described precisely in the following definition. The term *high-jump cardinal* comes from [AH99]. However, these cardinals were previously defined in [SRK78, p.111], where they were given the designation A_4.

[2] In many cases, the English usage rule for the punctuation of compound adjectives is to hyphenate compound adjectives coming before a noun, but not compound adjectives coming after a noun. Hence, I will write that κ is a high-jump cardinal, but also that the cardinal κ is high jump.

Definition 52. The cardinal κ is a **high jump cardinal** if and only if there exists a cardinal θ and an elementary embedding $j : V \to M$ with critical point κ and clearance θ such that $M^\theta \subseteq M$.

An embedding witnessing that κ is high jump is called a **high-jump embedding** for κ. A normal fine measure on $P_\kappa\theta$ generating an ultrapower embedding that is a high-jump embedding is called a **high-jump measure**.

The clearance of an embedding has strong properties, as I will show in the next section. In particular, I will show in corollary 65 that if θ is the clearance of *any* elementary embedding $j : V \to N$ with critical point κ, then $N_\theta \prec N_{j(\kappa)}$.

I wish to establish a combinatorial characterization of high-jump measures. In order to do so, the following two lemmas will be useful. These lemmas will continue to be useful throughout the chapter.

Lemma 53. *Suppose the diagram of elementary embeddings below commutes, let the critical point of j be κ, and let $\theta > \kappa$ be the critical point of k. Let $f : \kappa \to V_\kappa$ be a function. Then*

1. $\mathrm{rank}_\in(h(f)(\kappa)) \leq \mathrm{rank}_\in(j(f)(\kappa))$, *and*

2. *if $j(f)(\kappa) \in V_\theta$, then $h(f)(\kappa) = j(f)(\kappa)$.*

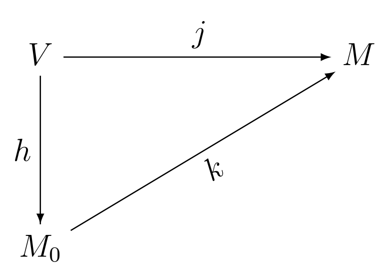


Figure 2.2: Factor embeddings

Proof. Since the critical point of k is above κ, it follows that

$$j(f)(\kappa) = (k \circ h)(f)(\kappa) = \big(k(h(f))\big)(k(\kappa)) = k(h(f)(\kappa)).$$

The first conclusion of the lemma follows from the elementarity of k. Next, suppose $j(f)(\kappa) \in V_\theta$. By the first part of the lemma, $h(f)(\kappa) \in V_\theta$ as well, and so $h(f)(\kappa)$ is fixed by k, and the second part of the lemma follows. □

This next lemma analyzes a special case of the situation from lemma 53 where the embedding j is high jump.

Lemma 54. *Let $j : V \to M$ be a high-jump embedding with critical point κ. Let θ be some cardinal such that $M^\theta \subseteq M$ and such that the clearance of j is at most θ. Let U be the normal fine measure on $P_\kappa\theta$ given by $A \in U \iff j\text{"}\,\theta \in j(A)$. Let h be the θ-supercompactness embedding generated by U, and let the embedding k be defined so that the diagram below commutes. Let $f : \kappa \to V_\kappa$ be a function in V. Then $h(f)(\kappa) = j(f)(\kappa)$. In particular, the measure U is a high-jump measure, the map h is a high-jump embedding, and the clearance of h is the same as the clearance of j.*

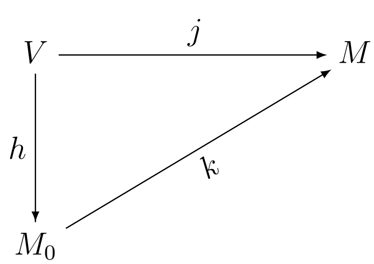


Figure 2.3: Factor embeddings of a high-jump embedding

Proof. The critical point of k is above θ and $j(f)(\kappa) < \theta$, so the proof follows immediately from lemma 53. □

Next, I provide a combinatorial characterization of high-jump measures, using the Łoś theorem. The exact details of this combinatorial characterization are not really important, though I do prove their correctness. Additionally, lemma 55 shows that the cardinal κ is high-jump if and only if there is a high-jump measure on $P_\kappa\theta$ for some cardinal θ.

Lemma 55. *Given an ordered pair of cardinals (κ, θ), the following are equivalent.*

1. *There exists a high-jump embedding $j : V \to M$ with critical point κ such that $M^\theta \subseteq M$ and the clearance of j is at most θ.*

2. *There exists a normal fine measure U on $P_\kappa\theta$ such that for every function $f : \kappa \to \kappa$, the set $\{ A \in P_\kappa\theta \mid f(\mathrm{ot}(A \cap \kappa)) < \mathrm{ot}(A) \}$ is a member of U. (The operator* ot *denotes order type.)*

Furthermore, any high-jump measure on $P_\kappa\theta$ (i.e. any normal fine measure generating a high-jump embedding) is of the form described in statement 2.

Proof. To prove that (2) $\implies$ (1), suppose that a measure of the type described in statement 2 exists, and let the function $f : \kappa \to \kappa$ be arbitrary. According to the theory of supercompact cardinals, the ultrapower $j_U : V \to M$ derived from U witnesses the θ-supercompactness of κ. Furthermore, the U-equivalence classes of the functions $c_f, g, h : P_\kappa\theta \to V$ given by $c_f(A) = f$, $f(A) = \mathrm{ot}(A) \cap \kappa$, and $g(A) = \mathrm{ot}(A)$ represent in M the sets $j(f)$, κ, and θ

respectively. Therefore, the requirement given on the measure guarantees that the clearance of j_U is at most θ.

Conversely, to prove that (1) $\implies$ (2), suppose there exists a high-jump embedding $j : V \to M$ with critical point κ and clearance at most θ. Let U be the normal fine measure on $P_\kappa\theta$ defined by $A \in U \iff j\text{ " }\theta \in j(A)$. Let $j_U : V \to M_U$ be the factor embedding of j generated by U, and let h be such that $h \circ j_U = j$.

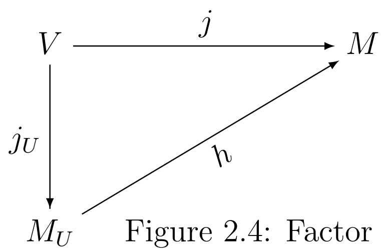


Figure 2.4: Factor embeddings of a high-jump embedding

Let $f : \kappa \to \kappa$ be an arbitrary function. By lemma 54, the factor embedding j_U is a high-jump embedding. Once again, the classes in the ultrapower M_U represented by c_f, f, and g from the previous paragraph represent $j(f)$, κ, and θ respectively, so it follows that the measure U satisfies statement 1.

More generally, if U is any normal fine measure on $P_\kappa\theta$ generating a high-jump embedding $j : V \to M_U$, then the classes in the ultrapower M_U represented by c_f, f, and g from the previous paragraph represent $j(f)$, κ, and θ respectively, so the measure U must satisfy statement 2. $\square$

Next, I define the almost-high-jump cardinals by a slight weakening of the closure property used for defining high-jump cardinals. An almost-high-jump cardinal is to a high-jump cardinal as an almost-huge cardinal is to a huge cardinal.

Definition 56. A cardinal κ is **almost high jump** if and only if there exists an elementary embedding $j : V \to M$ with critical point κ and clearance θ such that $M^{<\theta} \subseteq M$. Such an embedding is called an almost-high-jump embedding for κ.

Another way to look at the definition of an almost-high-jump cardinal is as follows. The cardinal κ is almost high jump if and only if there exists an elementary embedding $j : V \to M$ with critical point κ such that for every function $f : \kappa \to \kappa$, the closure property $M^{j(f)(\kappa)} \subseteq M$ holds.

The almost-high-jump cardinals have a combinatorial characterization in terms of coherent sequences of normal fine measures. As with the combinatorial characterization of high-jump measures, the details of the combinatorics are not so important; what matters is that the characterization exists and has a certain size and height. That being said, here are some of the details. By a *coherent* sequence of normal fine measures, I mean a sequence $\langle U_\lambda \mid \kappa \leq \lambda < \theta \rangle$, such that whenever $\kappa \leq \lambda \leq \delta < \theta$, the ultrafilter formed by chopping off U_δ at λ, given formally by $U_\delta \restriction \lambda = \{\, \{\, x \cap \lambda \mid x \in X \,\} \mid X \in U_\delta \,\}$, concentrates on U_λ. Such a coherent sequence of ultrafilters generates a directed system of elementary embeddings, with the maps $k_{\lambda\delta} : M_\lambda \to M_\delta$ given by $k([f]_{U_\lambda}) = [f']_{U_\delta}$, where f' is obtained from f by setting $f'(x) = f(x \cap \lambda)$.

Lemma 57. *A cardinal κ is almost high jump if and only if there exists a cardinal θ and a coherent sequence of normal fine measures $\langle U_\lambda \mid \kappa \leq \lambda < \theta \rangle$ such that for each λ, the measure U_λ concentrates on $P_\kappa\lambda$, and furthermore, the elementary embedding given by the direct limit of the system generated by this coherent sequence has clearance θ.*

Proof. The proof is essentially the same as that for the combinatorial characterization of almost-huge cardinals, which is given in [Kan04, Theorem 24.11], I will not give the details here. □

Weakening the definition of an almost-high-jump cardinal to allow for distinct embeddings to witness closure with respect to distinct functions $f : \kappa \to \kappa$ produces the definition of a Shelah-for-supercompactness cardinal. The analogue of this definition for strongness (in place of supercompactness) was originally formulated by Shelah.

Definition 58. A cardinal κ is **Shelah for supercompactness** if and only if for every function $f : \kappa \to \kappa$, there is an elementary embedding $j : V \to M$ such that $M^{j(f)(\kappa)} \subseteq M$.

Note that an almost-high-jump cardinal is a uniform version of a Shelah for supercompactness cardinal — with an almost-high-jump cardinal, one embedding must be the witness for every f uniformly, whereas with a Shelah-for-supercompactness cardinal, each function f may have a separate witnessing embedding.

One might want to define an almost-Shelah-for-supercompactness cardinal by tweaking the above definition to require that the closure of the target model is only $<j(f)(\kappa)$. However, this definition is actually equivalent to a Shelah-for-supercompactness cardinal, because of the following argument. Let $g : \kappa \to \kappa$ be given by $g(\alpha) = f(\alpha)^+$. If $j : V \to M$ is an elementary embedding with critical point κ such that $M^{<j(g)(\kappa)} \subseteq M$, then $M^{j(f)(\kappa) \subseteq M}$ as well.

In [Ham98, p.201], Hamkins defines a high-jump function as follows. A **high-jump function** for a (partially supercompact) cardinal κ is a function $f : \kappa \to \kappa$ such that

$j(f)(\kappa) > \lambda$ whenever j is a λ-supercompactness embedding on κ.[3] I will extend this definition to the vacuous case where κ is not partially supercompact and say that in this case, there exists a high-jump function for κ. The following proposition shows that the existence of a high-jump function for a cardinal κ is actually an anti-large-cardinal property.

Proposition 59. *Let κ be a cardinal. Then there exists a high-jump function for κ if and only if κ is not Shelah for supercompactness.*

Proof. The proof follows immediately from the definitions. The cardinal κ is Shelah for supercompactness if and only if

$$(\forall f : \kappa \to \kappa)(\exists j : V \to M \text{ with critical point } \kappa) \text{ such that } M^{j(f)(\kappa)} \subseteq M$$

The logical negation of this statement is

$$(*) \qquad (\exists f : \kappa \to \kappa)(\forall j : V \to M \text{ with critical point } \kappa)\ M^{j(f)(\kappa)} \nsubseteq M$$

I claim that the statement (*) is equivalent to the existence of a high-jump function for κ. In the forward direction, if (*) holds, then f is clearly a high-jump function for κ. The converse is also immediate if a λ-supercompactness embedding is taken to be simply an embedding in which the target model is λ-closed.

If a λ-supercompactness embedding is taken to mean an embedding generated by a normal fine measure on $P_\kappa\lambda$, then a factor embedding argument suffices as follows. Suppose that there exists a high-jump function, f, for κ. Let $j : V \to M$ be an elementary embedding (but

[3] Hamkins is unclear as to whether, by a λ-supercompactness embedding, he means an embedding generated by a normal fine measure on $P_\kappa\lambda$ or simply an embedding such that the target model is closed under λ sequences. I will handle both of these definitions in proposition 59 below.

not necessarily a supercompactness embedding) with critical point κ, and suppose towards a contradiction that $M^{j(f)(\kappa)} \subseteq M$. Let j_0 be the $j(f)(\kappa)$-supercompactness factor embedding of j generated by the normal fine measure on $P_\kappa(j(f(\kappa)))$ induced by j using $j \text{ " } j(f)(\kappa)$ as a seed. Then $j_0(f)(\kappa) = j(f)(\kappa)$, and so the embedding j_0 witnesses that the function f is not a high-jump function. □

The Shelah-for-supercompactness cardinals have an ultrafilter characterization similar to that for high-jump cardinals, given by the following corollary to lemma 55.

Corollary 60. *A cardinal κ is Shelah-for-supercompactness if and only if for every function $f : \kappa \to \kappa$, there is a cardinal θ and a normal fine measure U on $P_\kappa\theta$ such that the set $\{\, A \in P_\kappa\theta \mid f(\text{ot}(A \cap \kappa)) < \text{ot}(A) \,\}$ is a member of U.*

Proof. The proof is very similar to that of lemma 55, but I will repeat the details for the sake of clarity.

Let $f : \kappa \to \kappa$ be arbitrary. Let U be a measure on $P_\kappa\theta$ corresponding to f as given by the hypothesis of the corollary.

According to the theory of supercompact cardinals, the ultrapower $j_U : V \to M$ derived from U witnesses the θ-supercompactness of κ. Furthermore, the U-equivalence classes of the functions $c_f, g, h : P_\kappa\theta \to V$ given by $c_f(A) = f$, $f(A) = \text{ot}(A) \cap \kappa$, and $g(A) = \text{ot}(A)$ represent in M the sets $j(f)$, κ, and θ respectively. Therefore, the requirement given on the measure guarantees that j_U witnesses that κ is Shelah for supercompactness with respect to the function f. Since f was arbitrary, the cardinal κ is Shelah for supercompactness.

Conversely, suppose κ is Shelah for supercompactness, witnessed by the elementary embedding $j : V \to M$. Fix a function $f : \kappa \to \kappa$. θ be the maximum of $j(f)(\kappa)$ and κ. Let U be the normal fine measure on $P_\kappa\theta$ defined by $A \in U \iff j\,"\,\theta \in j(A)$. Let $j_U : V \to M_U$ be the factor embedding of j generated by U, and let h be such that $h \circ j_U = j$. Let $f : \kappa \to \kappa$ be an arbitrary function. From lemma 53, it follows that $j_U(f)(\kappa) = j(f)(\kappa)$, and so the embedding j_U witnesses that the cardinal κ is Shelah for supercompactness with respect to the function f. Once again, the classes in the ultrapower M_U represented by c_f, f, and g from the previous paragraph represent $j(f)$, κ, and θ respectively, so it follows that the measure U satisfies the combinatorial characterization given in the lemma with respect to the function f. The function f was chosen arbitrarily, so such a measure U exists for every function $f : \kappa \to \kappa$. □

2.3 Fiddling around with definitions

Three of the definitions from the previous section, those of an almost-high-jump cardinal, a Shelah-for-supercompactness cardinal, and a high-jump function, are closely related — they are obtained by rearranging the same basic clauses.

The cardinal κ is almost high jump if and only if

$$(\exists j : V \to M \text{ with } \operatorname{crit}(j) = \kappa)(\forall f : \kappa \to \kappa)\ M^{j(f)(\kappa)} \subseteq M.$$

(In this section, the expression $j : V \to M$ always denotes an elementary embedding.)

The cardinal κ is Shelah for supercompactness if and only if

$$(\forall f : \kappa \to \kappa)(\exists j : V \to M \text{ with } \operatorname{crit}(j) = \kappa)\ M^{j(f)(\kappa)} \subseteq M.$$

The cardinal κ has a high-jump function if and only if it is not Shelah for supercompactness, that is to say,

$$(\exists f:\kappa\to\kappa)(\forall j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)\ M^{j(f)(\kappa)}\not\subseteq M.$$

For the sake of thoroughness, it makes sense to investigate all the possible ways of rearranging these clauses. The first two quantified clauses can be arranged in either of two orders, and each can be existentially or universally quantified. The final clause can be negated or not. This leads to twelve formulas, as follows. Note that formulas 7 through 12 are the negations of formulas 1 through 6 respectively. That is to say, formula $n+6$ is the negation of formula n.

1. $(\exists j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)(\forall f:\kappa\to\kappa)\ M^{j(f)(\kappa)}\subseteq M$
2. $(\forall j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)(\exists f:\kappa\to\kappa)\ M^{j(f)(\kappa)}\subseteq M$
3. $(\exists j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)(\exists f:\kappa\to\kappa)\ M^{j(f)(\kappa)}\subseteq M$
4. $(\forall j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)(\forall f:\kappa\to\kappa)\ M^{j(f)(\kappa)}\subseteq M$
5. $(\forall f:\kappa\to\kappa)(\exists j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)\ M^{j(f)(\kappa)}\subseteq M$
6. $(\exists f:\kappa\to\kappa)(\forall j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)\ M^{j(f)(\kappa)}\subseteq M$
7. $(\forall j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)(\exists f:\kappa\to\kappa)\ M^{j(f)(\kappa)}\not\subseteq M$
8. $(\exists j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)(\forall f:\kappa\to\kappa)\ M^{j(f)(\kappa)}\not\subseteq M$
9. $(\forall j:V\to M \text{ with } \operatorname{crit}(j)=\kappa)(\forall f:\kappa\to\kappa)\ M^{j(f)(\kappa)}\not\subseteq M$

10. $(\exists j : V \to M \text{ with } \operatorname{crit}(j) = \kappa)(\exists f : \kappa \to \kappa)\ M^{j(f)(\kappa)} \not\subseteq M$

11. $(\exists f : \kappa \to \kappa)(\forall j : V \to M \text{ with } \operatorname{crit}(j) = \kappa)\ M^{j(f)(\kappa)} \not\subseteq M$

12. $(\forall f : \kappa \to \kappa)(\exists j : V \to M \text{ with } \operatorname{crit}(j) = \kappa)\ M^{j(f)(\kappa)} \not\subseteq M$

Taking κ to be an arbitrary cardinal, the formulas have the following meanings.

As discussed above, formula 1 is equivalent to "The cardinal κ is almost high jump," and so formula 7 is equivalent to "The cardinal κ is not almost high jump."

Formulas 2, 3, and 6 are trivially true, taking f to be the constant function given by $f(\alpha) = 0$. It follows that their negations, formulas 8, 9, and 12 are trivially false.

Formula 4 is vacuously true if there is no elementary embedding with critical point κ, i.e. if κ is not measurable. Otherwise, formula 4 is false — for a counterexample, take j to be an ultrapower by a normal fine measure on κ and f to be the function given by $f(\alpha) = 2^\alpha$. It follows that formula 10 is true if and only if κ is not measurable.

As discussed previously, formula 5 is equivalent to "The cardinal κ is Shelah for supercompactness," and its negation, formula 11, is equivalent to "there exists a high jump function for κ."

This exhaustive analysis shows that we are not missing out on analyzing any interesting notions. Of course, *interesting* has no rigorous definition, but it seems reasonable to say that the interesting notions that can be obtained from rearranging these clauses are an almost-high-jump cardinal, a Shelah for supercompactness cardinal, and a high-jump function.

2.4 The clearance, superstrongness embeddings, and related embeddings

Recall from definition 51 that the clearance of an elementary embedding $j : M \to N$ is defined as $\sup\{\, j(f)(\kappa) \mid f : \kappa \to \kappa \,\}$. In the large cardinal literature, a cardinal κ is **superstrong** if and only if there exists an elementary embedding $j : V \to M$ such that $V_{j(\kappa)} \subseteq M$. Such an embedding is called a superstrongness embedding for κ. A cardinal κ is **huge** if and only if there exists an elementary embedding $j : V \to M$ such that $\mathrm{M}^{j(\kappa)} \subseteq M$. A cardinal κ is **almost huge** if and only if there exists an elementary embedding $j : V \to M$ such that $M^{<j(\kappa)} \subseteq M$.

In section 2.5, I will show that an almost-huge cardinal is much stronger in consistency strength than a high-jump cardinal. Remarkably, the analogous situation does not hold in the case of strongness. In theorem 63, I will show that a superstrong cardinal is equivalent to a high-jump-for-strongness cardinal.

Before proving this result, I will prove some facts about the clearance of an embedding and about almost-high-jump embeddings. I begin with the following lemma.

Lemma 61. *Let $j : V \to M$ be an elementary embedding with critical point κ and clearance θ. Then the following conclusions are true.*

- *There is no function $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \theta$.*
- *The ordinal θ is a $\beth$ fixed point in M, that is to say, $\beth^M_\theta = \theta$.*
- *The inequality $\kappa^+ \leq \mathrm{cof}(\theta) \leq 2^\kappa$ holds in V.*

Proof. To prove the first conclusion, suppose to the contrary that f is a function such that $j(f)(\kappa) = \theta$. Let $g : \kappa \to \kappa$ be defined by $g(\alpha) = f(\alpha) + 1$. Then $j(g)(\kappa) = \theta + 1 > \theta$, contradicting the definition of the clearance.

Next, I will show that $\beth^M_\beta < \theta$ for all ordinals $\beta < \theta$, so that $\beth^M_\theta = \theta$. Let $\beta < \theta$. Then there exists a function $f : \kappa \to \kappa$ such that $j(f)(\kappa) \geq \beta$. Let the function $g : \kappa \to \kappa$ be given by $g(\alpha) = \beth_{f(\alpha)}$. Then $\beth^M_\beta \leq j(g)(\kappa) < \theta$. It follows that θ is a $\beth$ fixed point in M.

The cofinality of the clearance θ must be at most 2^κ, because the clearance is defined as the supremum of a set indexed by functions from κ to κ, of which there are 2^κ many.

Finally, I show that the cofinality of θ is at least κ^+ by a diagonalization argument. Suppose to the contrary that the cofinality of θ is at most κ. Then there is a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions on κ such that $\theta = \sup\{\, j(f_\alpha)(\kappa) \mid \alpha < \kappa \,\}$. Define a function $g : \kappa \to \kappa$ diagonalizing over these functions. That is to say, given $\beta < \kappa$, let $g(\beta) = \sup\{\, f_\alpha(\beta) + 1 \mid \alpha \leq \beta \,\}$. Then $j(f_\alpha)(\kappa) < j(g)(\kappa) < \theta$ for every $\alpha < \kappa$, contradicting the assumption that $\theta = \sup\{\, j(f_\alpha)(\kappa) \mid \alpha < \kappa \,\}$. □

More generally, the technique from lemma 61 shows that the clearance θ is a closure point not only of the $\beth$ function, but also of every function on the ordinals that is definable in N without parameters.

The next lemma applies the result of lemma 61 in the case that j is an almost-high-jump embedding.

Lemma 62. *Suppose $j : V \to M$ is an almost-high-jump embedding with critical point κ and clearance θ. Then the following conclusions are true in both V and M.*

- *The cardinal θ is a singular $\beth$ fixed point.*
- *The inequality $\kappa^+ \leq \text{cof}(\theta) \leq 2^\kappa$ holds.*
- *The cardinal exponentiation identity $\theta^\kappa = \theta$ holds.*

Proof. The proof follows from lemma 61, along with the fact that M is sufficiently closed so that it agrees with V on cofinalities less than θ and on cardinal exponentiation below θ.

To show that $\theta^\kappa = \theta$ in both V and M, note that θ is a strong limit in both V and M and $\text{cof}(\theta) > \kappa$ in both V and M. The fact that $\theta^\kappa = \theta$ in both V and M then follows from a basic theorem of cardinal arithmetic (see [Jec03, theorem 5.20]). □

With these preliminaries out of the way, I now state the main theorem of this section.

Theorem 63. *Let κ be a cardinal. The following are equivalent.*

1. *The cardinal κ is superstrong. That is to say, there exists an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(\kappa)} \subseteq M$.*
2. *There exists an elementary embedding $j : V \to M$ with critical point κ and clearance θ such that $V_\theta \prec M_{j(\kappa)}$.*
3. *The cardinal κ is high jump for strongness. That is to say, there exists an elementary embedding $j : V \to M$ with critical point κ and clearance θ such that $V_\theta \subseteq M$.*

In particular, every embedding of the type given in item 3 has a superstrongness factor embedding.

Proof. The proof of theorem 63 will hinge on a characterization of the clearance of the embedding j in terms of a certain seed hull. Consider an elementary embedding $j : V \to M$ with critical point κ and clearance θ. The seed hull of θ in M, denoted by X_θ, is defined by

$$X_\theta = \{\, j(f)(\alpha_0, \ldots \alpha_n) \mid \alpha < \theta \text{ and } f \in V \text{ is a function} \,\}.$$

Since θ is a cardinal in M, finite sequences of ordinals less than θ can be coded by Gödel coding using a single ordinal less than θ, and so

$$X_\theta = \{\, j(f)(\alpha) \mid \alpha < \theta \text{ and } f \in V \text{ is a function} \,\}.$$

The seed hull X_θ is an elementary substructure of M, and setting M' equal to its Mostowski collapse yields the following commutative diagram of elementary embeddings of models of set theory, where k is the inverse of the collapse map.

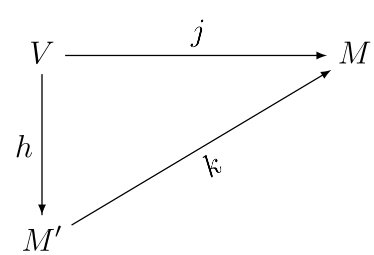


Figure 2.5: Factor embeddings of an embedding j with clearance θ

Lemma 64. *In the diagram described immediately above, the critical point of the embedding k is θ, and $k(\theta) = j(\kappa)$. Furthermore, $M_\theta \subseteq M'$.*

Proof of lemma 64. To prove that $\mathrm{crit}(k) = \theta$ and $k(\theta) = j(\kappa)$, it suffices to show that the supremum of the ordinals of X_θ below $j(\kappa)$ is θ. Towards this end, let $f : \kappa \to \kappa$ be a function. Let $\alpha < \theta$ be an ordinal. It suffices to show that $j(f)(\alpha) < \theta$.

Since $\alpha < \theta$, there exists a function $g : \kappa \to \kappa$ such that $\alpha < j(g)(\kappa)$. Define another function $h : \kappa \to \kappa$ by

$$h(\beta) = \sup\{\, f(\gamma) \mid \gamma < g(\beta) \,\}.$$

Since κ is inaccessible, the function h does indeed map from κ into κ. It follows from the definition of the clearance that $j(h)(\kappa) < \theta$. Furthermore, by elementarity of j, it follows that

$$j(h)(\kappa) = \sup\{\, j(f)(\gamma) \mid \gamma < j(g)(\kappa) \,\}.$$

In particular, from the case $\gamma = \alpha$, it follows that $j(f)(\alpha) \leq j(h)(\kappa) < \theta$, and so the proof of the first part of the lemma is complete.

Towards proving that $M_\theta \subseteq M'$, let $f : \kappa \to V_\kappa$ be an enumeration of V_κ in V such that whenever $\alpha < \kappa$, it follows that $V_\alpha \subseteq f\,``\,\beth_\alpha$. By lemma 61, the ordinal θ is a $\beth$ fixed point in M, and so it follows from the definition of f that $M_\theta \subseteq X_\theta$. Since M' is the Mostowski collapse of X_θ in M, it follows that $M_\theta \subseteq M'$. This completes the proof of the lemma. □

The bulk of the work of proving theorem 63 is now complete. It is time to put together the details.

To prove the implication (1) $\implies$ (2), note that, given (1), the map k in the diagram above is pointwise constant on V_θ and maps V_θ elementarily into $M_{j(\kappa)}$.

The implication (2) $\implies$ (3) is immediate.

Finally, to prove the implication (3) $\implies$ (1), assume that the map j is a high-jump-for-strongness embedding. It follows from lemma 64 that $V_\theta = M_\theta = M'_\theta$ and $k(\theta) = \kappa$. Since

$k \circ h = j$, it follows that $h(\kappa) = \theta$. It follows that h is a superstrongness embedding for κ. □

A few easy corollaries follow.

Corollary 65. *Let $j : V \to M$ be an elementary embedding with critical point κ and clearance θ. Then $M_\theta \models \mathrm{ZFC}$ and $M_\theta \prec M_{j(\kappa)}$.*

Proof. In the proof of lemma 64, the elementary embedding k fixes M_θ pointwise and maps M_θ elementarily into $M_{j(\kappa)}$. Since the cardinal κ is the critical point of an elementary embedding, it must be measurable, so that $V_\kappa \models \mathrm{ZFC}$, and so it follows that $M_{j(\kappa)}$ and M_θ satisfy ZFC as well. □

Lemma 66 is a key fact about almost-high-jump embeddings, and it will be used in many places in this chapter.

Lemma 66. *Let $j : V \to M$ be an almost-high-jump embedding for κ with clearance θ. Then $V_\theta \models \mathrm{ZFC}$ and $V_\theta \prec M_{j(\kappa)}$.*

Proof. By lemma 62, the clearance is a $\beth$ fixed point, so $|V_\theta| = \theta$ and $V_\theta = M_\theta$. Therefore, lemma 66 follows from corollary 65. □

Corollary 67. *Every almost-high-jump embedding has a factor embedding that is a superstrongness embedding.*

Proof. This follows immediately from theorem 63, since every almost-high-jump embedding is also a high-jump-for-strongness embedding. □

This final corollary demonstrates another closure property of the clearance. Informally speaking, it says that, given $j : V \to M$ with critical point κ, properties that hold up to the clearance in M hold all the way up to $j(\kappa)$ in M, allowing as parameters elements of V_θ and elements of the range of j. In case the embedding is almost high jump, this lemma becomes quite powerful, as many properties that hold in V up to the clearance will hold in N up to $j(\kappa)$.

Corollary 68. *Let $j : V \to M$ be an elementary embedding with critical point κ and clearance θ. Suppose that for some formula φ and some parameters $q, j(p)$, with $q \in V_\theta$,*

$$M \models (\forall\gamma < \theta)\varphi(\gamma, q, j(p)).$$

Then

$$M \models (\forall\gamma < j(\kappa))\varphi(\gamma, q, j(p)).$$

Proof. I will prove the contrapositive of the corollary. Suppose

$$M \models (\exists\gamma < j(\kappa))\neg\varphi(\gamma, q, j(p)).$$

Refer to the diagram accompanying theorem 63. The parameter q is fixed by the map k, since $q \in V_\theta$. By the elementarity of the map k, it follows that

$$M' \models (\exists\gamma < \theta)\neg\varphi(\gamma, q, h(p)).$$

In particular, this statement holds in M' for some particular $\gamma_0 < \theta$. But since the critical point of k is θ, it follows that

$$M \models \neg\varphi(\gamma_0, q, j(p)).$$

□

As a closing observation, note that analogues of many of the results in this section can be proven when V is replaced by a more general model, N.

2.5 Large cardinals strictly above a Vopěnka cardinal

In the next few sections, I define the remaining large cardinals mentioned in the chart from the introduction, and I prove results about their consistency and implicational strengths. The sections are organized in order of strength in the large cardinal hierarchy. In the present section, I consider cardinals stronger than a Vopěnka cardinal but no stronger than an almost-huge cardinal.

The method of reflection, which I will review here briefly, is a common technique in consistency proofs. Suppose that $j : V \to M$ is elementary, and suppose that in the model M, the cardinal κ satisfies some large cardinal property (or some other formula), $\varphi(\kappa, j(p))$. Then $\kappa \in j(\{ \alpha < \kappa \mid \varphi(\alpha, p) \})$, and so it follows that measure-1 many α below κ have property φ with parameter p.

I begin by defining the large cardinal notions that I will be analyzing in this section, starting with the high-jump order and the super-high-jump cardinals. These definitions are somewhat analogous to the definitions of the many times huge and superhuge cardinals, which are defined in [BDT84].

Definition 69. Given an ordinal η, the cardinal κ has **high-jump order** η if and only if there exists a strictly increasing sequence $\langle \theta_\alpha \mid \alpha < \eta \rangle$ of ordinals such that for each ordinal $\alpha < \eta$, there exists a high-jump embedding for κ with clearance θ_α. The cardinal κ is **super high jump** if and only if there exist high-jump embeddings for κ of arbitrarily high clearance. (In other words, a super-high-jump cardinal κ has high-jump order ORD.)

The almost-high-jump order and the super-almost-high-jump cardinals are defined similarly to the high-jump order and the super-high-jump cardinals, as follows.

Definition 70. Given an ordinal η, the cardinal κ has **almost-high-jump order** η if and only if there exists a strictly increasing sequence $\langle \theta_\alpha \mid \alpha < \eta \rangle$ of cardinals such that for each ordinal $\alpha < \eta$, there exists an almost-high-jump embedding for κ with clearance θ_α. The cardinal κ is **super almost high jump** if and only if there exist almost-high-jump embeddings of arbitrarily high clearance for κ.

It will also be interesting to consider high-jump embeddings with **excess closure**, that is, embeddings $j : V \to M$ with clearance θ such that the target model M is closed under sequences of length greater than θ. For instance, high-jump embeddings with clearance θ where the target model is closed under sequences of length 2^θ will be fruitful objects of study. An extreme example of excess closure is as follows.

Definition 71. The cardinal κ is **high jump with unbounded excess closure** if and only if for some fixed clearance θ, for all cardinals $\lambda \geq \theta$, there is a high-jump measure on $P_\kappa\lambda$ generating an embedding with clearance θ.

With all of the above definitions given, the time has come to prove many of the simpler consistency strength relations shown on the chart in the introduction, along with some additional related consistency strength relations that are not shown on the chart.

I begin with the following proposition, which involves a high-jump embedding with a little bit of excess closure. This proposition is a simple example of the use of lemma 66, which will be used in more complicated arguments later.

Proposition 72. *Suppose that there exists a pair of cardinals (κ, θ) such that there is a high-jump embedding $j : V \to M$ with critical point κ and clearance θ and such that $M^{2^\theta} \subseteq M$. Then the cardinal κ is super-high-jump in the model V_θ, and the cardinal κ has high-jump order θ in V. Furthermore, there are many super-high-jump cardinals in the model V_κ.*

Proof. By lemma 54, there is a factor embedding, h, of j such that h has clearance θ and is generated by a high-jump measure U on $P_\kappa\theta$. By lemma 62, the cardinal exponentiation identity $\theta^\kappa = \theta$ holds. It follows that the model M is sufficiently closed so that $U \in M$.

In the model $M_{j(\kappa)}$, consider the set of cardinals λ such that there is a high-jump measure generating an embedding with critical point κ and clearance λ. By lemma 66, the elementarity relation $V_\theta \prec M_{j(\kappa)}$ holds. It follows that if this set of cardinals is bounded in the model $M_{j(\kappa)}$, then this bound is below θ. But θ is an element of this set, since $U \in M$. Therefore, the set is unbounded in both V_θ and $M_{j(\kappa)}$. By reflection, there are many super-high-jump cardinals in the model V_κ. Finally, since $V_\theta \models$ ZFC and since every high-jump measure of V_θ is also a high-jump measure in V, it follows that the cardinal κ has high-jump order θ in V. □

Note that the hypothesis of proposition 72 is equivalent to the hypothesis that there for some pair (κ, θ), such that there is a high-jump measure on $P_\kappa 2^\theta$. This alternative hypothesis follows immediately from the hypothesis of proposition 72. For the converse, given a pair (κ, θ) such that there is a high-jump measure on $P_\kappa 2^\theta$, the clearance of the corresponding embedding must be at most θ. If the clearance of this embedding is some $\theta' < \theta$, then take a $2^{\theta'}$-supercompactness factor embedding and apply lemma 54.

Next, I will consider elementary embeddings for which the closure of the target model is extremely large compared with the clearance of the embedding, beginning with the high-jump cardinals with unbounded excess closure.

Proposition 73. *Suppose the cardinal κ is almost huge. Then in the model V_κ, there are many cardinals δ such that δ is high jump with unbounded excess closure*

Proof. Suppose κ is almost huge, witnessed by an elementary embedding $j : V \to M$ with clearance θ. In particular, the embedding j is also a high-jump embedding. Let λ be a cardinal such that $\theta \leq \lambda < j(\kappa)$. The cardinal $j(\kappa)$ is a strong limit cardinal. Therefore, by lemma 54, the embedding j has a λ-supercompactness factor embedding with clearance θ generated by a high-jump measure on $P_\kappa \lambda$. This high-jump measure is an element of $M_{j(\kappa)}$. The conclusion of the theorem follows by reflection. □

Consider a cardinal κ such that for all sufficiently large cardinals λ, there is a high-jump measure on $P_\kappa \lambda$. It may be possible that such a cardinal is not high jump with unbounded excess closure, because the high-jump measures may not all generate embeddings with the same closure. However, the following proposition shows that these two types of cardinals are equiconsistent.

Proposition 74. *The following two large cardinal axioms are equiconsistent over* ZFC*.*

1. *There exists a cardinal κ such that for all sufficiently large cardinals λ, there is a high-jump measure on $P_\kappa\lambda$.*

2. *There exists a cardinal that is high jump with unbounded excess closure*

In particular if there are high-jump measures on $P_\kappa\lambda$ for all sufficiently large cardinals λ, then either κ is high jump with unbounded excess closure or else there is a cardinal θ such that κ is high jump with unbounded excess closure in the model V_θ.

Proof. It is immediate from the definitions that if κ is high jump with unbounded excess closure, then for all sufficiently large λ, there is a high-jump measure on $P_\kappa\lambda$.

For the converse, suppose that for all sufficiently large λ, there is a high-jump measure on $P_\kappa\lambda$, but the cardinal κ is not high jump with unbounded excess closure. Let θ_0 be the minimal cardinal such that for all cardinals $\lambda \geq \theta_0$, there is a high-jump measure on $P_\kappa\lambda$. Since the cardinal κ is not high jump with unbounded excess closure, these high-jump measures do not all generate embeddings with clearance θ_0.

None of these measures generates a high-jump embedding with clearance less than θ_0. If it did, then the minimality of θ_0 would be contradicted by taking factor embeddings and applying lemma 54.

Accordingly, let θ_1 be the least cardinal above θ_0 such that there is a high-jump embedding for κ with clearance θ_1. Let $j : V \to M$ be a high-jump embedding for κ with clearance θ_1.

Then the model V_{θ_1} satisfies ZFC by lemma 66, and in this model, the cardinal κ is high jump with unbounded excess closure with respect to the clearance θ_0. □

Note that in the case from the proof above where the cardinal κ is high jump with unbounded excess closure in V_{θ_1}, it follows from lemma 66 that for any high-jump embedding $j : V \to M$ with critical point κ and clearance θ_1, the cardinal κ is high jump with unbounded excess closure in the model $M_{j(\kappa)}$, so by reflection there are many cardinals in the model V_κ that are high jump with unbounded excess closure.

The next two propositions follow the same basic line of reasoning as proposition 72. Informally speaking, the they show that the degrees of excess closure of high-jump embeddings form a hierarchy of consistency strength.

Proposition 75. *Suppose that for some cardinals κ and θ and for some ordinal $\alpha < \theta$, there exists a high-jump embedding $j : V \to M$ with critical point κ and clearance θ such that the model M is closed under sequences of length $2^{\aleph_{\theta+\alpha}^{<\kappa}}$. Then in the model $M_{j(\kappa)}$, there are unboundedly many cardinals λ such that there is a high-jump measure on $P_\kappa(\aleph_{\lambda+\alpha})$ generating a high-jump embedding with critical point κ and clearance λ.*

Proof. In the model $M_{j(\kappa)}$, consider the set of cardinals λ such that there is a high-jump measure on $P_\kappa(\aleph_{\lambda+\alpha})$ generating a high-jump embedding with critical point κ and clearance λ. The model $M_{j(\kappa)}$ is sufficiently closed to see that θ is an element of this set. By lemma 66, the elementarity relation $V_\theta \prec M_{j(\kappa)}$ holds, so it follows that this set is unbounded in $M_{j(\kappa)}$. □

By reflection, the conclusion of proposition 75 implies that there are many cardinals in the model V_κ with the same property that κ has in $M_{j(\kappa)}$.

The next proposition moves even further up the hierarchy of high-jump cardinals with excess closure. Any cardinal that is high jump with unbounded excess closure satisfies the hypothesis of the proposition 76. Note that the conclusion of proposition 76 does not imply that in $M_{j(\kappa)}$ there are high-jump measures on $P_\kappa\delta$ for all sufficiently large δ — in general there will be gaps in the set of such δ.

Proposition 76. *Suppose that for some cardinals κ and θ, there exists a high-jump embedding $j : V \to M$ with critical point κ and clearance θ such that $M^{\beth_{\theta+\theta}} \subseteq M$. Then in the model $M_{j(\kappa)}$, for all cardinals $\gamma < j(\kappa)$, there are unboundedly many cardinals λ such that there is a high-jump measure on $P_\kappa(\beth_{\lambda+\gamma})$ generating a high-jump embedding with critical point κ and clearance λ.*

Proof. In the model $M_{j(\kappa)}$, fix some cardinal γ, and consider the set of cardinals λ such that there is a high-jump measure on $P_\kappa(\beth_{\lambda+\gamma})$ generating a high-jump embedding with critical point κ and clearance λ. The model M is sufficiently closed to see that θ is an element of this set. By lemma 66, the elementarity relation $V_\theta \prec M_{j(\kappa)}$ holds, so it follows that this set is unbounded in $M_{j(\kappa)}$. □

By reflection, the conclusion of proposition 76 implies that many cardinals in the model V_κ have the same property that κ has in $M_{j(\kappa)}$.

Suppose that κ was chosen to be minimal such that there exists a θ so that the hypothesis of proposition 76 is satisfied. In that case, for every $\gamma < j(\kappa)$, the corresponding cardinals λ

satisfying the conclusion of the proposition must be greater than γ. This further emphasizes that there is lots of space in the large cardinal hierarchy between a cardinal satisfying the hypothesis of proposition 76 and a high-jump cardinal with unbounded excess closure.

I chose propositions 76 and 75 to give an idea of the flavor of this hierarchy of high-jump cardinals with excess closure. Many similar propositions could be proven. One way of proving similar propositions would be to use other functions in place of the $\aleph$ and $\beth$ functions.

Another possibility would be to look at target models that are even more closed in comparison to the clearance, for instance, closed under sequences of length $\beth_{\theta\cdot\omega}$ or $\beth_{\theta^+}$ or something even bigger.

Next, I move on to prove some results lower on the hierarchy of high-jump cardinals and related cardinals.

Proposition 77. *Let η and η' be ordinals such that $\eta < \eta'$. Suppose the cardinal κ has high-jump order η'. Then there is an elementary embedding $j : V \to M$ with critical point κ such that the cardinal κ has high-jump order η in $M_{j(\kappa)}$.*

Proof. The cardinal κ has high-jump order η', and this is witnessed by a sequence of clearances $\langle \theta_\alpha \mid \alpha < \eta' \rangle$. Let $j : V \to M$ be a high-jump embedding for κ with clearance θ for some θ_α such that $\alpha \geq \eta$. Then the model M is sufficiently closed so that in $M_{j(\kappa)}$, the cardinal κ has high-jump order η. □

Note that the hypothesis of proposition 77 is really much stronger than needed: the embedding j could have just been a supercompactness embedding generated by a normal fine measure on $P_\kappa(\theta_\alpha)$. Also, note that in case the ordinal η is fixed by the embedding j, then in the model V_κ there are many cardinals with high-jump order η.

I now move further down the large cardinal hierarchy, to the almost-high-jump cardinals.

Proposition 78. *Suppose there is a high-jump embedding with critical point κ and clearance θ. Then κ has almost-high-jump order θ, and in the models V_θ, $M_{j(\kappa)}$, and V_κ, there are many super-almost-high-jump cardinals.*

Proof. Suppose $j : V \to M$ is a high-jump embedding with critical point κ and clearance θ. It follows immediately from definitions that the embedding j also witnesses that κ is almost high jump. By corollary 62, the cardinal θ is a strong limit, and so it follows that the coherent sequence of measures witnessing that there is an almost-high-jump embedding for κ with clearance θ is an element of H_{θ^+}. This coherent sequence of measures is also an element of M, by the closure of M. Therefore, the cardinal κ is almost high jump in M with clearance θ. Consider the set $\{\, \delta \mid M_{j(\kappa)} \models \kappa \text{ almost high jump with clearance } \delta \,\}$. By theorem 65, if this set has a bound in $M_{j(\kappa)}$, then the bound must be less than θ. It follows that the set is unbounded in $M_{j(\kappa)}$, and so κ is super almost high jump in the model $M_{j(\kappa)}$ and also in the model V_θ. The other conclusions of the theorem follow immediately. □

Proposition 79. *Let $\eta < \eta'$ be ordinals. Suppose the cardinal κ has almost-high-jump order η'. Then there is an elementary embedding $j : V \to M$ with critical point κ such that the cardinal κ is has almost-high-jump order η in $M_{j(\kappa)}$.*

Proof. The proof follows the same reasoning as the proof of proposition 77, replacing high-jump embeddings with almost-high-jump embeddings and high-jump measures with sequences of measures of the sort described in lemma 57. □

Finally, I reach the Shelah-for-supercompactness cardinals.

Proposition 80. *Suppose the cardinal κ is almost-high-jump. Then there are many cardinals below κ that are Shelah for supercompactness.*

Proof. Suppose $j : V \to M$ is an almost-high-jump embedding for κ with clearance θ. I will show that κ is Shelah for supercompactness in M. Let $f : \kappa \to \kappa$ be a function in M. Let $j_0 : V \to M_0$ be the λ-supercompactness factor embedding induced by j, where λ is the maximum of $j(f)(\kappa)$ and κ. From corollary 62, it follows that $2^{\lambda^{<\kappa}} < \theta$. Therefore, the λ-supercompactness measure U that generates j_0 is an element of M. From lemma 53, it follows that $j(f)(\kappa) = j_0(f)(\kappa)$. Let j_0^M be the elementary embedding generated by U in M. The measure U is an element of $V_\theta = M_\theta$, which satisfies ZFC by theorem 65. It follows that $j_0 \restriction V_\theta = j_0^M \restriction M_\theta$. Therefore, in M, the elementary embedding j_0^M witnesses that κ is Shelah for supercompactness with respect to the function f. Since f was arbitrary, it follows that κ is Shelah for supercompactness in M. □

I wind up the section with a few miscellaneous propositions. Proposition 81 is easy to prove but worth stating.

Proposition 81. *Suppose $j : V \to M$ is an elementary embedding with clearance θ witnessing that κ is almost-high-jump. Then in the model V_κ there are many supercompact cardinals.*

Proof. The embedding j witnesses that the cardinal κ is λ-supercompact for every $\lambda < \theta$. The closure of M, along with the fact from lemma 62 that θ is a strong limit, guarantees that κ is supercompact in M_θ, and so by theorem 65, κ is supercompact in $M_{j(\kappa)}$, and the lemma is proven. □

Proposition 82 shows that several direct implications are lacking from the large cardinal hierarchy.

Proposition 82. *The least high-jump cardinal is not Σ_2-reflecting. In particular, it is not supercompact and not even strong. The same is true for the least almost-huge cardinal, the least almost-high-jump cardinal, and the least Shelah-for-supercompactness cardinal.*

Proof. All of these cardinals can be characterized by Σ_2 definitions — they are characterized by a measure or a set of measures with certain combinatorial properties, all of which can be seen from within a particular V_α. Since supercompact and strong cardinals are Σ_2-reflecting, the theorem follows. □

The definition of super high jump makes it tempting to think that every cardinal that is both supercompact and high jump is super high jump. However, the following simple proposition shows that this is not the case.

Proposition 83. *If κ is the least cardinal that has high-jump order 2, then in V_κ, there are many cardinals that are both supercompact and high jump but not super high jump.*

Proof. Let $j : V \to M$ witness that κ has high-jump order 2, and let U be the normal measure on κ given by $A \in U \iff \kappa \in j(A)$

By lemma 81, the set of supercompact cardinals of V_κ is a member of U. By proposition 77, the set of cardinals that are high jump in V_κ is a member of U. It follows that the set of cardinals that are both supercompact and high jump in V_κ is a member of U. However, since κ is the least cardinal that has high-jump order 2 , none of these cardinals has high-jump order 2, and hence none is super high jump in V_κ. □

In particular, this shows that below a cardinal of high-jump order 2, there are many cardinals that are both high jump and tall. If a cardinal κ is both high jump and tall, then there are high-jump embeddings for κ such that $j(\kappa)$ is arbitrarily large — this can be seen by taking a high-jump embedding for κ followed by a tallness embedding for $j(\kappa)$. It is easy to see that below a cardinal that is both high-jump and supercompact, there are many high-jump cardinals. But it is an open question whether the existence of a cardinal that is both high-jump and tall is equiconsistent with the existence of a high-jump cardinal.

Many more definitions could be made along the lines of the ones given in this section, and these definitions would lead to many more questions of consistency strength. In light of proposition 72, the following question comes to mind.

Question 84. *What is the consistency strength of the existence of a high-jump embedding with critical point κ and clearance θ generated by a high-jump measure on $P_\kappa\theta^+$? Is the existence of such an embedding equiconsistent with the existence of a high-jump embedding with critical point κ and clearance θ generated by a high-jump measure on $P_\kappa 2^\theta$?*

Of course, under GCH, these two types of embeddings are equivalent.

2.6 The equivalence of Vopěnka cardinals and Woodin-for-supercompactness cardinals

In this section, I define a Woodin-for-supercompactness cardinal and show that it is strictly weaker than a Shelah-for-supercompactness cardinal. I then show that a cardinal is Woodin for supercompactness if and only if it is a Vopěnka cardinal.

The definition of a Woodin-for-supercompactness cardinal is as follows. This definition is taken from [AS07, 2], although they have also been studied by Foreman [For07, p.31] and by Fuchs [Fuc09, p.1043] under the name of Woodinized supercompact cardinals.

A Woodin-for-supercompactness cardinal is the analogue of a Woodin cardinal, with supercompactness in place of strongness.

Definition 85. A cardinal δ is **Woodin for supercompactness** if and only if for every function $f : \delta \to \delta$, there exists a cardinal $\kappa < \delta$ such that κ is a closure point of f (i.e. $f\text{"}\,\kappa \subseteq \kappa$), and there exists an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(f)(\kappa)} \subseteq M$.

Note in particular that every Woodin-for-supercompactness cardinal is also a Woodin cardinal.

Apter and Sargsyan require that f be defined so that $f(\alpha)$ is always a cardinal and so that the elementary embedding j is generated by a supercompactness measure on $P_\kappa\lambda$ for some $\lambda < \delta$, but this definition is equivalent to the definition given here. The first requirement does not make their definition any weaker, because any function f can be replaced with a function f' given by $f'(\alpha) = |f(\alpha)|^+$. I will show below in theorem 88 that the second requirement does not make their definition any stronger.

This next result shows that a Shelah-for-supercompactness cardinal is strictly stronger than a Woodin-for-supercompactness cardinal. The proof is an improvement on lemma 1.1 of [AS07] and uses a similar line of reasoning.

Theorem 86. *Suppose the cardinal κ is Shelah for supercompactness. Then κ is Woodin for supercompactness, and there are many cardinals below κ that are Woodin for supercompactness in both V_κ and V.*

Proof. Let κ be Shelah for supercompactness. The main difficulty is to show that κ is Woodin for supercompactness. Once this has been shown, it is immediate that there are many cardinals below κ that are Woodin for supercompactness in the model V_κ, because "The cardinal κ is Woodin for supercompactness" is Π^1_1-definable over V_κ, and Shelah-for-supercompactness cardinals are Π^1_1-indescribable, since they are weakly compact. Furthermore, it is easily seen that any cardinal that is Woodin for supercompactness in the model V_κ must also be Woodin for supercompactness in V.

Towards showing that κ is Woodin for supercompactness, let $f : \kappa \to \kappa$ be an arbitrary function. I will show that κ satisfies the definition of Woodin for supercompactness with respect to f. Without loss of generality, I assume that f is nowhere regressive, that is to say, for all $\alpha < \kappa$ the inequality $\alpha \leq f(\alpha)$ holds. It follows that $\kappa \leq j(f)(\kappa)$. Let $g : \kappa \to \kappa$ be given by $g(\alpha) = 2^{f(\alpha)^{<\alpha}}$. Let $j : V \to M$ witness that κ is Shelah for supercompactness with respect to the function g, that is, the embedding j has critical point κ and $M^{j(g)(\kappa)} \subseteq M$.

Note that $j(g)(\kappa) = 2^{(j(f)(\kappa))^{<\kappa}}$, as calculated in both M and V.

Let U be the normal fine measure on $P_\kappa\big(j(f)(\kappa)\big)$ given by $A \in U \iff j"j(f)(\kappa) \in j(A)$. Let $j_U : V \to N$ be the $j(f)(\kappa)$-supercompactness factor embedding induced by U. Let $k : N \to M$ be the elementary embedding such that $k \circ h = j$. The model M is sufficiently closed so that $U \in M$, and the closure of M further guarantees that every function from $P_\kappa\big(j(f)(\kappa)\big)$ to M is an element of M. It follows that the elementary embedding induced by U in M, as calculated in M, is equal to $j_U \restriction M$, as calculated in V.

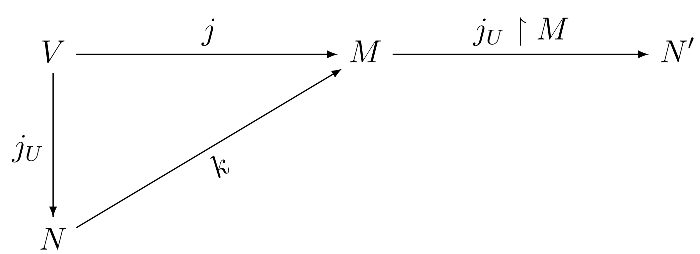


Figure 2.6: Shelah-for-supercompactness is stronger than Woodin-for-supercompactness

Next, I will show that

$$j(f)(\kappa) = j_U(j(f))(\kappa) \tag{2.1}$$

First of all, $f = j(f) \restriction \kappa$. Applying j_U to both sides, it follows that $j_U(f) = j_U(j(f)) \restriction j_U(\kappa)$, and in particular, $j_U(f)(\kappa) = j_U(j(f))(\kappa)$. Furthermore, from lemma 53, it follows that $j(f)(\kappa) = j_U(f)(\kappa)$, and so statement 2.1 is proven.

The embedding $j_U \restriction M$ is generated in M by the measure $U \in M$. Therefore, in M, the cardinal $j(\kappa)$ satisfies the definition of Woodin for supercompactness with respect to the function $j(f)$ — this fact is witnessed by the embedding $j_U \restriction M$ along with statement 2.1, since κ is a closure point of $j(f)$. It follows from the elementarity of j that in V, the cardinal κ satisfies the definition of Woodin for supercompactness with respect to the function f. But f was chosen arbitrarily, so κ is Woodin for supercompactness in V. □

Like Woodin cardinals, Woodin-for-supercompactness cardinals have several alternative characterizations, as described below in theorem 88. The proof that these characterizations hold is essentially the same as for the case of Woodin cardinals, (see [Kan04, Theorem 26.14]). In order to state them, I need the following definition of (γ, A)-supercompactness, which is sometimes also called γ-supercompactness for A.

Definition 87. Given a set A and a cardinal γ, the cardinal κ is **$(\boldsymbol{\gamma}, \mathbf{A})$-supercompact** if and only if there is an elementary embedding $j : V \to M$ with critical point κ such that

1. $\gamma < j(\kappa)$,
2. $M^\gamma \subseteq M$, and
3. $A \cap V_\gamma = j(A) \cap V_\gamma$.

Given cardinals κ and δ such that $\kappa < \delta$, the notation κ *is* $({<}\delta, A)$*-supercompact* denotes that κ is (γ, A)-supercompact for all cardinals $\gamma < \delta$. The alternative characterizations of Woodin-for-supercompactness cardinals are as follows.

Theorem 88. *Given a cardinal δ, the following are equivalent.*

1. *For every function $f : \delta \to \delta$, there exists a cardinal $\kappa < \delta$ such that κ is a closure point of f (i.e. $f\,"\,\kappa \subseteq \kappa$), and there exists an elementary embedding $h : V \to M$ with*

critical point κ such that $M^{h(f)(\kappa)} \subseteq M$. Furthermore, the embedding h is generated by a normal fine measure on $P_\kappa\lambda$ for some cardinal $\lambda < \delta$.

2. *The cardinal δ is Woodin for supercompactness. That is to say, for every function $f : \delta \to \delta$, there exists a cardinal $\kappa < \delta$ such that κ is a closure point of f (i.e. $f\,"\,\kappa \subseteq \kappa$), and there exists an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(f)(\kappa)} \subseteq M$.*

3. *For every set $A \subseteq V_\delta$, the set*

$$\{\,\kappa < \delta \mid \kappa \textit{ is } ({<}\delta, A)\textit{-supercompact}\,\}$$

is stationary in δ.

4. *For every set of ordinals $A \subseteq \delta$, the set*

$$\{\,\kappa < \delta \mid \kappa \textit{ is } ({<}\delta, A)\textit{-supercompact}\,\}$$

is nonempty.

Proof. The implication (1) $\implies$ (2) is immediate. I begin by proving the implication (2) $\implies$ (3).

Assume the cardinal δ is Woodin for supercompactness, and fix a set $A \subseteq V_\delta$ and a club $C \subseteq \delta$. Define a function $f : \delta \to \delta$ as follows. If ξ is not $({<}\delta, A)$-supercompact, then let $f(\xi)$ be the least inaccessible cardinal in C above ξ and above the failure of the $({<}\delta, A)$-supercompactness of ξ. Otherwise, define $f(\xi)$ arbitrarily — for the sake of concreteness, take $f(\xi) = 0$.

Since δ is Woodin for supercompactness, there exists a cardinal $\kappa < \delta$ and an elementary embedding $j : V \to M$ with critical point κ such that $f " \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$. Furthermore, the definition of f, along with the fact that κ is a closure point of f, ensures that $C \cap \kappa$ is a club in κ. It follows that $\kappa \in j(C)$. Now, it suffices to show that

$$M \models \kappa \text{ is } \big({<}j(\delta),\, j(A)\big)\text{-supercompact},$$

and then hypothesis 3 will follow by reflection.

Suppose to the contrary that for some minimal cardinal λ such that $\kappa \leq \lambda < j(\delta)$,

$$M \models \kappa \text{ is not } \big(\lambda,\, j(A)\big)\text{-supercompact}.$$

By the definition of f, it must be the case that $2^{\lambda^{<\kappa}} < j(f)(\kappa)$. Let the measure U be the λ-supercompactness measure on $P_\kappa\lambda$ induced by j via the seed $j " \lambda$. Let j_U be the corresponding factor embedding, and let k be defined so that $k \circ j_U = j$. The closure of the model M ensures that the measure U is an element of M and that every function from $P_\kappa\lambda$ to M is an element of M. It follows that the ultrapower induced by U on M, as calculated in M, is equal to $j_U \restriction M$, as calculated in V.

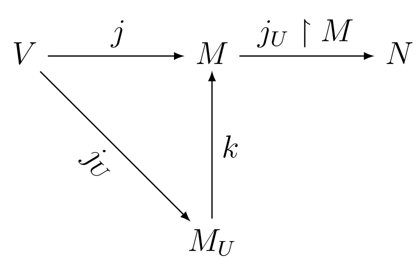


Figure 2.7: Factor embeddings of a Woodin-for-supercompactness embedding

In M, the map $j_U \restriction M$ is a λ-supercompactness embedding with critical point κ, so it only remains to verify the equality

$$j(A) \cap M_\lambda = j_U(j(A)) \cap M_\lambda.$$

The critical point of the embedding k is above λ, so for every set X it must be the case that $j(X) \cap V_\lambda = j_U(X) \cap V_\lambda$, and so this is true in particular when $X = A \cap V_\kappa$. It follows that

$$\begin{aligned} j(A) \cap M_\lambda &= j(A \cap V_\kappa) \cap M_\lambda \\ &= j_U(A \cap V_\kappa) \cap M_\lambda \\ &= j_U(j(A) \cap V_\kappa) \cap M_\lambda \\ &= j_U(j(A)) \cap M_\lambda \end{aligned}$$

This completes the proof of (2) $\implies$ (3). The implication (3) $\implies$ (4) is immediate. So finally, I will prove that (4) $\implies$ (1).

Assume hypothesis (4) holds, and let $f : \delta \to \delta$ be an arbitrary function. I will show that δ satisfies the hypothesis (1) with respect to the function f. Let $A_f \subseteq \delta$ encode f by Gödel pairing. Hypothesis (4) implies that there exists $\kappa < \delta$ such that the cardinal κ is $({<}\delta,\, j(A))$-supercompact. I will first show that such a κ must be a closure point of f, that is to say, $f\,''\,\kappa \subseteq \kappa$.

Let $\xi < \kappa$, and let $j : V \to M$ witness that κ is (θ, A_f) supercompact for some sufficiently large θ such that $\theta > f(\xi)$ and such that $j(f)(\xi) = f(\xi)$.[4] Since j is a (θ, A_f)-

[4] To be precise, this can be done as follows: let the ordinal β be the Gödel code for the ordered pair $(\xi, f(\xi))$, and take $\theta > \beta$. Then $j(A_f) \cap V_\theta = A_f \cap V_\theta$, and so $\beta \in j(A_f)$. Furthermore, by elementarity, $j(A_f)$ encodes $j(f)$ by Gödel pairing, and so $j(f)(\xi) = f(\xi)$.

supercompactness embedding for κ, it follows that

$$j(\kappa) > \theta > f(\xi) = j(f)(\xi).$$

Furthermore,

$$j(f)(\xi) = j(f)(j(\xi)) = j(f(\xi)),$$

since $\xi < \kappa = \mathrm{crit}(j)$. Thus $j(f(\xi)) < j(\kappa)$, and it follows by elementarity that $f(\xi) < \kappa$. Therefore, $f \text{ "} \kappa \subseteq \kappa$ as claimed.

Finally, let $j' : V \to M$ witness that κ is (θ', A_f)-supercompact, where $\theta' \geq f(\kappa)$ is chosen large enough so that $j'(f)(\kappa) = f(\kappa)$. It follows that $M^{j'(f)(\kappa)} \subseteq M$. Let U be the normal fine measure on $P_\kappa\big(j'(f)(\kappa)\big)$ given by $A \in U \iff j' \text{ "} j'(f)(\kappa) \in j'(A)$. Let $h : V \to N$ be the ultrapower embedding generated by U. Then $h(f)(\kappa) \leq j'(f)(\kappa)$ by lemma 53, so the embedding h witnesses that κ is Woodin for supercompactness with respect to the function f. Since $j'(f)(\kappa) = f(\kappa) < \delta$, it follows that h witnesses that hypothesis (1) is satisfied. □

I now shift my attention to the Vopěnka cardinals, which I will eventually show are equivalent to the Woodin-for-supercompactness cardinals.

Definition 89. The cardinal δ is **Vopěnka** if and only if for every δ-sequence of model-theoretic structures $\langle M_\alpha \mid \alpha < \delta \rangle$ over the same language, with each structure M_α an element of V_δ, there exists an elementary embedding $j : M_\alpha \to M_\beta$ for some ordinals $\alpha < \beta < \delta$.

It is convenient to have a characterization of Vopěnka cardinals in terms of a limited class of structures. Towards this end, following [Kan04], I make the following definition.

Definition 90. Let δ be a cardinal. A sequence of model-theoretic structures, $\langle M_\alpha \mid \alpha < \delta \rangle$ is a **natural δ-sequence** if and only if the following properties are satisfied. There is a function $f : \delta \to \delta$ such that the domain of M_α is $V_{f(\alpha)}$ and such that whenever $\alpha < \beta < \delta$, are ordinals, it follows that $\alpha < f(\alpha) \leq f(\beta) < \delta$. Furthermore, each structure M_α is of the form $(V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha)$, for some unary relation $R_\alpha \subseteq V_{f(\alpha)}$.

Note that without loss of generality, R_α may be taken to encode finitely many constants, functions, and relations of any finite arity on $V_{f(\alpha)}$. I will show in proposition 93 below that it suffices to consider only natural sequences when determining whether the cardinal κ is Vopěnka.

It is immediately clear that any sequence of the type specified in definition 89 can be encoded as a natural sequence if the language involved is countable. For larger languages, one might be concerned that the critical point of the embedding would be small enough to mess up the encoding.

To clear up this concern, I will next analyze the critical point of an elementary embedding between two elements of a natural sequence. This analysis will also play an important role in the proof of the equivalence between Vopěnka cardinals and Woodin-for-supercompactness cardinals. The inclusion of the constant $\{\alpha\}$ in the definition of a natural sequence ensures that any elementary embedding between members of a natural sequence has a critical point.

I begin the analysis of these critical points with the definition of the Vopěnka filter, given below in definition 91. The Vopěnka filter is actually a filter on δ if and only if δ is a Vopěnka cardinal. In this case, the Vopěnka filter is a normal filter and contains every club. These facts are proven in [Kan04, pp.336-337] and [SRK78, proposition 6.3], but I will give some of the proofs below.

Definition 91 ([Kan04, p.336]). Let δ be an inaccessible cardinal. Then the set $X \subseteq \delta$ is a member of the **Vopěnka filter** on δ if and only if there exists a natural δ-sequence $\langle M_\alpha \mid \alpha < \delta \rangle$ such that whenever $j : M_\alpha \to M_\beta$ is an elementary embedding, then the critical point of j is an element of X.

Let F denote the Vopěnka filter. First of all, if δ is not Vopěnka, then $\emptyset \in F$, and so F is not a filter. Next, suppose that δ is Vopěnka. It is immediate that $\delta \in F$ and that $A \in F$ and $A \subseteq B$ imply $B \in F$. One can see that F is closed under binary intersections by using one natural sequence to encode the pointwise amalgamation of two different natural sequences. Thus, F is a filter.

The argument that the filter F is normal is given in [Kan04, proposition 24.14]. The proof involves amalgamating together δ-many natural sequences into a single natural sequence.

The arguments above do not show that F contains every tail of δ, so in order to show that F contains every club, it does not suffice to know that F is normal; instead, an explicit argument is required. This argument is given in [SRK78, proposition 6.3], but I will reproduce it below, since I make use of its result in the proofs of proposition 93 and theorem 94.

Proposition 92 ([SRK78, proposition 6.3]). *Let δ be a Vopěnka cardinal. Then the Vopěnka filter, F, on δ contains every closed unbounded subset $C \subseteq \delta$.*

Proof. Let $C \subseteq \delta$ be a club. Consider the natural δ-sequence given by

$$M_\alpha = \big(V_{f(\alpha)}, \in, \{\alpha\}, C \cap f(\alpha)\big),$$

where $f(\alpha)$ is the least limit point of C above α. Let $j : M_\alpha \to M_\beta$ be an elementary embedding. Let κ be the critical point of j, and suppose towards a contradiction that $\kappa \notin C$, and so in particular, the cardinal κ is not a limit point of C. Let ρ be the greatest element of C below κ, and let η be the least element of C above κ. The ordinal η is definable in M_α from ρ, which is below the critical point, κ of j. It follows that $j(\eta + 2) = \eta + 2$, and $\eta > \kappa$. Thus, the restriction $j \restriction V_{\eta+2} : V_{\eta+2} \to V_{\eta+2}$ is an elementary embedding with a critical point, and this contradicts the Kunen inconsistency. □

Given an elementary embedding $j : V_\alpha \to V_\beta$, the critical point of j must be inaccessible, and so if δ is a Vopěnka cardinal, then the set of inaccessible cardinals below δ is a member of the Vopěnka filter on δ. In particular, this implies that δ is Mahlo, since the Vopěnka filter contains every club.

With proposition 92 in place, I am ready to prove that it suffices to look only at normal sequences in order to define the Vopěnka cardinals.

Proposition 93. *The cardinal δ is Vopěnka if and only if for every normal δ-sequence $\langle M_\alpha \mid \alpha < \delta\rangle$, there is an elementary embedding $j : M_\alpha \to M_\beta$ for some ordinals $\alpha, \beta < \delta$.*

Proof. The forwards direction is immediate. For the backwards direction, let $\langle N_\alpha \mid \alpha < \delta\rangle$ be a sequence of model-theoretic structures over a language $\mathcal{L}$. with each $N_\alpha \in V_\delta$. Encode the structures N_α in a natural sequence $\langle M_\alpha \mid \alpha < \delta \rangle$ given by

$$M_\alpha = (V_{f(\alpha)}, \in, \{\alpha\}, N_\alpha).$$

By proposition 92, there are elementary embeddings $j : M_\alpha \to M_\beta$ with arbitrarily high critical points. In particular, the critical point can be chosen to be larger than the size of the language $\mathcal{L}$, and so j induces an elementary embedding from N_α to N_β (considered as $\mathcal{L}$-structures rather than subsets of $V_{f(\alpha)}$), provided that the structures N_α were encoded in a reasonable way. □

Kanamori suggests the equivalence between a Woodin-for-supercompactness cardinal and a Vopěnka cardinal in [Kan04, p.364]. However, he does not formally define a Woodin-for-supercompactness cardinal, nor does he work out the details of the equivalence.

Theorem 94. *The cardinal δ is Woodin for supercompactness if and only if δ is a Vopěnka cardinal. Furthermore, if δ is a Vopěnka cardinal, then for every set $A \subseteq V_\delta$, the set $\{\kappa < \delta \mid \kappa \text{ is } ({<}\delta, A)\text{-supercompact}\}$ is a member of the Vopěnka filter on δ.*

Proof. For the forward direction, let δ be Woodin for supercompactness, and let $A = \langle A_\alpha \mid \alpha < \delta \rangle$ be a natural δ-sequence. I will show that for some ordinals $\alpha < \beta < \delta$, there exists an elementary embedding $j : A_\alpha \to A_\beta$.

Since δ is Woodin for supercompactness, there is a cardinal $\kappa < \delta$ such that κ is $({<}\delta, A)$-supercompact. Therefore, by choosing a large enough degree of A-supercompactness, there is an elementary embedding $j : V \to N$ such that $j(A)_\kappa = A_\kappa$ and $j \restriction A_\kappa \in N$. In N, the map $j \restriction A_\kappa$ is an elementary embedding from $j(A)_\kappa$ to $j(A)_{j(\kappa)}$. So in N, there exists an elementary embedding between two elements of $j(A)$. By the elementarity of j, there exists an elementary embedding between two elements of A in V. It follows that δ is Vopěnka.

The proof of the converse direction uses some of the same ideas as the proof of proposition 24.14 of [Kan04], which shows that V_δ contains many extendible cardinals if δ is Vopěnka. Suppose that the cardinal δ is Vopěnka, and let $A \subseteq V_\delta$. I will show that there exists a cardinal $\kappa < \delta$ such that κ is $({<}\delta, A)$-supercompact, thereby showing that the cardinal δ is Woodin for supercompactness. Indeed, I will show that the set of such κ is an element of the Vopěnka filter on δ.

Let $g : \delta \to \delta$ be the variation of the failure-of-A-supercompactness function described as follows. Given $\xi < \delta$, let $g(\xi)$ be the least cardinal $\eta > \xi$ such that ξ is not (η, A)-supercompact. In case no such η exists, then set $g(\xi) = \xi$.

Let $C \subseteq \delta$ be the club of closure points of g. That is to say, $C = \{\, \rho < \delta \mid g\,"\,\rho \subseteq \rho \,\}$. Since C is a club, it follows from proposition 92 that C is an element of the Vopěnka filter on δ. Therefore, there exists a natural δ-sequence $\langle M_\alpha \mid \alpha < \delta \rangle$ such that whenever $j : M_\alpha \to M_\beta$ is an elementary embedding, the critical point of j is an element of C.

For each ordinal $\alpha < \delta$, let γ_α be the least inaccessible element of C above all the ordinals of M_α, and for each ordinal $\alpha < \delta$, let

$$N_\alpha = (V_{\gamma_\alpha}, \in, \{\alpha\}, M_\alpha, C \cap \gamma_\alpha, A \cap V_{\gamma_\alpha}).$$

Let $j : N_\alpha \to N_\beta$ be an elementary embedding. It suffices to show that the critical point, κ, of j is $({<}\delta, A)$-supercompact.

Assume to the contrary that κ is not $({<}\delta, A)$-supercompact. Then $\kappa < g(\kappa)$. Furthermore, $g(\kappa) < \gamma_\alpha$, because $\gamma_\alpha \in C$.

Since M_α is encoded in N_α, it follows from the definition of the sequence $\langle M_\alpha \rangle$ that $\kappa \in C$. By the elementarity of j, it follows that $j(\kappa) \in C$ as well.

Let U be the normal fine measure on $P_\kappa\big(g(\kappa)\big)$ generated by j using $j \text{ " } g(\kappa)$ as a seed.

Using the fact that γ_α is inaccessible, the theory of factor embeddings shows that there exists a model N and an elementary embedding k so that the following diagram commutes. The map j_U is the ultrapower generated by U in V.

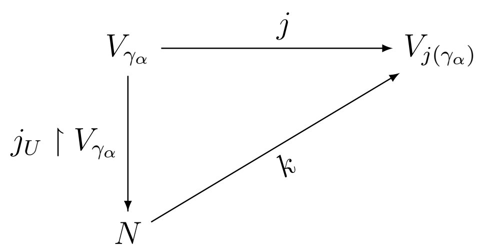


Figure 2.8: Factor embeddings of a Vopěnka embedding

I claim that the map $j_U : V \to M$ witnesses that κ is $(g(\kappa), A)$-supercompact. This will contradict the definition of g, thereby completing the proof. Clearly, this map is a $g(\kappa)$-supercompactness embedding with critical point κ, and so it suffices to show that $j_U(A) \cap V_{g(\kappa)} = A \cap V_{g(\kappa)}$.

First of all, since $A \cap V_{\gamma_\alpha}$ is encoded in N_α, it follows that $j(A) \cap V_{g(\kappa)} = A \cap V_{g(\kappa)}$. Since the critical point of k is an inaccessible cardinal above $g(\kappa)$, it follows that $j(A) \cap V_{g(\kappa)} = j_U(A) \cap V_{g(\kappa)}$. $\square$

For the sake of completeness, I conclude this section with a basic nonimplication result, which is well-established in the large cardinal literature.

Proposition 95. *The least cardinal that is Vopěnka is not weakly compact.*

Proof. "The cardinal κ is Vopěnka" is definable by a Π^1_1 formula over V_κ, but weakly compact cardinals are Π^1_1-indescribable. □

In particular, this result implies that there is no direct implication between a Vopěnka cardinal and a hypercompact cardinal or an enhanced supercompact cardinal, since the latter two cardinals (which will be defined in sections 2.8 and 2.9) are measurable.

2.7 Universal high-jump functions

In this section, the expression "$j : V \to M$" will always denote an elementary embedding. Recall from sections 2.2 and 2.3 that a cardinal κ has a high-jump function if and only if κ is not Shelah for supercompactness, which is true if and only if

$$(\exists f : \kappa \to \kappa)(\forall j : V \to M \text{ with } \operatorname{crit}(j) = \kappa)\ M^{j(f)(\kappa)} \not\subseteq M.$$

In section 2.3, I rearranged the clauses and quantifiers in this definition in many different ways and concluded that the only nontrivial formulations that came up were almost-high-jump cardinals, Shelah-for-supercompactness cardinals, and their negations. But there is another way of rearranging this definition that I have not considered. Instead of limiting consideration to a single cardinal κ, one could consider the behavior of a function $f : \mathrm{ORD} \to \mathrm{ORD}$ with respect to all measurable cardinals κ. One interesting formulation is as follows. I will call a function f satisfying the following statement a **universal high-jump function**.

$$(\exists f : \mathrm{ORD} \to \mathrm{ORD})(\forall (j,\kappa) \text{ such that } j : V \to M \text{ with } \operatorname{crit}(j) = \kappa)$$
$$f \text{"}\, \kappa \subseteq \kappa \text{ and } M^{j(f)(\kappa)} \not\subseteq M$$

Formally, this may be considered as a proposition in GBC set theory. Alternatively, by restricting the embeddings in question to those generated by supercompactness measures, one may ask in a first-order way whether a particular definable function $f : \mathrm{ORD} \to \mathrm{ORD}$ is a universal high-jump function. A factor embedding argument similar to the one from proposition 59 shows that the added requirement that j be generated by a supercompactness measure produces an equivalent definition.

Given a cardinal δ such that $V_\delta \models \mathrm{ZFC}$, one can also ask whether there exists a universal high-jump function for V_δ, that is, does there exist a function $f : \delta \to \delta$ such that $(V_\delta, f) \models f$ is a universal high-jump function?

Like the existence of a high-jump function, the existence of a universal high-jump function is an anti-large-cardinal property. If there are no measurable cardinals, then any function $f : \mathrm{ORD} \to \mathrm{ORD}$ is a universal high-jump function. If there is no cardinal κ that is κ^+-supercompact, then the function given by $\alpha \mapsto \alpha^+$ is a universal high-jump function.

Proposition 96. *If the cardinal δ is Woodin for supercompactness, then there is no universal high-jump function for V_δ.*

Proof. The proof follows immediately from the definitions. Since δ is Woodin for supercompactness, it follows that for every function $f : \delta \to \delta$, there is a closure point κ of f and an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(f)(\kappa)} \subseteq M$. Therefore, the function f is not a universal high-jump function. □

If there exists a universal high-jump function f, then there cannot exist a Shelah-for-supercompactness cardinal, because for any measurable cardinal κ, the function $f \restriction \kappa$ is a high-jump function for κ. However, the existence of a universal high-jump function is not provably equivalent to the nonexistence of a Shelah-for-supercompactness cardinal. For instance, if δ is the least Woodin-for-supercompactness cardinal, then in the model V_δ, there are no Shelah-for-supercompactness cardinals by theorem 86, but there is also no universal high-jump function for V_δ by proposition 96.

A similar line of reasoning shows that the converse of proposition 96 is not provable in ZFC. For instance, if δ is inaccessible but not Woodin for supercompactness, and some cardinal $\kappa < \delta$ is Shelah for supercompactness in V_δ, then there is no universal high-jump function for V_δ, but δ is not Woodin for supercompactness. One would not expect to achieve this example simply by cutting off the universe at the least inaccessible cardinal δ above a Shelah-for-supercompactness cardinal κ, because κ may no longer be Shelah for supercompactness in V_δ. However, if δ is the least strong cardinal above κ, then cutting off the universe at V_δ achieves the example. In this case, the cardinal κ is still Shelah for supercompactness in V_δ — by taking a large enough strongness embedding $j : V \to M$ with critical point δ, the cardinal κ is Shelah for supercompactness in $M_{j(\delta)}$, and so κ is also Shelah for supercompactness in V_δ. Furthermore, the cardinal δ is not Woodin for supercompactness. Otherwise, the $<\delta$-strong cardinals would be unbounded below δ, and these cardinals would be fully strong since the cardinal δ is strong. This would contradict the definition of δ as the least strong cardinal above κ.

However, perhaps there is an equiconsistency, as I suggest in question 97, even if there is not an equivalence.

Question 97. *Are the following equiconsistent?*

1. *There exists a Woodin-for-supercompactness cardinal.*
2. *There exists a cardinal δ such that V_δ is a model of* ZFC*, but in V_δ there is no universal high-jump function.*

As I did in section 2.3 for high-jump functions, I next ask whether any further interesting definitions can be obtained by rearranging the definition of a universal high-jump function. I will consider rearrangements of the form

$$(\Box f : \mathrm{ORD} \to \mathrm{ORD})(\Box(j,\kappa) \text{ such that } j : V \to M \text{ with } \mathrm{crit}(j) = \kappa)$$
$$(\neg)(f \,"\, \kappa \subseteq \kappa) \text{ and/or } (\neg)(M^{j(f)(\kappa)} \subseteq M)$$

and their negations, where each $\Box$ is either an existential quantifier or a universal quantifier. The symbol $(\neg)$ means that the clause can possibly be negated. Overall, there are 5 binary choices: two quantifiers, two possible negations, and one option between *and* and *or*. So the total number of rearrangements of this form is $2^5 = 32$. Below, I list and analyze 16 of these rearrangements. The remaining 16 rearrangements are the negations of the first 16, so they do not require additional analysis. To save space, I use "$\Box j : V \to M$" as an abbreviation for "$\Box(j,\kappa)$ such that $j : V \to M$ with $\mathrm{crit}(j) = \kappa$)"

Statement 2 is equivalent to "ORD is Woodin for supercompactness." Statement 11 is by definition equivalent to the existence of a universal high-jump function. The remaining statements are all trivial, except for statement 15.

1. $(\forall f : \mathrm{ORD} \to \mathrm{ORD})(\forall j : V \to M) f \,''\, \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   If there are no measurable cardinals, then statement 1 is vacuously true. Otherwise, it is false — a counterexample is produced by taking f to be the function $\alpha \mapsto 2^\alpha$.

2. $(\forall f : \mathrm{ORD} \to \mathrm{ORD})(\exists j : V \to M) f \,''\, \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   Statement 2 is precisely the definition of "ORD is Woodin for supercompactness." Thus it is weaker in consistency strength than a Woodin-for-supercompactness cardinal: if the cardinal κ is Woodin for supercompactness, then the model V_κ satisfies that ORD is Woodin for supercompactness.

3. $(\exists f : \mathrm{ORD} \to \mathrm{ORD})(\forall j : V \to M) f \,''\, \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   Statement 3 is true. This is witnessed by taking f to be the constant zero function.

4. $(\exists f : \mathrm{ORD} \to \mathrm{ORD})(\exists j : V \to M) f \,''\, \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   If there are no measurable cardinals, then statement 4 is vacuously false. Otherwise, the truth of statement 4 follows immediately from the truth of statement 3,

5. $(\forall f : \mathrm{ORD} \to \mathrm{ORD})(\forall j : V \to M) f \,''\, \kappa \nsubseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   If there are no measurable cardinals, then statement 5 is vacuously true. Otherwise, it is false. For a counterexample, take f to be the constant zero function. Then $f \,''\, \kappa \subseteq \kappa$ for every cardinal κ.

6. $(\forall f : \mathrm{ORD} \to \mathrm{ORD})(\exists j : V \to M) f \,''\, \kappa \nsubseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   Statement 6 is false. For a counterexample, take f to be the constant zero function.

7. $(\exists f : \mathrm{ORD} \to \mathrm{ORD})(\forall j : V \to M) f \text{ " } \kappa \not\subseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   If the measurable cardinals are stationary in ORD, then statement 7 is false, because the closure points of the function f are a club, and so some measurable cardinal must be a closure point of f. Otherwise, statement 7 is true. To see this, let $C \subseteq \mathrm{ORD}$ be a club that avoids the measurable cardinals. Define the function f as follows. If α is a limit ordinal, then set $f(\alpha) = 0$. Otherwise, let $f(\alpha)$ be the least element of C above α. If the cardinal κ is a closure point of f, then κ is a limit point of C, and therefore κ cannot be measurable. But $j(f)(\kappa)$ is always 0 for every elementary embedding j with critical point κ, because κ is a limit ordinal.

8. $(\exists f : \mathrm{ORD} \to \mathrm{ORD})(\exists j : V \to M) f \text{ " } \kappa \not\subseteq \kappa$ and $M^{j(f)(\kappa)} \subseteq M$

   If there are no measurable cardinals, then statement 8 is vacuously false. Otherwise, let κ be a measurable cardinal, and let f be defined as follows.

$$\begin{cases} f(0) = \kappa + 1 & \\ f(\alpha) = 0 & \alpha \neq 0 \end{cases}$$

   Then the function f and any elementary embedding generated by a normal measure on κ witness that statement 8 is true.

9. $(\forall f : \mathrm{ORD} \to \mathrm{ORD})(\forall j : V \to M) f \text{ " } \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \not\subseteq M$

   If there are no measurable cardinals, then statement 9 is vacuously true. Otherwise, it is false — a counterexample is given by taking f to be the constant zero function.

10. $(\forall f : \mathrm{ORD} \to \mathrm{ORD})(\exists j : V \to M) f \text{ " } \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \not\subseteq M$

   Statement 10 is false — a counterexample is given by taking f to be the constant zero function.

11. $(\exists f : \text{ORD} \to \text{ORD})(\forall j : V \to M) f \,``\, \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \not\subseteq M$

    By definition, statement 11 states that the function f is a universal high-jump function.

12. $(\exists f : \text{ORD} \to \text{ORD})(\exists j : V \to M) f \,``\, \kappa \subseteq \kappa$ and $M^{j(f)(\kappa)} \not\subseteq M$

    If there are no measurable cardinals, then statement 12 is vacuously false. Otherwise, it is true — let the function f be given by $\alpha \mapsto 2^\alpha$.

13. $(\forall f : \text{ORD} \to \text{ORD})(\forall j : V \to M) f \,``\, \kappa \not\subseteq \kappa$ and $M^{j(f)(\kappa)} \not\subseteq M$

    If there are no measurable cardinals, then statement 13 is vacuously true. Otherwise, it is false — for a counterexample, take f to be the constant zero function.

14. $(\forall f : \text{ORD} \to \text{ORD})(\exists j : V \to M) f \,``\, \kappa \not\subseteq \kappa$ and $M^{j(f)(\kappa)} \not\subseteq M$

    Statement 14 is false — for a counterexample, take f to be the constant zero function.

15. $(\exists f : \text{ORD} \to \text{ORD})(\forall j : V \to M) f \,``\, \kappa \not\subseteq \kappa$ and $M^{j(f)(\kappa)} \not\subseteq M$

    The analysis of this statement is similar to the analysis of statement 7. If the measurable cardinals are stationary in ORD, then statement 15 false, because the closure points of the function f are a club, and so some measurable cardinal must be a closure point of f.

    If the measurable cardinals are not stationary in ORD but there is a universal high-jump function, g, then statement 15 is true. To see this, let $C \subseteq \text{ORD}$ be a club that avoids all measurable cardinals, and define f as follows. Given an ordinal α, let

$f(\alpha)$ be the next member of C above α and above $g(\alpha)$. Then for every elementary embedding j with critical point κ, it follows that $j(f)(\kappa) \geq j(g)(\kappa)$. But κ cannot be a closure point of f, as otherwise κ would be a limit point of C.

If there is no universal high-jump function, then statement 15 may be satisfied anyway, but I do not know enough to provide a complete analysis. In particular, statement 15 does not seem to contradict the existence of a Shelah-for-supercompactness cardinal.

It would also be interesting to consider the modification of statement 15 where the clause $f\,``\,\kappa \subseteq \kappa$ is removed.

16. $(\exists f : \mathrm{ORD} \to \mathrm{ORD})(\exists j : V \to M) f\,``\,\kappa \nsubseteq \kappa$ and $M^{j(f)(\kappa)} \nsubseteq M$

If there are no measurable cardinals, then statement 16 is vacuously false. Otherwise, let κ be the least measurable cardinal, and let f be the function with constant value $(2^\kappa)^+$. Then the function f and any elementary embedding generated by a normal measure on κ witness that statement 16 is true.

Of course, these 16 statements do not constitute an exhaustive analysis of every potentially interesting statement related to the definition of universal high-jump functions. One can always dream up additional ways of tweaking definitions to form new definitions, so there are other ways to generate potentially interesting statements along these lines.

2.8 Excessively hypercompact cardinals are inconsistent

In definition 1.2 of [Apt12], Apter defined an excessively hypercompact cardinal, as follows.[5]

Definition 98 (Apter, [Apt12])**.** A cardinal κ is excessively 0-hypercompact iff κ is supercompact. For $\alpha > 0$, a cardinal κ is excessively α-hypercompact iff for any cardinal $\delta \geq \kappa$, there is an elementary embedding $j : V \to M$ witnessing the δ-supercompactness of κ (i.e. $\mathrm{cp}(j) = \kappa, j(\kappa) > \delta$, and $M^\delta \subseteq M$) generated by a supercompact ultrafilter over $P_\kappa(\delta)$ such that $M \models$ "κ is excessively β-hypercompact for every $\beta < \alpha$". A cardinal κ is **excessively hypercompact** iff κ is excessively α-hypercompact for every ordinal α.

Postulating the existence of an excessively hypercompact cardinal leads to a contradiction.

Theorem 99. *There are no excessively hypercompact cardinals. In particular, there is no cardinal κ such that κ is excessively $(2^\kappa)^+$-hypercompact.*

Proof. Suppose towards a contradiction that κ is least such that κ is excessively $(2^\kappa)^+$-hypercompact. Apply the definition of excessive hypercompactness in the case $\delta = \kappa$ to obtain an elementary embedding $j : V \to M$ which is witnessed by a normal fine measure on $P_\kappa\kappa$ (which is isomorphic to a normal measure on κ) such that κ is excessively α-hypercompact in M for all $\alpha < \left((2^\kappa)^+\right)^V$. This includes all $\alpha < j(\kappa)$, since $j(\kappa)$ has cardinality 2^κ in V. In particular, it includes the case of $\alpha = \left((2^\kappa)^+\right)^M$, since this is less than

[5] At that time, Apter used called these cardinals hypercompact rather than excessively hypercompact. But in light of theorem 99, we now call them excessively hypercompact.

$j(\kappa)$. By reflection, there are many $\gamma < \kappa$ such that γ is excessively $(2^\gamma)^+$-hypercompact, and this contradicts the minimality of κ. □

In definition 100, I describe a hypercompact cardinal. Apter had erroneously believed that this definition was equivalent to the definition of an excessively hypercompact cardinal.[6] However, the existence of a hypercompact cardinal is strictly weaker in consistency strength than the existence of a Woodin-for-supercompactness cardinal; I prove this fact in theorem 101. The proofs in [Apt12] all work using hypercompact cardinals in place of excessively hypercompact cardinals, so the error in the definition given in that paper did not have severe consequences.[7]

Definition 100. The hypercompact cardinals are defined recursively as follows. Given any ordinal α, the cardinal κ is α-hypercompact if and only if for every ordinal $\beta < \alpha$ and for every cardinal $\lambda \geq \kappa$, there exists a cardinal $\lambda' \geq \lambda$ and there exists an elementary embedding $j : V \to M$ generated by a normal fine measure on $P_\kappa \lambda'$ such that the cardinal κ is β-hypercompact in M. (In particular, every cardinal is 0-hypercompact, and 1-hypercompact is equivalent to supercompact.) The cardinal κ is **hypercompact** if and only if it is β-hypercompact for every ordinal β.

The key difference between the definitions of hypercompact and excessively hypercompact is that in the definition of hypercompact, the embedding j need not be witnessed by a normal fine measure on $P_\kappa \lambda$, but can be witnessed instead by a larger supercompactness measure.[8]

[6] personal communication with Apter, 2012

[7] personal communication with Apter, 2012.

[8] An additional minor difference is that the definition of hypercompact handles limit stages differently

Note that both the hypercompact cardinals and the excessively hypercompact cardinals are first-order definable in ZFC. Formally, the definition of a hypercompact cardinal is by recursion on κ as follows. Assuming recursively that the set

$$\mathrm{HC}_{<\kappa} := \{\, (\alpha, \eta) \mid \eta \text{ is } \alpha\text{-hypercompact and } \eta < \kappa \,\}$$

is already defined, define that κ is α-hypercompact if and only if for every $\beta < \alpha$ and for every $\lambda \geq \kappa$ there exists $\lambda' \geq \lambda$ and there exists an elementary embedding $j : V \to M$ generated by a normal fine measure on $P_\kappa \lambda'$ such that $(\beta, \kappa) \in j(H_{<\kappa})$. This in turn can be stated formally as a first-order proposition using the Łoś theorem, without referring explicitly to the embedding j.

I now establish the consistency of a hypercompact cardinal relative to a Woodin-for-supercompactness cardinal. The bold part of the proof emphasizes why the proof would not work to establish the consistency of an excessively hypercompact cardinal.

Theorem 101. *If the cardinal δ is Woodin for supercompactness, then in the model V_δ, there is a proper class of hypercompact cardinals.*

Proof. Suppose δ is Woodin for supercompactness. Suppose towards a contradiction that the hypercompact cardinals of V_δ are bounded above by some cardinal η. Let the function $f : \delta \to \delta$ be the failure-of-hypercompactness function as defined in the model V_δ. That is to say, for an ordinal $\xi < \delta$, let $f(\xi)$ be the least ordinal β such that ξ is not β-hypercompact in V_δ if such a β exists, and let $f(\xi) = 0$ otherwise.

from the definition of excessively hypercompact. I made this change in order to unify the definition for the successor and limit stages, and also to define hypercompact cardinals analogously to the Mitchell order.

By theorem 88, there is a cardinal $\kappa > \eta$ such that κ is $(< \delta, f)$-supercompact, and this fact is witnessed by a collection of elementary embeddings $j_\gamma : V \to M_\gamma$ for $\gamma < \delta$. (The subscripted γ serves to index the target model, not to refer to a rank-initial cut thereof.) **If γ is taken to be sufficiently large,** then $(\kappa, f(\kappa)) \in j_\gamma(f)$, and so $j_\gamma(f)(\kappa) = f(\kappa)$. That is to say, in M_γ, the cardinal κ is β-hypercompact for every $\beta < f(\kappa)$. By taking a factor embedding if necessary, assume that j_γ is generated by a normal fine measure U on $P_\kappa\gamma$ such that $U \in V_\delta$. Thus in V_δ, the collection of embeddings $\langle j_\gamma \rangle$ witness that κ is $f(\kappa)$-hypercompact, contradicting the definition of f. This contradiction completes the proof. □

Finally, I consider the extent to which the definitions of the β-hypercompactness and excessive β-hypercompactness hierarchies coincide for particular small values of β.

Theorem 102. *Let κ be a cardinal, and let $\beta \leq \kappa^+$ be an ordinal. If κ is β-hypercompact, then for every ordinal $\alpha < \beta$ and for every cardinal $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ generated by a normal fine measure on $P_\kappa\lambda$ such that κ is α-hypercompact in M. Thus, the β-hypercompactness and excessive β-hypercompactness hierarchies align below κ^+.*[9]

Proof. The proof is by induction on ordinals β. Suppose that κ is β-hypercompact, and that the theorem is true for all $\alpha < \beta$. Let $\lambda \geq \kappa$ be a cardinal, and let $\alpha < \beta$. It suffices to show that there is an elementary embedding $j : V \to M$ generated by a normal fine measure in V

[9] Actually, this alignment is off by one, because the definitions of these hierarchies handle limit stages differently. But this fact is a technical detail not germane to the main idea.

on $P_\kappa\lambda$ such that in M, the cardinal κ is α-hypercompact.

By hypothesis, the cardinal κ is β-hypercompact. So for some cardinal $\theta \geq \lambda$, there exists an elementary embedding $j : V \to M$ such that in M, the cardinal κ is α-hypercompact.

Let $j_\lambda : V \to M_\lambda$ be the λ-supercompactness factor embedding induced by j, and let $k : M_\lambda \to M$ be the elementary embedding such that $k \circ j_\lambda = j$, as in the following commutative diagram. To be precise, the embedding j_λ is the ultrapower generated by U_λ, where U_λ is the normal fine measure on $P_\kappa\lambda$ given by $A \in U \iff j\text{ " }\lambda \in j(A)$. (The subscript λ in M_λ serves to index the model M_λ, not to denote a level of its cumulative hierarchy.)

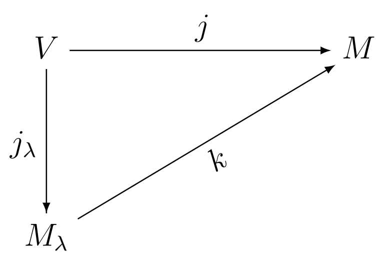


Figure 2.9: Factor embeddings of a hypercompactness embedding

If $M_\lambda = M$, then the existence of the embedding j_λ suffices to complete the proof. If $M_\lambda \neq M$, then the elementary embedding k must be nontrivial, and its critical point must be an greater than κ and inaccessible in M_λ. The model M_λ agrees with V on κ^+, so this critical point must be greater than κ^+. Therefore, it follows from the elementarity of k that κ is α-hypercompact in M_λ.

□

2.9 Enhanced supercompact cardinals

In this brief section, I analyze the consistency strength of an enhanced supercompact cardinal. The definition of an enhanced supercompact cardinal comes from Arthur Apter's paper, [Apt08].

Definition 103. A cardinal κ is **enhanced supercompact** if and only if there exists a strong cardinal $\theta > \kappa$ such that for every cardinal $\lambda > \theta$, there exists a λ-supercompactness embedding $j : V \to M$ such that θ is strong in M.

Apter required that the embedding j be generated by a normal fine measure on $P_\kappa\lambda$,. This requirement provides a first-order characterization, but it adds no strength, because one can take a factor embedding.

This next theorem shows that the consistency strength of an enhanced supercompact cardinal is strictly weaker than that of a Woodin-for-supercompactness cardinal.

Theorem 104. *Suppose the cardinal δ is Woodin for supercompactness. Then there are unboundedly many cardinals $\kappa < \delta$ such that κ is a limit of cardinals η such that there exists an inaccessible cardinal β such that $\eta < \beta < \kappa$, and*

$$V_\beta \models \eta \text{ is enhanced supercompact.}$$

Proof. The proof follows the same general line of reasoning as theorem 5 of [Apt08]. Suppose δ is Woodin for supercompactness. Let $f : \delta \to \delta$ be given by taking $f(\alpha)$ to be the second strong cardinal of V_δ greater than α. This function is well-defined, since the strong cardinals of V_δ are unbounded, since δ is Woodin.

Let κ be a closure point of f, and let $j : V \to M$ be an elementary embedding such that $M^{j(f)(\kappa)} \subseteq M$ and $j(f)(\kappa) < \delta$, i.e. the embedding j witnesses that δ is Woodin for supercompactness with respect to the function f. By theorem 88, assume without loss of generality that the embedding j is generated by a normal fine measure on $P_\kappa\lambda$ for some cardinal $\lambda < \delta$. It follows that $j(\delta) = \delta$. By the definition of f and the elementarity of j, there is a cardinal κ_0 such that $\kappa < \kappa_0 < j(f)(\kappa)$, and the cardinal κ_0 is strong in the model $M_{j(\delta)} = M_\delta$, and furthermore, the cardinal $j(f)(\kappa)$ is strong in the model M_δ.[10]

For each cardinal λ such that $\kappa_0 < \lambda < j(f)(\kappa)$, let U_λ be the normal fine measure on $P_\kappa\lambda$ given by $A \in U \iff j\,"\,\lambda \in j(A)$. Let $j_\lambda : V \to M_\lambda$ be the λ-supercompactness embedding generated by U_λ, and let $i : M_\lambda \to M$ be the elementary embedding such that $i \circ j_\lambda = j$. (The subscripted λ serves to index the model M_λ, not to denote a level of its cumulative hierarchy.)

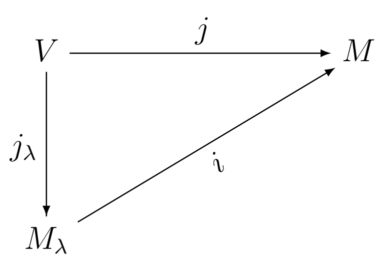


Figure 2.10: Factor embeddings of a Woodin-for-supercompactness embedding

Suppose towards a contradiction that for some cardinal γ with $\kappa_0 < \gamma < j(f)(\kappa)$, and for some cardinal λ such that $\kappa_0 < \lambda < j(f)(\kappa)$,

$$M_\lambda \models \kappa_0 \text{ is not } \gamma\text{-strong.}$$

[10]Actually, it suffices for the proof that $j(f)(\kappa)$ is inaccessible.

Then by elementarity,

$$M \models i(\kappa_0) \text{ is not } i(\gamma)\text{-strong.}$$

But i fixes κ_0, and so this contradicts the fact that the cardinal κ_0 is strong in M_δ, since $i(\gamma) < i(j(f)(\kappa)) \leq j(j(f)(\kappa)) < \delta$. From this contradiction, I conclude that for all cardinals γ and λ, if $\kappa_0 < \gamma < j(f)(\kappa)$ and $\kappa_0 < \lambda < j(f)(\kappa)$ then

$$M_\lambda \models \kappa_0 \text{ is } \gamma\text{-strong.}$$

Finally, from the closure of M, it follows that $U_\lambda \in M_{j(f)(\kappa)}$ for each cardinal λ such that $\kappa_0 < \lambda < j(f)(\kappa)$. Furthermore, for each such cardinal λ, the elementary embedding generated by U_λ in the model M is equal to $j_\lambda \upharpoonright M$. Since λ was taken to be an arbitrary cardinal between κ_0 and $j(f)(\kappa)$, it follows that in the model $M_{j(f)(\kappa)}$, the cardinal κ is enhanced supercompact.

By reflection, in V_κ, there are unboundedly many cardinals η such that for some inaccessible cardinal β with $\eta < \beta < \kappa$,

$$V_\beta \models \eta \text{ is enhanced supercompact.}$$

By a simple modification to the function f, the cardinal κ can be made arbitrarily large below δ. The conclusion of the theorem follows. □

2.10 High-jump cardinals and forcing

In this section, I prove some results about the preservation and destruction of high-jump cardinals by forcing.

Suppose $j : V \to M$ is a high-jump embedding, and $V[G]$ is a forcing extension of V. Under what conditions does j lift to a high-jump embedding $j^* : V[G] \to M[H]$? The conditions under which a supercompactness embedding lifts to a supercompactness embedding have been well-studied in the literature. The following lemma extends these conditions to provide conditions for which a high-jump embedding lifts to a high-jump embedding.

Lemma 105. *Suppose $j : V \to M$ is a high-jump embedding for κ with clearance θ. Let $V[G]$ be a forcing extension of V, and suppose that j lifts to a θ-supercompactness embedding $j^* : V[G] \to M[H]$.*[11] *Let U be the induced normal measure on κ given by $A \in U \iff \kappa \in j(A)$. If the family of functions $(\kappa^\kappa)^V$ is $\leq_U$-unbounded in $(\kappa^\kappa)^{V[G]}$, then the lifted embedding j^* is a high-jump embedding. Furthermore, if $M[H]^{\theta^+} \not\subseteq M[H]$ in $V[G]$, then the conclusion can be strengthened to a biconditional: the lifted embedding j^* is a high-jump embedding if and only if the family of functions $(\kappa^\kappa)^V$ is $\leq_U$-unbounded in $(\kappa^\kappa)^{V[G]}$.*

Proof. Note that since U is an ultrafilter, the family of functions $(\kappa^\kappa)^V$ is $\leq_U$-unbounded in $(\kappa^\kappa)^{V[G]}$ if and only this family is a dominating family, which is true if and only if the forcing does not add a U-dominating function.

To prove the first part of the theorem, assume that the family of functions $(\kappa^\kappa)^V$ is $\leq_U$-unbounded in $(\kappa^\kappa)^{V[G]}$. In $V[G]$, let $f : \kappa \to \kappa$. It suffices to show that $j^*(f)(\kappa) < \theta$. Since $(\kappa^\kappa)^V$ is a dominating family, there is a function $g \in (\kappa^\kappa)^V$ such that $f \leq_U g$. It follows that

$$j(f)(\kappa) \leq j(g)(\kappa) < \theta,$$

[11] By a θ-supercompactness embedding, I simply mean that $M[H]$ is sufficiently closed, not that the embedding is generated by a normal fine measure.

and so the lifted embedding is a high-jump embedding.

To prove the second part of the theorem, suppose that $M[H]^{\theta+} \nsubseteq M[H]$ in $V[G]$, and that $(\kappa^\kappa)^V$ is $\leq_U$-bounded by some function $g \in (\kappa^\kappa)^{V[G]}$. Then $j^*(g)(\kappa) \geq j^*(f)(\kappa)$ for every $f \in (\kappa^\kappa)^V$, and so in particular $j^*(g)(\kappa) \geq \theta$, and so the function g witnesses that j^* is not a high-jump embedding. □

One particular important instance where $(\kappa^\kappa)^V$ will be unbounded in $(\kappa^\kappa)^V[G]$ is if the forcing satisfies the κ-chain condition.

The biconditional version of the lemma actually holds even holds in many cases where $M[H]$ is closed under sequences of length greater than θ — given a g such that $j^*(g)(\kappa) \geq \theta$, one can easily modify the function g to produce another function h such that $j^*(h)(\kappa)$ is much larger than θ. For instance, let $h(\alpha)$ be the least measurable cardinal above $g(\alpha)$, so that $j^*(h)(\kappa)$ is the least measurable cardinal of $M[H]$ above θ.

The next theorem addresses the preservation of high-jump cardinals in the downwards direction.

Theorem 106. *Suppose $V \subseteq \overline{V}$ satisfies the δ approximation and cover properties, and for some cardinals $\kappa, \theta > \delta$ there is a high-jump measure U on $P_\kappa\theta$ in $\overline{V}$. Then there is a high-jump measure on $P_\kappa\theta$ in V as well.*

Proof. Let $j : \overline{V} \to \overline{N}$ be the elementary embedding generated by U in $\overline{V}$. By the proof of corollary 26 of [Ham03], the restricted embedding $j \restriction V : V \to N$ is amenable with V, and $N^\theta \subseteq N$ in V. In particular, $j \restriction V$ is a high-jump embedding. Let $j_0 : V \to M$ be the

θ-supercompactness factor embedding induced by $j \restriction V$ via the seed $j\,"\,\theta$. Let $f : \kappa \to \kappa$ be a function. It follows from lemma 53 applied in V to the embedding $j \restriction V$ that j_0 is a high-jump embedding. Furthermore, the factor embedding construction ensures that j_0 is generated by a measure that is an element of V, so the proof is complete. □

Next, I show that the previous two results together prove the analogue of the Levy-Solovay theorem for high-jump cardinals.

Theorem 107. *Let $\mathbb{P}$ be a forcing notion such that $|\mathbb{P}| < \kappa$. Let $G \subseteq \mathbb{P}$ be V-generic. Then in $V[G]$, the cardinal κ is high jump if and only if κ is high jump in V.*

Proof. Since the forcing is small, in particular it satisfies the κ-chain condition, and so every function $f : \kappa \to \kappa$ in $V[G]$ is bounded by such a function in V. Thus, the upwards direction of the proof follows from lemma 105. By lemma 13 of [Ham03], the forcing $\mathbb{P}$ satisfies the δ approximation and cover properties for some cardinal $\delta < \kappa$. Thus, the downwards direction of the proof follows immediately from theorem 106. □

Next, I apply lemma 105 to show that the canonical forcing of the GCH preserves high-jump cardinals.

Theorem 108. *Every high-jump cardinal is preserved by the canonical forcing $\mathbb{P}$ of the* GCH. *To be precise, the forcing $\mathbb{P}$ is defined as the Easton support product over all infinite cardinals δ of* $\mathrm{Add}(\delta^+, 1)$.

Proof. Let $G \subseteq \mathbb{P}$ be V-generic. In V, let U be a high-jump measure on $P_\kappa\theta$ for some cardinals κ and θ. Let j_U be the high-jump embedding generated by U. By lemma 62, the cardinal θ is a $\beth$ fixed point in V, the initial segment $\mathbb{P}_\theta$ is a subset of V_θ, and $\mathrm{cof}(\theta)^V > \kappa$. Furthermore, the forcing $\mathbb{P}$ does not change cofinalities, so it follows that $\mathrm{cof}(\theta)^{V[G]} > \kappa$ as well. With these preliminaries requirements satisfied, it follows from the proof of theorem 105 of [Ham], that the embedding j_U lifts to a θ-supercompactness embedding $j^*_U : V[G] \to M[H]$. To complete the proof that high-jump cardinals are preserved by $\mathbb{P}$, it suffices to show that every function on $f : \kappa \to \kappa$ in $V[G]$ is dominated by such a function in V and then apply lemma 105. Towards this end, note that the forcing $\mathbb{P}$ factors as $\mathbb{P}_{<\kappa} * \mathbb{P}_{\geq\kappa}$. The first factor satisfies the κ-chain condition, and so every function $f : \kappa \to \kappa$ added by it is dominated by a ground model function. The second factor is $\leq\kappa$-closed, and so it adds no new function $f : \kappa \to \kappa$. □

High-jump cardinals are in general much more fragile than supercompact cardinals, as is shown by the following theorem.

Theorem 109. *Let κ be a high-jump cardinal. Then after forcing with* $\mathrm{Add}(\kappa, 1)$ *or with Add$(\kappa^+, 1)$, the cardinal κ is no longer a high-jump cardinal.*

Proof. A recent theorem of Bagaria, Hamkins, and Tsaprounis shows that superstrong cardinals are destroyed by these forcings, among others [BHT]. By corollary 67, every high-jump cardinal is also superstrong, so it follows that these forcings also destroy high-jump cardinals. □

Finally, I show that if the cardinal κ is high jump, then there is a forcing extension where κ is still high jump but is not supercompact.

Theorem 110. *Suppose there exists a high-jump measure on $P_\kappa\theta$, and furthermore, the cardinal κ is supercompact. Let $\mathbb{P}$ be any forcing smaller than κ. Let $g \subseteq \mathbb{P}$ be V-generic. Let $\mathbb{Q}$ be any nontrivial forcing that is $\leq 2^{\theta^{<\kappa}}$-closed in $V[g]$, and let $G \subseteq \mathbb{Q}$ be $V[g]$-generic. Then in $V[g][G]$, there is still a high-jump measure on $P_\kappa\theta$, but the cardinal κ is not supercompact.*

Proof. Since the forcing $\mathbb{P}$ is small relative to κ, the cardinal κ is still both supercompact and high jump in $V[g]$. Because of the closure condition on the forcing $\mathbb{Q}$, this forcing does not add any subsets or elements to $P_\kappa\theta$, nor does it add any new functions $f : \kappa \to \kappa$. Therefore, in $V[g][G]$, the cardinal κ is still high jump. However, since the forcing $\mathbb{Q}$ is nontrivial, there is a cardinal $\lambda > 2^{\theta^{<\kappa}}$ such that $\mathbb{Q}$ adds a subset to λ. By a theorem of Hamkins and Shelah ([HS98, p.551]), the cardinal κ is no longer λ-supercompact in $V[g][G]$. □

Some open questions on the topics of this section are as follows.

Question 111. *Suppose $j : V \to M$ is a high-jump embedding for κ with clearance θ. What types of forcing, if any, preserve the θ-supercompactness of κ while destroying the high-jump cardinal property of κ?*

Question 111 can be further refined to a question about individual embeddings rather than about cardinals.

Question 112. *Let $j : V \to M$ be a high-jump embedding for κ generated by a high-jump measure. Let $\mathbb{P}$ be a forcing notion, and suppose that the embedding j lifts over $\mathbb{P}$ such that the lift is a supercompactness embedding. Under what conditions does the lift fail to be a high-jump embedding?*

2.11 Laver functions for high-jump cardinals

In this section, I establish the existence, under suitably strong hypotheses, of Laver functions for super-high-jump and related cardinals. Laver functions were originally defined for supercompact cardinals in [Lav78].

Given a supercompact cardinal κ, a supercompactness Laver function for κ is a partial function $\ell \vdots \kappa \to V_\kappa$ such that for every cardinal λ and for every set $x \in H_{\lambda^+}$ there is a λ-supercompactness embedding generated by a normal fine measure on $P_\kappa\lambda$ such that $j(\ell)(\kappa) = x$. One can also put additional requirements on the domain of a supercompactness Laver function. For instance, one can require that each $\gamma \in \mathrm{dom}(\ell)$ is an inaccessible cardinal such that the closure property $\ell\,``\,\gamma \subseteq V_\gamma$ is satisfied. A super-high-jump Laver function is defined similarly to a supercompactness Laver function, as follows.

Definition 113. Given a super-high-jump cardinal κ, a **super-high-jump Laver function** for κ is a partial function $\ell \vdots \kappa \to V_\kappa$ satisfying the following properties. For every set x, for unboundedly many cardinals δ, there is high-jump embedding with critical point κ and clearance δ, generated by a high-jump measure, such that $j(\ell)(\kappa) = x$. Furthermore, for every ordinal $\gamma \in \mathrm{dom}(\ell)$, the closure property $\ell\,``\,\gamma \subseteq V_\gamma$ holds.

Every supercompact cardinal has a supercompactness Laver function anticipating every set, and whenever the cardinal κ is $2^{\theta^{<\kappa}}$-supercompact, there is a supercompactness Laver function for κ anticipating every set in H_θ^+.

The analysis for high-jump cardinals is more complicated than in the case of supercompact cardinals, because a supercompactness factor embedding of a high-jump embedding is not in general a high-jump embedding. For this reason, the high-jump cardinals with excess closure are a useful tool.

As a warm-up exercise to reading the proof of the existence of super-high-jump Laver functions, the reader may wish to review proposition 72. This proposition shows that if there is a high-jump embedding $j : V \to M$ with critical point κ and clearance θ such that $M^{2^\theta} \subseteq M$, then the cardinal κ is super-high-jump in M_θ. In theorem 114, I will show from the same hypothesis that there is a super-high-jump Laver function in the model V_θ.

Theorem 114. *Let κ be a cardinal. Then there exists a partial function $\ell \,\vdots\, \kappa \to V_\kappa$ such that for all cardinals θ, if there is a high-jump measure on $P_\kappa 2^\theta$ generating an ultrapower embedding with clearance θ, then in the model V_θ, the function ℓ is a super-high-jump Laver function for κ.*

Proof. Define the function ℓ recursively as follows. Suppose that $\ell \restriction \gamma$ has been defined. Define $\ell(\gamma)$ as described in the next paragraph if the relevant hypotheses hold. Otherwise, leave γ out of the domain of ℓ.

Suppose that $\ell\,``\,\gamma \subseteq V_\gamma$ and that furthermore, in the model V_κ, some set x witnesses that the function $\ell \restriction \gamma$ is not a super-high-jump Laver function for γ. *That is to say, in the model V_κ, there is a cardinal δ_0 such that for all cardinals $\delta > \delta_0$, there is no elementary embedding $j : V \to M$ with critical point γ and clearance δ, generated by a high-jump measure, such that $j(\ell \restriction \gamma)(\gamma) = x$.* Then pick a set $x \in V_\kappa$ of minimal $\in$-rank among all sets with this property, and let $\ell(\gamma) = x$.

I now verify that the function ℓ has the desired feature. Suppose not. Then there is some cardinal θ such that there is a high-jump measure μ on $P_\kappa 2^\theta$ generating an ultrapower embedding with clearance θ, but in the model V_θ, some set x witnesses that the function ℓ fails to be a super-high-jump Laver function for κ. Let $j : V \to M$ be the ultrapower generated by μ. By lemma 66, the elementarity relation $V_\theta \prec M_{j(\kappa)}$ holds. Therefore, in the model $M_{j(\kappa)}$, the set x witnesses that the function ℓ is not a super-high-jump Laver function for κ. That is to say, in the model $M_{j(\kappa)}$, there is some cardinal δ_0 such that for all cardinals $\delta > \delta_0$, there does not exist a high-jump embedding h, generated by a high-jump measure, with critical point κ and clearance δ, such that $h(\ell)(\kappa) = x$.

Accordingly, since $j(\ell) \upharpoonright \kappa = \ell$, it follows from the definition of the function ℓ and from the elementarity of the embedding j that $j(\ell)(\kappa)$ is defined and equal to some set $y \in M_{j(\kappa)}$ such that in the model $M_{j(\kappa)}$, the set y witnesses that the function ℓ is not a super-high-jump Laver function for κ. Furthermore, this set y is of minimal $\in$-rank, and so $y \in V_\theta$, since $V_\theta \prec M_{j(\kappa)}$.

Let U be the θ-supercompactness measure on $P_\kappa\theta$ induced by j via the seed $j\,"\,\theta$. Let $j_U : V \to N$ be the supercompactness embedding generated by U, and let the elementary embedding k be such that the diagram below commutes.

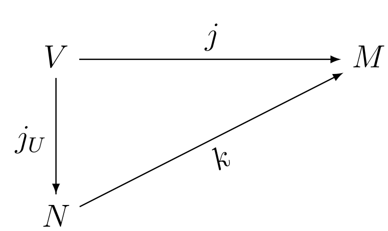


Figure 2.11: Factor embeddings of a high-jump embedding with excess closure

By lemma 54, the measure U is a high-jump measure, the clearance of j_U is θ, and $j_U(\ell)(\kappa) = y$. Furthermore, since the model $M_{j(\kappa)}$ is closed in V under sequences of length 2^θ, and since $\theta^\kappa = \theta$ by lemma 62, it follows that $U \in M_{j(\kappa)}$, and the model $M_{j(\kappa)}$ correctly computes that $j_U(\ell)(\kappa) = y$. (In the previous sentence, j_U denotes the embedding generated by the measure U in the model $M_{j(\kappa)}$.) However, $\theta > \delta_0$, so this computation contradicts the fact that y is not anticipated in $M_{j(\kappa)}$ by ℓ with respect to any high-jump embedding with clearance greater than δ_0 that is generated by a high-jump measure. □

Corollary 115. *Suppose that for some cardinal κ, there is an unbounded set of cardinals θ such that there is a high-jump measure on $P_\kappa 2^\theta$ generating an ultrapower embedding with clearance θ. Then there exists a super-high-jump Laver function for κ in V.*

Proof. Define the function ℓ as in theorem 114. It follows from theorem 114 that for arbitrarily large cardinals θ, the function ℓ is a super-high-jump Laver function for V_θ. Furthermore, for such a cardinal θ, if $U \in V_\theta$ is a high-jump measure, generating an embedding $j_U : V \to M$, then the embedding generated by the measure U in the model V_θ is $j_U \restriction V_\theta$. It follows that ℓ is a super-high-jump Laver function for κ. □

The next theorem generalizes theorem 114 to consider Laver functions that anticipate sets via high-jump embeddings with excess closure.

Theorem 116. *Let κ be a cardinal. Then there is a function $\ell : \kappa \to V_\kappa$ such that for every cardinal θ and for every ordinal $\xi < \theta$, if there is a high-jump measure on $P_\kappa(\beth^{<\kappa}_{\theta+\xi+1})$ generating an ultrapower embedding with clearance θ, then in the model V_θ, the function ℓ is* *a high-jump Laver function for* (κ, ξ) *in the following sense.*

For every set $x \in V_\theta$*, there are unboundedly many cardinals* δ *such that there is a high-jump embedding with critical point* κ *and clearance* δ*, generated by a high-jump measure on* $P_\kappa(\beth_{\delta+\xi})$ *such that* $j(\ell)(\kappa) = x$*.*

Proof. The proof is a generalization of the proof of theorem 114. Suppose that $\ell \upharpoonright \gamma$ has been defined. Define $\ell(\gamma)$ recursively as described in the next paragraph if the relevant hypotheses hold. Otherwise, leave γ out of the domain of ℓ.

Suppose that $\ell " \gamma \subseteq V_\gamma$ and that furthermore, in the model V_κ, for some minimal ordinal $\xi < \kappa$, some set x witnesses that the function $\ell \upharpoonright \gamma$ is not a high-jump Laver function for (γ, ξ) in the model V_κ. *That is to say, in the model* V_κ*, there is a cardinal* δ_0 *such that for all cardinals* $\delta > \delta_0$*, there is no high-jump embedding* $j : V \to M$ *with critical point* γ *and clearance* δ*, generated by a high-jump measure on* $P_\kappa(\beth_{\delta+\xi})$*, such that* $j(\ell \upharpoonright \gamma)(\gamma) = x$*.* Then pick a set $x \in V_\kappa$ of minimal $\in$-rank among all sets with this property, and let $\ell(\gamma) = x$.

I now verify that the function ℓ has the desired feature. Suppose not. Then there is some minimal ordinal ξ and some cardinal θ such that $\xi < \theta$ and such that there is a high-jump measure μ on $P_\kappa(\beth^{<\kappa}_{\theta+\xi+1})$ generating an ultrapower embedding with clearance θ, but some set x witnesses that the function ℓ fails to be a Laver function for (κ, ξ). Let $j : V \to M$ be the ultrapower generated by μ. By lemma 66, the elementarity relation $V_\theta \prec M_{j(\kappa)}$ holds. Therefore, in the model $M_{j(\kappa)}$, there is some cardinal δ_0 such that for all cardinals $\delta > \delta_0$, there does not exist a high-jump embedding h with critical point κ and clearance δ, generated by a high-jump measure on $P_\kappa(\beth_{\delta+\xi})$, satisfying $h(\ell)(\kappa) = x$.

Accordingly, since $j(\ell) \upharpoonright \kappa = \ell$, it follows from the definition of the function ℓ and from the elementarity of the embedding j that $j(\ell)(\kappa)$ is defined and equal to some set $y \in M_\theta$ such that in the model $M_{j(\kappa)}$, the set y witnesses that the function ℓ is not a high-jump Laver function for (κ, ξ).

Let U be the $\beth_{\theta+\xi}$-supercompactness measure on $P_\kappa(\beth_{\theta+\xi})$ given by $A \in U \iff j\,''\,\beth_{\theta+\xi} \in A$. Let $j_U : V \to N$ be the supercompactness embedding generated by U, and let the elementary embedding k be such that the diagram below commutes.

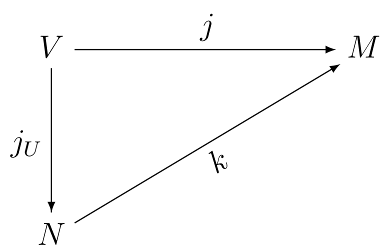


Figure 2.12: Factor embeddings of a high-jump embedding with excess closure

By lemma 54, the measure U is a high-jump measure, the clearance of j_U is θ, and $j_U(\ell)(\kappa) = y$. Furthermore, the model $M_{j(\kappa)}$ is sufficiently closed such that $U_\lambda \in M_{j(\kappa)}$, and the model $M_{j(\kappa)}$ correctly computes that $j_U(\ell)(\kappa) = y$. These observations contradict the definition of the set y. $\square$

Theorem 116 could be modified to define other sorts of Laver functions anticipating high-jump embeddings with excess closure where the excess closure is defined in other ways than the $\beth$ function.

I close the section with a question.

Question 117. *Is it possible to prove the existence or the consistency of a super-high-jump Laver function from a hypothesis weaker than that of theorem 114?*

For instance, it may be possible to force the existence of a super-high-jump Laver function for κ, beginning in a model where κ is only super-high-jump.

2.12 Ideas for further research

In this section, I review some of the areas for further research discussed in previous sections, and I also suggest a few additional areas for further research.

One relationship between cardinals in the chart are unresolved. I do not know the relationship between enhanced supercompact cardinals and weakly hypercompact cardinals. One established large cardinal is conspicuously missing from my analysis. An extendible cardinal is known to be intermediate in consistency strength between a supercompact cardinal and a Vopěnka cardinal. But I don't know the relationship between an extendible cardinal and a hypercompact cardinal or an enhanced supercompact cardinal. Furthermore, the $C^{(n)}$-extendible cardinals, introduced by Bagaria in [Bag12], fall into the large cardinal hierarchy between an extendible cardinal and a Vopěnka cardinal.

Another possible direction for further research would be to define more large cardinal notions by modifying the definitions that I have already given. One possibility would be to modify the definition of a high-jump cardinal so that M is closed under $j(f)(\kappa)$-sequences for all $f : \kappa \to ORD$ rather than just $f : \kappa \to \kappa$. Such a cardinal would be huge, to say the least. This is witnessed by the case where f is the function with constant value κ.

The universal high-jump functions of section 2.7 are another interesting topic for further study, along with statement 15 of that section.

As discussed in the conclusion of section 2.10, more work can also be done on the relationships between high-jump cardinals and forcing. More generally, there is more work to be done proving forcing results for all the cardinals discussed in this chapter, and in particular, many of the results from section 2.10 could be extended to apply to other large cardinals in this chapter.

Keeping in mind that many of the large cardinals in this chapter were first applied towards universal indestructibility results, it is an interesting goal to use the new large cardinals to weaken the hypotheses for universal indestructibility results, with the goal of eventually proving an equiconsistency between a universal indestructibility result and a large cardinal notion. One could also prove new universal indestructibility results. More generally, the large cardinals that I have studied here could be used to weaken the hypotheses or find equiconsistencies for other set-theoretic results as well.

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