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GRAPHS WITH MAXIMAL EVEN GIRTH

by

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1. Introduction

We consider ordinary (i.e., finite, undirected, with at most one edge joining a pair of nodes, and no arc joining a node to itself) graphs. A path in a graph of length k joining nodes x and y is a sequence of nodes $x = x_0, x_1, \ldots, x_k = y$, in which x_i and x_{i+1} are joined by an arc. The distance between two nodes is the smallest integer k such that there is a path of length k joining them. The diameter of a graph is d if d is the largest distance between any pair of nodes. A circuit of length k is a path of positive length for which $x = y$ and all nodes $x = x_0, x_1, \ldots, x_k = y$ are distinct except for $x = y$. The girth of a graph is the smallest integer k such that the graph contains a circuit of length k.

In this paper, we focus our attention on graphs of diameter $d > 1$ whose girth is $2d$, i.e., the girth is a maximum (given the diameter) under the stipulation that the girth is even. We impose also the requirement that there is a number $t > 1$ such that any two nodes at distance d from each other are contained in exactly $\binom{t}{2}$ circuits of length $2d$. And it is our task to investigate properties of this class of graphs.

We first show that every such graph is regular of valency (say) v (every node is adjacent to exactly v other nodes). If we let G stand for the class of all graphs being considered, we use the symbol $g(d, t, v, n)$ to denote a graph with n nodes and parameters d, t, v. The proper subclass $B \subset G$ consisting of graphs $b(d, t, t, n)$

was considered in [1], [2].

By studying properties of the powers of the adjacency matrix of a graph $g(d, t, v, n)$ we obtain information on the characteristic roots of the adjacency matrix, which enables us to prove a variety of necessary conditions connecting the parameters d, t, v, n. For example, it is shown that if $d = 2$, $t \neq 2, 4, 6$ then there are only a finite number of graphs $g(2, t, v, n)$. Other necessary conditions, using different arguments, are also found. In addition, connections between graphs in G and block designs are explored.

Finally, we exhibit several instances of the parameters for which the graphs do exist, and prove uniqueness. The most interesting of these cases is $g(2, 2, 10, 56)$, which has a relation to group theory.

A related problem, dealing with graphs of diameter d and girth $2d+1$ was considered in [6]. An application of those graphs to the construction of transmission networks was given in [10], and the same concept can be easily modified to apply to the graphs considered here. Assume, for instance, a network in which any node (= station) has valency at most v (is connected directly with at most v other stations), and in which two stations not directly connected are connected by at least t intermediate links. The number of stations in such a network is at most $v^2 + (t-1)v + t$, and equality is obtained precisely for the graphs $g(2, t, v, n) = (v^2 + (t-1)v + t)t^{-1}$.

2. Regularity

Throughout this paper we assume that g is a graph of diameter $d > 1$, girth $2d$ and that every pair of nodes at distance d from each other is contained in exactly $\binom{t}{2}$ circuits of length $2d$, $t > 1$. In this section we prove that these hypotheses imply that g is regular (i.e., each node has the same valency).

We write $d(i, j) = k$ if the distance from node i to node j is k. We write (i, j) if i is adjacent to j, and $(i, \ldots, x, \ldots, j)$ for a path from i to j containing x. A path, unless otherwise specified, has no retraced arcs and each node is in exactly 2 arcs, except for i and j which are in 1 arc.

<u>Lemma 2.1</u>: Let i be a node of g. Then there is a node j of g such that $d(i, j) = d$ and there are precisely t distinct paths from i to j of length d.

<u>Proof</u>: If $d(i, j) = d$ then that there are exactly t paths of length d from i to j is immediate from the hypothesis on the number of circuits joining i and j. We therefore need only show, given i, the existence of a j such that $d(i, j) = d$.

Let i be given and let ℓ be a circuit in g of length the girth. If $i \in \ell$, take $j \in \ell$ such that $d(i, j) = d$. If $i \notin \ell$ then for all $x \in \ell$ $d(i, x) \le d$ by the hypothesis on the diameter of g. For some $x \in \ell$ let $d(i, x) = k$. If $k = d$ we are finished. If not, $k < d$ and suppose $k + r = d$. Let $z \in \ell$ be such that $d(x, z) = r$. Then $d(i, z) = d$.

Observe there is no shorter path from i to z or else the girth condition would be violated. Q. E. D.

We write v_i for the valency of the node i.

Lemma 2.2: If $d(i, j) = d - 1$ then $v_i = v_j$.

Let $X = \{x_k : (x_k, i) \text{ and } d(x_k, j) = d\}$

$Y = \{y_s : (y_s, j) \text{ and } d(y_s, i) = d\}$

clearly $|X| = v_i - 1$

$|Y| = v_j - 1$

We observe that the stated path (of length $d - 1$) joining i and j is unique, otherwise there would be a circuit of length less than the girth. We distinguish the nodes of this unique path by $i = i_1, i_2, \ldots, i_{d-1} = j$. For each k the path $(x_k, i_1, \ldots, i_{d-1})$ has the length d. Thus by Lemma 2.1 there are $t-1$ additional paths from x_k to $i_{d-1} = j$. The node i_{d-c} $c = 1, \ldots, d-1$ cannot be in such a path because if it were we would have the circuit $(x_k, \ldots, i_{d-c}, \ldots, i, x_k)$ whose length would be $2(d-c) + 1 \leq 2(d-1) + 1 < 2d$, a contradiction. Thus the $t-1$ paths must be of the form $(x_k, \ldots, y_s, j)$. Since this holds for all $k = 1, \ldots, v_i - 1$ there are $(t - 1)(v_i - 1)$ such paths. By the same reasoning there are $(t - 1)(v_j - 1)$ paths of length d of the form $(y_s, \ldots, x_k, i)$. But, each of the above numbers is the number of paths of length $d-1$ joining nodes in X with nodes in Y. Thus $(t - 1)(v_i - 1) = (t - 1)(v_j - 1)$ and since $t \geq 2$ $v_i - 1 = v_j - 1$ and therefore $v_i = v_j$. Q. E. D.

Lemma 2.3: Let $(i_0, i_1, \ldots, i_{2d-1}, i_0)$ be a circuit of length 2d.

Then (a) $v_{i_j} = v_{i_k}$ $\quad j, k = 0, 1, \ldots, 2d-1$ $\quad$ if d is even

$$\text{(b)} \quad \left.\begin{aligned} v_{i_j} &= v_{i_k} \quad j, k = 0, 2, \ldots, 2d-2 \\ v_{i_s} &= v_{i_r} \quad s, r = 1, 3, \ldots, 2d-1 \end{aligned}\right\} \text{ if } d \text{ is odd}$$

Proof of (a): By Lemma 2.2 we have (d = 2 is immediate). For d > 2

$$2.1) \qquad v_{i_0} = v_{i_{d-1}} = \ldots = v_{i_{md-m}} = \ldots$$

But d-1 is relatively prime to 2d if d is even. Hence the numbers m(d-1), m = 0, 1, 2, ... exhaust the residue classes modulo 2d. Thus (a) follows from 2.1).

Proof of (b): If d is odd then the numbers m(d-1), m = 0, 1, 2, ... exhaust the even residue classes modulo 2d and the numbers 1+m(d-1) exhaust the odd residue classes modulo 2d. Thus (b) also follows from 2.1). Q.E.D.

Theorem 2.1: Let $g \in G$. Then g is regular.

Proof: There are two cases to consider.

(a) d is even

(b) d is odd

Proof of (a): Let $(i_0, \ldots, i_{2d-1}, i_0) = \ell$ be a circuit of length 2d. By Lemma 2.3 each node in ℓ has the same valency, say v. If $\ell = g$ we are finished. If not, let $x \in g$, $x \notin \ell$. As in the proof of Lemma 2.1 x is contained in a circuit of length 2d which also contains some node of ℓ. Therefore $v_x = v$.

Proof of (b): We know from Lemma 2.3 that, if ℓ is a circuit of length 2d, then there are numbers v_1 and v_2 such that every node of ℓ has valency v_1 or v_2; and, if i and j are adjacent nodes of ℓ and $v_i = v_1$, then $v_j = v_2$. We now show that if i and j are any adjacent nodes of g then their respective valencies are v_1 and v_2 or v_2 and v_1. Let (x_1, x_2), $x_1, x_2 \in g$ and let $\ell = (i_0, \ldots, i_{2d-1}, i_0)$.

Case 1: $x_1 \notin \ell$, $x_2 \in \ell$. Then as in the proof of Lemma 2.1 x_1 and x_2 are adjacent nodes in a circuit of length 2d which includes at least 2 nodes of ℓ. Hence $v_{x_1} = v_1$ and $v_{x_2} = v_2$ or $v_{x_1} = v_2$ and $v_{x_2} = v_1$.

Case 2: $x_1, x_2 \notin \ell$ and $d(x_2, \ell) = d - j$ $j = 1, \ldots, d-1$. If $j = 2, \ldots, d-1$ then x_1, x_2 are adjacent nodes in a circuit of length 2d containing at least 2 nodes of ℓ and the result follows from Case 1. We may now assume, without loss of generality, that $d(x_2, i_0) = d-1$ and $v_{i_0} = v_2$. Thus $(x_2, x_3, \ldots, i_0, i_1)$ is a path of length d, and thus part of a circuit ℓ_1 of length 2d, containing 2 adjacent nodes of ℓ and thus $v_{x_2} = v_{i_0} = v_2$ and $v_{x_3} = v_{i_1} = v_1$. Now, since $(x_1, x_2, x_3, \ldots, i_0)$ is a path of length d containing at least 2 nodes of ℓ_1 we have $v_{x_1} = v_{x_3} = v_1$.

Case 3: $x_1, x_2 \notin \ell$ and $d(x_1, i_j) = d(x_2, i_j) = d$ $j = 0, \ldots, 2d-1$. Let $(x_2, x_3, x_4, \ldots, i_0)$ be a path of length d. Then by Case 2 $v_{x_2} = v_{x_4} = v_{i_1} = v_1$. Similarly let $(x_1, x_5, x_6, \ldots, i_1)$ be a path of length d. Then, again, by Case 2 $v_{x_1} = v_{x_6} = v_{i_0} = v_2$.

We now use this fact to count the nodes of g in two ways. From the fact that both ways count the same number, we will infer $v_1 = v_2$. We define (see [6]) a hierarchy of g as follows. Pick a node of g which we will call the distinguished node and identify it by 0. We will say that 0 is on level 0 (tier 0) of the hierarchy (still to be defined). the valency of 0 is v_j $j = 1, 2$. Then the nodes adjacent to 0 would have valency v_{j+1} where $j+1$ is an index modulo 2. These nodes are identified as $1, 2, \ldots, v_j$ and are said to be on level 1 of the hierarchy. Each node on level 1 is connected to $v_{j+1} - 1$ nodes other than 0 and this collection of nodes is said to be level 2 of the hierarchy. Clearly, in order not to violate the girth condition, the nodes on level i cannot be connected to each other unless $i = d$. If the arcs connecting nodes on level d to each other are removed, the residual graph is called a hierarchy of g. Clearly there are at most 2 hierarchies, one with distinguished node having valency v_j, the other with distinguished node having valency v_{j+1}. We propose now to show $v_j = v_{j+1}$ and thus there is but one hierarchy (and of course g will be regular). We display the hierarchy as Figure 2.1.

Let ℓ_i be the set of nodes at distance i from the distinguished node (see Figure 2.1). If we assume the distinguished node has valency v_j, $j = 1, 2$, and index $j+1$ is thought of modulo 2 then from our preceding discussion

2.4) $$|\ell_0| = 1$$

2.5) $$|\ell_i| = v_j (v_j - 1)^{\frac{i-2}{2}} (v_{j+1} - 1)^{\frac{i}{2}} \quad \text{if } i \text{ is even } 2 \leq i \leq d-1$$

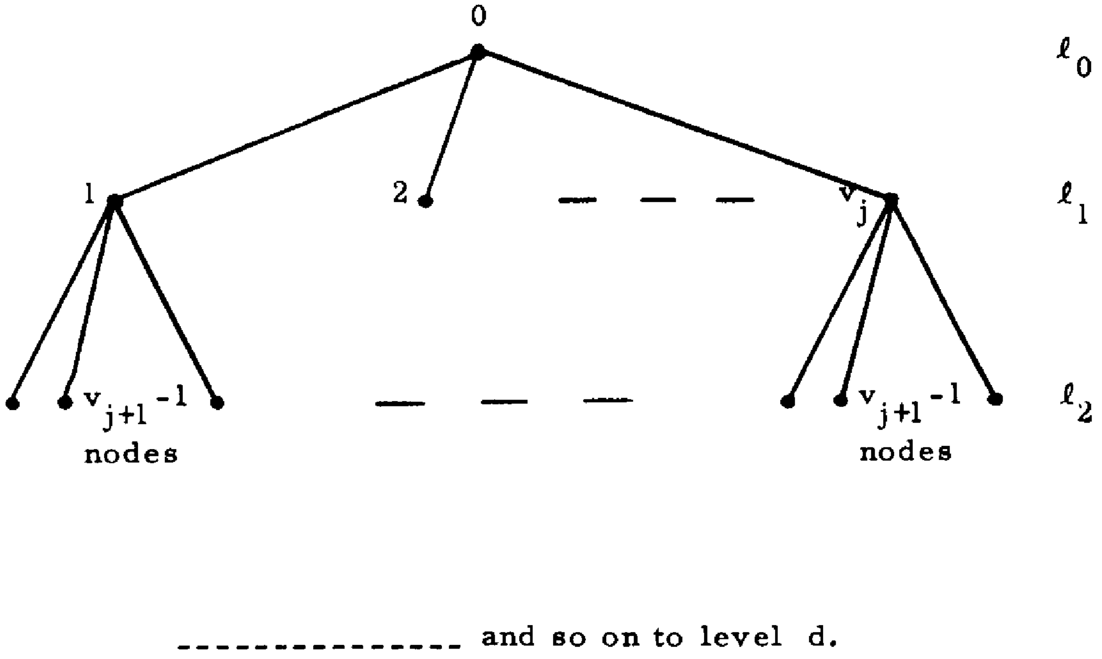
0
1
2
v_j
ℓ_0
ℓ_1
ℓ_2
$v_{j+1}-1$ nodes
$v_{j+1}-1$ nodes
-------------- and so on to level d.

Figure 2.1

2.6) $$|\ell_i| = v_j(v_j-1)^{\frac{i-1}{2}}(v_{j+1}-1)^{\frac{i-1}{2}} \quad \text{if } i \text{ is odd } 1 = i = d-2$$

Since every node of ℓ_d is at distance d from the distinguished node t of the edges from ℓ_{d-1} must go to each node of ℓ_d and thus

2.7) $$|\ell_d| = v_j(v_j-1)^{\frac{d-1}{2}}(v_{j+1}-1)^{\frac{d-1}{2}}\, t^{-1}$$

We now show that the sum of the cardinalities implied by 2.4), 2.5), 2.6) is the same regardless of whether v_j or v_{j+1} is picked as the valency of the distinguished node. To show this we need only show $|\ell_{2h-1}| + |\ell_{2h}|$ $h = 1, \ldots, \frac{d-1}{2}$ is the same regardless of whether the distinguished node has valency v_j or v_{j+1}. Let the distinguished node have valency v_j. Then

2.8) $$\begin{aligned} |\ell_{2h-1}| + |\ell_{2h}| &= v_j(v_j-1)^{h-1}(v_{j+1}-1)^{h-1} + v_j(v_j-1)^{h-1}(v_{j+1}-1)^{h} \\ &= v_j(v_j-1)^{h-1}(v_{j+1}-1)^{h-1}\, v_{j+1} \end{aligned}$$

If the distinguished node has valency v_{j+1} then

2.9) $$\begin{aligned} |\ell_{2h-1}| + |\ell_{2h}| &= v_{j+1}(v_{j+1}-1)^{h-1}(v_j-1)^{h-1} + v_{j+1}(v_{j+1}-1)^{h-1}(v_j-1)^{h} \\ &= v_{j+1}(v_{j+1}-1)^{h-1}(v_j-1)^{h-1}\, v_j . \end{aligned}$$

But, 2.8) and 2.9) are the same. Thus, since the left-hand sides of 2.4) and 2.7) add up to the number of nodes of g, we have from 2.7) $v_j(v_j-1)^{\frac{d-1}{2}}(v_{j+1}-1)^{\frac{d-1}{2}}\, t^{-1} = v_{j+1}(v_{j+1}-1)^{\frac{d-1}{2}}(v_j-1)^{\frac{d-1}{2}}\, t^{-1}$. Thus $v_j = v_{j+1} = v$. Q.E.D.

Corollary 2.1: $|\ell_0| = 1 \quad |\ell_i| = v(v-1)^{i-1} \quad 1 \leq i \leq d-1$

$$|\ell_d| = v(v-1)^{d-1}\, t^{-1}$$ Q. E. D.

3. Some necessary d, t, v, n conditions

We now have that if $g \in G$ with diameter $d > 1$, girth $2d$, and satisfies the condition that if $d(i, j) = d$ then there are t distinct paths from i to j of length d, then G is regular of valency (say) v. We let n represent the number of nodes of g and we write $g = g(d, t, v, n)$. Clearly $t \leq v$. The case $t = v$ was thoroughly investigated by Singleton. See [1], [5] and also [2]. We define $B \subset G$ to be the class studied by Singleton and R to be the complimentary class, $R = G - B$. We use the symbols $g(d, t, v, n)$, $b(d, t, t, n)$, $r(d, t, v, n)$ for elements of G, B and R respectively and $g(d, t, v, n)$ if the class is unspecified. Elements of G are bipartite if they belong to B, that is, using the language of [6], there are no reentering arcs (arcs which connect nodes of ℓ_d to each other) in the graph. The principal result of this section is that if $r(d, t, v, n)$ exists then $v > 2t-1$. We prove some additional theorems and lemmas for later reference.

Theorem 3.1: A necessary condition for the existence of $g(d, t, v, n)$ where $v \neq 2$ is

$$3.1) \qquad t(2-v)n = 2t - [v^2 + v(2-t)](v-1)^{d-1}$$

Proof: $$n = \sum_{0}^{d-1} |\ell_i| = 1 + v\sum_{0}^{d-2} (v-1)^i + v(v-1)^{d-1} t^{-1}$$

$$= 1 + v\left[\frac{1-(v-1)^{d-1}}{1-(v-1)} + \frac{(v-1)^{d-1}}{t}\right]$$

$$= 1+v[t(1-(v-1)^{d-1}) + (2-v)(v-1)^{d-1}]t^{-1}(2-v)^{-1}$$

Q. E. D.

Let $i = \{1,\ldots,v\}$ then by $\ell_{d-j}(i)$ $j = 0,\ldots,d-1$ we mean the set of nodes which are connected to node i of ℓ_1 via the hierarchy arcs only (see Figure 3.1).

Lemma 3.1: Let $m, n \in \ell_{d-1}(j)$ and $x \in \ell_d(i)$.

Then (m,x) and (n,x) implies $m = n$.

Proof: If $m \neq n$ then $(j,\ldots,m,x,n,\ldots,j)$ has length $2d-2$ which is a contradiction. Q. E. D.

Lemma 3.2: $|\ell_{d-j}(i)| = (v-1)^{d-1-j}$ $i = 1,\ldots,v$ $j = 0,\ldots,d-1$

Proof: By regularity and the tree structure of the hierarchy

$|\ell_{d-j}(i)| = v^{-1}|\ell_{d-j}| = v^{-1}v(v-1)^{d-1-j} = (v-1)^{d-1-j}$ Q. E. D.

Lemma 3.3: Let $i,j = 1,\ldots,v$ $i \neq j$. Then,

(a) $|\ell_d(i) \cap \ell_d(j)| = (t-1)(v-1)^{d-2}$

(b) $|\ell_d(i) \cup \ell_d(j)| = (2v-t-1)(v-1)^{d-2}$

Proof: Let $m \in \ell_{d-1}(j)$. Then $d(m,i) = d$ (via the distinguished node). Thus there must exist $t-1$ other paths of length d, from m to i. Thus m must be connected to $(t-1)$ nodes of $\ell_d(i)$. $|\ell_{d-1}(j)| = (v-1)^{d-2}$ and thus by Lemma 3.1 (a) follows. For (b) we have $|\ell_d(i) \cup \ell_d(j)| =$ $|\ell_d(i)| + |\ell_d(j)| - |\ell_d(i) \cap \ell_d(j)| = 2(v-1)^{d-1} - (t-1)(v-1)^{d-2}$. Q. E. D.

Lemma 3.4: Let $x \in \bigcap_{i \in I} \ell_d(i)$ $I \subset \{1,\ldots,v\}$ and let $y \in \ell_d$. Then

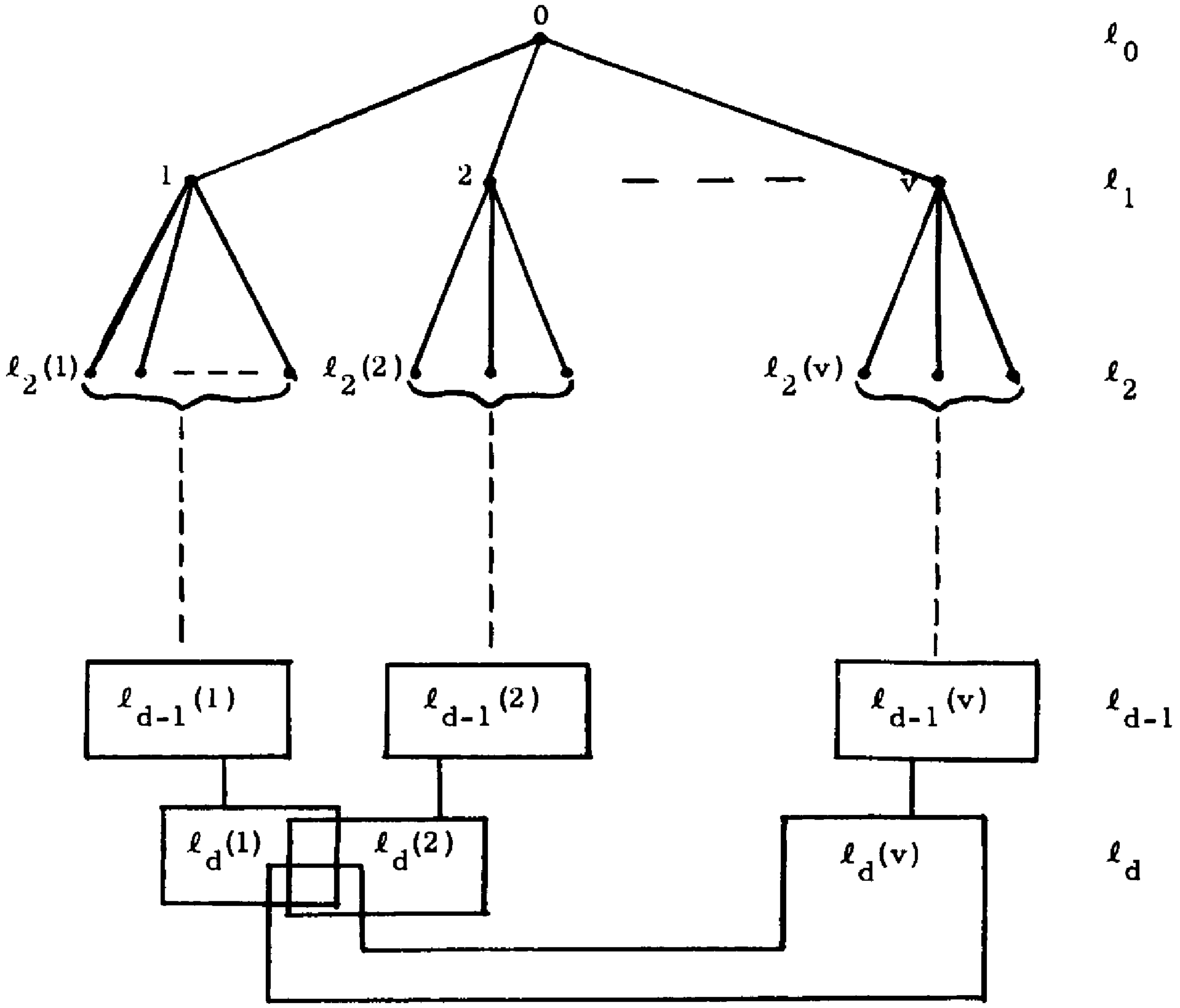
0
1
2
v
ℓ_0
ℓ_1
ℓ_2
ℓ_{d-1}
ℓ_d
$\ell_2(1)$
$\ell_2(2)$
$\ell_2(v)$
$\ell_{d-1}(1)$
$\ell_{d-1}(2)$
$\ell_{d-1}(v)$
$\ell_d(1)$
$\ell_d(2)$
$\ell_d(v)$

Figure 3.1

(x, y) implies $y \notin \ell_d(i)$ for all $i \in I$.

Proof: If x and y are in $\ell_d(j)$ then the length of $(j, \dots, x, y, \dots, j)$ is $2d-1$, a contradiction. Q. E. D.

Lemma 3.5: $x \in \ell_d$ implies $x \in \bigcap_{j=1}^{t} \ell_d(i_j)$ $\quad i_j \in \{1, \dots, v\}$.

Proof: There must be t paths of length d from x to 0. Q. E. D.

Lemma 3.6: Let h be the subgraph of g whose nodes are the nodes $x \in \ell_d$ (the arcs are the reentering arcs).

Then the valency of x, in h, is $v-t$, and if we define $\langle x \rangle = \{y \in \ell_d : (x, y)\}$ then $|\langle x \rangle| = v-t$.

Proof: Obvious. Q. E. D.

Theorem 3.2: A necessary condition for the existence of $r(d, t, v, n)$ is $v > 2t-1$.

Proof: Let $x \in \bigcap_{j=1}^{t} \ell_d(i_j)$ and suppose $y \in \ell_d$ is adjacent to x. (We know y exists since our graph is in R. Observe that this argument is not valid for B graphs). Then by Lemma 3.4 $y \notin \ell_d(i_j)$ $j = 1, \dots, t$. By Lemma 3.5 $y \in \ell_d(k)$, $k \in \{1, \dots, v\}$ for t distinct values of k. Thus $v \geq \Sigma j + \Sigma k = t + t = 2t$. Q. E. D.

Theorem 3.3: If t is a prime then a necessary condition for the existence of $r(d, t, v, n)$ is $v = mt$ or $v = mt + 1$ $\quad m > 1$ an integer. (Note: if $m = 1$, then either $v = t$, which is the case for B graphs, or $v = t+1 \leq 2t-1$, which means, from Theorem 3.2, no graph.)

Proof: $|\ell_d| = v(v-1)^{d-1} t^{-1}$ and when t is a prime the expression for $|\ell_d|$ is integral only if $t|v$ or $t|v-1$. Thus $mt = v$ or $mt = v-1$. Q. E. D.

4. The polynomial of the graph

In this section we establish, following the line of argument in [1], [6] and [9], the existence of certain matrix polynomials. We will associate, with each graph $g = g(d, t, v, n) \in G$, one of these polynomials $J = P(A)$ where A is the $n \times n$ adjacency matrix, soon to be defined, of g and J is the $n \times n$ matrix of all ones.

Let $A = (a_{ij})$ be the $n \times n$ adjacency matrix of $g \in G$, that is let $a_{ij} = 1$ if (i, j) and $a_{ij} = 0$ if i is not adjacent to j. Since (i, j) implies (j, i), A is symmetric and since (i, i) does not exist, $tr(A) = 0$. We also note that if $u' = (1, 1, \ldots, 1)$ then $Au = vu$ so that u is an eigenvector, and v the corresponding eigenvalue, of A. Let $J = uu'$ be the $n \times n$ matrix of all ones. Let I be the identity matrix (we will sometimes write $A^0 = I$).

We have that $A^p = (A^p)_{ij}$ $p = 0, 1, \ldots, d$ has the property that $(A^p)_{ij} = c$ if there are c paths, including paths in which arcs are retraced, of length p, from node i to node j. We observe that if $d(i, j) = d$ then $(A^d)_{ij} = t$. This follows from Lemma 2.1 and the above statement.

Theorem 4.1: A necessary condition for the existence of $g(2, t, v, n)$ with adjacency matrix A is

$$4.1) \qquad A^2 + tA + (t-v)I = tJ$$

Proof: If $d = 2$, then for any $g \in G$ we have an adjacency matrix A.

A^2 has the following three properties:

1) If $d(i, j) = 2$ then $(A^2)_{ij} = t$

2) If $d(i, j) = 1$ then $(A^2)_{ij} = 0$ (note that in this case $a_{ij} = 1$)

3) $(A^2)_{ii} = v$ for all i

The theorem now follows immediately. Q. E. D.

The roots of A (see [4] page 23) can now be found. We postpone doing so (to section 5) and proceed to establish, for all g, the necessary polynomial. We define

$$F_0(A) = I \qquad G_0(A) = I$$

$$F_1(A) = A \qquad G_1(A) = A + I$$

$$F_2(A) = A^2 - vI$$

$$F_{i+1}(A) = AF_i(A) - (v-1) F_{i-1}(A) \qquad i \geq 2$$

$$G_{i+1}(A) = AG_i(A) - (v-1) G_{i-1}(A) \qquad i \geq 1$$

Observe

$$G_i(A) = \sum_{j=0}^{i} F_j(A) \qquad i \geq 0 \tag{4.2}$$

Theorem 4.2: $F_k(A) = \left(f_{ij}^{(k)}\right)$ has the property that $f_{ij}^{(k)}$ is equal to the number of paths of length k from node i to node j.

Proof: See [1] page 315.

Theorem 4.3: A necessary condition for the existence of $g(d, t, v, n)$ is given by

$$F_d(A) + t G_{d-1}(A) = t J \tag{4.3}$$

Proof: By Lemma 2.1 $$f_{ij}^{(d)} = \begin{cases} t & \text{if } d(i,j) = d \\ 0 & \text{otherwise} \end{cases}$$

From equation 4.2), Theorem 4.2, and the hypothesis for $d(i,j) < d$ we have $$g_{ij}^{(d-1)} = \begin{cases} 1 & \text{if } f_{ij}^{(d)} = 0 \\ 0 & \text{if } f_{ij}^{(d)} = t \end{cases}$$

The theorem follows. Q.E.D.

Since $Ju = nu$ we have that n is the eigenvalue of J which corresponds to the eigenvalue v of A. The other eigenvalue of J is 0, with multiplicity $n-1$ and therefore, from 4.3) the other $n-1$ roots α, of A, must satisfy

4.4) $$F_d(\alpha) + tG_{d-1}(\alpha) = 0.$$

5. In this section we examine, in detail, 4.1) in order to obtain certain necessary conditions for the existence of $g(2, t, v, n)$. In particular we prove that for a given t (with the possible exception of $t = 2, 4$, or 6) there are a finite number, possibly zero, of $r(2, t, v, n)$ graphs.

From 4.3) or rewriting 4.1) we have

5.1) $$A^2 + tA + (t-v)I = tJ$$

and thus

5.2) $$v^2 + (t-1)v + t = tn$$

(which could have been obtained directly from 3.1)).

From 4.4) we have

5.3) $$\alpha^2 + t\alpha + (t-v) = 0$$

and therefore $\alpha = \left[-t \pm (t^2 - 4t + 4v)^{\frac{1}{2}}\right]2^{-1}$.

Lemma 5.1: $(t^2 - 4t + 4v)^{\frac{1}{2}} = a$ is integral.

Proof: If not then $(-t+a)2^{-1}$ and $(-t-a)2^{-1}$ have the same multiplicity x as roots of A. Since $tr(A) = 0$ we have $v + (-t+a)2^{-1}x + (-t-a)2^{-1}x = 0$ giving $x = vt^{-1}$ (note that v has multiplicity 1). Since the total number of roots of A is n we have $1 + x + x = n$ or $1 + vt^{-1} + vt^{-1} = n$ and therefore $tn = 2v + t$. Substituting in 5.2) gives

5.4) $$v^2 + (t-3)v = 0$$

if $t = 2$ $v^2 = v$ implying $v = 0$ or $v = 1$ which is impossible

if $t = 3$ $v^2 = 0$ implying $v = 0$

if $t > 3$ $v = 0$ or $v = 3-t < 0$ a contradiction, thus a is integral.

We have $a^2 = t^2 - 4t + 4v$ and thus

$$5.5) \qquad 4v = a^2 - (t^2 - 4t).$$

Theorem 5.2: Let $d = 2$ and t be fixed. Then there are a finite number of $r(2, t, v, n)$ graphs, except, possibly, for the cases $t = 2, 4,$ or 6.

Proof: Let x represent the multiplicity of $(-t+a)2^{-1}$ as a root. Therefore, since there are n roots and v has multiplicity 1, we have $(-t-a)2^{-1}$ is a root with multiplicity $n-x-1$. Since $tr(A) = 0$ $v+(-t+a)2^{-1}x + (-t-a)2^{-1}(n-1-x) = 0$ or

$$5.6) \qquad 2ax = (n-1)(t+a) - 2v$$

multiplying through by $16t$ and using 5.5) we get $32tax =$ $(t+a)[(a^2 - (t^2 - 4t))^2 + (4t-4)(a^2 - (t^2 - 4t))] - 8t(a^2 - (t^2 - 4t))$ or

$$5.7) \quad 32tax = a^5 + ta^4 - (2t^2 - 12t + 4)a^3 - (2t^3 - 12t^2 + 12t)a^2 + (t^4 - 12t^3 + 36t^2 - 16t)a$$
$$+ t^2(t-2)(t-4)(t-6).$$

We proved, in Lemma 5.1, that a is integral. The integral solutions a of 5.7) must be the factors of $t^2(t-2)(t-4)(t-6)$, (we think of $32tx$ as part of the coefficient of a), unless $t = 2, 4,$ or 6, in which case this constant term is 0. If $t^2(t-2)(t-4)(t-6) \neq 0$ then there are at most a finite number of factors and the theorem follows. Q. E. D.

6. In this section we consider the existence of elements of G of the form g(2, 2, v, n). Of course we have b(2, 2, 2, 4) and there are no other $\underline{B}$ graphs with d = 2, t = 2. We will show the existence and uniqueness of r(2, 2, 5, 16) and r(2, 2, 10, 56) and establish some necessary conditions for other values of v and n.

We get, by letting t = 2 in 5.1)

6.1) $$A^2 + 2A + (2-v)I = 2J$$

and from 5.2)

6.2) $$v^2 + v + 2 = 2n.$$

Letting t = 2 in Lemma 5.1 and factoring we see that $(-1+v)^{\frac{1}{2}} = \bar{a}$ must be integral and thus

6.3) $$v = \bar{a}^2 + 1.$$

Substituting in 5.7) gives

6.4) $$4x = \bar{a}^4 + \bar{a}^3 + 3\bar{a}^2 + \bar{a} + 2$$

and thus x is integral except when $\bar{a} \equiv 0\ (4)$. We have previously mentioned

6.5) b(2, 2, 2, 4)

the complete bipartite graph on 4 nodes and

6.6) r(2, 2, 5, 16)

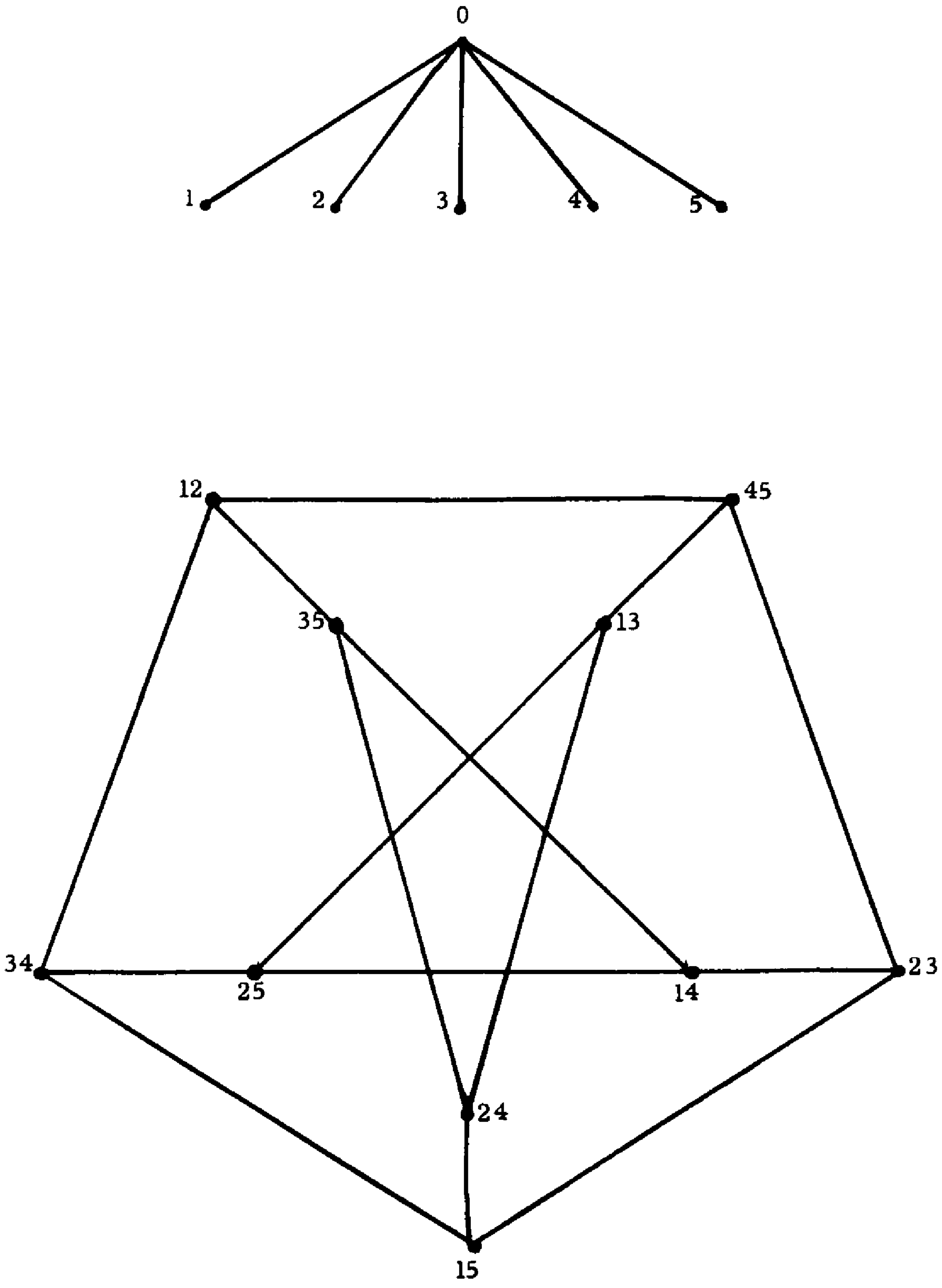

Figure 6.1

(see [5]) shown in Figure 6.1. 6.6) can be viewed as a 5 claw above a Petersen graph (Moore graph of type (3, 2), see [6]) where the nodes of the Petersen graph are identified by the claw numbers which connect to them.

Theorem 6.1: $r(2, 2, 10, 56)$ exists and is unique.

Proof: We exhibit the hierarchy as Figure 6.2, the adjacency matrix A as Figure 6.3 and A^2 as Figure 6.4. We identify the nodes of ℓ_1 as $1, 2, \ldots, 9, T$. By the lemmas of section 3 we see that the identification of the nodes of ℓ_2 is unique. We identify a node in ℓ_2 by the identifications of the 2 nodes in ℓ_1 which connect to it. This identification is unordered but in proving the theorem we will often establish order, as a convenience only. The bulk of the following argument will be concerned with the subgraph which consists of the nodes of ℓ_2 and the reentering arcs. In particular we will examine the adjacencies of $12, 13, \ldots, 1T$, in this subgraph. Clearly, in this subgraph $v_{1j} = 8 \quad j = 2, \ldots, T$. We will show that the nodes adjacent to 12 must be 34, 45, 56, 36, 78, 89, 9T, 7T (two 4-cycles as opposed to the other alternates of an 8-cycle or a 3-cycle and a 5-cycle). Once this is established the rest of the proof follows. In order to establish this result we will have to look at the 5 adjacencies to $34, 45, \ldots, 7T$ which do not have a 1 or a 2 in their identification. We note that the nodes of ℓ_2 can be viewed as the blocks of a balanced incomplete block design [3] with parameters $b = 45$, $k = 2$, $\lambda = 1$, $v = 10$, $r = 9$. We return to the discussion of bibd's in section 7.

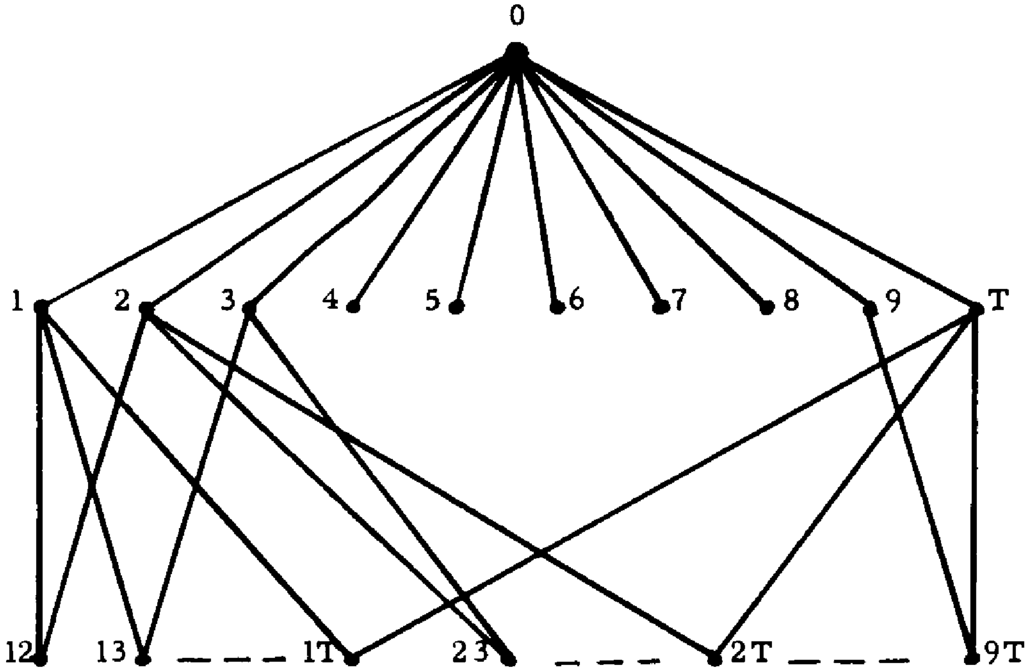

Figure 6.2

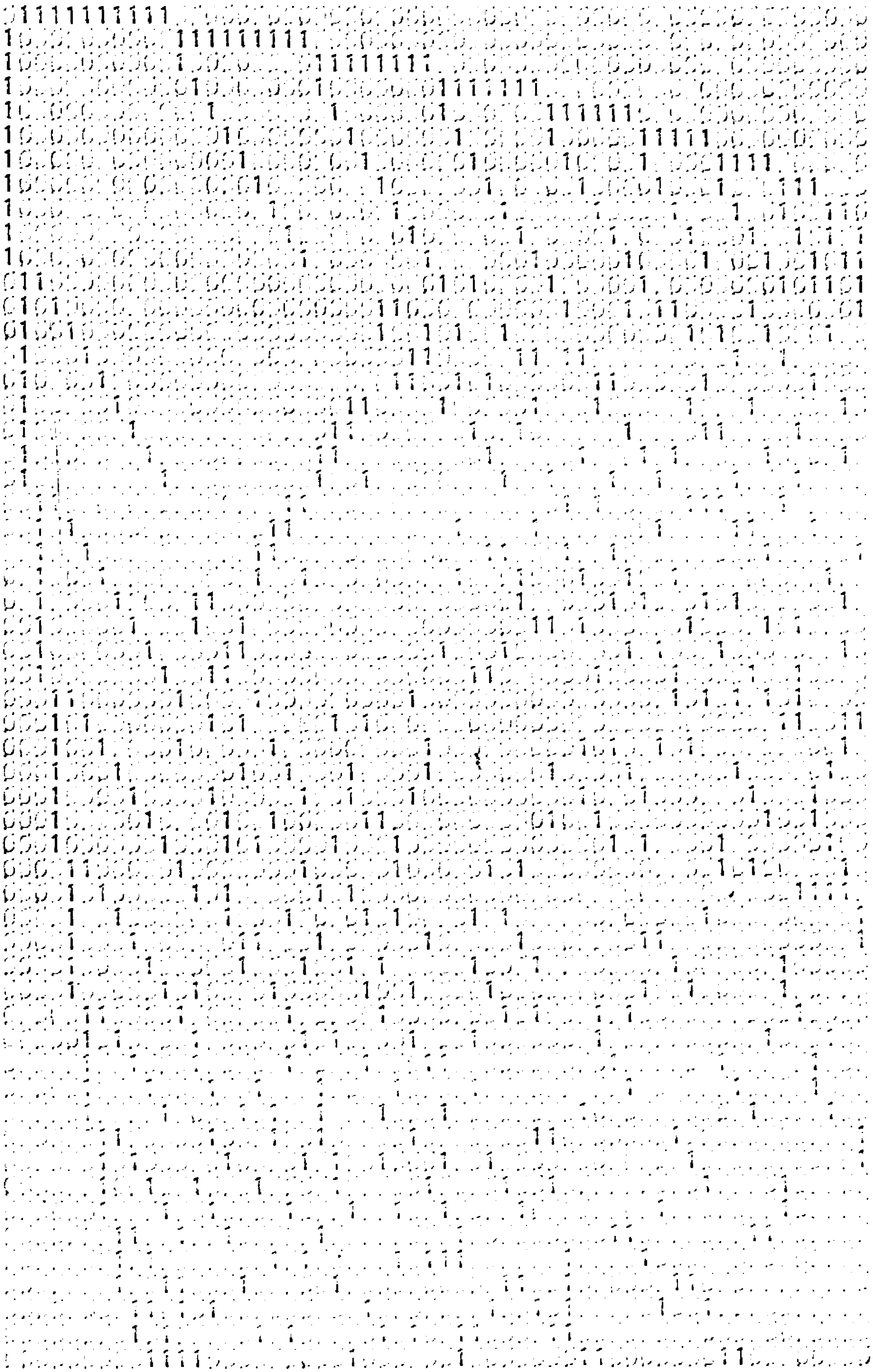

Figure 6.3

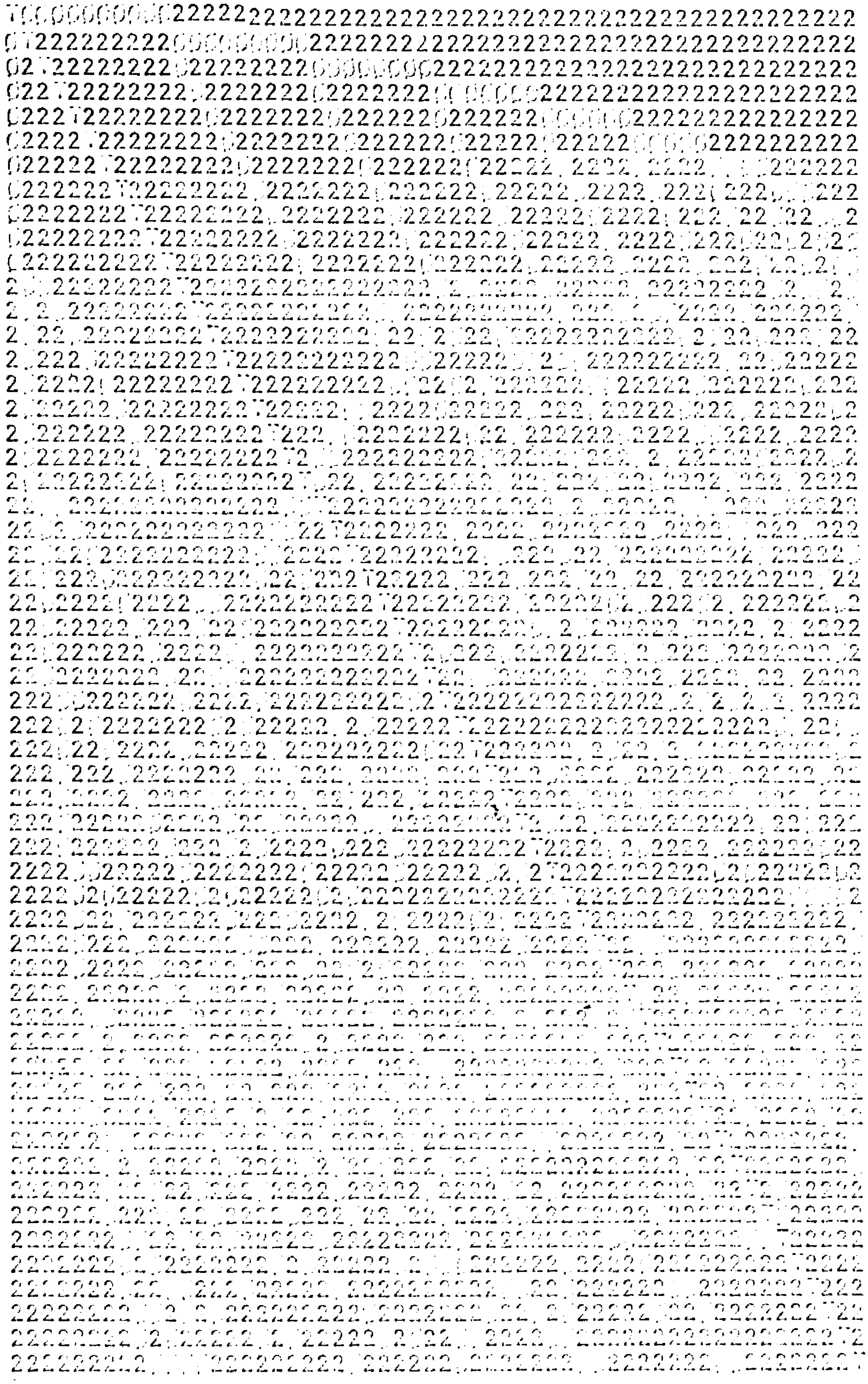

Figure 6.4

Let B be the submatrix of A which is the adjacency matrix of ℓ_2.

Let $\langle ij \rangle = \{k\ell : b_{ij,\,k\ell} = 1\}$ (see Lemma 3.6).

Lemma 6.1: $\langle ij \rangle$ has the following properties

a) $k\ell \in \langle ij \rangle$ implies $k, \ell \neq i, j$

b) $k\ell \in \langle ij \rangle$ implies there is an $m \neq \ell$ and an $s \neq k$ such that $km, s\ell \in \langle ij \rangle$

c) Suppose $k\ell, km \in \langle ij \rangle$, $\ell \neq m$. Then for all $r \neq \ell, m$ $kr \notin \langle ij \rangle$

d) If $x \neq i, j$ then there exist m, n $m \neq n$ such that xm, $xn \in \langle ij \rangle$.

Proof:

a) Lemma 3.4

b) $(k, k\ell, ij)$ is a path of length 2 from k to ij and therefore there must exist a second such. Clearly this can only be accomplished using a reentering arc and thus there must exist a km $m \neq \ell$ such that (k, km, ij) is a path of length 2. Similarly $(\ell, k\ell, ij)$ exists and by an identical argument we get the existence of $s\ell$ giving $(\ell, s\ell, ij)$. Thus $km, s\ell \in \langle ij \rangle$.

c) If $kr \in \langle ij \rangle$ then $(k, k\ell, ij)$ (k, km, ij) and (k, kr, ij) would be 3 distinct paths of length 2 from k to ij contradicting $t = 2$.

d) By Lemma 3.6 $|\langle ij \rangle| = v-2$. By a) there are $v-2$ sets

$\ell_1(k)$ $k \neq i, j$ available to form the elements of $\langle ij \rangle$. By c) no set $\ell_1(k)$ can be used 3 times and thus each set must be used twice. Q.E.D.

Lemma 6.2:

a) If $k\ell \in \langle ij \rangle$ then there is an m such that $k\ell \in \langle im \rangle$

b) If $s \neq m, j$ then $k\ell \notin \langle is \rangle$

c) Let $xy \in \ell_2$ then for every $i \neq x, y$ there is a j and a k such that $xy \in \langle ij \rangle$ $xy \in \langle ik \rangle$

d) If $k \neq j$ $|\langle ij \rangle \cap \langle ik \rangle| = 1$

Proof:

a) $(i, ij, k\ell)$ is a path of length 2 and thus there must be a second one, say $(i, im, k\ell)$

b) If $k\ell \in \langle is \rangle$ then $\langle i, is, k\ell \rangle$ would be a third such path, a contradiction.

c) By Lemma 3.2 $|\ell_2(i)| = v-1$ and for each element $ij \in \ell_2(i)$ $|\langle ij \rangle| = v-2$ by Lemma 3.6. Therefore $\sum_j |\langle ij \rangle| = (v-1)(v-2)$. $|\ell_2 - \ell_2(i)| = \binom{v-1}{2} = \frac{(v-1)(v-2)}{2}$ which means every element of $\ell_2 - \ell_2(i)$ must appear in exactly 2 of the sets $\langle ij \rangle$ otherwise some element would appear 3 times contradicting b).

d) (ij, i, ik) is a path of length 2 and thus there must be a second such. Since this cannot happen in the hierarchy it must be through a reentering arc. If the path is $(ij, mn, k\ell)$ then $mn \in \langle ij \rangle$ $mn \in \langle k\ell \rangle$. Q.E.D.

Let $ij \in \langle k\ell \rangle$ then with respect to $\langle k\ell \rangle$ we define $\langle \overline{ij} \rangle =$ $\langle ij \rangle - \{k\ell, kx, \ell y : kx, \ell y \in \langle ij \rangle\}$.

Lemma 6.3:

a) $|\langle \overline{ij} \rangle| = v-5$

b) Let ij, $im \in \langle k\ell \rangle$. Then $\langle \overline{ij} \rangle \cap \langle \overline{im} \rangle = \phi$.

Proof:

a) $|\langle \overline{ij} \rangle| = |\langle ij \rangle| - 3 = v - 2 - 3 = v - 5$

b) By Lemma 6.2 $|\langle ij \rangle \cap \langle im \rangle| = 1$ $k\ell \in \langle ij \rangle$ $k\ell \in \langle im \rangle$ and $k\ell \notin \langle \overline{ij} \rangle$ $k\ell \notin \langle \overline{im} \rangle$ thus if $\langle \overline{ij} \rangle \cap \langle \overline{im} \rangle \neq \phi$ we would contradict Lemma 6.2. Q.E.D.

Lemma 6.4: Let $\langle xy \rangle$ be given ij, $k\ell$, $ms \in \langle xy \rangle$ $i, j \neq k, \ell$ then

a) $|\langle \overline{ij} \rangle \cap \langle \overline{k\ell} \rangle| = 1$

b) $|(\langle \overline{ij} \rangle \cap \langle \overline{k\ell} \rangle) \cap \langle \overline{ms} \rangle| = 0$

Proof:

a) $(ij, xy, k\ell)$ is a path of length 2 from ij to $k\ell$ and therefore there must be a second (which cannot be through the hierarchy). Thus there is a pq such that $(ij, pq, k\ell)$ is a path of length 2. Neither p or q can be x or y because suppose $p = x$. Then $ij, k\ell \in \langle xq \rangle$ $ij, k\ell \in \langle xy \rangle$ which implies $|\langle xq \rangle \cap \langle xy \rangle| = 2$ contradicting Lemma 6.2. Therefore $pq \in \langle \overline{ij} \rangle$ $pq \in \langle \overline{k\ell} \rangle$. There is no other such point or else there would be 3 paths of length 2 from ij to $k\ell$.

b) If m or s is equal to i or j or k or ℓ the result follows from Lemma 6.3. Otherwise let $pq = \langle\overline{ij}\rangle \cap \langle\overline{k\ell}\rangle$. Therefore (xy, ij, pq) and $(xy, k\ell, pq)$ are 2 paths of length 2. If $pq \in \langle\overline{ms}\rangle$ then (xy, ms, pq) would be a 3'd such path. A contradiction. Q. E. D.

Lemma 6.5: A necessary condition for the existence of $r(2, 2, 10, 56)$ is that $\langle xy\rangle$ be one of the following:

a) $\{ij, jk, k\ell, \ell m, mn, np, pq, iq\}$

b) $\{ij, jk, ik, \ell m, mn, np, pq, \ell q\}$

c) $\{ij, jk, k\ell, i\ell, mn, np, pq, mq\}$

Proof: Consequence of Lemma 6.1. Q. E. D.

There is no loss in generality in assuming that when $\langle ab\rangle = \langle 12\rangle$ we have $i = 3$, $j = 4$, $k = 5$, $\ell = 6$, $m = 7$, $n = 8$, $p = 9$, $q = T$.

Lemma 6.6: $\langle 12\rangle \neq \{34, 45, 35, 67, 78, 89, 9T, 6T\}$.

Proof: The elements of ℓ_2 available to fill the sets $\langle\overline{34}\rangle$ $\langle\overline{45}\rangle$ $\langle\overline{35}\rangle$ $\langle\overline{67}\rangle$ $\langle\overline{78}\rangle$ $\langle\overline{89}\rangle$ $\langle\overline{9T}\rangle$ $\langle\overline{6T}\rangle$ are

$$\left.\begin{array}{llllll}
36 & 46 & 56 & & & \\
37 & 47 & 57 & & & \\
38 & 48 & 58 & 68 & & \\
39 & 49 & 59 & 69 & 79 & \\
3T & 4T & 5T & & 7T & 8T
\end{array}\right\} \text{displayed this way for clarity}$$

since by definition no elements of the form $1x$ or $2y$ can be in the sets and clearly neither can elements of $\langle 12\rangle$ otherwise there would be triangles. By Lemma 6.3 we know $|\langle\overline{34}\rangle| = |\langle\overline{35}\rangle| = 5$. We show that to meet this cardinality condition must cause a contradiction of Lemma 6.1. $\langle\overline{34}\rangle \subset \langle 34\rangle$. Thus, by Lemma 6.1, no elements of the form $3x$ or $4x$ are in $\langle\overline{34}\rangle$. Also at most 2 elements of the form $5x$ are in $\langle\overline{34}\rangle$ and thus at least 3 of 68, 69, 79, 7T, 8T are in $\langle\overline{34}\rangle$. By Lemma 6.3 this means at most 2 of 68, 69, 79, 7T, 8T are in $\langle\overline{35}\rangle$ and since no elements of the form $3x$ or $5x$ are in $\langle\overline{35}\rangle$ at least 3 elements of the form $4x$ must be contradicting Lemma 6.1. Q.E.D.

Lemma 6.7: $\langle 12\rangle \neq \{34, 45, 56, 67, 78, 89, 9T, 3T\}$.

Proof: The elements of ℓ_2 available to fill the sets $\langle\overline{34}\rangle$ $\langle\overline{45}\rangle$ $\langle\overline{56}\rangle$ $\langle\overline{67}\rangle$ $\langle\overline{78}\rangle$ $\langle\overline{89}\rangle$ $\langle\overline{9T}\rangle$ $\langle\overline{3T}\rangle$ are

$$\left.\begin{array}{llllll} 35 & & & & & \\ 36 & 46 & & & & \\ 37 & 47 & 57 & & & \\ 38 & 48 & 58 & 68 & & \\ 39 & 49 & 59 & 69 & 79 & \\ & 4T & 5T & 6T & 7T & 8T \end{array}\right\} \begin{array}{l}\text{for the same reasons as}\\ \text{outlined in Lemma 6.6.}\end{array}$$

Case 1: $34 \in \langle 15\rangle$ (or $\langle 25\rangle$, the argument is identical)

Observe that Lemma 6.2 guarantees the existence of an x and a z such that $34 \in \langle 1x\rangle$ $34 \in \langle 1z\rangle$. There is at most one element of the form $5x \in \langle\overline{34}\rangle$ (observe there might be none since $5x$ could be in $\langle 34\rangle - \langle\overline{34}\rangle$). By Lemma 6.1 no elements of the form $3x$ or $4x$ are in $\langle\overline{34}\rangle$ and thus at least 4 of $68, 69, 6T, 79, 7T, 8T$ must be. Therefore by Lemma 6.3 $\langle\overline{45}\rangle$ can have at most 2 of these 6 elements, by Lemma 6.1 none of the form $4x$ or $5x$, and thus at least 3 of the form $3x$. A contradiction.

<u>Case 2</u>: $34 \in \langle 1T\rangle$ or $\langle 2T\rangle$

There is at most one element of the form $xT \in \langle\overline{34}\rangle$. By Lemma 6.1 no elements of the form $3x$ or $4x$ are in $\langle\overline{34}\rangle$ and thus at least 4 of $57, 58, 59, 68, 69, 79$ are. Thus by Lemma 6.3 $\langle\overline{3T}\rangle$ can have at most 2 of these 6 elements, by Lemma 6.1 none of the form $3x$ or xT and thus at least 3 of the form $4x$. A contradiction.

<u>Case 3</u>: $34 \in \langle 16\rangle$ or $\langle 26\rangle$

At most one of $68, 69, 6T \in \langle\overline{34}\rangle$. At most two elements of the form $5x \in \langle\overline{34}\rangle$. Therefore at least 2 of $79, 7T, 8T \in \langle\overline{34}\rangle$. Thus $\langle\overline{45}\rangle$ contains at most one of $79, 7T, 8T$, no elements of the form $4x$ or $5x$ and therefore 2 of $68, 69, 6T$ and 2 of $37, 38, 39$. Therefore by Lemma 6.3 $\langle\overline{56}\rangle$ contains at most one of $37, 38, 39$ and at most one of $79, 7T, 8T$ by Lemma 6.3 and 6.4. Thus $\langle\overline{56}\rangle$ must contain at least 3 elements of the form $4x$. A contradiction.

<u>Case 4</u>: $34 \in \langle 19\rangle$ or $\langle 29\rangle$

The argument is exactly as in Case 3 with the following replacements:

a) 68, 69, 6T by 59, 69, 79

b) 79, 7T, 8T by 57, 58, 68

c) 2 elements of the form 5x by 2 elements of the form xT

d) $\langle\overline{45}\rangle$ by $\langle\overline{3T}\rangle$

e) 37, 38, 39 by 46, 47, 48

f) $\langle\overline{56}\rangle$ by $\langle\overline{9T}\rangle$

g) 3 elements of the form 4x by 3 elements of the form 3x

Case 5: $34 \in \langle 17\rangle$ and $34 \in \langle 27\rangle$

It is clear that the permutation (3 4 5 6 7 8 9 T) will not change the above arguments and thus we have

Proposition 6.1: $45 \notin \langle a\ b\rangle$ $a = 1, 2$ $b = 3, 6, 7, T$
$56 \notin \langle a\ b\rangle$ $a = 1, 2$ $b = 3, 4, 7, 8$
$67 \notin \langle a\ b\rangle$ $a = 1, 2$ $b = 4, 5, 8, 9$
$78 \notin \langle a\ b\rangle$ $a = 1, 2$ $b = 5, 6, 9, T$
$89 \notin \langle a\ b\rangle$ $a = 1, 2$ $b = 3, 6, 7, T$
$9T \notin \langle a\ b\rangle$ $a = 1, 2$ $b = 3, 4, 7, 8$
$3T \notin \langle a\ b\rangle$ $a = 1, 2$ $b = 4, 5, 8, 9$

Since 57, 79, 7T cannot be in $\langle\overline{34}\rangle$ (Lemma 6.2) 8T must or else there would be 3 elements of the form 5x or 3 of the form 6x.

Subcase 5a: 58, 59, 69, 6T, $8T \in \langle\overline{34}\rangle$

In this case since $\langle\overline{45}\rangle \cap \langle\overline{34}\rangle = \phi$ and there cannot be 3 elements of the form $3x \in \langle\overline{45}\rangle$ we have 68, 79, $7T \in \langle\overline{45}\rangle$. Similarly 57, 68, $79 \in \langle\overline{3T}\rangle$ therefore $|\langle\overline{3T}\rangle \cap \langle\overline{45}\rangle| = 2$ contradicting Lemma 6.4.

Since it is also clear that 59 and 69 must both be in $\langle\overline{34}\rangle$ all other subcases must have

6.7) $59, 5T, 68, 69, 8T \in \langle\overline{34}\rangle$

and therefore

6.8) $3x, 3y, 6T, 79, 7T \in \langle\overline{45}\rangle \qquad x, y = 6, 8, 9$

Subcase 5b: $x = 8,\ y = 9$ in 6.8)

Contradiction of Proposition 6.1 since this would require $45 \in \langle 16\rangle$ or $45 \in \langle 26\rangle$.

Since $|\langle\overline{3T}\rangle \cap \langle\overline{45}\rangle| = 1$ we see from 6.8) that $79 \in \langle\overline{3T}\rangle$ and since $|\langle\overline{34}\rangle \cap \langle\overline{3T}\rangle| = \phi$ we see that $57, 58 \in \langle\overline{3T}\rangle$ otherwise 3 elements of the form $4x$ would be required. We then have

6.9) $4z, 4w, 57, 58, 79 \in \langle\overline{3T}\rangle \qquad z, w = 6, 8, 9$

Subcase 5c: $z = 6$ in 6.9)

Then $w = 8$ or $w = 9$ and therefore $3T \in \langle 19\rangle$ (or $\langle 29\rangle$) in the first instance or $3T \in \langle 18\rangle$ (or $\langle 28\rangle$) in the second in either case a contradiction of Proposition 6.1.

We now have

6.10) $48, 49, 57, 58, 79 \in \langle\overline{3T}\rangle$.

Subcase 5d: in 6.8) let $x = 6 \quad y = 8$

By Lemmas 6.1 and 6.4 $57, 5T \in \langle \overline{89} \rangle$. Therefore $35 \notin \langle \overline{89} \rangle$ and thus by Proposition 6.1 $36, 37 \in \langle \overline{89} \rangle$. For a fifth element there is either 46 or 4T but in either case Proposition 6.1 is contradicted.

Subcase 5e: in 6.8) let $x = 6$ $y = 9$

Same argument as in Subcase 5d.

Case 6: $34 \in \langle 18 \rangle$ and $34 \in \langle 28 \rangle$

By the symmetry of the cycle this case is precisely the same as Case 5, in the same sense that Case 4 is the same as Case 3 (or Case 2 the same as Case 1).

Case 7: $34 \in \langle 17 \rangle$ and $34 \in \langle 28 \rangle$ or ($34 \in \langle 18 \rangle$ and $34 \in \langle 27 \rangle$)

If $57 \in \langle \overline{34} \rangle$ then clearly 8T must also or else 3 elements of the form 5x or 3 of the form 6x would be required. Thus $58, 68 \notin \langle \overline{34} \rangle$, therefore $69, 6T \in \langle \overline{34} \rangle$ and this forces $59 \in \langle \overline{34} \rangle$. This means there is only one way to place elements in $\langle \overline{34} \rangle$ if $57 \in \langle \overline{34} \rangle$.

Subcase 7a: $57, 59, 69, 6T, 8T \in \langle \overline{34} \rangle$

By the usual arguments using Lemmas 6.1 to 6.4 we have $68, 79, 7T \in \langle \overline{45} \rangle$ and $58, 68, 79 \in \langle \overline{3T} \rangle$ contradicting Lemma 6.4. If $58 \in \langle \overline{34} \rangle$ then 8T cannot and 68 cannot. Thus $69, 6T \in \langle \overline{34} \rangle$. There are therefore 2 possible subcases with $58 \in \langle \overline{34} \rangle$.

Subcase 7b: $58, 59, 69, 6T, 7T \in \langle \overline{34} \rangle$

Then $68, 79, 8T \in \langle \overline{45} \rangle$ $57, 68, 79 \in \langle \overline{3T} \rangle$ contradicting Lemma 6.4.

Subcase 7c: $58, 5T, 69, 6T, 79 \in \langle\overline{34}\rangle$

In this case $68, 7T, 8T \in \langle\overline{45}\rangle$ but this violates Proposition 6.1 and Cases 5, 6 which together imply $45 \in \langle 18\rangle$ and $45 \in \langle 29\rangle$ (or in $\langle 19\rangle$ and $\langle 28\rangle$). The only remaining subcases have $59, 5T \in \langle\overline{34}\rangle$. We then have

6.11) $59, 5T, 68, 69, 7T \in \langle\overline{34}\rangle$

or

6.12) $59, 5T, 68, 6T, 79 \in \langle\overline{34}\rangle$.

Subcase 7d: 6.11) holds.

Then $57, 58, 79 \in \langle\overline{3T}\rangle$ which by Proposition 6.1 and Cases 5, 6 cannot be since $3T \in \langle 16\rangle$ and $\langle 27\rangle$ (or $\langle 17\rangle$ and $\langle 26\rangle$). If 6.12) holds then $57, 58, 69 \in \langle\overline{3T}\rangle$ and therefore in order not to violate Proposition 6.1 and Cases 5 and 6 $48, 49 \in \langle\overline{3T}\rangle$ giving

6.13) $48, 49, 57, 58, 79 \in \langle\overline{3T}\rangle$.

Subcase 7e: 6.12) and 6.13) hold.

Then $68 \in \langle\overline{9T}\rangle$ which by Proposition 6.1 and Cases 5, 6 means $46 \notin \langle\overline{9T}\rangle$ thus $47, 48 \in \langle\overline{9T}\rangle$ and $\langle\overline{9T}\rangle \cap \langle\overline{3T}\rangle \neq \phi$. A contradiction. Q. E. D.

Lemma 6.8: $\langle 12\rangle = \{34, 45, 56, 36, 78, 89, 9T, 7T\}$

Proof: The elements of ℓ_2 available to fill $\langle\overline{34}\rangle$ $\langle\overline{45}\rangle$ $\langle\overline{56}\rangle$ $\langle\overline{36}\rangle$ $\langle\overline{78}\rangle$ $\langle\overline{89}\rangle$ $\langle\overline{9T}\rangle$ $\langle\overline{7T}\rangle$ are

35					
	46				
37	47	57	67		
38	48	58	68		
39	49	59	69	79	
3T	4T	5T	6T		8T

(Note: we really have that Lemma 6.5 c must apply)

Case 1: $34 \in \langle 15 \rangle$ (or $\langle 25 \rangle$)

Using the standard lemmas 5x, 6y, 6z, 79, $8T \in \langle\overline{34}\rangle$ thus 79, $8T \notin \langle\overline{36}\rangle$ and therefore $\langle\overline{36}\rangle$ contains either 3 elements of the form 4x or 3 of the form 5x. A contradiction.

Case 2: $34 \in \langle 16 \rangle$ (or $\langle 26 \rangle$)

As in Case 1 79, $8T \in \langle\overline{34}\rangle$ implying 79, $8T \notin \langle\overline{36}\rangle$ giving the same contradiction of Lemma 6.1.

Case 3: $34 \in \langle 1T \rangle$ and $34 \in \langle 2T \rangle$

Note that this is the same as the cases that would arise if we replace T by 7, 8 or 9. We will list those as Cases 4, 5, 6 for completeness. It is of interest to note that in all 4 of these cases the sets $\langle\overline{ab}\rangle$ can be filled. An example of this will be shown at the end of Case 6. Of course the proofs will therefore be of a different type than those previously shown. It is clear that $79 \in \langle\overline{34}\rangle$ and thus either 69 or 59 is but not both and therefore we have

Case 3a 57, 58, 68, 69, $79 \in \langle\overline{34}\rangle$

Case 3b 58, 59, 67, 68, $79 \in \langle \overline{34} \rangle$.

In both cases we show that $\langle 1T \rangle$ cannot be filled without a contradiction.

Case 3a:

We have $34 \in \langle 1T \rangle$ and thus if $xy \in \langle \overline{34} \rangle$, then $xy \notin \langle 1T \rangle$ otherwise (34, xy, 1T) would be a triangle. Also since $\langle 12 \rangle \cap \langle 1T \rangle$ contains 34 this intersection can have no other elements by Lemma 6.2. Thus the only elements of the form $5x (= x5)$ that can belong to $\langle 1T \rangle$ are 25, 35, 59 and by Lemma 6.1 2 of these must belong to $\langle 1T \rangle$. By Lemma 6.5c this pair must be 35, 59 which together with 34 imply 49. The only elements of the form $6x$ available are 26, 46, 67. Clearly $46 \notin \langle 1T \rangle$ and thus $26, 67 \in \langle 1T \rangle$. Therefore $28, 78 \in \langle 1T \rangle$ but then $|\langle 12 \rangle \cap \langle 1T \rangle| = 2$. A contradiction.

Case 3b:

Using the same arguments as above $35, 57 \in \langle 1T \rangle$ and therefore $47 \in \langle 1T \rangle$. This implies $26, 69 \in \langle 1T \rangle$ and thus $28, 89 \in \langle 1T \rangle$ giving same contradiction.

Case 4: $34 \in \langle 19 \rangle$ and $34 \in \langle 29 \rangle$

Use the permutation (78)(9T) and the proof is as in Case 3.

Case 5: $34 \in \langle 18 \rangle$ and $34 \in \langle 28 \rangle$

Use (8T)(79)

Case 6: $34 \in \langle 17 \rangle$ and $34 \in \langle 27 \rangle$

Use (7T)(89)

For Case 6 we show the completed $\langle \overline{ab} \rangle$ sets:

$58, 59, 69, 6T, 8T \in \langle\overline{34}\rangle$

$38, 39, 67, 68, 79 \in \langle\overline{45}\rangle$

$37, 3T, 47, 49, 8T \in \langle\overline{56}\rangle$

$48, 4T, 57, 5T, 79 \in \langle\overline{36}\rangle$

$39, 3T, 46, 4T, 59 \in \langle\overline{78}\rangle$

$35, 47, 5T, 67, 6T \in \langle\overline{89}\rangle$

$37, 38, 46, 57, 58 \in \langle\overline{9T}\rangle$

$35, 48, 49, 68, 69 \in \langle\overline{7T}\rangle$

This illustrates

Proposition 6.2: The existence of all sets $\langle\overline{ij}\rangle$ with respect to given $\langle ab\rangle$ is not sufficient for the existence of $r(d, t, v, n)$ graphs.

Proposition 6.3: $45 \notin \langle 1x\rangle \quad x = 3, 6$

$56 \notin \langle 1y\rangle \quad y = 3, 4$

$36 \notin \langle 1z\rangle \quad z = 4, 5$

Proposition 6.4: $xy \notin \langle 1a\rangle$ and $\langle 2a\rangle \quad x, y = 3, 4, 5, 6 \quad a = 7, 8, 9, T$

Case 7: $34 \in \langle 17\rangle$ and $34 \in \langle 28\rangle$

There are 4 distinct ways to fill $\langle\overline{34}\rangle$:

6.14) $58, 5T, 69, 6T, 79 \in \langle\overline{34}\rangle$

6.15) $59, 5T, 68, 6T, 79 \in \langle\overline{34}\rangle$

6.16) $57, 59, 69, 6T, 8T \in \langle\overline{34}\rangle$

6.17) $59, 5T, 67, 69, 8T \in \langle\overline{34}\rangle$.

If $45 \in \langle 19 \rangle$ and $45 \in \langle 2T \rangle$ or $45 \in \langle 1T \rangle$ $45 \in \langle 29 \rangle$ we have when 6.14) holds

6.18) $37, 39, 67, 68, 8T \in \langle \overline{45} \rangle$.

Case 7a: Let 6.14) and 6.18) hold and let $45 \in \langle 19 \rangle$ $45 \in \langle 2T \rangle$.

As in Case 3 we have that the only elements of the form 5x or 6x that can belong to $\langle 17 \rangle$ are 25, 35, 59 and 26, 46, 68. To avoid cycles greater than 4 we see that 35, 59 and 26, 68 are the only possibilities and thus

6.19) $\langle 17 \rangle = \{34, 35, 59, 49, 26, 68, 8T, 2T\}$.

Those elements of the form 6x, 7x, xT which can be used for $\langle 19 \rangle$ are 26, 46, 6T and 27, 47, 57 and 2T, 4T, 5T, 6T (we repeat 6T to show it with both its groupings). If 26, 46 or 46, 6T are in $\langle 19 \rangle$ then there cannot be 2 elements in $\langle 19 \rangle$ of the form 7x so suppose $26, 6T \in \langle 19 \rangle$. Then the other element of the form xT is either 2T which gives a 3 cycle or one of 4T or 5T which together with 45 creates a 5 cycle. A contradiction.

Case 7b: Let 6.14) and 6.18) hold and let $45 \in \langle 1T \rangle$ $45 \in \langle 29 \rangle$

The allowable elements for $\langle 1T \rangle$ of the form

6x are 26, 46, 69

3x are 23, 35, 38

9x are 29, 49, 59, 69, 79.

That 26, 46 cannot be used as a pair is shown by our standard technique. If $46, 69 \in \langle 1T \rangle$ then $(46, 69, 45, 59)$ and $(23, 38, 78, 27)$ would be 2 cycles of length 4 all elements of $\langle 1T \rangle$. But then $\langle 1T \rangle \cap \langle 17 \rangle = \phi$ contradicting Lemma 6.2. Observe $\langle 17 \rangle$ is the same as in Case 7a. When $26, 69 \in \langle 1T \rangle$ we get

6.20) $\{26, 27, 35, 38, 45, 48, 69, 79\} = \langle 1T \rangle$.

We now consider subcases of 7b, that is we have 6.14), 6.18), 6.19), 6.20) and $45 \in \langle 1T \rangle$ $45 \in \langle 29 \rangle$.

<u>Case 7b1</u>: Let $56 \in \langle 19 \rangle$ and $56 \in \langle 2T \rangle$ (notice $59 \notin \langle 1T \rangle$ since 45 is). Then $\langle \overline{56} \rangle$ is given by

6.21) $38, 3T, 47, 48, 79 \in \langle \overline{56} \rangle$.

We examine $\langle 19 \rangle$ and observe that the only elements allowable of the

form 3x are 23, 35, 37

4x are 24, 46, 4T

xT are 2T, 4T, 5T, 6T, 8T.

That the pair 26, 46 cannot be used is immediate. If 46, 4T are used then $46, 4T, 5T, 56, 35, 37, 58, 78 \in \langle 19 \rangle$ but then $|\langle 12 \rangle \cap \langle 19 \rangle| = 2$. A contradiction. If 24, 4T are used then $24, 4T, 8T, 28, 35, 37, 56, 67 \in \langle 19 \rangle$ which means $(\langle 17 \rangle \cap \langle 19 \rangle) \cap \langle 1T \rangle = 35$. A contradiction of Lemma 6.2.

<u>Case 7b2</u>: Let $56 \in \langle 18 \rangle$ and $56 \in \langle 2T \rangle$

6.22) $38, 3T, 47, 49, 79 \in \langle \overline{56} \rangle$

We examine $\langle 18 \rangle$ and observe that the only elements allowable of the

form 4x are 24, 46, 4T
3x are 23, 35, 37, 39
x9 are 29, 39, 59, 69
xT are 2T, 4T, 5T, 6T

The only pairs with form 4x that allow $\langle 18 \rangle$ to be filled are 46, 4T giving $\langle 18 \rangle = \{46, 4T, 56, 5T, 23, 39, 27, 79\}$ and thus $|\langle 18 \rangle \cap \langle 1T \rangle| = 2$. A contradiction.

Case 7b 3: $56 \in \langle 19 \rangle$ $56 \in \langle 27 \rangle$

6.23) $\langle \overline{56} \rangle = \{38, 3T, 48, 4T, 79\}$

We examine $\langle 19 \rangle$ and observe that the only elements allowable of the

form 4x are 24, 46, 47
7x are 27, 37, 47, 57, 67
8x are 28, 58, 68, 8T
Tx are 2T, 5T, 6T, 8T.

That the pair 24, 46 cannot be used is immediate. If 46, 67 are used we must have $2T, 8T \in \langle 19 \rangle$ and thus $|\langle 17 \rangle \cap \langle 19 \rangle| \geq 2$. A contradiction. If 24, 47 are used then $\langle 19 \rangle$ is one of the 2 sets shown below $\langle 19 \rangle = \{24, 47, 23, 37, 56, 6T, 58, 8T\}$ or

$\langle 19 \rangle = \{24, 47, 23, 37, 56, 68, 5T, 8T\}$

in both cases $\langle 19 \rangle \cap \langle 1T \rangle = \phi$. A contradiction.

Case 7b 4: $56 \in \langle 18\rangle$ $56 \in \langle 27\rangle$

6.24) $38, 3T, 49, 4T, 79 \in \langle\overline{56}\rangle$

We examine $\langle 18\rangle$ and observe that the only elements allowable of the form $4x$ are $24, 46, 47$

xT are $2T, 5T, 6T$

That $\langle 18\rangle$ cannot be filled is immediate. There are no other pairs of sets that can contain 56 without contradicting Lemma 6.2 and therefore Case 7b is complete.

Proposition 6.5: If 6.14) holds $45 \notin \langle x8\rangle$ $x = 1, 2$

Proof: $8T \in \langle\overline{45}\rangle$ or else $\langle\overline{45}\rangle$ would contain 3 elements of the form $3x$ or 3 of the form $6x$. Thus $68 \notin \langle\overline{45}\rangle$ and since $69, 6T \in \langle\overline{34}\rangle$ only 67 can be in $\langle\overline{45}\rangle$. A contradiction.

Case 7c: Let 6.14) hold and let $45 \in \langle 19\rangle$ $45 \in \langle 27\rangle$

Then $\langle\overline{45}\rangle = \{39, 3T, 67, 68, 8T\}$ as is seen by standard techniques. Then $36 \in \langle 18\rangle$ and $36 \in \langle 2T\rangle$ which means $56 \in \langle 1T\rangle$ and $56 \in \langle 28\rangle$. Therefore $\langle\overline{56}\rangle = \{37, 38, 49, 4T, 79\}$ and thus $\langle\overline{36}\rangle = \{47, 48, 57, 59, 8T\}$ implying $18, 48, 8T \in \langle 36\rangle$ contradicting Lemma 6.2.

Case 7d: Let 6.14) hold and let $45 \in \langle 1T\rangle$ $45 \in \langle 27\rangle$

Then $\langle\overline{45}\rangle = \{38, 39, 67, 68, 8T\}$. A contradiction. When 6.15) holds if $45 \in \langle 19\rangle$ and $45 \in \langle 2T\rangle$ or $45 \in \langle 1T\rangle$ $45 \in \langle 29\rangle$ we have

6.25) $37, 38, 67, 69, 8T \in \langle\overline{45}\rangle$.

If then $56 \in \langle 18\rangle$ $56 \in \langle 27\rangle$ we have

6.26) $39, 3T, 48, 4T, 79 \in \langle\overline{56}\rangle$

and since $36 \in \langle 1T\rangle$ $36 \in \langle 29\rangle$ we also have

6.27) $47, 49, 57, 58, 8T \in \langle\overline{36}\rangle$

Note that if 6.25) holds then $56 \in \langle 18\rangle$ and $56 \in \langle 29\rangle$ is impossible and if $56 \in \langle 1T\rangle$ and $56 \in \langle 27\rangle$ then $36 \in \langle 18\rangle$ and $36 \in \langle 29\rangle$ which is also impossible and also $56 \in \langle 1T\rangle$ and $56 \in \langle 29\rangle$ is impossible so that if 6.15) and 6.25) then 6.26) and 6.27) must follow.

<u>Case 7e</u>: Let 6.15) (6.25) and $45 \in \langle 19\rangle$ $45 \in \langle 2T\rangle$

We examine $\langle 19\rangle$ and observe that the only allowable elements

of the form 3x are 23, 35, 3T
7x are 27, 47, 57

which together with $45 \in \langle 19\rangle$ give a 3 cycle (45, 47, 57). A contradiction.

<u>Case 7f</u>: Let 6.15), 6.25), 6.26), 6.27) and $45 \in \langle 1T\rangle$ $45 \in \langle 29\rangle$ hold.

Then

6.28) $\langle 17\rangle = \{34, 46, 39, 69, 25, 2T, 58, 8T\}$ and

$\langle 1T\rangle = \{23, 27, 39, 79, 45, 46, 58, 68\}$.

Thus $|\langle 17\rangle \cap \langle 1T\rangle| = 3$. A contradiction. Let 6.15) still hold and let $45 \in \langle 18\rangle$ $45 \in \langle 27\rangle$ then

6.29) $39, 3T, 67, 69, 8T \in \langle\overline{45}\rangle$

6.30) $37, 38, 48, 4T, 79 \in \langle\overline{56}\rangle$

6.31) $47, 49, 57, 58, 8T \in \langle\overline{36}\rangle$.

Case 7g: Let 6.15), 6.29), 6.30), 6.31) hold and let $56 \in \langle 19\rangle$ $56 \in \langle 2T\rangle$. We still have 6.28) and from 6.30) and $56 \in \langle 19\rangle$ we get $\langle 19\rangle = \{23, 28, 3T, 8T, 46, 47, 56, 57\}$ thus $\langle 17\rangle \cap \langle 19\rangle = \{46, 8T\}$. A contradiction.

Case 7h: Same as Case 7g but let $56 \in \langle 1T\rangle$ and $56 \in \langle 29\rangle$

Then it is immediate that $\langle 1T\rangle$ cannot be filled since $56 \in \langle 1T\rangle$ and the only allowable elements of the form $3x$ are 23, 35, 39
$8x$ are 28, 58, 68.

Let 6.15) still hold and let $45 \in \langle 18\rangle$ $45 \in \langle 29\rangle$ then

6.32) $37, 3T, 67, 69, 8T \in \langle\overline{45}\rangle$.

It is immediate that $56 \notin \langle 19\rangle$ and $\langle 2T\rangle$ and thus $56 \in \langle 1T\rangle$ $56 \in \langle 27\rangle$ and $36 \in \langle 19\rangle$ $36 \in \langle 2T\rangle$ giving

6.33) $38, 39, 48, 4T, 79 \in \langle\overline{56}\rangle$

6.34) $47, 49, 57, 58, 8T \in \langle\overline{36}\rangle$.

Case 7i: Let 6.15), 6.32), 6.33) and 6.34) hold

That $\langle 1T\rangle$ cannot be filled is easily seen since $56 \in \langle 1T\rangle$ and 23, 35, 37 are only possible elements of form $3x$ and 28, 58, 68 only possible of form $8x$.

Case 7j: Let 6.15) still hold and let $45 \in \langle 18\rangle$ $45 \in \langle 2T\rangle$

Then

6.35) $37, 39, 67, 69, 8T \in \langle\overline{45}\rangle$.

Therefore $\langle 18 \rangle = \{45, 46, 5T, 6T, 23, 3T, 7T, 27\}$ which means $|\langle 12 \rangle \cap \langle 18 \rangle| = 2$. A contradiction. Let 6.15) still hold and let $45 \in \langle 19 \rangle$ $45 \in \langle 27 \rangle$. Then

6.36) $38, 3T, 67, 69, 8T \in \langle \overline{45} \rangle$.

Now $56 \in \langle 18 \rangle$ $56 \in \langle 2T \rangle$ or $56 \in \langle 1T \rangle$ $56 \in \langle 29 \rangle$. In the second instance one immediately sees that $\langle \overline{56} \rangle$ cannot be filled. In the first instance we also know $36 \in \langle 1T \rangle$ $36 \in \langle 29 \rangle$. We then get

6.37) $37, 39, 48, 4T, 79 \in \langle \overline{56} \rangle$

6,38) $47, 49, 57, 58, 8T \in \langle \overline{36} \rangle$.

<u>Case 7k</u>: Let 6.15), 6.36), 6.37), 6.38) hold

From 6.36) we get $\langle 19 \rangle = \{35, 45, 37, 47, 28, 2T, 68, 6T\}$. From 6.38) we get $\langle 1T \rangle = \{36, 46, 38, 48, 25, 27, 59, 79\}$. Or $\langle 19 \rangle \cap \langle 1T \rangle = \phi$. A contradiction.

<u>Case 7ℓ</u>: Let 6.15) still hold and let $45 \in \langle 1T \rangle$ $45 \in \langle 27 \rangle$.

Then

6.39) $38, 39, 67, 69, 8T \in \langle \overline{45} \rangle$

and therefore $\langle 1T \rangle = \{45, 46, 58, 68, 23, 29, 37, 79\}$. But then from 6.28) $|\langle 17 \rangle \cap \langle 1T \rangle| = 2$. A contradiction. Let 6.16) hold. Then the permutation (12)(78)(9T) gives 6.14) which we have already considered. Let 6.17) hold then the same permutation gives 6.15) which finishes Case 7. We observe that the permutation (12) settles the case $34 \in \langle 18\rangle$ $34 \in \langle 27\rangle$. Whether $\langle 17\rangle\langle 28\rangle$ or $\langle 18\rangle\langle 29\rangle$ or $\langle 19\rangle\langle 2T\rangle$ or $\langle 1T\rangle\langle 27\rangle$ was the pair used for Case 7 is clearly unimportant and so we have

Proposition 6.5: If $r(2,2,10,56)$ exists then for $x \neq y$ $z \neq w$ we must have

$$\begin{array}{lll} 34 \in \langle 1x\rangle \quad 34 \in \langle 2y\rangle & x, y = 7, 9 \\ 45 \in \langle 1z\rangle \quad 45 \in \langle 2w\rangle & z, w = 7, 9 \text{ or } 8, T \\ 56 \in \langle 1y\rangle \quad 56 \in \langle 2x\rangle & \text{if } z, w \neq 7, 9 \text{ or } 56 \in \langle 18\rangle\langle 2T\rangle \\ 36 \in \langle 1w\rangle \quad 36 \in \langle 2z\rangle & \text{if } z, w \neq 7, 9 \text{ or } 36 \in \langle 1T\rangle\langle 28\rangle \end{array}$$

up to a permutation of the following types

a) (12)

b) (12)(78)(9T)

c) (78)(9T)

d) (79)

e) (8T)

Case 8: $34 \in \langle 17\rangle$ $34 \in \langle 29\rangle$. The possibilities are

6.40) $58, 5T, 68, 6T, 79 \in \langle \overline{34}\rangle$

6.41) $57, 58, 69, 6T, 8T \in \langle \overline{34}\rangle$

6.42) $57, 59, 68, 6T, 8T \in \langle \overline{34}\rangle$

6.43) $57, 5T, 68, 69, 8T \in \langle \overline{34}\rangle$

6.44) $58, 59, 67, 6T, 8T \in \langle \overline{34}\rangle$

6.45) $58, 5T, 67, 69, 8T \in \langle\overline{34}\rangle$

6.46) $59, 5T, 67, 68, 8T \in \langle\overline{34}\rangle$

6.47) $37, 39, 67, 69, 8T \in \langle\overline{45}\rangle$

6.48) $39, 3T, 67, 68, 79 \in \langle\overline{45}\rangle$

6.49) $38, 3T, 67, 69, 79 \in \langle\overline{45}\rangle$

6.50) $38, 3T, 67, 69, 8T \in \langle\overline{45}\rangle$

6.51) $38, 39, 67, 6T, 79 \in \langle\overline{45}\rangle$

6.52) $37, 3T, 68, 69, 79 \in \langle\overline{45}\rangle$

6.53) $38, 3T, 68, 6T, 79 \in \langle\overline{45}\rangle$

6.54) $37, 38, 69, 6T, 79 \in \langle\overline{45}\rangle$

6.55) $37, 39, 68, 6T, 79 \in \langle\overline{45}\rangle$

6.56) $38, 3T, 47, 49, 79 \in \langle\overline{56}\rangle$

6.57) $37, 38, 49, 4T, 8T \in \langle\overline{56}\rangle$

6.58) $37, 39, 47, 49, 8T \in \langle\overline{56}\rangle$

6.59) $37, 39, 48, 4T, 79 \in \langle\overline{56}\rangle$

6.60) $37, 3T, 48, 49, 8T \in \langle\overline{56}\rangle$

6.61) $38, 39, 47, 4T, 8T \in \langle\overline{56}\rangle$

6.62) $39, 3T, 47, 48, 8T \in \langle\overline{56}\rangle$

6.63) $37, 39, 48, 4T, 8T \in \langle\overline{56}\rangle$

6.64) $38, 3T, 47, 49, 8T \in \langle \overline{56} \rangle$

6.65) $38, 3T, 48, 4T, 79 \in \langle \overline{56} \rangle$

6.66) $48, 4T, 57, 59, 8T \in \langle \overline{36} \rangle$

6.67) $47, 48, 59, 5T, 79 \in \langle \overline{36} \rangle$

6.68) $48, 4T, 58, 5T, 79 \in \langle \overline{36} \rangle$

6.69) $47, 49, 57, 59, 8T \in \langle \overline{36} \rangle$

6.70) $47, 4T, 58, 59, 79 \in \langle \overline{36} \rangle$

6.71) $48, 49, 57, 5T, 79 \in \langle \overline{36} \rangle$

6.72) $48, 4T, 57, 59, 79 \in \langle \overline{36} \rangle$

6.73) $49, 4T, 57, 58, 79 \in \langle \overline{36} \rangle$

6.74) $47, 49, 58, 5T, 79 \in \langle \overline{36} \rangle$

We now consider as subcases those combinations of 6.40) to 6.74) which meet the conditions of Proposition 6.5.

Case 8a: Let 6.40), 6.47), 6.56), 6.66) hold.

Then $36 \in \langle 19 \rangle$ and therefore

6.75) $\langle 17 \rangle = \{34, 4T, 8T, 38, 25, 59, 69, 29\}$

6.76) $\langle 19 \rangle = \{36, 46, 47, 37, 28, 58, 5T, 2T\}$

implying $\langle 17 \rangle \cap \langle 19 \rangle = \phi$. A contradiction.

Case 8b: Let 6.41), 6.48), 6.57), 6.67) hold.

Then $56 \in \langle 19 \rangle$ and therefore

6.77) $\langle 17 \rangle = \{34, 46, 68, 38, 2T, 5T, 59, 29\}$

6.78) $\langle 19 \rangle = \{56, 35, 3T, 6T, 28, 48, 47, 27\}$

implying $\langle 17 \rangle \cap \langle 19 \rangle = \phi$. A contradiction.

Case 8c: Let 6.42), 6.49), 6.58), 6.68) hold.

Then $36 \in \langle 19 \rangle$ and therefore

6.79) $\langle 17 \rangle = \{34, 46, 69, 39, 2T, 5T, 58, 28\}$

and one of the following hold

6.80) $\langle 19 \rangle = \{36, 24, 47, 57, 25, 68, 8T, 3T\}$

or

6.81) $\langle 19 \rangle = \{36, 24, 47, 57, 25, 6T, 8T, 38\}$.

In either case $\langle 17 \rangle \cap \langle 19 \rangle = \phi$. A contradiction.

Case 8d: Let 6.40), 6.50), 6.59), 6.69) hold.

Then $45 \in \langle 19 \rangle$ and either 6.75) holds or

6.82) $\langle 17 \rangle = \{34, 25, 59, 69, 26, 48, 8T, 3T\}$

also

6.83) $\langle 19 \rangle = \{45, 35, 37, 47, 28, 68, 6T, 2T\}$.

In either case $\langle 17\rangle \cap \langle 19\rangle = \phi$.

<u>Case 8e</u>: Let 6.43), 6.51), 6.60), 6.70) hold.

Then $56 \in \langle 19\rangle$ and therefore

6.84) $\langle 17\rangle = \{34, 46, 6T, 3T, 28, 58, 59, 29\}$

6.85) $\langle 19\rangle = \{56, 35, 38, 68, 2T, 4T, 47, 27\}$

implying $\langle 17\rangle \cap \langle 19\rangle = \phi$.

<u>Case 8f</u>: Let 6.44), 6.52), 6.61), 6.71) hold.

Then $56 \in \langle 19\rangle$ and

6.86) $\langle 17\rangle = \{34, 35, 5T, 4T, 28, 68, 67, 27\}$

6.87) $\langle 19\rangle = \{56, 46, 48, 58, 2T, 3T, 37, 27\}$.

We observe that $45 \in \langle 18\rangle$ and $45 \in \langle 2T\rangle$ or $45 \in \langle 1T\rangle$ and $45 \in \langle 28\rangle$. If $45 \in \langle 18\rangle$ we get

6.88) $\langle 18\rangle = \{45, 35, 39, 49, 27, 67, 2T, 6T\}$.

If $45 \in \langle 1T\rangle$ we get

6.89) $\langle 1T\rangle = \{45, 46, 67, 57, 29, 39, 38, 28\}$.

If 6.88) then $|\langle 17\rangle \cap \langle 18\rangle| = 3$. A contradiction. If 6.89) then $|\langle 17\rangle \cap \langle 1T\rangle| = 2$. A contradiction.

<u>Case 8g</u>: Let 6.45), 6.53), 6.58), 6.72) hold.

Then $45 \in \langle 19 \rangle$ and therefore

6.90) $\langle 17 \rangle = \{34, 35, 59, 49, 2T, 6T, 28, 68\}$

and one of the following hold

6.91) $\langle 19 \rangle = \{45, 23, 37, 67, 26, 58, 8T, 4T\}$

or

6.92) $\{45, 23, 37, 67, 26, 5T, 8T, 48\}$.

In either case $\langle 17 \rangle \cap \langle 19 \rangle = \phi$. A contradiction.

Case 8h: Let 6.46), 6.54), 6.62), 6.73) hold.

Then $56 \in \langle 19 \rangle$ and therefore

6.93) $\langle 17 \rangle = \{34, 35, 48, 58, 2T, 6T, 29, 69\}$

6.94) $\langle 19 \rangle = \{56, 46, 4T, 5T, 28, 38, 27, 37\}$

implying $\langle 17 \rangle \cap \langle 19 \rangle = \phi$. A contradiction.

Case 8i: Let 6.42), 6.49), 6.63), 6.74) hold.

Then $56 \in \langle 19 \rangle$ and therefore

6.95) $\langle 17 \rangle = \{34, 46, 69, 39, 28, 58, 2T, 5T\}$

6.96) $\langle 19 \rangle = \{56, 35, 37, 67, 2T, 4T, 28, 48\}$

implying $|\langle 17 \rangle \cap \langle 19 \rangle| = 3$. A contradiction.

Case 8j: Let 6.45), 6.55), 6.64), 6.72) hold.

Then $56 \in \langle 19 \rangle$ and therefore 6.90) and 6.96) hold implying $|\langle 17 \rangle \cap \langle 19 \rangle| = 3$. A contradiction.

Case 8k: Let 6.40), 6.47), 6.65), 6.69) hold.

We now have $45 \in \langle 1T \rangle$ $45 \in \langle 28 \rangle$

$56 \in \langle 19 \rangle$ $56 \in \langle 27 \rangle$

$36 \in \langle 18 \rangle$ $36 \in \langle 2T \rangle$

and it is immediate that

6.97) $\langle 17 \rangle = \{34, 3T, 48, 8T, 25, 26, 59, 69\}$

6.98) $\langle 18 \rangle = \{24, 25, 4T, 5T, 36, 39, 67, 79\}$

6.99) $\langle 19 \rangle = \{23, 24, 37, 47, 56, 58, 6T, 8T\}$

6.100) $\langle 1T \rangle = \{23, 26, 38, 68, 45, 49, 57, 79\}$.

By the symmetry of the two 4 cycles which constitute $\langle 12 \rangle$ and by the result of Case 8 we have

Proposition 6.6:

a) $78 \in \langle 14 \rangle$, $78 \in \langle 26 \rangle$ or

b) $78 \in \langle 16 \rangle$, $78 \in \langle 24 \rangle$ or

c) $78 \in \langle 13 \rangle$, $78 \in \langle 25 \rangle$ or

d) $78 \in \langle 15 \rangle$, $78 \in \langle 23 \rangle$.

Case 8k1: Let Case 8k hold and let Proposition 6.6a hold. Then

6.101) $39, 3T, 59, 5T, 46 \in \langle\overline{78}\rangle$

6.102) $47, 4T, 67, 6T, 35 \in \langle\overline{89}\rangle$

6.103) $37, 38, 57, 58, 46 \in \langle\overline{9T}\rangle$

6.104) $48, 49, 68, 69, 35 \in \langle\overline{7T}\rangle$

and therefore

6.105) $\langle 14\rangle = \{78, 29, 69, 6T, 2T, 35, 38, 57\}$

and since we have 6.98) $\langle 14\rangle \cap \langle 18\rangle = \phi$. A contradiction. (Also from 6.100) $|\langle 14\rangle \cap \langle 1T\rangle| = 2$.)

Case 8k2: Let Case 8k hold and let Proposition 6.6b hold. Then we still have 6.101) to 6.104) giving

6.106) $\langle 16\rangle = \{78, 29, 49, 2T, 4T, 35, 37, 58\}$

and therefore $\langle 16\rangle \cap \langle 17\rangle = \phi$. A contradiction. ($|\langle 16\rangle \cap \langle 19\rangle| = 2$)

Case 8k3: Let Case 8k hold and let Proposition 6.6c hold and suppose $89 \in \langle 14\rangle$ $89 \in \langle 26\rangle$. Then

6.107) $49, 4T, 69, 6T, 35 \in \langle\overline{78}\rangle$

6.108) $37, 3T, 57, 5T, 46 \in \langle\overline{89}\rangle$

6.109) $47, 48, 67, 68, 35 \in \langle\overline{9T}\rangle$

6.110) $38, 39, 58, 59, 46 \in \langle\overline{7T}\rangle$.

Therefore

6.111) $\langle 13\rangle = \{78, 59, 29, 5T, 2T, 68, 46, 47\}$

6.112) $\langle 15\rangle = \{9T, 27, 37, 38, 28, 46, 69, 4T\}$

6.113) $\langle 14\rangle = \{89, 27, 67, 2T, 6T, 35, 39, 58\}$

and since 6.99) holds we have $|\langle 14\rangle \cap \langle 19\rangle| = 2$. A contradiction.

Case 8k4: Let Case 8k hold and let Proposition 6.6c hold and suppose $89 \in \langle 16\rangle$ $89 \in \langle 24\rangle$. Then 6.107) to 6.110) still hold and

6.114) $\langle 16\rangle = \{89, 27, 47, 2T, 4T, 35, 39, 58\}$

which together with 6.99) gives $|\langle 16\rangle \cap \langle 19\rangle| = 2$. A contradiction.

Case 8k5: Let Case 8k hold and let Proposition 6.6d hold and suppose $89 \in \langle 16\rangle$ $89 \in \langle 24\rangle$. Then 6.107) to 6.110) still hold and

6.115) $\langle 13\rangle = \{9T, 27, 57, 28, 58, 46, 4T, 69\}$

6.116) $\langle 15\rangle = \{78, 29, 39, 2T, 3T, 46, 47, 68\}$

6.117) $\langle 16\rangle = \{89, 35, 38, 59, 27, 47, 2T, 4T\}$.

Therefore $|\langle 15\rangle \cap \langle 16\rangle| = |\langle 13\rangle \cap \langle 16\rangle| = 2$. A contradiction.

Case 8k6: Let Case 8k hold and let Proposition 6.6d hold and suppose $89 \in \langle 14\rangle$ $89 \in \langle 26\rangle$. Then 6.107) to 6.110) still hold and 6.115) holds, 6.116) holds.

6.118) $\langle 14\rangle = \{89, 35, 38, 59, 27, 67, 2T, 6T\}$

6.119) $\langle 16\rangle = \{7T, 28, 29, 48, 49, 35, 5T, 37\}$.

We now have the sets $\langle 1x\rangle$ $x = 2, \ldots, T$ given by

6.120) $\langle 12\rangle = \{34, 45, 56, 36, 78, 89, 9T, 7T\}$

6.115), 6.118), 6.116), 6.119), 6.97), 6.98), 6.99), 6.100). The number of edges in the graph is $\frac{56 \cdot 10}{2} = 280$ [7]. Of these 260 are now accounted for as follows:

1) 10 given by $(0, x)$ $x = 1, \ldots, T$
2) 90 given by (x, ab) $x = 1, \ldots, T$ $a, b = 1, \ldots, T$ $a \neq b$ iff $x = a$ or b
3) 40 given by the sets $\langle\overline{ab}\rangle$ with respect to $\langle 12\rangle$
4) 72 given by the sets $\langle 1x\rangle$ $x = 2, \ldots, T$
5) 48 given by the sets $\langle 2x\rangle$ $x = 1, 3, \ldots, T$ (those not counted in 4)

All nodes have valency 10 except the 20 nodes contained in the sets $\langle\overline{ab}\rangle$ with respect to $\langle 12\rangle$. These nodes have valency 8 and the 20 interconnections between them give the additional 20 edges required. These connections can be made in one and only one way as shown by the submatrix of A in Figure 6.5.

For fixed i $i = 2, \ldots, T$ the sets $\langle ij\rangle$ $j = 1, \ldots, T$ $j \neq i$ are listed below.

A - sets of the form $\langle 2j\rangle$

$\langle 21\rangle = \langle 12\rangle$ is given by 6.120).

	35	37	38	39	3T	46	47	48	49	4T	57	58	59	5T	67	68	69	6T	79	8T
35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
37	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
38	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
39	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
3T	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
47	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
48	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
49	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
4T	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
57	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
58	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
59	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
5T	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
67	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
68	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
69	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
6T	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
79	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8T	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 6.5

6.121) $\langle 23 \rangle = \{19, 59, 1T, 5T, 46, 48, 67, 78\}$

6.122) $\langle 24 \rangle = \{18, 19, 68, 69, 35, 3T, 57, 7T\}$

6.123) $\langle 25 \rangle = \{17, 18, 37, 38, 46, 49, 6T, 9T\}$

6.124) $\langle 26 \rangle = \{17, 1T, 47, 4T, 35, 39, 58, 89\}$

6.125) $\langle 27 \rangle = \{13, 14, 39, 49, 56, 5T, 68, 8T\}$

6.126) $\langle 28 \rangle = \{13, 16, 3T, 6T, 45, 47, 59, 79\}$

6.127) $\langle 29 \rangle = \{15, 16, 57, 67, 34, 38, 4T, 8T\}$

6.128) $\langle 2T \rangle = \{14, 15, 48, 58, 36, 37, 69, 79\}$

B - sets of the form $\langle 3j \rangle$

$\langle 31 \rangle = \langle 13 \rangle$ is given by 6.115).

$\langle 32 \rangle = \langle 23 \rangle$ is given by 6.121).

6.129) $\langle 34 \rangle = \{12, 17, 29, 79, 58, 5T, 68, 6T\}$

6.130) $\langle 35 \rangle = \{14, 16, 24, 26, 78, 79, 8T, 9T\}$

6.131) $\langle 36 \rangle = \{12, 18, 2T, 8T, 47, 49, 57, 59\}$

6.132) $\langle 37 \rangle = \{16, 19, 68, 89, 25, 2T, 45, 4T\}$

6.133) $\langle 38 \rangle = \{14, 1T, 47, 7T, 25, 29, 56, 69\}$

6.134) $\langle 39 \rangle = \{15, 18, 45, 48, 26, 27, 6T, 7T\}$

6.135) $\langle 3T \rangle = \{15, 17, 56, 67, 24, 28, 49, 89\}$

C - sets of the form $\langle 4j \rangle$

$\langle 41 \rangle = \langle 14 \rangle$ is given by 6.118).

$\langle 42 \rangle = \langle 24 \rangle$ is given by 6.122).

$\langle 43 \rangle = \langle 34 \rangle$ is given by 6.129).

6.136) $\langle 45 \rangle = \{12, 1T, 28, 8T, 37, 39, 67, 69\}$

6.137) $\langle 46 \rangle = \{13, 15, 23, 25, 79, 89, 7T, 8T\}$

6.138) $\langle 47 \rangle = \{15, 19, 5T, 9T, 26, 28, 36, 38\}$

6.139) $\langle 48 \rangle = \{16, 17, 56, 57, 23, 2T, 39, 9T\}$

6.140) $\langle 49 \rangle = \{16, 1T, 36, 3T, 25, 27, 58, 78\}$

6.141) $\langle 4T \rangle = \{13, 18, 37, 78, 26, 29, 56, 59\}$

D - sets of the form $\langle 5j \rangle$

We write down only those sets where $6 \leq j \leq T$

6.142) $\langle 56 \rangle = \{12, 19, 27, 79, 38, 3T, 48, 4T\}$

6.143) $\langle 57 \rangle = \{13, 1T, 36, 6T, 24, 29, 48, 89\}$

6.144) $\langle 58 \rangle = \{13, 19, 34, 49, 26, 2T, 67, 7T\}$

6.145) $\langle 59 \rangle = \{14, 17, 4T, 7T, 23, 28, 36, 68\}$

6.146) $\langle 5T \rangle = \{16, 18, 69, 89, 23, 27, 34, 47\}$

E - sets of the form $\langle 6j \rangle$ $7 \le j \le T$

6.147) $\langle 67 \rangle = \{14, 18, 45, 58, 23, 29, 3T, 9T\}$

6.148) $\langle 68 \rangle = \{15, 1T, 59, 9T, 24, 27, 34, 37\}$

6.149) $\langle 69 \rangle = \{13, 17, 38, 78, 24, 2T, 45, 5T\}$

6.150) $\langle 6T \rangle = \{14, 19, 34, 39, 25, 28, 57, 78\}$

F - sets of the form $\langle 7j \rangle$ $8 \le j \le T$

6.151) $\langle 78 \rangle = \{12, 15, 23, 35, 49, 4T, 69, 6T\}$

6.152) $\langle 79 \rangle = \{18, 1T, 28, 2T, 34, 35, 46, 56\}$

6.153) $\langle 7T \rangle = \{12, 16, 24, 46, 38, 39, 58, 59\}$

G - sets of the form $\langle 8j \rangle$ $j = 9, T$

6.154) $\langle 89 \rangle = \{12, 14, 26, 46, 37, 3T, 57, 5T\}$

6.155) $\langle 8T \rangle = \{17, 19, 27, 29, 35, 36, 45, 46\}$

H - sets of the form $\langle 9j \rangle$ $j = T$

6.156) $\langle 9T \rangle = \{12, 13, 25, 35, 47, 48, 67, 68\}$

I - sets of the form $\langle Tj \rangle$

All listed previously. Q. E. D.

C. Sims, in the course of his study of primitive groups has also discovered $r(2,2,10,56)$ (private correspondence). This author has shown the two representations to be the same. Sim's representation is the following. Call the distinguished node $*$. The nodes of ℓ_1 are the 10 Sylow 3-subgroups of A_6. The nodes of ℓ_2 are the 45 involutions of A_6. A node of ℓ_2 is connected to a node of ℓ_1 if the node of ℓ_2 normalizes the node of ℓ_1. The reentering arcs are defined by the following rule. Let $x, y \in \ell_2$. Then (x, y) if, as involutions, the product xy has order 4. We show as Figure 6.6 the nodes as just defined and the corresponding node from Case 8k 6 above. One notes that applying the permutation (12)(79)(8T) to the graph of 8k 6 would again give a graph isomorphic to the original. (In the Sim's representation interchange the numbers 5 and 6.)

ℓ_0	*	0
ℓ_1	(125)(346)	1
	(126)(345)	2
	(156)(234)	3
	(124)(356)	4
	(134)(256)	5
	(123)(456)	6
	(145)(236)	7
	(135)(246)	8
	(146)(235)	9
	(136)(245)	T
ℓ_2	(12)(34)	12
	(15)(34)	13
	(12)(36)	14
	(25)(34)	15
	(12)(46)	16
	(15)(36)	17
	(15)(46)	18
	(25)(46)	19
	(25)(36)	1T
	(16)(34)	23
	(12)(35)	24
	(26)(34)	25
	(12)(45)	26
	(26)(45)	27
	(26)(35)	28
	(16)(35)	29
	(16)(45)	2T
	(24)(56)	34
	(34)(56)	35
	(23)(56)	36
	(15)(23)	37
	(15)(24)	38
	(16)(23)	39
	(16)(24)	3T
	(14)(56)	45
	(12)(56)	46
	(14)(36)	47
	(24)(35)	48
	(14)(35)	49
	(24)(36)	4T
	(13)(56)	56
	(14)(26)	57
	(13)(26)	58
	(14)(25)	59
	(13)(25)	5T
	(23)(45)	67
	(13)(46)	68
	(23)(46)	69
	(13)(45)	6T
	(15)(26)	78
	(14)(23)	79
	(36)(45)	7T
	(35)(46)	89
	(13)(24)	8T
	(16)(25)	9T

Figure 6.6

7. In this section we establish some possible parameters for $r(2, 3 \leq t \leq 10, v, n)$ graphs by use of 5.2), 5.3), 5.5) and 5.7). We will in fact have shown that for parameters not listed in this section $r(2, 3 \leq t \leq 10, v, n)$ cannot exist. We note, in particular, that for $t = 3, 5$ or 8 there is only one undecided case. Examination of 5.7) shows that for other values of t there is more than one undecided case. For $t = 3$ we go through the proof and for $4 \leq t \leq 10$ we list the results. In sections 8 and 9, using other techniques, we will settle some of the undecided cases.

Theorem 7.1: $r(2, 3, 21, 162)$ is the only possible $r(2, 3, v, n)$ graph.

Proof: We have by substitution in 5.2), 5.3) 5.5) and 5.7)

7.1) $A^2 + 3A + (3-v)I = 3J$

7.2) $v^2 + 2v + 2 = 3n$

7.3) $a^2 + 3 = 4v$

7.4) $96ax = a^5 + 3a^4 + 14a^3 + 18a^2 + 33a + 27.$

The values of a for which integral x are possible are the factors of 27 namely 1, 3, 9, 27. From 7.3) if $a = 1$ then $v = 1$ which does not give a graph. If $a = 3$ $(x = 4)$ $v = 3$ and $n = 6$ which we know is $b(2, 3, 3, 6)$. When $a = 9$ $(x = 105)$ we have $v = 21$ and $n = 162$. If $a = 27$ then $x = \frac{18788}{3}$ which is not integral and therefore $a = 27$ cannot be used. Q. E. D.

Theorem 7.2: For $4 \leq t \leq 10$ necessary parameters for the existence of $r(2, 4 \leq t \leq 10, v, n)$ graphs are:

a) For $t = 4$

7.5) $v = a^2 \qquad a > 1$

7.6) $v^2 + 3v + 4 = 4n$

b) For $t = 5$

7.7) $r(2, 5, 25, 650)$

is the only possible $r(2, 5, v, n)$ graph.

c) For $t = 7$

7.8) $v = a^2 - 3 \qquad a \geq 3 \qquad a \not\equiv 0, 4, 8 \pmod{12}$

7.9) $v^2 + 5v + 6 = 6n$

d) For $t = 7$

7.10) $r(2, 7, 105, 1666)$

7.11) $r(2, 7, 301, 12202)$

7.12) $r(2, 7, 2646, 1002457)$

are the only possible $r(2, 7, v, n)$ graphs.

e) For $t = 8$

7.13) r(2, 8, 136, 2432)

is the only possible r(2, 8, v, n) graph.

f) For t = 9

7.14) r(2, 9, 45, 266)

7.15) r(2, 9, 99, 1178)

7.16) r(2, 9, 171, 3402)

7.17) r(2, 9, 495, 27666)

7.18) r(2, 9, 981, 107802)

7.19) r(2, 9, 2745, 839666)

7.20) r(2, 9, 8919, 8846658)

7.21) r(2, 9, 24795, 68134018)

are the only possible r(2, 9, v, n) graphs.

g) For t = 10

7.22) r(2, 10, 21, 64)

7.23) r(2, 10, 85, 800)

7.24) r(2, 10, 385, 15170)

7.25) $r(2, 10, 885, 74720)$

7.26) $r(2, 10, 3585, 1288450)$

are the only possible $r(2, 10, v, n)$ graphs. Q. E. D.

We observe in passing that when $t = 11$ there are 10 possible r graphs and when $t = 12$ there are 6 possibilities.

8. Balanced incomplete block designs

In this section we examine the relationship that exists between the nodes of ℓ_d and balanced incomplete block designs (BIBD's). A BIBD can be thought of as a collection of b sets (blocks) with k elements (varieties) in each set, the varieties to be picked from a set with v elements, each variety to appear in exactly r blocks, and each pair of varieties to appear together in exactly λ blocks.
v, b, k, r, λ are called the parameters of the BIBD. It is well known, [8], that the parameters of a BIBD satisfy

8.1) $vr = bk$

8.2) $r(k-1) = \lambda(v-1)$.

Hanani [3] proved that 8.1) and 8.2) are sufficient for $k = 3$ or 4 and any λ, and also for $k = 5$ and $\lambda = 4$. We view $x \in \ell_d$ as a block of a BIBD, whose varieties are the t nodes i_j of Lemma 3.5. By Corollary 2.1 and the lemmas of section 3 we have

Lemma 8.1: The nodes $x \in \ell_d$ are the blocks of a BIBD with parameters b, k, v, r and λ where

1) $v = v$ the valency of g

2) $b = |\ell_d| = v(v-1)^{d-1} t^{-1}$

3) $k = t$

4) $r = |\ell_d(i)| = (v-1)^{d-1}$

5) $\lambda = (t-1)(v-1)^{d-2}$ Q. E. D.

Corollary 8.1: If $d = 2$, the nodes of ℓ_2 are the blocks of a BIBD with parameters given by

8.3) $v, \ r = v-1, \ k = t, \ \lambda = k-1, \ b = v(v-1)t^{-1}$ Q. E. D.

If the nodes of some given ℓ_d give rise to a BIBD then the BIBD will be called an associated design of ℓ_d. Many associated designs of ℓ_d can exist, for a given ℓ_d. If a design is an associated design of ℓ_d we will write BIBD (ℓ_d).

In sections 6 and 7 we considered the case for $d = 2$. If $t = 2$ it is clear that associated designs of ℓ_2 exist. When $t = 3$, 4 or 5, BIBD (ℓ_2) exist by theorems of Hanani. We now proceed to show that neither the parameters suggested by the polynomial technique, nor the existence of a BIBD (ℓ_d) is sufficient for the existence of g. (In fact we will be showing, at the same time, the insufficiency of Theorems 3.2 and 3.3.)

Theorem 8.1: $r(2, 4, 9, 28) \notin R$

Proof: (Note that the eigenvalue argument suggests [see 7.5) and 7.6)] that the parameters 2, 4, 9, 28 may possibly be those of an r graph.) The BIBD parameters are $v = 9$, $b = 18$, $k = 4$, $r = 8$ and $\lambda = 3$. A BIBD (ℓ_2) is given below. The actual proof will not depend on any particular associated design of ℓ_2.

(1, 2, 3, 4)
(1, 2, 6, 9)
(1, 2, 7, 8)
(1, 3, 5, 6)
(1, 3, 7, 9)
(1, 4, 5, 7)
(1, 4, 6, 8)
(1, 5, 8, 9)
(2, 3, 5, 9)
(2, 3, 6, 8)
(2, 4, 5, 8)
(2, 4, 7, 9)
(2, 5, 6, 7)
(3, 4, 6, 7)
(3, 4, 8, 9)
(3, 5, 7, 8)
(4, 5, 6, 9)
(6, 7, 8, 9)

Let (abcd) be an arbitrary node of ℓ_2. We have, from Corollary 2.1

8.4) $|\ell_2| = 18$

and from Lemma 3.3

8.5) $|\ell_2(a) \cup \ell_2(b)| = 13$

and from Lemma 3.2

8.6) $|\ell_2(c)| = 8.$

We know, again from Lemma 3.3, that

8.7) $|\ell_2(c) \cap \ell_2(i)| = 3$ for all $i \neq c$

and in particular 8.7) holds if $i = a$ or b. One of the nodes in the intersection $\ell_2(c) \cap \ell_2(a)$ is (abcd) and thus there are exactly 2 other nodes in this intersection, say α and β. Similarly (abcd) $\in$ $\ell_2(c) \cap \ell_2(b)$ and thus there are exactly 2 other nodes in this intersection, (possibly α and β), call them δ and γ. In any case there are at most 5 nodes in $\ell_2(c)$ that have the letters a or b in their identification, namely (abcd), α, β, δ, and γ. Thus, there are at least, by 8. 6), 3 nodes of $\ell_2(c)$ which do not have a or b in their identification. This fact, together with 8. 5), gives

8.8) $\quad |[\ell_2(a) \cup \ell_2(b)] \cup \ell_2(c)| \geq 16.$

Therefore from 8. 4) the number of nodes in ℓ_2 of the form (efgh) where $e, f, g, h \neq a, b, c$ is 0, 1 or 2.

Now from Lemma 3. 6 we have

8.9) $\quad |\langle abcd \rangle| = 5$

and from Lemma 3. 4 $(efgh) \in \langle abcd \rangle$ implies $e, f, g, h \neq a, b, c, d$. By the statement following 8. 8) there are at most 2 such nodes and thus 8. 9) cannot be satisfied. Q. E. D.

Given the parameters $v = 16$, $r = 15$, $k = 4$, $\lambda = 3$, and $b = 60$ we know a BIBD exists. In fact, in [3] Hanani gives a construction technique. Using this method one gets a design with the following property. Distinguish a variety 1 and let the other varieties be b, c, d, e, . . . , p. Then the blocks (lbcd), (lefg), (lhij), (lkℓm), (lnop) are each repeated 3 times. Using such a design it is trivial to show that $r(2, 4, 16, 77) \notin R$ even though the eigenvalue argument (7. 5) and 7. 6)) suggests their use as parameters. In fact, we will show in Theorem 8. 2 that a much weaker condition on the BIBD will insure the nonexistence of $r(2, 4, 16, 77)$. It is a conjecture of this author, that the weaker condition, as stated in Theorem 8.2, is in fact a necessary condition for the existence of BIBD's with the given parameters.

Theorem 8.2: If (abcd) and (abce) $\in \ell_2$ then $r(2, 4, 16, 77) \notin R$.

Proof: As in Theorem 8.1 we can construct a design but it will not affect the proof.

Using the techniques of Theorem 8.1, we have

8.10) $|\langle abci\rangle| = 12$ (in particular for $i = d, e$)

8.11) $|\ell_2(a) \cup \ell_2(b)| = 27$

8.12) $|\ell_2| = 60$

8.13) $|\ell_2(i)| = 15$

8.14) $|[(\ell_2(a) \cup \ell_2(b)) \cup \ell_2(c)] \cup \ell_2(i)| \geq 46$ for $i = d$ or e.

From 8.14) and 8.12) we have that the number of nodes that can belong to $\langle abcd\rangle$ is no more than 14. Suppose that the 12 nodes required for $\langle abcd\rangle$ by 8.10) have been selected from the 14 available and now let us examine the set $\langle abce\rangle$. Again, from 8.14) and 8.12) there are at most 14 nodes that can belong to $\langle abce\rangle$.

The number of paths, of length 2, from (abcd) to (abce), via the hierarchy nodes, is 3 (via a, b, and c). Thus there must be another path of length 2 between these nodes and thus there is at most one node of ℓ_2 adjacent to (abcd) and (abce). (None, if $d = e$.)

This node, say α, is certainly in $\langle abcd \rangle$. There are, in $\langle abcd \rangle$, exactly 4 nodes of the form (exyz). The other 7 (or 8) nodes of $\langle abcd \rangle$ do not have e in their identification and thus were counted in the 14 nodes that were potentially members of $\langle abce \rangle$. However, since they are nodes in $\langle abcd \rangle$ and we already have α as the only node in both $\langle abcd \rangle$ and $\langle abce \rangle$ we see that there are only 7 (or 6) other possible nodes for $\langle abce \rangle$, which implies $|\langle abce \rangle| \leq 8$ contradicting 8.10). Q. E. D.

Prior to the statement of Theorem 8.2 this author conjectured the existence of blocks (abcd) and (abce) in every BIBD with parameters $v = 16$, $r = 15$, $k = 4$, $\lambda = 3$ and $b = 60$. An additional conjecture is that this condition need not be in the hypothesis of Theorem 8.2, namely, that the theorem is true without the requirement (abcd) and (abce) $\epsilon\ \ell_2$.

If $v = 25$, $r = 24$, $k = 4$, $\lambda = 3$ and $b = 150$ (see Figure 8.1), Hanani's construction criteria again forces the distinguishing of a variety 1 and the repeating of each block containing this 1, 3 times, as above. That is, if the other varieties are a, b, c, d, . . . , x, then the blocks which are repeated 3 times are (labc), (ldef), (lghi), (ljkℓ), (lmno), (lpqr), (lstu), (lvwx). We will now designate any BIBD with sets of repeated blocks as shown, as a design of Hanani type. We then have

<u>Theorem 8.3</u>: A necessary condition for the possible existence of $r(2, 4, 25, 176)$ is that the nodes of ℓ_2 not be blocks of a Hanani type

design.

Proof: Once again note 7.5) and 7.6) suggest the possible existence of this graph.

To prove the theorem one need only observe that the 3 distinct nodes each labeled (1abc) can have no common adjacencies (other than in the hierarchy) since they are connected to each other by 4 paths of length 2 via the hierarchy nodes 1, a, b, and c. Thus, for instance, the 3 distinct sets $\langle 1234 \rangle$ $\langle 1234 \rangle$ $\langle 1234 \rangle$ contain 63 distinct nodes. There are (see Figure 8.1) 87 nodes with 1, 2, 3, or 4 in their identifications and thus there is no choice in picking the 63 nodes. Similarly, if we examine $\langle 1567 \rangle$ $\langle 1567 \rangle$ $\langle 1567 \rangle$ we see that there is no choice, but using the methods of Theorems 8.1 and 8.2 we see that some of the 63 possible nodes are in a $\langle 1234 \rangle$ set and cannot be used for a $\langle 1567 \rangle$ set. Q. E. D.

One might note that the full strength of the hypothesis was not used. The hypothesis could have been weakened to include all BIBD's which have (1abc) and (1def) repeated 3 times each as blocks.

Theorem 8.4: $r(2, 6, 22, 100)$ and $r(2, 10, 21, 64) \notin R$.

Proof: See Theorem 8.1. Q. E. D.

Note again that the eigenvalue argument does not eliminate $r(2, 6, 22, 100)$ (see 7.8) and 7.9)) or $r(2, 10, 21, 64)$ (see 7.22)) as graphs. Some BIBD(ℓ_2) have been constructed for $r(2, 3, 21, 162)$. It is not known if $r(2, 3, 21, 162)$ exists. In fact, it is not known

1, 2, 3, 4
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1, 8, 9, 10
1, 11, 12, 13
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1, 11, 12, 13
1, 14, 15, 16
1, 14, 15, 16
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1, 20, 21, 22
1, 20, 21, 22
1, 20, 21, 22
1, 23, 24, 25
1, 23, 24, 25
1, 23, 24, 25

2, 5, 9, 13
2, 5, 15, 19
2, 5, 21, 25
2, 6, 8, 12
2, 6, 14, 18
2, 6, 20, 24
2, 7, 10, 11
2, 7, 16, 17
2, 7, 22, 23
2, 8, 15, 22
2, 8, 18, 25
2, 9, 14, 21
2, 9, 17, 24
2, 10, 16, 20
2, 10, 19, 23
2, 11, 15 25
2, 11, 18, 22
2, 12, 14, 24
2, 12, 17, 21
2, 13, 16, 23
2, 13, 19, 20

3, 5, 8, 11
3, 5, 14, 17
3, 5, 20, 23
3, 6, 10, 13
3, 6, 16, 19
3, 6, 22, 25
3, 7, 9, 12
3, 7, 15, 18
3, 7, 21, 24
3, 8, 14, 20
3, 8, 17, 23
3, 9, 16, 22
3, 9, 19, 25
3, 10, 15, 21
3, 10, 18, 24
3, 11, 14, 23
3, 11, 17, 20
3, 12, 16, 25
3, 12, 19, 22
3, 13, 15, 24
3, 13, 18, 21

4, 5, 10, 12
4, 5, 16, 18
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4, 6, 9, 11
4, 6, 15, 17
4, 6, 21, 23
4, 7, 8, 13
4, 7, 14, 19
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4, 10, 14, 22
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10, 11, 22, 24
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10, 13, 14, 19
10, 13, 20, 25
14, 17, 21, 25
14, 18, 20, 24
14, 19, 22, 23
15, 17, 20, 23
15, 18, 22, 25
15, 19, 21, 24
16, 17, 22, 24
16, 18, 21, 23
16, 19, 20, 25

Figure 8.1

whether any of the BIBD (ℓ_2) that have been constructed would be the ones that lead to a graph, if the graph did exist. The known (to this author) BIBD (ℓ_2)'s that have been constructed are all composed of 2 distinct (and in one case disjoint) Steiner triple systems [8]. A conjecture is that if $r(2, 3, 21, 162)$ exists then the BIBD (ℓ_2) for the graph will not be decomposable into 2 separate Steiner triple systems.

9. In this section we consider the $g(d \geq 3, t, v, n)$ graphs and in particular the $r(d \geq 3, t, v, n)$ graphs. We exhibit the unique graph $r(3, 2, 4, 35)$. We have

9.1) $$F_3(A) + tG_2(A) = tJ.$$

Therefore

9.2) $$A^3 + tA^2 + (t - 2v + 1)A + (t - tv)I = tJ$$

and thus

9.3) $$v^3 + (t-2)v^2 + v + t = tn.$$

If $t = 2$ using the techniques of previous sections, we get

Lemma 9.1: Necessary conditions for the existence of $g(3, 2, v, n)$ are

9.4) $$v^3 + v + 2 = 2n$$

9.5) $$8v = a^2 + 7 \qquad a \equiv 1, 3, 5, 7 \ (8)$$

Theorem 9.1: $r(3, 2, 4, 35)$ exists and is unique. We define the hierarchy in the following manner:

The distinguished node is 0.

The nodes of ℓ_1 are named $1, 2, 3, 4$.

The nodes of $\ell_2(i)$ are named $i1, i2, i3 \qquad i = 1, \ldots, 4$.

The nodes of ℓ_3 are named as follows. We observe that each element of ℓ_3 is connected to 2 elements of ℓ_2. If the 2 elements of ℓ_2 are ia, jb $i, j = 1, \ldots, 4$ $a, b = 1, \ldots, 3$ then the element of ℓ_3 is given by kia mjb $k, m = 1, \ldots, 3$. We display the hierarchy, the reentering arc subgraph (note that this subgraph is bipartite and consists of 3 disjoint circuits of length the girth) and the matrices A, A^2 and A^3 in Figures 9.1, 9.2, 9.3 and 9.4 respectively. That the adjacencies between ℓ_2 and ℓ_3 are correct is given by the lemmas of section 3. That they are unique is clear, if we note that, given the adjacencies of the nodes $11, 21 \in \ell_2$ in ℓ_3, if 22 was adjacent to any node of ℓ_3 of the form 112abc or 113xyz then there would be 3 paths of length 3 from 2 to 11. That the adjacencies using the reentering arcs are the bipartite ones shown is the result of the lemmas of section 3. Q. E. D.

From 9.3) if $t = 3$ we have that a necessary condition for the existence of $g(3, 3, v, n)$ is

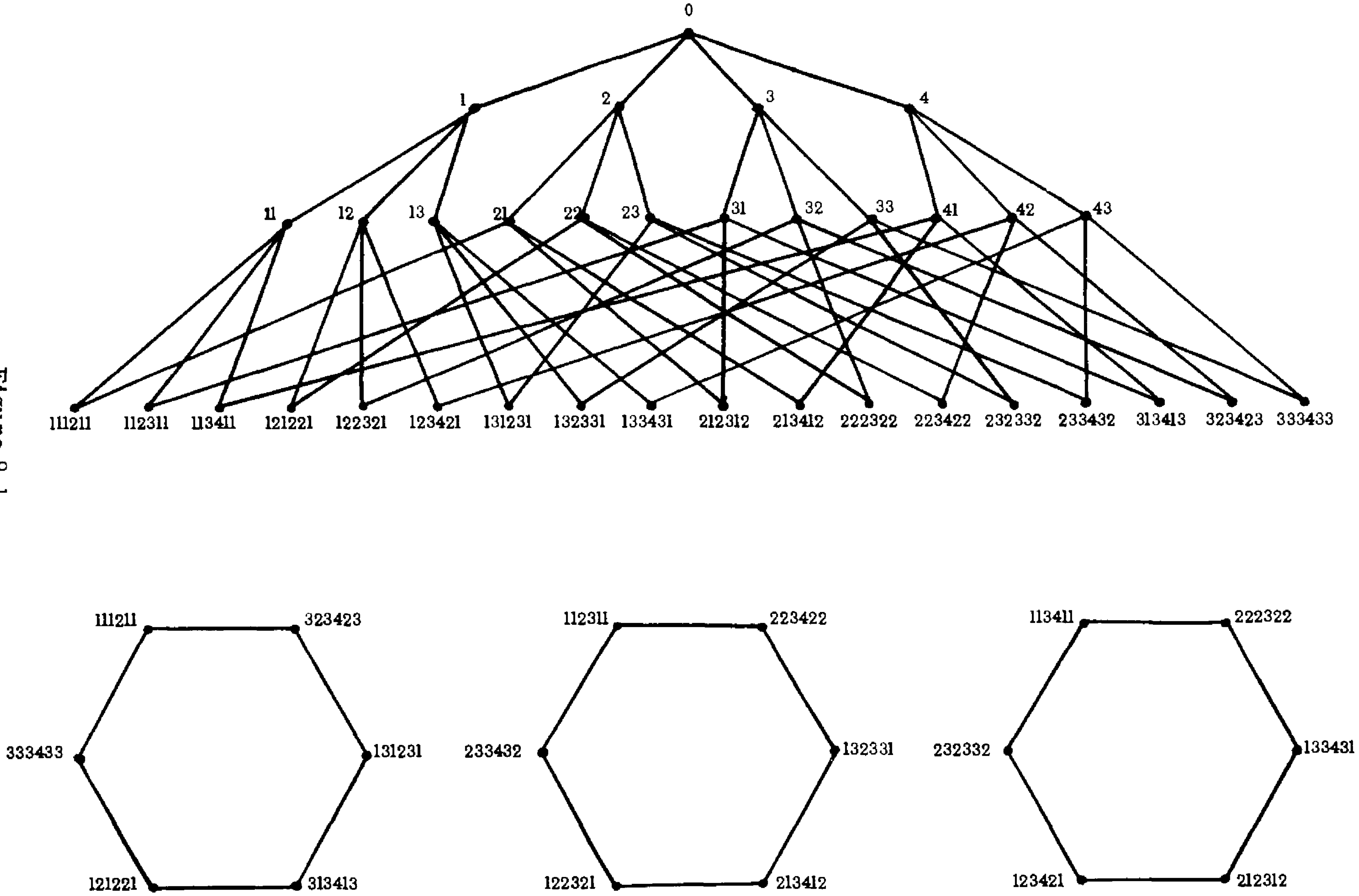
9.6) $$v^3 + v^2 + v + 3 = 3n$$

which implies $v \equiv 0, 1 \ (3)$. From Theorem 3.2 we deduce

9.7) $$r(3, 3, 4, 29) \notin R$$

(even though the parameters satisfy 9.6)).

For $d > 3$ we know [1], [2] that B graphs exist for $d = 4$ and $d = 6$ and for no other values of d, provided $t > 2$. Of course for $t = 2$ we know $b(d, 2, 2, 2d)$ always exists. For $d = 4$ we have,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313413
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Figure 9.1

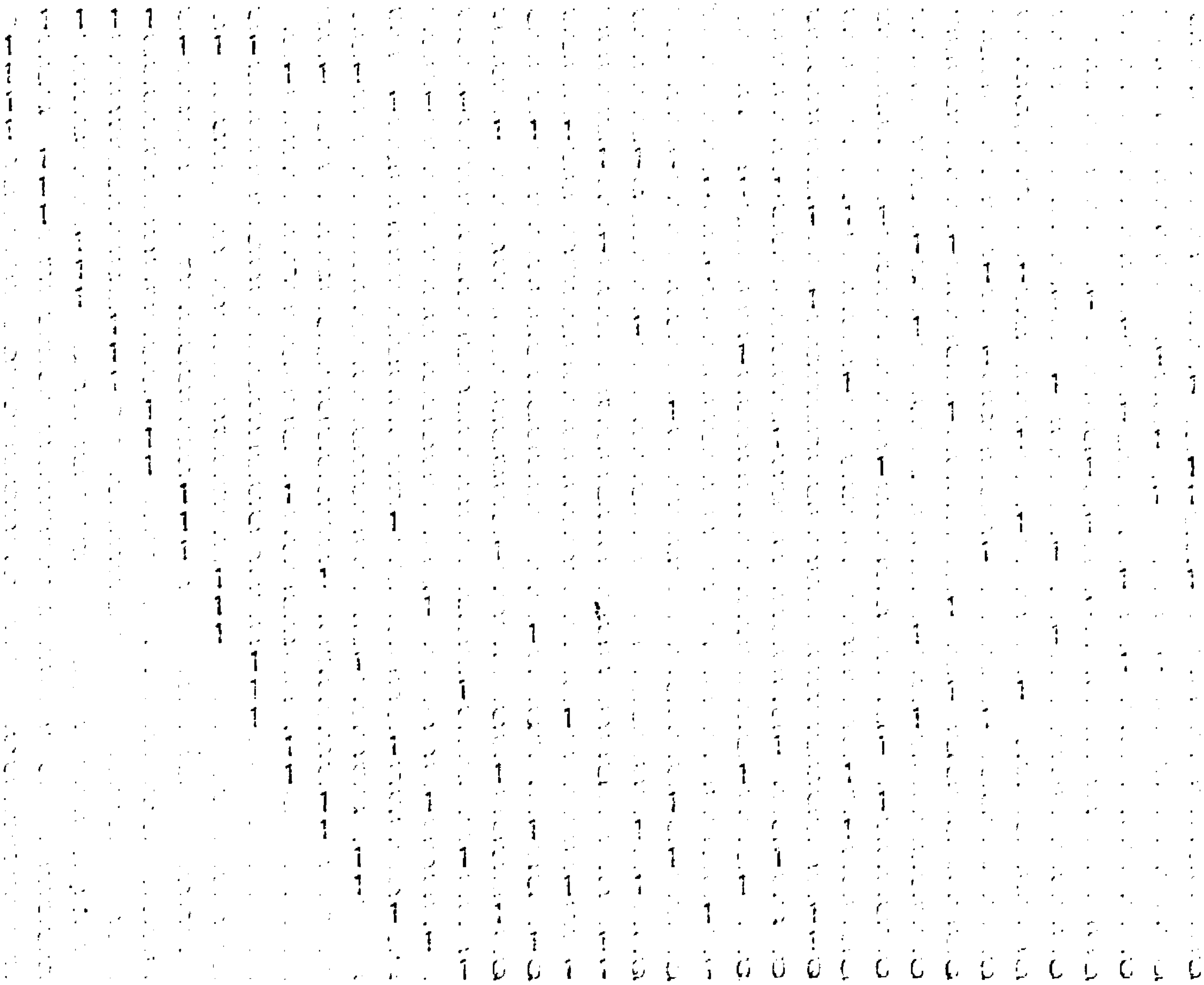

Figure 9.2

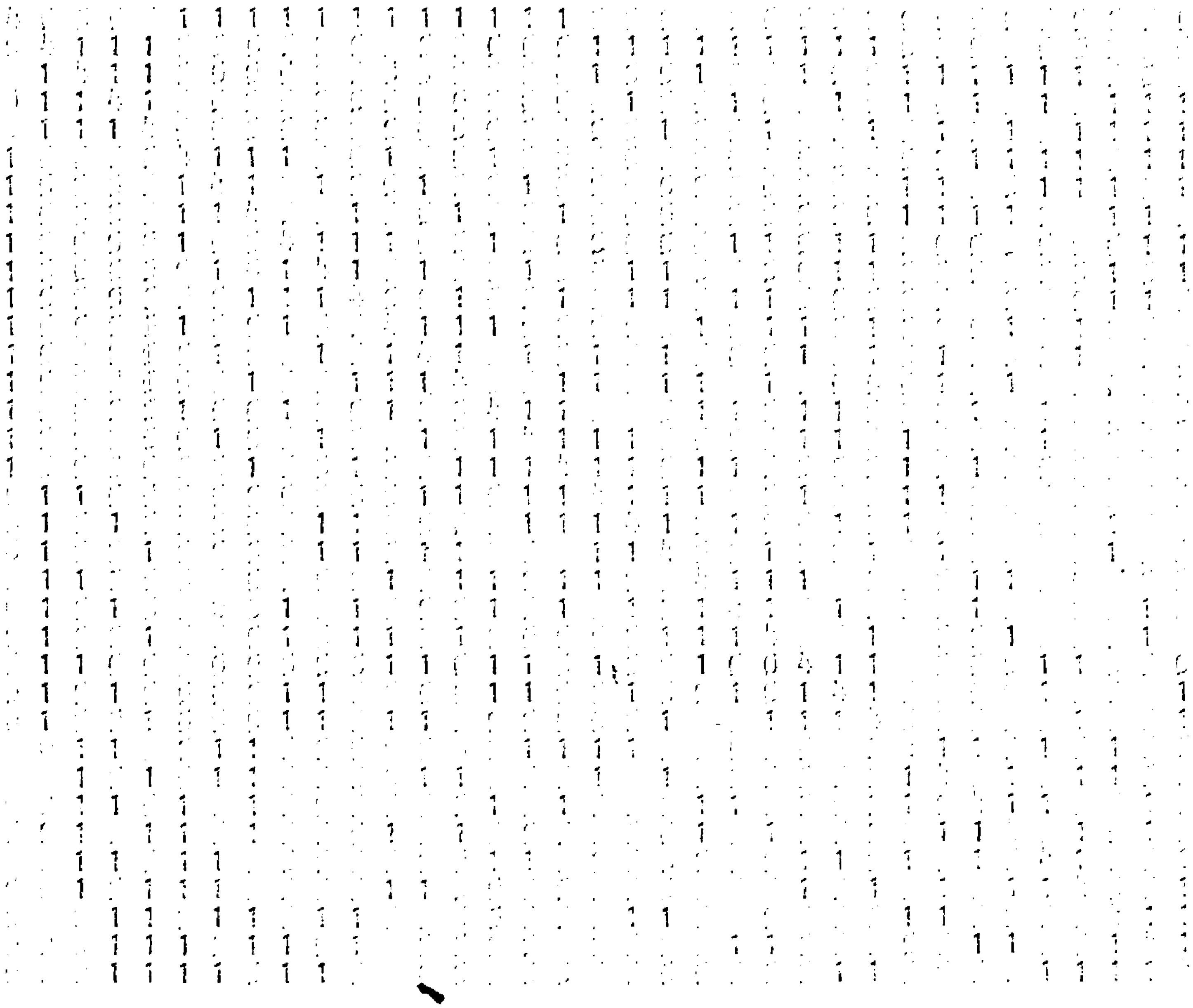

Figure 9.3

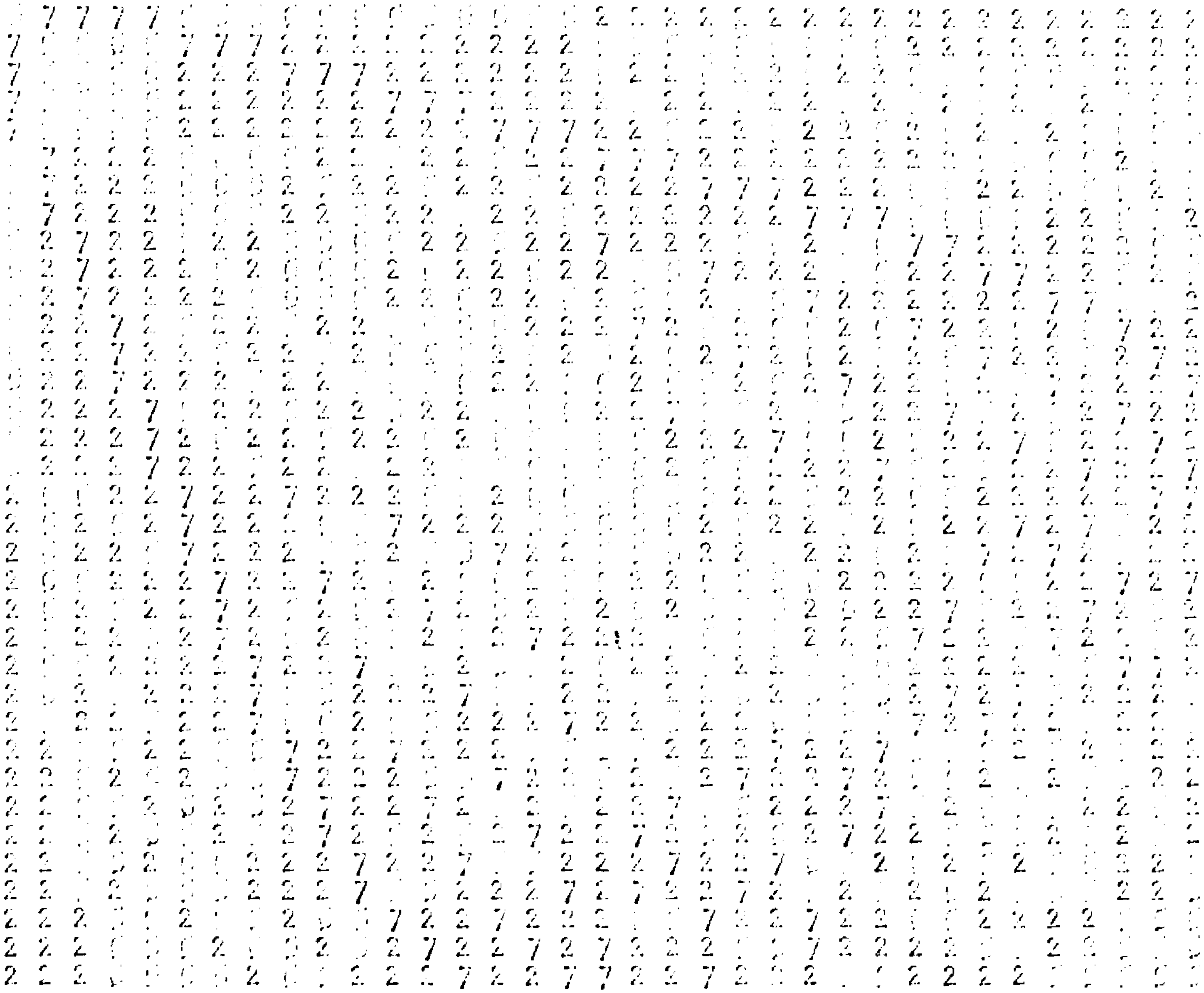

Figure 9.4

by Theorem 3.1, that a necessary condition for the existence of $r(4, t, v, n)$ is

9.8) $$v^4 + (t-3)v^3 + (3-t)v^2 + (t-1)v + t = tn.$$

Thus if $t = 2$ a necessary condition for the existence of $r(4, 2, v, n)$ is

9.9) $$v^4 - v^3 + v^2 + v + 2 = 2n.$$

However, by Theorem 3.2 we have

9.10) $$r(4, 2, 3, 34) \notin R.$$

Similarly a necessary condition for $r(4, 3, v, n)$ to exist is

9.11) $$v^4 + 2v + 3 = 3n$$

and thus $v \equiv 0, 1 \ (3)$.

Similar necessary conditions can be written for any d and t. Further study of the polynomials of the graphs is indicated as being a way to impose stiffer necessary conditions.

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AUTOBIOGRAPHICAL STATEMENT

Allan Gewirtz was born in New York City on May 30, 1931. He received his elementary education in the New York City public school system, graduating from Brooklyn Technical High School in 1948. After working and serving in the United States Air Force, he entered Brooklyn College in 1954 and received a B.S. cum laude with honors in mathematics in 1959 and an M. A. in 1964. He is a member of PI MU EPSILON and SIGMA XI.

He was a Standards Engineer for Western Electric from 1956 to 1959. He was Electronics Project Leader for Vacuum Electronics from 1959 to 1964 where he was engaged in the study and design of low current detection devices and mass spectrometry. Since that time, as an engineering consultant, he has invented various instruments in the vacuum measurement field which are marketed for him by Logatorr Company of Roseland, New Jersey. From 1959 to 1965 he taught mathematics in the School of General Studies at Brooklyn College. Since 1964 he has been Assistant Professor of Mathematics at Pace College in New York City and will chair Pace's Math Department at Pace College, Westchester, commencing in September 1967.