Description of homology and cohomology theories of connected algebras as quotients of ideals of free connected algebras

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Abstract

DESCRIPTION OF HOMOLOGY AND COHOMOLOGY THEORIES OF CONNECTED ALGEBRAS AS QUOTIENTS OF IDEALS OF FREE CONNECTED ALGEBRAS

by Avraham Goldstein

Advisor: Professor Martin Bendersky

In this work we give develop technics of calculations with the ideals of the free connected algebras. We use those technics to calculate the homology and the cohomology vector spaces and the kernels and the co-kernels of the product of the cohomology spaces of connected algebras as quotients of ideals of the "overlying" free connected algebras.

This permits us to prove some results about connected algebras with certain cohomology structure. We also demonstrate our methods at calculating the cohomology structure of Mod 2 Steenrod algebra and some of its sub-algebras.

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1 Introduction

In this work we introduce definitions and results about connected algebras. Particularly, we give an explicit formula for the homology groups, $H_n(A)$, of a connected algebra, A, in terms of its generators and relations. The main reference for this material is [14]. A connected algebra, A, over a field F, is an F-algebra, graded by non-negative integers, s.t. it's 0^{th} grading, A_0, is isomorphic to F. This isomorphism is given as a part of the structure of connected algebra, hence one says that $A_0 = F$ [see Definition 2.1].

The augmentation ideal, I, of a connected algebra A is defined as $\bigcap_{i=1}^{\infty} A_i$.

A set $G \subset I$ is called a generating set of A if every element of A can be represented as a sum of coefficients from F with products of elements of G, [i.e. $a = f_1 g_{1,1} \cdot \ldots \cdot g_{1,k_1} + \ldots + f_m g_{m,1} \cdot \ldots \cdot g_{m,k_m}$] G is called minimal generating set if none of its proper subsets is a generating set.

For a connected algebra A and a fixed generating set G of A, we denote by A' a free algebra on a set G, i.e. the elements of A' are sums of coefficients from F with products of elements of G, where each such sum represents a *different* element in A'. Clearly, A' is a connected algebra and it's augmentation ideal is denoted by I'.

There is a natural projection $proj : A' \rightarrow A$. The kernel of this projection is called the relations ideal and is denoted by $Rel \subset A'$.

We define [Definition 4.7 and Lemma 4.19] $\{Z_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ -ideals of A':

$$Z'_n := \bigcap_{\sum_i(\alpha_i+2\beta_i)=n} (I'^{\alpha_1} \cdot Rel^{\beta_1} \cdot I'^{\alpha_2} \cdot Rel^{\beta_2} \cdot \ldots \cdot I'^{\alpha_{n+1}} \cdot Rel^{\beta_{n+1}}) \subset A'$$

$$B'_n := I' \cdot Z'_n + Z'_n \cdot I' + [\sum_{i=1}^{n} (Z'_{i-1} \cdot Rel \cdot Z'_{n-i})] \text{ – Will be proved in Corollary 4.15 to be } \subset Z'_n$$

<u>The main new result of this work is:</u>

[Theorem 4.18] $H_n(A) = Z'_n / B'_n$

If G is a minimal generating set, then $Rel \subset I' \cdot I'$. This fact simplifies the formulae for Z'_n and B'_n. In this case one gets, by directly plugging into the above formulae:

$$Z'_0 = A' \quad , \quad B'_0 = I'$$

$$Z'_1 = I' \quad , \quad B'_1 = I' \cdot I'$$

Hence $H_0(A) = A'/I' = F$ and $H_1(A) = I'/(I' \cdot I') = I/(I \cdot I)$. Those are classical results in the theory of connected algebras [14]. Also

$$Z'_2 = Rel$$

$$B'_2 = (I' \cdot Rel) + (Rel \cdot I')$$

Hence $H_2(A) = Rel/[(I' \cdot Rel) + (Rel \cdot I')]$, which is a restatement of Wall's result from [14] in our terminology.

All other formulae i.e. formula for $H_2(A)$, using a general [i.e. not necessarily minimal] generating set G, and formulae for higher $H_n(A)$, using a minimal or a general generating set G, which are given in Example 4.22 are new.

In Section 5 we introduce Yoneda cup-product

$$Ext_A^{s,t}(M,M') \otimes Ext_A^{s',t'}(M',M'') \to Ext_A^{s+s',t+t'}(M,M'')$$

A new result, developed in Section 5, is Definition 5.4 of parings

$$\rho_{i,j} : Z'_{i+j} \to Z'_i \otimes Z'_j$$

such that [due to Lemmas 5.2 and 5.3] $\rho_{i,j}(B'_{i+j}) \subset [(B'_i \otimes Z'_j) + (Z'_i \otimes B'_j)]$.

Thus we define in Lemma 5.5, by passing to quotients, parings

$$\nu_{i,j} : H'_{i+j}(A) \to H'_i(A) \otimes H'_j(A)$$

In **Theorem 5.6** we establish duality between $TR\circ\nu_{j,i}$ and Yoneda cup-product in the case $M = M' = M'' = F$ and $s = i, s' = j$ [TR is unoriented transpose, $TR(x \otimes y) := y \otimes x$]. Using this duality, we produce, in **Theorem 5.8**, explicit formulae for kernels and co-kernels of Yoneda cup-product maps in terms of generators and relations of A.

In Lemmas 4.20 and 4.21 we find all the connected algebras A, for which there is certain commutativity relations between the ideals I' and Rel of A'. In Theorem 5.9 we combine these results with Theorem 5.8 to find all algebras, in which certain Yoneda cup-products have zero kernel.

In order to achieve our results on the homology of a connected algebra A, we introduce in Definition 2.19 and construct in Section 3 the minimal resolution. We then translate this construction, in Section 4, into the language of ideals of A'.

In Section 6 we define Hopf algebras and discuss their properties. A Hopf algebra A is a connected F-algebra, equipped with an algebra homomorphism $\nu_A : A \rightarrow A \otimes A$, called co-product, such, that for all $a \in A$ the projection of $\nu_A(a)$ onto $A_0 \otimes A$ is $1 \otimes a$ and the projection onto $A \otimes A_0$ is $a \otimes 1$. A Hopf algebra A is called co-associative if $[Id_A \otimes \nu_A] \circ \nu_A = [\nu_A \otimes Id_A] \circ \nu_A$. For a co-associative Hopf algebra A, we construct ν_{H_n}, the cup-co-products on homology of A. This construction uses ν_A. It is known from [2] that

$$(\nu_{H_n} = \bigoplus_{i+j=n} \nu_{i,j}) : H_n(A) \rightarrow \bigoplus_{i+j=n} H_i(A) \otimes H_j(A)$$

In other words the *construction* of ν_{H_n} uses ν_A, but ν_{H_n} does not depend on ν_A

Section 6 also introduces Liulevicius's [from [6]] SQ^n, $n = 0, 1, \ldots$ operations on co-homology of co-commutative [see Definition 6.3] Hopf algebras over a field with characteristics 2.

For a comprehensive treatment of this material see [11]

We use our results in Section 7, to calculate the H_3 of the *mod* 2 Steenrod algebra, A, and of some of it's finite sub-algebras. We produce some relations on the products of the elements of $Ext^1(A)$ and $Ext^1(A_{[1]})$ and $Ext^1(A_{[2]})$.

Those calculations are then compared with the literature. [1], [10].

2 Definitions and Notations

We assume some fixed field F throughout this article. We recall the definitions of Z-tensor product, F-vector space, F-linear map and F-tensor product. For H_1, H_2 - two commutative groups, the Z-tensor product, $H_1 \otimes_Z H_2$, is defined as a free commutative group on all pairs (v, x), modulo all the relations $([v+w], x) - (v, x) - (w, x)$ and $(v, [x+y]) - (v, x) - (v, y)$, for all $v, w \in H_1,\ x, y \in H_2$. F-vector space is a commutative group V, equipped with a map $h : F \otimes_Z V \rightarrow V$, s.t. $h([f_1 \otimes_Z v_1] + [f_2 \otimes_Z v_2]) = h(f_1 \otimes_Z v_1) + h(f_2 \otimes_Z v_2)$ and $h(f_1 \otimes_Z h(f_2 \otimes_Z v)) = h([f_1 f_2] \otimes_Z v)$ for all $f_1, f_2 \in F,\ v_1, v_2 \in V$. Throughout this work we write fv for $h(f \otimes_Z v)$ and we call f - the coefficient of v in fv. Let V and W be two F-vector spaces. The map $g : V \rightarrow W$ is called F-linear if $g(f_1 v_1 + f_2 v_2) = f_1 g(v_1) + f_2 g(v_2)$. The F-tensor product, $V \otimes_F W$, is defined as a commutative group $V \otimes_Z W$, modulo all the relations $([fv] \otimes_Z w) - (v \otimes_Z [fw])$, equipped with a map $h : F \otimes_Z (V \otimes_F W) \rightarrow V \otimes_F W$, given by $h(f \otimes_Z [v \otimes_F w]) := (fv) \otimes_F w$. All the tensor products in this work are F-tensor products and we omit subscript from them. Since for any F-vector space V, $V \otimes F$ and $F \otimes V$ are naturally isomorphic to V, by isomorphism $v \leftrightarrow v \otimes 1 \leftrightarrow 1 \otimes v$, we regard them as the same object.

Definition 2.1 *A graded, connected F-algebra A is a disjoin union of F-vector spaces A_n, $n \geq 0$, such that $A_0 = F$, equipped with the product map $\mu_A = \bigcup_{i,j\geq 0} \mu_{A_i,A_j} : A_i \otimes A_j \rightarrow A_{i+j}$, where each μ_{A_i,A_j} is F-linear and μ_{A_i,A_0} and μ_{A_0,A_j} are all identity maps, such, that*

$$[\mu_{A_{i+j},A_k} \circ (\mu_{A_i,A_j} \otimes Id_{A_k}) = \mu_{A_i,A_{j+k}} \circ (Id_{A_i} \otimes \mu_{A_j,A_k})] : A_i \otimes A_j \otimes A_k \rightarrow A_{i+j+k}$$

The product μ_A is regarded as a map from $A \otimes A$ into A. All algebras in this work are assumed to be connected F-algebras.

Where possible, we will omit the subscript from the μ_A and we also write $a \cdot b$ for $\mu(a \otimes b)$.

Definition 2.2 *Let A and B be two F-algebras. Then the algebra homomorphism $f : B \rightarrow A$ is union of F-linear maps $f_i : B_i \rightarrow A_i$, such that $f_0 = Id_F$ and:*

$$[f \circ \mu_B = \mu_A \circ (f \otimes f)] \ : B \otimes B \rightarrow A$$

Let V be a disjoint union of F-vector spaces V_i, $i \geq 0$ and A be an F-algebra.

Definition 2.3 *If one fixes a map $f = \bigcup_{i,j\geq 0} f_{A_i,V_j} : A_i \otimes V_j \rightarrow V_{i+j}$, where each f_{A_i,V_j} is F-linear and f_{A_0,V_j} are all identity maps, such that $f \circ (\mu \otimes Id_V) = f \circ (Id_A \otimes f) : A \otimes A \otimes V \rightarrow V$, we say that $\{V, f\}$ is a left A-module and call f a left action of A on $\{V, f\}$. The action f is regarded as a map from $A \otimes V$ into V.*

Definition 2.4 *If one fixes a map $f = \bigcup_{i,j\geq 0} f_{V_i,A_j} : V_i \otimes A_j \to V_{i+j}$, where each f_{V_i,A_j} is F-linear and f_{V_i,A_0} are all identity maps, such that $f \circ (Id_V \otimes \mu) = f \circ (f \otimes Id_A) : V \otimes A \otimes A \to V$, we say that $\{V, f\}$ is a right A-module and call f a right action of A on $\{V, f\}$. The action f is regarded as a map from $V \otimes A$ into V.*

In both cases of modules and actions we will omit, where possible, the mention of f and write just V for $\{V, f\}$, $a \cdot v$ for $f(a \otimes v)$ and $v \cdot a$ for $f(v \otimes a)$. The field F is also regarded as left and as right A-module with $F_0 = F$ and $F_{n>0} = 0$. The actions are given by $f \cdot a = a \cdot f = fa$ if $a \in A_0 = F$ and $f \cdot a = a \cdot f = 0$ otherwise.

If V is a graded F-vector space with additional structure, then $[V]^F$ denotes the underlying graded F-vector space.

Definition 2.5 *Let V be a graded F-vector space. Then, $\mathcal{F}^A(V)$, the free left A-module, generated by V, is a vector space $[A]^F \otimes V$ with A-action defined by $a_1 \cdot (a_2 \otimes v) := (a_1 \cdot a_2) \otimes v$.*

Let A be a connected F-algebra.

Definition 2.6 *Let M be a left A module and $S \subset M$ be any subset of M. Then $S^F \subset [M]^F$ is an F-vector space $\{x \in [M]^F \mid x = \sum_i f_i \cdot s_i\}$ and $S^A \subset M$ is a left A-module $\{x \in M \mid x = \sum_i a_i \cdot s_i\}$. Here $f_i \in F$ and $a_i \in A$.*

Definition 2.7 *$B \subset A$ is called a left ideal of A if for any $a_1, a_2 \in A$ and $b_1, b_2 \in B,\ a_1b_1 + a_2b_2 \in B$.*

Definition 2.8 *$B \subset A$ is called a right ideal of A if for any $a_1, a_2 \in A$ and $b_1, b_2 \in B,\ b_1a_1 + b_2a_2 \in B$.*

Definition 2.9 *$I_A := \bigcup_{i>0} A_i$ is called an augmentation ideal of A. I_A is both right and left ideal of A. The subscript will be omitted where possible.*

Definition 2.10 *The set $G \subset I_A$ is called a generating set of A if every element $x \in I_A$ is expressible [not necessarily uniquely] as $x = \sum_i f_i(g_{i,1} \cdot \ldots \cdot g_{i,t_i})$, where $f_* \in F$ and $g_{*,*} \in G$. The generating set G is called minimal, if none of it's proper subsets are generating sets.*

Definition 2.11 *Let G be a graded set. The free algebra on G, Fr_G, is defined as F in degree 0 and in degree n as F-vector space $Fr_{G,n}$, spanned by all expressions $\sum_i (g_{i,1} \cdot \ldots \cdot g_{i,t_i})$, where $g_{*,*} \in G$ and $deg(g_{i,1}) + \ldots + deg(g_{i,t_i}) = n$. $\mu_{Fr_G} : Fr_G \otimes Fr_G \to Fr_G$ is an additive extension of: [$f_1, f_2 \in F$, $g_{*,*} \in G$]*

$$\mu_{Fr_G}(f_1(g_{1,1} \cdot \ldots \cdot g_{1,t_1}) \otimes f_2(g_{2,1} \cdot \ldots \cdot g_{2,t_2})) := f_1f_2(g_{1,1} \cdot \ldots \cdot g_{1,t_1} \cdot g_{2,1} \cdot \ldots \cdot g_{2,t_2})$$

Definition 2.12 *Let A be a connected F-algebra and $G_A \subset I_A$ be a generating set of A. If G_A is clear from the context, we will write A' for Fr_{G_A}. I' will denote the augmentation ideal of A'. For each $g_t \in G$, g'_t will denote the corresponding element in A' and $G' := \{g'_t\} \subset I'$. The map $g'_t \to g_t$ extends to an algebra homomorphism $proj : A' \to A$, which is onto. It's kernel $Rel \subset A'$ is called relation ideal [relative to G_A]. Rel is both right and left ideal.*

Definition 2.13 *Let A be an F-algebra and X and Y be two left [right] A-modules. Then F-linear map $f : X \to Y$ is called left [right] A-module homomorphism if $f(a \cdot x) = a \cdot f(x)$ [$f(x \cdot a) = f(x) \cdot a$] for all $a \in A$ and $x \in X$.*

Any bi-graded F-vector space $X_{i,j}$ can be regarded as a diagonally graded by $X_n = \bigoplus_{i+j=n} X_{i,j}$. Hence, for two graded F-vector spaces, V and W, the product $V \otimes W$ is regarded as a graded F-vector space.

Definition 2.14 *Let A be an F-algebra, X be a right A-module and Y be a left A-module. Then F-vector space $X \otimes_A Y$ is defined as a quotient of F-vector space $[X]^F \otimes [Y]^F$ by the F-vector sub-space, spanned by all $(x \cdot a) \otimes y - x \otimes (a \cdot y)$, $a \in A, x \in X, y \in Y$.*

Definition 2.15 *Let A be an F-algebra. The left [right] A-module P is called projective if for any left [right] A-modules X and Y, and for any left [right] A-module homomorphisms f from X onto Y and g from P into Y, exists [at least one] left [right] A-module homomorphism $h : P \to X$, such that $(f \circ h = g) : P \to Y$.*

It is a classical result in homological algebra that, for any graded F-vector space V, $\mathcal{F}^A(V)$ is projective left A-module. See [5] and [15].

Definition 2.16 *Let A be an F-algebra and M be left [right] A-module. Then the long exact sequence $\ldots \xrightarrow{\partial_n} P_n \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \xrightarrow{\partial_{-1}} M \to 0$, where all P_n are projective left [right] A-modules and all ∂_n are left [right] A-module homomorphisms, is called a projective resolution of M.[Hence the sequence $\ldots \xrightarrow{\partial_n} P_n \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \to 0$ has $[M]^F$ as its 0^{th} homology F-vector space]*

Definition 2.17 *Let A be an F-algebra. Then the functor $\mathcal{T}$ from the category of left A-modules and left A-module homomorphisms to the category of F-vector spaces and F-linear maps is defined as follows: $\mathcal{T}(M) := F \otimes_A M = [M]^F/[I \cdot M]^F$ and $\mathcal{T}(f) := Id_F \otimes_A f$.*

F-vector spaces $\mathcal{T}(F)$ and F are naturally isomorphic, by $1 \otimes_A 1 \leftrightarrow 1$.

Definition 2.18 *Let A be an F-algebra. Then the n^{th} homology F-vector space, $H_n(A)$, of*

$$\cdots \xrightarrow{\delta_2:=\mathcal{T}(\partial_2)} \mathcal{T}(P_2) \xrightarrow{\delta_1:=\mathcal{T}(\partial_1)} \mathcal{T}(P_1) \xrightarrow{\delta_0:=\mathcal{T}(\partial_0)} \mathcal{T}(P_0) \xrightarrow{\delta_{-1}:=0} 0$$

where $\{P_i, \partial_i\}$ is [any] projective resolution of F, i.e. $Ker(\delta_{n-1})/Image(\delta_n)$, is called the n^{th} homology F-vector space of A and it's dual F-vector space, $H^n(A) := Hom(H_n(A), F)$, is called the n^{th} cohomology F-vector space of A.

One also uses $Tor_n(A)$ for $H_n(A)$ and $Ext^n(A)$ for $H^n(A)$. Another definition of $Ext^n(A)$ is the $Ker(\delta_n^*)/Image(\delta_{n-1}^*)$ for

$$0 \xrightarrow{\delta_{-1}^*:=0} Hom_A(P_0, F) \xrightarrow{\delta_0^*:=-\circ\partial_0} Hom_A(P_1, F) \xrightarrow{\delta_1^*:=-\circ\partial_1} Hom_A(P_2, F) \xrightarrow{\delta_2^*:=-\circ\partial_2} \cdots$$

Since each P_n is a graded A-module, the F-vector spaces $H_n(A)$ and $H^n(A)$ are bi-graded. We will use the notation $P_{n,k}$, $H_{n,k}$, $H^{n,k}$ for the sub-object of n^{th} object (n^{th} homology degree) of inner degree k [and similarly for $Tor_{n,k}(A)$ and $Ext^{n,k}(A)$].

Definition 2.19 *Let A be an F-algebra and M any left A-module. Then the projective resolution of M is called minimal if $\partial_n(P_{n+1}) \subset I \cdot P_n$ for $n \geq 0$.*

The other description of minimal resolution is that $\mathcal{T}(\partial_n) = 0$ for all $n \geq 0$. Hence, if $\{P_i, \partial_i\}$ is a minimal resolution of F, then

$$H_n(A) = \mathcal{T}(P_n) = [P_n]^F / [I \cdot P_n]^F$$

Definition 2.20 *Let A be an F-algebra and M any left A-module. Then the subset $S \subset M$ is called a generating set of M if every $x \in M$ is expressible (not necessarily uniquely) as $x = \sum_i a_i \cdot s_i$, where $a_* \in A, s_* \in S$, i.e. $M = S^A$. A generating set S is called minimal in none of it's proper subsets is a generating set.*

We observe that if S is a minimal generating set of M then

$$[M]^F = S^F \oplus [I \cdot M]^F$$

Also, in this case, S is a basis of S^F.

We also observe that every [minimal] generating set of A, as in Definition 2.10, is [minimal] generating set of I, regarded as left A-module, as in Definition 2.20

3 C.T.C. Wall's Construction and it's extension.

We will produce here [extended] Wall's construction from [14], of minimal resolution of F with respect to an F-algebra A. Out methods are easily extended for a general left A-module M.

We start by taking $P_0 := A$, with A acting on itself by multiplication from the left, and $\partial_{-1}(a) := a$ if $a \in A_0 = F$ and $\partial_{-1}(a) := 0$ otherwise. Clearly, $Ker(\partial_{-1}) = I \subset A = P_0$. Let now G be some minimal generating set of left A-module I. Then $[I]^F = G^F \oplus [I \cdot I]^F$. Take $P_1 := \mathcal{F}^A(G^F)$ and $\partial_0(a \otimes (\sum_i f_i \cdot g_i)) := \sum_i (f_i\, a \cdot g_i)$. In general, if one already constructed P_n and ∂_{n-1}, one chooses S_n as some minimal generating set of left A-module $Ker(\partial_{n-1})$, hence $[Ker(\partial_{n-1})]^F = S_n^F \oplus [I \cdot Ker(\partial_{n-1})]^F$, and takes $P_{n+1} := \mathcal{F}^A(S_n^F)$ and $\partial_n(a \otimes (\sum_i f_i \cdot x_i)) := \sum_i (f_i\, a \cdot x_i) \in Ker(\partial_{n-1}) \subset P_n$, where $x_i \in S_n$. One proves $\partial_n(P_{n+1}) \subset I \cdot P_n$ by induction on n. It is the same, as to prove, that $S_n \subset I \cdot P_n$. We already have it for $n = 0$. Suppose $S_m \subset I \cdot P_m$ for all $m < n$. Choose $x = \sum_i a_i \otimes s_i \in P_n$, s.t. $\partial_{n-1}(x) = \sum_i a_i \cdot s_i = 0$. Here $a_i \in A, s_i \in S_{n-1}$, $s_i = s_j \Leftrightarrow i = j$. If $x \notin I \cdot P_n$, i.e. some $a_k \in A_0 = F$, we can solve $s_k = \sum_{i \neq k} (\frac{a_i}{a_k}) \cdot s_i$, which contradicts the minimality of S_{n-1}. So $Ker(\partial_{n-1}) \subset I \cdot P_n$, hence $S_n \subset I \cdot P_n$. One gets that

$$H_{n+1}(A) = S_n^F$$

In that notation S_0 is just G, a minimal generating set of A and any $x \in Ker(\partial_0)$ is some $a_1 \otimes g_1 + ... + a_k \otimes g_k$, $a_* \in A, g_* \in G$, s.t. $a_1 \cdot g_1 + ... + a_k \cdot g_k = 0$. So S_1 is just a minimal generating set of "relations" on G, i.e. such set of "relations", that any "relation" can be represented (not uniquely) as $a_1 \cdot s_1 + ... + a_t \cdot s_t$, $a_* \in A, s_* \in S_1$, and no proper subset of S_1 has that property. By the analogy, S_2 is a minimal generating set of "relations" on S_1, etc. This description of $H_2(A)$ as "minimal relations" is due to Wall. See [14].

We put the word relation in quotations, because the "relations" in Wall's sense are a left A-module and a different object from the relations of Definition 2.12, where the relations constitute an ideal in some free algebra "over" A. Hence the non uniqueness of the representation of "relations" by the elements of a minimal generating set S_1.

Definition 3.1 *The left A-module map* $d_{n,i} : A^{\otimes(n+1)} \longrightarrow A^{\otimes n}$ *defined by additively extending*

$$d_{n,i}(a_1 \otimes ... \otimes a_i \otimes a_{i+1} \otimes ... \otimes a_n \otimes a_{n+1}) := a_1 \otimes ... \otimes (a_i \cdot a_{i+1}) \otimes ... \otimes a_n \otimes a_{n+1}$$

is called i^{th} *face map. Here* $1 \leq i \leq n$. *All* $d_{n,i}$ *are clearly surjective (onto).*

By abuse of notation we will write also $d_{n,i} : I^{\otimes(n+1)} \longrightarrow I^{\otimes n}$. In terms of this construction $S_n \subset I^{\otimes(n+1)}$ and for $n > 0$,

$$S_n \subset Ker(d_{n,1}) \cap (I \otimes S^F_{n-1}) \subset \bigcap_{i=1}^{n} Ker(d_{n,i}) \subset I^{\otimes(n+1)} \tag{3.1}$$

We define $S_{-1} := 1$, $X_{-1} := J_{-1} := A$, $S_0 := G$, $X_0 := J_0 := I$ and left A-modules

$$X_n := [Ker(d_{n,1}) \cap (I \otimes S^F_{n-1})] \subset I^{\otimes n+1} \quad , \quad J_n := [\bigcap_{i=1}^{n} Ker(d_{n,i})] \subset I^{\otimes n+1} \tag{3.2}$$

Notice that, $X_n = S^A_n = Ker(\partial_{n-1}) \subset P_n$.

Notice also that $H_0(A) = [P_0]^F/[I \cdot P_0]^F = [A]^F/[I \cdot A]^F = [A]^F/[I]^F = F$.

One has that $\underline{H_{n+1}(A) = S^F_n = [X_n]^F/[I \cdot X_n]^F}$ for all $n \geq -1$.

$$\begin{tikzcd}
J_5 \subset I \otimes J_4 \subset I^{\otimes 6} & J_4 \subset I \otimes J_3 \subset I^{\otimes 5} & J_3 \subset I \otimes J_2 \subset I^{\otimes 4} & J_2 \subset I \otimes J_1 \subset I^{\otimes 3} & J_1 \subset I \otimes I & J_0 = I & \\
\cup & \cup & \cup & \cup & \cup & \cup & \\
Ker(\partial_4) = X_5 = S^A_5 \arrow[d] & Ker(\partial_3) = X_4 = S^A_4 \arrow[d] & Ker(\partial_2) = X_3 = S^A_3 \arrow[d] & Ker(\partial_1) = X_2 = S^A_2 \arrow[d] & Ker(\partial_0) = X_1 = S^A_1 \arrow[d] & Ker(\partial_{-1}) = X_0 = S^A_0 \arrow[d] & \\
A \otimes S^F_4 = P_5 \arrow[r, "\partial_4 = d_{5,1}"] \arrow[ur] & A \otimes S^F_3 = P_4 \arrow[r, "\partial_3 = d_{4,1}"] \arrow[ur] & A \otimes S^F_2 = P_3 \arrow[r, "\partial_2 = d_{3,1}"] \arrow[ur] & A \otimes S^F_1 = P_2 \arrow[r, "\partial_1 = d_{2,1}"] \arrow[ur] & A \otimes S^F_0 = P_1 \arrow[r, "\partial_0 = d_{1,1}"] \arrow[ur] & A = P_0 \arrow[r] \arrow[d, "\partial_{-1}"] & 0 \\
 & & & & & F &
\end{tikzcd}$$

4 Calculations with ideals of A'

For an algebra A fix some generating set G and take $A' = Fr_G(A)$. In this section we shall "lift" the construction, described by (3.1) and (3.2) to $I' \otimes ... \otimes I'$ and then "multiply it out" onto ideals of A'. This will enable us, in Theorem 4.18, to express homology groups of A in terms of ideals of A'. We begin developing some results concerning A'. [Note, $A'^{\otimes 0} := I'^{\otimes 0} := F$]

On page 19 we provide a full page diagram, illustrating our definitions and constructions from this section.

Definition 4.1 *The element $x \in A'$ is called homogeneous, if $x = fg'_1 \cdot ... \cdot g'_t$ or $x = f$, where f is any element of F and g'_* are any elements of G'. If $x = fg'_1 \cdot ... \cdot g'_t$, we say that the weight of x, $w(x)$, is t, and if $x = f$, we say that the weight of x, $w(x)$, is 0. So the weight of a homogeneous element is the number of generators in it.*

Definition 4.2 *The element $x \in A'^{\otimes n} = A' \otimes ... \otimes A'$ is called homogeneous, if $x = x_1 \otimes ... \otimes x_n$, with each factor x_i being homogenous. We say that the weight of x, $w(x) := w(x_1) + ... + w(x_n)$. So the weight of a homogeneous element of $A'^{\otimes n}$ is the total number of generators in it.*

By definition of A', every element $x \in A'$ is a unique sum of homogeneous elements, which are called homogeneous summands of x. Similarly, every element $x \in A'^{\otimes n}$ is a unique sum of homogeneous elements, which are called homogeneous summands of x.

Definition 4.3 *The left A'-module map $d'_{n,i} : A'^{\otimes(n+1)} \longrightarrow A'^{\otimes n}$ defined by additively extending*

$$d'_{n,i}(a'_1 \otimes \ldots \otimes a'_i \otimes a'_{i+1} \otimes \ldots \otimes a'_n \otimes a'_{n+1}) := a'_1 \otimes \ldots \otimes (a'_i \cdot a'_{i+1}) \otimes \ldots \otimes a'_n \otimes a'_{n+1}$$

is called i^{th} face map. Here $1 \leq i \leq n$. Every face map is surjective (onto) and takes a homogeneous element to a homogeneous element of the same weight.

Definition 4.4 *We define F-linear map $\pi_m : [A'^{\otimes n}]^F \to [A'^{\otimes n}]^F$. On each $x \in A'^{\otimes n}$, $\pi_m(x)$ is the sum of all homogeneous summands of x of weight m. We also define F-linear maps $\pi_{<m}(x) := \sum_{i<m} \pi_i(x)$ and $\pi_{\leq m}(x) := \sum_{i \leq m} \pi_i(x)$. We also use $w_{min}(x)$ for the minimal weight of homogeneous summands of x and $\pi_{min}(x) := \pi_{w_{min}(x)}(x)$, i.e. the sum of all homogeneous summands of x of minimal weight. NOTE that π_{min} is NOT an additive map.*

The functions π_m , $\pi_{<m}$ and $\pi_{\leq m}$ provide some kind of "filtration" on the elements of $[A'^{\otimes n}]^F$.

By the abuse of notation we will apply the functions π_m , $\pi_{<m}$ and $\pi_{\leq m}$ to elements of $A'^{\otimes n}$.

It is clear that if $x \in A'^{\otimes n}$ is of degree (grading) d, then $\pi_{\leq d}(x) = x$.

We are now ready to "translate" (3.1) and (3.2) into the (two sided) ideals of A'. Let

$$\Omega'_{-1} := F \quad , \quad X'_{-1} := J'_{-1} := A'$$

$$X'_0 := J'_0 := I'$$

$$J'_n := \bigcap_{i=1}^{n} d'^{-1}_{n,i}(I'^{\otimes(i-1)} \otimes Rel \otimes I'^{\otimes(n-i)})$$

$$\Omega'_n = proj^{-1}(S^F_n) \cap J'_n \quad , \quad X'_n = proj^{-1}(X_n) \cap J'_n$$

where $proj : I' \otimes ... \otimes I' \to I \otimes ... \otimes I$ is the natural projection.

We now want to show that $proj(J'_n) = J_n$ for all $-1 \leq n$. It is clear, that $proj(J'_n) \subset J_n$ and it is clear, that $proj(J'_{-1}) = J_{-1}$ and $proj(J'_0) = J_0$. So, assume that $1 \leq n$.

Let $\{1, i_1, ...\} \subset A$ be some F-basis of $[A]^F$. Then every element $x \in J_n$ is represented as

$$x = \sum_{t_1,...,t_{n+1}} (f_{(i_{t_1},...,i_{t_{n+1}})} i_{t_1} \otimes ... \otimes i_{t_{n+1}})$$

where each $f_{(i_{t_1},...,i_{t_{n+1}})} \in F$ and only finitely many of them are nonzero.

For all $1 \leq k \leq n$, we have that, for all $t_1, ..., t_{k-1}, t_{k+2}, ..., t_{n+1}$,

$$i_{t_1} \otimes ... \otimes i_{t_{k-1}} \otimes (\sum_{t_k, t_{k+1}} (f_{(i_{t_1},...,i_{t_{n+1}})}(i_{t_k} \cdot i_{t_{k+1}}))) \otimes i_{t_{k+2}} \otimes ... \otimes i_{t_{n+1}} = 0$$

$$\sum_{t_k, t_{k+1}} (f(i_{t_1}, ..., i_{t_{n+1}})(i_{t_k} \cdot i_{t_{k+1}})) = 0$$

Fix some $i'_1 \in proj^{-1}(i_1), i'_2 \in proj^{-1}(i_2), ...$ and define

$$x' := \sum_{t_1,...,t_{n+1}} (f_{(i_{t_1},...,i_{t_{n+1}})} i'_{t_1} \otimes ... \otimes i'_{t_{n+1}})$$

We have that

$$proj(\sum_{t_k, t_{k+1}} (f_{(i_{t_1},...,i_{t_{n+1}})}(i'_{t_k} \cdot i'_{t_{k+1}}))) = \sum_{t_k, t_{k+1}} (f(i_{t_1}, ..., i_{t_{n+1}})(i_{t_k} \cdot i_{t_{k+1}})) = 0$$

$$\sum_{t_k, t_{k+1}} (f_{(i_{t_1},...,i_{t_{n+1}})}(i'_{t_k} \cdot i'_{t_{k+1}})) \in Rel$$

Hence $x' \in J'_n$. But $proj(x') = x$. So $proj(J'_n) = J_n$.

Definition 4.5 *The left A'-module homomorphism $pr^n : A'^{\otimes(n+1)} \to A'$ is defined by extending linearly $pr^n(a'_1 \otimes ... \otimes a'_{n+1}) := a'_1 \cdot ... \cdot a'_{n+1}$.*
Or $[pr^n := d'_{1,1} \circ d'_{2,1} \circ ... \circ d'_{n,1}] : A'^{\otimes(n+1)} \to A'$.

Thus for $x \in A'^{\otimes(n+1)}$, the map pr^n just turns every tensor product in x into the multiplication in A'.

Lemma 4.6 *For any F-vector space $B' \subset [A']^F$ one can choose F-basis $\{c'_1, c'_2, ...\}$ of B', s.t. every c'_m has a homogeneous summand, called $sgn(c'_m)$, not appearing, as a homogeneous summand, in any other c'_n.*

Proof: If B' has nonzero elements of grading 0, we take $c'_1 = 1 \in F$.

For any fixed grading d, $d = 1, 2, ...$, $\{\pi_1(b') \mid b' \in B'_d\}$ is some F-vector subspace, $B'_d[1]$, of F-vector space $[G'_d]^F$, spanned by all $g'_{i_1} \in G'_d$.

Choose $b'_{1,1}, ..., b'_{1,k_1} \in B'_d$, s.t. $\pi_1(b'_{1,1}), ..., \pi_1(b'_{1,k_1})$ is a basis of $B'_d[1]$. By simple matrix inverting argument from linear algebra, we can construct $b''_{1,1}, ..., b''_{1,k_1}$, each $b''_{1,j}$ is an F-linear combination of $b'_{1,1}, ..., b'_{1,k_1}$, s.t. each $b''_{1,j}$ has some summand g'_{i_1} appearing only in it, and each $b'_{1,h}$ is F-linear combination of $b''_{1,1}, ..., b''_{1,k_1}$. Furthermore, for every $b' \in B'_d$, we can subtract some unique F-linear combination of $b''_{1,1}, ..., b''_{1,k_1}$ from b', to "annihilate" its $\pi_1(b'_m)$ summands. The result of this subtraction is called $\sigma_1(b')$.

All $\{\pi_2(\sigma_1(b')) \mid b' \in B'_d\}$ spans some F-vector subspace, $B'_d[2]$, of F-vector space $[\sum_{d_1+d_2=d} G'_{d_1} \cdot G'_{d_2}]^F$, spanned by all $g'_{i_1} \cdot g'_{i_2}$, where $g'_{i_1} \in G'_{d_1}$, $g'_{i_2} \in G'_{d_2}$ and $d_1 + d_2 = d$.

Choose $b'_{2,1}, ..., b'_{2,k_2} \in B'_d$, s.t. $\pi_2(\sigma_1(b'_{2,1})), ..., \pi_2(\sigma_1(b'_{2,k_2}))$ is a basis of $B'_d[2]$. Again, by matrix inverting argument, we can construct $b''_{2,1}, ..., b''_{2,k_2}$, each $b''_{2,j}$ is an F-linear combination of $b'_{2,1}, ..., b'_{2,k_2}$, s.t. each $\sigma_1(b''_{2,j})$ has some $g'_{i_1} g'_{i_2}$, which appears only in it, and each $b'_{2,h}$ is F-linear combination of $b''_{2,1}, ..., b''_{2,k_2}$. Furthermore, for any $b' \in B'_d$, we can subtract some unique F-linear combination of $\sigma_1(b''_{2,1}), ..., \sigma_1(b''_{2,k_2})$ from $\sigma_1(b')$, to "annihilate" the $\pi_2(\sigma_1(b'))$ part of $\sigma_1(b')$. The result of this subtraction is called $\sigma_2(b')$.

We then repeat the process for weights $3, ..., d$. It is clear that $\sigma_d(b') = 0 \mid \forall b' \in B'_d$. Hence the subset of $\{b''_{1,1}, ..., b''_{1,k_1}, \sigma_1(b''_{2,1}), ..., \sigma_1(b''_{2,k_2}), ..., \sigma_{d-1}(b''_{d,1}), ..., \sigma_{d-1}(b''_{d,k_d})\}$, consisting of all nonzero elements, is a F-basis of B'_d and every element in it has a unique homogeneous summand.

The basis $c'_1, c'_2, ...$ is the disjoin union of the above constructed basises for all gradings d, for which $B'_d \neq 0$.

Definition 4.7 *Here and elsewhere* $I'^{0} := Rel^{0} := A'$, $I'^{\infty} := Rel^{\infty} := 0$.

$$Z'_n := \bigcap_{\sum_i(\alpha_i+2\beta_i)=n} (I'^{\alpha_1} \cdot Rel^{\beta_1} \cdot I'^{\alpha_2} \cdot Rel^{\beta_2} \cdot \ldots \cdot I'^{\alpha_{n+1}} \cdot Rel^{\beta_{n+1}}) \subset A'$$

$$B'_n := I' \cdot Z'_n + [\sum_{i=1}^{n}(Z'_{i-1} \cdot Rel \cdot Z'_{n-i})]$$

Theorem 4.8 $pr^n(J'_n) \subset Z'_{n+1}$. *Hence we can define*

$$\lambda_n : X'_n \to Z'_{n+1}$$

as restriction of pr^n *to* X'_n.

Proof: By induction on n. $X'_0 = I' = Z'_1$ and $pr^1(X'_1) \subset (Rel \cap (I' \cdot I')) = Z'_2$, so we have the Theorem for $n = 0, 1$. Assume the Theorem for all numbers, till $n - 1$.

Any element $x' = \sum_t f_t(i'_{1,t} \otimes \ldots \otimes i'_{n+1,t}) \in J'_n$, where all $i'_{*,*}$ are homogeneous and $f_* \in F$, can be written as $i'_{1,1} \otimes w'_{1,1} + \ldots + i'_{1,k} \otimes w'_{1,k}$, where all $w'_{1,v}$ are in J'_{n-1} and all $i'_{1,u}$ are different. Hence, by induction hypothesis, $pr^n(J'_n) \subset I' \cdot Z'_n$.

Let $c'_1, c'_2, \ldots$ be a F-basis of $[Rel]^F$, as in Lemma 4.6. $d_{n,1}(x') \in Rel \otimes I'^{\otimes n-1}$ can be represented as

$$\sum_h (c'_h \otimes [\sum_j f_{j,h}(i'_{3,j,h} \otimes \ldots \otimes i'_{n+1,j,h})])$$

Again all $i'_{*,*,*}$ are homogeneous. But

$$\sum_{i'_{1,t}\cdot i'_{2,t}=sgn(c'_h)} f_t(i'_{1,t}\otimes i'_{2,t}\otimes \dots \otimes i'_{n+1,t}) \in I'\otimes I'\otimes J'_{n-2}$$

Hence $d_{n,1}(x') \in Rel\otimes J'_{n-2}$ and, by induction, $pr^n(J'_n)\subset Rel\cdot Z'_{n-1}$

Hence $pr^n(J'_n)\subset (I'\cdot Z'_n)\cap(Rel\cdot Z'_{n-1})\subset Z'_{n+1}$ - last inclusion constructed by induction.

Lemma 4.9 *Every* $x'\in Ker(proj : J'_n \to J_n)$ *can be written as:*

$$x' = \sum_{k=1}^{n+1} x'_k \quad , \quad x'_k \in J'_{k-2}\otimes Rel\otimes J'_{n-k}$$

Proof: Let $\{c'_1, c'_2, ...\}$ be an F-basis of $[Rel]^F$, as in Lemma 4.6, and $\{i'_1, i'_2, ...\}$ be the set of all homogeneous elements of I' [no repetitions], different from $sgn(c'_1), sgn(c'_2), \dots$. Hence $\{sgn(c'_1), sgn(c'_2), ..., i'_1, i'_2, ...\}$ is the set of all homogeneous elements of I', with no repetitions, and, thus, an F-basis of $[I']^F$. Clearly, $\{c'_1, c'_2, ..., i'_1, i'_2, ...\}$ is an F-basis of $[I']^F$ as well. Denote $\Phi := [\{i'_1, i'_2, ...\}]^F$. Clearly, $proj(\Phi) = [I]^F$ and the kernel of $proj$, restricted to Φ, is 0.

Represent $x' = \sum_t c'_t\otimes u'_t + \sum_t i'_t \otimes y'_t$. Then each u'_t has to be in J'_{n-1}.

Define $x'_1 := \sum_t c'_t \otimes u'_t$. Now take $x'' := x' - x'_1$.

Clearly, $x'_1 \in Rel\otimes J'_{n-1}\subset Ker(proj : J'_n \to J_n)$, so $x''\in Ker(proj : J'_n\to J_n)$.

Represent: $$x'' = \sum_t v''_t \otimes c'_t \otimes u''_t + \sum_{t_1,t_2} i'_{t_1} \otimes i'_{t_2} \otimes y''_{t_1,t_2}$$

Again, each v''_t has to be in J'_0 and each u''_t has to be in J'_{n-2}. Define

$$x'_2 := \sum_t v''_t \otimes c'_t \otimes u''_t$$

Now take $x''' := x'' - x'_2 = x' - (x'_1 + x'_2)$.

Clearly, $x'_2 \in J'_0 \otimes Rel \otimes J'_{n-2} \subset Ker(proj : J'_n \to J_n)$, so

$x''' \in Ker(proj : J'_n \to J_n)$.

After repeating this process $n+1$ times, we are left with $x'^{(n+1)} := x' - (x'_1 + x'_2 + ... + x'_{n+1})$, which is an element of $Ker(proj : J'_n \to J_n)$. But $x'^{(n+1)} \in \Phi^{\otimes(n+1)}$. Since $proj$ has zero kernel on $\Phi^{\otimes(n+1)}$, $x'^{(n+1)} = 0$ and $x' = x'_1 + x'_2 + ... + x'_{n+1}$.

Theorem 4.10 *pr^n maps $Ker(proj : J'_n \to J_n)$ into B'_{n+1}.*

Proof: Every $x' \in Ker(proj : J'_n \to J_n)$ can be written $x' = x'_1 + x'_2 + ... + x'_{n+1}$, as in the previous Lemma.

But $pr^n(x'_k) \in pr^{k-2}(J'_{k-2}) \cdot Rel \cdot pr^{n-k}(J'_{n-k}) \subset B'_{n+1}$ for each $1 \leq k \leq n+1$.

Lemma 4.11 *Let $L' \subset A'$ be a left ideal, and $l'_1, l'_2, \ldots$ be some minimal generating set of L'. Then there is only trivial solution to [y'_k are homogeneous elements of A']*

$$\sum_k y'_k \cdot l'_k = \sum_k f_k(g'_{1,k} \cdot \ldots \cdot g'_{h_k,k}) \cdot l'_k = 0 \;, \quad \textit{here } l'_i \textit{ might be equal to } l'_j \textit{ for } i \neq j, \textit{ but } y'_i \neq y'_j$$

Proof: Any non-trivial solution must have $h_k > 0$ for all k, since this generating set is minimal [if $h_k = 0$ one can solve for l'_k]. Choose a non-trivial solution with minimal total amount of $g'_{i,j}$. Fix $g' = g'_{1,1}$. Now

$$\sum_{g'_{1,t}=g'} f_t(g'_{1,t} \cdot g'_{2,t} \cdot \ldots \cdot g'_{h_t,t}) \cdot l_k = 0$$

$$\sum_{g'_{1,t}=g'} f_t(g'_{2,t} \cdot \ldots \cdot g'_{h_t,t}) \cdot l_k = 0$$

This contradicts the minimality of total amount of $g'_{i,j}$ in the chosen solution.

Corollary 4.12 *Let $L' \subset A'$ be a left ideal, and $M', N' \subset A'$ be right ideals, then*

$$(M' \cap N') \cdot L' = (M' \cdot L') \cap (N' \cdot L')$$

Proof: Let $l'_1, \ldots, l'_k$ be a minimal generating set of L'. $x' \in M' \cdot L'$ can be represented by $x' = m'_1 \cdot l'_1 + \ldots + m'_k \cdot l'_k$, $m'_j \in M'$, and $x' \in N' \cdot L'$ can be represented by $x' = n'_1 \cdot l'_1 + \ldots + n'_k \cdot l'_k$, $n'_j \in N'$. By previous Lemma, $m'_1 = n'_1, \ldots, m'_k = n'_k$.

Obviously, in 4.11 and in 4.12, the "left ideal" and "right ideal" can be interchanged.

Theorem 4.13 *$pr^n : J'_n \to Z'_{n+1}$ is onto.*

Proof: For $n = 0$, $J'_0 = I' = Z'_1$. For $n = 1$ take any $\sum_j i'_j \cdot g'_j \in Z'_2 = (I' \cdot I') \cap Rel$. Then for $x' := \sum_j i'_j \otimes g'_j \in J'_1$, $pr^1(x') = \sum_j i'_j \cdot g'_j$. We proceed by induction on n.

Assume chosen $w'_{n-1,1} \in J'_{n-1}, \ldots, w'_{n-1,k} \in J'_{n-1}$ and $w'_{n-2,1} \in J'_{n-2}, \ldots, w'_{n-2,h} \in J'_{n-2}$, s.t. $pr^{n-1}(w'_{n-1,1}), \ldots, pr^{n-1}(w'_{n-1,k})$ are all different and form a minimal generating set of Z'_n, as left A' ideal, and $pr^{n-2}(w'_{n-2,1}), \ldots, pr^{n-2}(w'_{n-2,h})$ are all different and form a minimal generating set of Z'_{n-1}, as left A' ideal, with each

$$w'_{n-1,j} = u'_{n-1,j,1} \otimes w'_{n-2,1} + \ldots + u'_{n-1,j,h} \otimes w'_{n-2,h} \quad | \quad u'_{n-1,*,*} \in I'$$

Making these choices will be done in this proof inductively, and is trivial for $n = 1$ [i.e. $w'_{0,j} := g'_j$, $w'_{-1,1} := 1$ and $u'_{0,j,1} := g'_j$].

Remark: We here abuse the notation and identify A' with $A' \otimes 1$.

For any element $x' \in Z'_{n+1} \subset I' \cdot Z'_n$ pick a representation

$$x' = i'_1 \cdot pr^{n-1}(w'_{n-1,1}) + \ldots + i'_k \cdot pr^{n-1}(w'_{n-1,k})$$

Here $i'_1, ..., i'_k \in I'$ are not assumed to be homogeneous. We define

$$y' := i'_1 \otimes w'_{n-1,1} + ... + i'_k \otimes w'_{n-1,k} = v'_1 \otimes w'_{n-2,1} + ... + v'_h \otimes w'_{n-2,h}$$

where $v'_* \in I' \otimes I'$. Since all $\{r_\alpha \cdot pr^{n-2}(w'_{n-2,p}) \; | p = 1, ..., h \; ; \; r_\alpha \in Rel \cap (I' \cdot I')\}$ generate a left A'-ideal $[Rel \cap (I' \cdot I')] \cdot Z'_{n-1}$, we can choose some minimal generating set $\{r_{q,p} \cdot pr^{n-2}(w'_{n-2,p})\}$ of $Z'_{n+1} \subset [Rel \cap (I' \cdot I')] \cdot Z'_{n-1}$.

Since representation of $x' = pr^1(v'_1) \cdot pr^{n-2}(w'_{n-2,1}) + ... + pr^1(v'_h) \cdot pr^{n-2}(w'_{n-2,h}) \in Z'_{n+1} \subset A'$ in terms of the $\{r_{q,p} \cdot pr^{n-2}(w'_{n-2,p})\}$ is unique, we get $pr^1(v'_1) \in Rel, ..., pr^1(v'_h) \in Rel$. Clearly, $y' \in J'_n$ and $pr^n(y') = x'$.

Now choose any minimal generating set $\{z'_t\}_{t=1,...,e}$ of Z'_{n+1} and represent each

$$z'_t = u'_{n,t,1} \cdot pr^{n-1}(w'_{n-1,1}) + ... + u'_{n,t,k} \cdot pr^{n-1}(w'_{n-1,k})$$

We now define each

$$w'_{n,t} := u'_{n,t,1} \otimes w'_{n-1,1} + ... + u'_{n,t,k} \otimes w'_{n-1,k}$$

That finishes the proof of the Theorem.

Corollary 4.14 $\lambda_n : X'_n \to Z'_{n+1}$ *is onto.*

Proof: For any $x' \in Z'_{n+1}$, take $y' \in J'_n$, s.t. $pr^n(y') = x'$. Since, $J'_n \subset I'^{\otimes n} \otimes I' = I'^{\otimes n} \otimes X'_0$, we have $y' \in I'^{\otimes n} \otimes X'_0$. Assume that, for some $0 \leq K \leq n$, $y' \in I'^{\otimes k} \otimes X'_{n-k}$, i.e. $y' = \sum_t i'_{t,1} \otimes ... \otimes i'_{t,k} \otimes x'_t$, where all $i'_{t,j} \in I'$ and all $x'_t \in X'_{n-k}$.

Choose elements $\{s'_u \in \Omega'_{n-k} \mid proj(s'_u) \in (S_{n-k} \cup \{0\})\}_{u=1,2,...}$ s.t. every x'_t can be represented as $x'_t = \sum_u a'_{t,u} \cdot s'_u$.

$$\text{Define} \quad y'' := \sum_u [\sum_t i'_{t,1} \otimes ... \otimes i'_{t,k} \cdot a'_{t,u}] \otimes s'_u$$

Then, $y'' \in J'_n \cap (I'^{\otimes(k-1)} \otimes X'_{n+1-k})$ and $pr^n(y'') = pr^n(y') = x'$.

So, after repeating this process k times, we get

$y'^k \in X'_n$, s.t. $\lambda_n(y'^k) := pr^n(y'^k) = x'$.

Corollary 4.15 $B'_n \subset Z'_n$

Proof: For any $z'_1 \cdot r \cdot z'_2 \in B'_n$, s.t. $z'_1 \in Z'_{i-1}$, $r \in Rel$, $z'_2 \in Z'_{n-i}$, choose $y'_1 \in X'_{i-2}$ such that $\lambda_{i-2}(y'_1) = z'_1$ and $y'_2 \in X'_{n-i-1}$ such that $\lambda_{n-i-1}(y'_2) = z'_2$. Then $y'_1 \otimes r \otimes y'_2 \in J'_n$, hence $z'_1 \cdot r \cdot z'_2 = pr^n(y'_1 \otimes r \otimes y'_2) \in Z'_n$

Actually, if $y'_2 = \sum_u a'_u \cdot s'_u$, where $\{s'_u\}$ are selected as in the proof of Corollary 4.15, then for $q' = \sum_u y'_1 \otimes (r \cdot a'_u) \otimes s'_u$ is an element of X'_n, and we have that $proj(q') = 0$ and $\lambda_n(q') = z'_1 \cdot r \cdot z'_2$.

Corollary 4.16 *For any $x' \in X'_n$, s. t. $\lambda_n(x') \in B'_{n+1}$, one can find some $x'' \in X'_n$, s. t. $proj(x'') = proj(x')$ and $\lambda_n(x'') \in I' \cdot Z'_{n+1}$.*

Proof: The element $\lambda_n(x') \in B'_{n+1}$ is sum of some element $i' \cdot z' \in B'_{n+1}$, where $i' \in I'$ and $z' \in Z'_{n+1}$, and elements of $Z'_k \otimes Rel \otimes Z'_{n+1-k}$. But for each element $z'_1 \cdot r \cdot z'_2 \in B'_n$ exists $q' \in X'_n$, s.t. $proj(q') = 0$ and $\lambda_n(q') = z'_1 \cdot r \cdot z'_2$. Select $y' \in X'_n$, s.t. $\lambda_n(y') = z'$. Now take x'' to be sum of $i' \cdot y'$ and the elements q'.

Theorem 4.17 *If for $x \in X_n$, exists $x' \in X'_n$, s. t. $proj(x') = x$ and $\lambda_n(x') \in I' \cdot Z'_{n+1}$, then one can find $x'' \in I' \cdot X'_n$, s. t. $proj(x'') = x$. In other words $x \in I \cdot X_n$.*

Proof: By induction on n.

For $n = -1$, $(\lambda_{-1} = Id_{A'}) : (X'_{-1} = A') \to (Z'_0 = A')$.

For $n = 0$, $(\lambda_0 = Id_{I'}) : (X'_0 = I') \to (Z'_1 = I')$. Assume the Theorem for all numbers, till $n - 1$.

Since $\lambda_n : X'_n \to Z'_{n+1}$ is onto, we need only to prove the Theorem for the case $\lambda_n(x') = 0$.

For each $s_e \in S_{n-1}$, choose $s'_e \in \Omega'_{n-1}$, s.t. $proj(s'_e) = s_e$. Then any $u' \in \Omega'_{n-1}$ can be represented [uniquely - due to the fact, that $s_1, s_2, \ldots$ are F-basis of S^F_{n-1}] as an F-linear combination of s'_h plus some $u'' \in \Omega'_{n-1}$, $proj(u'') = 0$. Hence any $x' \in X'_n$ can be represented as

$$x' = i'_1 \otimes s'_1 + \ldots + i'_k \otimes s'_k + i'' \otimes s'' \quad , \quad s'' \in \Omega'_{n-1} \mid proj(s'') = 0$$

Assume that

$$pr^{n-1}(i'_1 \cdot s'_1 + \ldots + i'_k \cdot s'_k + i'' \cdot s'') = pr^n(x') = 0 \ \text{ and } \ i'_1 \neq 0$$

[Since if at least one of the $i'_1, i'_2, \ldots$ is nonzero, we can assume that i'_1 is nonzero.] Then fix some $f_1 g'_1 \cdot \ldots \cdot g'_h$ homogeneous summand of i'_1. For all $y' \in I', g' \in G'$, define [F-linear map] $\beta_{g'}(y')$ as sum of all homogeneous summands of y', which start with g', with the initial g' changed to $1 \in F$ in them [Note. If y' has no such summands, $\beta_{g'}(y') = 0$. We applied $\beta_{g'_{1,1}}$, without defining or mentioning it, in the proof of Lemma 4.11]. Now

$$pr^{n-1}(\beta_{g'_1}(i'_1) \cdot s'_1 + \ldots + \beta_{g'_1}(i'_k) \cdot s'_k + \beta_{g'_1}(i'') \cdot s'') = 0$$

$$pr^{n-1}(\beta_{g'_2}(\beta_{g'_1}(i'_1)) \cdot s'_1 + \ldots + \beta_{g'_2}(\beta_{g'_1}(i'_k)) \cdot s'_k + \beta_{g'_2}(\beta_{g'_1}(i'')) \cdot s'') = 0$$

After repeating this process h times, one will get

$$pr^{n-1}(f_1 s_1' + a_2' \cdot s_2' + ... + a_k' \cdot s_k' + a'' \cdot s'') = 0$$

By induction hypothesis,

$$f_1 s_1 + a_2 \cdot s_2 + ... + a_k \cdot s_k = proj(f_1 s_1' + a_2' \cdot s_2' + ... + a_k' \cdot s_k' + a'' \cdot s'') = y_1 \cdot s_1 + ... + y_k \cdot s_k \ \in I \cdot X_{n-1}$$

Here $y_1, ..., y_k \ \in I$. But that contradicts the fact that S_{n-1} is a *minimal* generating set of X_{n-1}.

Hence $i_1' = ... = i_k' = 0$, so $x' = i'' \otimes s''$ and $x = proj(x') = 0$. We can take $x'' := 0$.

Theorem 4.18 *λ_n induces isomorphism γ_{n+1} from $H_{n+1}(A) = X_n/(I \cdot X_n)$ to Z_{n+1}'/B_{n+1}'*

Proof: For any $[x] \in X_n/(I \cdot X_n)$, $-1 \leq n$, take any representative $x \in X_n$, choose any $x' \in X_n'$, s.t. $proj(x') = x$, and define

$$\gamma_{n+1}([x]) := [\lambda_n(x')] \in Z_{n+1}'/B_{n+1}'$$

If one chooses some other x_1 to represent $[x]$, then $y := x_1 - x \in I \cdot X_n$. So $y = i \cdot u$ for some $i \in I$ and $u \in X_n$. Choose any $i' \in I'$ and $u' \in X_n'$, s.t. $proj(i') = i$ and $proj(u' = u)$. Then $x' + i' \cdot u' \ \in X_n'$ and $proj(x' + i' \cdot u') = x_1$. But $\lambda_n(x' + i' \cdot u') - \lambda_n(x') = \lambda_n(i' \cdot u') \ \in B_{n+1}'$.

If one chooses some other $x'_1 \in X'_n$, s.t. $proj(x'_1) = x$, then $proj(x'_1 - x') = 0$ and $\lambda_n(x'_1) - \lambda_n(x') = \lambda_n(x'_1 - x') \in B'_{n+1}$. So, γ_{n+1} is well defined.

We have also proved that λ_n is onto. Hence γ_{n+1} is onto.

If $\lambda_n(x') \in B'_{n+1}$, then $x = proj(x') \in I \cdot X_n$, hence $[x] = 0$, so γ_{n+1} is one-to-one.

Lemma 4.19 *$Z'_n \cdot I' \subset B'_n$, hence*

$$B'_n = I' \cdot Z'_n + Z'_n \cdot I' + [\sum_{i=1}^{n}(Z'_{i-1} \cdot Rel \cdot Z'_{n-i})]$$

Proof: For $n = 0$, $Z'_0 \cdot I' = A' \cdot I' = I' \cdot A' = I' \cdot Z'_0$. For $n = 1$, $Z'_1 \cdot I' = I' \cdot I' = I' \cdot Z'_1$.

For $2 \le n$, $Z'_n \subset Z'_{n-2} \cdot Rel$, hence $Z'_n \cdot I' \subset Z'_{n-2} \cdot Rel \cdot I' = Z'_{n-2} \cdot Rel \cdot Z'_1$

Lemma 4.20 *If $I' \cdot Rel \subset Rel \cdot I'$, then $Rel = I'^m$ for some $1 \le m \le \infty$ [Note. $I'^\infty := 0$]*

Proof: Choose some non-zero element $r \in Rel$, with smallest minimal weight m. I.E. this r has some homogeneous summand, with m generators, and all other summands of all other nonzero elements of Rel have m generators or more [If $Rel = 0$, take $m = \infty$].

For any $g_1' \in G'$ of smallest grading, $g_1' \cdot r = \sum_k r_k \cdot i_k'$ with all i_k' homogeneous and different. But, due to min/max conditions, for some k, r_k has minimal weight m and $i_k' \in G'$. Every homogeneous summand of weight m of every such r_k has to start with g_1'. Define $r' := r_k$.

For any $g_2' \in G'$ of smallest grading, $g_2' \cdot r' = \sum_t r_t' \cdot i_t'$. But, due to min/max conditions, for some t, r_t' has minimal weight m and $i_t' \in G'$. Every homogeneous summand of weight m of every such r_t' has to start with $g_2' \cdot g_1'$.

Repeating this process for m steps gives us: $g_m' \cdot \ldots \cdot g_1' \in Rel$ for all $g_1', \ldots, g_m'$ of minimal grading in G'. But now, for any $g_1'' \in G'$, $g_1'' \cdot g_m' \cdot \ldots \cdot g_1' \in Rel \cdot I'$, hence $g_1'' \cdot g_m' \cdot \ldots \cdot g_2' \in Rel$. Repeating this process m times gives us: $g_m'' \cdot \ldots \cdot g_1'' \in Rel$ for all $g_1'', \ldots, g_m'' \in G'$.

Of cause, if $I' \cdot Rel \supset Rel \cdot I'$, then again $Rel = I'^m$. And if $Rel = I'^m$, then $I' \cdot Rel = Rel \cdot I'$.

Lemma 4.21 *If G is minimal and $I' \cdot Rel \cdot I' \subset Rel \cdot Rel$, then $Rel = I'^2$ or $Rel = 0$.*

Proof: Choose any $g'_1 \in G'$, with minimal grading. From all nonzero elements of Rel, with minimal grading, choose any r with minimal amount of homogeneous summands. Let $\{r_1, r_2, ...\}$ be an F-basis of $[Rel]^F$ as in Lemma 4.6. Then $g'_1 \cdot r \cdot g'_1 = \sum_{t,k} r_t \cdot r_k$. But, due to min/max conditions, the grading of r has to be twice the grading of g'_1 and r must have only one homogenous summand. So $r = g'_3 \cdot g'_4$. But then $\sum_{t,k} r_t \cdot r_k = (g'_1 \cdot g'_3) \cdot (g'_4 \cdot g'_2)$. So, for some t, $r_t = g'_1 \cdot g'_3$. Now choose any $g'_2 \in G'$, with minimal grading and repeat this process with r_t. One gets $g'_2 \cdot g'_1 \in Rel$ for any $g'_1, g'_2 \in G'$, with minimal grading.

Now, for any $g''_1 \in G'$, $\;g''_1 \cdot (g'_1 \cdot g'_1) \cdot g'_1 \in Rel \cdot Rel$, hence $g''_1 \cdot g'_1 \in Rel$. For any $g''_2 \in G'$, $\;g''_2 \cdot (g''_1 \cdot g'_1) \cdot g'_1 \in Rel \cdot Rel$, hence $g''_2 \cdot g''_1 \in Rel$.

If $Rel = I'^2$ or $Rel = 0$, then, clearly, $I' \cdot Rel \cdot I' = Rel \cdot Rel$.

Example 4.22 *Let us calculate Z'_n and B'_n for several values of n. If G is a minimal generating set of A then $Rel \subset I' \cdot I'$. For G - any generating set of A, we get:*

$$Z'_0 = I'^0 \cdot Rel^0 := A' \;\;,\;\; B'_0 = (I' \cdot Z'_0) + (Z'_0 \cdot I') = I'$$

$$Z'_1 = (I'^1 \cdot Rel^0 \cdot I'^0 \cdot Rel^0) \cap (I'^0 \cdot Rel^0 \cdot I'^1 \cdot Rel^0) = I'$$

$$B'_1 = (I' \cdot Z'_1) + (Z'_0 \cdot Rel \cdot Z'_0) + (Z'_1 \cdot I') = (I' \cdot I') + (A' \cdot Rel \cdot A') + (I' \cdot I') = (I' \cdot I') + Rel$$

And for G-minimal $\qquad B'_1 = I' \cdot I'$

$$Z'_2 = (I'^2 \cdot Rel^0 \cdot I'^0 \cdot Rel^0 \cdot I'^0 \cdot Rel^0) \cap (I'^0 \cdot Rel^1 \cdot I'^0 \cdot Rel^0 \cdot I'^0 \cdot Rel^0) \cap \ldots = (I' \cdot I') \cap Rel$$

$$B'_2 = (I' \cdot Z'_2) + (Z'_0 \cdot Rel \cdot Z'_1) + (Z'_1 \cdot Rel \cdot Z'_0) + (Z'_2 \cdot I') =$$

$$= (I' \cdot [(I' \cdot I') \cap Rel]) + (A' \cdot Rel \cdot I') + (I' \cdot Rel \cdot A') + ([(I' \cdot I') \cap Rel] \cdot I') = (I' \cdot Rel) + (Rel \cdot I')$$

And for G-minimal $\qquad Z'_2 = Rel$

$$Z'_3 = \ldots = (I' \cdot Rel) \cap (I' \cdot I' \cdot I') \cap (Rel \cdot I')$$

$$B'_3 = (I' \cdot Z'_3) + (Z'_3 \cdot I') + \sum_{i=0}^{2} (Z'_i \cdot Rel \cdot Z'_{2-i}) = (I' \cdot Rel \cdot I') + ([Rel \cdot Rel] \cap [(Rel \cdot I' \cdot I') + (I' \cdot I' \cdot Rel)])$$

And for G-minimal $\quad Z'_3 = (I' \cdot Rel) \cap (Rel \cdot I')$, $B'_3 = (I' \cdot Rel \cdot I') + (Rel \cdot Rel)$

$$Z'_4 = \ldots = (I' \cdot I' \cdot Rel) \cap (I' \cdot Rel \cdot I') \cap (Rel \cdot I' \cdot I') \cap (I' \cdot I' \cdot I' \cdot I') \cap (Rel \cdot Rel)$$

$$B'_4 = (I' \cdot Z'_4) + (Z'_4 \cdot I') + \sum_{i=0}^{3} (Z'_i \cdot Rel \cdot Z'_{3-i}) =$$

$$= (Rel \cdot [(I' \cdot Rel) \cap (I' \cdot I' \cdot I') \cap (Rel \cdot I')]) + (I' \cdot Rel \cdot [Rel \cap (I' \cdot I')]) + ([Rel \cap (I' \cdot I')] \cdot Rel \cdot I') +$$

$$+ ([(I' \cdot Rel) \cap (I' \cdot I' \cdot I') \cap (Rel \cdot I')] \cdot Rel)])$$

And for G-minimal $\quad Z'_4 = (I' \cdot Rel \cdot I') \cap (Rel \cdot Rel)$

$$B'_4 = (I' \cdot Rel \cdot Rel) + (Rel \cdot Rel \cdot I')$$

For G-minimal we also have:

$$Z_5' = (Rel \cdot Rel \cdot I') \cap (Rel \cdot I' \cdot Rel) \cap (I' \cdot Rel \cdot Rel)$$

$$B_5' = (Rel \cdot Rel \cdot Rel) + ([I' \cdot Rel \cdot Rel \cdot I'] \cap [(I' \cdot Rel \cdot I' \cdot Rel) + (Rel \cdot I' \cdot Rel \cdot I')])$$

$$Z_6' = (Rel \cdot Rel \cdot Rel) \cap (Rel \cdot I' \cdot Rel \cdot I') \cap (I' \cdot Rel \cdot Rel \cdot I') \cap (I' \cdot Rel \cdot I' \cdot Rel)$$

$$B_6' = ([Rel \cdot Rel \cdot Rel \cdot I'] \cap [(Rel \cdot Rel \cdot I' \cdot Rel) + (I' \cdot Rel \cdot I' \cdot Rel \cdot I')])+$$

$$+([I' \cdot Rel \cdot Rel \cdot Rel] \cap [(Rel \cdot I' \cdot Rel \cdot Rel) + (I' \cdot Rel \cdot I' \cdot Rel \cdot I')])$$

Hence: $H_0(A) = F$, $H_1(A) = (I'/Rel)/([(I' \cdot I') + Rel]/Rel) = I/(I \cdot I)$ *which are classical results.*

For G-minimal, $H_2(A) = Rel/([I' \cdot Rel] + [Rel \cdot I'])$ *- is stated in [14] using different terminology.*

$$X'_n := J'_n \cap proj^{-1}(X^F_n) \ , \ \Omega'_n := J'_n \cap proj^{-1}(S^F_n)$$

$$\begin{tikzcd}
& & Rel \otimes I'^{\otimes 3} & & Rel \otimes I'^{\otimes 2} & & Rel \otimes I' & & Rel & & \\
& I'^{\otimes 5} \supset I' \otimes J'_3 \arrow[d, phantom, "\cup"] & & I'^{\otimes 4} \supset I' \otimes J'_4 \arrow[d, phantom, "\cup"] & & I'^{\otimes 3} \supset I' \otimes J'_1 \arrow[d, phantom, "\cup"] & & I'^{\otimes 2} \arrow[d, phantom, "\cup"] & & I' \arrow[d, equal] & \\
& J'_4 \arrow[rr, "d'_{4,1}"] \arrow[uur, "d'_{3,1}"] \arrow[dd] \arrow[dl, phantom, "\cup"] \arrow[ddr, phantom, "\downarrow proj"] & & J'_3 \arrow[rr, "d'_{3,1}"] \arrow[uur, "d'_{2,1}"] \arrow[dd] \arrow[dl, phantom, "\cup"] & & J'_2 \arrow[rr, "d'_{2,1}"] \arrow[uur, "d'_{1,1}"] \arrow[dd] \arrow[dl, phantom, "\cup"] & & J'_1 \arrow[rr, "d'_{1,1}"] \arrow[uur, "d'_{3,1}"] \arrow[dd] \arrow[dl, phantom, "\cup"] & & J'_0 \arrow[r, "inclusion"] \arrow[dd] \arrow[dl, phantom, "\cup"] \arrow[ddr, phantom, "\downarrow proj"] & J'_{-1} = A' \arrow[d, equal] \\
X'_4 \arrow[dd] & & X'_3 \arrow[dd] & & X'_2 \arrow[dd] & & X'_1 \arrow[dd] & & X'_0 \arrow[dd] & & X'_{-1} \arrow[d] \\
& J_4 \subset I^{\otimes 5} \arrow[dl, phantom, "\cup"] & & J_3 \subset I^{\otimes 4} \arrow[dl, phantom, "\cup"] & & J_2 \subset I^{\otimes 3} \arrow[dl, phantom, "\cup"] & & J_1 \subset I^{\otimes 2} \arrow[dl, phantom, "\cup"] & & J_0 = I \arrow[dl, phantom, "\cup"] & J_{-1} = A \arrow[d, equal] \\
X_4 = S^A_4 \arrow[d] & & X_3 = S^A_3 \arrow[d] & & X_2 = S^A_2 \arrow[d] & & X_1 = S^A_1 \arrow[d] & & X_0 = S^A_0 \arrow[d] & & X_{-1} = A \arrow[d, "0"] \\
A \otimes S^F_3 = P_4 \arrow[rr, "d_{4,1}"] \arrow[urr] & & A \otimes S^F_2 = P_3 \arrow[rr, "d_{3,1}"] \arrow[urr] & & A \otimes S^F_1 = P_2 \arrow[rr, "d_{2,1}"] \arrow[urr] & & A \otimes S^F_0 = P_1 \arrow[rr, "d_{1,1}"] \arrow[urr] & & A = P_0 \arrow[rr, "0"] \arrow[urr, "Id_A"] \arrow[d, "\partial_{-1}"] & & 0 \\
& & & & & & & & F & &
\end{tikzcd}$$

5 Pairings and cup-co-products in homology

We start with reviewing Yoneda cup-product [Note: we drop grading indexes from μ]

$$\mu_{M,M',M''} : Ext_A^{s,t}(M,M') \otimes Ext_A^{s',t'}(M',M'') \to Ext_A^{s+s',t+t'}(M,M'')$$

Let $\{P_{M,j}, \partial_{M,j}\}_{j=0,\ldots}$ and $\{P_{M',j}, \partial_{M',j}\}_{j=0,\ldots}$ be some projective resolutions of left A-modules M and M'.

Any element $[x] \in Ext_A^{s,t}(M,M')$ can be represented by some left A-module homomorphism $x : P_{M,s} \to M'$, s.t. $(x \circ \partial_{M,s}) : P_{M,s+1} \to M'$ is a zero map [x is a co-cycle]. Co-cycle x reduces gradings by t.

Since $P_{M',0} \xrightarrow{\partial_{M',-1}} M'$ is onto and $P_{M,s}$ is projective, one can lift x to some $\overline{x}_0 : P_{M,s} \to P_{M',0}$.

Since $\partial_{M',-1} \circ (\overline{x}_0 \circ \partial_{M,s}) = x \circ \partial_{M,s} = 0$, exists left A-module homomorphism $u_0 : P_{M,s+1} \to Ker(\partial_{M',-1}) = Im(\partial_{M',0})$, s.t. $(inclusion) \circ u_0 = \overline{x}_0 \circ \partial_{M,s}$. Since $P_{M,s+1}$ is projective, u_0 can be lifted to some $\overline{x}_1 : P_{M,s+1} \to P_{M',1}$. One sees that

$$\partial_{M',0} \circ \overline{x}_1 = \overline{x}_0 \circ \partial_{M,s}$$

Repeating this process n times, $n = 1, 2, \ldots$, produces left A-module homomorphisms $\overline{x}_n : P_{M,s+n} \rightarrow P_{M',n}$, s.t.

$$\partial_{M',n-1} \circ \overline{x}_n = \overline{x}_{n-1} \circ \partial_{M,s+n-1}$$

Now, any element $[y] \in Ext_A^{s',t'}(M', M'')$ can be represented by a co-cycle $y : P_{M',s'} \rightarrow M''$ and we define

$$\mu_{M,M',M''}([x] \otimes [y]) := [y \circ \overline{x}_{s'}]$$

To justify this definition, we must show that, $y \circ \overline{x}_{s'}$ is indeed a co-cycle, and that making different choices in our construction will produce a difference by co-boundary in $y \circ \overline{x}_{s'}$ [the bi-linearity of $[y \circ \overline{x}_{s'}]$ relative to $[x]$ and $[y]$ is obvious].

But $(y \circ \overline{x}_{s'}) \circ \partial_{M,s+s'} = y \circ \partial_{M',s'} \circ \overline{x}_{s'+1} = (y \circ \partial_{M',s'}) \circ \overline{x}_{s'+1} = 0$, so $y \circ \overline{x}_{s'}$ is a co-cycle.

If y is a co-boundary, i.e. $y = w \circ \partial_{M',s'-1}$, for $s' > 0$, then $y \circ \overline{x}_{s'} = w \circ \partial_{M',s'-1} \circ \overline{x}_{s'} = w \circ \overline{x}_{s'-1} \circ \partial_{M',s'-1}$, which is a co-boundary. So, the construction does not depend on the choice of the representative y for $[y]$. For $s' = 0$, the only co-boundary is $y = 0$.

Assume that x is a co-boundary, i.e. $x = k \circ \partial_{M,s-1}$. Then, for $k : P_{M,s-1} \to M'$, construct $\overline{k}_0 : P_{M,s-1} \to P_{M',0}$, s.t. $\partial_{M',-1} \circ \overline{k}_0 = k$. One sees that $\overline{x}_0$ can be chosen to be $= \overline{k}_0 \circ \partial_{M,s}$. Constructing higher $\overline{k}_n$, like we did for $\overline{x}_n$, gives us $\overline{x}_n = \overline{k}_n \circ \partial_{M,s+n}$. Hence $y \circ \overline{x}_{s'} = y \circ \overline{k}_{s'} \circ \partial_{M,s+s'}$, which is a co-boundary.

Also, if one chooses another lifting $\overline{x}'_0$ for x, or $\overline{x}'_{n+1}$ for $\overline{x}_n$ then $\partial_{M',m-1} \circ (\overline{x}'_m - \overline{x}_m) = 0$ [Here m is 0 or $n+1$]. So, suppose $\partial_{M',m-1} \circ \overline{x}_m = 0$. Then $\overline{x}_m = \partial_{M',m} \circ h_m$, for some h_m. But $\partial_{M',m} \circ \overline{x}_{m+1} = \overline{x}_m \circ \partial_{M,s+m} = \partial_{M',m} \circ h_m \circ \partial_{M,s+m}$. So one can choose $\overline{x}_{m+1} := h_m \circ \partial_{M,s+m}$. But now $\partial_{M',m+1} \circ \overline{x}_{m+2} = \overline{x}_{m+1} \circ \partial_{M,s+m+1} = h_m \circ \partial_{M,s+m} \circ \partial_{M,s+m+1} = 0$. So $\overline{x}_{m+2} = \partial_{M',m+2} \circ h_{m+2}$. Etc. So $y \circ \overline{x}_{s'} = y \circ \partial_{M',s'} \circ h_{s'} = 0$ or $y \circ \overline{x}_{s'} = y \circ h_{s'-1} \circ \partial_{M,s+s'-1}$, which is a co-boundary.

To show that Yoneda cup-product is associative, i.e.

$$\mu_{M,M',M'''} \circ (Id_{Ext_A^{s,t}(M,M')} \otimes \mu_{M',M'',M'''}) = \mu_{M,M'',M'''} \circ (\mu_{M,M',M''} \otimes Id_{Ext_A^{s'',t''}(M'',M''')})$$

$$: Ext_A^{s,t}(M,M') \otimes Ext_A^{s',t'}(M',M'') \otimes Ext_A^{s'',t''}(M'',M''') \longrightarrow Ext_A^{s+s'+s'',t+t'+t''}(M,M''')$$

one uses the fact that $\overline{y}_{s''} \circ \overline{x}_{s'+s''}$ can be chosen for $\overline{(y \circ \overline{x}_{s'})}_{s''}$ [Note: We use here the same notations as on the previous page]. One uses standard arguments from homological algebra to show that the Yoneda cup-product does not depend on the choices of resolutions and is natural in M, M', M''.

Lemma 5.1 *Let $N' \subset A'$ be a right A' ideal and $M' \subset A'$ be a left A' ideal. Then there exists one-to-one F-linear map*

$$\psi : [N' \cdot M']^F \longrightarrow [N' \otimes M']^F$$

Proof: Fix $n'_1, n'_2, \ldots$ - a minimal generating set of N', as right A' ideal, and $m'_1, m'_2, \ldots$ - a minimal generating set of M', as left A' ideal. Any $x' \in N' \cdot M'$ can be represented as $x' = \sum_{i,j} n'_i \cdot a'_{i,j} \cdot m'_j$. Suppose, that $x' = \sum_{i,j} n'_i \cdot \alpha'_{i,j} \cdot m'_j$. Then

$$\sum_{i,j} n'_i \cdot a'_{i,j} \cdot m'_j = \sum_{i,j} n'_i \cdot \alpha'_{i,j} \cdot m'_j$$

$$\forall i \; , \quad \sum_{j} a'_{i,j} \cdot m'_j = \sum_{j} \alpha'_{i,j} \cdot m'_j$$

so $a'_{i,j} = \alpha'_{i,j}$ for all i, j. Hence, one defines [relative to the selected $\{n_i\}$ and $\{m_i\}$]

$$\psi(x') := \sum_{i,j} (n'_i \cdot a'_{i,j}) \otimes m'_j = \sum_{j} ((\sum_{i} n'_i \cdot a'_{i,j}) \otimes m'_j) \tag{5.3}$$

Lemma 5.2 *Let $N'' \subset N' \subset A'$ be right A' ideals and $M'' \subset M' \subset A'$ be left A' ideals, s.t. $N' \cdot I' \subset N''$ and $I' \cdot M' \subset M''$. Then ψ can be selected, to induce isomorphism of F-vector spaces*

$$\overline{\psi} : [N' \cdot M']^F / [(N' \cdot M'') + (N'' \cdot M')]^F \longrightarrow [N'/N'']^F \otimes [M'/M'']^F$$

Proof: Let $n'_1, n'_2, \ldots, n''_1, n''_2, \ldots$ be a minimal generating set of N', as right A' ideal, with $\{n'_1, n'_2, \ldots\}^F \cap [N'']^F = \{0\}$ and $n''_1, n''_2, \ldots \in N''$

and let $m'_1, m'_2, ..., m''_1, m''_2, ...$ be a minimal generating set of M', as left A' ideal, with $\{m'_1, m'_2, ...\}^F \cap [M'']^F = \{0\}$ and $m''_1, m''_2, ... \in M''$. One constructs such minimal generating set, by taking any generating set of N'', as right A'-ideal, union with the set of representatives, in N', of any F-vector space basis of $[N'/N'']^F$. Since this union generates N', as right A'-ideal, one can choose from it a minimal generating set. [And similarly for M'].

Now, any $x' \in N' \cdot M'$ can be uniquely represented as [$a'_{i,j}, b'_{i,j}, c'_{i,j}, d'_{i,j} \in A'$]

$$x' = \sum_{i,j} n'_i \cdot a'_{i,j} \cdot m'_j + \sum_{i,j} n'_i \cdot b'_{i,j} \cdot m''_j + \sum_{i,j} n''_i \cdot c'_{i,j} \cdot m'_j + \sum_{i,j} n''_i \cdot d'_{i,j} \cdot m''_j$$

If $x' \in (N' \cdot M'') + (N'' \cdot M')$, then all $a'_{i,j} \in I'$, hence $\psi(x') \in (N' \otimes M'') + (N'' \otimes M')$. So $\overline{\psi}$ is well defined. Since $\mu_{A'} \circ \psi = Id_{N' \cdot M'}$ and $\mu_{A'}((N' \otimes M'') + (N'' \otimes M')) \subset (N' \cdot M'') + (N'' \cdot M')$, $\overline{\psi}$ is one-to-one. [Note. We abuse the notation, by not specifying the restrictions in range and domain of $\mu_{A'}$]

Take any $y' \in [N'/N'']^F \otimes [M'/M'']^F$. Then $y' = \sum_{t,k} f_{t,k}([n'_t] \otimes [m'_k])$, where all $f_{t,k} \in F = A'_0$. Take $x' := \sum_{t,k} f_{t,k}(n'_t \cdot m'_k)$. Clearly, $\overline{\psi}([x']) = y'$, so $\overline{\psi}$ is onto.

Lemma 5.3 $B'_{i+j} \subset (Z'_i \cdot B'_j) + (B'_i \cdot Z'_j)$

Proof: For any $k < i$, $Z'_k \cdot Rel \cdot Z'_{n-k-1} \subset Z'_k \cdot Rel \cdot Z'_{i-k-1} \cdot Z'_{n-k} \subset B'_i \cdot Z'_j$.

For any $i \leq k$, $Z'_k \cdot Rel \cdot Z'_{n-k-1} \subset Z'_i \cdot\ Z'_{k-i} \cdot Rel \cdot Z'_{n-k-1} \subset Z'_i \cdot B'_j$.

$$I' \cdot Z'_{i+j} \subset I' \cdot Z'_i \cdot Z'_j \subset B'_i \cdot Z'_j \quad , \quad Z'_{i+j} \cdot I' \subset Z'_i \cdot Z'_j \cdot I' \subset Z'_i \cdot B'_j$$

Definition 5.4 *Define, for each choice of* $\psi : Z'_i \cdot Z'_j \to Z'_i \otimes Z'_j$ *, the corresponding*

$$\rho_{i,j} : Z'_{i+j} \xrightarrow{inclusion} Z'_i \cdot Z'_j \xrightarrow{\psi} Z'_i \otimes Z'_j$$

Lemma 5.5 *One can choose* $\rho_{i,j}$*, which induces some*

$$\nu_{i,j} : H_{i+j}(A) \to H_i(A) \otimes H_j(A)$$

$\nu_{i,j} :=$ *isomorphism* $\circ$ *quotient* $\circ$ *inclusion , in the following diagram:*

$$\begin{tikzcd}
Z'_{i+j} \arrow[rrr, "\rho_{i,j}"] \arrow[d, "projection"] & & & Z'_i \otimes Z'_j \arrow[d, "projection"] \\
H_{i+j} = Z'_{i+j}/B'_{i+j} \arrow[r, "inclusion"] & (Z'_i \cdot Z'_j)/B'_{i+j} \arrow[r, "quotient"] & (Z'_i \cdot Z'_j)/[(Z'_i \cdot B'_j) + (B'_i \cdot Z'_j)] \arrow[r, "isom"] & (Z'_i/B'_i) \otimes (Z'_j/B'_j) = H_i \otimes H_j
\end{tikzcd}$$

Theorem 5.6 *There is a choice of* $\nu_{j,i} : H_{j+i,\, d_2+d_1}(A) \to H_{j,\, d_2}(A) \otimes H_{i,\, d_1}(A)$*, s.t.* $TR \circ \nu_{j,i}$ *is dual to Yoneda pairing* $\mu_{F,F,F} : Ext_A^{i,\, d_1}(F,F) \otimes Ext_A^{j,\, d_2}(F,F) \to Ext_A^{i+j,\, d_1+d_2}(F,F)$ *. I.E.*

$$< ([x] \otimes [y]), TR(\nu_{j,i}([z])) >=< \mu_{F,F,F}([x] \otimes [y]), [z] > \tag{5.4}$$

$$\forall [x] \in Ext_A^{i,\, d_1}(F,F), \forall [y] \in Ext_A^{j,\, d_2}(F,F), \forall [z] \in H_{i+j,\, d_1+d_2}(A)$$

TR stands for unoriented transpose, i.e. $TR(a \otimes b) := b \otimes a$ *for* $a \in V, b \in W$*, where* V *and* W *are* F*-vector spaces.*

Proof: Select some F-basis $\{[x^1], [x^2], ...\}$ of $Ext_A^{i,d_1}(F, F)$ and some representatives $x^1 \in Hom_A^{d_1}(P_{F,i}, F), x^2 \in Hom_A^{d_1}(P_{F,i}, F), ...$ for all elements of this basis.

If one lifts each $s_t \in S_{i-1,d_1}$ to some $s'_t \in \Omega'_{i-1,d_1}$ and then applies pr^{i-1} to it, one gets elements $z'_t \in Z'_{i,d_1}$, s.t. $\{[z'_1], ...\}$ is a basis of $[Z'_{i,d_1}/B'_{i,d_1}]^F$. One can choose S_{i-1,d_1} in such a way, that $[x^h]([z'_t]) := [x^h]([1 \otimes s_t])$ is 1 if $h = t$ and 0 if $h \neq t$.

[Take any minimal generating set Θ, of $Ker(\partial_{i-1})_{d_1}$ Since $[x^1] \neq 0$, exists some $\theta_q \in \Theta$, s.t. $[x^1]([\theta_q]) = f \neq 0$. Assume, that $q = 1$. Then define $\theta'_1 := \frac{\theta_1}{f}$. Clearly, $[x^1]([\theta'_1]) = 1$. Now define $\theta'_t := \theta_t - [x^1]([\theta_t])\,\theta'_1$ for $t \neq 1$. Clearly, $[x^1]([\theta'_t]) = 0$. Each θ_t is a unique F-linear combination of $\theta'_1, ...$ and vice versa, hence $\{\theta'_1, ...\}$ is also a minimal generating set of $Ker(\partial_{i-1})_{d_1}$. Now, since $[x^2] \notin \{[x^1]\}^F$, exists some θ'_{q_2}, $q_2 > 1$, s.t. $[x^2]([\theta'_{q_2}]) = f_2 \neq 0$. Assume, that $q_2 = 2$. Define $\theta''_2 := \frac{\theta'_2}{f_2}$. Etc.]

With respect to the above mentioned choice of S_{i-1,d_1}, $\overline{(x^h)_0}(\sum_t a_t \otimes s_t) = a_h$. We define left A-module map $\xi_0 : A \otimes J_{-1} \to P_{F,0}$ by $(\xi_0 := \mu_A) : (A \otimes A) \to A$.

$$\text{Then} \quad \overline{(x^h)_0}(\sum_t a_t \otimes s_t) = \xi_0(a_h \otimes 1)$$

We now select some $\{\theta_1, ...\}$ - an F-basis of $[I]^F$ and select, for each θ_q , a representation $\theta_q = \sum_u (f_u g_{u,1} \cdot \ldots \cdot g_{u,k(u)})$. We define

$$\varphi(\theta_q) := \sum_u (f_u g_{u,1} \cdot \ldots \cdot g_{u,k(u)-1}) \otimes g_{u,k(u)}$$

We now extend F-linearly to $\varphi : [I]^F \to [A]^F \otimes G^F$.

For every $\sum_t a_t \otimes i_t \in A \otimes J_0$ we define a left A-module map

$$\xi_1(\sum_t a_t \otimes i_t) := \sum_t a_t \cdot \varphi(i_t) \quad \in P_{F,1}$$

[Notice, that $\partial_0 \circ \xi_1 = \xi_0 \circ (-\otimes 1) \circ d_{1,1}$]. Now, for every $(\sum_t a_t \otimes i_t \otimes s_t) \in P_{F,i+1}$, we can select

$$\overline{(x^h)}_1(\sum_t a_t \otimes i_t \otimes s_t) := \xi_1(a_h \otimes i_h)$$

Assume, that for all $m = 0, \ldots, n-1$, exists some left A-module map $\xi_m : (A \otimes J_{m-1}) \to P_{F,m}$, such that

$$(\partial_{m-1} \circ \xi_m = \xi_{m-1} \circ d_{m,1}) : (A \otimes J_{m-1}) \to P_{F,m-1}$$

and, for all $w \in P_{F,i+m}$, represented as $w = \sum_t (a_t \otimes v_t \otimes s_t)$, where $a_t \in A$, $v_t \in J_{m-1}$,

$$\overline{(x^h)}_m(\sum_t a_t \otimes v_t \otimes s_t) := \xi_m(a_h \otimes i_h)$$

We will make the inductive step, by constructing a left A-module map ξ_n, with these two properties. For $w \in J_{n-1} \subset A \otimes J_{n-1}$, we have

$$\partial_{n-2} \circ \xi_{n-1}(w) = \xi_{n-2} \circ d_{n-1,1}(w) = 0$$

Choose any F-basis $\{w_1, w_2, ...\}$ of $[J_{n-1}]^F$. For each w_u select $\vartheta_u \in P_{F,n}$, s. t. $\partial_{n-1}(\vartheta_u) = \xi_{n-1}(w_u)$. Now define

$$\xi_n(\sum_u a_u \otimes w_u) := \sum_u a_u \cdot \vartheta_u$$

$$\partial_{n-1} \circ \xi_n(\sum_u a_u \otimes w_u) = \partial_{n-1}(\sum_u (a_u \cdot \vartheta_u)) = \sum_u a_u \cdot \partial_{n-1}(\vartheta_u) =$$

$$= \sum_u a_u \cdot \xi_{n-1}(\vartheta_u) = \xi_{n-1}(\sum_u a_u \cdot w_u) = \xi_{n-1} \circ d_{n,1}(\sum_u a_u \otimes w_u)$$

It is clear, that $\overline{(x^h)}_n$ for $\sum_t (a_t \otimes v_t \otimes s_t) \in P_{F,i+n}$, where $v_t \in J_{n-1}$, can be defined as

$$\overline{(x^h)}_n(\sum a_t \otimes v_t \otimes s_t) := \xi_n(a_h \otimes v_h)$$

For any $w \in A \otimes J_0$, one can choose $w' \in proj^{-1}(w) \cap (A' \otimes J_0')$ and $w'' \in proj^{-1}(\xi_1(w)) \cap (A' \otimes \Omega_0')$ so, that $pr^1(w') = pr^1(w'')$. [For example, for $a \in A_0$, select $a' := a$, for each $a \in G$ fix some $a' \in G'$ and for $a = \theta_q$ select $a' := \sum_u (f_u g'_{u,1} \cdot ... \cdot g'_{u,k(u)})$. Now extend this lifting F-linearly to all $a \in A \otimes A$.] Assume, that for any $w \in A \otimes J_{n-1}$ one can choose $w' \in proj^{-1}(w) \cap (A' \otimes J'_{n-1})$ and $w'' \in proj^{-1}(\xi_n(w)) \cap (A' \otimes \Omega'_{n-1})$ so, that $pr^n(w') = pr^n(w'')$. We will now make the inductive step.

Let $\{w_1, w_2, ...\}$ be an F-basis of $[J_n]^F$, $u'_1, u'_2, ...$ be some choice of liftings of elements of S_n to Ω'_n and $u''_1, u''_2, ... \in \Omega'_n$ $[proj(u''_*) = 0]$ be such, that $\{u'_1, ..., u''_1, ...\}$ is an F-basis of Ω'_n. Then

$$w''_k = \sum_m b'_{k,m} \cdot u'_m + \sum_m b''_{k,m} \cdot u''_m$$

where $b'_{k,m}$ and $b''_{k,m}$ are in A'. Define

$$\vartheta'_k := \sum_m b'_{k,m} \otimes u'_m + \sum_m b''_{k,m} \otimes u''_m$$

Clearly, $\partial_n(proj(\vartheta'_k)) = \xi_n(w_k)$. So, for any $w = \sum_k a_k \otimes w_k$, we choose $w'' := \sum_k a'_k \cdot \vartheta'_k$ and $w' := \sum_k a'_k \otimes w'_k$. Clearly, $pr^{n+1}(w') = pr^{n+1}(w'')$.

Thus, for $[z] = [\sum_t 1 \otimes v_t \otimes s_t]$ in $H_{j+i,d_2+d_1}(A)$, where $v_t \in J_{j-1}$ and $s_t \in S_{i-1}$,

$$< ([x^h] \otimes [y]), TR(\nu_{j,i}([\sum_t 1 \otimes v_t \otimes s_t])) >= [y]([pr^n(v'_h)]) = [y]([pr^n(v''_h)]) =$$

$$= y(\xi_j(1 \otimes v_h)) = y(\overline{(x^h)}_j(\sum_t 1 \otimes v_t \otimes s_t)) =< \mu_{F,F,F}([x^h] \otimes [y]), [\sum_t 1 \otimes v_t \otimes s_t] >$$

where $v'_h \in proj^{-1}(1 \otimes v_h) \cap (A' \otimes J'_{j-1})$, $v''_h \in proj^{-1}(\xi_j(1 \otimes v_h)) \cap (A' \otimes \Omega'_{j-1})$.

The first equality in this formula will hold if selected minimal generating set of left A-module Z_i is: $\{z'_1, z'_2, ...\}$ from grading d_1, [we have defined them in the beginning of the proof] $\bigcup$ some set of elements of Z_i from other gradings $\bigcup$ some set of elements of B_i.

Remark 5.7 *$Ext^0_A(F,F) = F$ and one can show, that*

$$\mu_{F,F,F} : Ext^i_A(F,F) \otimes Ext^0_A(F,F) \to Ext^i_A(F,F)$$

and

$$\mu_{F,F,F} : Ext^0_A(F,F) \otimes Ext^j_A(F,F) \to Ext^j_A(F,F)$$

are identity maps.

From Theorem 5.6 and Lemma 5.5 follows, that the kernel and the co-kernel of

$$\mu_{F,F,F} : Ext^i_A(F,F) \otimes Ext^j_A(F,F) \to Ext^{i+j}_A(F,F)$$

are dual to, respectively, co-kernel and kernel of the composition

$$Z'_{j+i}/B'_{j+i} \xrightarrow{inclusion} (Z'_j \cdot Z'_i)/B'_{j+i} \xrightarrow{quotient} (Z'_j \cdot Z'_i)/[(Z'_j \cdot B'_i) + (B'_j \cdot Z'_i)]$$

Theorem 5.8 *Co-kernel of Yoneda cup-product*

$$\mu_{F,F,F} : Ext^i_A(F,F) \otimes Ext^j_A(F,F) \to Ext^{i+j}_A(F,F)$$

is equal to

$$Hom_F([(Z'_{j+i} \cap [(Z'_j \cdot B'_i) + (B'_j \cdot Z'_i)])/(B'_{j+i} \cap [(Z'_j \cdot B'_i) + (B'_j \cdot Z'_i)])]^F, F)$$

and kernel is equal to

$$Hom_F([(Z'_j \cdot Z'_i)/[Z'_{j+i} + (Z'_j \cdot B'_i) + (B'_j \cdot Z'_i)]]^F, F)$$

Theorem 5.9 *Assume that* G *is a minimal generating set of* A *and, hence,* $Rel \subset I' \cdot I'$.

1) If kernel of $\mu_{F,F,F} : Ext^1_A(F,F) \otimes Ext^1_A(F,F) \rightarrow Ext^2_A(F,F)$ *is* 0, *then*

$$I' \cdot I' = Z'_1 \cdot Z'_1 = Z'_2 + (Z'_1 \cdot B'_1) + (B'_1 \cdot Z'_1) = Rel$$

2) If kernel of $\mu_{F,F,F} : Ext^2_A(F,F) \otimes Ext^1_A(F,F) \rightarrow Ext^3_A(F,F)$ *is* 0, *then*

$$I' \cdot Rel = Z'_1 \cdot Z'_2 = Z'_3 + (Z'_1 \cdot B'_2) + (B'_1 \cdot Z'_2) = [(I' \cdot Rel) \cap (Rel \cdot I')] + (I' \cdot I' \cdot Rel)$$

In this case, $Rel = I'^m$ *for some* $2 \leq m \leq \infty$.

The same result follows if kernel of

$\mu_{F,F,F} : Ext^1_A(F,F) \otimes Ext^2_A(F,F) \rightarrow Ext^3_A(F,F)$ *is* 0.

3) If kernel of $\mu_{F,F,F} : Ext^2_A(F,F) \otimes Ext^2_A(F,F) \rightarrow Ext^4_A(F,F)$ *is* 0, *then*

$$Rel \cdot Rel = Z'_2 \cdot Z'_2 = Z'_4 + (Z'_2 \cdot B'_2) + (B'_2 \cdot Z'_2) = [(Rel \cdot Rel) \cap (I' \cdot Rel \cdot I')] + (Rel \cdot I' \cdot Rel)$$

This is the same as $(Rel \cdot Rel) \subset [(I' \cdot Rel \cdot I') + (Rel \cdot I' \cdot Rel)]$

Proof: We will only prove that, if $I' \cdot Rel = [(I' \cdot Rel) \cap (Rel \cdot I')] + (I' \cdot I' \cdot Rel)$, then $Rel = I'^m$. Clearly,

$([(I' \cdot Rel) \cap (Rel \cdot I')] + (I' \cdot I' \cdot Rel)) \subset [(Rel \cdot I') + (I' \cdot I' \cdot Rel)]$.

Choose some non-zero element $r \in Rel$, with smallest minimal weight m. For any $g_1' \in G'$ of smallest grading, $g_1' \cdot r = \sum_k r_{1,k} \cdot i_{1,k}' + \sum_h (i_{2,h}' \cdot i_{3,h}' \cdot r_{2,h})$ with all $i_{1,k}'$ homogeneous and different. But, due to min/max conditions, for some k, $r_{1,k}$ has minimal weight m and $i_{1,k}' \in G'$. Every homogeneous summand of weight m of every such $r_{1,k}$ has to start with g_1'. Define $r' := r_{1,k}$.

Continuing this process, like in the proof of the Lemma 4.20, yields $Rel = I'^m$.

For any F-vector space $V \subset [I']^F$, define $\pi_n(V) := \{\pi_n(v) \mid v \in V\}$. If we assume that G is a minimal generating set of A, then G' is a minimal generating set of Z_1' as right and as left A'-module and also is a representative set for an F-basis of $[Z_1'/B_1']^F$. For this case:

Definition 5.10 *We define $\kappa_n : H_n \to \pi_n([Z_n']^F)$ as follows: For any $[x] \in H_n$ take any representative z' in Z_n' of $\gamma_n([x])$. Since any element of B_n' has a minimal weight of $n+1$ or more, $\pi_n(z_n')$ does not depend on the choice of z'. Define $\kappa_n([x]) := \pi_n(z')$.*

One sees, that that for $[x] \in H_n(A)$, $\kappa_n([x])$ describes uniquely the $n-1$-iterated pairing

$$\nu_{1,1,\ldots,1}([x]) \in H_1(A) \otimes \ldots \otimes H_1(A)$$

and that and vise-versa this pairing fully describes $\kappa_n([x])$.

6 Hopf Algebras and Cup-co-Products in Homology

Definition 6.1 *Let A, B be connected F-algebras. We define the algebra structure on $A \otimes B$ by the formula*

$$(a1 \otimes b1) \cdot (a2 \otimes b2) = -1^{deg(a2)deg(b1)}(a1 \cdot a2) \otimes (b1 \cdot b2)$$

Definition 6.2 *Let A be a connected F-algebra and $\nu : A \longrightarrow A \otimes A$ be an algebra homomorphism, such, that for all $a \in A$ the projection of $\nu(a)$ into $A_0 \otimes A$ is $1 \otimes a + a \otimes 1$ and the projection into $A \otimes A_0$ is $a \otimes 1$. Then we call $\{A, \nu\}$ - a connected F-Hopf algebra. ν is called co-product of this Hopf algebra. We will suppress the mention of ν where possible. Likewise we will suppress the word "connected".*

Definition 6.3 *Let A be a Hopf algebra. Then: If $(Id_A \otimes \nu)(\nu(a)) = (\nu \otimes Id_A)(\nu(a))$, then A is called co-associative. If, for all $a \in A$, $\nu(a) = \sum_j a1_j \otimes a2_j = \sum_j -1^{deg(a1_j)deg(a2_j)} a2_j \otimes a1_j$ then A is called co-commutative. [Note: Unless otherwise stated, we do not assume Hopf algebras to be co-commutative or co-associative.]*

Definition 6.4 *Let A be a Hopf algebra and $I \subset A$ be a left [right] ideal of the underlying algebra. Then I is called a left [right] Hopf ideal of A if*
$\nu(I) \subset I \otimes A + A \otimes I$.

Definition 6.5 *Let* $\{A, \nu_A\}$ *and* $\{B, \nu_B\}$ *be Hopf algebras. If algebra* B *is a quotient algebra of* A*, with projection map* $proj : A \longrightarrow B$*, then* $\{B, \nu_B\}$ *is called quotient Hopf algebra of* $\{A, \nu_A\}$*, if* $(proj \otimes proj)(\nu_A(a)) = \nu_B(proj(a))$ *for all* $a \in A$.

Let $\{A, \nu_A\}$ be a Hopf algebra.

Lemma 6.6 *Given short exact sequence of left [right]* A*-modules*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

Then: I *is a Hopf ideal of* A $\Longleftrightarrow$ B *is a quotient Hopf algebra of* A

Definition 6.7 *Let* M, N *be left [right]* A*-modules. Then* $M \otimes N$ *becomes left [right]* A *module with left action*

$$a \cdot (m \otimes n) := \sum_j -1^{deg(a2_j)deg(m)}(a1_j \cdot m) \otimes (a2_j \cdot n)\ , \quad \nu(a) = \sum_j a1_j \otimes a2_j$$

[right action is $(m \otimes n) \cdot a := \sum_j -1^{deg(a1_j)deg(n)}(m \cdot a1_j) \otimes (n \cdot a2_j)$*] We will use* $\overline{\otimes}$ *to denote this diagonal action of* A.

Definition 6.8 *Let* A *be a free algebra on some [graded] set* G*. Then any degree preserving map of sets* $f : G \longrightarrow A \otimes A$*, such that the projection of* $f(g)$ *onto* $(A_0 \otimes A) \oplus (A \otimes A_0)$ *is* $(1 \otimes g) + (g \otimes 1)$*, extends to a unique co-product* $\nu^f : A \longrightarrow A \otimes A$.

Definition 6.9 *Let A be a Hopf algebra, G be a minimal generating set of A, as algebra, and $Fr_G(A)$ be a free algebra on G. Then, for each $g_j \in G$, we lift $g_j \longrightarrow \nu(g_j)$ to some $g'_j \longrightarrow x'_j = 1 \otimes g'_j + g'_j \otimes 1 + \sum_k i1'_{j,k} \otimes i2'_{j,k}$, where $(proj \otimes proj)(x_i) = \nu(g_i)$, and extend this lifting to some co-product on $Fr_G(A)$. Once such co-product has been selected, we call the resulting Hopf algebra A'. Rel, defined as Kernel of $proj : A' \longrightarrow A$, is, thus, a Hopf ideal of A'.*

We now investigate the conditions on a Hopf Algebra A, under which the product $M\overline{\otimes}N$ of two projective left A-modules M and N is a projective left A-module. We, first state some general results from homological algebra.

Lemma 6.10 *A left A-module P is projective if and only if it is a direct summand in some free left A-module.*

Proof: Suppose that P is a projective left A-module. Consider a free left A-module $V := A \otimes [P]^F$. There is a left A-module epimorphism $f : V \longrightarrow P$, defined by $f(\sum_i a_i \otimes p_i) := \sum_i a_i \cdot p_i$. Define left A-module homomorphism $g := Id_P : P \longrightarrow P$. Hence, exists left A-module homomorphism $h : P \longrightarrow V$, such, that $f \circ h = Id_P$. So P is a direct summand in V. Now suppose that P is a direct summand in some free left A-module V, with $projec : V \longrightarrow P$ and $incl : P \longrightarrow V$.

For any two left A-modules X and Y, A-module epimorphism $f : X \longrightarrow Y$ and A-module homomorphism $g : P \longrightarrow Y$, define $g1 := g \circ projec : V \longrightarrow Y$. There exists A-module homomorphism $h1 : V \longrightarrow X$, such, that $f \circ h1 = g1$. Define $h := h1 \circ incl : P \longrightarrow X$. Now $f \circ h = f \circ h1 \circ incl = g1 \circ incl = g \circ projec \circ incl = g$ So P is projective.

Lemma 6.11 *Let A be a Hopf algebra. If two left A-modules M^1 and M^2 are direct summands in left A-modules N^1 and N^2, then $M^1 \overline{\otimes} M^2$ is a direct summand in $N^1 \overline{\otimes} N^2$.*

Lemma 6.12 *Let A be a Hopf algebra. For any two free left A-modules $V^1 = A \otimes {S^1}^F$ and $V^2 = A \otimes {S^2}^F$, $V^1 \overline{\otimes} V^2$ is isomorphic, as left A-module, to $V = (A \overline{\otimes} A) \otimes (S^1 \times S^2)^F$ with A action given by*

$$b \cdot (a1 \overline{\otimes} a2) \otimes s := \sum_i -1^{deg(b2_i) deg(a1)} ((b1_i \cdot a1) \overline{\otimes} (b2_i \cdot a2)) \otimes s \ , \quad \nu(b) = \sum_i b1_i \otimes b2_i$$

Theorem 6.13 *Let A be a Hopf algebra. The product $M \overline{\otimes} N$ of two projective left A-modules M and N is a projective left A-module if and only if $A \overline{\otimes} A$ is a projective left A-module.*

Proof: The necessity of $A \overline{\otimes} A$ being projective is obvious, since $M = N = A$ is a projective left A-module.

Assume now that $A \overline{\otimes} A$ is projective, hence a direct summand in some free left A-module $A \otimes S^F$.

For any two projective left A-modules M^1 and M^2, exist some free left A-modules $V^1 = A \otimes S^{1^F}$ and $V^2 = A \otimes S^{2^F}$, in which M^1 and M^2 are, respectively, direct summands. So, $M^1 \overline{\otimes} M^2$ is a direct summand in $V^1 \overline{\otimes} V^2 = (A \overline{\otimes} A) \otimes (S^1 \times S^2)^F$, which is a direct summand in $(A \otimes S^F) \otimes (S^1 \times S^2)^F = A \otimes (S \times S^1 \times S^2)^F$, which is free. Hence $M^1 \overline{\otimes} M^2$ is a projective left A-module.

Theorem 6.14 *Let A be a co-associative Hopf algebra. Exists a unique F-linear map $\chi : [A]^F \to [A]^F$, called an antipode of A, such, that*

$$\mu_A \circ (Id_A \otimes \chi) \circ \nu_A = \mu_A \circ (\chi \otimes Id_A) \circ \nu_A = projection : A \to A_0$$

[Note: Here and elsewhere we abuse the notations by applying χ to A, rather than to $[A]^F$]

Proof: We will construct χ on elements of each A_n. For $a \in A_0$ define $\chi(a) := a$. Assume then χ was defined through A_n. For $a \in A_{n+1}$ one defines $\chi(a) := -a - \sum_i a1_i \cdot \chi(a2_i)$, where $\nu(a) = 1 \otimes a + a \otimes 1 + \sum_i a1_i \otimes a2_i$. One sees that $(Id_A \otimes \chi)(\nu(a)) = 1 \otimes (-a - \sum_i a1_i \cdot \chi(a2_i)) + \sum_i a1_i \otimes \chi(a2_i) + a \otimes 1$, and $\mu \circ (Id_A \otimes \chi) \circ \nu(a) = 0 = proj(a)$.

Now, for $a \in A_0$, $\mu \circ (Id_A \otimes \chi) \circ \nu(a) = \mu \circ (\chi \otimes Id_A) \circ \nu(a)$. Assume this equality until $a \in A_n$. Take any $a \in A_{n+1}$.

$a = \mu \circ (\mu \otimes Id_A) \circ (Id_A \otimes \chi \otimes Id_A) \circ (\nu \otimes Id_A) \circ \nu(a) =$

$\mu \circ (Id_A \otimes \mu) \circ (Id_A \otimes \chi \otimes Id_A) \circ (Id_A \otimes \nu) \circ \nu(a) =$ [by induction hypothesis] $= a + \mu \circ (\chi \otimes Id_A) \circ \nu(a)$. Hence $proj(a) = 0 = \mu \circ (\chi \otimes Id_A) \circ \nu(a)$.

The uniqueness of χ is obvious for $a \in A_0$ and is proved by induction on grading.

Theorem 6.15 *For any co-associative Hopf algebra A, $A\overline{\otimes}A$ is isomorphic, as left A-module, to a free left A-module $A \otimes [A]^F$.*

Proof: Consider the F-linear maps $f : A\overline{\otimes}A \to A \otimes [A]^F$, $f(a\overline{\otimes}b) := \sum_i a1_i \otimes (\chi(a2_i) \cdot b)$, and $g : A \otimes [A]^F \to A\overline{\otimes}A$, $g(a \otimes b) := \sum_i a1_i \otimes (a2_i \cdot b)$. Here $\nu(a) = \sum_i a1_i \otimes a2_i$. Since

$$a = \mu \circ (\mu \otimes Id_A) \circ (Id_A \otimes \chi \otimes Id_A) \circ (\nu \otimes Id_A) \circ \nu(a) =$$

$$= \mu \circ (Id_A \otimes \mu) \circ (Id_A \otimes \chi \otimes Id_A) \circ (Id_A \otimes \nu) \circ \nu(a)$$

one gets that $(g \circ f)(a\overline{\otimes}b) = a\overline{\otimes}b$ and $(f \circ g)(a \otimes b) = a \otimes b$.

We have thus showed that if A is a co-associative Hopf algebra, then the $\overline{\otimes}$ product of projective left A-modules is projective. The antipode χ has a crucial role in the theory of Hopf algebras. If one requires the Hopf algebra to be commutative or co-commutative, then the antipode behaves very much like the inverse map in the group theory. See [11, pages 5-13 and 70-75] and [8]. Actually, if one studies homology or cohomology theory of an H-space [for example, of a topological group] over a field, [which is naturally a Hopf algebra over that field] then the antipode χ is indeed induced by the inverse map of this H-space [topological group]. See [12, pages 25-26] and [16, pages 142-155].

Let A be a co-associative Hopf algebra, M and N be two left-A modules and $\{P_i^M, \partial_i^M\}$ and $\{P_i^N, d_i^N\}$ be some projective resolutions of M and N. We will construct a projective resolution on $M\overline{\otimes}N$. Define:

$$P_i^{M\overline{\otimes}N} := \bigoplus_{k=0}^{i} P_k^M \overline{\otimes} P_{i-k}^N$$

$$\partial_i^{M\overline{\otimes}N} := \bigoplus_{k=0}^{i} (-1^{k+1} Id_{P_k^M} \overline{\otimes} \partial_{i-k}^N + \partial_k^M \overline{\otimes} Id_{P_{i-k}^N})$$

$$\partial_{-1}^{M\overline{\otimes}N} := \partial_{-1}^M \overline{\otimes} \partial_{-1}^N$$

One checks by direct calculations that these formulae produce a long exact sequence of projective left A-modules and left A-homomorphisms.

We are now interested in the case, where $M = N = F$.

The isomorphism $\nu_F : F \to F\overline{\otimes}F$, which is defined by $\nu_F(f) := f(1\overline{\otimes}1)$, gives rise to epimorphism $(\nu_F \circ \partial_{-1}^F) : P_0^F \to F\overline{\otimes}F$, which can be lifted to some $\nu_{P_0^F} : P_0^F \to P_0^{F\overline{\otimes}F}$.

The image of the homomorphism $\nu_{P_0^F} \circ \partial_0^F : P_1^F \to P_0^{F\overline{\otimes}F}$ is a subset of $Ker(\partial_{-1}^{F\overline{\otimes}F}) = Im(\partial_0^{F\overline{\otimes}F})$, so $(\nu_{P_0^F} \circ \partial_0^F) : P_1^F \to P_0^{F\overline{\otimes}F}$ lifts to some $\nu_{P_1^F} : P_1^F \to P_1^{F\overline{\otimes}F}$.

Repeating this process yields family of left A-module homomorphisms $\nu_{P_n^F} : P_n^F \to P_n^{F\overline{\otimes}F}$, s.t.

$$\partial_{n-1}^{F\overline{\otimes}F} \circ \nu_{P_n^F} = \nu_{P_{n-1}^F} \circ \partial_{n-1}^F$$

Definition 6.16 *We will call any choice of family of homomorphisms* $\{\nu_{P_n^F} : P_n^F \to P_n^{F\overline{\otimes}F}\}$ *"cup-co-product" or "cup-co-product on the chain level".*

Let $\{\nu 1_{P_n^F} : P_n^F \to P_n^{F\overline{\otimes}F}\}$ and $\{\nu 2_{P_n^F} : P_n^F \to P_n^{F\overline{\otimes}F}\}$ represent any two families of "cup-co-product". Then

$$(\partial_{-1}^{F\overline{\otimes}F} \circ \nu 1_{P_0^F} = \nu_F \circ \partial_{-1}^F = \partial_{-1}^{F\overline{\otimes}F} \circ \nu 2_{P_0^F}) : P_0^F \to F\overline{\otimes}F$$

So, exists some left A-module homomorphism $\Gamma_{P_0^F} : P_0^F \to P_1^{F\overline{\otimes}F}$, s.t.

$$\partial_0^{F\overline{\otimes}F} \circ \Gamma_{P_0^F} = (\nu 2_{P_0^F} - \nu 1_{P_0^F}) : P_0^F \to P_0^{F\overline{\otimes}F}$$

Now

$$(\partial_0^{F\overline{\otimes}F} \circ (\nu 2_{P_1^F} - \nu 1_{P_1^F}) = (\nu 2_{P_0^F} - \nu 1_{P_0^F}) \circ \partial_0^F = \partial_0^{F\overline{\otimes}F} \circ \Gamma_{P_0^F} \circ \partial_0^F) : P_0^F \to F\overline{\otimes}F$$

and one selects some left A-module homomorphism $\Gamma_{P_1^F} : P_1^F \to P_2^{F\overline{\otimes}F}$, s.t.

$$\partial_1^{F\overline{\otimes}F} \circ \Gamma_{P_1^F} = \nu 2_{P_0^F} - \nu 1_{P_0^F} - \Gamma_{P_0^F} \circ \partial_0^F$$

Inductively continuing this process, one constructs left A-homomorphisms $\Gamma_{P_n^F} : P_n^F \to P_{n+1}^{F\overline{\otimes}F}$, such, that $\partial_n^{F\overline{\otimes}F} \circ \Gamma_{P_n^F} = \nu 2_{P_{n-1}^F} - \nu 1_{P_{n-1}^F} - \Gamma_{P_{n-1}^F} \circ \partial_{n-1}^F$ or, shortly $\partial \circ \Gamma + \Gamma \circ \partial = \nu 2 - \nu 1$.

The composition homomorphism

$$P_n^F \xrightarrow{\nu_{P_n^F}} P_n^{F\overline{\otimes}F} \xrightarrow{projection} \bigoplus_{k=0}^{n} (P_k^F/(I \cdot P_k^F))\overline{\otimes}(P_{n-k}^F/(I \cdot P_{n-k}^F))$$

is 0 on elements from $I \cdot P_n^F$, hence induces

$$\overline{\nu_{P_n^F}} : (P_n^F/(I \cdot P_n^F)) \to \bigoplus_{k=0}^{n} (P_k^F/(I \cdot P_k^F))\overline{\otimes}(P_{n-k}^F/(I \cdot P_k^F))$$

By taking the minimal resolution P_n^F, we define

$$(\nu_{H_n} := \overline{\nu_{P_n^F}}) : H_n \to \bigoplus_{k=0}^{n} H_k \otimes H_{n-k}$$

Since $\partial_n^{F\overline{\otimes}F} \circ \Gamma_{P_n^F} + \Gamma_{P_{n-1}^F} \circ \partial_{n-1}^F$ takes elements of P_n^F into $I \cdot P_n^{F\overline{\otimes}F}$, the ν_{H_n} is independent of the choices, made for $\nu_{P_n^F}$.

Definition 6.17 *The family of homomorphisms*

$$\{\nu_{H_n} : H_n \to \bigoplus_{k=0}^{n} H_k \otimes H_{n-k}\}$$

is called cup-co-product.

One could construct this cup-co-product in homology from any resolution, but that would require additional technics from homological algebra.

Lemma 6.18 $\nu_{H_n} : H_n \to \bigoplus_{k=0}^{n} H_k \otimes H_{n-k}$ *is co-commutative and co-associative.*

Proof: If $\nu_{P_n^F}(x) = \bigoplus_j x1_j \overline{\otimes} x2_j$, then $\nu_{P_n^F}^{TR}(x) := \bigoplus_j -1^{deg(x1_j)deg(x2_j)}(x2_j \overline{\otimes} x1_j)$ is also a "cup-co-product", hence must induce the same ν_{H_n}.

Hence co-commutativity of ν_{H_n}.

The associativity follows from the fact, that $\{\bigoplus_{k1+k2+k3=n} P_{k1}^F \overline{\otimes} P_{k2}^F \overline{\otimes} P_{k3}^F\}$, with appropriate differentials, is also a projective resolution of F. One repeats our argument to show that any two choices for the family of left A-module homomorphisms

$$\nu_{P_n^F}^2 : P_n^F \longrightarrow \bigoplus_{k1+k2+k3=n} P_{k1}^F \overline{\otimes} P_{k2}^F \overline{\otimes} P_{k3}^F$$

differ by some $d \circ \Gamma + \Gamma \circ d$, hence induce the same $\nu^2_{H_n}$. So $(\bigoplus_{k=0}^{n} Id_{P_k^F} \overline{\otimes} \nu_{P_{n-k}^F}) \circ \nu_{P_n^F}$ and $(\bigoplus_{k=0}^{n} \nu_{P_{n-k}^F} \overline{\otimes} Id_{P_k^F}) \circ \nu_{P_n^F}$ induce the same $\nu^2_{H_n}$. Hence co-associativity of ν_{H_n} .

We now proceed with briefly describing the Steenrod Squaring operations in the cohomology of co-associative and co-commutative Hopf algebras over a field with characteristics 2. Those results are due to [6]. The comprehensive treatment of this subject is given in [11].

Let $\nu_{P_n^F}(x) = \bigoplus_j x1_j \overline{\otimes} x2_j$ be a "cup-co-product". Then

$$TR(\nu_{P_n^F}(x)) = \bigoplus_j x2_j \overline{\otimes} x1_j$$

is also a "cup-co-product", and

$$\partial_{-1}^{F\overline{\otimes}F}(\nu_{P_0^F}(x) + TR(\nu_{P_0^F}(x))) =$$

$$= \partial_{-1}^{F\overline{\otimes}F}(\nu_{P_0^F}(x)) + \partial_{-1}^{F\overline{\otimes}F}(TR(\nu_{P_0^F}(x))) = \partial^F_{-1}(x)(1\overline{\otimes}1) + \partial^F_{-1}(x)(1\overline{\otimes}1) = 0$$

Hence we can choose some $D_{1,P_0^F} : P_0^F \to P_1^{F\overline{\otimes}F}$, s.t.

$$(\partial_0^{F\overline{\otimes}F} \circ D_{1,P_0^F} := \nu_{P_0^F} + TR(\nu_{P_0^F})) : P_0^F \to P_0^{F\overline{\otimes}F}$$

Hence

$$\partial_0^{F\overline{\otimes}F}(\nu_{P_1^F}(x) + TR(\nu_{P_1^F}(x)) + D_{1,P_0^F}(\partial_0^F(x))) = \partial_0^{F\overline{\otimes}F}(\nu_{P_1^F}(x)) + \partial_0^{F\overline{\otimes}F}(TR(\nu_{P_1^F}(x))) +$$

$$+ (\nu_{P_0^F}(\partial_0^F(x)) + TR(\nu_{P_0^F}(\partial_0^F(x)))) = \nu_{P_1^F}(\partial_0^F(x)) + TR(\nu_{P_1^F}(\partial_0^F(x))) + \nu_{P_0^F}(\partial_0^F(x)) +$$

$$+ TR(\nu_{P_0^F}(\partial_0^F(x))) = 0$$

Now we can choose some $D_{1,P_1^F} : P_1^F \to P_2^{F\overline{\otimes}F}$, s.t.

$$(\partial_1^{F\overline{\otimes}F} \circ D_{1,P_1^F} := \nu_{P_1^F} + TR(\nu_{P_1^F}) + D_{1,P_0^F} \circ \partial_0^F) : P_1^F \to P_1^{F\overline{\otimes}F}$$

By induction, define $D_{1,P_n^F} : P_n^F \to P_{n+1}^{F\overline{\otimes}F}$, s.t.

$$(\partial_n^{F\overline{\otimes}F} \circ D_{1,P_n^F} := \nu_{P_n^F} + TR(\nu_{P_n^F}) + D_{1,P_{n-1}^F} \circ \partial_{n-1}^F) : P_n^F \to P_n^{F\overline{\otimes}F}$$

Such a family $\{D_{1,P_n^F}\}_{n=0}^{\infty}$ is called a cup$_1$-co-product.

Observe that

$$\partial_0^{F\overline{\otimes}F} \circ D_{1,P_0^F} = \nu_{P_0^F} + TR(\nu_{P_0^F}) = \partial_0^{F\overline{\otimes}F} \circ TR \circ D_{1,P_0^F}$$

Hence

$$\partial_0^{F\overline{\otimes}F} \circ (D_{1,P_0^F} + TR \circ D_{1,P_0^F}) = 0$$

And exists some $D_{2,P_0^F} : P_0^F \to P_2^{F\overline{\otimes}F}$, s.t.

$$\partial_1^{F\overline{\otimes}F} \circ D_{2,P_0^F} = D_{1,P_0^F} + TR \circ D_{1,P_0^F}$$

One then proceeds to construct a family $\{D_{2,P_n^F}\}_{n=0}^{\infty}$, which is called cup$_2$-co-product. In a general case the cup$_k$-co-products are constructed inductively [first induction on n and then on k] by $D_{k,P_{-1}^F} := 0$ and

$$\partial_{n+k-1}^{F\overline{\otimes}F} \circ D_{k,P_n^F} := D_{k-1,P_n^F} + TR \circ D_{k-1,P_n^F} + D_{k,P_{n-1}^F} \circ \partial_{n-1}^F$$

Here $D_{0,P_n^F} := \nu_{P_n^F}$

For $[x] \in H^n(A)$ define

$$SQ^t([x]) := [(x \otimes x) \circ D_{n-t,P^F_{n+t}}] \quad \in H^{n+t}(A)$$

It can be proved that the SQ^t are well defined, independent of all the choices, made in cup$_k$-co-products, and F-linear.

Remark 6.19 *Let B be a co-associative and co-commutative co-algebra over a finite field F with characteristics 2. [i.e. F-vector space, equipped with co-associative and co-commutative F-linear co-product map and with left and right co-unit]. If one selects $b_1, b_2 \ldots$, an F-basis of $[B]^F$, one can represent the co-product as:*

$$\nu_B(b_j) = \sum_{k,t} f_{j,k,t}(b_k \otimes b_t)$$

One defines

$$V(b_j) := \sum_{m} \sqrt{f_{j,m,m}}\, b_m$$

and extends it F_2-linearly to all $[B]^F$.

Note $\sqrt{x}$ is well defined and F-linear for a finite field with char. 2

Then for all $x \in B$,

$$\nu_B(x) = V(x) \otimes V(x) \ + y \otimes z + z \otimes y$$

for some $y, z \in B$. The F-linear map V is called ***Verschiebung****. It is not degree preserving. It can be shown, that $V : H_0(A) \to H_0(A)$ is dual to $SQ^0 : H^0(A) \to H^0(A)$. See [11]*

7 Applications to the Mod 2 Steenrod Algebra.

In this Section we always assume $F = F_2$, and A is the Steenrod algebra of stable cohomology operations for the ordinary cohomology theory. A is given by the [not minimal] set of generators $\{Sq^n \in A_n\}_{n=0}^{\infty}$, where $Sq^0 = 1$, and relations

$$Sq^nSq^m + \sum_{k=0}^{[n/2]} \left(\frac{(m-k-1)!}{(n-2k)!(m+k-n-1)!} \right) Sq^{n+m-k}Sq^k = 0 \ , \quad n < 2m$$

called *Adem Relations.* Here $[-]$ represents the integer part. The dot is omitted in all the multiplications.

The standard references in the literature for the construction of the Steenrod algebra and for the development of it's properties are: [9], [7], [12], [13, Chapter 16], [14].

The Steenrod Algebra A is also equipped with the co-product $\nu_A : A \longrightarrow A \otimes A$, given by the *Cartan formula*

$$\nu_A(Sq^n) := \sum_{i=0}^{n} Sq^i \otimes Sq^{n-i}$$

It is a tedious calculation, requiring some combinatorial identities, to show, by direct plugging in, that the Adem relations generate Hopf ideal. In the most of the above mentioned literature, however, this fact is proved by utilizing the topological background, leading to the construction of the Steenrod algebra.

Note that the co-product, induced by the Cartan formula is co-commutative and co-associative.

By analyzing Adem relations, one can show that any element $Sq^{2^m} \in I$ is not decomposable as a sum of products of other Squares, i.e. $Sq^{2^m} \notin I \cdot I$, and any other Sq^n is decomposable as a sum of such products.

By filtering the Steenrod algebra by "degree": $d(Sq^{i_1}Sq^{i_2}...Sq^{i_n}) := i_1 + i_2 + ... + i_n$ and applying "moment": $m(Sq^{i_1}Sq^{i_2}...Sq^{i_n}) := 1i_1 + 2i_2 + ... + ni_n$, and "length": weight in our article, to the products of Squares, one shows that the "admissible sequences" : $Sq^{i_1}Sq^{i_2}...Sq^{i_n}$ with $i_k \geq 2i_{k+1}\ \forall k$, generate A as F_2-vector space. It can be shown by induction on the length [weight] of admissible sequences, using the Cartan formula, that admissible sequences are F_2-linearly independent. One defines the "excess" for an admissible sequence

$$e(Sq^{i_1}Sq^{i_2}...Sq^{i_n}) := 2i_1 - d(Sq^{i_1}Sq^{i_2}...Sq^{i_n}) =$$

$$= (i_1 - 2i_2) + (i_2 - 2i_3) + ... + (i_{n-1} - 2i_n) + (i_n - 0)$$

Following Milnor's approach, one then studies the *dual* algebra $A^* := Hom^*(A, F_2)$. Since the product of A^* is dual to the co-product of A [i.e. $(\mu_{A^*}(x \otimes y))(a) := (x \otimes y)(\nu_A(a))$, $x, y \in A^*$, $a \in A$] , it is commutative. One then shows that A^* is a polynomial algebra on generators ξ_k, where $\xi_k \in A^*$ is dual to $Sq^{2^{k-1}} Sq^{2^{k-2}} ... Sq^1$ with respect to the F_2-additive basis of admissible monomials, i.e. $\xi_k(Sq^{i_i}...Sq^{i_n}) = 1$ if and only if $i_1 = 2^{k-1}$, ... , $i_n = 1$.

Since the products $\xi_1^{n_1}\xi_2^{n_2}...\xi_k^{n_k}$ form an F_2-additive basis of A^*, their duals $Sq^{(n_1,n_2,...,n_k)}$ form an F_2-additive basis of A. It follows from Cartan formula that $\xi_1^n(Sq^n) = (\xi_1^{\otimes n})(\nu_A^{n-1}(Sq^n)) = 1$ and $\xi_1^m(Sq^n) = (\xi_1^{\otimes m})(\nu_A^{m-1}(Sq^n)) = 0$ if $m \neq n$. It can be shown that $\xi_k(Sq^m) = 0$ if $k > 1$. So $Sq^{(n)} = Sq^n$.

The co-product of A^*, which is dual to the product of A, is given by

$$\nu_{A^*}(\xi_n) = \sum_{i=0}^{n} \xi_{k-i}^{2^i} \otimes \xi_i$$

Milnor in [7] uses this fact to develop the formula for the product of $Sq^{(n_1,...,n_i)}$ and $Sq^{(m_1,...,m_j)}$. This formula is also re-printed in [14].

For $t > 0$, let $J_{[t-1]}$ denote the two-sided ideal of A^*, generated by $\{\xi_1^{2^t}, \xi_2^{2^{t-1}}, ..., \xi_t^2, \xi_{t+1}, ...\}$. By applying the co-product to the generators of $J_{[t-1]}$ one sees that $J_{[t-1]}$ is a Hopf ideal. So $C_{[t-1]} := A^*/J_{[t-1]}$, which is a finite commutative algebra, generated by $\{\xi_1, \xi_2, ..., \xi_t\}$, with relations $\xi_k^{2^{t+1-k}} = 0$, is a quotient Hopf algebra of A^*. Hence $A_{[t-1]}$, the dual of $C_{[t-1]}$, is a sub Hopf algebra of A. Now $Sq^i \in A_{[t-1]}$ if and only if $\xi_1^i \notin J_{[t-1]}$. It can be shown that $A_{[t-1]}$ is generated, as algebra, by all Sq^{2^i}, $0 \le i \le t-1$.

Lemma 7.1 *[9],[12] Every element in A is nilpotent, i.e. $\forall a \in A$, $a^n = 0$ for some number n.*

Proof: Every element $a \in A$ belongs to some $A_{[m]}$. But every $A_{[m]}$ is finite, hence is 0 from some grading on.

Lemma 7.2 *Let $x \in A^*$ and $Sq^{i_1}...Sq^{i_n} \in A$, with all i_k are even. Then $(x^2)(Sq^{i_1}...Sq^{i_n}) = x(Sq^{(i_1/2)}...Sq^{(i_n/2)})$ and $(x^2)(Sq^{i_1}...Sq^{i_n}) = 0$ if there is a k with odd i_k.*

Proof: $(x^2)(Sq^{i_1}...Sq^{i_n}) = (x \otimes x)(\nu_A(Sq^{i_1}...Sq^{i_n}) =$

$= \sum_{j1_k+j2_k=i_k} x(Sq^{j1_1}...Sq^{j1_n})x(Sq^{j2_1}...Sq^{j2_n})$.

Since $F = F_2$, all the terms will cancel, except then $j1_k = j2_k = \frac{i_k}{2}$.

Since A^* is commutative, the Frobenius map $\alpha^* : A^* \to A^*$, defined by $\alpha^*(x) := x^2$ is an algebra homomorphism. It commutes with any algebra homomorphism $f : A^* \to B$, i.e. $f(\alpha^*(x)) = f(x)^2$, hence it commutes with coproduct of A^*. So α^* is a Hopf algebra homomorphism. The dual Hopf algebra homomorphism $\alpha : A \to A$, due to the Lemma 7.2, is given by $\alpha(Sq^{(2i)}) = Sq^i$ and $\alpha(Sq^{(2i+1)}) = 0$. It can be shown that α is precisely, the annihilator of the two-sided ideal, generated by Sq^1. [See [12, pages 24,25].] From the Cartan formula it is clear that α is a Verschiebung V of the Steenrod algebra.

If one chooses $\{Sq^{2^n}\}_{n=0}^{\infty}$ as the generating set of A, the minimal generating set of relations was shown by Wall in [14] to be:

$$(A) \qquad (Sq^{2^j}Sq^{2^i} + Sq^{2^i}Sq^{2^j}) \in A_{[i-1]}\,, \quad 0 \le j \le i-2 \qquad (7.5)$$

$$(B) \quad (Sq^{2^i}Sq^{2^i} + Sq^{2^{i-1}}Sq^{2^i}Sq^{2^{i-1}} + Sq^{2^{i-1}}Sq^{2^{i-1}}Sq^{2^i}) \in A_{[i-1]} \qquad (7.6)$$

The arguments of [14, Sections 4,5] can be repeated for $A_{[n]}$ to show that $A_{[n]}$ has $\{Sq^{2^j}\}_{j=0}^{n}$ as a minimal generating set and relations (A) and (B), with $i \le n$, as a minimal [generating] set of relations.

We now define A' to be the free algebra generated by [graded] generators $\{s'_0, ..., s'_{2^i}, ...\}$ where s'_{2^i} is of the grading 2^i. The projection $proj : A' \to A$ is defined by $proj(s'_{2^i}) = Sq^{2^i}$. We also define $A'_{[n]} \subset A'$ as a free algebra on $s'_1, ..., s'_{2^n}$. The projection $proj$ restricts to $proj : A'_{[n]} \to A_{[n]}$.

In is customary in the literature to denote by $h_k \in H^{1,2^k}(A)$ a class in a first cohomology group of A, of grading 2^k, which is 1 on $[1 \otimes Sq^{2^k}] \in H_1(A)$ and 0 on $[1 \otimes Sq^{2^m}] \in H_1(A)$, $m \neq k$. h_k are also regarded [by abuse of notations] as elements of $H^{1,2^k}(A_{[n]})$, $k \leq n$.

Furthermore, due to the first (B) relation $Sq^1 Sq^1 = 0$, the element

$$1 \otimes Sq^1 \otimes Sq^1 \otimes ... \otimes Sq^1 \in A \otimes I^{\otimes m}$$

will belong to P_m, for all m, by induction on m. This element is not in $I \cdot P_m$, hence it represents nonzero class in $H_m(A)$ [and, similarly, in all $H_m(A_{[n]})$]. Since it is of weight m and since it is the unique element of degree m, it is dual to h_1^m. In other words

$$\nu_{1,1,...,1}([1 \otimes Sq^1 \otimes Sq^1 \otimes ... \otimes Sq^1]) = [1 \otimes Sq^1] \otimes [1 \otimes Sq^1] \otimes ... \otimes [1 \otimes Sq^1]$$

We will now apply our methods to $A_{[1]}$ and $A_{[2]}$. Their cohomology algebras are given in [10] on page 107 and pages 109,110 respectively.

$$H^*(A_{[1]}) = F_2[h_0,\ h_1,\ u_0,\ \omega_0]/(h_0 h_1,\ h_1^3,\ h_1 u_0,\ u_0^2 + h_0 \omega_0)$$

Here the generators $u_0 \in H^{3,7}(A_{[1]})$ and $\omega_0 \in H^{4,12}(A_{[1]})$.

Note: The representation $u_0 = a_2^2 b + a_1 b^2$, given in [10], means that, if one takes any $x \in P_3$, and denotes by $\sum_i x1_i \overline{\otimes} x2_i \overline{\otimes} x3_i$ the $P_1 \overline{\otimes} P_1 \overline{\otimes} P_1$ direct summand of $\nu^2_{P_3}(x)$, for any choice of $\nu^2_{P_m}$, then

$$\sum_i (\xi_1^2(x1_i))(\xi_1^2(x2_i))(\xi_2(x3_i)) + \sum_i (\xi_1(x1_i))(\xi_2^2(x2_i))(\xi_2^2(x3_i)) = u_0([x])$$

We now proceed to construct all the elements of $H_2(A_{[1]})$ and $H_3(A_{[1]})$.

The elements $r1 := s'_1 s'_1 \in Rel$ and $r2 := s'_2 s'_2 + s'_1 s'_2 s'_1 \in Rel$ are a minimal generating set of *Rel* [by Wall's arguments]. So, they correspond to two different, nonzero elements of $H_2(A_{[1]})$. Also $\kappa_2([r1]) = s'_1 s'_1$ and $\kappa_2([r2]) = s'_2 s'_2$, hence $\nu_{1,1}([r1]) = [s'_1] \otimes [s'_1]$ and $\nu_{1,1}([r2]) = [s'_2] \otimes [s'_2]$.

In $I' \cdot Rel \cap Rel \cdot I'$ there are following two elements:

$$q1 := s'_1 s'_1 s'_1 = s'_1(s'_1 s'_1) = (s'_1 s'_1) s'_1$$

and

$$q2 := s'_1 s'_2 (s'_2 s'_2 + s'_1 s'_2 s'_1) + s'_1 (s'_2 s'_2 + s'_1 s'_2 s'_1) s'_2 + s'_2 s'_1 s'_2 (s'_1 s'_1) =$$

$$= s'_2 (s'_2 s'_2 + s'_1 s'_2 s'_1) s'_1 + (s'_2 s'_2 + s'_1 s'_2 s'_1) s'_2 s'_1 + (s'_1 s'_1) s'_2 s'_1 s'_2 =$$

$$= s'_2 s'_1 s'_2 s'_1 s'_1 + s'_1 s'_1 s'_2 s'_1 s'_2 + s'_1 s'_2 s'_1 s'_2 s'_1$$

To show that $q2 \notin I' \cdot Rel \cdot I' + Rel \cdot Rel$, one notices that the only "part" of the homogeneous term $s'_1 s'_2 s'_1 s'_2 s'_1$, which appears as a homogeneous part of $r1$ or $r2$ is $s'_1 s'_2 s'_1$. Multiplying it from both sides by elements of I' or multiplying it from any side by any homogeneous term of any element of *Rel* will not produce $s'_1 s'_2 s'_1 s'_2 s'_1$. Hence $q2$ represents nonzero element in $H_3(A_{[1]})$. It's degree is 7. Hence this element is the dual of u_0. Since $\kappa_3([q2]) = 0$, $\nu^2_{H_3}([q2])$ has 0 as it's $H_1 \otimes H_1 \otimes H_1$ direct summand. So, dually, u_0 is not a product of h_0's and h_1's.

To show that $h_1 h_1 h_1 = 0$ in $H^{3,6}(A_{[1]})$, one, dually, shows that does not exist $[x] \in H_{3,6}(A_{[1]})$, s.t. $\kappa_3([x]) = s'_2 s'_2 s'_2 + \ldots$ In other words, for all $q \in (I' \cdot Rel) \cap (Rel \cdot I')$, $\pi_3(q) \neq s'_2 s'_2 s'_2 + \ldots$ But $s'_2 s'_2$ appears as a homogeneous part of $r2 \in Rel$ *only*. And $s'_2 r2 \in I' \cdot Rel$ will have a homogeneous term $s'_2 s'_1 s'_2 s'_1$. The only "part" of this term, which appears in any relation from *Rel* is $s'_1 s'_2 s'_1$ in $r2$, so $s'_2 s'_1 s'_2 s'_1$ will not be present in any element of $Rel \cdot I'$.

The cohomology of $A_{[2]}$ is described in [10]. $H^3(A_{[2]})$ is spanned, as F-vector space by:

$$h_0^3 \, , \; h_1^3 = h_0^2 h_2 \, , \; \alpha_1 \, , \; \alpha_2 \, , \; \alpha_3$$

The image of the element u_0 in $H^{3,7}(A_{[2]})$ is 0 and one says that u_0 does not "survive" to the cohomology of $A_{[2]}$.

The element $\alpha_1 \in H^{3,11}(A_{[2]})$ is also known to "survive" to a nonzero element $c_0 \in H^{3,11}(A)$. We also have that $h_2^3 = 0$.

We first notice two new elements $r3 := s'_4s'_1 + s'_1s'_4 + s'_2s'_1s'_2$ and $r4 := s'_4s'_4 + s'_2s'_4s'_2 + s'_2s'_2s'_4$ in the minimal generating set of *Rel.*

So

$$q2 + s'_1(s'_4s'_1 + s'_1s'_4 + s'_2s'_1s'_2)s'_1 = s'_1s'_4s'_1s'_1 + s'_2s'_1s'_2s'_1s'_1 + s'_1s'_1s'_2s'_1s'_2 + s'_1s'_1s'_4s'_1 =$$

$$= (s'_4s'_1+s'_1s'_4+s'_2s'_1s'_2)(s'_1s'_1)+(s'_1s'_1)(s'_4s'_1+s'_1s'_4+s'_2s'_1s'_2)+s'_4(s'_1s'_1)s'_1+s'_1(s'_1s'_1)s'_4$$

This shows that $q2 \in (I' \cdot Rel \cdot I') + (Rel \cdot Rel)$, hence represents 0 class in homology. Hence the non-"survival" of u_0.

We now prove that $h_0h_2h_0 = h_2h_0^2 = h_0^2h_2 = h_1^3 \neq 0$. The element

$$q3 := s'_1(s'_4s'_1 + s'_1s'_4 + s'_2s'_1s'_2) + s'_4(s'_1s'_1) + s'_2(s'_2s'_2 + s'_1s'_2s'_1) =$$

$$= s'_1s'_4s'_1 + s'_1s'_1s'_4 + s'_1s'_2s'_1s'_2 + s'_4s'_1s'_1 + s'_2s'_2s'_2 + s'_2s'_1s'_2s'_1 =$$

$$= (s'_4s'_1 + s'_1s'_4 + s'_2s'_1s'_2)s'_1 + (s'_1s'_1)s'_4 + (s'_2s'_2 + s'_1s'_2s'_1)s'_2$$

is in $(I' \cdot Rel) \cap (Rel \cdot I')$ and $\kappa_3(q3) = s'_1s'_4s'_1 + s'_4s'_1s'_1 + s'_1s'_1s'_4 + s'_2s'_2s'_2$. Hence $\nu^2_{H_3}([q3])$ has

$$[1 \otimes s'_1] \otimes [1 \otimes s'_4] \otimes [1 \otimes s'_1] + [1 \otimes s'_4] \otimes [1 \otimes s'_1] \otimes [1 \otimes s'_1] +$$

$$+ [1 \otimes s'_1] \otimes [1 \otimes s'_1] \otimes [1 \otimes s'_4] + [1 \otimes s'_2] \otimes [1 \otimes s'_2] \otimes [1 \otimes s'_2]$$

in it's $H_1 \otimes H_1 \otimes H_1$ direct summand. It is the only nonzero element of $H_{3,6}(A_{[2]})$ and its dual is $h_0 h_2 h_0 = h_2 h_0^2 = h_0^2 h_2 = h_1^3$

To see, that $h_2^3 = 0$ notice, that $s'_4 s'_4 s'_4$ can appear as a homogenous term of some $x \in I' \cdot Rel$ only from $s'_4(s'_4 s'_4 + s'_2 s'_4 s'_2 + s'_2 s'_2 s'_4)$. But in this case x will have $s'_4 s'_2 s'_4 s'_2$ as it's homogeneous term. Since no "part" of this term, except $s'_2 s'_4 s'_2$ in $r4$, appears in any element of Rel, $x \notin Rel \cdot I'$.

We now describe the representative $q4 \in (I' \cdot Rel) \cap (Rel \cdot I')$ of the unique nonzero element of $H_{3,11}(A_{[2]})$. This element is dual to α_1.

$$s'_2 s'_1[s'_4 s'_4 + s'_2 s'_4 s'_2 + s'_2 s'_2 s'_4] + s'_2 s'_4[s'_1 s'_4 + s'_4 s'_1 + s'_2 s'_1 s'_2] + s'_2[s'_4 s'_4 + s'_2 s'_4 s'_2 + s'_2 s'_2 s'_4] s'_1 +$$

$$s'_4[s'_1 s'_4 + s'_4 s'_1 + s'_2 s'_1 s'_2] s'_2 + s'_1 s'_2[s'_1 s'_4 + s'_4 s'_1 + s'_2 s'_1 s'_2] s'_2 s'_1 + s'_1 s'_2 s'_4[s'_2 s'_2 + s'_1 s'_2 s'_1] +$$

$$s'_2 s'_1[s'_1 s'_2 s'_1 + s'_2 s'_2] s'_4 + s'_2[s'_1 s'_1] s'_2 s'_1 s'_4 + s'_1[s'_1 s'_2 s'_1 + s'_2 s'_2] s'_4 s'_2 +$$

$$s'_2 s'_2 s'_2[s'_1 s'_4 + s'_4 s'_1 + s'_2 s'_1 s'_2] + s'_2[s'_2 s'_2 + s'_1 s'_2 s'_1] s'_1 s'_4 + s'_2 s'_1 s'_2[s'_1 s'_1] s'_4 +$$

$$s'_2 s'_2[s'_1 s'_4 + s'_4 s'_1 + s'_2 s'_1 s'_2] s'_2 + s'_1 s'_2[s'_1 s'_1] s'_4 s'_2 + s'_1 s'_2[s'_1 s'_1] s'_2 s'_4 + s'_4 s'_2 s'_1[s'_2 s'_2 + s'_1 s'_2 s'_1] + s'_4 s'_2[s'_1 s'_1] s'_2 s'_1 +$$

$$s'_1[s'_2 s'_2 + s'_1 s'_2 s'_1] s'_1 s'_2 s'_2 s'_1 + s'_1 s'_1 s'_2[s'_1 s'_1] s'_2 s'_2 s'_1 + s'_2 s'_2[s'_2 s'_2 + s'_1 s'_2 s'_1] s'_1 s'_2 + s'_2 s'_2 s'_1 s'_2[s'_1 s'_1] s'_2 +$$

$$s'_2[s'_2 s'_2 + s'_1 s'_2 s'_1] s'_1 s'_2 s'_2 + s'_2 s'_1 s'_2[s'_1 s'_1] s'_2 s'_2 =$$

$$s'_2s'_1s'_4s'_4 + s'_2s'_1s'_2s'_4s'_2 + s'_2s'_1s'_2s'_2s'_4 + s'_2s'_4s'_1s'_4 + s'_2s'_4s'_4s'_1 + s'_2s'_4s'_2s'_1s'_2+$$

$$s'_2s'_4s'_4s'_1 + s'_2s'_2s'_4s'_2s'_1 + s'_2s'_2s'_2s'_4s'_1 + s'_4s'_1s'_4s'_2 + s'_4s'_4s'_1s'_2 + s'_4s'_2s'_1s'_2s'_2+$$

$$s'_1s'_2s'_1s'_4s'_2s'_1 + s'_1s'_2s'_4s'_1s'_2s'_1 + s'_1s'_2s'_2s'_1s'_2s'_2s'_1 + s'_1s'_2s'_4s'_2s'_2 + s'_1s'_2s'_4s'_1s'_2s'_1+$$

$$s'_2s'_1s'_1s'_2s'_1s'_4 + s'_2s'_1s'_2s'_2s'_4 + s'_2s'_1s'_1s'_2s'_1s'_4 + s'_1s'_1s'_2s'_1s'_4s'_2 + s'_1s'_2s'_2s'_4s'_2+$$

$$s'_2s'_2s'_2s'_1s'_4+s'_2s'_2s'_2s'_4s'_1+s'_2s'_2s'_2s'_2s'_1s'_2+s'_2s'_2s'_2s'_1s'_4+s'_2s'_1s'_2s'_1s'_1s'_4+s'_2s'_1s'_2s'_1s'_1s'_4+$$

$$s'_2s'_2s'_1s'_4s'_2 + s'_2s'_2s'_4s'_1s'_2 + s'_2s'_2s'_2s'_1s'_2s'_2 + s'_1s'_2s'_1s'_1s'_4s'_2 + s'_1s'_2s'_1s'_1s'_2s'_4+$$

$$s'_4s'_2s'_1s'_2s'_2 + s'_4s'_2s'_1s'_1s'_2s'_1 + s'_4s'_2s'_1s'_1s'_2s'_1 + s'_1s'_2s'_2s'_1s'_2s'_2s'_1 + s'_1s'_1s'_2s'_1s'_1s'_2s'_2s'_1+$$

$$s'_1s'_1s'_2s'_1s'_1s'_2s'_2s'_1 + s'_2s'_2s'_2s'_2s'_1s'_2 + s'_2s'_2s'_1s'_2s'_1s'_1s'_2 + s'_2s'_2s'_1s'_2s'_1s'_1s'_2+$$

$$s'_2s'_2s'_2s'_1s'_2s'_2 + s'_2s'_1s'_2s'_1s'_1s'_2s'_2 + s'_2s'_1s'_2s'_1s'_1s'_2s'_2 =$$

$$s'_2s'_1s'_4s'_4 + s'_2s'_1s'_2s'_4s'_2 + s'_2s'_4s'_1s'_4 + s'_2s'_4s'_2s'_1s'_2 + s'_2s'_2s'_4s'_2s'_1+$$

$$s'_4s'_1s'_4s'_2 + s'_4s'_4s'_1s'_2 + s'_1s'_2s'_1s'_4s'_2s'_1 + s'_1s'_2s'_4s'_2s'_2+$$

$$s'_1s'_1s'_2s'_1s'_4s'_2 + s'_1s'_2s'_2s'_4s'_2 + s'_2s'_2s'_1s'_4s'_2 + s'_2s'_2s'_4s'_1s'_2 =$$

$$s'_4s'_4s'_1s'_2 + s'_2s'_4s'_2s'_1s'_2 + s'_2s'_2s'_4s'_1s'_2 + s'_1s'_4s'_4s'_2 + s'_4s'_1s'_4s'_2 + s'_2s'_1s'_2s'_4s'_2 +$$

$$s'_2s'_1s'_4s'_4 + s'_2s'_4s'_1s'_4 + s'_2s'_2s'_1s'_2s'_4 + s'_1s'_4s'_4s'_2 + s'_1s'_2s'_4s'_2s'_2 + s'_1s'_2s'_2s'_4s'_2 +$$

$$s'_1s'_1s'_2s'_1s'_2s'_4 + s'_2s'_2s'_4s'_2s'_1 + s'_1s'_2s'_1s'_4s'_2s'_1 + s'_2s'_2s'_1s'_4s'_2 + s'_1s'_2s'_1s'_1s'_4s'_2 +$$

$$s'_2s'_2s'_1s'_2s'_4 + s'_1s'_2s'_1s'_1s'_2s'_4 + s'_1s'_1s'_2s'_1s'_2s'_4 + s'_1s'_1s'_2s'_1s'_4s'_2 =$$

$$[s'_4s'_4 + s'_2s'_4s'_2 + s'_2s'_2s'_4]s'_1s'_2 + [s'_1s'_4 + s'_4s'_1 + s'_2s'_1s'_2]s'_4s'_2 + s'_2[s'_1s'_4 + s'_4s'_1 + s'_2s'_1s'_2]s'_4 +$$

$$s'_1[s'_4s'_4 + s'_2s'_4s'_2 + s'_2s'_2s'_4]s'_2 + [s'_1s'_1]s'_2s'_1s'_2s'_4 + [s'_2s'_2 + s'_1s'_2s'_1]s'_4s'_2s'_1 +$$

$$[s'_2s'_2 + s'_1s'_2s'_1]s'_1s'_4s'_2 + [s'_2s'_2 + s'_1s'_2s'_1]s'_1s'_2s'_4 + [s'_1s'_1]s'_2s'_1s'_2s'_4 + [s'_1s'_1]s'_2s'_1s'_4s'_2$$

To show that this class is nonzero one has to look at all elements of $[Rel \cdot Rel + I' \cdot Rel \cdot I']^{F_2}$ of degree 11. There are much less then 4^{10} of such elements.

Adams proved in [1, Section 6] that the products of h_i in the third cohomology group of A are subject only to the following relations:

$$h_i h_{i+1} h_j = 0$$

$$h_i^2 h_{i+2} = h_{i+1}^3$$

$$h_i h_{i+2}^2 = 0$$

$$h_i h_j h_k = h_i h_k h_j = h_j h_i h_k$$

The first relation follows from multiplying $h_i h_{i+1} = 0$ by h_j. The last pair states commutativity. We will prove the other two.

We define an algebra homomorphism $\alpha' : A' \rightarrow A'$ by $\alpha'(s_1') := 0$, $\alpha(s_{2n}') := s_n'$. This homomorphism carries *Rel* ideal into itself [since α was proved to be A algebra homomorphism]. Since α' carries I' into itself, it also carries $I' \cdot Rel \cap Rel \cdot I'$ into itself and $I' \cdot Rel \cdot I' + Rel \cdot Rel$ into itself. So α' induces a F-linear map $SQ_0 : H_3(A) \rightarrow H_3(A)$.

The dual F-linear map $SQ^0 : H^3(A) \rightarrow H^3(A)$ takes each h_i into h_{i+1} [this is precisely the SQ^0 from the end of Section 6]. Hence to prove Adam's results, it is sufficient to prove that $h_0^2 h_2 = h_1^3$ [we have already done that] and $h_0 h_2^2 = 0$.

Suppose $w \in (I' \cdot Rel) \cap (Rel \cdot I')$ and $h_0 h_2^2([w]) = 1$. Then $\kappa_3([w])$ must have $s'_1 s'_4 s'_4 + s'_4 s'_1 s'_4 + s'_4 s'_4 s'_1$ in it. The homogeneous term $s'_4 s'_1 s'_4$ can be a summand in some $s'_k \cdot r_1$, where r_1 is from the minimal generating set of Rel, only if $s'_k = s'_4$ and $r_1 = s'_1 s'_4 + s'_4 s'_1 + s'_2 s'_1 s'_2$. But the homogeneous term $s'_4 s'_2 s'_1 s'_2$, which occurs in $s'_4(s'_1 s'_4 + s'_4 s'_1 + s'_2 s'_1 s'_2)$ can not occur in any other $s'_m \cdot r_2$ or $r_2 \cdot s_m$, where r_2 is from the minimal generating set of Rel. [For grading reasons, one has to check only the Relations from grading *less* then 9.] Hence such w does not exist.

We will now prove that $h_i h_j h_k \neq 0$ if $i + 1 < j < k - 1$. Define

$$w1 := s'_{2^i}[s'_{2^j} s'_{2^k} + s'_{2^k} s'_{2^j} + x1] + s'_{2^j}[s'_{2^i} s'_{2^k} + s'_{2^k} s'_{2^i} + x2] + s'_{2^k}[s'_{2^j} s'_{2^i} + s'_{2^i} s'_{2^j} + x3] \in I' \cdot Rel$$

$$w2 := [s'_{2^j} s'_{2^k} + s'_{2^k} s'_{2^j} + x1] s'_{2^i} + [s'_{2^i} s'_{2^k} + s'_{2^k} s'_{2^i} + x2] s'_{2^j} + [s'_{2^j} s'_{2^i} + s'_{2^i} s'_{2^j} + x3] s'_{2^k} \in Rel \cdot I'$$

Due to relations (B), $w1 + w2$ is a relation of height (See [14, page 432]) less then $H_k H_j H_i$, so is reducible in terms of minimal relations of smaller height. In other words, one can define

$$\beta := w1 + \sum_i y_i \cdot r_i = w2 \neq 0$$

Then $h_i h_j h_k([\beta]) = 1$.

There is a conjecture by Davis, stating that ALL the relations on the products of h_i are induced from $h_0 h_1 = 0$ and $h_0 h_2 h_2 = 0$ by SQ^k operations. See [4].

References

[1] Adams Frank J., *On the Structure and Applications of the Steenrod Algebra*, Commentarii Mathematici Helvetici, vol 32, 1958.

[2] Adams Frank J., *On the non-existence of elements of Hopf Invariant one*, Annals of Mathematics, vol 72, no. 1, July 1960.

[3] Henry Cartan and Samuel Eilenberg, *Homological Algebra* Princeton: Princeton University Press 1956

[4] Davis Donald M., *An infinite family in the cohomology of the Steenrod Algebra*, Journal of Pure and Applied Algebra 21 (1981) pg.145-150

[5] Hilton Peter J. and Stammbach Urs, *A course in Homological Algebra*. Berlin, Heidelberg, New York: Springer-Verlag 1971

[6] Liulevicius Arunas L., *The factorization of cyclic reduced powers by secondary cohomology operations.*, Memoires of American Mth. Soc. 42 (1962).

[7] Milnor John W., *The Steenrod Algebra and it's dual.*, Ann. of Math. 67 (1958), pg.150-171

[8] Milnor John W. and Moore John C., *On the structure of Hopf Algebras.*, Ann. of Math. (2)81 (1965), pg.211-264

[9] Mosher Robert E. and Tangora Martin C., *Cohomology Operations and Applications in Homotopy Theory.*: Harper and Row, 1968

[10] Nobuo Shimada and Akira Iwai, *On the Cohomology of some Hopf Algebras.*, Kyoto University Journal of Math. 1967, pg.103-111

[11] Singer William M., *Steenrod Squares in Spectral Sequences: The Cohomology of Hopf Algebra Extensions and Classifying Spaces.*: Monograph

[12] Steenrod Norman E. and Epstein D.B.A., *Cohomology Operations.* Princeton: Princeton University Press 1962

[13] Switzer Robert M., *Algebraic Topology-Homotopy and Homology.* Berlin, Heidelberg, New York: Springer-Verlag 1975

[14] Wall C.T.C., *Generators and Relations for the Steenrod Algebra.*, Ann. of Math. (3)72 (1960), pg.429-444

[15] Weibel Charles A., *An introduction to Homological Algebra.*: Cambridge University Press 1994

[16] Whitehead George W., *Elements of Homotopy Theory.* Berlin, Heidelberg, New York: Springer-Verlag 1978