# Synchronization Algorithm for Stream Control Transmission Protocol (SCTP) Network

**By:**

**Hussein Elsayed**

**A dissertation submitted to the Graduate Faculty in Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy.**

**The City University of New York**

**2003**

UMI Number: 3103107





ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

June 12, 2003 — Prof. Tarek Saadawi [signature]

Date — Chair of Examining Committee

June 12, 2003 — Dean. Mumtaz Kassir [signature]

Date — Executive officer

Prof. Umit Uyar (EE Dept., CCNY, CUNY)

Prof. Myung Lee (EE Dept., CCNY, CUNY)

Prof. Michael Conner (EE Dept., CCNY, CUNY)

Prof. Mehmet Ulema (CIS Dept., Manhattan College)

(External examiner)

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

# Abstract


## Synchronization Algorithm for Stream Control Transmission Protocol (SCTP) Network

**By**

**Hussein Elsayed**

**Advisor: Professor Tarek Saadawi**

Synchronization has been a problem in communication networks especially for real time applications. Synchronization can be done in different layers of the ISO model. In this Thesis, a synchronization algorithm is introduced and applied on Stream Control Transmission Protocol (SCTP), which is a layer four protocol. However, the current SCTP standard, RFC 2960, doesn't specify any synchronization mechanisms. We examine the Transfer Delay (TD) and the Interarrival Time (IT) as network performance measure. The Algorithm is applied on periodic and non-periodic traffic and shows significant synchronization improvement.

The periodic traffic algorithm has three levels depending on the reference delay, namely, "Maximum Delay Algorithm", "Accumulative Average Algorithm", and "Average Over n Algorithm". The Maximum Delay Algorithm adjusts the jitter based on the maximum delay of the preceding packets. The Accumulative Average Algorithm keeps track of the average delay and considers it as the reference delay to adjust the upcoming packets. Finally, the Average Over n Algorithm takes the average over n preceding packets as the reference for jitter adjustment. Our results show good deal of

jitter improvement. It also shows that the improvement depends on the input traffic distribution and the amount of its jitter.

This Thesis also provides an analytical model for the periodic traffic synchronization algorithm. The analysis uses a basic principle in probability, which is called "transformation of random variables". Since the analytical model depends on the probability distribution function, we analyze the uniform and normal distribution to show how much improvement our algorithm provides. Other distribution can be analyzed but they come with similar conclusion. We also found upper and lower limits for the mean and variance of the output traffic delay independent of the distribution.

## Acknowledgements

I would take this opportunity to thank to my supervisor Professor Tarek Saadawi for his invaluable academic advice. He devoted much of his time to discussing details of this research with me and helping me in so many ways. I have been greatly influenced by his methodology in the field of scientific research, and have benefited from his detailed comments on every aspect related the research in this thesis.

I would also like to thank the Electrical Engineering faculty and specially the examining committee for their support and for giving me the chance to achieve my goals.

I will never forget the people who helped me either direct or indirect to have this effort completed, special thanks to Bisrat Bulcha, Kris Vanderbei, and Pete Klopchin.

Finally, I would like to thank my family for their support all the time. They were always there and gave me the love and support when I needed them the most. Very special thanks for my wife who helped me a lot in so many ways. Without her, this work would not have been possible.

This dissertation is dedicated to my loving parents, wife, son and daughter.

# Table of contents

# Table of figures

# 1 Chapter I: Introduction

Delay and jitter have been problems in communication networks especially in Quality of Service (QoS) networks. ATM is the most famous QoS network as it was built to carry such traffic. There has been an effort to pass QoS traffic through Frame relay and IP networks. Multi-Protocol Label Switching (MPLS) and Stream Control Transmission Protocol (SCTP) are other type of network that carry IP traffic.

IP protocol is the most popular networking protocol. Originally, it wasn't implemented to carry non-real-time traffic. Since then, many research activities enhanced the IP protocol to carry real time traffic but jitter is still an issue. Our algorithm can be implemented at the next high layer, namely transport layer, to improve the real time applications' end-to-end performance over an IP network.

Multi-Protocol Label Switching combines layer 2 and layer 3 functions to provide more flexible and efficient protocol. One of the important MPLS features is "merging", which classifies data based on the classes of service (CoS). Allocation of special treatment facilities for packets associated with a Forwarding Equivalence Class (FEC) that is to receive some form of preferential treatment is done at Label Switched Path (LSP) setup. In addition, it is possible to perform policing and/or traffic shaping at ingress to an LSP (as opposed to at each hop).

Merging is an essential feature in getting MPLS to scale to at least as large as a typical routed network [1]. With merge capability available and in use at every node, it is possible to setup multi-point to point LSPs such that only a single label is consumed per FEC at each Label Switched Router (LSR) including all egress LSRs.

Different levels of merge capability are defined to provide means for LSRs to support at least partial merge capability even when full merge capability is particularly hard to do given the switching hardware (as is the case with many ATM switches). The two popular merging types are “Frame Merge” and “VC Merge”. Frame merge is the capability typical of standard routing and is a natural successive of transport media that encapsulate an entire layer 3 packet inside a layer 2 frame. In this case, full merging occurs naturally and no action is required of the LSR. This is typically the case with non-ATM layer 2 technologies.

VC merge is the name applied to any technique that, when used with an ATM switch, allows it to effectively perform frame merge. Typically, this requires queuing cells associated with a single AAL, ATM Adaptation Layer, frame (if they are not actually re-assembled) until the last one has been received. Then those cells are transmitted in the same order in which they were received while being careful not to interleave them with cells from any other AAL frame being transmitted on the same VC. Interleaving cells using different VCIs is permissible however cells associated with the same VCI on any input interface must be transmitted without interleaving with cells received on other input interfaces (or the same interface using a different VCI) which will be transmitted using the same VCI. Figure 1 shows the different types of MPLS merge. It also shows how the merge causes jitter.

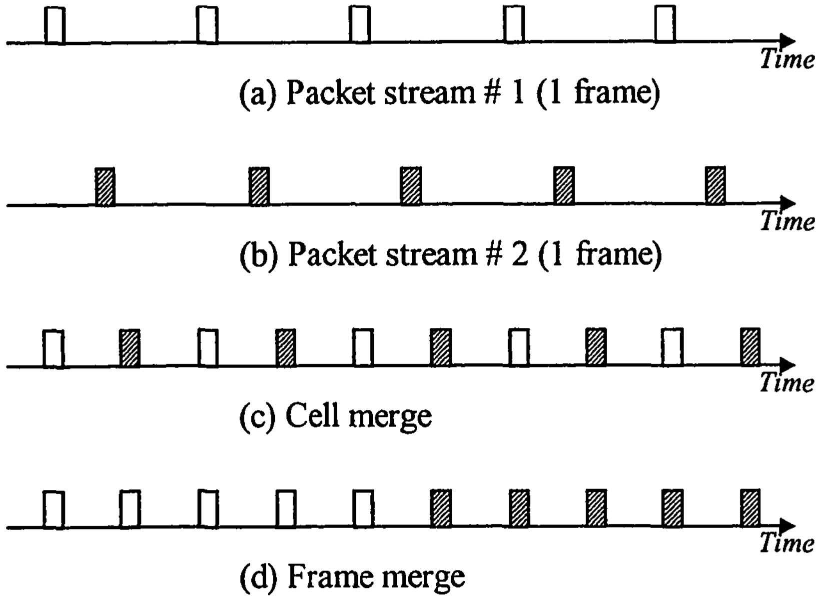



**Figure 1 MPLS merge**

### *1.1* Delay types:

This section contains an overview of various aspects that affect the end-to-end delay as well as jitter in a QoS network. The delay depends on the type of service and access speed. The following subsections provide the various types of delay with brief description.

The following is a list of the various types of delay that contribute in the end-to-end delay:

- Propagation delay that depends on the link mileage whether it is access or trunks. ($T_p$)

- Transmission delay which depends on the link speed 56K, T1, T3, OC3, ... ($T_t$)
- Queuing delay at each node (switches, routers, ...etc.), which also include buffering and processing delay. ($T_q$)
- Insertion delay due to the added header ($T_i$).
- Conversion delay (assembly and reassembly) ($T_c$)

The total delay is the sum of all of various delay types as follows;

$$T_{Total} = T_t + T_p + T_q + T_i + T_c \quad (1)$$

### 1.1.1 Conversion delay

The key point is that the transmitted message can't be reassembled and send to the other end unless all the packets are received. The time between packets depends on the connection rate. The technical term for this time is interarrival time (IT), like Cell Interarrival Time (CIT) in ATM networks. If the message consists of 3 packets, then there is a delay of 2 interarrival intervals. In general, if the packet consists of n packets, then the conversion delay is (n-1)*IT.

#### 1.1.1.1 Case study.

Assuming FR service over ATM network. Assume also that the ping size is 100 bytes and the Sustainable Cell Rate (SCR) is 100 cells/sec.

The ATM to FR conversion delay can be found as follows. In the FR side, the 100 bytes are sent in one frame. In the ATM side, the 100 bytes are divided into 3 ATM cells (48 data bytes/cell). So, 3 ATM cell are needed to reassemble the FR frame before it sent to the FR side. Three ATM cells means two CIT intervals. For SCR of 100 cells/sec, the CIT is 10 msec. That means that the ATM to FR conversion is 2*10 which is 20 msec.

### 1.1.2 Access delay

This is the transmission delay caused by the access link. As an example, at 56Kbps, the transmission time is 1/56, which is 0.018 msec/bit. Since the byte consists of 8 bits, then, the transmission delay is 8*0.018, which is 0.14 msec/byte.

It means that at 56 Kbps, one byte takes 0.14 msec to access the link. Depending on the packet size, the access delay can be calculated.

### 1.1.3 Network delay

The network delay includes queuing, transmission, and propagation delay. The queuing delay depends on the switches type, background load, and number background sources. The transmission and propagation delay depend on the trunk type and the distance.

### 1.1.4 Total End-to-End delay

The total End-to-End delay is the sum of conversion delay, access delay, and network delay.

$$T_{Total} = T_t + T_p + T_q + T_i + T_c$$

Each of these components affects the delay. Figure 2 summarizes the various types of delay.

### 1.1.5 Other Factors.

There are many factors that affect the network performance, depending on the network type and the operating conditions. Factors that may affect the delay, which aren't included in the previous description, are:

- Packet header.
- Protocol conversion method.
- Encapsulation type.
- Background load.
- Peak hour load.
- Traffic exceeding the connection rate.
- Traffic shaping

Some of the above factors have minor effect on the delay.

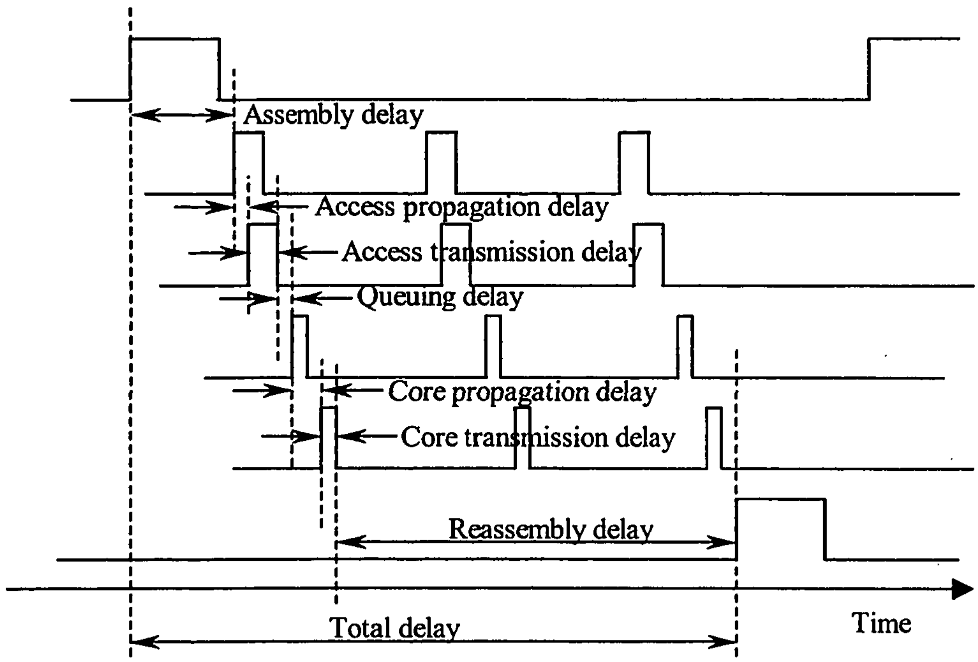



**Figure 2 Various types of delay.**

### *1.2 The ISO Model*

Figure 3 shows the ISO 7-layer model, which is one of the standard layering architecture. The model description is outside the scope of our discussion but we will discuss it from the synchronization point of view. We will start with the data link layer, which includes point-to-point protocols. As the data travels from one point to another, jitter can be introduced. In its simplest form, jitter is the deviation from the ideal timing of a packet. It can also be defined as unwanted variations of packet timing (the interval between successive packets). So that the jitter needs to be eliminated or reduced as possible to be able to recover the original packet timing information.

One of the data like layer functions is to equalize the jitter. The importance of the data link layer synchronization appears at the network-to-network interfaces. Most of the network-to-network communication is performed within the first three layers. The communication is also done based on an opened contract between the two networks to describe the traffic characteristics (e.g., rate, burst,...etc). If one of the networks starts to exceed the contract, traffic may be lost. So that the sending gateway needs to equalize the jitter before sending the traffic to the other network. Also, within the same network, the traffic may be lost if one of the links exceeds certain value of the jitter. Good example of layer-two traffic that requires synchronization is the ATM CBR traffic.

Layer four is another example of the protocol sets that required synchronization. Since layer four looks at the traffic from end-to-end point of view, it is necessary to equalize the jitter independent on the core network before handling the traffic to the application layer. SCTP is one of the new developed layer four protocol and deals with synchronization issues. In this thesis, we focus on SCTP as our main protocol for the new synchronization algorithm.

All of the synchronization purpose is to send the traffic from one node to another and from one layer to another to reach the application layer within specified characteristics. In other words, the application layer is the one that determines the needs of traffic synchronization. Some applications don't require synchronization, like the e-mail and ftp applications. Other applications are jitter sensitive like audio and video applications.

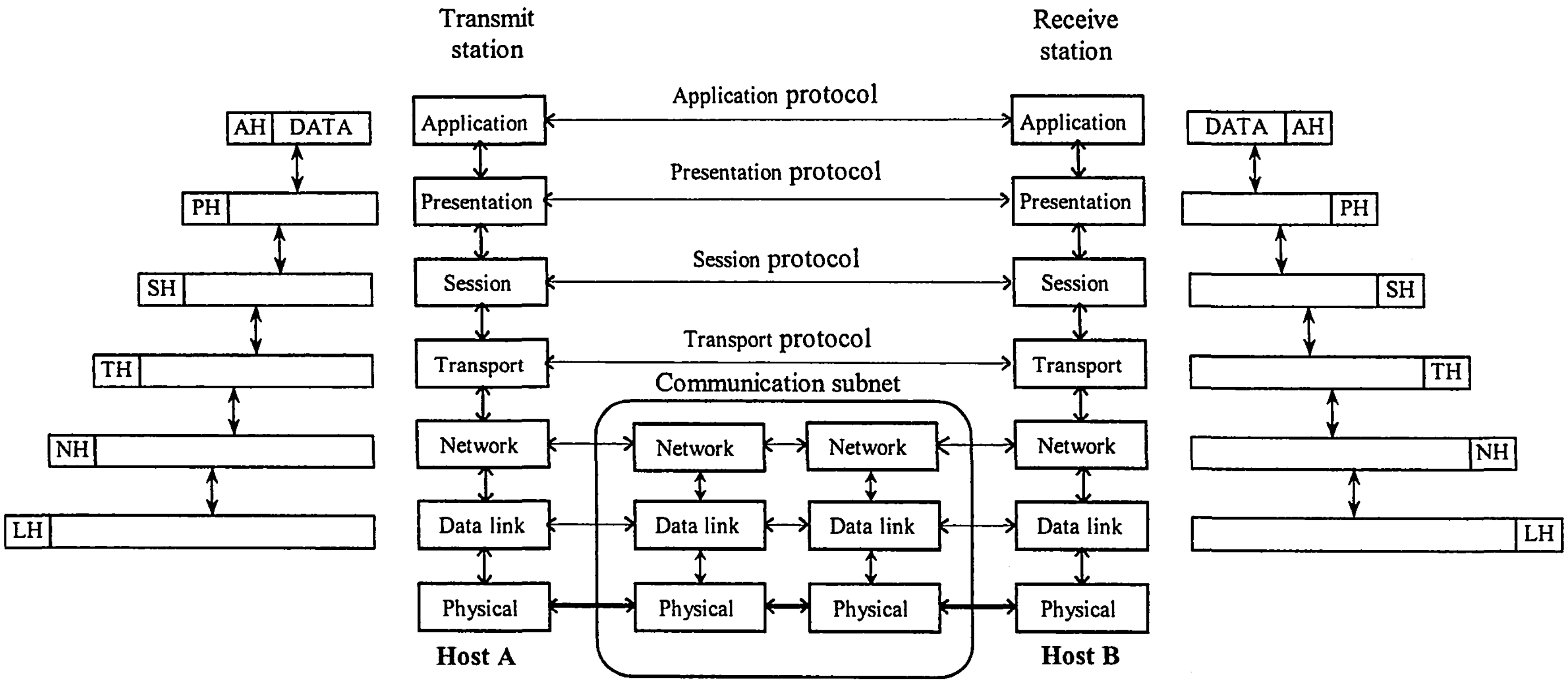



**Figure 3 The network architecture used in ISO model**

| | | | |
|---|---|---|---|
| *AH* | *Application header* | *PH* | *Presentation header* |
| *SH* | *Session header* | *TH* | *Transport header* |
| *NH* | *Network header* | *LH* | *Data Link header* |

## *1.3 SCTP Protocol*

Stream Control Transmission Protocol (SCTP) is a reliable transport protocol operating on top of a potentially unreliable connectionless packet service such as IP. It offers acknowledged error-free non-duplicated transfer of SCTP packets (messages). Detection of data corruption and loss or duplication of data is achieved by using checksums and sequence numbers. A selective retransmission mechanism is applied to correct loss and errors.

While TCP is byte stream oriented, SCTP transfers data units, which are called "chunks". SCTP performs Path-MTU discovery, i.e. determines the maximum possible size at which IP packets can be sent to a destination without being segmented. SCTP segments large messages into chunks that fit into such an IP packet. Several small chunks resulting from small messages may be multiplexed into one IP packet (bundling).

While TCP performs a strict in-sequence delivery of data per connection, SCTP has a more flexible delivery scheme. SCTP distinguishes different "streams" of messages within one SCTP association, as shown in Figure 4. This enables a delivery scheme where the message sequence is only maintained per stream (partial in-sequence delivery) to reduce unnecessary head-of-line blocking among independent streams. Furthermore, SCTP provides a bypassing mechanism, such that message are delivered to SCTP user as soon as they have been completely received (order-of-arrival delivery).

Flow and congestion control in SCTP have been designed to assure that SCTP traffic behaves in the same way as TCP traffic does. This allows for seamless introduction of SCTP services into existing IP networks.

To provide fault tolerance, SCTP supports multihoming at the transport layer. A host is multihomed if it can be identified by multiple IP addresses, as would be the case when the host has multiple network interfaces. Network layer redundancy allows access to a host at a time when its primary IP address becomes unreachable; packets are routed dynamically to one of the host's alternate IP addresses. An SCTP association spans over all possible source/destination address combinations between the involved nodes. Thus, each multi-homed node can be reached from another node via several paths. The status of each path is monitored by SCTP with respect to reachability, delay, and number of consecutive retransmissions. Path monitoring, the usage of an alternate path for retransmissions, and status dependent path selection, make SCTP more robust against partial network failures than TCP.

### 1.3.1 SCTP packet format

An SCTP packet is formed of a common header and chunks as shown in Figure 5. Multiple chunks may be multiplexed into one packet up to path-MTU size. A chunk contains either control information or user data.

The common header consists of 12 bytes. For the identification of an association, SCTP uses the same port concept of TCP and UDP. For the detection of transmission errors, each SCTP packet is protected by a 32-bit checksum (Adler-32 algorithm), which is more robust than the 16-bit checksum of TCP and UDP. SCTP with an invalid

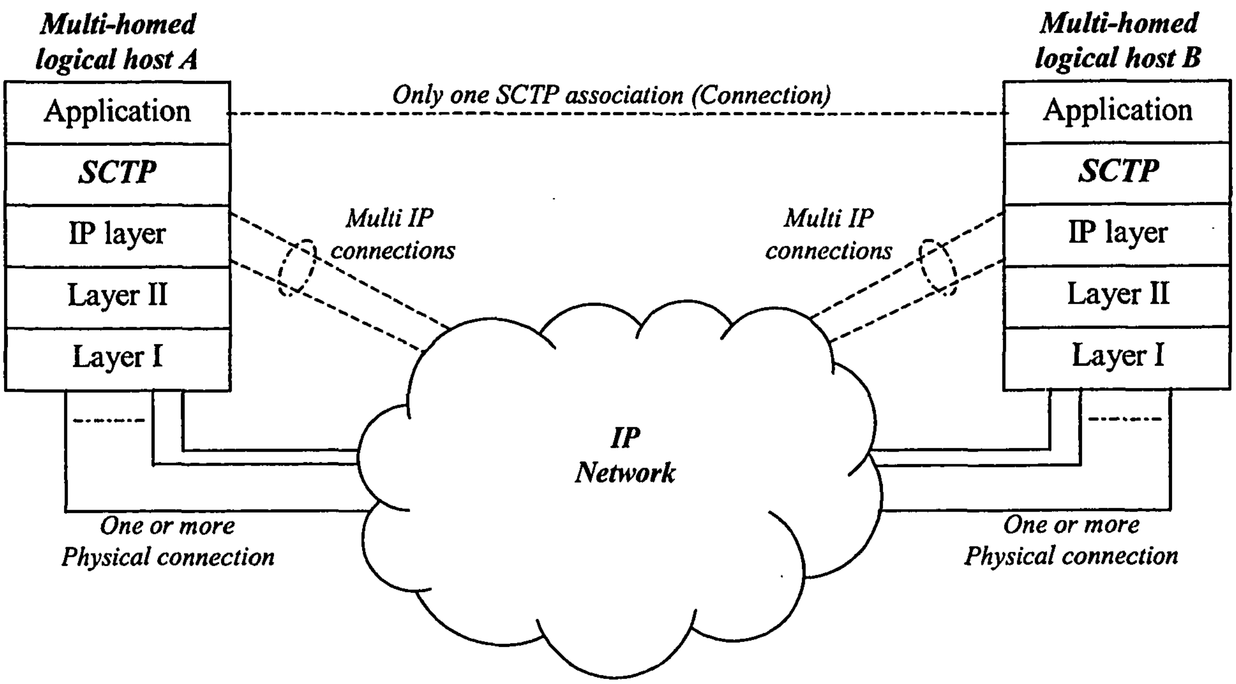



**Figure 4 SCTP end-to-end network**

check sum are silently discarded. The common header contains a verification tag to validate the end of the association.

Each chunk begins with chunk type field, which is used to distinguish data chunks and different types of control chunks, followed by chunk specific flags and chunk length field needed because chunks have a variable length. The value field contains the actual payload of the chunk.

### 1.3.2 Multihoming

An essential property of SCTP is the support of multihomed nodes, i.e. nodes that can be reached under several IP addresses. If the SCTP nodes and the IP network are configured such that traffic from one node to another follows physically different paths for different destination IP addresses, associations become tolerant against physical network failures and similar problems.

During the association initiation, SCTP entities exchange their IP addresses SCTP is able to handle IP version 4 and IP version 6 addresses (even mixed). Also host name can be used instead of one or more IP addresses. An SCTP entity considers each IP address of its peer to identify the endpoint of one "transmission path" toward this node.

An SCTP entity monitors all transmission paths to the per entity of an association. HEARTBEAT chunks are sent over all paths, which are currently not used for the transmission of data chunks. Each HEARTBEAT chunk has to be acknowledged by a HEARTBEAT-ACK chunk. Each path is assigned a state, which is either "active" or "inactive". A path is "active" if it has been used in the recent past to transmit as arbitrary SCTP packet, which has been acknowledged by the peer. If transmissions on a certain path fail repeatedly, this path is considered "inactive".

### *1.4 SCTP Synchronization Algorithm*

As it was mentioned earlier, SCTP is a layer 4 protocol and it takes care of the

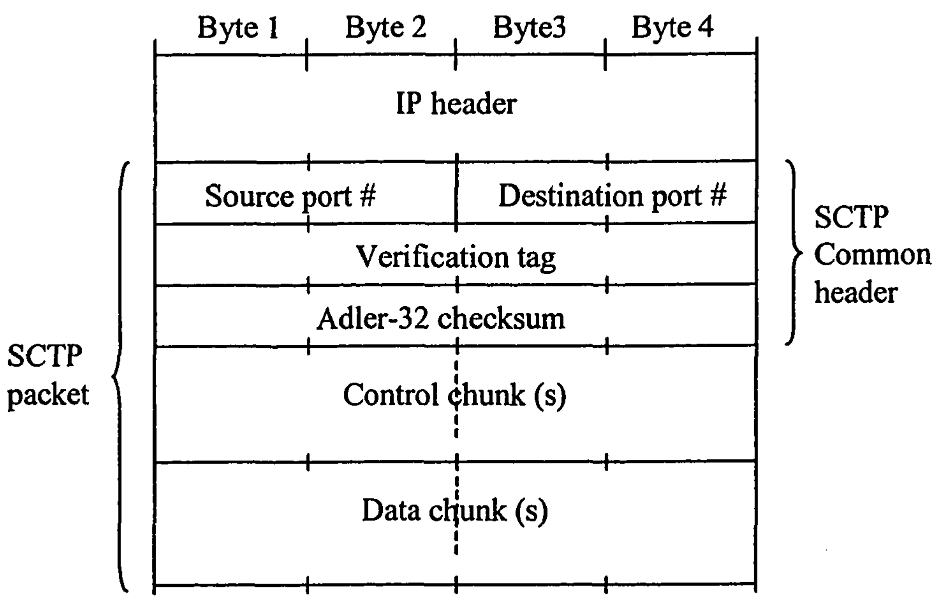



**Figure 5 SCTP packet format**

end-to-end synchronization. This section introduced an overview of the SCTP protocol. Chapter 2 provides the synchronization algorithm in case of periodic traffic. It also describes the proposed algorithm and provides the simulation results to show how much the proposed algorithm improves the jitter problem. The non-periodic traffic algorithm and its simulation results are provided in chapter 3. Chapter 4 provides analytical model to the periodic traffic synchronization algorithm. Finally, the conclusion is introduced in chapter 5.

## 1.5 Introduction

The Stream Control Transmission Protocol is a new reliable transport protocol operating on top of the Internet Protocol (IP). It is designed to transport telecommunications signaling traffic over an IP network and has been recently standardized by the Internet Engineering Task Force (IETF). "Multi-homing" is one of the key features of the protocol. It provides SCTP with a certain network level fault tolerance by the use of network address redundancy. Multi-homing improves the reliability of the transmission protocol but it caused burst of jitter and may be packet loss. Our algorithm improves the SCTP multi-homing feature by significantly decreasing the introduced jitter. SCTP protocol is our focus in this thesis.

### 1.5.1 Synchronization in SCTP networks

SCTP network, like other types of networks, require synchronization. Since SCTP is a layer four protocol, SCTP synchronization equalizes the end-to-end jitter. There are several reasons of traffic jitter in SCTP network. Some of these reasons are, regular path jitter, multihoming jitter, waiting for acknowledgement, retransmission, fragmentation, and timers' jitter. It is difficult to differentiate between different types of jitter at the destination end. But synchronization mission is to equalize the jitter independent on the jitter source.

Research has been done and we found that there is a tremendous need for dejittering at different locations in the network. For example at the network-to-network interfaces, where the traffic is handed off to another network. Now a days, networks are getting bigger and bigger and there are usually gateways that checks the traffic from other networks before accepting it. Traffic that exceeds the opened contract is eligible for tagging or discarding. Like in ATM and Frame Relay networks, there is what is called DE (Discard Eligible) flag that is set to one if the traffic exceeds the contacted rate. Another example dejittering request is at the end point, where the traffic is delivered to the final destination. Our proposed algorithm can be applied at the network-to-network interface to eliminate or reduce the jitter to an acceptable value.

To explain the effect of jitter, let us assumed that we have periodic traffic, then the packet interarrival time should be constant with acceptable deviation. Due to buffering, scheduling algorithm and other factors, periodic traffic will have larger deviation at the network egress. Shuaib in [10] performed lab experiments to show the jitter doesn't only depend on number pf hops but also on the background load and number of background sources.

The idea of dejittering can be explained as follows. If we assumed that there is no jittering, then each packet comes on time. Due to the network jitter, some packets come late and some come early. The early packets can be delayed but the late ones can't be controlled. So, The proposed algorithm buffers the early packets and takes the late packets as reference. In other words, the early packets are delayed according to the late packets. The problem here is that we can't predict how late are the late packets. The introduced algorithm overcomes this problem by memorizing the history of the previous packets.

As the algorithm improves the jitter performance, like any other algorithm, it has a drawback. The drawback is that the average transfer delay (TD) is increased. That drawback is also analyzed and the results show that it has a limit. In chapters 2 and 4, the output traffic transfer delay boundaries are found and the results show that the algorithm doesn't have much effect on the average TD. In either case, the jitter improvement and the drawback acceptance depend on the application. Most of the real-time applications are jitter sensitive.

Many researches have been studying solution for the jittering and synchronization problems. Synchronization algorithm for real time multimedia conferencing is in [2]. Xie [3] introduced another synchronization scheme for multimedia traffic. Series of researches have been done to analyze the delay and jitter in different network conditions. Privalov [4] analyzed the homogenous and heterogeneous CBR traffic. Walker [5] has proved that the SRTS inherently generates jitter. Analyzing CBR traffic whose delay under non-real-time traffic background was introduced in [6]. Qiong [7] proposed a novel delay-boundary prediction algorithm based on a deviation-lag function to characterize the end-to-end delay variations. Bennett [8] found boundaries for periodic traffic.

## *1.6 SCTP modification*

Figure 6 shows high-level representation of the required SCTP modification. Two modules are required, one at the source node and another one at the destination node to communicate the timing information. Figure 7 shows the sender building blocks. The shaded block is the sender module that takes the synchronization paramagnets from the higher layers and communicates with the synchronization module at the destination node. Those parameters will be explained later in chapter 2. The destination diagram is shown in Figure 8 where the shaded block is added to equalize the jitter. The following subsections describe how it works.

### 1.6.1 Synchronization parameters

Since the first phase of this thesis is considering periodic traffic, there is only one parameter that is required from the higher-layers. That parameter is the Interarrival Time or IT. The destination module will reconstruct the packet timing based on the

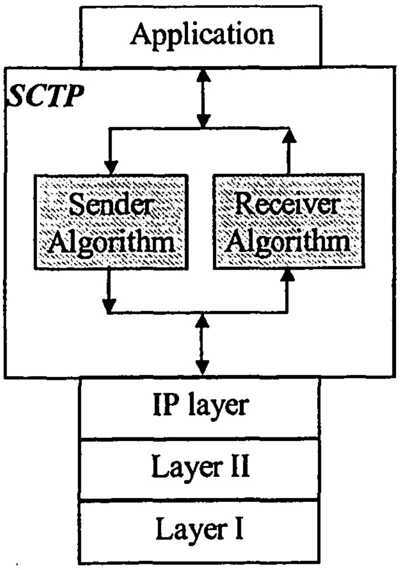



**Figure 6 High level of SCTP synchronization algorithm.**

interarrival time and timing information that is sent in the timestamp.

### 1.6.2 Timestamps issues

Currently, timestamp is used by SCTP in the associattion setup phase only. The current SCTP specification doesn't send timestamp with the data chunk as described in [9]. On the other hand, SCTP handles up to 256 chunk types. Only 16 types are defined and the remaining 240 are still available. The proposed algorithm requires sending timestamp, so that a new chunk type will be defined to indicate data chunk with timestamp in it as shown in Figure 9. Absolute or differential time can be used. The timestamp field length is 32 bits because of two reasons. The first one is that 32 bits is the basic unit for SCTP. Secondly, 32 bit timestamp is used in the associaition setup.

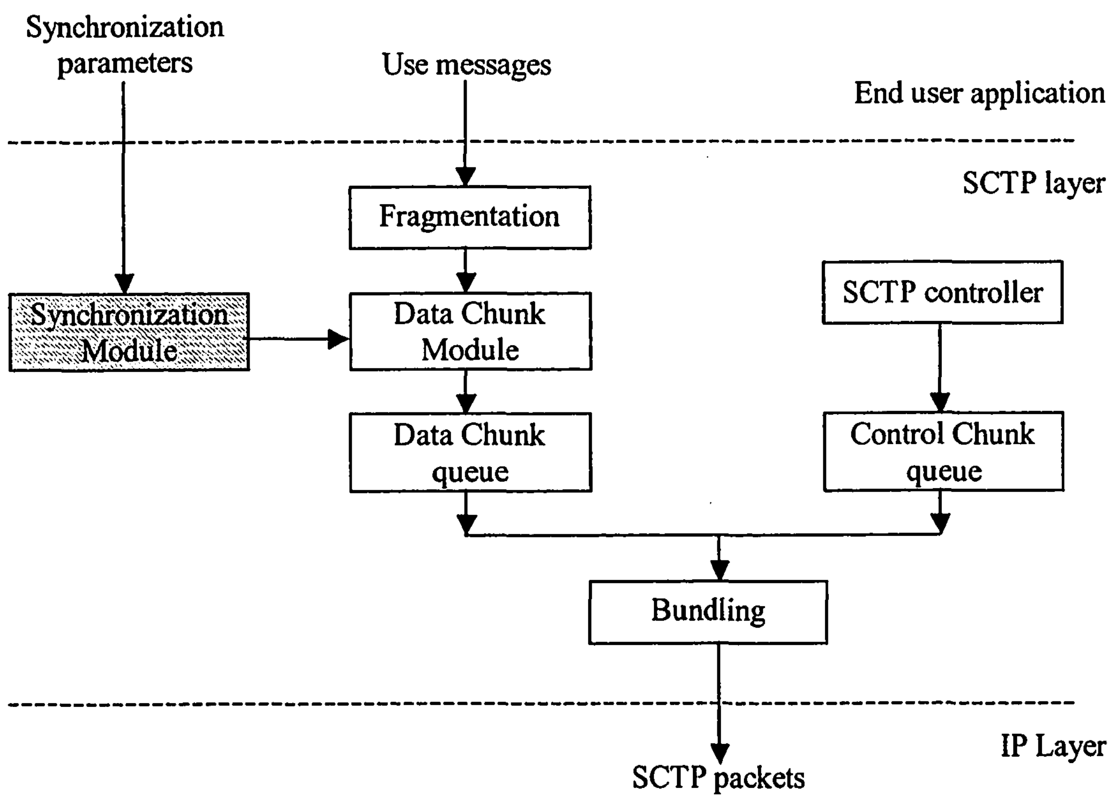



**Figure 7 SCTP sender modules**

# 2 Chapter II Synchronization Algorithm for periodic traffic

The proposed synchronization algorithm reconstruct the packet timing at the destination node based on the interarrival time. It stores the packets and delivers them to the higher layers with interval T. The following subsections describe three levels of the algorithm.

## *2.1 Maximum delay algorithm.*

In periodic traffic, the IT is constant. So, the incoming packets can be buffered and controlled. The output IT can be adjusted according to the maximum delay of the

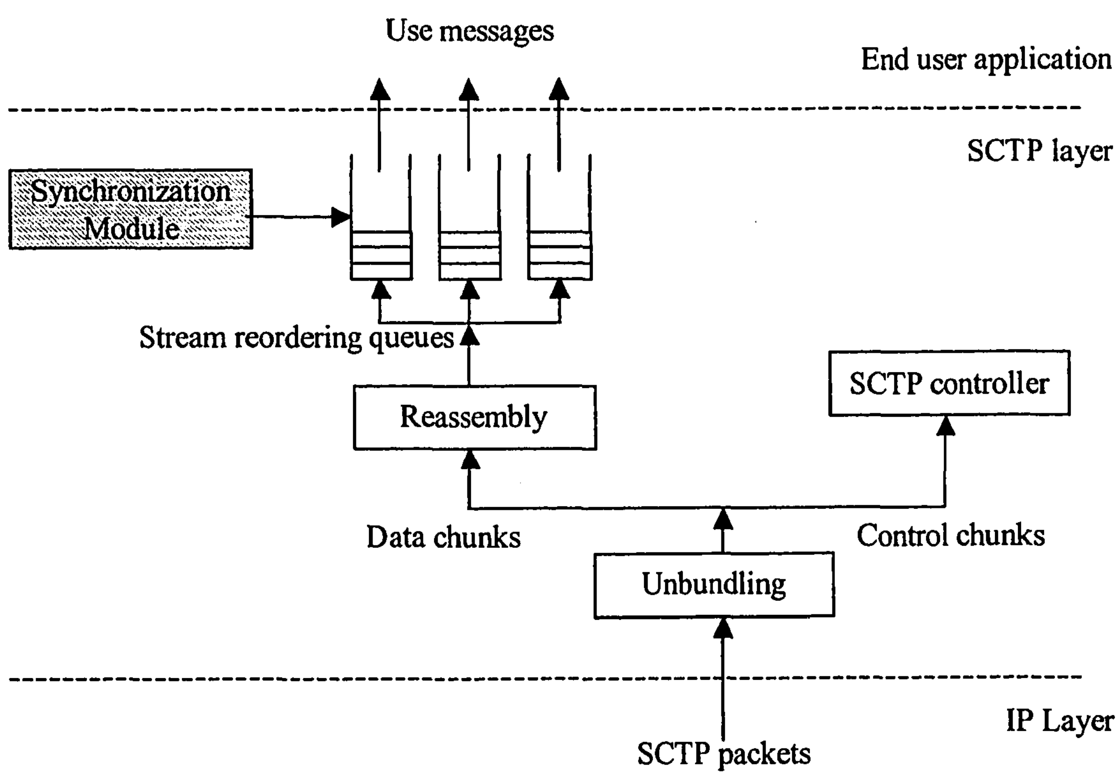



**Figure 8 SCTP receiver modules**

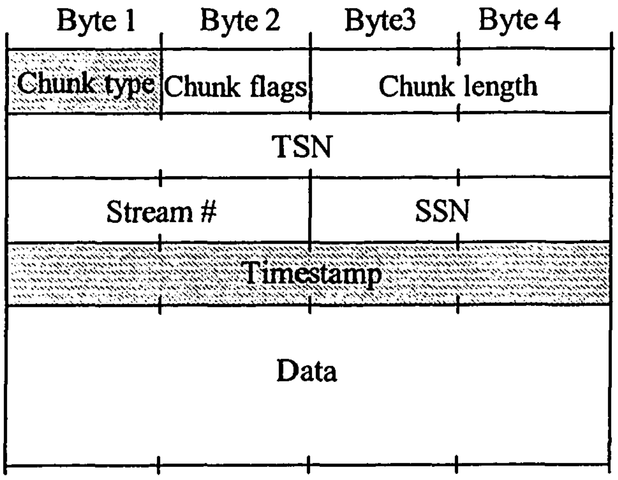



**Figure 9 Data chunk format with timestamp**

incoming packets.

The algorithm is simple and can be explained as follows. If a packet comes very late, all of the successive packets well be delayed with the same amount of delay. If the packet comes early, it will be delayed according to the previous mostly delayed one.

The advantage of this method is that the output IT is controllable and can be adjusted to be periodic traffic. The only disadvantaged of this method is that every packet will be delayed with the maximum delay even some of them may come early i.e. TD will be increased. Delaying the packets also requires buffer that depends on the amount of delay. Without going into details, the buffer size equals the difference between the maximum and minimum jitter. This disadvantage won't have a great effect because the jitter is always much less than TD. If the transfer delay of packet $i$ is $t_i$. The algorithm adds delay $D$, which can be expressed as follows,

$$\begin{aligned}
&\mathit{if} \quad t_i > t_{i-1} \\
&\quad \mathit{Then} \quad D_i = 0 \\
&\qquad\qquad t_{\max(i)} = t_i \\
&\mathit{Else} \quad D_i = t_i - t_{\max(i-1)}
\end{aligned} \tag{1}$$

Where $t_{max(i)}$ keeps track of the maximum transfer delay.

## 2.2 Average Delay Algorithm

Instead of dealing with the maximum delay and affecting the TD, we can consider the average value of the IT. This is a great idea also because not all of the packets come late but some of them come early and others come late. Hence, early packets will be delayed if the surrounding packets are late but they won't be delayed if the surrounding packets are early. The average can be accumulative average or average over n packets as described in the following subsections.

### 2.2.1 Accumulative Average Delay Algorithm

In this case we consider the average of TD of all packets, i.e. long-term average. The average will be taken as a reference to adjust the output IT. Considering the same variables in the previous section, the algorithm can be described as follows,

$$t_{a(i)} = \frac{(i-1)t_{a(i-1)} + t_i}{i}$$

Where $t_{a(i)}$ is the average delay that is used instead $t_{max(i)}$ in equation (1) The algorithm can be expressed in the following if statement,

$$\begin{aligned}
&\textit{if} \quad t_i > t_{a(i-1)} \\
&\textit{Then} \\
&\qquad D_i = 0 \\
&\textit{Else} \\
&\qquad D_i = t_{a(i-1)} - t_i
\end{aligned}$$

### 2.2.2 Average Over n Packets Algorithm

This algorithm is exactly like the above algorithm but the TD average is taken over a certain number of precious packets not over all packets. It can also be called moving average. In this case the algorithm can be described as follows,

$$t_{a(i)} = \frac{1}{n} \sum_{i-n}^{i} t_j$$

Again, $t_{a(i)}$ is the average delay over n packets, which is used instead $t_{max(i)}$ in equation (1). In this case, the algorithm can be expressed in the following if statement,

*if* $t_i > t_{a(i-1)}$
*Then*
$D_i = 0$
*Else*
$D_i = t_{a(i-1)} - t_i$

### 2.3 Simulation results

Simulation has been done to find the performance for each of the synchronization solution algorithms. MATLAB is used to analysis the algorithm. In our simulation, the input traffic is captured from a experimental testbed, which consists of four ATM switches act as core network and a single B-ISDN ATM test set, used as both a source and sink of the traffic stream. This system provides a controlled environment that allows us to emulate networks of different size (with up to 16 ATM nodes/ switches), and to examine the introduced jitter under almost any set of conditions that we considered important for our study. Thus, our end-to-end delay and interarrival measurements and the performance analysis should quite adequately depict the behavior of a typical WAN.

During our course of study, OPNET is also used to simulate an SCTP network to examine the delay and jitter. The jittery traffic information is saved in a file that is being read by MATLAB. The synchronization algorithm itself is implemented by MATLAB and is applied on the OPNET results in different cases. There is a noticeable decrease in the output jitter. Those results aren't shown in this chapter since the improvement is similar to the one of the experimental results. The following subsections introduce the simulation results in details.

### 2.3.1 Max Delay Algorithm

Figure 10 (a) and (b) show the TD and IT in case of maximum delay algorithm, respectively. Figure 11 shows the TD and IT histograms of the traffic before and after the algorithm. The TD and IT histograms are sharper but the TD is delayed more that the others. That is because all of the packets will be delayed based on the one that has maximum jitter. This is not a drawback because the jitter usually is much less than the TD. So that increasing the delay by small amount can be accepted. However, the following section describes another algorithm that overcomes this problem but doesn't provide as much improvement as this algorithm.

### 2.3.2 Average Delay Algorithm

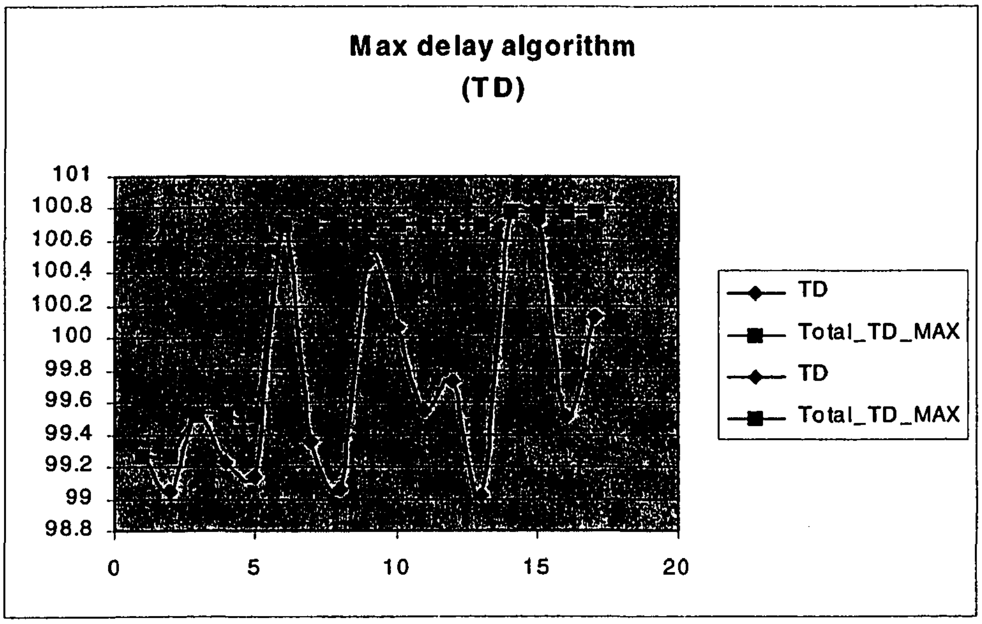
#### 2.3.2.1 Accumulative Average Algorithm

As in the Max delay algorithm, the TD and IT have been improved as shown in Figure 12. But still the improvement is less than the Max delay algorithm improvements. The average delay algorithm doesn't cause as much delay as the Max delay algorithm.

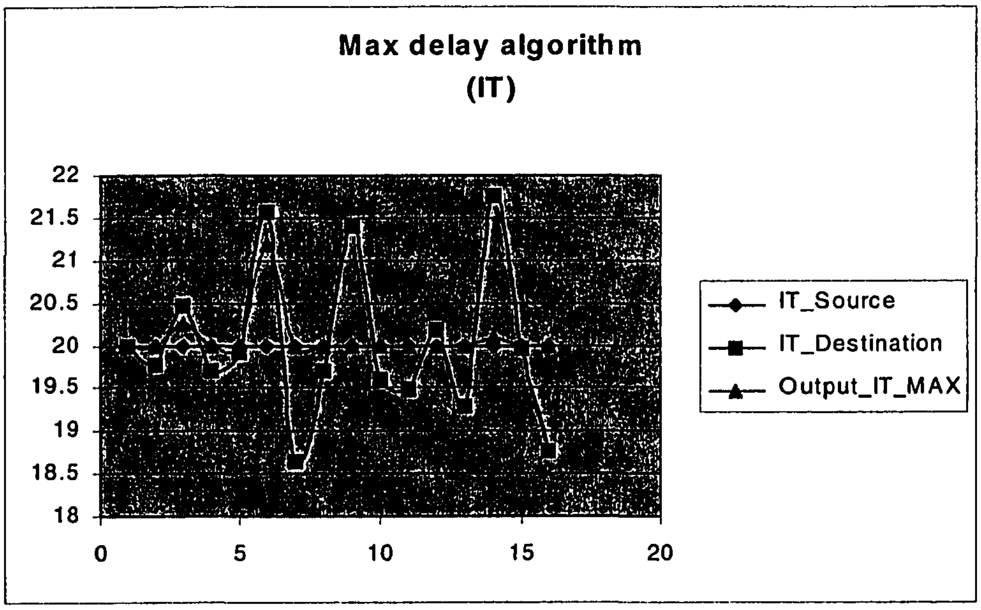
#### 2.3.2.2 Average Over n packets Algorithm

Figure 10 (c), (d), (e), and (f) show the timing diagram of TD and IT in two different cases. The first case is when the average is taken over two packets. The second case is when the average is taken over 5 packets. The second case give better jitter improvement where the IT deviation around the mean is less than that of the average over two packets case, as shown in Figure 11.

To determine how far should n go, we changed n and found the mean and standard deviation of TD and IT. Figure 12 shows the effect of TD and IT mean and standard deviation as a function of n. It is clear that they reach saturation as n increases. That means that in the average of n packets algorithm, we don't have to use the average over large number of packets.

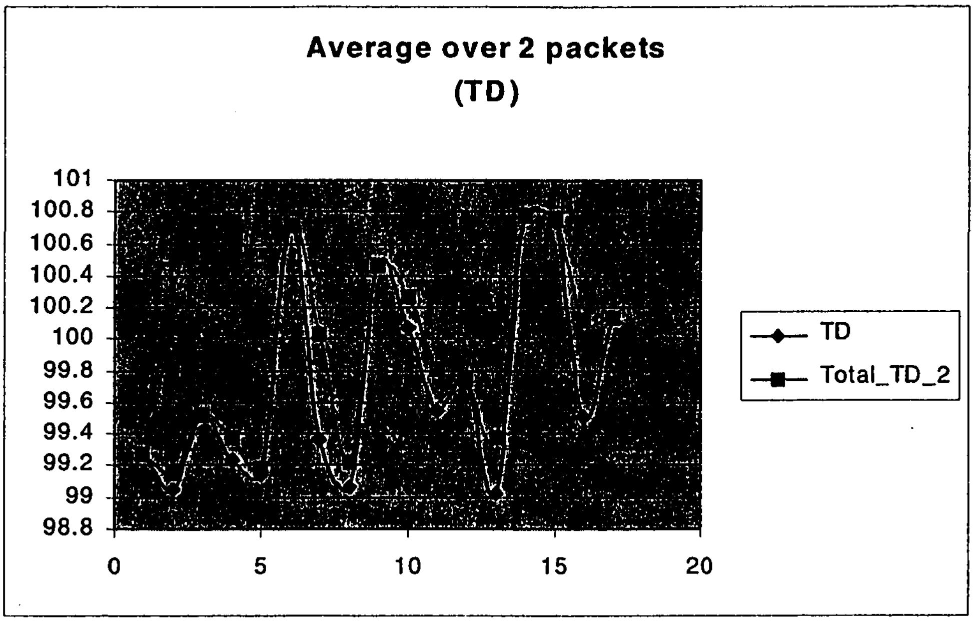



(a) TD of Max delay algorithm

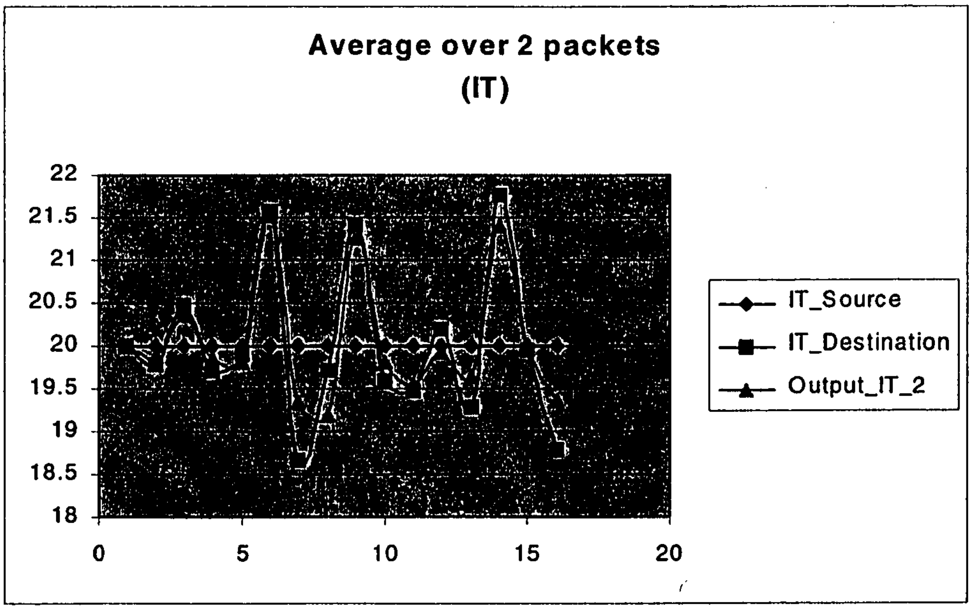



(b) IT of Max delay algorithm

**Figure 10 Timing diagram of the simulation results**

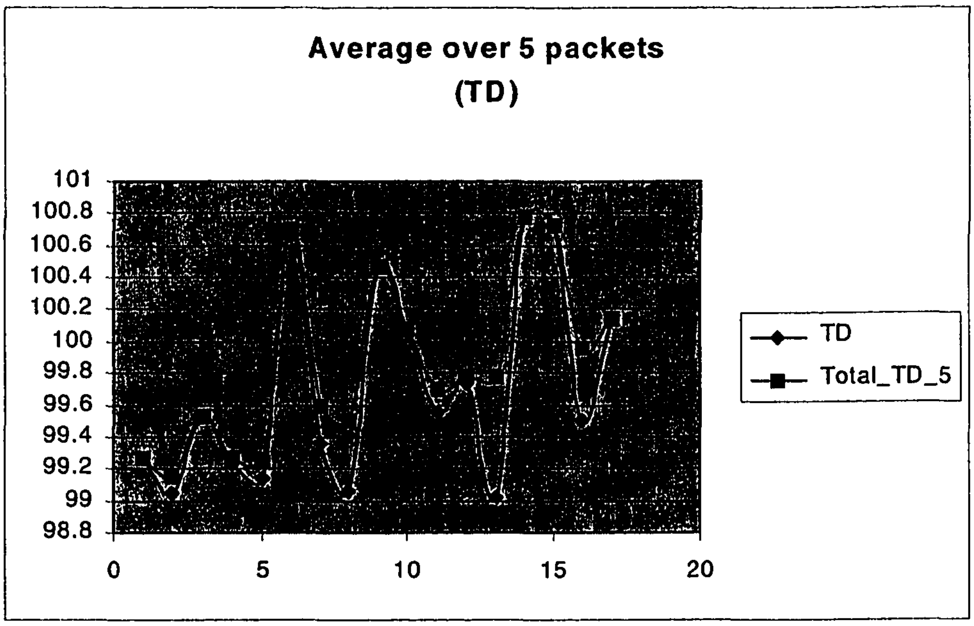



(c) TD of Average over 2 packets

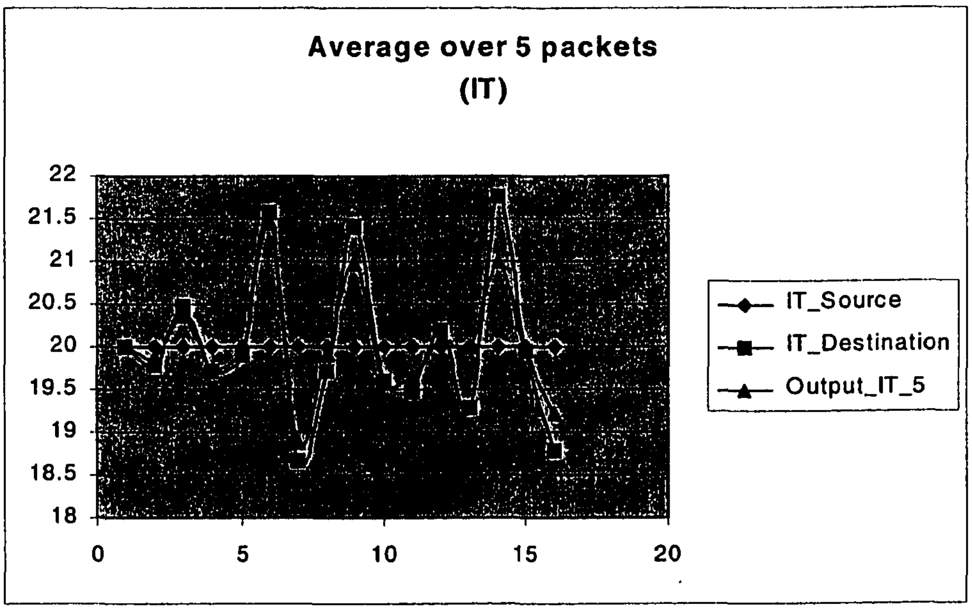



(d) IT of Average over 2 packets

**Figure 10 Timing diagram of the simulation results (Cont.)**

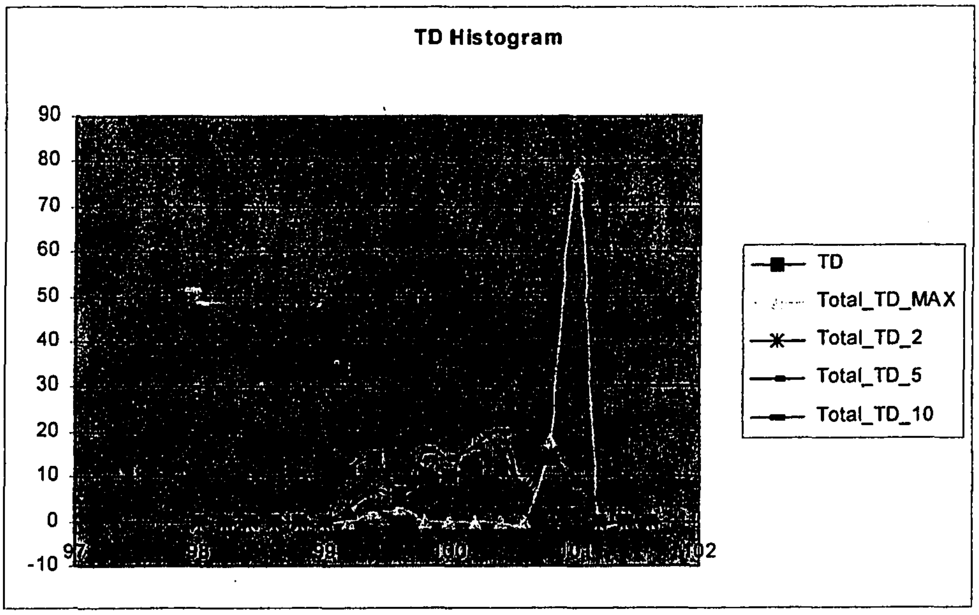



(e) TD of Average over 5 packets

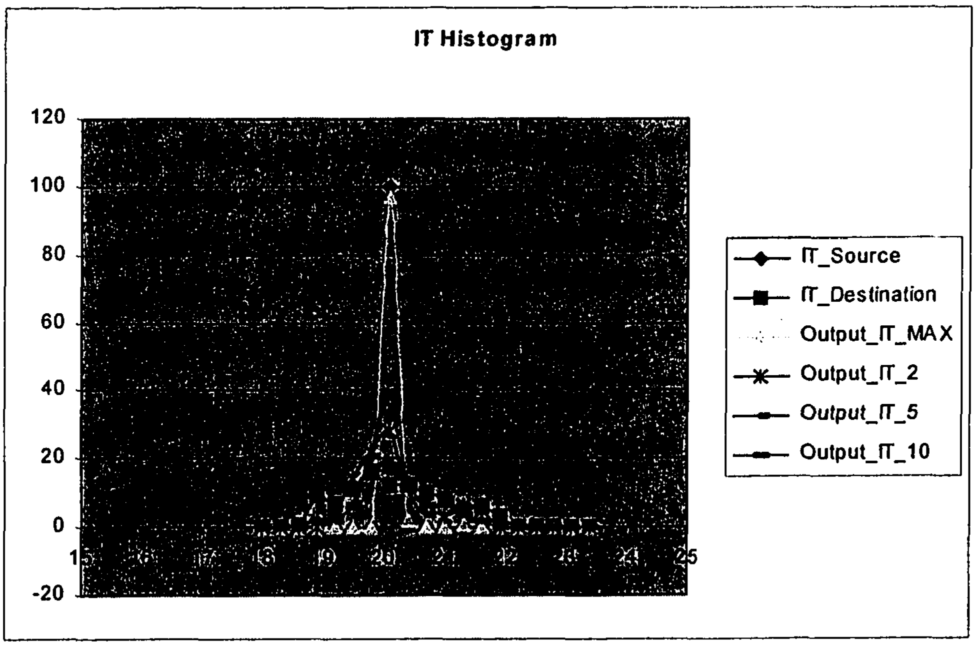



(f) IT of Average over 5 packets

**Figure 10 Timing diagram of the simulation results (Cont.)**

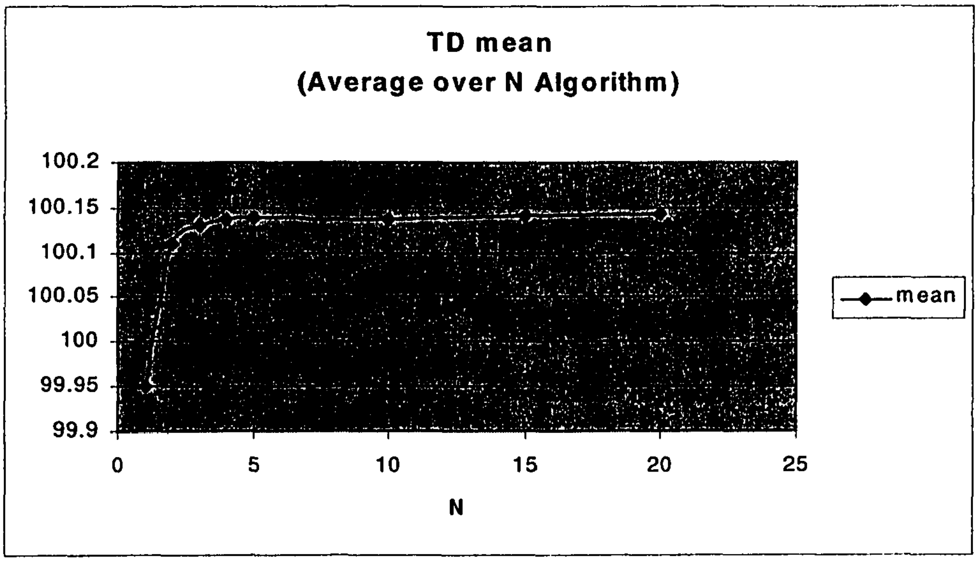



(a) TD histogram

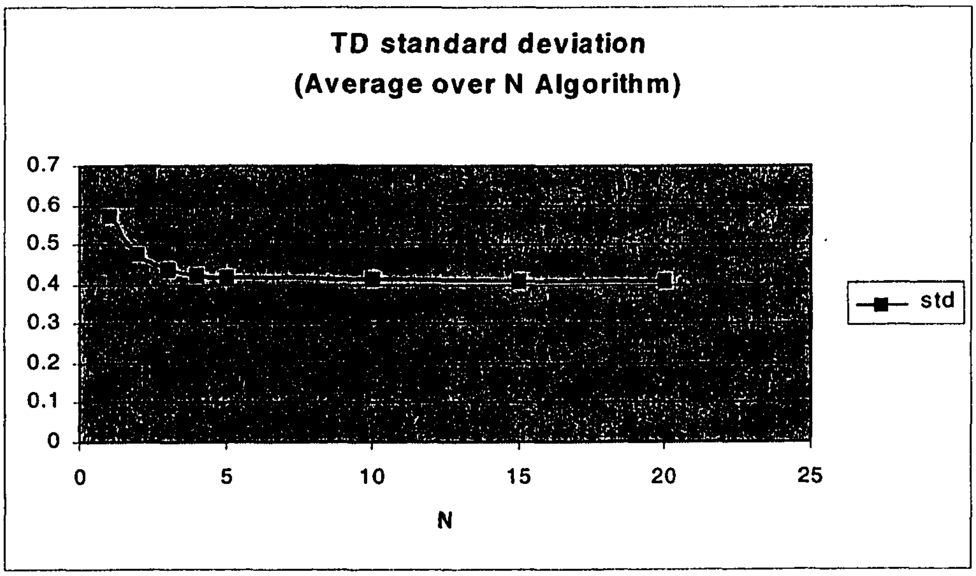



(b) IT histogram

**Figure 11 IT and TD histogram**

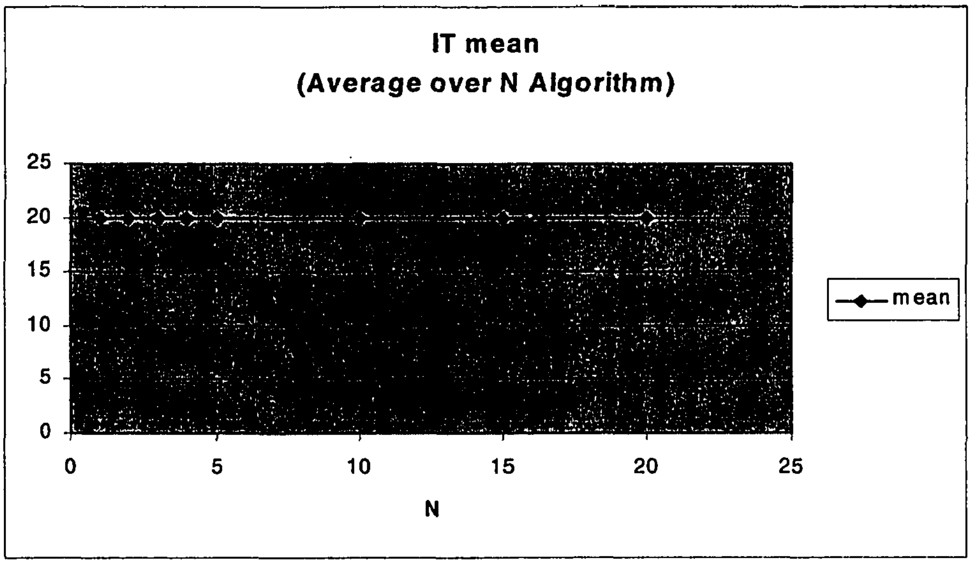



(a) TD mean

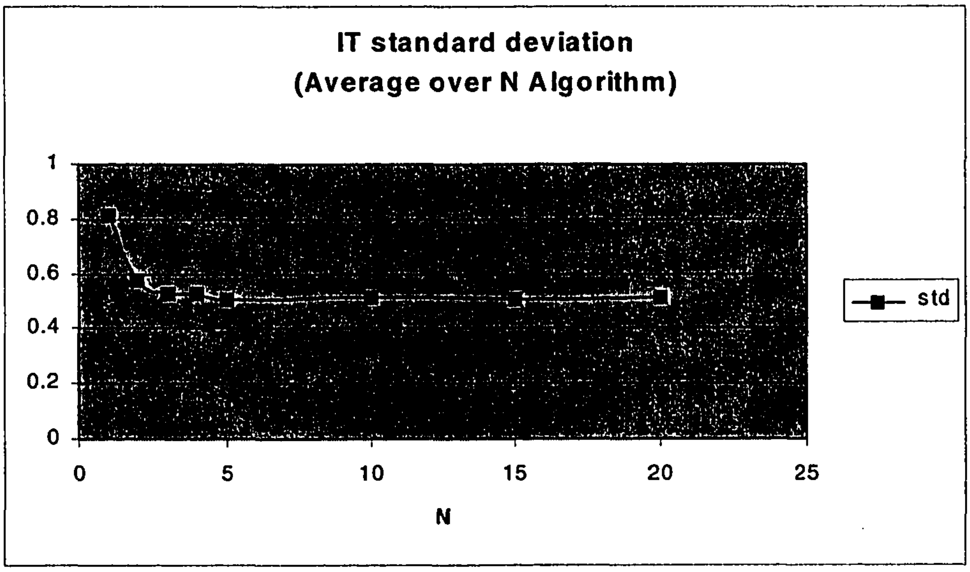



(b) TD standard deviation

**Figure 12 mean and standard deviation of TD and IT vs. n in case of average over n algorithm**




(c) IT mean




(d) IT standard deviation

**Figure 12 mean and standard deviation of TD and IT vs. n in case of average over n algorithm (Cont.)**

# 3 Chapter III Non-periodic Traffic Synchronization Algorithm

Unlike the periodic traffic, non-periodic traffic doesn't have constant interarrival time (IT). Furthermore, in most cases, there is no pattern for the IT change. But it is still important to some application to recover the timing pattern at the receiver end. For example, the video traffic contains the line and frame timing information. The synchronization between the video and audio signals is also important. Research has been done and found that there is no previous thesis to solve the non-periodic case.

Our thesis to solve the synchronization problem in case of non-periodic traffic is to send timestamp with the data packet, which is not described in the SCTP standards [9]. Figure 9 above shows the SCTP packet with the timestamp field. A modification is also introduced to improve the network performance and reduce the overhead. After some study and investigation, we found that the protocol efficiency can be improved by modifying the transmitter and receiver modules for non-periodic traffic transmission. Sending the timestamp with each and every chunk is redundant and can be improved. The idea can be explained using Nyquist theory. It states that any signal can be recovered if it is sampled at rate equals to or greater than Nyquist rate.

By looking at the IT as a regular function of time, we can send timestamp with only Nyquist rate. The destination can recover the IT function and reassign it to its corresponding TSN (Transmission Sequence Number). The following subsections describe the transmitter and receiver module required modification to implement the proposed theory.

### 3.1.1 Transmitter Module

We are going to describe the transmitter module function in case of using Nyquist theory. Sending the timestamp with each chunk has been discussed in an early section.

The transmitter module looks to be simple by sending the timestamp with chunks at a rate equals to or greater than Nyquist rate, but it is not that simple. The original chunks represent signal at rate, which is not periodic or follow predetermined pattern. Nyquist theory requires sampling at periodic rate. To solve this problem, the transmitter calculates the corresponding IT at mT using interpolation as shown in Figure 13, where T is the sampling rate and m is the sample number. The IT at mT will be sent with the next chunk. The destination should note that the received timestamp is not the IT of the received TSN but it is the IT at mT. The transmitter module steps can be summarized as follows,

- Determine the max frequency component in the IT pattern.
- Determine the sampling rate, which equals to or greater than Nyquist rate depending on the application.
- Calculates the corresponding IT at mT using interpolation.
- Send the calculated IT in the next chunk timestamp.

Figure 14 shows the flow chart of the transmitter module. It deals with IT as a regular function of time and determines the maximum frequency using Fourier transform. Then it determines the Nyquist rate, which is the minimum sampling rate necessary to reconstruct the IT function at the receiver. That is not indeed the rate that will be sent. The application may require any rate greater than Nyquist rate. For example if the communication link encounter high loss rate, like in wireless communications, the application may require more timing information. Once the transmitter determines the sampling rate, say m packets per second, it adds the timestamp at packet with SSN is multiple of m. The timestamp contains the IT at the previous *nT* period. Finally, it checks if there is nay more packet ready for transmission, if not, it ends the process.

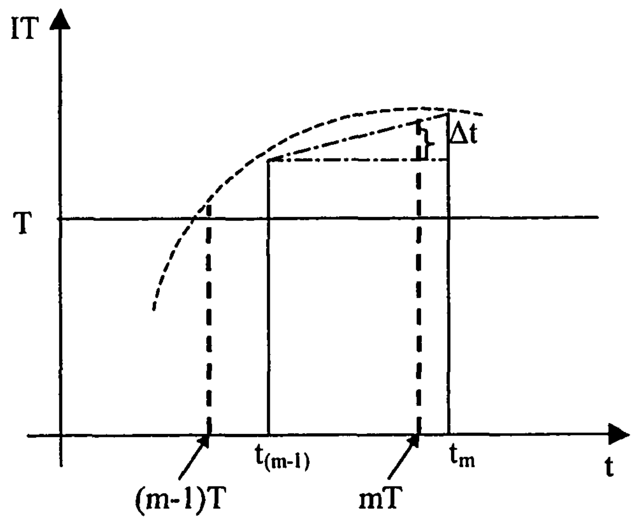



**Figure 13 Transmitter module**

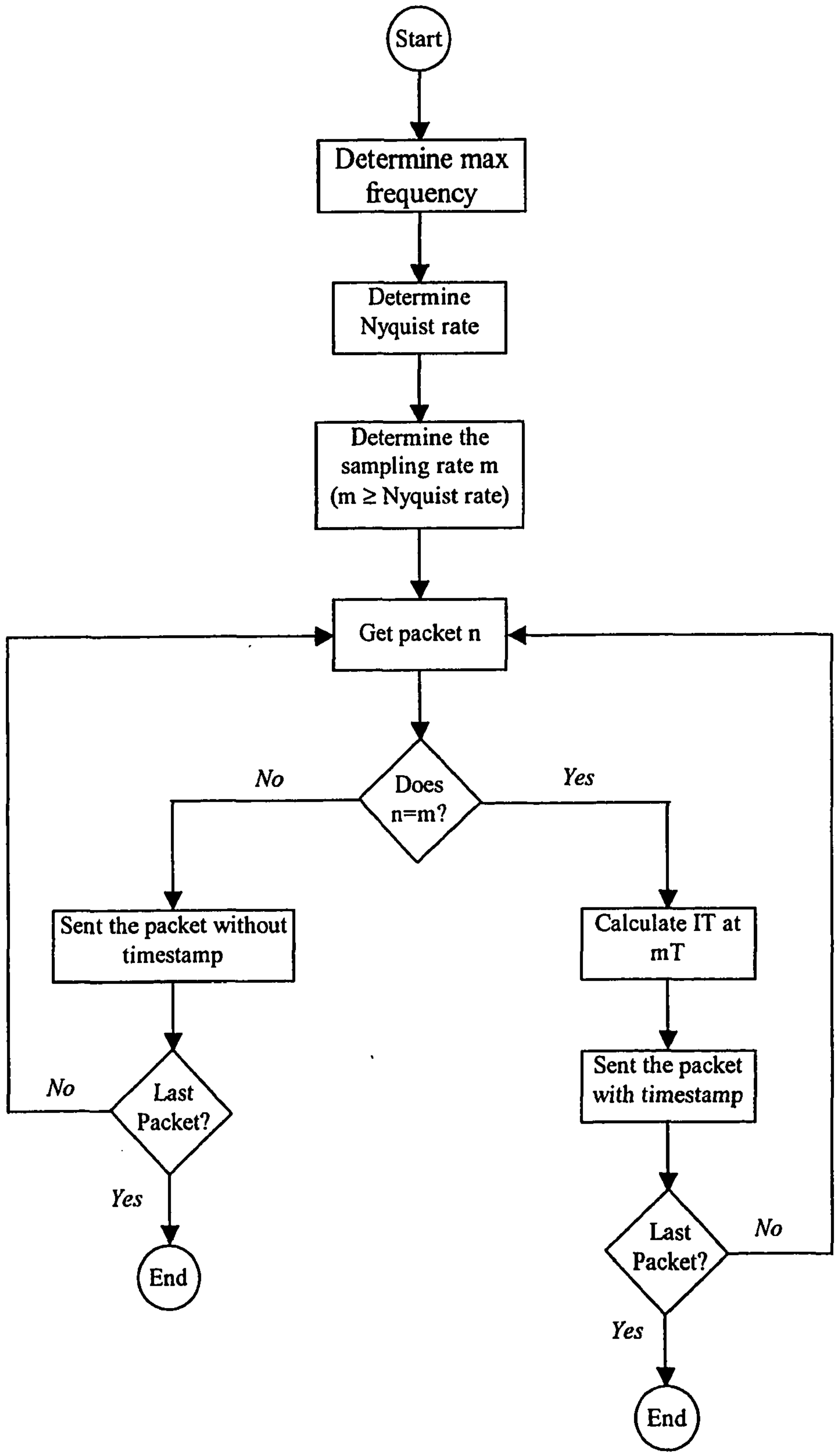



**Figure 14 Flow chart of the transmitter module**

### 3.1.2 Receiver Module

The main function of the receiver module is to recover the IT timing function. The transmitted timing information, at *nT*, can be easy recovered using low pass filter and the IT at mT. Since the SCTP packet with TSN n is jittered and is received before or after the correct time, which is not necessary *nT*, the receiver module function is to recover that IT value.

Figure 15 shows the two SCTP packets number (n-1) and n. Packet (n-1) is received without jitter and doesn't need any adjustment. Packet n is jitter by +Δt, which is represented by the solid line. To recover the correct IT for packet n, draw a line with 45° and find its intersection with the IT diagram. This is the correct IT value for packet n, which can be explained as follows: The x-axis represents the time. If the jitter is +Δt, then, the packet will be shifted by +Δt. In the mean time, the y-axis represents the IT. Again, if the jitter is +Δt, then, the IT is incremented +Δt. That means that the slope of the line connecting the original and the new IT values is 45°.

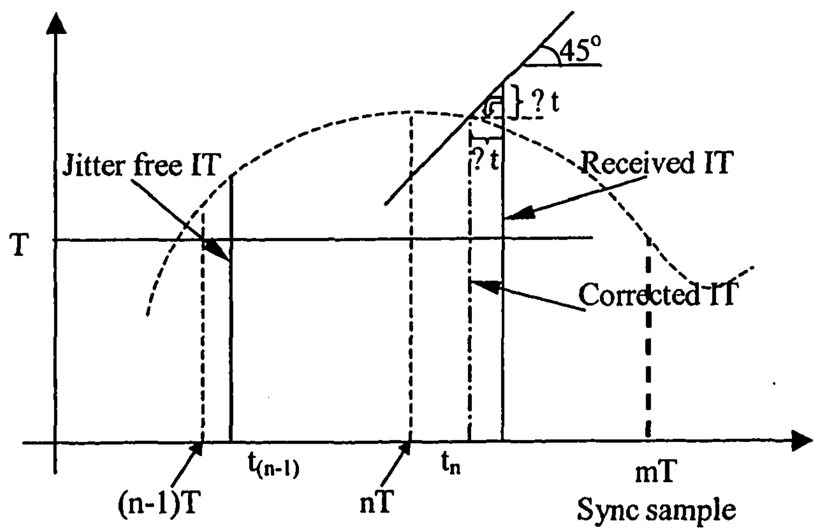



**Figure 15 Receiver module**

The flow chart of the receiver module is shown in Figure 16. The receiver first determines whether the received packet has timestamp or not using the chunk type field. Then there are two main processes to recover the IT. The first one is executed *if* the packet has timestamp to recover IT information at *nT*. It captures IT at the sampling rate, mT, from the received packets. The IT information at *nT* can be reconstructed by obtaining Fourier transform of IT at *mt*, then the inverse Fourier transform at *nT*. That is like LPF (Low Pass Filter) operation.

The second process is to find IT at the packet time, $t_n$, by finding the intersection between the IT curve ate *nT* and 45o line drawn at the received IT (the jittery one) as explained above.

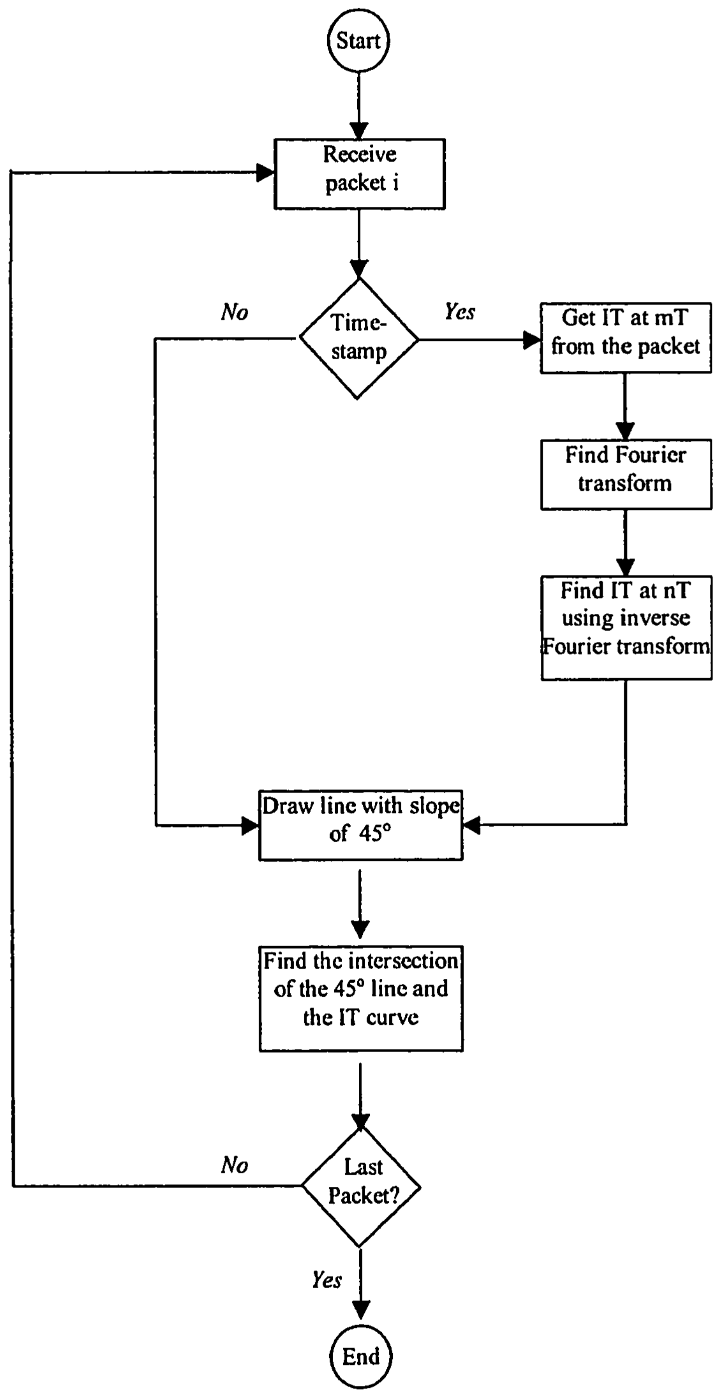



**Figure 16 Flow chart of the receiver module**

### 3.1.3 Non-periodic traffic simulation results

We have simulated the above algorithm to recover the non-periodic traffic IT using MATLAB since it is capable of performing the required numerical mathematical manipulation. In our simulation, we considered the non-periodic traffic with different patterns. In this thesis, the sine wave one is introduced. We Also considered that the transfer delay is 100 msec and introduced jitter is 2 msec. Those values are typical ones and as shown by laboratory results in [10]

Figure 17 shows the timing diagram of at the source and destination. Because of the jitter, the received IT fluctuates around the transmitted IT diagram. In this simulation, we send the IT timestamp at four times the maximum frequency in the original IT. The IT is completely recovered and we are able to determine the IT at any time spot on the diagram.

Figure 18 shows a zoom of the timing diagram to illustrate the theory. The timestamp is sent every 5 packets. If we use Nyquist rate (double the maximum frequency) instead of four times, we could have sent one timestamp each ten packets. It depends also on the rate of IT change, i.e. if the IT change is slow with respect to the packet rate, our theory saves more overheads.

As it was mention in earlier, the IT recovery consists of two processes. Figure 17 shows the first process to reconstruct the IT at the periodic rate *nT*. The second process is shown in Figure 19, which shows the recovery of IT of the original packets (non-periodic packets). Using the 45o line, The IT is recovered and it is identical to the original IT at the transmitter before the jitter is introduced. "IT_Destination_Algorithm" is the inter-arrival time at the destination produced by the algorithm. To make it clear, Figure 20 shows zoom around the second packet.

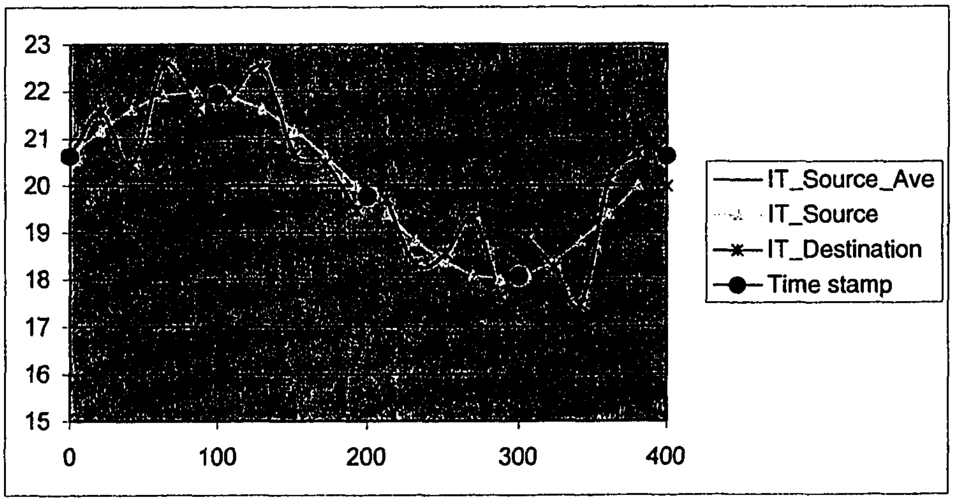



**Figure 17 Non-periodic timing diagram**

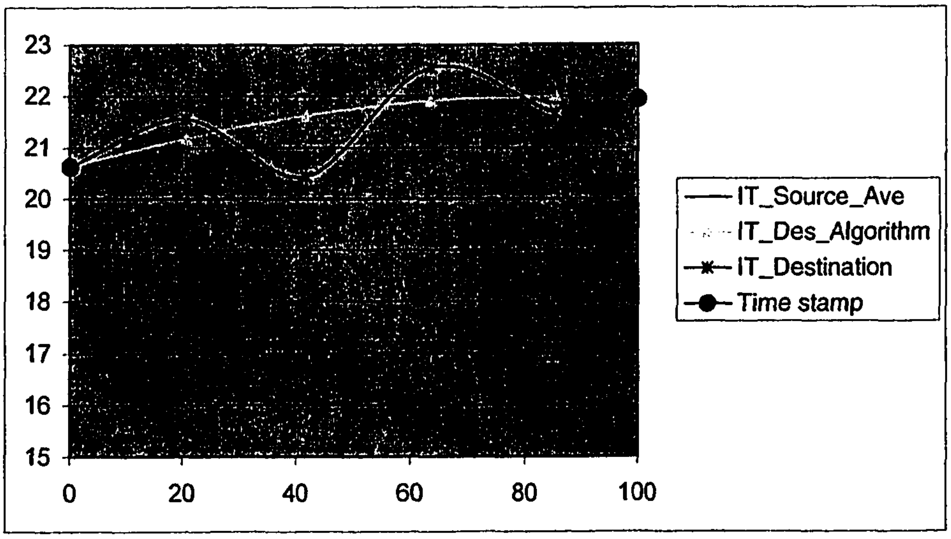



**Figure 18 Zoom of timing recovery**

It is worth mentioning that the 45° line may intersect with the IT curve at more that one point. In most cases the inappropriate points can be detected by the out of sequence SSN. In other words, wrong intersection is located between two SSN numbers away from the one under process. If the points are located in the right SSN location, the nearest IT is considered. Even if it is not the right IT, it will not propagate the error to the successive packets since they are treated separately. It is not only that, but even in the worst case when a packet is lost, it will not affect any other packet timing recovery since the IT recovery of any packet is independent of the others. It depends only on the 45° line at its arrival time and the IT curve. This is also considered as a strong advantage of our algorithm.

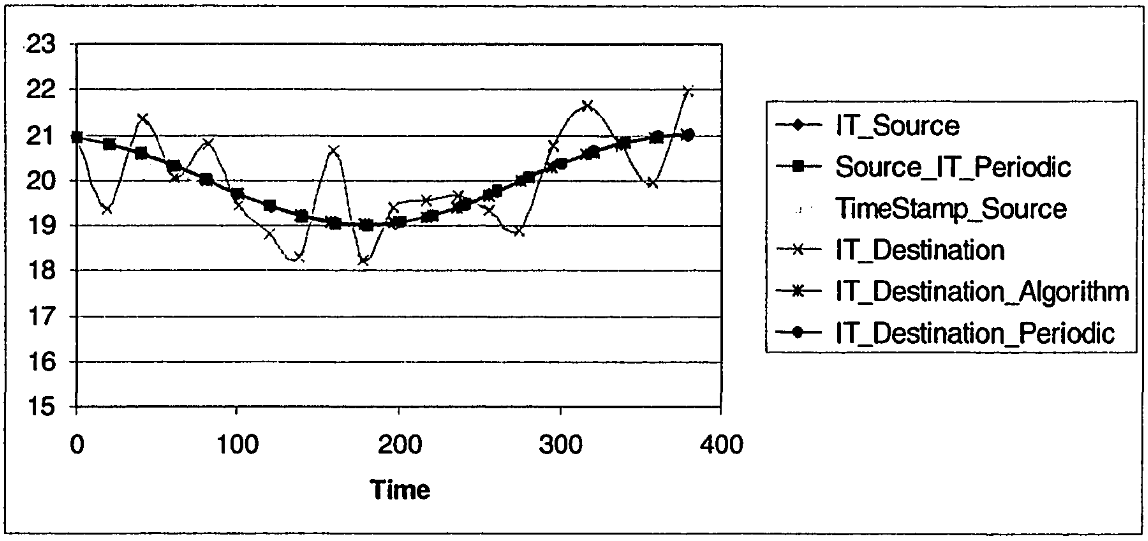



**Figure 19 Non-periodic IT recovery**

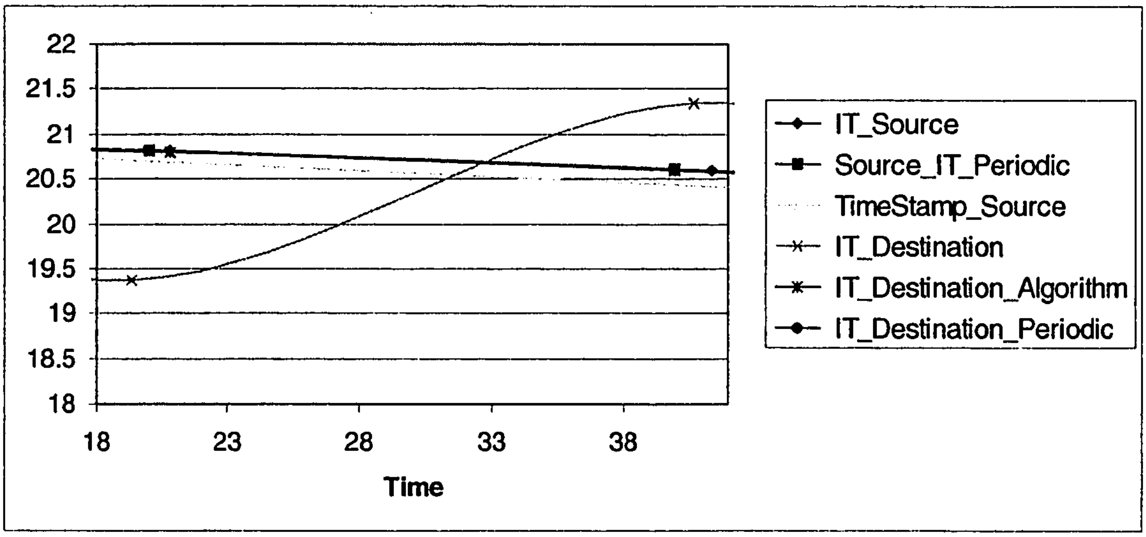



**Figure 20 Zoom of non-periodic IT recovery**

# 4 Chapter IV Synchronization Algorithm Analytical Model

Since there are three different proposed algorithms, there are three flavors of the analytical model. First, we analyze the maximum delay algorithm case, which is somehow simple. They will extend the analysis to the accumulative delay algorithm, which is very much similar to the maximum delay algorithm but different approach. Finally, we analyze the average over n algorithm, which will take most of the effort.

In all of our analysis we consider that the jitter has upper limit. In other words, the jitter does reach infinity, which is a practical assumption. That means that the inter-arrival time (IT) has lower and upper limits, $J_l$, and $J_u$ respectively. We use $\tau$ as the average inter-arrival time. The main goal of the analysis is to find the mean and variance of the output traffic.

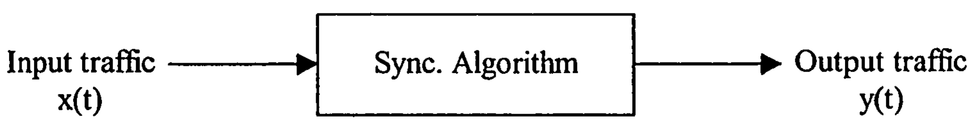



**Figure 21 Synchronization algorithm representation.**

## 4.1 Probability overview.

Our analysis is based on transformation of random variable principle. We are going to use some fact and theory of probability. In this section, we will discuss important terms that help our analysis.

### 4.1.1 Transformation of random variable

Transformation of random variable (RV) is the main idea to find analytical model for the proposed algorithm. Our concern is two types of transformation of RVs. The first one is the transformation of one RV to another based on certain function. The second type is the transformation from multiple RV to one RV. The first type of transformation of RV can be described as follows,

Knowing $f_x(x)$ and $y=g(x)$, what is the value of $f_y(y)$. We are not going to prove the following equation since it is hugely used in probability.

$$f_y(y) = f_x(y)\left|\frac{dy}{dx}\right|_{x=y} \tag{2}$$

#### 4.1.1.1 Sum of Random Variables

The second type of transformation of RV, which is transformation from multiple RVs to one, is different. Knowing the pfd of certain random variables, $x_1$, $x_2$, ... , $x_n$, and the relation between those random variables and another random variable, y, what is the pdf of y. The following questions describe the purpose,

Knowing $f_{x_1 x_2 \cdots x_n}(x_1, x_2, x_3, \cdots, x_n)$,

and $y = g(x_1, x_2, x_3, \cdots, x_n)$

what is the pdf of y, $f_y(y)$

Since the above definition is general one, we will consider the cases, which help our analysis. The two important cases are the sum and average of random variable. Let us consider first the sum of random variables described in the following equation,

$$y_n = x_1 + x_2 + x_3 + \cdots + x_n \tag{3}$$

It is not easy to fine the pfd of y from equation (3) direct. So, we will divide it into parts as follows, find first the pfd of adding the first two items, then add the result to the third item and find the pdf. Continue doing that till the last item is reached. So, the first two items are,

$$y_2 = x_1 + x_2$$

$$\begin{aligned} f_{y_2}(y_2) &= f_{x1}(x_1) * f_{x2}(x_2) = \int f_{x1}(x_1) f_{x2}(y_2 - x_1) dx_1 \\ &= f_{x2}(x_2) * f_{x1}(x_1) = \int f_{x1}(y_2 - x_2) f_{x2}(x_2) dx_2 \end{aligned} \tag{4}$$

#### 4.1.1.2Average of Random Variables

In this section, we will consider the average of random variable expressed in equation (5) below.

$$y_{an} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \tag{5}$$

We can notice that equation (5) is nothing but the sum divided by the number of RVs $n$. That means that we don't have to redo the analysis but will use the result in section 4.1.1.1 as follows,

$$y_{an} = \frac{y_n}{n} \tag{6}$$

Knowing the pdf of $y_n$, the pdf of $y_{an}$ can be found using equation (2) as follows,

$$f_{y_{an}}(y_{an}) = f_{y_n}(y_{an}) \left| \frac{dy_n}{dy_{an}} \right|$$

Where,

$$\frac{dy_n}{dy_{an}} = n$$

Then,

$$f_{y_{an}}(y_{an}) = nf_{y_n}(y_{an}) \tag{7}$$

### 4.1.2 Central limit theory

Central limit theory is one of the probability background that helps our analysis as the number of RVs increases. Given that $x_1$, $x_2$, ... and ,$x_n$, are $n$ independent RVs, with arbitrary pfd distributions, and (mean, variance) as ($\mu_1$, $\sigma_1^2$), ($\mu_2$, $\sigma_2^2$), ... , ($\mu_n$, $\sigma_n^2$). The pfd of $w_n$, given by equation (8), is normal distribution as $n$ tends to be infinity. The mean and variance $\mu_{wn}$, $\sigma_{wn}^2$ are described by the following equations,

$$w_n = x_1 + x_2 + x_3 + \cdots + x_n \tag{8}$$

$$\mu_{wn} = \mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n$$

and,

$$\sigma_{wn}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \cdots + \sigma_n^2$$

An important special case is when all of the RVs, $x_1$, $x_2$, ... and ,$x_n$, have the same pfd, mean and variance namely, $\mu_x$, $\sigma_x^2$,

$$\mu_{an} = n\mu_x$$

and,

$$\sigma_{an}^2 = n\sigma_x^2$$

The above two equations will be used as they are applicable to our analysis.

### 4.1.3 Taylor Series

Taylor series is a polynomial expansion of a give function. It states that the derivative function can be expanded around a point a in the following polynomial form,

$$f(x) = f(a) + f^{/}(a)(x-a) + \frac{f^{//}(a)(x-a)^2}{2!} + ... + \frac{f^{(n)}(a)(x-a)^n}{n!} + ...$$

The polynomial expansion is useful in case of difficult mathematical representation. For example, if the integration is unresolved one, a polynomial approximation with a valid assumption may help. Here, we are going to introduce a polynomial expansion of exponential and normal functions, which are going to be useful for our analysis.

#### 4.1.3.1 Exponential expansion

The exponential function and its expansion are as follows

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} \qquad (9)$$

#### 4.1.3.2 Normal expansion

The Normal distribution function and its expansion are as follows

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{x^2}{2} + \frac{x^4}{2! * 2^2} - \frac{x^6}{3! * 2^3} + ... + (-1)^n \frac{x^{2n}}{n! * 2^n} + ... \right) \qquad (10)$$

The above distribution is the standard normal distribution with zero mean and variance of 1. If the normal distribution under discussion is not standard, a transformation of variables can be used to get the standard normal distribution described above. We will leave this whenever is needed in the analysis.

## *4.2 Maximum delay algorithm.*

The maximum delay algorithm is an important one since it provides the best results. This case analysis is not difficult if there is good understanding of the algorithm. Figure 21 shows the timing diagram of input and output traffic, x(t) and y(t) respectively. Since the maximum delay algorithm keeps track of the maximum value of x(t), we can divide the timing diagram into two regions. The first region is the transit part where the maximum hasn't been reached. The second region is where it reaches the maximum, which represents the steady state part of the timing diagram. Figure 23 shows the histogram diagram of the steady state response, which is our concern. The mean and various of the output traffic y(t) can simply be found as follows,

$$\mu_y = J_u$$

$$\sigma_y = 0$$

The probability density function of y(t) can be described as follows,

$$f(y) = \delta(y - J_u) \tag{11}$$

As it was mentioned earlier, the maximum delay algorithm may require large buffer if the traffic burst is too high. So, it is important to find out the buffer size in this case. Figure 22 and Figure 23 show that the incoming traffic is stored if it below the maximum, which is always the case. Then, the maximum buffer size is required when the earliest packet arrives, which comes at $J_l$. They also show that the output packets have the same end-to-end delay independent on the network jitter. That means that the output traffic has the same IT as the original traffic at the transmitter end. The maximum delay algorithm may not work well if the network introduces large burst of jitter, which means $J_u$ tends to be infinity. That is not a considerable drawback since it is not always the case in real networks as indicated in[10]. The following subsection provides a case study of this algorithm where the network jitter is uniform distribution.

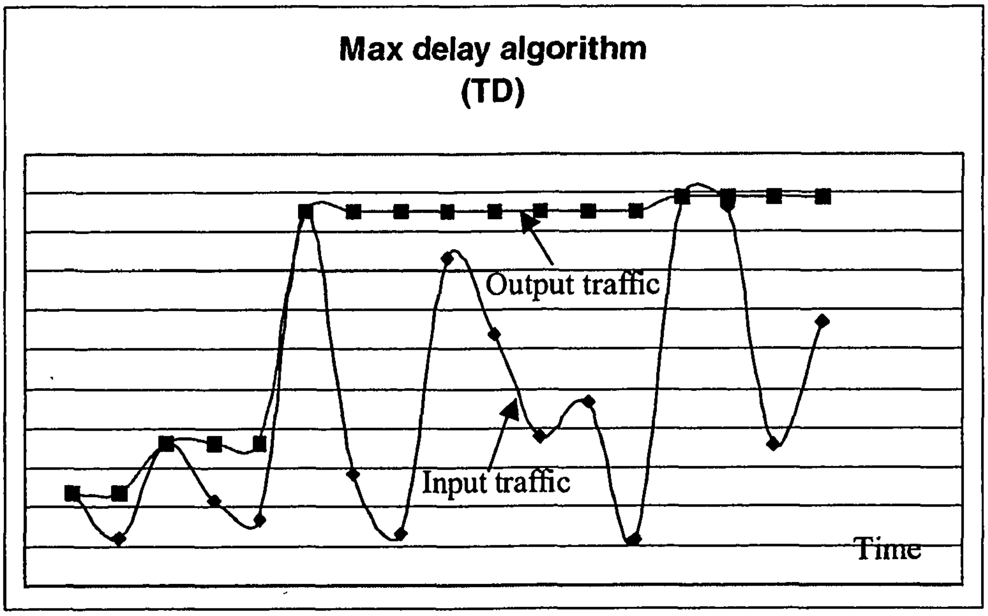



**Figure 22 Maximum delay algorithm illustration**

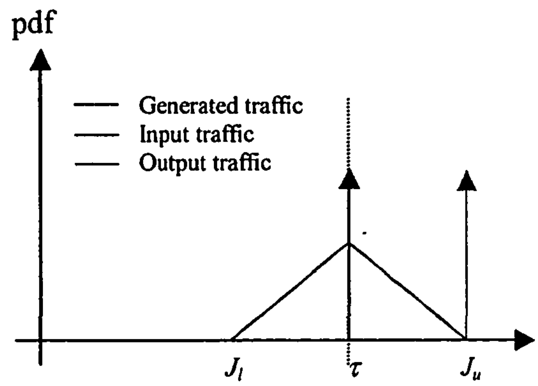



**Figure 23 Packet histogram for maximum delay algorithm**

### 4.2.1 Case study (Uniform distribution)

The uniform distribution is one of the common pdf distributions in data networks. Its importance in the synchronization world comes from the upper and lower limits of the distribution, which represents the real traffic pattern. So that it will be our focus in this thesis. The uniform distribution is shown in Figure 24 and it is described as follows,

$$f_X(x) = \frac{1}{J_u - J_l}\left[U(x - J_l) - U(x - J_u)\right] \tag{12}$$

Where, *U(x)* is the unit step function. The output traffic pdf can be expressed as an impulse function as described in equation (11) above. Figure 24 shows the input and output traffic pfd functions. The output pdf is a delta function at $J_u$ with magnitude of one, which is the area under input traffic pdf curve. It is worth mentioning that in the maximum delay algorithm, the output traffic pdf doesn't depend on the input distribution but depends only on the upper limit of the input traffic jitter $J_u$. That is not the case in the average delay algorithm as it will be shown in later sections.

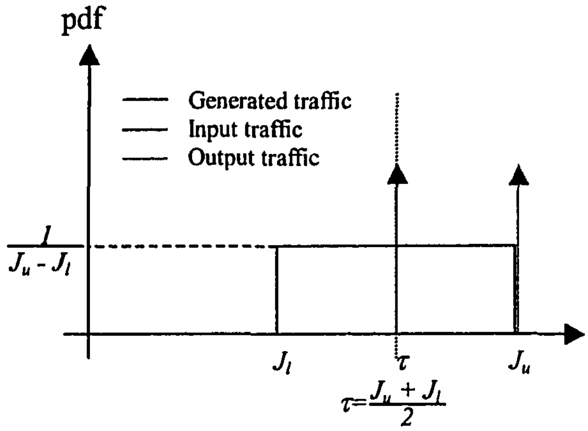



**Figure 24 Packet histogram for maximum delay algorithm**
**Case study (uniform distribution)**

## 4.3 Accumulative Average algorithm

In this case, the accumulative average is taken as the synchronization reference. If the input packet delay is more than the average, it will be delivered once it arrives. If the packet comes early, it will be delayed to the average delay as shown in Figure 25. Like the maximum delay algorithm, the output traffic timing diagram can be divided into two regions, transit and steady state. Again, our concern is the steady state region, where the average is constant and equals to $\tau$. The accumulative average algorithm has special importance as it represents the extreme case of the average over n algorithm. In other words, when n tends to be infinity.

Figure 26 shows the histogram of the input and output traffic in case of accumulative delay algorithm. The packets below the average are delayed to the average value so that there is an impulse at τ. The packets above the average are delivered once they arrive. The output probability density function can be expressed as follows,

$$f_y(y) = f_x(y)U(y-\tau) + F_x(\tau)\delta(y-\tau) \tag{13}$$

Where, $U(y\text{-}\tau)$ and $\delta(y\text{-}\tau)$ are the unit step and delta function delayed by τ respectively. The mean and variance of $y(t)$ can be found as follows,

$$\begin{aligned}\mu_y = \overline{yf_y(y)} &= \int_{J_l}^{J_u} yf_y(y)dy \\ &= \int_{J_l}^{J_u} y\left[f_x(y)U(y-\tau) + F_x(\tau)\delta(y-\tau)\right]dy \\ &= \int_{\tau}^{J_u} yf_x(y)dy + \int_{\tau}^{J_u} yF_x(\tau)\delta(y-\tau)dy \\ &= \int_{\tau}^{J_u} yf_x(y)dy + \tau F_x(\tau)\end{aligned} \tag{14}$$

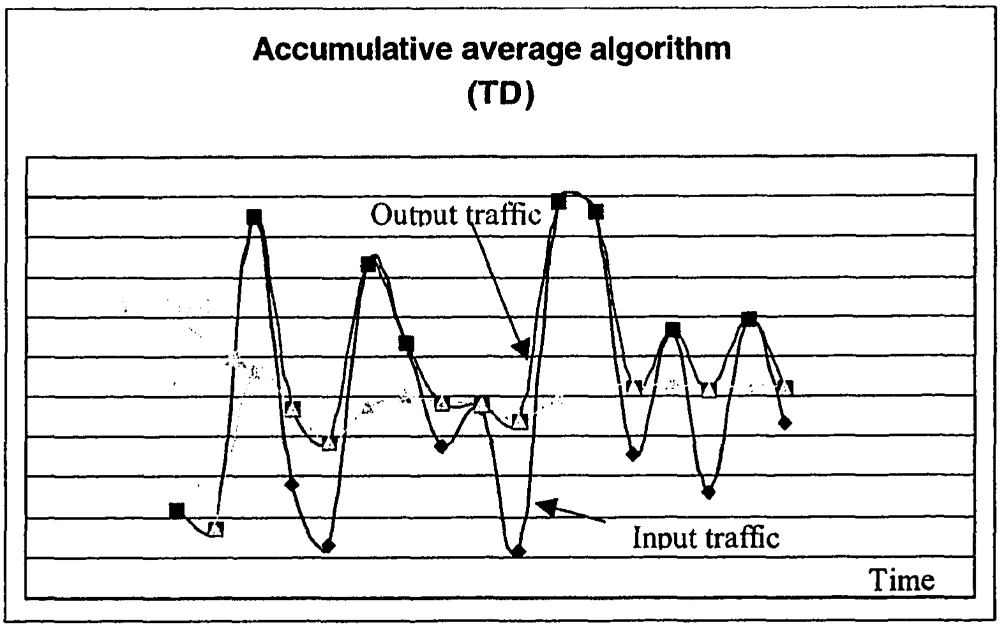



**Figure 25 Timing diagram in of the accumulative average algorithm**

By adding and subtracting the term $\int_{J_l}^{\tau} y f_x(y) dy$, then,

$$\begin{aligned}\mu_y &= \int_{\tau}^{J_u} y f_x(y)dy + \tau F_x(\tau) + \int_{J_l}^{\tau} y f_x(y)dy - \int_{J_l}^{\tau} y f_x(y)dy \\ &= \int_{J_l}^{J_u} y f_x(y)dy + \tau F_x(\tau) - \int_{J_l}^{\tau} y f_x(y)dy \\ &= \tau + \tau F_x(\tau) - \int_{J_l}^{\tau} y f_x(y)dy \\ &= \tau[1 + F_x(\tau)] - \int_{J_l}^{\tau} y f_x(y)dy \end{aligned} \tag{15}$$

Which leads to,

$$\mu_y \le \tau[1 + F_x(\tau)]$$

Figure 26 shows also that,

$$\tau \le \mu_y \le J_u$$

Then, $\mu_y$ boundary can be found as follows,

$$\tau \le \mu_y \le Min[J_u, \tau(1 + F_x(\tau))] \tag{16}$$

Inequality (16) provides upper and lower limits of output traffic mean. It indicates that the output traffic mean is somewhere between the input traffic average, $\tau$, and the maximum input traffic jitter, $J_u$. This result enforces our discussion earlier since this algorithm increases the delay of the early packets, which, in turn, increases the overall average of the output traffic delay. In most of the cases, the input traffic jitter is less much less than its average delay as indicated in [10], which means that the output traffic delay doesn't increase too much.

Similarly, the variance can be found as follows,

$$\sigma_y^2 = \overline{(y-\mu_y)^2 f_Y(y)} = \int_{J_l}^{J_u} (y-\mu_y)^2 f_Y(y)dy$$
$$= \int_{J_l}^{J_u} (y-\mu_y)^2 \left[f_X(y)U(y-\tau) + F_X(\tau)\delta(y-\tau)\right]dy$$
$$= \int_{\tau}^{J_u} (y-\mu_y)^2 f_X(y)dy + \int_{\tau}^{J_u} (y-\mu_y)^2 F_X(\tau)\delta(y-\tau)dy$$
$$= \int_{\tau}^{J_u} (y-\mu_y)^2 f_X(y)dy + (\tau-\mu_y)^2 F_X(\tau) \quad (17)$$

Equations (15) and (17) provide the mean and variance of the output traffic depending on the distribution. In the following subsections, we are going to consider two of the most important traffic distributions in the network, namely, uniform and normal distributions. Our goal is to find the amount of increase in the output traffic average delay and decrease in its variance. The variance decrease is our indication of the reduced jitter.

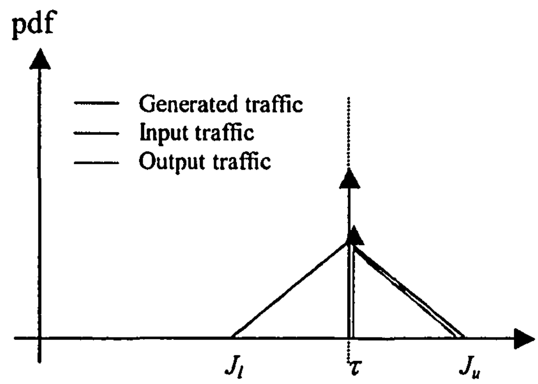



**Figure 26 Packet histogram for accumulative average algorithm**

### 4.3.1 Case study (Uniform distribution)

Here, we are going to consider that the input traffic has constant distribution as shown in Figure 27. The main purpose is to fine the pfd, mean, and variance of the output traffic in case of accumulative delay algorithm, using equations (13), (16), and (17) respectively. First, let us find the mean and variance of the input traffic, $\mu_x$, $\sigma_x$.

$$f_y(x) = \frac{1}{J_u - J_l}\left[U(y - J_l) - U(y - J_u)\right]$$

$$\mu_x = \int x f_x(x)dx = \int_{J_l}^{J_u} \frac{x}{J_u - J_l} dx = \frac{1}{2}\frac{J_u^2 - J_l^2}{J_u - J_l} = \frac{1}{2}(J_u + J_l) = \tau$$

Then,

$$\mu_x = \tau \tag{18}$$

$$\sigma_x^2 = \overline{x^2 f_x(x)} - \mu_x^2 = \int_{J_l}^{J_u} \frac{x^2}{J_u - J_l} dx - \tau^2 = \frac{1}{3}\frac{J_u^3 - J_l^3}{(J_u - J_l)} = \frac{1}{3}\left(J_u^2 + J_u J_l + J_l^2\right) - \tau^2$$

But, $J_l = 2\tau - J_u$

Then,

$$\begin{aligned} \sigma_x^2 &= \frac{1}{3}\left(J_u^2 + 2\tau J_u - J_u^2 + 4\tau^2 - 4\tau J_u + J_u^2\right) - \tau^2 \\ &= \frac{1}{3}(\tau^2 - 2\tau J_u + J_u^2) \end{aligned} \tag{19}$$

Now, let us find the mean and variance of the output traffic *y(t)*,

$$F_x(\tau) = \int_{J_l}^{\tau} \frac{1}{J_u - J_l} dy = \frac{\frac{J_u + J_l}{2} - J_l}{J_u - J_l} = \frac{1}{2}$$

$$f_y(y) = \frac{1}{J_u - J_l}\left[U(y - J_l) - U(y - \tau)\right] + \frac{1}{2}\delta(y - \tau)$$

The mean can be found from equation (14) as follows,

$$\mu_y = \int_\tau^{J_u} y f_x(y)dy + \tau F_x(\tau)$$
$$= \int_\tau^{J_u} \frac{y}{J_u - J_l} dy + \frac{1}{2}\tau = \frac{1}{2}\frac{J_u^2 - \tau^2}{J_u - J_l} + \frac{1}{2}\tau$$
$$= \frac{1}{2}\frac{(J_u+\tau)(J_u-\tau)}{J_u - J_l} + \frac{1}{2}\tau = \frac{1}{2}\left[\frac{(J_u+\tau)}{2} + \tau\right]$$

Then,

$$\mu_y = \frac{1}{4}(J_u + 3\tau) \tag{20}$$

The above equation gives the output traffic mean as a function of the input traffic mean and maximum jitter. The lower limit of the maximum jitter is zero at which $J_u$ equal to $\tau$. By using that substitution, equation (20) shows that lower limit of the output traffic mean is $\tau$, which is consistent with inequality (16)

Similarly, the variance can be found as follows,

$$\sigma_y^2 = \overline{y^2 f_y(y)} - \mu_y^2 = \int y^2\left[\frac{1}{J_u - J_l}[U(y-J_l) - U(y-\tau)] + \frac{1}{2}\delta(y-\tau)\right]dy - \mu_y^2$$
$$= \int_\tau^{J_u} y^2 (\frac{1}{J_u - J_l})dy + \frac{1}{2}\tau^2 - \mu_y^2$$
$$= \frac{1}{3}\frac{J_u^3 - \tau^3}{J_u - J_l} + \frac{1}{2}\tau^2 - \mu_y^2 = \frac{1}{3}\frac{J_u^3 - \tau^3}{J_u - J_l} + \frac{1}{2}\tau^2 - \frac{1}{16}(J_u + 3\tau)^2$$
$$= \frac{1}{6}(J_u^2 + \tau J_u + \tau^2) + \frac{1}{2}\tau^2 - \frac{1}{16}(J_u^2 + 6\tau J_u + 9\tau^2)$$
$$= \frac{1}{48}\left(5J_u^2 - 10\tau J_u + 5\tau^2\right)$$

Then,

$$\sigma_y^2 = \frac{5}{48}(J_u - \tau)^2 \tag{21}$$

To find out the effect of the algorithm on the mean and variance, let us consider that the percentage of the input traffic jitter is η, which is described as follows,

$$\eta = \frac{J_u - \tau}{\tau} \quad \Rightarrow \quad J_u = (1+\eta)\tau \tag{22}$$

Equations (18), (20), and (22) lead to,

$$\frac{\mu_y}{\mu_x} = \frac{\frac{1}{4}(J_u + 3\tau)}{\tau} = \frac{1}{4}\frac{((1+\eta)\tau + 3\tau)}{\tau}$$

Then,

$$\frac{\mu_y}{\mu_x} = 1 + \frac{\eta}{4} \tag{23}$$

Equation (23) gives increase in the percentage of increase in mean value of packet transfer delay. It indicates that the increase is one quarter of the maximum jitter.

Now, let us find the effect on the variance using equations, (19), (21), and (22),

$$\frac{\sigma_y^2}{\sigma_x^2} = \frac{\frac{5}{48}(J_u - \tau)^2}{\frac{1}{3}(\tau^2 - 2\tau J_u + J_u^2)} = \frac{\frac{5}{48}\eta^2\tau^2}{\frac{1}{3}(\tau^2 - 2\tau^2(1+\eta) + \tau^2(1+\eta)^2)}$$

$$= \frac{5\eta^2}{16(1 - 2(1+\eta) + (1 + 2\eta + \eta^2))}$$

$$\frac{\sigma_y^2}{\sigma_x^2} = \frac{5}{16} = 0.3125 \tag{24}$$

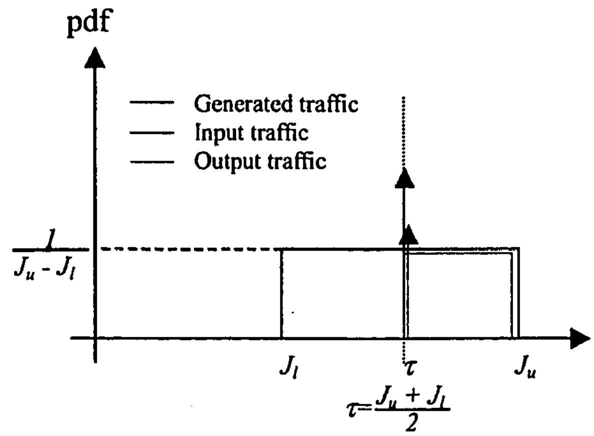



**Figure 27 Packet histogram for accumulative average algorithm Case study (uniform distribution)**

Equation (24) shows the ration of jitter improvement, which is represented by the variance, is constant. It also shows that there is a significant decrease n the jitter, almost 70%.

### 4.3.2 Case study (Normal distribution)

Section 4.3.1 showed the jitter improvement in case of uniform distribution. Here, we are going to consider another common case, which is normal distribution. It can be easily proved that the normal distribution represents good deal of jitter in the network. Since jitter is cased by different sources (i.e. different pfd distributions), and the overall jitter is the sum of those individual ones, then by using the central limit theory, the overall jitter tends to be normal distribution. Anyway, The normal distribution jitter is represented by the following function,

$$f_X(x)=\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

The accumulative average algorithm lead to an output traffic pdf, $f_y(y)$, described as follows,

$$f_y(y)=\frac{1}{2}\delta(y-\mu_x)+\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}}U(y-\mu_x)$$

Now, we can find the mean as follows,

$$\mu_y=\int yf_y(y)dy=\int y\left[\frac{1}{2}\delta(y-\mu_x)+\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}}U(y-\mu_x)\right]dy$$

$$=\frac{\mu_x}{2}+\int_{\mu_x}^{\infty}\frac{y}{\sigma_x\sqrt{2\pi}}e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}}dy=\frac{\mu_x}{2}+\int_{\mu_x}^{\infty}\frac{(y-\mu_x+\mu_x)}{\sigma_x\sqrt{2\pi}}e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}}dy$$

$$=\frac{\mu_x}{2}+\int_{\mu_x}^{\infty}\frac{(y-\mu_x)}{\sigma_x\sqrt{2\pi}}e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}}dy+\int_{\mu_x}^{\infty}\frac{\mu_x}{\sigma_x\sqrt{2\pi}}e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}}dy$$

$$=\frac{\mu_x}{2}-\frac{\sigma_x}{\sqrt{2\pi}}e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}}\Bigg|_{\mu_x}^{\infty}+\frac{\mu_x}{2}=\mu_x+\frac{\sigma_x}{\sqrt{2\pi}} \qquad (25)$$

The ratio of the output and input traffic mean values is,

$$\frac{\mu_y}{\mu_x}=\frac{\mu_x+\frac{\sigma_x}{\sqrt{2\pi}}}{\mu_x}=1+\frac{\sigma_x}{\mu_x\sqrt{2\pi}}$$

$$\frac{\mu_y}{\mu_x}=1+\frac{\sigma_x}{\mu_x\sqrt{2\pi}} \qquad (26)$$

Equation (26), shows the amount of increase in the mean by applying the accumulative average algorithm on a normal distribution jitter.

Similarly, the variance can be found. Let us first find the mean of $y^2$,

$$\overline{y^2} = \int y^2 f_y(y) dy$$

$$= \int y^2 \left[ \frac{1}{2}\delta(y-\mu_x) + \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} U(y-\mu_x) \right] dy$$

$$= \int y^2 \frac{1}{2}\delta(y-\mu_x) dy + \int_{\mu_x}^{\infty} y^2 \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} dy$$

$$= \frac{1}{2}\mu_x^2 + \int_{\mu_x}^{\infty} \frac{(y-\mu_x+\mu_x)^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} dy$$

$$= \frac{1}{2}\mu_x^2 + \int_{\mu_x}^{\infty} \left[ \frac{(y-\mu_x)^2 + 2(y-\mu_x)\mu_x + \mu_x^2}{\sigma_x\sqrt{2\pi}} \right] e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} dy$$

$$= \frac{1}{2}\mu_x^2 + \int_{\mu_x}^{\infty} \frac{(y-\mu_x)^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} dy + \int_{\mu_x}^{\infty} \frac{2(y-\mu_x)\mu_x}{\sigma_x\sqrt{2\pi}} e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} dy +$$

$$+ \int_{\mu_x}^{\infty} \frac{\mu_x^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} dy$$

To find the first integration, we know that,

$$\sigma_x^2 = \int_{-\infty}^{\infty} \frac{(x-\mu_x)^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx = \int_{-\infty}^{\infty} \frac{z^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_x^2}} dz$$

$$= \int_{-\infty}^{0} \frac{z^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_x^2}} dz + \int_{0}^{\infty} \frac{z^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_x^2}} dz$$

From the symmetry and by taking into consideration that the value under the integration sign is always positive, then,

$$\int_{-\infty}^{0} \frac{z^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_x^2}} dz = \int_{0}^{\infty} \frac{z^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_x^2}} dz$$

Then,

$$\int_{0}^{\infty} \frac{z^2}{\sigma_x\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_x^2}} dz = \frac{\sigma_x^2}{2}$$

Then,

$$\begin{aligned}
\overline{y}^2 &= \frac{1}{2}\mu_x^2 + \frac{\sigma_x^2}{2} - \frac{2\sigma_x\mu_x}{\sqrt{2\pi}} e^{-\frac{(y-\mu_x)^2}{2\sigma_x^2}} \Bigg|_{\mu_x}^{\infty} + \frac{\mu_x^2}{2} \\
&= \frac{1}{2}\mu_x^2 + \frac{\sigma_x^2}{2} + \frac{2\sigma_x\mu_x}{\sqrt{2\pi}} + \frac{\mu_x^2}{2} \\
&= \mu_x^2 + \frac{2\sigma_x\mu_x}{\sqrt{2\pi}} + \frac{\sigma_x^2}{2}
\end{aligned}$$

Now, the output traffic variance can be found as follows,

$$\begin{aligned}
\sigma_y^2 &= \overline{y}^2 - \mu_x^2 \\
&= \mu_x^2 + \frac{2\sigma_x\mu_x}{\sqrt{2\pi}} + \frac{\sigma_x^2}{2} - \left(\mu_x + \frac{\sigma_x}{\sqrt{2\pi}}\right)^2 \\
&= \mu_x^2 + \frac{2\sigma_x\mu_x}{\sqrt{2\pi}} + \frac{\sigma_x^2}{2} - \left(\mu_x^2 + \frac{2\sigma_x\mu_x}{\sqrt{2\pi}} + \frac{\sigma_x^2}{2\pi}\right) \\
&= \frac{\sigma_x^2}{2} - \frac{\sigma_x^2}{2\pi} = \sigma_x^2\left(\frac{1}{2} - \frac{1}{2\pi}\right) \\
&= 0.34\sigma_x^2
\end{aligned} \tag{27}$$

The ratio of the output traffic to the input traffic variance is simply found as,

$$\frac{\sigma_y^2}{\sigma_x^2} = 0.34 \tag{28}$$

This result is very useful and indicates that traffic variance is only 34% of the input traffic variance. That means that improvement is 66%, which is good result. We also saw a better improvement in case of uniform distribution in section 4.3.1. In general, the improvement depends on the input traffic distribution.

### 4.4 Average over n algorithm.

The average over n algorithm is the third level of our thesis. Its performance is between the maximum delay algorithm and the accumulative average algorithm. Figure 28 shows a generalized pdf distribution of the input, output, and average over n traffic. Figure 29 shows the transformation of random variable mapping function. It is helps to fine the out pfd, $f_y(y)$, as follows,

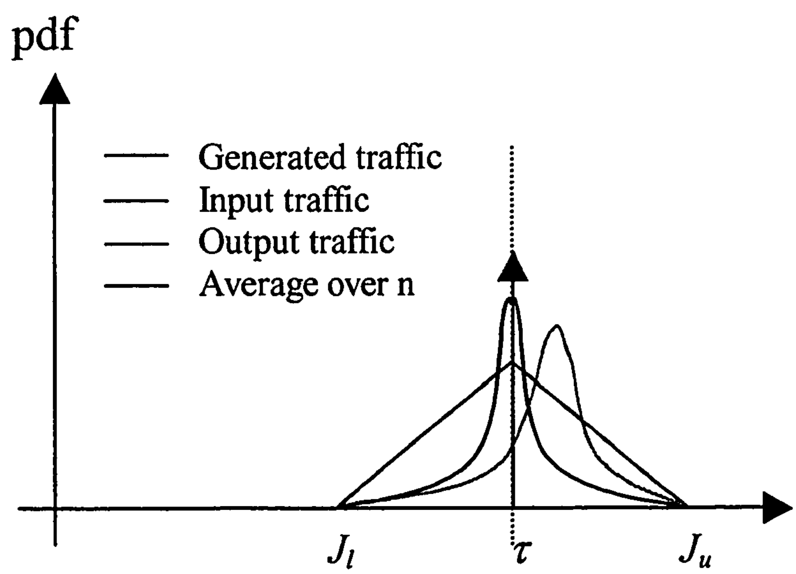



**Figure 28 Packet histogram for average over n algorithm**

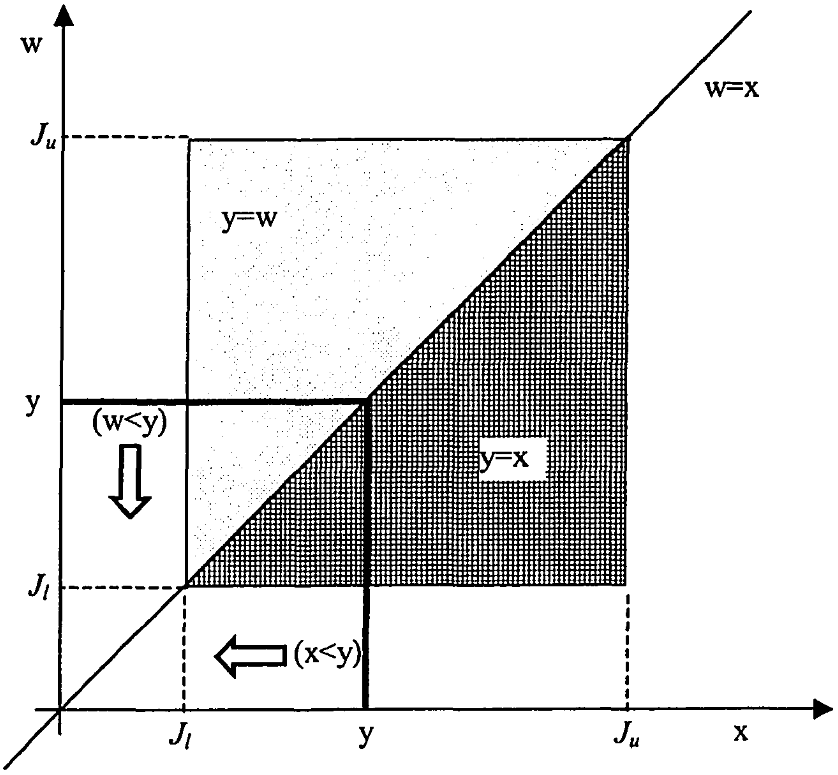



**Figure 29 Transformation of RV mapping function**

$$y = Max(x, w)$$

$$F_y(y) = F_{wx}(y, y) = F_w(y).F_x(y)$$

$$f_y(y) = f_w(y).F_x(y) + F_w(y)f_x(y) \tag{29}$$

$$\begin{aligned} \mu_y = \overline{yf_y(y)} &= \overline{yf_w(y).F_x(y)} + \overline{yF_w(y)f_x(y)} \\ &\leq \overline{yf_w(y)} + \overline{yf_x(y)} \end{aligned} \tag{30}$$

$$\mu_y \leq \mu_w + \mu_x$$

$$\mu_w = \mu_x$$

$\mu_y \geq \mu_x$

$$\mu_x \leq \mu_y \leq 2\mu_x \tag{31}$$

$$\begin{aligned}\sigma_y^2 &= \overline{(y-\mu_y)^2 f_y(y)} = \overline{(y-\mu_y)^2 f_w(y).F_x(y)} + \overline{(y-\mu_y)^2 F_w(y) f_x(y)} \\ &= \overline{(y-\mu_y+\mu_w-\mu_w)^2 f_w(y).F_x(y)} + \overline{(y-\mu_y+\mu_x-\mu_x)^2 F_w(y) f_x(y)} \\ &= \overline{[(y-\mu_w)^2 - 2(y-\mu_w)(\mu_y-\mu_w) + (\mu_y-\mu_w)^2] f_w(y).F_x(y)} \\ &\qquad + \overline{[(y-\mu_w)^2 - 2(y-\mu_x)(\mu_y-\mu_x) + (\mu_y-\mu_x)^2] F_w(y) f_x(y)}\end{aligned} \tag{32}$$

Since,

$F_x(y) \leq 1, and$

$Fw(y) \leq 1$

Then,

$$\begin{aligned}\sigma_y^2 \leq\ & \overline{[(y-\mu_w)^2 - 2(y-\mu_w)(\mu_y-\mu_w) + (\mu_y-\mu_w)^2] f_w(y)} \\ & + \overline{[(y-\mu_w)^2 - 2(y-\mu_x)(\mu_y-\mu_x) + (\mu_y-\mu_x)^2] f_x(y)}\end{aligned}$$

And since,

$\overline{(y-\mu_x)^2 f_x(y)} = \sigma_x^2,$

$\overline{(y-\mu_w)^2 f_w(y)} = \sigma_w^2,$

$\overline{(y-\mu_x) f_x(y)} = 0,$

$\overline{(y-\mu_w) f_w(y)} = 0,$

$\overline{f_x(y)} = 1, and$

$\overline{f_w(y)} = 1$

$$\sigma_y^2 \leq \sigma_x^2 + \sigma_w^2 + (\mu_y-\mu_x)^2 + (\mu_y-\mu_w)^2$$

Since,

$$\sigma_w^2 = \frac{\sigma_x^2}{n}, and$$

$$\mu_w = \mu_x$$

Then,

$$\sigma_y^2 \leq (1+\frac{1}{n})\sigma_x^2 + 2(\mu_y - \mu_x)^2 \tag{33}$$

### 4.4.1 Case study (Uniform distribution)

We are going to consider that the input jittery traffic has uniform distribution as shown in Figure 30. To find the output pdf, mean, and variance, we need to find the pfd of the average over n first, $f_w(w)$. The average over n depends on the degree of n, so will consider n equals 2 and 3. At the end, we will consider large value of n (large enough to use the central limit theory).

#### 4.4.1.1Case of n=2

Let us first find the pdf in case of n=2, $f_w(w)$. Equation (4), indicates that $f_w(w)$ is the convolution of the two functions. Those two functions are the input traffic at time $t$ and $(t-\tau)$, which have uniform distribution. Let us name the two RVs as $x_1$, and $x_2$, and the pfd functions as $f_{x1}(x_1)$, and $f_{x2}(x_2)$ as shown in Figure 31. As described in section 4.1.1.2, let us find the pdf of the sum of the two RVs, $f_{w2}(w_2)$. Equation (4) leads to,

$$w_2 = x_1 + x_2$$

$$f_{w_2}(w_2) = f_{x1}(x_1) * f_{x2}(x_2) = \int f_{x1}(x_1) f_{x2}(w_2 - x_1) dx_1$$

$$\begin{aligned} f_x(x) &= \frac{1}{J_u - J_l}\left[U(x-J_l) - U(x-J_u)\right] \\ &= \frac{1}{2(J_u - \tau)}\left[U(x-J_l) - U(x-J_u)\right] \end{aligned} \tag{34}$$

The following equations are useful in finding $f_{w2}(w_2)$,

$$\tau = \frac{J_u + J_l}{2}$$
$$J_l = 2\tau - J_u, and$$
$$J_u - J_l = 2(J_u - \tau)$$

$f_{w2}(w_2)$ can be found from Figure 31 as follows,

(i) **$2J_l<w_2<2\tau$**

Figure 31 (a) (b) show the convolution of $f_{x1}(x_1)$ and $f_{x2}(x_2)$ in the range from $2J_l$ to $2\tau$. $f_{w2}(w_2)$ can be found as follows,

$$f_{w_2}(w_2) = \int_{J_l}^{-\tau+w_2+(\tau-J_l)} \left(\frac{1}{J_u - J_l}\right)^2 dx_1$$
$$= \left(\frac{1}{J_u - J_l}\right)^2 (w_2 - 2J_l)$$
$$= \frac{1}{4(J_u - \tau)^2}(w_2 - 4\tau + 2J_u)$$

(ii) **$2\tau<w_2<2J_u$**

Figure 31 (a) (d) show the convolution of $f_{x1}(x_1)$ and $f_{x2}(x_2)$ in the range from $2\tau$ to $2J_l$. $f_{w2}(w_2)$ can be found as follows,

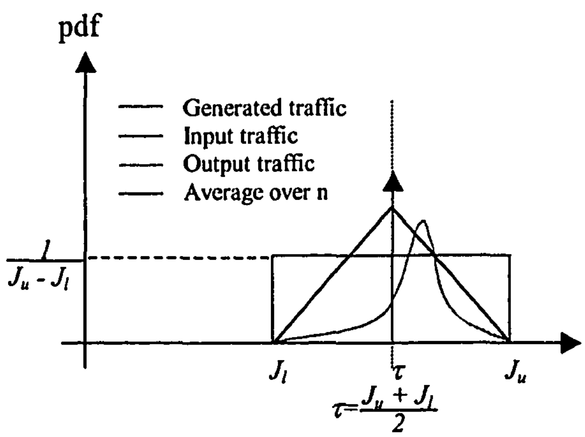



**Figure 30 Packet histogram for average over n algorithm (n=2)**
**Case study (uniform distribution)**

$$f_{w_2}(w_2)=\int_{-\tau+w_2-(J_u-\tau)}^{J_u}\left(\frac{1}{J_u-J_l}\right)^2 dx_1$$
$$=\left(\frac{1}{J_u-J_l}\right)^2 (2J_u-w_2)$$
$$=\frac{1}{4(J_u-\tau)^2}(2J_u-w_2)$$

Then, $f_{w2}(w_2)$ is expressed as follows,

$$f_{w_2}(w_2)=\begin{cases}\frac{1}{4(J_u-\tau)^2}(w_2-4\tau+2J_u) & 2J_l<w_2<2\tau \\ \frac{1}{4(J_u-\tau)^2}(2J_u-w_2) & 2\tau<w_2<2J_u \\ 0 & elsewhere\end{cases} \quad (35)$$

Having that and,

$$w=\frac{w_2}{2}$$

Then, using equation (2), $f_w(w)$ can be found as follows,

$$f_w(w)=f_{w_2}(w)\frac{dw_2}{dw}=2f_{w_2}(w)$$

$$f_w(w)=\begin{cases}\frac{1}{(J_u-\tau)^2}(w-2\tau+J_u) & J_l<w<\tau \\ \frac{1}{(J_u-\tau)^2}(J_u-w) & \tau<w<J_u \\ 0 & elsewhere\end{cases} \quad (36)$$

The output traffic pdf, mean and variance can be found using equations (29), (30), and (32) respectively. Let us first find the output traffic pfd, $f_y(y)$, using equation (29) in conjunction with equations (34) and (36) as follows,

$$f_y(y)=f_w(y).F_x(y)+F_w(y)f_x(y)$$

$$F_x(x)=\int f_x(x)dx=\int\frac{1}{2(J_u-\tau)}[U(x-J_l)-U(x-J_u)]dx$$

$$=\int_{J_l}^{x}\frac{1}{2(J_u-\tau)}dx=\frac{x-J_l}{2(J_u-\tau)} \qquad J_l<x<J_u$$

Then, as shown in Figure 32,

$$F_x(x)=\begin{cases}0 & x<J_l \\ \frac{x-2\tau+J_u}{2(J_u-\tau)} & J_l<x<J_u \\ 1 & J_u<x\end{cases} \quad (37)$$

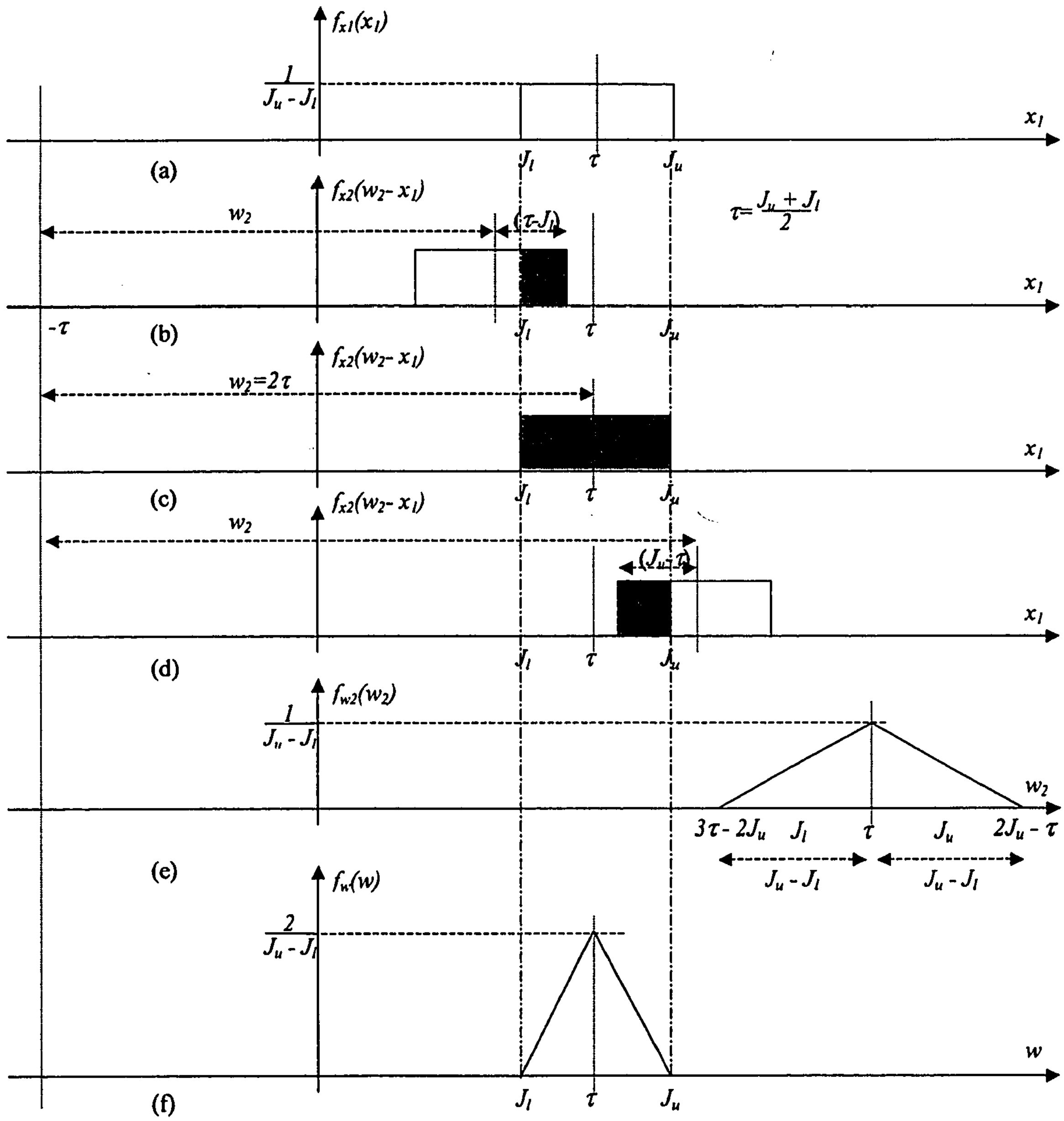



**Figure 31 Convolution to find $f_{w2}(w_2)$, and $f_w(w)$**

In the other side Fw(w), Figure 32, can be found as follows,

**(i) $J_l \le w < \tau$**

$$F_w(w)=\int_{J_l}^{w}\frac{1}{(J_u-\tau)^2}(w-2\tau+J_u)dw$$

$$=\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}w^2-(2\tau-J_u)w\right]_{2\tau-J_u}^{w}$$

$$=\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}w^2-(2\tau-J_u)w\right)-\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}(2\tau-J_u)^2-(2\tau-J_u)^2\right)$$

$$=\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}w^2-(2\tau-J_u)w+\frac{1}{2}(2\tau-J_u)^2\right) \qquad (38)$$

Equation (38) leads to,

$$F_w(J_l)=\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}(2\tau-J_u)^2-(2\tau-J_u)(2\tau-J_u)+\frac{1}{2}(2\tau-J_u)^2\right)=0$$

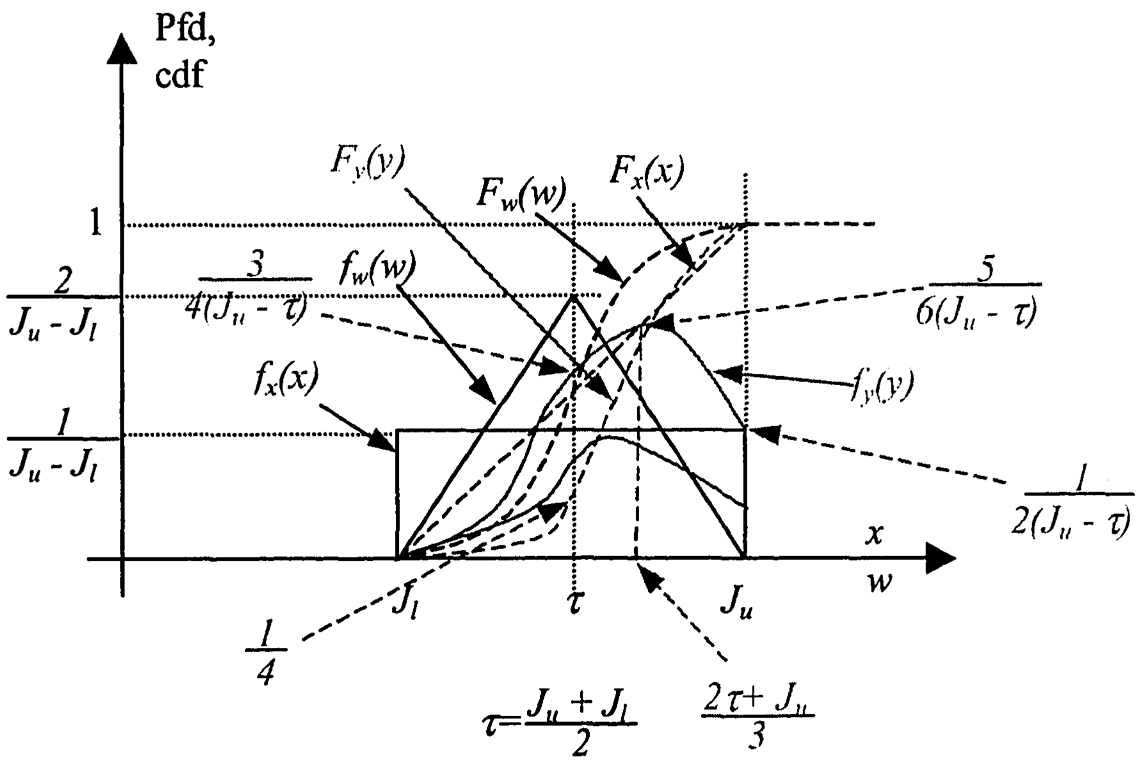



**Figure 32 pfd and cdf of the input and average traffic.**

And,

$$F_w(\tau)=\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}\tau^2-(2\tau-J_u)\tau+\frac{1}{2}(2\tau-J_u)^2\right)$$
$$=\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}\tau^2-2\tau^2+J_u\tau+2\tau^2-2J_u\tau+\frac{1}{2}J_u^2\right)$$
$$=\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}(J_u-\tau)^2\right)=\frac{1}{2}$$

**(ii)** **<u>$\tau$<w< $J_u$</u>**

$$F_w(w)=\int f_w(w)dw=\int_{-\infty}^{\tau} f_w(w)dw+\int_{\tau}^{w}\frac{1}{(J_u-\tau)^2}(J_u-w)dw$$
$$=\frac{1}{2}+\frac{1}{(J_u-\tau)^2}\left(J_u w-\frac{1}{2}w^2\right)\Bigg|_{\tau}^{w}$$
$$=\frac{1}{2}+\frac{1}{(J_u-\tau)^2}\left(J_u w-\frac{1}{2}w^2\right)-\frac{1}{(J_u-\tau)^2}\left(J_u\tau-\frac{1}{2}\tau^2\right)$$
$$=\frac{1}{2}+\frac{1}{(J_u-\tau)^2}\left(J_u w-\frac{1}{2}w^2-J_u\tau+\frac{1}{2}\tau^2\right) \tag{39}$$

Equation (39) leads to,

$$F_w(\tau)=\frac{1}{2}+\frac{1}{(J_u-\tau)^2}\left(J_u\tau-\frac{1}{2}\tau^2-J_u\tau+\frac{1}{2}\tau^2\right)=\frac{1}{2}$$

And,

$$F_w(J_u)=\frac{1}{2}+\frac{1}{(J_u-\tau)^2}\left(J_uJ_u-\frac{1}{2}J_u^{\,2}-J_u\tau+\frac{1}{2}\tau^2\right)$$
$$=\frac{1}{2}+\frac{1}{2(J_u-\tau)^2}\left(J_u^{\,2}-2J_u\tau+\tau^2\right)=1$$

Finally, $F_w(w)$ can be expressed as follows,

$$F_w(w)=\begin{cases}0 & w<J_l\\[2ex] \dfrac{1}{(J_u-\tau)^2}\left(\dfrac{1}{2}w^2-(2\tau-J_u)w+\dfrac{1}{2}(2\tau-J_u)^2\right) & J_l<w<\tau\\[2ex] \dfrac{1}{2}+\dfrac{1}{(J_u-\tau)^2}\left(J_u w-\dfrac{1}{2}w^2-J_u\tau+\dfrac{1}{2}\tau^2\right) & \tau<w<J_u\\[2ex] 1 & J_u<w\end{cases} \tag{40}$$

Now, our goal to fine $f_y(y)$ can be achieved. Using equations (29), (34), (36), (37), and (40) as follows,

**(i)** <u>$J_l$<w<τ</u>

Here, the output traffic pdf , $f_y(y)$, in the rage of y is from $J_l$ to τ, as follows,

$$\begin{aligned}f_y(y)&=f_w(y).F_x(y)+F_w(y)f_x(y)\\ &=\left[\frac{1}{(J_u-\tau)^2}(y-2\tau+J_u)\right]\left[\frac{y-2\tau+J_u}{2(J_u-\tau)}\right]+\\ &\qquad\left[\frac{1}{(J_u-\tau)^2}\left(\frac{1}{2}y^2-(2\tau-J_u)y+\frac{1}{2}(2\tau-J_u)^2\right)\right]\left[\frac{1}{2(J_u-\tau)}\right]\\ &=\frac{1}{2(J_u-\tau)^3}\left[(y-2\tau+J_u)^2+\left(\frac{1}{2}y^2-(2\tau-J_u)y+\frac{1}{2}(2\tau-J_u)^2\right)\right]\\ &=\frac{1}{2(J_u-\tau)^3}\left[y^2-2y(2\tau-J_u)+(2\tau-J_u)^2+\frac{1}{2}y^2-(2\tau-J_u)y+\frac{1}{2}(2\tau-J_u)^2\right]\\ &=\frac{1}{2(J_u-\tau)^3}\left[\frac{3}{2}y^2-3y(2\tau-J_u)+\frac{3}{2}(2\tau-J_u)^2\right]\\ &=\frac{3}{4(J_u-\tau)^3}(y-2\tau+J_u)^2\end{aligned}$$

Then,

$$f_y(y) = \frac{3}{4(J_u - \tau)^3}(y - 2\tau + J_u)^2 \qquad (41)$$

Now, we can find $f_y(y)$ at the two edges, $J_l$ to τ, as follows,

$$f_y(J_l) = \frac{3}{4(J_u - \tau)^3}(J_l - 2\tau + J_u)^2 = 0$$

$$f_y(\tau) = \frac{3}{4(J_u - \tau)^3}(\tau - 2\tau + J_u)^2 = \frac{3}{4(J_u - \tau)}$$

To get better view on how this curve is, let us find the maximum and minimum of $f_y(y)$, by differentiation as follows,

$$\frac{df_y(y)}{dy} = \frac{3}{2(J_u - \tau)^3}(y - 2\tau + J_u) = 0$$

then,

$$y_{\min} = 2\tau - J_u = J_l$$

That means that $f_y(y)$ has its minimum at $J_l$, and that minimum equals to *zero*.

**(ii)** **$\tau < w < J_u$**

Here, the output traffic pdf , $f_y(y)$, in the rage of y is from τ to $J_u$, as follows,

$$f_y(y) = f_w(y).F_x(y) + F_w(y) f_x(y)$$
$$= \left[\frac{1}{(J_u - \tau)^2}(J_u - y)\right]\left[\frac{y - 2\tau + J_u}{2(J_u - \tau)}\right] +$$
$$\left[\frac{1}{2} + \frac{1}{(J_u - \tau)^2}\left(J_u y - \frac{1}{2}y^2 - J_u\tau + \frac{1}{2}\tau^2\right)\right]\left[\frac{1}{2(J_u - \tau)}\right]$$

$$=\frac{1}{2(J_u-\tau)^3}\left[(J_u-y)(y-2\tau+J_u)+\left(J_u y-\frac{1}{2}y^2-J_u\tau+\frac{1}{2}\tau^2\right)\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{1}{2(J_u-\tau)^3}\left[J_u y-J_u(2\tau-J_u)-y^2+y(2\tau-J_u)+J_u y-\frac{1}{2}y^2-J_u\tau+\frac{1}{2}\tau^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+y(2\tau+J_u)-\frac{3}{2}y^2\right]+\frac{1}{4(J_u-\tau)}$$

Then,

$$f_y(y)=\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+y(2\tau+J_u)-\frac{3}{2}y^2\right]+\frac{1}{4(J_u-\tau)} \tag{42}$$

Now, we can find $f_y(y)$ at the boundaries, $\tau$ and $J_u$, as follows,

$$f_y(\tau)=\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+\tau(2\tau+J_u)-\frac{3}{2}\tau^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{1}{2(J_u-\tau)^3}\left[J_u^2-2J_u\tau+\tau^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{(J_u-\tau)^2}{2(J_u-\tau)^3}+\frac{1}{4(J_u-\tau)}=\frac{3}{4(J_u-\tau)}$$

And,

$$f_y(J_u)=\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+J_u(2\tau+J_u)-\frac{3}{2}J_u^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{1}{2(J_u-\tau)^3}\left[\frac{1}{2}J_u^2-J_u\tau+\frac{1}{2}\tau^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{(J_u-\tau)^2}{4(J_u-\tau)^3}+\frac{1}{4(J_u-\tau)}=\frac{1}{2(J_u-\tau)}$$

Again, to get better view on how this curve is, let find the maximum and minimum of $f_y(y)$, by differentiation as follows,

$$\frac{df_y(y)}{dy}=\frac{1}{2(J_u-\tau)^3}[(2\tau+J_u)-3y]=0$$

Then,

$$y_{\max}=\frac{1}{3}(2\tau+J_u)$$

And $f_{ymax}(y)$ is,

$$f_{y\max}(y_{\max})=\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+\frac{1}{3}(2\tau+J_u)(2\tau+J_u)-\frac{1}{6}(2\tau+J_u)^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+\frac{1}{6}(2\tau+J_u)^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{1}{12(J_u-\tau)^3}\left[6J_u^2-18J_u\tau+3\tau^2+4\tau^2+4\tau J_u+J_u^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{1}{12(J_u-\tau)^3}\left[7J_u^2-14J_u\tau+7\tau^2\right]+\frac{1}{4(J_u-\tau)}$$

$$=\frac{7(J_u-\tau)^2}{12(J_u-\tau)^3}+\frac{1}{4(J_u-\tau)}=\frac{5}{6(J_u-\tau)}$$

Now, $f_y(y)$ is shown in Figure 32 and can be expressed as follows,

$$f_y(y)=\begin{cases}\frac{3}{4(J_u-\tau)^3}(y-2\tau+J_u)^2 & J_l<y<\tau\\ \frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+y(2\tau+J_u)-\frac{3}{2}y^2\right]+\frac{1}{4(J_u-\tau)} & \tau<y<J_u\\ 0 & elsewhere\end{cases}\qquad(43)$$

The CDF, $F_y(y)$, can be found as follows,

**(i) $J_l<w<\tau$**

$$F_y(y)=\int f_y(y)dy=\int_{J_l}^{y}\frac{3}{4(J_u-\tau)^3}(y-2\tau+J_u)^2dy$$

$$=\frac{1}{4(J_u-\tau)^3}(y-2\tau+J_u)^3\Big|_{2\tau-J_u}^{y}$$

$$=\frac{1}{4(J_u-\tau)^3}(y-2\tau+J_u)^3$$

$F_y(y)$ at the boundaries, $J_l$ and $\tau$, can be found as follows,

$$F_y(J_l)=\frac{1}{4(J_u-\tau)^3}(J_l-2\tau+J_u)^3=0$$

And,

$$F_y(\tau)=\frac{1}{4(J_u-\tau)^3}(\tau-2\tau+J_u)^3=\frac{1}{4}$$

**(ii) $\tau<w<J_u$**

$$F_y(y)=\int f_y(y)dy=\int_{-\infty}^{\tau}f_y(y)dy+\int_{\tau}^{y}f_y(y)dy$$

$$=F_y(\tau)+\int_{\tau}^{y}\left[\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+y(2\tau+J_u)-\frac{3}{2}y^2\right]+\frac{1}{4(J_u-\tau)}\right]dy$$

$$=\frac{1}{4}+\left[\frac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)y+\frac{1}{2}y^2(2\tau+J_u)-\frac{1}{2}y^3\right]+\frac{y}{4(J_u-\tau)}\right]\Bigg|_{\tau}^{y}$$

$$=\frac{1}{4}+\left[\frac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)y+\frac{1}{2}y^2(2\tau+J_u)-\frac{1}{2}y^3\right]+\frac{y}{4(J_u-\tau)}\right]-$$

$$\frac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)\tau+\frac{1}{2}\tau^2(2\tau+J_u)-\frac{1}{2}\tau^3\right]-\frac{\tau}{4(J_u-\tau)}$$

$$=\frac{1}{4}+\left[\frac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)y+\frac{1}{2}y^2(2\tau+J_u)-\frac{1}{2}y^3\right]+\frac{y}{4(J_u-\tau)}\right]-$$
$$\frac{1}{2(J_u-\tau)^3}\left[J_u^2\tau-3J_u\tau^2+\frac{1}{2}\tau^3+\tau^3+\frac{1}{2}\tau^2J_u-\frac{1}{2}\tau^3\right]-\frac{\tau}{4(J_u-\tau)}$$
$$=\frac{1}{4}+\frac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)y+\frac{1}{2}y^2(2\tau+J_u)-\frac{1}{2}y^3-J_u^2\tau+\frac{5}{2}\tau^2J_u-\tau^3\right]+$$
$$\frac{y-\tau}{4(J_u-\tau)}$$

*$F_y(y)$* at the boundaries, τ and $J_l$, can be found as follows,

$$F_y(\tau)=\frac{1}{4}+\frac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)\tau+\frac{1}{2}\tau^2(2\tau+J_u)-\frac{1}{2}\tau^3-J_u^2\tau+\frac{5}{2}\tau^2J_u-\tau^3\right]+$$
$$\frac{\tau-\tau}{4(J_u-\tau)}$$
$$=\frac{1}{4}+\frac{1}{2(J_u-\tau)^3}\left[J_u^2\tau-3J_u\tau^2+\frac{1}{2}\tau 3+\tau^3+\frac{1}{2}\tau^2J_u-\frac{1}{2}\tau^3-J_u^2\tau+\frac{5}{2}\tau^2J_u-\tau^3\right]$$
$$=\frac{1}{4}$$

$$F_y(J_u)=\frac{1}{4}+\frac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)J_u+\frac{1}{2}J_u^2(2\tau+J_u)-\frac{1}{2}J_u^3-J_u^2\tau+\frac{5}{2}\tau^2J_u-\tau^3\right]+$$
$$\frac{J_u-\tau}{4(J_u-\tau)}$$
$$=\frac{1}{4}+\frac{1}{2(J_u-\tau)^3}\left[J_u^3-3J_u^2\tau+\frac{1}{2}J_u\tau^2+J_u^2\tau+\frac{1}{2}J_u^3-\frac{1}{2}J_u^3-J_u^2\tau+\frac{5}{2}\tau^2J_u-\tau^3\right]+\frac{1}{4}$$
$$=\frac{1}{2}+\frac{1}{2(J_u-\tau)^3}\left[J_u^3-3J_u^2\tau+3J_u\tau^2-\tau^3\right]$$
$$=\frac{1}{2}+\frac{1}{2(J_u-\tau)^3}(J_u-\tau)^3$$
$$=1$$

*$F_y(y)$* is shown in Figure 32 and can be expressed as follows,

$$F_y(y)=\begin{cases} 0 & y<J_l \\ \dfrac{1}{4(J_u-\tau)^3}(y-2\tau+J_u)^3 & J_l<y<\tau \\ \dfrac{1}{2(J_u-\tau)^3}\left[\left(J_u^2-3J_u\tau+\dfrac{1}{2}\tau^2\right)y+\dfrac{1}{2}y^2(2\tau+J_u)-\dfrac{1}{2}y^3-J_u^2\tau+\dfrac{5}{2}\tau^2J_u-\tau^3\right. \\ \qquad\qquad +\dfrac{1}{4}+\dfrac{y-\tau}{4(J_u-\tau)} & \tau<y<J_u \\ 1 & J_u<y \end{cases} \quad (44)$$

Finally, the mean and variance of the output traffic, $\mu_y$ and $\sigma_y$, can be found using equation (43). Let us first find the mean as follows,

$$\mu_y=\int yf_y(y)dy$$
$$=\int_{J_l}^{\tau} y\left[\frac{3}{4(J_u-\tau)^3}(y-2\tau+J_u)^2\right]dy+$$
$$\int_{\tau}^{J_u} y\left[\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+y(2\tau+J_u)-\frac{3}{2}y^2\right]+\frac{1}{4(J_u-\tau)}\right]dy$$

The above integration is provided in Appendix A, which shows that the mean can be is described by the following equation,

$$\mu_y=\tau\left(1+\frac{7\eta}{24}\right) \quad (45)$$

Now, we can find the ratio on the output traffic mean to the input traffic mean as follows,

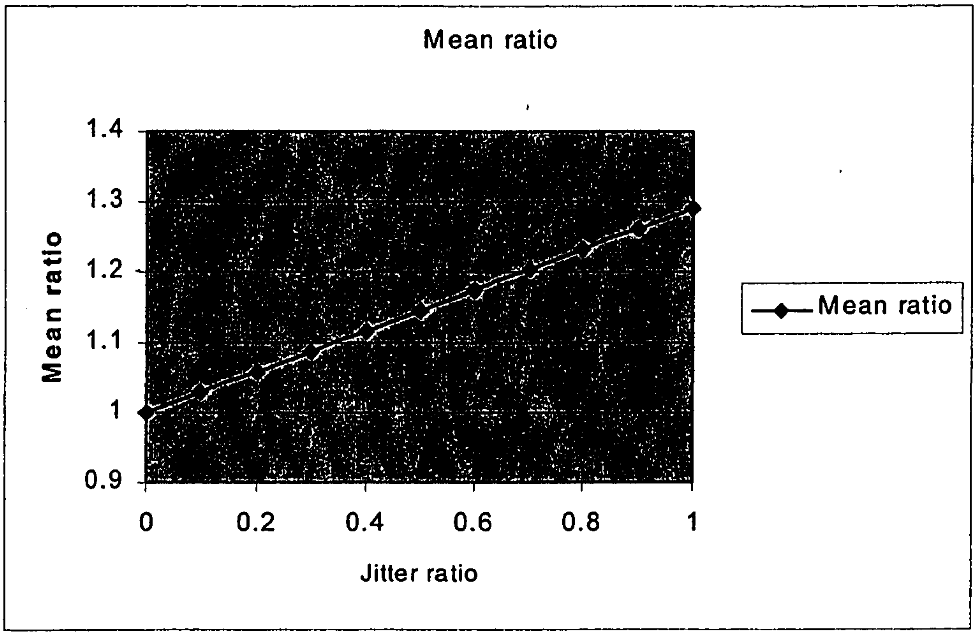



**Figure 33 Mean ratio in case of average over 2 algorithm (Uniform distribution)**

$$\boxed{\frac{\mu_y}{\mu_x} = 1 + \frac{7\eta}{24}} \tag{46}$$

By comparing this equation with equation (23), it is clear that the increase in the average is less in this case. That means that less buffer size is needed. Figure 33 shows the increase in the output traffic mean as a function of $\eta$

$$\sigma_y^2 = \overline{y^2} - \mu_y^2 = \int y^2 f_y(y) dy - \mu_y^2$$

Let use first find $\overline{y^2}$

$$\overline{y^2} = \int y^2 f_y(y) dy = \int_{J_l}^{\tau} y^2 \left[ \frac{3}{4(J_u - \tau)^3} (y - 2\tau + J_u)^2 \right] dy +$$

$$+ \int_{\tau}^{J_u} y^2 \left[ \frac{1}{2(J_u - \tau)^3} \left[ J_u^2 - 3J_u\tau + \frac{1}{2}\tau^2 + y(2\tau + J_u) - \frac{3}{2}y^2 \right] + \frac{1}{4(J_u - \tau)} \right] dy$$

The above integration is long but the result descriptive and it is used to find the output traffic variance. The proof of the above integration is shown in appendix B, which leads to the following equation,

$$\overline{y^2} = \frac{\tau^2}{12}\left[12 + 7\eta + 3\eta^2\right] \tag{47}$$

Now, the variance of y can be found as follows,

$$\begin{aligned}\sigma_y^2 &= \overline{y^2} - \mu_y^2 = \frac{\tau^2}{12}\left[12 + 7\eta + 3\eta^2\right] - \tau^2\left(1 + \frac{7\eta}{24}\right)^2 \\ &= \frac{\tau^2}{12}\left[12 + 7\eta + 3\eta^2\right] - \tau^2\left(1 + \frac{7\eta}{12} + \frac{49\eta^2}{576}\right) \\ &= \frac{95}{576}\tau^2\eta^2\end{aligned} \tag{48}$$

The ratio of the input and output traffic variance, using equations (19) and (48),is,

$$\begin{aligned}\frac{\sigma_y^2}{\sigma_x^2} &= \frac{\frac{95}{576}\tau^2\eta^2}{\frac{1}{3}(\tau^2 - 2\tau J_u + J_u^2)} \\ &= \frac{\frac{95}{192}\tau^2\eta^2}{(\tau^2 - 2\tau\tau(1+\eta) + \tau^2(1+\eta)^2)} = \frac{95\eta^2}{192(1 - 2 - 2\eta + 1 + 2\eta + \eta^2)} \\ &= \frac{95}{192}\end{aligned}$$

Then,

$$\frac{\sigma_y^2}{\sigma_x^2} = \frac{95}{192} = 0.4948 \tag{49}$$

Equation (49) shows that the jitter improvement is more that 50%. It also shows that the ration of the output traffic jitter to the one doesn't depend on the input traffic jitter.

### 4.4.1.2 Case of n=3

In this case, we take the average over three functions, which are represented by the input traffic $f_x(x)$, but shifted by $\tau$ (i.e. $t$ and $t-\tau$, and $t-2\tau$). Let us name the functions as $f_{x1}(x_1)$, $f_{x2}(x_2)$, and $f_{x3}(x_3)$ respectively. It is worth noticing that the first two functions are the ones that we have in case of $n$=2. As we did in section 4.4.1.1, we introduce the two functions $w$ and $w_3$ as follows,

$$w_3 = x_1 + x_2 + x_3$$

and,

$$w = \frac{w_3}{3}$$

$w_3$ can also be expressed as follows,

$$w_3 = w_2 + x_3$$

Since $w_2$ pdf is obtained in section 4.4.1.1, $w_3$ pdf can be obtained by the convolution of $w_2$ and $x_3$, using the following equations,

$$f_{w_3}(w_3) = f_{w2}(w_2) * f_{x3}(x_3) = \int f_{w2}(x_3) f_{x3}(w_3 - x_3) dx_3 \qquad (50)$$

$$f_{x_3}(x_3) = \frac{1}{J_u - J_l}[U(x_3 - J_l) - U(x_3 - J_u)] = \frac{1}{2(J_u - \tau)}[U(x_3 - J_l) - U(x_3 - J_u)] \qquad (51)$$

$$f_{w_2}(w_2)=\begin{cases}\dfrac{1}{4(J_u-\tau)^2}(w_2-4\tau+2J_u) & 2J_l<w_2<2\tau\\[2ex] \dfrac{1}{4(J_u-\tau)^2}(2J_u-w_2) & 2\tau<w_2<2J_u\\[2ex] 0 & elsewhere\end{cases} \tag{52}$$

Figure 34 shows the convolution process. To find $w_3$ pdf, let us first determine the different regions of $w_3$, as follows,

1. Region I where the convolution intersection is in the first section of $w_2$. The lower limit is determined by,

$$\tau+2J_l-(\tau-J_l)=3J_l$$

And the upper limit is

$$\tau+2J_l+(J_u-\tau)=2J_l+J_u=2(\tau-(J_u-\tau))+J_u=4\tau-J_u=3\tau-(J_u-\tau)$$

2. Region I where the convolution intersection is in the first and second sections of $w_2$. The lower limit is determined by,

$$\tau+2\tau-(\tau-J_l)=2\tau+J_l=3\tau-(J_u-\tau)$$

And the upper limit is

$$\tau+2\tau+(J_u-\tau)=3\tau-(J_u-\tau)$$

3. Region I where the convolution intersection is in the second sections of $w_2$. The lower limit is determined by,

$$\tau+2J_u-(\tau-J_l)=2J_u+J_l=2(\tau+(J_u-\tau))+(\tau-(J_u-\tau))=3\tau-(J_u-\tau)$$

And the upper limit is

$$\tau + 2J_u + (J_u - \tau) = 3J_u$$

Now, $w_3$ pdf, can be determined as follows,

(i) <u>$3J_l < w_3 < 3\tau - (J_u - \tau)$</u>

Figure 34 (b) and equations (50), (51), and (52) lead to,

$$f_{w_3}(w_3) = f_{w2}(w_2) * f_{x3}(x_3) = \int f_{w2}(x_3) f_{x3}(w_3 - x_3) dx_3$$
$$= \int_{2J_l}^{w_3 - J_l} \frac{1}{2(J_u - \tau)} \frac{1}{4(J_u - \tau)^2} (x_3 - 4\tau + 2J_u) dx_3$$
$$= \frac{1}{8(J_u - \tau)^3} \left( \frac{x_3^2}{2} - 2(2\tau - J_u) x_3 \right]_{2J_l}^{w_3 - J_l}$$

By defining η as follows,

$$\eta = \frac{J_u - \tau}{\tau} = \frac{\tau - J_l}{\tau}$$

*Then,*

$$J_u = \tau(1 + \eta) \quad and,$$
$$J_l = \tau(1 - \eta) \quad and,$$
$$J_u - \tau = \tau\eta \quad and,$$
$$2\tau + J_u = \tau(3 + \eta) \quad and,$$
$$2\tau - J_u = \tau(1 - \eta)$$

Then,

$$f_{w_3}(w_3) = \frac{1}{8\tau^3\eta^3} \left( \frac{x_3^2}{2} - 2\tau(1 - \eta) x_3 \right]_{2\tau(1-\eta)}^{w_3 - \tau(1-\eta)}$$

$$=\frac{1}{8\tau^3\eta^3}\left(\frac{(w_3-\tau(1-\eta))^2}{2}-2\tau(1-\eta)(w_3-\tau(1-\eta))\right)-$$

$$-\frac{1}{8\tau^3\eta^3}\left(\frac{4\tau^2(1-\eta)^2}{2}-2\tau(1-\eta)2\tau(1-\eta)\right)$$

$$=\frac{1}{8\tau^3\eta^3}\left(\frac{w_3^2-2\tau(1-\eta)w_3+\tau^2(1-\eta)^2}{2}-2\tau(1-\eta)w_3+2\tau^2(1-\eta)^2\right)+\frac{2(1-\eta)^2}{8\tau\eta^3}$$

$$=\frac{1}{16\tau^3\eta^3}\left(w_3^2-6\tau(1-\eta)w_3+5\tau^2(1-\eta)^2\right)+\frac{2(1-\eta)^2}{8\tau\eta^3}$$

$$=\frac{1}{16\tau^3\eta^3}\left(w_3^2-6\tau(1-\eta)w_3+9\tau^2(1-\eta)^2\right)$$

Then

$$f_{w_3}(w_3)=\frac{1}{16\tau^3\eta^3}\left(w_3^2-6\tau(1-\eta)w_3+9\tau^2(1-\eta)^2\right) \quad (53)$$

The boundaries, $3J_l$ and $[3\tau-(J_u-\tau)]$ can be expressed in terms of $\tau$ and $\eta$ as follows,

$$3J_l=3\tau(1-\eta)$$

$$3\tau-(J_u-\tau)=3\tau-\tau\eta$$

Now, we can find $f_{w3}(w_3)$ at the boundaries, $3J_l$ and $[3\tau-(J_u-\tau)]$ as follows,

$$f_{w_3}(3J_l)=\frac{1}{16\tau^3\eta^3}\left(9\tau^2(1-\eta)^2-6\tau(1-\eta)3\tau(1-\eta)+9\tau^2(1-\eta)^2\right)$$

$$=\frac{1}{16\tau^3\eta^3}\left(9\tau^2(1-\eta)^2-18\tau^2(1-\eta)^2+9\tau^2(1-\eta)^2\right)$$

$$=0$$

And,

$$f_{w_3}(3\tau-(J_u-\tau))=\frac{1}{16\tau^3\eta^3}\left((3\tau-\tau\eta)^2-6\tau(1-\eta)(3\tau-\tau\eta)+9\tau^2(1-\eta)^2\right)$$
$$=\frac{1}{16\tau\eta^3}\left(\left(9-6\eta+\eta^2\right)-6\left(3-4\eta+\eta^2\right)+9\left(1-2\eta+\eta^2\right)\right)$$
$$=\frac{1}{16\tau\eta^3}\left(4\eta^2\right)=\frac{1}{4\tau\eta}=\frac{1}{4(J_u-\tau)}=\frac{1}{2(J_u-J_l)}$$

**(ii)** <u>$3\tau-(J_u-\tau)<w_3<3\tau+(J_u-\tau)$</u>

Figure 34 (d) and equations (50), (51), and (52) lead to,

$$f_{w_3}(w_3)=f_{w2}(w_2)*f_{x3}(x_3)=\int f_{w2}(x_3)f_{x3}(w_3-x_3)dx_3$$
$$=\int_{w_3-J_u}^{2\tau}\frac{1}{2(J_u-\tau)}\frac{1}{4(J_u-\tau)^2}(x_3-4\tau+2J_u)dx_3+$$
$$+\int_{2\tau}^{w_3-J_l}\frac{1}{2(J_u-\tau)}\frac{1}{4(J_u-\tau)^2}(2J_u-x_3)dx_3$$
$$=\frac{1}{8\tau^3\eta^3}\left(\frac{x_3^2}{2}-2\tau(1-\eta)x_3\right)\Bigg]_{w_3-\tau(1+\eta)}^{2\tau}+\frac{1}{8\tau^3\eta^3}\left(2\tau(1+\eta)x_3-\frac{x_3^2}{2}\right)\Bigg]_{2\tau}^{w_3-\tau(1-\eta)}$$
$$=\frac{1}{8\tau^3\eta^3}\left\{\left(\frac{4\tau^2}{2}-2\tau(1-\eta)(2\tau)\right)-\left(\frac{(w_3-\tau(1+\eta))^2}{2}-2\tau(1-\eta)(w_3-\tau(1+\eta))\right)\right\}+$$
$$+\frac{1}{8\tau^3\eta^3}\left\{\left(2\tau(1+\eta)(w_3-\tau(1-\eta))-\frac{(w_3-\tau(1-\eta))^2}{2}\right)-\left(2\tau(1+\eta)(2\tau)-\frac{4\tau^2}{2}\right)\right\}$$
$$=\frac{1}{16\tau^3\eta^3}\left\{\begin{array}{l}4\tau^2-8\tau^2(1-\eta)-\\ \quad-\left(w_3^2-2\tau(1+\eta)w_3+\tau^2(1+\eta)^2-4\tau(1-\eta)w_3+4\tau^2(1+\eta)(1-\eta)\right)+\\ \quad+\left(4\tau(1+\eta)w_3-4\tau^2(1+\eta)(1-\eta)-w_3^2+2\tau(1-\eta)w_3-\tau^2(1-\eta)^2\right)-\\ \quad-\left(8\tau^2(1+\eta)-4\tau^2\right)\end{array}\right\}$$
$$=\frac{1}{16\tau^3\eta^3}\left\{\begin{array}{l}-2w_3^2+w_3\left[6\tau(1+\eta)+6\tau(1-\eta)\right]+\\ \quad+\left[\begin{array}{l}8\tau^2-8\tau^2(1-\eta)-\tau^2(1+\eta)^2-\tau^2(1-\eta)^2-\\ \quad-8\tau^2(1+\eta)(1-\eta)-8\tau^2(1+\eta)\end{array}\right]\end{array}\right\}$$

$$=\frac{1}{16\tau^3\eta^3}\left\{\begin{array}{l}-2w_3^2+12\tau w_3+\\ \quad+\tau^2\left[-\left(1+2\eta+\eta^2\right)-\left(1-2\eta+\eta^2\right)-8\left(1-\eta^2\right)-8\right]\end{array}\right\}$$

$$=\frac{1}{16\tau^3\eta^3}\left\{-2w_3^2+12\tau w_3+\tau^2\left[6\eta^2-18\right]\right\}=\frac{1}{8\tau^3\eta^3}\left\{3\tau^2\left(\eta^2-3\right)+6\tau w_3-w_3^2\right\}$$

Then,

$$f_{w_3}(w_3)=\frac{1}{8\tau^3\eta^3}\left\{3\tau^2\left(\eta^2-3\right)+6\tau w_3-w_3^2\right\} \tag{54}$$

The boundaries, $[3\tau-(J_u-\tau)]$ and $[3\tau+(J_u-\tau)]$ can be expressed in terms of $\tau$ and $\eta$ as follows,

$$3\tau-\left(J_u-\tau\right)=3\tau-\tau\eta$$

$$3\tau+\left(J_u-\tau\right)=3\tau+\tau\eta$$

Now, we can find $f_{w3}(w_3)$ at the boundaries, $[3\tau-(J_u-\tau)]$ and $[3\tau+(J_u-\tau)]$ as follows,

$$f_{w_3}(3\tau-(J_u-\tau))=\frac{1}{8\tau^3\eta^3}\left\{3\tau^2\left(\eta^2-3\right)+6\tau(3\tau-\tau\eta)-(3\tau-\tau\eta)^2\right\}$$

$$=\frac{1}{8\tau\eta^3}\left\{3\left(\eta^2-3\right)+6(3-\eta)-\left(9-6\eta+\eta^2\right)\right\}$$

$$=\frac{1}{8\tau\eta^3}\left(2\eta^2\right)=\frac{1}{4\tau\eta}=\frac{1}{4(J_u-\tau)}=\frac{1}{2(J_u-J_l)}$$

And,

$$f_{w_3}(3\tau+(J_u-\tau))=\frac{1}{8\tau^3\eta^3}\left\{3\tau^2\left(\eta^2-3\right)+6\tau(3\tau+\tau\eta)-(3\tau+\tau\eta)^2\right\}$$

$$=\frac{1}{8\tau\eta^3}\left\{3\left(\eta^2-3\right)+6(3+\eta)-\left(9+6\eta+\eta^2\right)\right\}$$

$$=\frac{1}{8\tau\eta^3}\left(2\eta^2\right)=\frac{1}{4\tau\eta}=\frac{1}{4(J_u-\tau)}=\frac{1}{2(J_u-J_l)}$$

**(iii)** **$3\tau+(J_u-\tau) < w_3 < 3J_u$**

Figure 34 (f) and equations (50), (51), and (52) lead to,

$$f_{w_3}(w_3) = f_{w2}(w_2) * f_{x3}(x_3) = \int f_{w2}(x_3) f_{x3}(w_3 - x_3)dx_3$$

$$= \int_{w_3-J_u}^{2J_u} \frac{1}{2(J_u-\tau)} \frac{1}{4(J_u-\tau)^2}(2J_u - x_3)dx_3$$

$$= \frac{1}{8\tau^3\eta^3}\left(2\tau(1+\eta)x_3 - \frac{x_3^2}{2}\right)\Bigg|_{w_3-\tau(1+\eta)}^{2\tau(1+\eta)}$$

$$= \frac{1}{8\tau^3\eta^3}\left\{\left(2\tau(1+\eta)2\tau(1+\eta) - \frac{4\tau^2(1+\eta)^2}{2}\right) - \left(2\tau(1+\eta)(w_3 - \tau(1+\eta)) - \frac{(w_3-\tau(1+\eta))^2}{2}\right)\right\}$$

$$= \frac{1}{8\tau^3\eta^3}\left\{2\tau^2(1+\eta)^2 - \left(2\tau(1+\eta)w_3 - 2\tau^2(1+\eta)^2 - \frac{\left(w_3^2 - 2\tau(1+\eta)w_3 + \tau^2(1+\eta)^2\right)}{2}\right)\right\}$$

$$= \frac{1}{16\tau^3\eta^3}\left\{8\tau^2(1+\eta)^2 - \left(4\tau(1+\eta)w_3 - w_3^2 + 2\tau(1+\eta)w_3 - \tau^2(1+\eta)^2\right)\right\}$$

$$= \frac{1}{16\tau^3\eta^3}\left\{9\tau^2(1+\eta)^2 - 6\tau(1+\eta)w_3 + w_3^2\right\}$$

Then,

$$f_{w_3}(w_3) = \frac{1}{16\tau^3\eta^3}\left\{9\tau^2(1+\eta)^2 - 6\tau(1+\eta)w_3 + w_3^2\right\} \qquad (55)$$

The boundaries, $[3\tau+(J_u-\tau)]$ and $3J_u$ can be expressed in terms of $\tau$ and $\eta$ as follows,

$$3\tau + (J_u - \tau) = 3\tau + \tau\eta$$

$$3J_u = 3\tau(1+\eta)$$

Now, we can find $f_{w3}(w_3)$ at the boundaries, $[3\tau+(J_u-\tau)]$ and $3J_u$ as follows,

$$f_{w_3}(w_3)=\frac{1}{16\tau^3\eta^3}\left\{9\tau^2(1+\eta)^2-6\tau(1+\eta)(3\tau+\tau\eta)+(3\tau+\tau\eta)^2\right\}$$
$$=\frac{1}{16\tau\eta^3}\left\{9\left(1+2\eta+\eta^2\right)-6\left(3+4\eta+\eta^2\right)+\left(9+6\eta+\eta^2\right)\right\}$$
$$=\frac{1}{16\tau\eta^3}\left(4\eta^2\right)=\frac{1}{4\tau\eta}=\frac{1}{4(J_u-\tau)}=\frac{1}{2(J_u-J_l)}$$

And,

$$f_{w_3}(w_3)=\frac{1}{16\tau^3\eta^3}\left\{9\tau^2(1+\eta)^2-6\tau(1+\eta)3\tau(1+\eta)+9\tau^2(1+\eta)^2\right\}$$
$$=\frac{1}{16\tau\eta^3}\left\{9(1+\eta)^2-18(1+\eta)^2+9(1+\eta)^2\right\}=0$$

Finally, from equations (53), (54), and (55), $f_{w3}(w_3)$ can be expressed as follows,

$$f_{w_3}(w_3)=\begin{cases}\frac{1}{16\tau^3\eta^3}\left(w_3^2-6\tau(1-\eta)w_3+9\tau^2(1-\eta)^2\right) & 3J_l<w_3<3\tau-(J_u-\tau)\\ \frac{1}{8\tau^3\eta^3}\left\{3\tau^2\left(\eta^2-3\right)+6\tau w_3-w_3^2\right\} & 3\tau-(J_u-\tau)<w_3<3\tau+(J_u-\tau)\\ \frac{1}{16\tau^3\eta^3}\left\{9\tau^2(1+\eta)^2-6\tau(1+\eta)w_3+w_3^2\right\} & 3\tau+(J_u-\tau)<w_3<3J_u\end{cases} \quad (56)$$

Having that and,

$$w=\frac{w_3}{3}$$

Then, using equation (2), $f_w(w)$ can be found as follows,

$$f_w(w)=f_{w_3}(w)\frac{dw_3}{dw}=3f_{w_3}(w)$$

$$f_w(w)=\begin{cases}\frac{3}{16\tau^3\eta^3}\left(9w^2-18\tau(1-\eta)w+9\tau^2(1-\eta)^2\right) & J_l<w<\tau-\frac{(J_u-\tau)}{3}\\ \frac{3}{8\tau^3\eta^3}\left\{3\tau^2\left(\eta^2-3\right)+18\tau w-9w^2\right\} & \tau-\frac{(J_u-\tau)}{3}<w<\tau+\frac{(J_u-\tau)}{3}\\ \frac{3}{16\tau^3\eta^3}\left\{9\tau^2(1+\eta)^2-18\tau(1+\eta)w+9w^2\right\} & \tau+\frac{(J_u-\tau)}{3}<w<J_u\end{cases}$$

$$f_w(w)=\begin{cases}\frac{27}{16\tau^3\eta^3}\left(w^2-2\tau(1-\eta)w+\tau^2(1-\eta)^2\right) & J_l<w<\tau-\frac{(J_u-\tau)}{3}\\ \frac{9}{8\tau^3\eta^3}\left\{\tau^2\left(\eta^2-3\right)+6\tau w-3w^2\right\} & \tau-\frac{(J_u-\tau)}{3}<w<\tau+\frac{(J_u-\tau)}{3}\\ \frac{27}{16\tau^3\eta^3}\left\{\tau^2(1+\eta)^2-2\tau(1+\eta)w+w^2\right\} & \tau+\frac{(J_u-\tau)}{3}<w<J_u\end{cases} \quad (57)$$

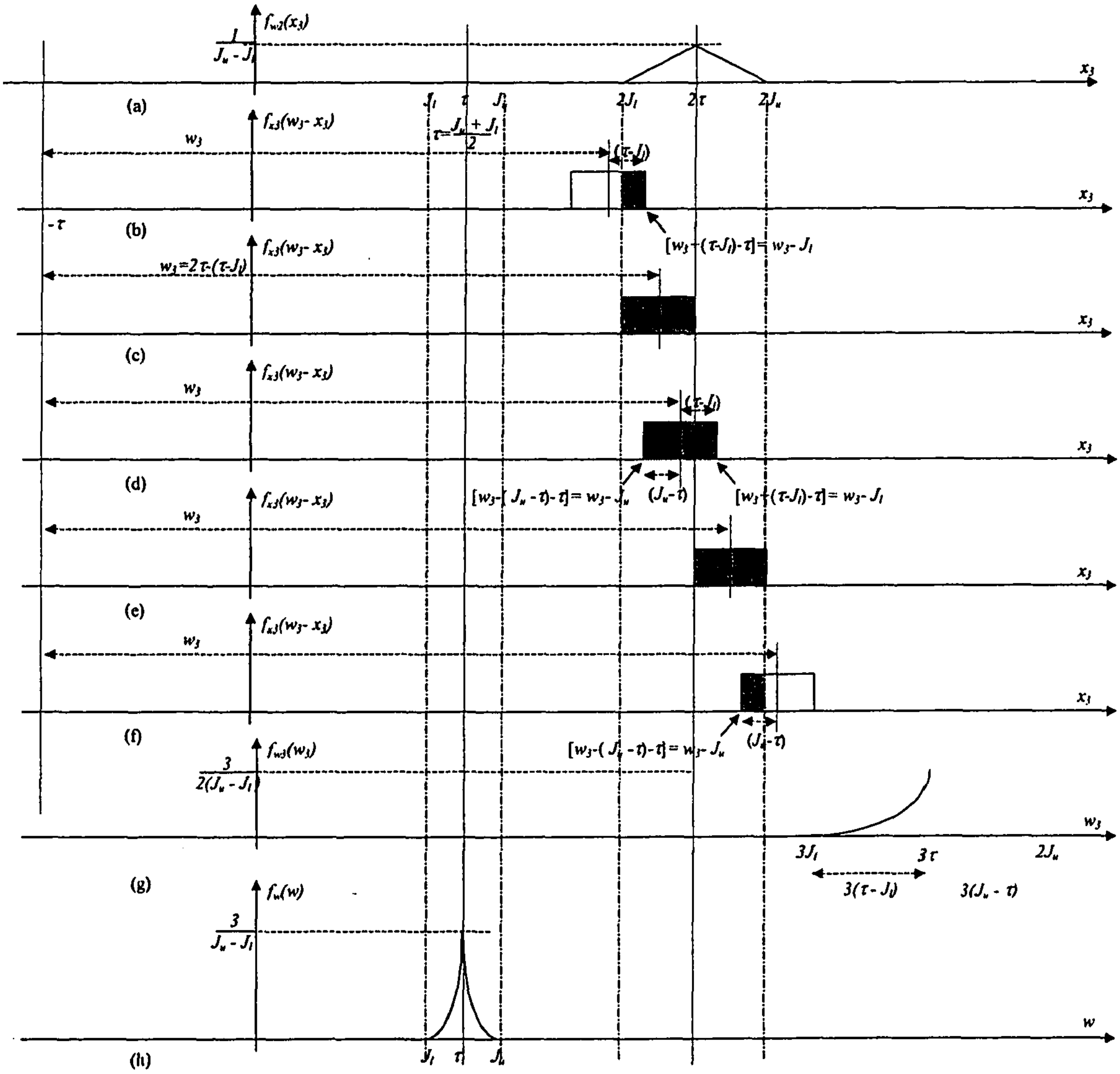


**Figure 34 Convolution to find $f_{w3}(w_3)$, and $f_w(w)$ in case of $n=3$**

The output traffic pdf, mean and variance can be found using equations (29), (30), and (32) respectively. Let us first find the output traffic pfd, $f_y(y)$, using equation (29) in conjunction with equations (34), (37), and (57) as follows,

$$f_y(y) = f_w(y).F_x(y) + F_w(y)f_x(y)$$

In the other side $F_w(w)$ can be found as follows,

**(i) $J_l < w < \tau - (J_u - \tau)/3$**

$$F_w(w) = \int f_w(w)dw = \int_{J_l}^{w} \frac{27}{16\tau^3\eta^3}\left(w^2 - 2\tau(1-\eta)w + \tau^2(1-\eta)^2\right)dw$$

$$= \frac{27}{16\tau^3\eta^3}\left(\frac{w^3}{3} - 2\tau(1-\eta)\frac{w^2}{2} + \tau^2(1-\eta)^2 w\right)\Bigg|_{\tau(1-\eta)}^{w}$$

$$= \frac{27}{16\tau^3\eta^3}\left\{\left(\frac{w^3}{3} - \tau(1-\eta)w^2 + \tau^2(1-\eta)^2 w\right) - \left(\frac{\tau^3(1-\eta)^3}{3} - \tau(1-\eta)\tau^2(1-\eta)^2 + \tau^2(1-\eta)^2\tau(1-\eta)\right)\right\}$$

$$= \frac{27}{16\tau^3\eta^3}\left\{\left(\frac{w^3}{3} - \tau(1-\eta)w^2 + \tau^2(1-\eta)^2 w\right) - \frac{\tau^3(1-\eta)^3}{3}\right\}$$

Then,

$$F_w(w) = \frac{27}{16\tau^3\eta^3}\left\{\left(\frac{w^3}{3} - \tau(1-\eta)w^2 + \tau^2(1-\eta)^2 w\right) - \frac{\tau^3(1-\eta)^3}{3}\right\} \qquad (58)$$

Now, we can find $F_w(w)$ at the boundaries, $J_l$ and $[\tau-(J_u-\tau)/3]$ as follows,

$$F_w(J_l)=F_w(\tau(1-\eta))=\frac{27}{16\tau^3\eta^3}\left\{\left(\frac{\tau^3(1-\eta)^3}{3}-\tau(1-\eta)\tau^2(1-\eta)^2+\tau^2(1-\eta)^2\tau(1-\eta)\right)-\frac{\tau^3(1-\eta)^3}{3}\right\}$$

$$=0$$

And,

$$F_w\left(\tau-\frac{(J_u-\tau)}{3}\right)=F_w\left(\tau-\frac{\eta\tau}{3}\right)$$

$$=\frac{27}{16\tau^3\eta^3}\left\{\left(\frac{\left(\tau-\frac{\eta\tau}{3}\right)^3}{3}-\tau(1-\eta)\left(\tau-\frac{\eta\tau}{3}\right)^2+\tau^2(1-\eta)^2\left(\tau-\frac{\eta\tau}{3}\right)\right)-\frac{\tau^3(1-\eta)^3}{3}\right\}$$

$$=\frac{1}{48\eta^3}\left\{\left((3-\eta)^3-9(1-\eta)(3-\eta)^2+27(1-\eta)^2(3-\eta)\right)-27(1-\eta)^3\right\}$$

$$=\frac{1}{48\eta^3}\left\{\left(27-27\eta+9\eta^2-\eta^3\right)-9(1-\eta)\left(9-6\eta+\eta^2\right)+27\left(1-2\eta+\eta^2\right)(3-\eta)-27\left(1-3\eta+3\eta^2-\eta^3\right)\right\}$$

$$=\frac{1}{48\eta^3}\left\{\left(27-27\eta+9\eta^2-\eta^3\right)-9\left(9-15\eta+7\eta^2-\eta^3\right)+27\left(3-7\eta+5\eta^2-\eta^3\right)-27\left(1-3\eta+3\eta^2-\eta^3\right)\right\}$$

$$=\frac{1}{48\eta^3}\left(8\eta^3\right)=\frac{1}{6}$$

**(ii)** **$\tau-(J_u-\tau)/3 < w < \tau+(J_u-\tau)/3$**

$$F_w(w)=\int f_w(w)dw=\int_{-\infty}^{\tau-\frac{(J_u-\tau)}{3}} f_w(w)dw+\int_{\tau-\frac{(J_u-\tau)}{3}}^{w} f_w(w)dw$$

$$=\frac{1}{6}+\int_{\tau-\frac{\eta\tau}{3}}^{w}\frac{9}{8\tau^3\eta^3}\left\{\tau^2\left(\eta^2-3\right)+6\tau w-3w^2\right\}dw$$

$$=\frac{1}{6}+\frac{9}{8\tau^3\eta^3}\left(\tau^2\left(\eta^2-3\right)w+3\tau w^2-w^3\right)\Big|_{\tau-\frac{\eta\tau}{3}}^{w}$$

$$= \frac{1}{6} + \frac{9}{8\tau^3\eta^3}\left\{ \begin{array}{l} \left(\tau^2\left(\eta^2-3\right)w + 3\tau w^2 - w^3\right) - \\ \qquad -\left(\tau^2\left(\eta^2-3\right)\left(\tau - \frac{\eta\tau}{3}\right) + 3\tau\left(\tau - \frac{\eta\tau}{3}\right)^2 - \left(\tau - \frac{\eta\tau}{3}\right)^3\right) \end{array} \right\}$$

$$= \frac{1}{6} + \frac{9}{8\tau^3\eta^3}\left\{ \begin{array}{l} \tau^2\left(\eta^2-3\right)w + 3\tau w^2 - w^3 - \\ \qquad -\frac{\tau^3}{27}\left[9\left(\eta^2-3\right)(3-\eta) + 9(3-\eta)^2 - (3-\eta)^3\right] \end{array} \right\}$$

$$= \frac{1}{6} + \frac{1}{24\tau^3\eta^3}\left\{ \begin{array}{l} 27\left[\tau^2\left(\eta^2-3\right)w + 3\tau w^2 - w^3\right] - \\ \qquad -\tau^3\left[ \begin{array}{l} 9\left(3\eta^2 - \eta^3 - 9 + 3\eta\right) + 9\left(9 - 6\eta + \eta^2\right) - \\ \qquad -\left(27 - 27\eta + 9\eta^2 - \eta^3\right) \end{array} \right] \end{array} \right\}$$

$$= \frac{1}{6} + \frac{1}{24\tau^3\eta^3}\left\{27\left[\tau^2\left(\eta^2-3\right)w + 3\tau w^2 - w^3\right] + \tau^3\left[27 - 27\eta^2 + 8\eta^3\right]\right\}$$

$$= \frac{1}{6} + \frac{1}{24\tau^3\eta^3}\left\{\tau^3\left[27 - 27\eta^2 + 8\eta^3\right] + 27\tau^2\left(\eta^2-3\right)w + 81\tau w^2 - 27w^3\right\}$$

Then,

$$F_w(w) = \frac{1}{6} + \frac{1}{24\tau^3\eta^3}\left\{\tau^3\left[27 - 27\eta^2 + 8\eta^3\right] + 27\tau^2\left(\eta^2-3\right)w + 81\tau w^2 - 27w^3\right\} \tag{59}$$

Now, we can find $F_w(w)$ at the boundaries, $[\tau-(J_u-\tau)/3]$ and $[\tau+(J_u-\tau)/3]$ as follows,

$$F_w\left(\tau - \frac{(J_u-\tau)}{3}\right) = F_w\left(\tau - \frac{\eta\tau}{3}\right)$$

$$= \frac{1}{6} + \frac{1}{24\tau^3\eta^3}\left\{ \begin{array}{l} \tau^3\left[27 - 27\eta^2 + 8\eta^3\right] + 27\tau^2\left(\eta^2-3\right)\left(\tau - \frac{\eta\tau}{3}\right) + \\ \qquad + 81\tau\left(\tau - \frac{\eta\tau}{3}\right)^2 - 27\left(\tau - \frac{\eta\tau}{3}\right)^3 \end{array} \right\}$$

$$= \frac{1}{6} + \frac{1}{24\eta^3}\left\{\left[27 - 27\eta^2 + 8\eta^3\right] + 9\left(\eta^2-3\right)(3-\eta) + 9(3-\eta)^2 - (3-\eta)^3\right\}$$

$$= \frac{1}{6} + \frac{1}{24\eta^3}\left\{ \begin{array}{l} \left[27 - 27\eta^2 + 8\eta^3\right] + 9\left(3\eta^2 - \eta^3 - 9 + 3\eta\right) + 9\left(9 - 6\eta + \eta^2\right) - \\ \qquad -\left(27 - 27\eta + 9\eta^2 - \eta^3\right) \end{array} \right\}$$

$$= \frac{1}{6} + \frac{1}{24\eta^3}\left\{\left[27 - 27\eta^2 + 8\eta^3\right] + \left[-27 + 27\eta^2 - 8\eta^3\right]\right\} = \frac{1}{6}$$

And,

$$F_w\left(\tau+\frac{(J_u-\tau)}{3}\right)=F_w\left(\tau+\frac{\eta\tau}{3}\right)$$

$$=\frac{1}{6}+\frac{1}{24\tau^3\eta^3}\left\{\begin{array}{c}\tau^3\left[27-27\eta^2+8\eta^3\right]+27\tau^2\left(\eta^2-3\right)\left(\tau+\frac{\eta\tau}{3}\right)+\\ +81\tau\left(\tau+\frac{\eta\tau}{3}\right)^2-27\left(\tau+\frac{\eta\tau}{3}\right)^3\end{array}\right\}$$

$$=\frac{1}{6}+\frac{1}{24\eta^3}\left\{\left[27-27\eta^2+8\eta^3\right]+9\left(\eta^2-3\right)(3+\eta)+9(3+\eta)^2-(3+\eta)^3\right\}$$

$$=\frac{1}{6}+\frac{1}{24\eta^3}\left\{\begin{array}{c}\left[27-27\eta^2+8\eta^3\right]+9\left(3\eta^2+\eta^3-9-3\eta\right)+9\left(9+6\eta+\eta^2\right)-\\ -\left(27+27\eta+9\eta^2+\eta^3\right)\end{array}\right\}$$

$$=\frac{1}{6}+\frac{1}{24\eta^3}\left\{\left[27-27\eta^2+8\eta^3\right]+\left[-27+27\eta^2+8\eta^3\right]\right\}$$

$$=\frac{1}{6}+\frac{1}{24\eta^3}\left(16\eta^3\right)=\frac{1}{6}+\frac{4}{6}$$

$$=\frac{5}{6}$$

**(iii)** $\tau+(J_u-\tau)/3<w<J_u$

$$F_w(w)=\int f_w(w)dw=\int_{-\infty}^{\tau+\frac{(J_u-\tau)}{3}}f_w(w)dw+\int_{\tau+\frac{(J_u-\tau)}{3}}^{w}f_w(w)dw$$

$$=\frac{5}{6}+\int_{\tau+\frac{\eta\tau}{3}}^{w}\frac{27}{16\tau^3\eta^3}\left\{\tau^2(1+\eta)^2-2\tau(1+\eta)w+w^2\right\}dw$$

$$=\frac{5}{6}+\frac{27}{16\tau^3\eta^3}\left(\frac{w^3}{3}-2\tau(1+\eta)\frac{w^2}{2}+\tau^2(1+\eta)^2w\right)\Bigg]_{\tau+\frac{\eta\tau}{3}}^{w}$$

$$=\frac{5}{6}+\frac{27}{16\tau^3\eta^3}\left\{\begin{array}{l}\left(\frac{w^3}{3}-\tau(1+\eta)w^2+\tau^2(1+\eta)^2w\right)-\\ \qquad -\left(\frac{\left(\tau+\frac{\eta\tau}{3}\right)^3}{3}-\tau(1+\eta)\left(\tau+\frac{\eta\tau}{3}\right)^2+\tau^2(1+\eta)^2\left(\tau+\frac{\eta\tau}{3}\right)\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{27}{16\tau^3\eta^3}\left\{\begin{array}{l}\left(\frac{w^3}{3}-\tau(1+\eta)w^2+\tau^2(1+\eta)^2w\right)-\\ \qquad -\frac{\tau^3}{81}\left((3+\eta)^3-9(1+\eta)(3+\eta)^2+27(1+\eta)^2(3+\eta)\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\tau^3\eta^3}\left\{\begin{array}{l}\left(27w^3-81\tau(1+\eta)w^2+81\tau^2(1+\eta)^2w\right)-\\ \qquad -\tau^3\left(\begin{array}{l}\left(27+27\eta+9\eta^2+\eta^3\right)-9(1+\eta)\left(9+6\eta+\eta^2\right)+\\ \quad +27\left(1+2\eta+\eta^2\right)(3+\eta)\end{array}\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\tau^3\eta^3}\left\{\begin{array}{l}\left(27w^3-81\tau(1+\eta)w^2+81\tau^2(1+\eta)^2w\right)-\\ \qquad -\tau^3\left(\begin{array}{l}\left(27+27\eta+9\eta^2+\eta^3\right)-9\left(9+15\eta+7\eta^2+\eta^3\right)+\\ \quad +27\left(3+7\eta+5\eta^2+\eta^3\right)\end{array}\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\tau^3\eta^3}\left\{\left(27w^3-81\tau(1+\eta)w^2+81\tau^2(1+\eta)^2w\right)-\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\right\}$$

Then,

$$F_w(w)=\frac{5}{6}+\frac{1}{24\tau^3\eta^3}\left\{\begin{array}{l}\left(27w^3-81\tau(1+\eta)w^2+81\tau^2(1+\eta)^2w\right)-\\ \qquad -\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\end{array}\right\} \tag{60}$$

Now, we can find $F_w(w)$ at the boundaries, $[\tau+(J_u-\tau)/3]$ and $J_u$ as follows,

$$F_w\left(\tau+\frac{(J_u-\tau)}{3}\right)=F_w\left(\tau+\frac{\eta\tau}{3}\right)$$

$$=\frac{5}{6}+\frac{1}{48\tau^3\eta^3}\left\{\begin{array}{l}\left(27\left(\tau+\frac{\eta\tau}{3}\right)^3-81\tau(1+\eta)\left(\tau+\frac{\eta\tau}{3}\right)^2+81\tau^2(1+\eta)^2\left(\tau+\frac{\eta\tau}{3}\right)\right)- \\ \qquad -\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\eta^3}\left\{\begin{array}{l}\left((3+\eta)^3-9(1+\eta)(3+\eta)^2+27(1+\eta)^2(3+\eta)\right)- \\ \qquad -\left(27+81\eta+81\eta^2+19\eta^3\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\eta^3}\left\{\begin{array}{l}\left(\left(27+27\eta+9\eta^2+\eta^3\right)-9(1+\eta)\left(9+6\eta+\eta^2\right)+27\left(1+2\eta+\eta^2\right)(3+\eta)\right)- \\ \qquad -\left(27+81\eta+81\eta^2+19\eta^3\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\eta^3}\left\{\begin{array}{l}\left(\left(27+27\eta+9\eta^2+\eta^3\right)-9\left(9+15\eta+7\eta^2+\eta^3\right)+27\left(3+7\eta+5\eta^2+\eta^3\right)\right)- \\ \qquad -\left(27+81\eta+81\eta^2+19\eta^3\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\eta^3}\left\{\left(27+81\eta+81\eta^2+19\eta^3\right)-\left(27+81\eta+81\eta^2+19\eta^3\right)\right\}$$

$$=\frac{5}{6}$$

And,

$$F_w(J_u)=F_w(\tau(1+\eta))$$

$$=\frac{5}{6}+\frac{1}{48\tau^3\eta^3}\left\{\begin{array}{l}\left(27\tau^3(1+\eta)^3-81\tau(1+\eta)\tau^2(1+\eta)^2+81\tau^2(1+\eta)^2\tau(1+\eta)\right)- \\ \qquad -\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\end{array}\right\}$$

$$=\frac{5}{6}+\frac{1}{48\eta^3}\left\{27\left(1+3\eta+3\eta^2+\eta^3\right)-\left(27+81\eta+81\eta^2+19\eta^3\right)\right\}$$

$$=\frac{5}{6}+\frac{1}{48\eta^3}\left\{8\eta^3\right\}$$

$$=1$$

Then, $F_w(w)$ can be described as follows,

$$F_w(w)=\begin{cases}\dfrac{27}{16\tau^3\eta^3}\left\{\begin{pmatrix}\dfrac{w^3}{3}-\tau(1-\eta)w^2+\tau^2(1-\eta)^2 w\end{pmatrix}-\\ -\dfrac{\tau^3(1-\eta)^3}{3}\end{array}\right\} & J_l<w<\tau-\dfrac{(J_u-\tau)}{3}\\ \dfrac{1}{6}+\dfrac{1}{24\tau^3\eta^3}\left\{\begin{array}{l}\tau^3\left[27-27\eta^2+8\eta^3\right]+\\ +27\tau^2(\eta^2-3)w+81\tau w^2-27w^3\end{array}\right\} & \tau-\dfrac{(J_u-\tau)}{3}<w<\tau+\dfrac{(J_u-\tau)}{3}\\ \dfrac{5}{6}+\dfrac{1}{24\tau^3\eta^3}\left\{\begin{array}{l}\left(27w^3-81\tau(1+\eta)w^2+81\tau^2(1+\eta)^2 w\right)-\\ -\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\end{array}\right\} & \tau+\dfrac{(J_u-\tau)}{3}<w<J_u\end{cases} \tag{61}$$

Now, our goal to fine $f_y(y)$ can be achieved. Using equations (29), (34), (57), (37), and (61) as follows,

$$\eta=\frac{J_u-\tau}{\tau}=\frac{\tau-J_l}{\tau}$$

*Then,*

$J_u=\tau(1+\eta)$ *and,*

$J_l=\tau(1-\eta)$ *and,*

$J_u-\tau=\tau\eta$ *and,*

$2\tau+J_u=\tau(3+\eta)$ *and,*

$2\tau-J_u=\tau(1-\eta)$

**(i) $J_l<y<\tau-(J_u-\tau)/3$**

Here, the output traffic pdf , $f_y(y)$, in the rage of y is from $J_l$ to $\tau-(J_u-\tau)/3$, as follows,

$$
\begin{aligned}
f_y(y) &= f_w(y).F_x(y) + F_w(y) f_x(y) \\
&= \frac{27}{16\tau^3\eta^3}\left(y^2 - 2\tau(1-\eta)y + \tau^2(1-\eta)^2\right)\frac{y-\tau(1-\eta)}{2\eta\tau} + \\
&\quad + \frac{27}{16\tau^3\eta^3}\left\{\begin{array}{l}\left(\frac{y^3}{3} - \tau(1-\eta)y^2 + \tau^2(1-\eta)^2 y\right) - \\ \qquad - \frac{\tau^3(1-\eta)^3}{3}\end{array}\right\}\frac{1}{2\eta\tau} \\
&= \frac{9}{32\tau^4\eta^4}\left\{\begin{array}{l}3\left[y^2 - 2\tau(1-\eta)y + \tau^2(1-\eta)^2\right][y-\tau(1-\eta)] + \\ \qquad + y^3 - 3\tau(1-\eta)y^2 + 3\tau^2(1-\eta)^2 y - \tau^3(1-\eta)^3\end{array}\right\} \\
&= \frac{9}{32\tau^4\eta^4}\left\{\begin{array}{l}3y^3 - 6\tau(1-\eta)y^2 + 3\tau^2(1-\eta)^2 y - \\ \qquad - 3\tau(1-\eta)y^2 + 6\tau^2(1-\eta)^2 y - 3\tau^3(1-\eta)^3 + \\ \qquad + y^3 - 3\tau(1-\eta)y^2 + 3\tau^2(1-\eta)^2 y - \tau^3(1-\eta)^3\end{array}\right\} \\
&= \frac{9}{32\tau^4\eta^4}\left\{4y^3 - 12\tau(1-\eta)y^2 + 12\tau^2(1-\eta)^2 y - 4\tau^3(1-\eta)^3\right\} \\
&= \frac{9}{8\tau^4\eta^4}\left\{y^3 - 3\tau(1-\eta)y^2 + 3\tau^2(1-\eta)^2 y - \tau^3(1-\eta)^3\right\} \\
&= \frac{9}{8\tau^4\eta^4}[y-\tau(1-\eta)]^3
\end{aligned}
$$

$$
f_y(y) = \frac{9}{8\tau^4\eta^4}[y-\tau(1-\eta)]^3 \tag{62}
$$

Now, we can find $fy(y)$ at the boundaries, $J_l$ and $[\tau\text{-}(J_u\text{-}\tau)/3]$ as follows,

$$
f_y(J_l) = f_y[\tau(1-\eta)] = \frac{9}{8\tau^4\eta^4}[\tau(1-\eta) - \tau(1-\eta)]^3 = 0
$$

And,

$$f_y\left(\tau-\frac{J_u-\tau}{3}\right)=f_y\left(\tau-\frac{\eta\tau}{3}\right)=\frac{9}{8\tau^4\eta^4}\left[\tau-\frac{\eta\tau}{3}-\tau(1-\eta)\right]^3$$
$$=\frac{9}{8\tau^4\eta^4}\left[-\frac{\eta\tau}{3}+\tau\eta\right]^3=\frac{9}{8\tau^4\eta^4}\left(\frac{8\eta^3\tau^3}{27}\right)$$
$$=\frac{1}{3\tau\eta}=\frac{1}{3(J_u-\tau)}=\frac{2}{3(J_u-J_l)}$$

**(ii)** $\tau-(J_u-\tau)/3<y<\tau+(J_u-\tau)/3$

Here, the output traffic pdf , $f_y(y)$, in the rage of y is from $\tau-(J_u-\tau)/3$ to $\tau+(J_u-\tau)/3$, as follows,

$$f_y(y)=f_w(y).F_x(y)+F_w(y)f_x(y)$$
$$=\frac{9}{8\tau^3\eta^3}\left\{\tau^2\left(\eta^2-3\right)+6\tau y-3y^2\right\}\frac{y-\tau(1-\eta)}{2\eta\tau}+$$
$$+\left\{\frac{1}{6}+\frac{1}{24\tau^3\eta^3}\left[\begin{array}{l}\tau^3\left(27-27\eta^2+8\eta^3\right)+\\ \quad+27\tau^2\left(\eta^2-3\right)y+81\tau y^2-27y^3\end{array}\right]\right\}\frac{1}{2\eta\tau}$$
$$=\frac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}27\left[\tau^2\left(\eta^2-3\right)+6\tau y-3y^2\right]\left[y-\tau(1-\eta)\right]+\\ \quad+\left[\begin{array}{l}\tau^3\left(27-27\eta^2+8\eta^3\right)+\\ \quad+27\tau^2\left(\eta^2-3\right)y+81\tau y^2-27y^3\end{array}\right]\end{array}\right\}+\frac{1}{12\eta\tau}$$
$$=\frac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}4\tau^3\eta^3+27\tau^2\left(\eta^2-3\right)y+162\tau y^2-81y^3-\\ \quad-27\tau^3(1-\eta)\left(\eta^2-3\right)-162\tau^2(1-\eta)y+81\tau(1-\eta)y^2+\\ \quad\tau^3\left(27-27\eta^2+8\eta^3\right)+27\tau^2\left(\eta^2-3\right)y+81\tau y^2-27y^3\end{array}\right\}$$
$$=\frac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}4\tau^3\eta^3+-108y^3+81\tau(4-\eta)y^2+\left[54\tau^2\left(\eta^2-3\right)-162\tau^2(1-\eta)\right]y-\\ \quad-27\tau^3\left(-3+3\eta+\eta^2-\eta^3\right)+\tau^3\left(27-27\eta^2+8\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}-108y^3+81\tau(4-\eta)y^2+54\tau^2\left[\eta^2+3\eta-6\right]y+\\ \quad+\tau^3\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$

$$f_y(y)=\frac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}-108y^3+81\tau(4-\eta)y^2+54\tau^2\left[\eta^2+3\eta-6\right]y+\\ \quad+\tau^3\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\} \tag{63}$$

Now, we can find $f_y(y)$ at the boundaries, $[\tau-(J_u-\tau)/3]$ and $[\tau+(J_u-\tau)/3]$ as follows,

$$f_y\left(\tau-\frac{J_u-\tau}{3}\right)=f_y\left(\tau-\frac{\eta\tau}{3}\right)$$
$$=\frac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}-108\left(\tau-\frac{\eta\tau}{3}\right)^3+81\tau(4-\eta)\left(\tau-\frac{\eta\tau}{3}\right)^2+54\tau^2\left[\eta^2+3\eta-6\right]\left(\tau-\frac{\eta\tau}{3}\right)+\\ +\tau^3\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-4(3-\eta)^3+9(4-\eta)(3-\eta)^2+18\left(-6+3\eta+\eta^2\right)(3-\eta)+\\ +\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-4\left(27-27\eta+9\eta^2-\eta^3\right)+9(4-\eta)\left(9-6\eta+\eta^2\right)+18\left(-6+3\eta+\eta^2\right)(3-\eta)+\\ +\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau\eta^4}\left\{27\eta-90\eta^2+43\eta^3+9\left(36-33\eta+10\eta^2-\eta^3\right)+18\left[-18+15\eta-\eta^3\right]\right\}$$
$$=\frac{1}{48\tau\eta^4}\left\{6\eta^3\right\}=\frac{1}{3\tau\eta}$$

And,

$$f_y\left(\tau+\frac{J_u-\tau}{3}\right)=f_y\left(\tau+\frac{\eta\tau}{3}\right)$$
$$=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-108\left(\tau+\frac{\eta\tau}{3}\right)^3+81\tau(4-\eta)\left(\tau+\frac{\eta\tau}{3}\right)^2+54\tau^2\left[\eta^2+3\eta-6\right]\left(\tau+\frac{\eta\tau}{3}\right)+\\ +\tau^3\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-4(3+\eta)^3+9(4-\eta)(3+\eta)^2+18\left[\eta^2+3\eta-6\right](3+\eta)+\\ +\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-4\left(27+27\eta+9\eta^2+\eta^3\right)+9(4-\eta)\left(9+6\eta+\eta^2\right)+\\ \quad+18\left(\eta^2+3\eta-6\right)(3+\eta)+\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-4\left(27+27\eta+9\eta^2+\eta^3\right)+9\left(36+15\eta-2\eta^2-\eta^3\right)+\\ \quad+18\left(-18+3\eta+6\eta^2+\eta^3\right)+\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}$$
$$=\frac{1}{48\tau\eta^4}\left(44\eta^3\right)=\frac{11}{12\tau\eta}$$

**(iii)** $\tau+(J_u-\tau)/3<y<J_u$

Here, the output traffic pdf , $f_y(y)$, in the rage of y is from $\tau+(J_u-\tau)/3$ to $J_u$, as follows,

$$
\begin{aligned}
f_y(y) &= f_w(y).F_x(y)+F_w(y)f_x(y) \\
&= \frac{27}{16\tau^3\eta^3}\left\{\tau^2(1+\eta)^2-2\tau(1+\eta)y+y^2\right\}\frac{y-\tau(1-\eta)}{2\eta\tau}+ \\
&\quad +\left\{\frac{5}{6}+\frac{1}{48\tau^3\eta^3}\left[\begin{matrix}\left(27y^3-81\tau(1+\eta)y^2+81\tau^2(1+\eta)^2y\right)- \\ -\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\end{matrix}\right]\right\}\frac{1}{2\eta\tau} \\
&= \frac{27}{32\tau^4\eta^4}\left\{\tau^2(1+\eta)^2-2\tau(1+\eta)y+y^2\right\}(y-\tau(1-\eta))+\frac{5}{12\eta\tau}+ \\
&\quad +\frac{1}{96\tau^4\eta^4}\left[\begin{matrix}\left(27y^3-81\tau(1+\eta)y^2+81\tau^2(1+\eta)^2y\right)- \\ -\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\end{matrix}\right] \\
&= \frac{1}{96\tau^4\eta^4}\left\{\begin{matrix}81\left[\tau^2(1+\eta)^2-2\tau(1+\eta)y+y^2\right](y-\tau(1-\eta))+40\tau^3\eta^3+ \\ +\left(27y^3-81\tau(1+\eta)y^2+81\tau^2(1+\eta)^2y\right)- \\ -\tau^3\left(27+81\eta+81\eta^2+19\eta^3\right)\end{matrix}\right\} \\
&= \frac{1}{96\tau^4\eta^4}\left\{\begin{matrix}81\left[\begin{matrix}\tau^2(1+\eta)^2y-2\tau(1+\eta)y^2+y^3- \\ -\tau^2(1+\eta)^2\tau(1-\eta)+2\tau(1+\eta)\tau(1-\eta)y-\tau(1-\eta)y^2\end{matrix}\right]+ \\ +\left(27y^3-81\tau(1+\eta)y^2+81\tau^2(1+\eta)^2y\right)- \\ -\tau^3\left(27+81\eta+81\eta^2-21\eta^3\right)\end{matrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{96\tau^4\eta^4}\left\{\begin{matrix}108y^3-81y^2\left[3\tau(1+\eta)+\tau(1-\eta)\right]+81y\left[2\tau^2(1+\eta)^2+2\tau^2\left(1-\eta^2\right)\right]- \\ -81\tau^2(1+\eta)^2\tau(1-\eta)-\tau^3\left(27+81\eta+81\eta^2-21\eta^3\right)\end{matrix}\right\} \\
&= \frac{1}{96\tau^4\eta^4}\left\{\begin{matrix}108y^3-81\tau y^2(4+2\eta)+162\tau^2y(2+2\eta)- \\ -81\tau^3\left(1+\eta-\eta^2-\eta^3\right)-\tau^3\left(27+81\eta+81\eta^2-21\eta^3\right)\end{matrix}\right\} \\
&= \frac{1}{96\tau^4\eta^4}\left\{108y^3-162\tau y^2(2+\eta)+324\tau^2y(1+\eta)-\tau^3\left(108+162\eta-102\eta^3\right)\right\}
\end{aligned}
$$

$$f_y(y)=\frac{1}{96\tau^4\eta^4}\left\{108y^3-162\tau y^2(2+\eta)+324\tau^2y(1+\eta)-\tau^3\left(108+162\eta-102\eta^3\right)\right\} \quad (64)$$

Now, we can find $f_y(y)$ at the boundaries, $[\tau+(J_u-\tau)/3]$ and $J_u$ as follows,

$$f_y\left(\tau+\frac{J_u-\tau}{3}\right)=f_y\left(\tau+\frac{\eta\tau}{3}\right)$$

$$=\frac{1}{96\tau^4\eta^4}\left\{\begin{array}{l}108\left(\tau+\frac{\eta\tau}{3}\right)^3-162\tau\left(\tau+\frac{\eta\tau}{3}\right)^2(2+\eta)+324\tau^2\left(\tau+\frac{\eta\tau}{3}\right)(1+\eta)- \\ -\tau^3\left(108+162\eta-102\eta^3\right)\end{array}\right\}$$

$$=\frac{1}{96\tau\eta^4}\left\{\begin{array}{l}4(3+\eta)^3-18(3+\eta)^2(2+\eta)+108(3+\eta)(1+\eta)- \\ -\left(108+162\eta-102\eta^3\right)\end{array}\right\}$$

$$=\frac{1}{96\tau\eta^4}\left\{\begin{array}{l}4\left(27+27\eta+9\eta^2+\eta^3\right)-18\left(9+6\eta+\eta^2\right)(2+\eta)+ \\ +108\left(3+4\eta+\eta^2\right)-\left(108+162\eta-102\eta^3\right)\end{array}\right\}$$

$$=\frac{1}{96\tau\eta^4}\left\{\begin{array}{l}4\left(27+27\eta+9\eta^2+\eta^3\right)-18\left(18+21\eta+8\eta^2+\eta^3\right)+ \\ +108\left(3+4\eta+\eta^2\right)-\left(108+162\eta-102\eta^3\right)\end{array}\right\}$$

$$=\frac{1}{96\tau\eta^4}\left(88\eta^3\right)=\frac{11}{12\tau\eta}$$

$$f_y(J_u)=f_y(\tau(1+\eta))$$

$$=\frac{1}{96\tau^4\eta^4}\left\{\begin{array}{l}108\tau^3(1+\eta)^3-162\tau\tau^2(1+\eta)^2(2+\eta)+324\tau^2\tau(1+\eta)(1+\eta)- \\ -\tau^3\left(108+162\eta-102\eta^3\right)\end{array}\right\}$$

$$=\frac{1}{96\tau\eta^4}\left\{\begin{array}{l}108\left(1+3\eta+3\eta^2+\eta^3\right)-162\left(2+5\eta+4\eta^2+\eta^3\right)+ \\ +324\left(1+2\eta+\eta^2\right)-\left(108+162\eta-102\eta^3\right)\end{array}\right\}$$

$$=\frac{1}{96\tau\eta^4}\left(48\eta^3\right)=\frac{1}{2\tau\eta}$$

Then, $f_y(y)$ can be described as follows,

$$
f_y(y)=\begin{cases}
\dfrac{9}{8\tau^4\eta^4}\left[y-\tau(1-\eta)\right]^3 & J_l < w < \tau-\dfrac{(J_u-\tau)}{3} \\[2ex]
\dfrac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}-108y^3+81\tau(4-\eta)y^2+ \\ +54\tau^2\left[\eta^2+3\eta-6\right]y+ \\ +\tau^3\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\} & \tau-\dfrac{(J_u-\tau)}{3} < w < \tau+\dfrac{(J_u-\tau)}{3} \\[2ex]
\dfrac{1}{96\tau^4\eta^4}\left\{\begin{array}{l}108y^3-162\tau y^2(2+\eta)+324\tau^2 y(1+\eta)- \\ -\tau^3\left(108+162\eta-102\eta^3\right)\end{array}\right\} & \tau+\dfrac{(J_u-\tau)}{3} < w < J_u
\end{cases} \quad (65)
$$

| [illegible] | $J_l$ | $\tau-\frac{\eta\tau}{\sqrt{3}}$ | $\tau+\frac{\eta\tau}{\sqrt{3}}$ | $J_u$ |
|---|---|---|---|---|
| $f_x$ | $\frac{1}{2\tau\eta}$ | $\frac{1}{2\tau\eta}$ | $\frac{1}{2\tau\eta}$ | $\frac{1}{2\tau\eta}$ |
| $F_x$ | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 |
| $f_w$ | 0 | $\frac{3}{4\tau\eta}$ | $\frac{3}{4\tau\eta}$ | 0 |
| $F_w$ | 0 | $\frac{1}{6}$ | $\frac{5}{6}$ | 1 |
| $f_y$ | 0 | $\frac{1}{3\tau\eta}$ | $\frac{11}{12\tau\eta}$ | $\frac{1}{2\tau\eta}$ |

Now, we would like to find the mean and variance of the output traffic pdf $f_y(y)$, to show the algorithm improvement. Equation (65) shows that $f_y(y)$ is a third order function. That mean that the mean and variance are the integration of forth and fifth functions and the results will be fifth and sixth order functions respectively. Then the manipulation will be very difficult and long to find the mean and variance. So that we will use the following transformation of variable to simplify the upcoming analysis, Let us define and variable z, such that,

$$y = \tau(1 + z\eta) \qquad (66)$$

Equation (65) lead to the following ones that will be used in our analysis,

$$z = \frac{y - \tau}{\tau\eta}$$

*and*,

$$dy = \tau\eta dz$$

By applying the above transformation on equation (65), then, $f_y(y)$ pdf can be expressed as a function of z as follows,

**(i) $J_l < y < \tau - (J_u - \tau)/3$**

Let us first find the boundaries of z as follows,

$$y = J_l \quad \Rightarrow \quad \tau(1 + z\eta) = J_l = \tau(1 - \eta)$$

*Then*,

$$z = -1$$

And,

$$y = \tau - \frac{(J_u - \tau)}{3} \quad \Rightarrow \quad \tau(1 + z\eta) = \tau - \frac{(J_u - \tau)}{3} = \tau - \frac{\eta\tau}{3}$$

*Then*,

$$z = -\frac{1}{3}$$

$$f_y(y) = f_y(\tau(1 + z\eta)) = f_z(z)$$

$$= \frac{9}{8\tau^4\eta^4}[\tau(1 + z\eta) - \tau(1 - \eta)]^3 = \frac{9(z+1)^3}{8\tau\eta}$$

$$f_y(y)=f_z(z)=\frac{9(z+1)^3}{8\tau\eta} \tag{67}$$

**(ii) τ-( $J_u$-τ)/3<y<τ+( $J_u$-τ)/3**

Let us first find the boundaries of z as follows,

$$y=\tau-\frac{(J_u-\tau)}{3} \quad\Rightarrow\quad \tau(1+z\eta)=\tau-\frac{(J_u-\tau)}{3}=\tau-\frac{\eta\tau}{3}$$

*Then,*

$$z=-\frac{1}{3}$$

And,

$$y=\tau+\frac{(J_u-\tau)}{3} \quad\Rightarrow\quad \tau(1+z\eta)=\tau+\frac{(J_u-\tau)}{3}=\tau+\frac{\eta\tau}{3}$$

*Then,*

$$z=\frac{1}{3}$$

$$\begin{aligned}
f_y(y)&=f_y(\tau(1+z\eta))=f_z(z)\\
&=\frac{1}{48\tau^4\eta^4}\left\{\begin{array}{l}-108\tau^3(1+z\eta)^3+81\tau(4-\eta)\tau^2(1+z\eta)^2+54\tau^2\left[\eta^2+3\eta-6\right]\tau(1+z\eta)+\\ \quad+\tau^3\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}\\
&=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-108\left(1+3z\eta+3z^2\eta^2+z^3\eta^3\right)+81(4-\eta)\left(1+2z\eta+z^2\eta^2\right)+\\ \quad+54\left[-6+3\eta+\eta^2\right](1+z\eta)+\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}\\
&=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}-108\left(1+3z\eta+3z^2\eta^2+z^3\eta^3\right)+81\left(4+8z\eta+4z^2\eta^2-\eta-2z\eta^2-z^2\eta^3\right)+\\ \quad+54\left(-6+3\eta+\eta^2-6z\eta+3z\eta^2+z\eta^3\right)+\left(108-81\eta-54\eta^2+39\eta^3\right)\end{array}\right\}\\
&=\frac{1}{48\tau\eta^4}\left\{\begin{array}{l}(-108+81*4-54*6+108)+\eta(-108*3z+81*8z-81+54*3-54*6z-81)+\\ \quad+\eta^2\left(-108*3z^2+81*4z^2-81*2z+54+54*3z-54\right)+\\ \quad+\eta^3\left(-108z^3-81z^2+54z+39\right)\end{array}\right\}\\
&=\frac{1}{48\tau\eta^4}\eta^3\left(-108z^3-81z^2+54z+39\right)\\
&=\frac{-108z^3-81z^2+54z+39}{48\tau\eta}
\end{aligned}$$

$$f_y(y) = f_z(z) = \frac{-108z^3 - 81z^2 + 54z + 39}{48\tau\eta} \tag{68}$$

**(iii)** $\tau + (J_u - \tau)/3 < y < J_u$

Let us first find the boundaries of z as follows,

$$y = \tau + \frac{(J_u - \tau)}{3} \quad \Rightarrow \quad \tau(1 + z\eta) = \tau + \frac{(J_u - \tau)}{3} = \tau + \frac{\eta\tau}{3}$$

*Then,*

$$z = \frac{1}{3}$$

And,

$$y = J_u \quad \Rightarrow \quad \tau(1 + z\eta) = \tau(1 + \eta)$$

*Then,*

$$z = 1$$

$$f_y(y) = f_y(\tau(1 + z\eta)) = f_z(z)$$

$$= \frac{1}{96\tau^4\eta^4}\left\{\begin{matrix} 108\tau^3(1+z\eta)^3 - 162\tau\tau^2(1+z\eta)^2(2+\eta) + 324\tau^2\tau(1+z\eta)(1+\eta) - \\ -\tau^3\left(108 + 162\eta - 102\eta^3\right) \end{matrix}\right\}$$

$$= \frac{1}{96\tau\eta^4}\left\{\begin{matrix} 108\left(1 + 3z\eta + 3z^2\eta^2 + z^3\eta^3\right) - 162\left(1 + 2z\eta + z^2\eta^2\right)(2+\eta) + \\ +324\left(1 + z\eta + \eta + z\eta^2\right) - \left(108 + 162\eta - 102\eta^3\right) \end{matrix}\right\}$$

$$= \frac{1}{96\tau\eta^4}\left\{\begin{matrix} 108\left(1 + 3z\eta + 3z^2\eta^2 + z^3\eta^3\right) - 162\left(2 + 4z\eta + 2z^2\eta^2 + \eta + 2z\eta^2 + z^2\eta^3\right) + \\ +324\left(1 + z\eta + \eta + z\eta^2\right) - \left(108 + 162\eta - 102\eta^3\right) \end{matrix}\right\}$$

$$= \frac{1}{96\tau\eta^4}\left\{\begin{matrix} (108 - 162*2 + 324 - 108) + \eta(108*3z - 162*4z - 162 + 324z + 324 - 162) + \\ +\eta^2\left(108*3z^2 - 162*2z^2 - 162*2z + 324z\right) \\ +\eta^3\left(108z^3 - 162z^2 + 102\right) \end{matrix}\right\}$$

$$= \frac{1}{96\tau\eta^4}\eta^3\left(108z^3 - 162z^2 + 102\right)$$

$$= \frac{\left(54z^3 - 81z^2 + 51\right)}{48\tau\eta}$$

$$f_y(y) = f_z(z) = \frac{54z^3 - 81z^2 + 51}{48\tau\eta} \tag{69}$$

Then, $f_z(z)$ is expressed as follows,

$$f_y(y) = f_z(z) = \begin{cases} \frac{9(z+1)^3}{8\tau\eta} & -1 < z < -\frac{1}{3} \\ \frac{-108z^3 - 81z^2 + 54z + 39}{48\tau\eta} & -\frac{1}{3} < z < \frac{1}{3} \\ \frac{54z^3 - 81z^2 + 51}{48\tau\eta} & \frac{1}{3} < z < 1 \end{cases} \tag{70}$$

Let us found the mean of the output traffic pfd, $f_y(y)$, $\mu_y$ as follows,

$$\begin{aligned} \mu_y &= \int_y y f_y(y)dy = \int_z \tau(1 + z\eta) f_z(z)\eta\tau dz \\ &= \int_{-1}^{-\frac{1}{3}} \eta\tau^2 (1 + z\eta) \frac{9(z+1)^3}{8\tau\eta} dz + \\ &\quad + \int_{-\frac{1}{3}}^{\frac{1}{3}} \eta\tau^2 (1 + z\eta) \left( \frac{-108z^3 - 81z^2 + 54z + 39}{48\tau\eta} \right) dz + \\ &\quad + \int_{\frac{1}{3}}^{1} \eta\tau^2 (1 + z\eta) \left( \frac{54z^3 - 81z^2 + 51}{48\tau\eta} \right) dz \end{aligned}$$

The above integration is provided in Appendix C, which shows that the mean can be is described by the following equation,

$$\mu_y = \tau\left(1 + \frac{5\eta}{18}\right) \tag{71}$$

Now, We can find the ratio on the output traffic mean to the input traffic mean as follows,

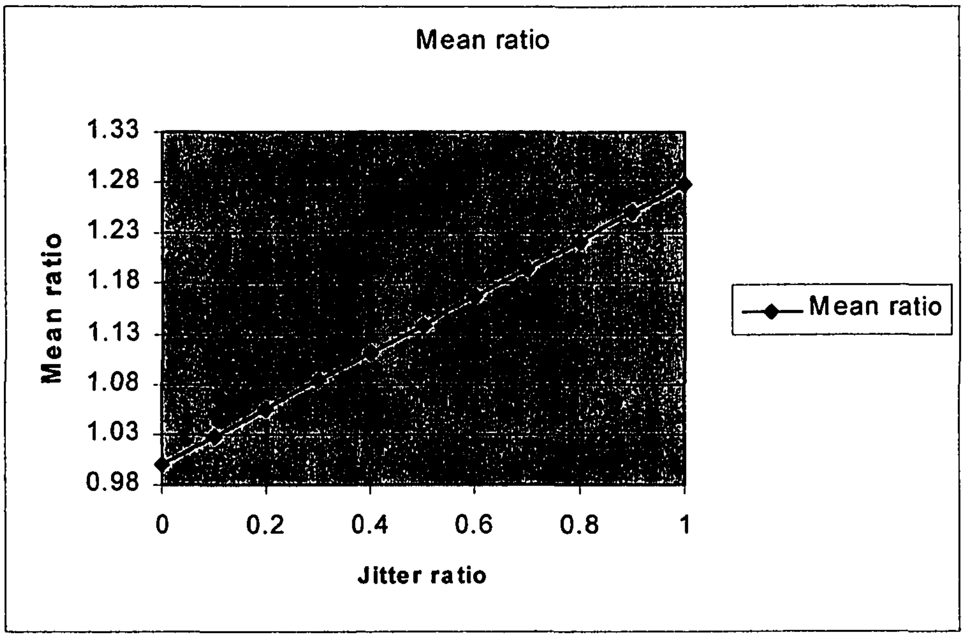



**Figure 35 Mean ratio in case of average over 3 algorithm (Uniform distribution)**

$$\boxed{\frac{\mu_y}{\mu_x} = 1 + \frac{5\eta}{18}} \qquad (72)$$

By comparing this equation with equation(46), it clear that the increase in the average is more that that of n=2 case. That means that more buffer size is needed. Figure 35shows the increase in the output traffic mean as a function of $\eta$

$$\sigma_y^2 = \overline{y^2} - \mu_y^2 = \int y^2 f_y(y)dy - \mu_y^2$$

Let use first find $\overline{y^2}$

$$\overline{y^2} = \int_y y f_y(y) dyz = \int_z \tau^2 (1+z\eta)^2 f_z(z) \eta\tau dz$$

$$= \int_{-1}^{-\frac{1}{3}} \eta\tau^3 (1+z\eta)^2 \frac{9(z+1)^3}{8\tau\eta} dz +$$

$$+ \int_{-\frac{1}{3}}^{\frac{1}{3}} \eta\tau^3 (1+z\eta)^2 \left( \frac{-108z^3 - 81z^2 + 54z + 39}{48\tau\eta} \right) dz +$$

$$+ \int_{\frac{1}{3}}^{1} \eta\tau^3 (1+z\eta)^2 \left( \frac{54z^3 - 81z^2 + 51}{48\tau\eta} \right) dz$$

The above integration is long and it is shown in appendix D, which leads to the following equation,

$$\overline{y^2} = \frac{\tau^2}{9} \left( 9 + 5\eta + 2\eta^2 \right) \tag{73}$$

Now, the variance of y can be found as follows,

$$\sigma_y^2 = \overline{y^2} - \mu_y^2 = \frac{\tau^2}{9} \left( 9 + 5\eta + 2\eta^2 \right) - \tau^2 \left( 1 + \frac{5\eta}{18} \right)^2$$

$$= \frac{\tau^2}{9} \left( 9 + 5\eta + 2\eta^2 \right) - \frac{\tau^2}{324} \left( 324 + 180\eta + 25\eta^2 \right)$$

$$= \frac{47}{324} \tau^2 \eta^2$$

$$\sigma_y^2 = \frac{47}{324} \tau^2 \eta^2 \tag{74}$$

The ratio of the input and output traffic variance , using equations (19) and (48),is,

$$\frac{\sigma_y^2}{\sigma_x^2}=\frac{\frac{47}{324}\tau^2\eta^2}{\frac{1}{3}(\tau^2-2\tau J_u+J_u^2)}$$
$$=\frac{47\tau^2\eta^2}{108(\tau^2-2\tau\tau(1+\eta)+\tau^2(1+\eta)^2)}$$
$$=\frac{47\eta^2}{108(1-2(1+\eta)+(1+\eta)^2)}=\frac{47\eta^2}{108(1-2-2\eta+1+2\eta+\eta^2)}$$
$$=\frac{47}{108}$$

$$\frac{\sigma_y^2}{\sigma_x^2}=\frac{47}{108}=0.4352 \qquad (75)$$

Equation (75) shows the variance improvement in case of average over 3 algorithm. By comparing this result and the one in case of n=2, equation (49), it is clear that n=3 provides better jitter improvement. Later in section 4.4.1.3, we will analysis the algorithm in case of large n. As we will see, the improvement reaches saturation, which useful conclusion. Based on that, the receiver doesn't need to memorize too far back. In other words, taking the average over few neighborhood packets is enough to reach reasonable jitter improvement.

#### 4.4.1.3 Large values of n

As it was discuss above and to complete the analysis, let us find the algorithm results in case of large n. Here we are going to use the central limit theory as described in section 4.1.2. In case of large n value, the average pdf, $f_w(w)$, can be expressed as normal distribution with mean and variance described as follows,

$$w_n = x_1 + x_2 + x_3 + \cdots + x_n$$

$$\begin{aligned}\mu_{wn} &= \mu_{x1} + \mu_{x2} + \mu_{x3} + \cdots + \mu_{xn} \\ &= n\mu_x\end{aligned}$$

And,

$$\begin{aligned}\sigma_{wn}^2 &= \sigma_{x1}^2 + \sigma_{x2}^2 + \sigma_{x3}^2 + \cdots + \sigma_{xn}^2 \\ &= n\sigma_x^2\end{aligned}$$

Then $f_{wn}(w_n)$ can be expressed as follows,

$$\begin{aligned}f_{w_n}(w_n) &= \frac{1}{\sqrt{2\pi}\sigma_{w_n}} e^{-\frac{(w_n - \mu_{w_n})^2}{2\sigma_{w_n}^2}} \\ &= \frac{1}{\sqrt{2\pi n}\sigma_x} e^{-\frac{(w_n - n\mu_x)^2}{2n\sigma_x^2}}\end{aligned}$$

Since we are interested in the average not the sum of the n RVs, $f_w(w)$, can be found as follows,

Since

$$w = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{w_n}{n}$$

Then using equation (2),

$$f_w(w) = f_{w_n}\left(\frac{w_n}{n}\right)\frac{dw_n}{dw}$$

$$= \frac{n}{\sqrt{2\pi n}\sigma_x} e^{-\frac{(nw - n\mu_x)^2}{2n\sigma_x^2}} = \frac{1}{\sqrt{2\pi}\frac{\sigma_x}{\sqrt{n}}} e^{-\frac{(w-\mu_x)^2}{2\frac{\sigma_x^2}{n}}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{(w-\mu_w)^2}{2\sigma_w^2}}$$

The above distribution is a normal distribution with mean and variance,

$$\mu_w = \mu_x$$

and,

$$\sigma_w^2 = \frac{\sigma_x^2}{n}$$

Knowing that,

$$\mu_w = \tau$$

And,

$$\sigma_x^2 = \frac{1}{3}(\tau^2 - 2\tau J_u + J_u^2) = \frac{1}{3}\left[4\tau^2 - 2\tau^2(1+\eta) + \tau^2(1+\eta)^2\right]$$

$$= \frac{1}{3}\left[\tau^2 - 2\tau^2 - 2\tau^2\eta + \tau^2\left(1 + 2\eta + \eta^2\right)\right] = \frac{\tau^2}{3}\left[3 + \eta^2\right]$$

$$= \frac{\eta^2}{3}\tau^2$$

Then,

$$f_w(w)=\frac{1}{\sqrt{\frac{2\pi\tau^2}{n}\frac{\eta^2}{3}}}e^{-\frac{(w-\tau)^2}{\frac{2\tau^2}{n}\frac{\eta^2}{3}}}$$

$$=\frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}}e^{-\frac{3n(w-\tau)^2}{2\tau^2\eta^2}}$$

Then,

$$f_w(w)=\frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}}e^{-\frac{3n(w-\tau)^2}{2\tau^2\eta^2}} \tag{76}$$

The CDF of the average RV, $F_w(w)$ is defined by the following normal integration,

$$F_w(w)=\int_w f_w(w)dw=\int_{-\infty}^{w}\frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}}e^{-\frac{3n(w-\tau)^2}{2\tau^2\eta^2}}dw \tag{77}$$

Now, the output traffic pdf , $f_y(y)$, can be found using equations (29), (34), (76), (37), and (77).

$$\begin{aligned} f_y(y) &= f_w(y).F_x(y)+F_w(y)f_x(y) \\ &=\left[\frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}}e^{-\frac{3n(y-\tau)^2}{2\tau^2\eta^2}}\right]\frac{y-\tau(1-\eta)}{2\eta\tau}+ \\ &\quad +\left[\int_{-\infty}^{y}\frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}}e^{-\frac{3n(w-\tau)^2}{2\tau^2\eta^2}}dw\right]\frac{1}{2\eta\tau} \end{aligned} \tag{78}$$

Since we are going to face unresolved integration to determine the mean and the variance, we are going to use a valid and reasonable assumption to find $f_y(y)$ at the neighborhood of the mean. The Taylor expansion of normal distribution described in equation (10), can be used here since we are considering large value of n. Let us first used the following transformation of variables to match the above equation with equation number (10)

$$\frac{3n(y-\tau)^2}{2\tau^2\eta^2}=\frac{z^2}{2}$$

Then,

$$\frac{3n(y-\tau)^2}{2\tau^2\eta^2}=\frac{z^2}{2}$$

$$y=z\sqrt{\frac{\tau^2\eta^2}{3n}}+\tau$$

And,

$$dy=\left(\sqrt{\frac{\tau^2\eta^2}{3n}}\right)dz$$

And,

$$z=\sqrt{\frac{3n}{\tau^2\eta^2}}(y-\tau)$$

Since $y$ ranges from $\tau(1\text{-}\eta)$ to $\tau(1+\eta)$, $z$ ranges from,

$$z=-\eta\sqrt{\frac{3n}{\eta^2}}$$

to

$$z=\eta\sqrt{\frac{3n}{\eta^2}}$$

It is worth noticing that z is not a RV but it is just used as a transformation of variables to simplify the analysis. Now, $f_w(y)$ and $F_w(y)$ are expressed as follows,

$$f_w(y) = f_z(z) = \frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}} e^{-\frac{z^2}{2}}$$

$$F_w(y) = F_z(z) = \int_{-\infty}^{z} \frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}} e^{-\frac{w^2}{2}} \left( \sqrt{\frac{\tau^2\eta^2}{3n}} \right) dz$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dz$$

To find the above integration, let us use Taylor expansion of equation (10) to the second degree,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \approx \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{x^2}{2} \right)$$

Then,

$$f_w(y) = f_z(z) = \frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}} \left( 1 - \frac{z^2}{2} \right)$$

$$F_w(y) = F_z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\eta\sqrt{\frac{3n}{\tau\eta^2}}}^{z} \left( 1 - \frac{w^2}{2} \right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \left( w - \frac{w^3}{6} \right) \Bigg|_{-\eta\sqrt{\frac{3n}{\eta^2}}}^{z}$$

$$=\frac{1}{\sqrt{2\pi}}\left(z-\frac{z^3}{6}\right)-\frac{1}{\sqrt{2\pi}}\left[-\eta\sqrt{\frac{3n}{\eta^2}}-\frac{1}{6}\left(-\eta\sqrt{\frac{3n}{\eta^2}}\right)^3\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left(z-\frac{z^3}{6}\right)+\eta\sqrt{\frac{3n}{2\pi\eta^2}}\left[1-\frac{1}{6}\left(\frac{3n\eta^2}{\eta^2}\right)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left(z-\frac{z^3}{6}\right)+\eta\sqrt{\frac{3n}{2\pi\eta^2}}\left[1-\frac{n\eta^2}{2\eta^2}\right]$$

Now, $f_y(y)$ can be found from equation (78) as follows,

$$f_y(y)=f_z(z)=\left[\frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}}\left(1-\frac{z^2}{2}\right)\right]\left\{\frac{1}{2\eta\tau}\left[z\sqrt{\frac{\tau^2\eta^2}{3n}}+\tau\right]-\frac{\tau(1-\eta)}{2\eta\tau}\right\}+$$

$$+\left\{\frac{1}{\sqrt{2\pi}}\left(z-\frac{z^3}{6}\right)+\eta\sqrt{\frac{3n}{2\pi\eta^2}}\left[1-\frac{n\eta^2}{2\eta^2}\right]\right\}\frac{1}{2\eta\tau}$$

$$=\left[\frac{\sqrt{3n}}{\sqrt{2\pi\tau^2\eta^2}}-\frac{z^2\sqrt{3n}}{2\sqrt{2\pi\tau^2\eta^2}}\right]\left\{\frac{z}{2\eta}\sqrt{\frac{\eta^2}{3n}}-\frac{\tau-\tau(1-\eta)}{2\eta\tau}\right\}+$$

$$+\frac{z}{2\eta\tau\sqrt{2\pi}}-\frac{z^3}{12\eta\tau\sqrt{2\pi}}+\frac{\sqrt{3n}}{2\sqrt{2\pi\tau^2\eta^2}}\left[1-\frac{n\eta^2}{2\eta^2}\right]$$

$$=\left[\frac{z}{2\eta\tau\sqrt{2\pi}}-\frac{z^3}{4\eta\tau\sqrt{2\pi}}-\frac{\sqrt{3n}}{2\sqrt{2\pi\tau^2\eta^2}}+\frac{z^2\sqrt{3n}}{4\sqrt{2\pi\tau^2\eta^2}}\right]+$$

$$+\frac{z}{2\eta\tau\sqrt{2\pi}}-\frac{z^3}{12\eta\tau\sqrt{2\pi}}+\frac{\sqrt{3n}}{2\sqrt{2\pi\tau^2\eta^2}}\left[1-\frac{n\eta^2}{2\eta^2}\right]$$

$$=-\frac{z^3}{3\eta\tau\sqrt{2\pi}}+\frac{z^2\sqrt{3n}}{4\sqrt{2\pi\tau^2\eta^2}}+\frac{z}{\eta\tau\sqrt{2\pi}}-\frac{n\eta^2\sqrt{3n}}{4\eta^2\sqrt{2\pi\tau^2\eta^2}}$$

Then,

$$f_y(y)=f_z(z)=\frac{z^2\sqrt{3n}}{4\sqrt{2\pi\tau^2\eta^2}}+\frac{z}{\eta\tau\sqrt{2\pi}}-\frac{z^3}{3\eta\tau\sqrt{2\pi}}-\frac{n\eta^2\sqrt{3n}}{4\eta^2\sqrt{2\pi\tau^2\eta^2}} \tag{79}$$

The above equation provides the output traffic pdf in the neighborhood of the mean. That equation can't be used to find out either the mean or the variance of the output traffic because they require the pdf definition in the whole range of y. To be able to find the mean and variance of $f_y(y)$, $\mu_y$ $\sigma_y$, we have to go back to equation (78). Since the integration equation (78) is not a resolvable one, let us used numerical analysis to fine the mean and variance described in the following equations receptivity,

$$\overline{y^2} = \int_y y f_y(y) dyz =$$

$$\sigma_y^2 = \overline{y^2} - \mu_y^2 = \int y^2 f_y(y) dy - \mu_y^2$$

We used MATLAB as a tool to find the mean and variance as a function of both the n and $\eta$. Figure 36 shows the output traffic mean as a function of $\eta$. The mean is greater than one, which means that thee output traffic delayed. That delay comes from the nature of the average over n algorithm. By comparing this results with the one for the maximum delay algorithm shown in equation (23), we conclude that the average delay algorithm induce less delay and eventually, needs less buffer. The variance as a function of $\eta$ is shown in Figure 37. The variance is the jitter improvement indication. As shown in the figure, the variance ratio is less than one, the less the variance the less the jitter.

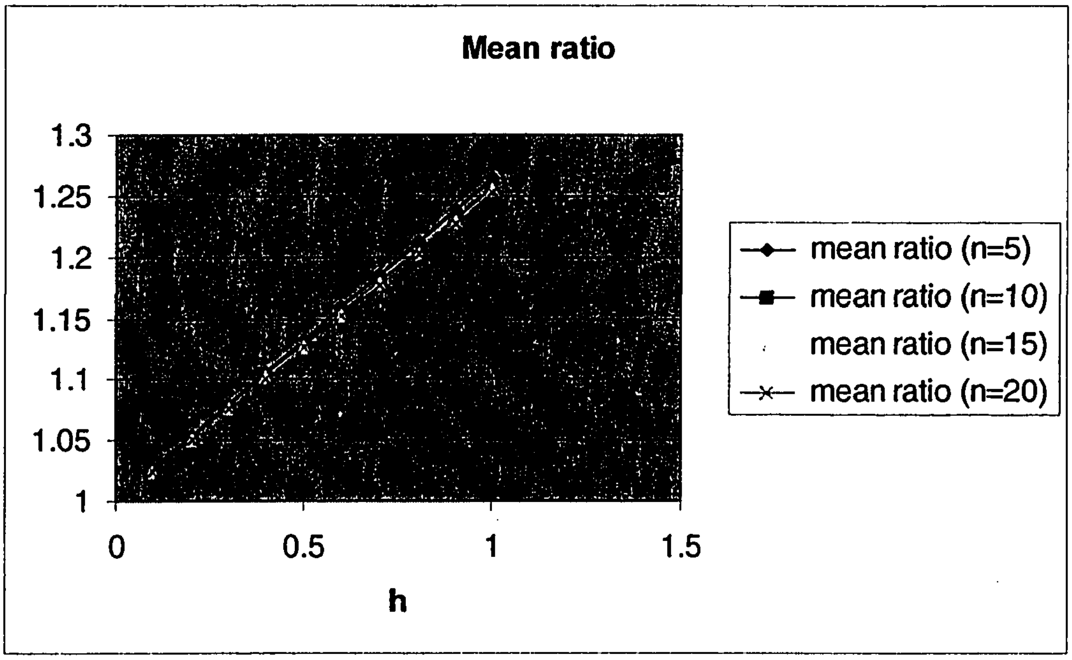



**Figure 36 Mean as a function of η (h is η)**

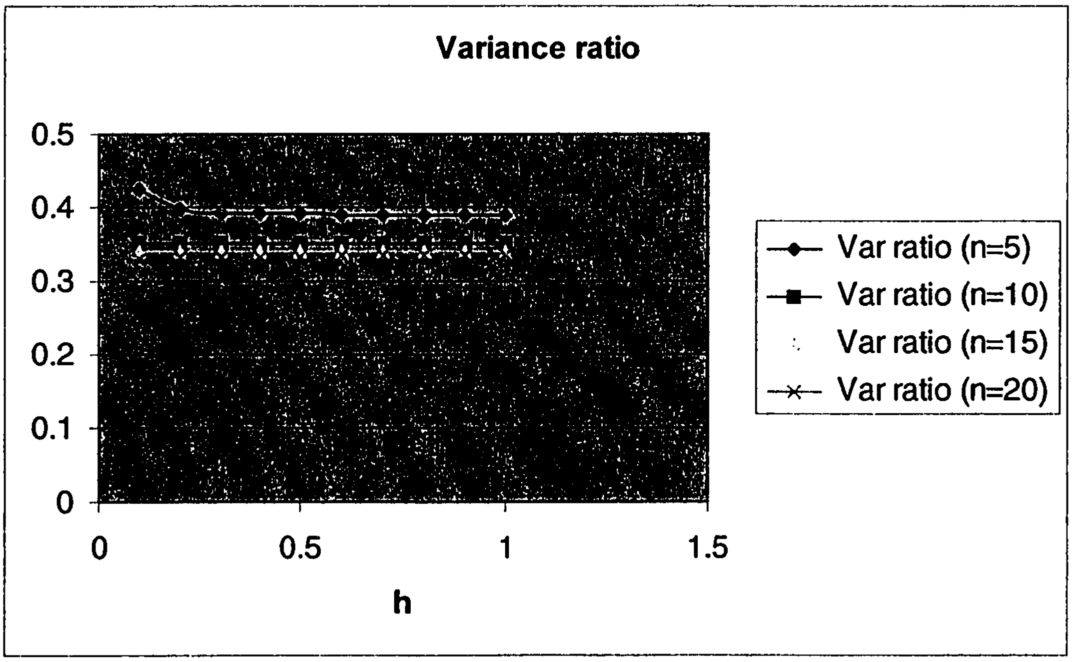



**Figure 37 Variance as a function of η (h is η)**

### 4.4.1.4 Mean and variance

Since we found the mean and variance in different cases, let us show the effect of n on those values. From mean and variance diagrams in sections 4.4.1.1 and 4.4.1.2, we found out the variance improvement in different cases of n. We can also find the mean increase, which represents the incremental increase in the e-to-end delay. Figure 39 shows the mean increase as a function of n for different values of $\eta$. The figure shows that the mean increases as n and/or $\eta$ increase but still not too much. It my be effective in some of the applications but it is reasonable in most of the cases.

Figure 38 show the variance improvement, which represents the jitter improvement. The figure shows that the variance significantly decreases as n increases. It also shows that the variance decrease as $\eta$ decrease, which is a positive point since $\eta$ represents the amount of jitter in the input traffic. The jitter is usually much less that the mean (i.e. small $\eta$).

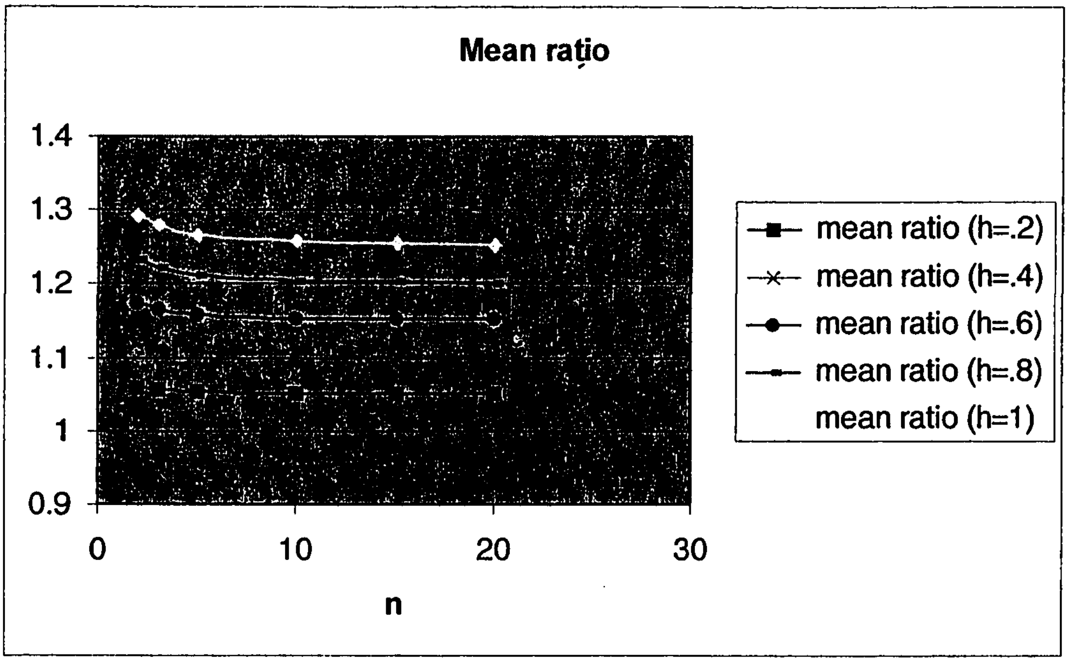



**Figure 38 Mean as a function of n (h is η)**

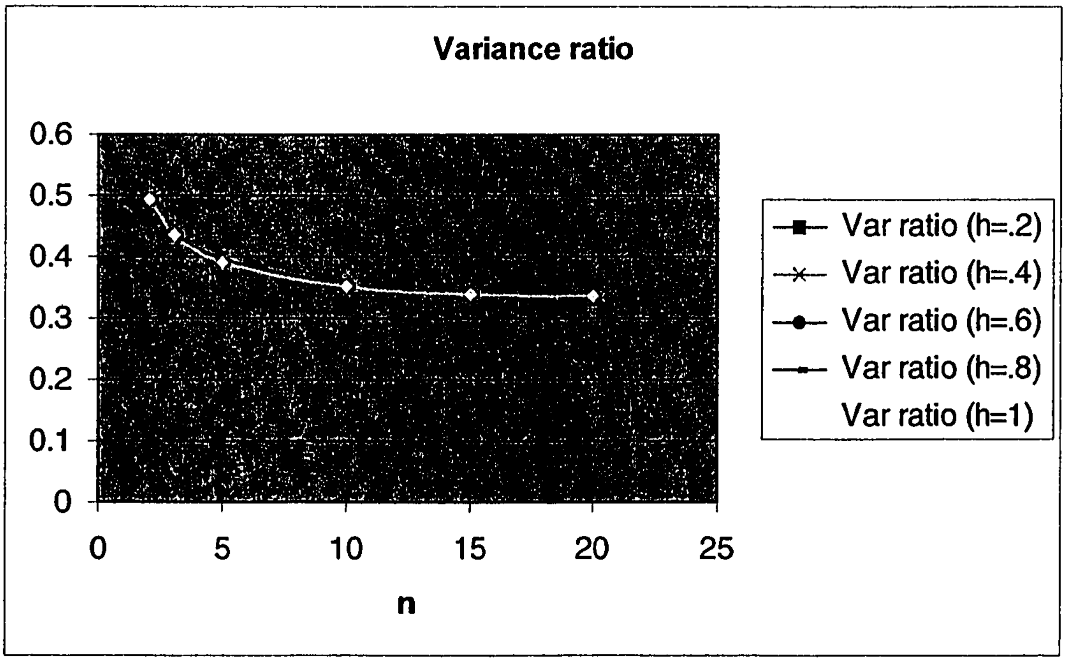



**Figure 39 Variance as a function of n (h is η)**

# 5 Chapter V: Conclusion

In this thesis, we have introduced a new algorithm that reduces the SCTP packet jitter in case of periodic and non-periodic traffic types. It requires only one little modification at the sender and another one at the destination. We have applied the algorithm on laboratory network captured traffic, which acts like a WAN. MATLAB is used to simulate the algorithms in case of periodic and non-periodic traffic. The results showed good improvement in the jitter performance. Thus, this algorithm can be implemented at any SCTP host to solve the end-to-end jitter problem.

In case of periodic traffic, different levels of the algorithm are introduced to cover different traffic and network conditions. The maximum delay algorithm provides better performance but it requires larger buffer to absorb sudden peaks of jitter. The average delay algorithm equalizes the jitter based on the neighborhood packets. In either case, this algorithm significantly improved the jitter.

The non-periodic algorithm uses Fourier transform the LPF principle. The IT is sent in the timestamp at only Nyquist rate instead of each packet. The receiver is able to recover the IT of each individual packet independently. So that if there is a lost or high jittered packet, it will not affect the recovery of any other packet's IT.

Finally, we were able to come out with analytical model that describes the performance of the periodic traffic algorithm. We used the transformation of random variable principle and some other probability background. The analysis showed significant jitter performance improvement.

# Appendix A:

## Proof of Equation (45)

This appendix provides the proof of the mean value of the output traffic pdf, y, in case of n = 2.

$$\mu_y = \int y f_y(y) dy$$

$$= \int_{J_l}^{\tau} y\left[\frac{3}{4(J_u-\tau)^3}(y-2\tau+J_u)^2\right]dy +$$

$$\int_{\tau}^{J_u} y\left[\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+y(2\tau+J_u)-\frac{3}{2}y^2\right]+\frac{1}{4(J_u-\tau)}\right]dy$$

$$= \int_{J_l}^{\tau}\left[\frac{3}{4(J_u-\tau)^3}\left[y^3-2y^2(2\tau-J_u)+y(2\tau-J_u)^2\right]\right]dy +$$

$$\int_{\tau}^{J_u}\left[\frac{1}{2(J_u-\tau)^3}\left[y\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)+y^2(2\tau+J_u)-\frac{3}{2}y^3\right]+\frac{y}{4(J_u-\tau)}\right]dy$$

$$= \left[\frac{3}{4(J_u-\tau)^3}\left[\frac{y^4}{4}-\frac{2}{3}y^3(2\tau-J_u)+\frac{y^2}{2}(2\tau-J_u)^2\right]\right]_{J_l}^{\tau} +$$

$$\left[\frac{1}{2(J_u-\tau)^3}\left[\frac{y^2}{2}\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)+\frac{y^3}{3}(2\tau+J_u)-\frac{3}{8}y^4\right]+\frac{y^2}{8(J_u-\tau)}\right]_{\tau}^{J_u}$$

As it was discussed above, let us define η as follows,

$$\left.\begin{aligned}
&\eta = \frac{J_u-\tau}{\tau} = \frac{\tau-J_l}{\tau}\\
&\textit{Then,}\\
&J_u = \tau(1+\eta) \quad and,\\
&J_l = \tau(1-\eta) \quad and,\\
&J_u-\tau = \tau\eta \quad and,\\
&2\tau+J_u = \tau(3+\eta) \quad and,\\
&2\tau-J_u = \tau(1-\eta)
\end{aligned}\right\} \qquad (80)$$

Then,

$$\mu_y = \frac{3}{48(J_u-\tau)^3}\left[3\tau^4 - 8\tau^3(2\tau - J_u) + 6\tau^2(2\tau - J_u)^2\right] -$$

$$-\frac{3}{48(J_u-\tau)^3}\left[3J_l^4 - 8J_l^3(2\tau - J_u) + 6J_l^2(2\tau - J_u)^2\right] +$$

$$+\frac{1}{48(J_u-\tau)^3}\left[12J_u^2\left(J_u^2 - 3J_u\tau + \frac{1}{2}\tau^2\right) + 8J_u^3(2\tau + J_u) - 9J_u^4\right] -$$

$$-\frac{1}{48(J_u-\tau)^3}\left[12\tau^2\left(J_u^2 - 3J_u\tau + \frac{1}{2}\tau^2\right) + 8\tau^3(2\tau + J_u) - 9\tau^4\right] + \frac{J_u+\tau}{8}$$

$$= \frac{3}{48\tau^3\eta^3}\left[3\tau^4 - 8\tau^4(1-\eta) + 6\tau^4(1-\eta)^2\right] -$$

$$-\frac{3}{48\tau^3\eta^3}\left[3\tau^4(1-\eta)^4 - 8\tau^4(1-\eta)^4 + 6\tau^4(1-\eta)^4\right] +$$

$$+\frac{1}{48\tau^3\eta^3}\left[\begin{array}{c}12\tau^2(1+\eta)^2\left(\tau^2(1+\eta)^2 - 3\tau(1+\eta)\tau + \frac{1}{2}\tau^2\right) \\ + 8\tau^4(1+\eta)^3(3+\eta) - 9\tau^4(1+\eta)^4\end{array}\right] -$$

$$-\frac{1}{48\tau^3\eta^3}\left[12\tau^2\left(\tau^2(1+\eta)^2 - 3\tau(1+\eta)\tau + \frac{1}{2}\tau^2\right) + 8\tau^4(3+\eta) - 9\tau^4\right] + \frac{\tau(2+\eta)}{8}$$

$$= \frac{3\tau}{48\eta^3}\left[3 - 8(1-\eta) + 6(1-\eta)^2\right] -$$

$$-\frac{3\tau}{48\eta^3}\left[3(1-\eta)^4 - 8(1-\eta)^4 + 6(1-\eta)^4\right] +$$

$$+\frac{\tau}{48\eta^3}\left[12(1+\eta)^2\left((1+\eta)^2 - 3(1+\eta) + \frac{1}{2}\right) + 8(1+\eta)^3(3+\eta) - 9(1+\eta)^4\right] -$$

$$-\frac{\tau}{48\eta^3}\left[12\left((1+\eta)^2 - 3(1+\eta) + \frac{1}{2}\right) + 8(3+\eta) - 9\right] + \frac{\tau(2+\eta)}{8}$$

$$= \frac{3\tau}{48\eta^3}\left[3 - 8(1-\eta) + 6(1-\eta)^2\right] - \frac{3\tau(1-\eta)^4}{48\eta^3} +$$

$$+\frac{\tau}{48\eta^3}\left[12(1+\eta)^4 - 36(1+\eta)^3 + 6(1+\eta)^2 + 16(1+\eta)^3 + 8(1+\eta)^4 - 9(1+\eta)^4\right] -$$

$$-\frac{\tau}{48\eta^3}\left[12(1+\eta)^2 - 36(1+\eta) + 6 + 16 + 8(1+\eta) - 9\right] + \frac{\tau(2+\eta)}{8}$$

$$=\frac{3\tau}{48\eta^3}\left[3-8(1-\eta)+6(1-\eta)^2\right]-\frac{3\tau(1-\eta)^4}{48\eta^3}+\frac{\tau}{48\eta^3}\left[11(1+\eta)^4-20(1+\eta)^3+6(1+\eta)^2\right]-$$

$$-\frac{\tau}{48\eta^3}\left[12(1+\eta)^2-28(1+\eta)+13\right]+\frac{\tau(2+\eta)}{8}$$

$$=\frac{\tau}{48\eta^3}\left[\begin{array}{c}9-24(1-\eta)+18(1-\eta)^2-3(1-\eta)^4+11(1+\eta)^4-20(1+\eta)^3+\\+6(1+\eta)^2-12(1+\eta)^2+28(1+\eta)-13\end{array}\right]+\frac{\tau(2+\eta)}{8}$$

$$=\frac{\tau}{48\eta^3}\left[\begin{array}{c}11(1+\eta)^4-3(1-\eta)^4-20(1+\eta)^3-6(1+\eta)^2+18(1-\eta)^2+\\+28(1+\eta)-24(1-\eta)-4\end{array}\right]+\frac{\tau(2+\eta)}{8}$$

$$=\frac{\tau}{48\eta^3}\left[\begin{array}{c}11\left(1+4\eta+6\eta^2+4\eta^3+\eta^4\right)-3\left(1-4\eta+6\eta^2-4\eta^3+\eta^4\right)-\\-20\left(1+3\eta+3\eta^2+\eta^3\right)-6\left(1+2\eta+\eta^2\right)+\\+18\left(1-2\eta+\eta^2\right)+28(1+\eta)-24(1-\eta)-4\end{array}\right]+\frac{\tau(2+\eta)}{8}$$

$$=\frac{\tau}{48\eta^3}\left[36\eta^3+8\eta^4\right]+\frac{\tau(2+\eta)}{8}$$

$$=\tau\left[\frac{3}{4}+\frac{1}{6}\eta\right]+\frac{\tau(2+\eta)}{8}$$

$$=\tau\left(1+\frac{7\eta}{24}\right)$$

Then,

$$\mu_y=\tau\left(1+\frac{7\eta}{24}\right)$$

# Appendix B:

## Proof of Equation (47)

This appendix provides the proof of the mean value of $y^2$ in case of n = 2. The result is used to find the variance of the output traffic pdf.

$$\overline{y^2} = \int y^2 f_y(y)dy = \int_{J_l}^{\tau} y^2\left[\frac{3}{4(J_u-\tau)^3}(y-2\tau+J_u)^2\right]dy +$$

$$+\int_{\tau}^{J_u} y^2\left[\frac{1}{2(J_u-\tau)^3}\left[J_u^2-3J_u\tau+\frac{1}{2}\tau^2+y(2\tau+J_u)-\frac{3}{2}y^2\right]+\frac{1}{4(J_u-\tau)}\right]dy$$

$$=\int_{J_l}^{\tau}\left[\frac{3}{4(J_u-\tau)^3}\left[y^4-2y^3(2\tau-J_u)+y^2(2\tau-J_u)^2\right]\right]dy +$$

$$+\int_{\tau}^{J_u}\left[\frac{1}{2(J_u-\tau)^3}\left[y^2\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)+y^3(2\tau+J_u)-\frac{3}{2}y^4\right]+\frac{y^2}{4(J_u-\tau)}\right]dy$$

$$=\left[\frac{3}{4(J_u-\tau)^3}\left[\frac{y^5}{5}-\frac{2}{4}y^4(2\tau-J_u)+\frac{y^3}{3}(2\tau-J_u)^2\right]\right]_{J_l}^{\tau} +$$

$$+\left[\frac{1}{2(J_u-\tau)^3}\left[\frac{y^3}{3}\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)+\frac{y^4}{4}(2\tau+J_u)-\frac{3}{10}y^5\right]+\frac{y^3}{12(J_u-\tau)}\right]_{\tau}^{J_u}$$

$$=\frac{3}{4(J_u-\tau)^3}\left[\frac{\tau^5}{5}-\frac{1}{2}\tau^4(2\tau-J_u)+\frac{\tau^3}{3}(2\tau-J_u)^2\right]-$$

$$-\frac{3}{4(J_u-\tau)^3}\left[\frac{J_l^5}{5}-\frac{1}{2}J_l^4(2\tau-J_u)+\frac{J_l^3}{3}(2\tau-J_u)^2\right]+$$

$$+\left[\frac{1}{2(J_u-\tau)^3}\left[\frac{J_u^3}{3}\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)+\frac{J_u^4}{4}(2\tau+J_u)-\frac{3}{10}J_u^5\right]+\frac{J_u^3}{12(J_u-\tau)}\right]-$$

$$-\left[\frac{1}{2(J_u-\tau)^3}\left[\frac{\tau^3}{3}\left(J_u^2-3J_u\tau+\frac{1}{2}\tau^2\right)+\frac{\tau^4}{4}(2\tau+J_u)-\frac{3}{10}\tau^5\right]+\frac{\tau^3}{12(J_u-\tau)}\right]$$

$$
\begin{aligned}
&= \frac{3}{120(J_u-\tau)^3}\left[6\tau^5 - 15\tau^4(2\tau - J_u) + 10\tau^3(2\tau - J_u)^2\right] - \\
&\quad - \frac{3}{120(J_u-\tau)^3}\left[6J_l^5 - 15J_l^4(2\tau - J_u) + 10J_l^3(2\tau - J_u)^2\right] + \\
&\quad + \frac{1}{120(J_u-\tau)^3}\left[20J_u^3\left(J_u^2 - 3J_u\tau + \frac{1}{2}\tau^2\right) + 15J_u^4(2\tau + J_u) - 18J_u^5\right] - \\
&\quad - \frac{1}{120(J_u-\tau)^3}\left[20\tau^3\left(J_u^2 - 3J_u\tau + \frac{1}{2}\tau^2\right) + 15\tau^4(2\tau + J_u) - 18\tau^5\right] \\
&\quad + \frac{1}{12}\left(J_u^2 + J_u\tau + \tau^2\right)
\end{aligned}
$$

Again, let us define η as follows,

$$\eta = \frac{J_u - \tau}{\tau} = \frac{\tau - J_l}{\tau}$$

*Then,*

$$
\begin{aligned}
&J_u = \tau(1+\eta) \quad \textit{and}, \\
&J_l = \tau(1-\eta) \quad \textit{and}, \\
&J_u - \tau = \tau\eta \quad \textit{and}, \\
&2\tau + J_u = \tau(3+\eta) \quad \textit{and}, \\
&2\tau - J_u = \tau(1-\eta)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\overline{y^2} &= \frac{3}{120\tau^3\eta^3}\left[6\tau^5 - 15\tau^4\tau(1-\eta) + 10\tau^3\tau^2(1-\eta)^2\right] - \\
&\quad - \frac{3}{120\tau^3\eta^3}\left[6\tau^5(1-\eta)^5 - 15\tau^4(1-\eta)^4\tau(1-\eta) + 10\tau^3(1-\eta)^3\tau^2(1-\eta)^2\right] + \\
&\quad + \frac{1}{120\tau^3\eta^3}\left[\begin{array}{l} 20\tau^3(1+\eta)^3\left(\tau^2(1+\eta)^2 - 3\tau(1+\eta)\tau + \frac{1}{2}\tau^2\right) + \\ \qquad + 15\tau^4(1+\eta)^4(2\tau + \tau(1+\eta)) - 18\tau^5(1+\eta)^5 \end{array}\right] - \\
&\quad - \frac{1}{120\tau^3\eta^3}\left[20\tau^3\left(\tau^2(1+\eta)^2 - 3\tau(1+\eta)\tau + \frac{1}{2}\tau^2\right) + 15\tau^4(2\tau + \tau(1+\eta)) - 18\tau^5\right] \\
&\quad + \frac{1}{12}\left(\tau^2(1+\eta)^2 + \tau(1+\eta)\tau + \tau^2\right)
\end{aligned}
$$

$$=\frac{3\tau^2}{120\eta^3}\left[6-15(1-\eta)+10(1-\eta)^2\right]-$$

$$-\frac{3\tau^2}{120\eta^3}\left[6(1-\eta)^5-15(1-\eta)^4(1-\eta)+10(1-\eta)^3(1-\eta)^2\right]+$$

$$+\frac{\tau^2}{120\eta^3}\left[20(1+\eta)^3\left((1+\eta)^2-3(1+\eta)+\frac{1}{2}\right)+15(1+\eta)^4(2+(1+\eta))-18(1+\eta)^5\right]-$$

$$-\frac{\tau^2}{120\eta^3}\left[20\left((1+\eta)^2-3(1+\eta)+\frac{1}{2}\right)+15(2+(1+\eta))-18\right]$$

$$+\frac{\tau^2}{12}\left((1+\eta)^2+(1+\eta)+1\right)$$

$$=\frac{3\tau^2}{120\eta^3}\left[6-15(1-\eta)+10(1-\eta)^2\right]-\frac{3\tau^2(1-\eta)^5}{120\eta^3}+$$

$$+\frac{\tau^2}{120\eta^3}\left[20(1+\eta)^5-60(1+\eta)^4+10(1+\eta)^3+30(1+\eta)^4+15(1+\eta)^5-18(1+\eta)^5\right]-$$

$$-\frac{\tau^2}{120\eta^3}\left[20(1+\eta)^2-60(1+\eta)+10+30+15(1+\eta)-18\right]$$

$$+\frac{\tau^2}{12}\left((1+\eta)^2+(1+\eta)+1\right)$$

$$=\frac{3\tau^2}{120\eta^3}\left[6-15(1-\eta)+10(1-\eta)^2-(1-\eta)^5\right]+\frac{\tau^2}{120\eta^3}\left[17(1+\eta)^5-30(1+\eta)^4+10(1+\eta)^3\right]-$$

$$-\frac{\tau^2}{120\eta^3}\left[20(1+\eta)^2-45(1+\eta)+22\right]+\frac{\tau^2}{12}\left((1+\eta)^2+(1+\eta)+1\right)$$

$$=\frac{\tau^2}{120\eta^3}\left[\begin{matrix}-4+45(1+\eta)-45(1-\eta)-20(1+\eta)^2+30(1-\eta)^2+\\ +10(1+\eta)^3-30(1+\eta)^4+17(1+\eta)^5-3(1-\eta)^5\end{matrix}\right]+\frac{\tau^2}{12}\left((1+\eta)^2+(1+\eta)+1\right)$$

$$=\frac{\tau^2}{120\eta^3}\left[\begin{matrix}-4+45(1+\eta)-45(1-\eta)-20\left(1+2\eta+\eta^2\right)+\\ +30\left(1-2\eta+\eta^2\right)+10\left(1+3\eta+3\eta^2+\eta^3\right)-\\ -30\left(1+4\eta+6\eta^2+4\eta^3+\eta^4\right)+\\ +17\left(1+5\eta+10\eta^2+10\eta^3+5\eta^4+\eta^5\right)-\\ -3\left(1-5\eta+10\eta^2-10\eta^3+5\eta^4-\eta^5\right)\end{matrix}\right]+\frac{\tau^2}{12}\left(\left(1+2\eta+\eta^2\right)+(1+\eta)+1\right)$$

$$= \frac{\tau^2}{120\eta^3}\left[90\eta^3 + 40\eta^4 + 20\eta^5\right] + \frac{\tau^2}{12}\left(3 + 3\eta + \eta^2\right)$$

$$= \frac{\tau^2}{12}\left[9 + 4\eta + 2\eta^2\right] + \frac{\tau^2}{12}\left(3 + 3\eta + \eta^2\right)$$

$$= \frac{\tau^2}{12}\left[12 + 7\eta + 3\eta^2\right]$$

Finally,

$$\overline{y^2} = \frac{\tau^2}{12}\left[12 + 7\eta + 3\eta^2\right]$$

# Appendix C:

# Proof of Equation (71)

This appendix provides the proof of the mean value of the output traffic pdf, y, in case of n = 3.

$$\mu_y = \int_y y f_y(y)dy = \int_z \tau(1+z\eta) f_z(z)\eta\tau dz$$

$$= \int_{-1}^{-\frac{1}{3}} \eta\tau^2(1+z\eta)\frac{9(z+1)^3}{8\tau\eta}dz +$$

$$+ \int_{-\frac{1}{3}}^{\frac{1}{3}} \eta\tau^2(1+z\eta)\left(\frac{-108z^3-81z^2+54z+39}{48\tau\eta}\right)dz +$$

$$+ \int_{\frac{1}{3}}^{1} \eta\tau^2(1+z\eta)\left(\frac{54z^3-81z^2+51}{48\tau\eta}\right)dz$$

$$= \frac{9\tau}{8}\int_{-1}^{-\frac{1}{3}} \left[(z+1)^3 + \eta\left(z+3z^2+3z^3+z^4\right)\right]dz +$$

$$+ \frac{\tau}{48}\int_{-\frac{1}{3}}^{\frac{1}{3}} \left[-108z^3-81z^2+54z+39+\eta\left(-108z^4-81z^3+54z^2+39z\right)\right]dz +$$

$$+ \frac{\tau}{48}\int_{\frac{1}{3}}^{1} \left[54z^3-81z^2+51+\eta\left(54z^4-81z^3+51z\right)\right]dz$$

$$= \frac{9\tau}{8}\left[\frac{1}{4}(z+1)^4 + \eta\left(\frac{1}{2}z^2+z^3+\frac{3}{4}z^4+\frac{1}{5}z^5\right)\right]\Bigg|_{-1}^{-\frac{1}{3}} +$$

$$+ \frac{\tau}{48}\left[-27z^4-27z^3+27z^2+39z+\eta\left(-\frac{108}{5}z^5-\frac{81}{4}z^4+18z^3+\frac{39}{2}z^2\right)\right]\Bigg|_{-\frac{1}{3}}^{\frac{1}{3}} +$$

$$+ \frac{\tau}{48}\left[\frac{27}{2}z^4-27z^3+51z+\eta\left(\frac{54}{5}z^5-\frac{81}{4}z^4+\frac{51}{2}z^2\right)\right]\Bigg|_{\frac{1}{3}}^{1}$$

$$=\frac{9\tau}{8}\left\{\frac{1}{4}\left[\left(\frac{2}{3}\right)^4-0\right]+\eta\left[\frac{1}{2}\left(\frac{1}{9}-1\right)+\left(\frac{-1}{27}+1\right)+\frac{3}{4}\left(\frac{1}{81}-1\right)+\frac{1}{5}\left(\frac{-1}{243}+1\right)\right]\right\}+$$

$$+\frac{\tau}{48}\left[\begin{array}{l}-27\left(\frac{1}{81}-\frac{1}{81}\right)-27\left(\frac{1}{27}+\frac{1}{27}\right)+27\left(\frac{1}{9}-\frac{1}{9}\right)+39\left(\frac{1}{3}+\frac{1}{3}\right)+\\ \quad+\eta\left(-\frac{108}{5}\left(\frac{1}{243}+\frac{1}{243}\right)-\frac{81}{4}\left(\frac{1}{81}-\frac{1}{81}\right)+18\left(\frac{1}{27}+\frac{1}{27}\right)+\frac{39}{2}\left(\frac{1}{9}-\frac{1}{9}\right)\right)\end{array}\right]+$$

$$+\frac{\tau}{48}\left\{\begin{array}{l}\frac{27}{2}\left(1-\frac{1}{81}\right)-27\left(1-\frac{1}{27}\right)+51\left(1-\frac{1}{3}\right)+\\ \quad+\eta\left[\frac{54}{5}\left(1-\frac{1}{243}\right)-\frac{81}{4}\left(1-\frac{1}{81}\right)+\frac{51}{2}\left(1-\frac{1}{9}\right)\right]\end{array}\right\}$$

$$=\frac{9\tau}{8}\left\{\frac{1}{4}\left(\frac{16}{81}\right)+\eta\left[\frac{1}{2}\left(\frac{-8}{9}\right)+\left(\frac{26}{27}\right)+\frac{3}{4}\left(\frac{-80}{81}\right)+\frac{1}{5}\left(\frac{242}{243}\right)\right]\right\}+$$

$$+\frac{\tau}{48}\left[-27\left(\frac{2}{27}\right)+39\left(\frac{2}{3}\right)+\eta\left(-\frac{108}{5}\left(\frac{2}{243}\right)+18\left(\frac{2}{27}\right)\right)\right]+$$

$$+\frac{\tau}{48}\left\{\frac{27}{2}\left(\frac{80}{81}\right)-27\left(\frac{26}{27}\right)+51\left(\frac{2}{3}\right)+\eta\left[\frac{54}{5}\left(\frac{242}{243}\right)-\frac{81}{4}\left(\frac{80}{81}\right)+\frac{51}{2}\left(\frac{8}{9}\right)\right]\right\}$$

$$=\frac{9\tau}{8}\left\{\frac{4}{81}+\eta\left[\frac{-4}{9}+\frac{26}{27}+\frac{-20}{27}+\frac{1}{5}\left(\frac{242}{243}\right)\right]\right\}+$$

$$+\frac{\tau}{48}\left[-2+26+\eta\left(-\frac{8}{45}+\frac{4}{3}\right)\right]+$$

$$+\frac{\tau}{48}\left\{\frac{40}{3}-26+34+\eta\left[\frac{2}{5}\left(\frac{242}{9}\right)-20+17\left(\frac{4}{3}\right)\right]\right\}$$

$$=\tau\left(1+\frac{5\eta}{18}\right)$$

Then,

$$\mu_y=\tau\left(1+\frac{5\eta}{18}\right)$$

# Appendix D:

# Proof of Equation (73)

This appendix provides the proof of the mean value of $y^2$ in case of n = 3. The result is used to find the variance of the output traffic pdf.

$$\overline{y^2} = \int_y y f_y(y) dyz = \int_z \tau^2 (1+z\eta)^2 f_z(z) \eta\tau dz$$

$$= \int_{-1}^{-\frac{1}{3}} \eta\tau^3 (1+z\eta)^2 \frac{9(z+1)^3}{8\tau\eta} dz +$$

$$+ \int_{-\frac{1}{3}}^{\frac{1}{3}} \eta\tau^3 (1+z\eta)^2 \left( \frac{-108z^3 - 81z^2 + 54z + 39}{48\tau\eta} \right) dz +$$

$$+ \int_{\frac{1}{3}}^{1} \eta\tau^3 (1+z\eta)^2 \left( \frac{54z^3 - 81z^2 + 51}{48\tau\eta} \right) dz$$

$$= \frac{9\tau^2}{8} \int_{-1}^{-\frac{1}{3}} \left(1 + 2z\eta + z^2\eta^2\right)(z+1)^3 dz +$$

$$+ \frac{\tau^2}{48} \int_{-\frac{1}{3}}^{\frac{1}{3}} \left(1 + 2z\eta + z^2\eta^2\right)\left(-108z^3 - 81z^2 + 54z + 39\right) dz +$$

$$+ \frac{\tau^2}{48} \int_{\frac{1}{3}}^{1} \left(1 + 2z\eta + z^2\eta^2\right)\left(54z^3 - 81z^2 + 51\right) dz$$

$$= \frac{9\tau^2}{8} \int_{-1}^{-\frac{1}{3}} \left[ (z+1)^3 + \left(2z\eta + 6\eta z^2 + 6\eta z^3 + 2\eta z^4\right) + \left(\eta^2 z^2 + 3\eta^2 z^3 + 3\eta^2 z^4 + \eta^2 z^5\right) \right] dz +$$

$$+ \frac{\tau^2}{48} \int_{-\frac{1}{3}}^{\frac{1}{3}} \left[ \begin{array}{l} \left(-108z^3 - 81z^2 + 54z + 39\right) + \left(-216\eta z^4 - 162\eta z^3 + 108\eta z^2 + 78\eta z\right) + \\ + \left(-108\eta^2 z^5 - 81\eta^2 z^4 + 54\eta^2 z^3 + 39\eta^2 z^2\right) \end{array} \right] dz +$$

$$+ \frac{\tau^2}{48} \int_{\frac{1}{3}}^{1} \left[ \begin{array}{l} \left(54z^3 - 81z^2 + 51\right) + \left(108\eta z^4 - 162\eta z^3 + 102\eta z\right) + \\ + \left(54\eta^2 z^5 - 81\eta^2 z^4 + 51\eta^2 z^2\right) \end{array} \right] dz$$

$$
=\frac{9\tau^2}{8}\left[\frac{(z+1)^4}{4}+\left(z^2\eta+2\eta z^3+\frac{6}{4}\eta z^4+\frac{2}{5}\eta z^5\right)+ +\left(\frac{1}{3}\eta^2z^3+\frac{3}{4}\eta^2z^4+\frac{3}{5}\eta^2z^5+\frac{1}{6}\eta^2z^6\right)\right]_{-1}^{-\frac{1}{3}}+
$$

$$
+\frac{\tau^2}{48}\left[\left(-27z^4-27z^3+27z^2+39z\right)+\left(-\frac{216}{5}\eta z^5-\frac{162}{4}\eta z^4+36\eta z^3+39\eta z^2\right)+ +\left(-18\eta^2z^6-\frac{81}{5}\eta^2z^5+\frac{54}{4}\eta^2z^4+13\eta^2z^3\right)\right]_{-\frac{1}{3}}^{\frac{1}{3}}+
$$

$$
+\frac{\tau^2}{48}\left[\left(\frac{54}{4}z^4-27z^3+51z\right)+\left(\frac{108}{5}\eta z^5-\frac{162}{4}\eta z^4+51\eta z^2\right)+ +\left(9\eta^2z^6-\frac{81}{5}\eta^2z^5+17\eta^2z^3\right)\right]_{\frac{1}{3}}^{1}
$$

$$=\frac{9\tau^2}{8}\left\{\begin{aligned}&\frac{1}{4}\left(\frac{2}{3}\right)^4+\left[\left(\frac{1}{9}-1\right)\eta+2\eta\left(\frac{-1}{27}+1\right)+\frac{6}{4}\eta\left(\frac{1}{81}-1\right)+\frac{2}{5}\eta\left(\frac{-1}{243}+1\right)\right]+\\&+\left[\frac{1}{3}\eta^2\left(\frac{-1}{27}+1\right)+\frac{3}{4}\eta^2\left(\frac{1}{81}-1\right)+\frac{3}{5}\eta^2\left(\frac{-1}{243}+1\right)+\frac{1}{6}\eta^2\left(\frac{1}{729}-1\right)\right]\end{aligned}\right\}+$$

$$+\frac{\tau^2}{48}\left\{\begin{aligned}&\left[-27\left(\frac{1}{81}-\frac{1}{81}\right)-27\left(\frac{1}{27}+\frac{1}{27}\right)+27\left(\frac{1}{9}-\frac{1}{9}\right)+39\left(\frac{1}{3}+\frac{1}{3}\right)\right]+\\&+\left[-\frac{216}{5}\eta\left(\frac{1}{243}+\frac{1}{243}\right)-\frac{162}{4}\eta\left(\frac{1}{81}-\frac{1}{81}\right)+36\eta\left(\frac{1}{27}+\frac{1}{27}\right)+39\eta\left(\frac{1}{9}-\frac{1}{9}\right)\right]+\\&+\left[\begin{aligned}&-18\eta^2\left(\frac{1}{729}-\frac{1}{729}\right)-\frac{81}{5}\eta^2\left(\frac{1}{243}+\frac{1}{243}\right)+\\&+\frac{54}{4}\eta^2\left(\frac{1}{81}-\frac{1}{81}\right)+13\eta^2\left(\frac{1}{27}+\frac{1}{27}\right)\end{aligned}\right]\end{aligned}\right\}+$$

$$+\frac{\tau^2}{48}\left\{\begin{aligned}&\left[\frac{54}{4}\left(1-\frac{1}{81}\right)-27\left(1-\frac{1}{27}\right)+51\left(1-\frac{1}{3}\right)\right]+\\&+\left[\frac{108}{5}\eta\left(1-\frac{1}{243}\right)-\frac{162}{4}\eta\left(1-\frac{1}{81}\right)+51\eta\left(1-\frac{1}{9}\right)\right]+\\&+\left[9\eta^2\left(1-\frac{1}{729}\right)-\frac{81}{5}\eta^2\left(1-\frac{1}{243}\right)+17\eta^2\left(1-\frac{1}{27}\right)\right]\end{aligned}\right\}$$

$$=\frac{9\tau^2}{8}\left\{\begin{aligned}&\frac{4}{81}+\left[\left(\frac{-8}{9}\right)\eta+2\eta\left(\frac{26}{27}\right)+\eta\left(\frac{-40}{27}\right)+\frac{2}{5}\eta\left(\frac{242}{243}\right)\right]+\\&+\left[\frac{1}{3}\eta^2\left(\frac{26}{27}\right)+\eta^2\left(\frac{-20}{27}\right)+\frac{3}{5}\eta^2\left(\frac{242}{243}\right)+\frac{1}{6}\eta^2\left(\frac{-728}{729}\right)\right]\end{aligned}\right\}+$$

$$+\frac{\tau^2}{48}\left\{[-2+26]+\left[-\frac{8}{5}\eta\left(\frac{2}{9}\right)+\eta\left(\frac{8}{3}\right)\right]+\left[-\frac{1}{5}\eta^2\left(\frac{2}{3}\right)+13\eta^2\left(\frac{2}{27}\right)\right]\right\}+$$

$$+\frac{\tau^2}{48}\left\{\begin{aligned}&\left[2\left(\frac{20}{3}\right)-26+34\right]+\left[\frac{4}{5}\eta\left(\frac{242}{9}\right)-\eta(40)+17\eta\left(\frac{8}{3}\right)\right]+\\&+\left[\eta^2\left(\frac{728}{81}\right)-\frac{1}{5}\eta^2\left(\frac{242}{3}\right)+17\eta^2\left(\frac{26}{27}\right)\right]\end{aligned}\right\}$$

$$= \frac{\tau^2}{48}\left\{\begin{aligned}&\left[\frac{8}{3}+24+\frac{40}{3}+8\right]+\eta\left[-48+104-80+\frac{968}{45}-\frac{16}{45}+\frac{8}{3}+\frac{968}{45}-40+\frac{136}{3}\right]+\\ &+\eta^2\left[\frac{52}{3}-40+\frac{484}{15}-\frac{728}{81}-\frac{2}{15}+\frac{26}{27}+\frac{728}{81}-\frac{242}{15}+\frac{442}{27}\right]\end{aligned}\right\}$$

$$= \frac{\tau^2}{48}\left\{48+\frac{80}{3}\eta+\frac{32}{3}\eta^2\right\} = \frac{\tau^2}{9}\left(9+5\eta+2\eta^2\right)$$

Then,

$$\overline{y^2} = \frac{\tau^2}{9}\left(9+5\eta+2\eta^2\right)$$

## References

[1] Eric W. Gray, "Multi-Protocol Label Switching: Implementing the Technology"

[2] P. Zarros, M. Lee, T. Saadawi, "Interparticipant Synchronization in Real-Time Multimedia Conference Using Feedback," IEEE/ACM Transactions on Networking, Vol. 4, No. 2, April 1996, pp. 173 -180.

[3] Y.Xie, C. Liu, M.Lee, and T.Saadawi, Multipoint multimedia teleconference system with adaptive synchronization, IEEE Journal on Selected Areas in Communications, September 1996.

[4] A. Privalov and K. Sohraby, "Per-Stream Jitter Analysis in CBR ATM Multiplexors," IEE/ACM Transactions on Networking, Vol. 6, No. 2, April, 1998, pp. 141-149.

[5] J. Walker and A. Cantoni, "Determining Parameters to Minimize Jitter generation in the SRTS Methods," IEEE Transaction on Networking, Vol. 46, No. 1, January, 1998, pp. 82-90.

[6] Iida, K., Takine, T., Sunahara, H., and Oie, Y., "Delay analysis for CBR traffic under static-priority scheduling," IEEE/ACM Trans. Networking, Vol. 9 pp. 177-185, April 2001.

[7] Qiong Li and Mills, D.L., "Jitter-based delay-boundary prediction of wide-area networks," IEEE/ACM Trans. Networking, Vol. 9 pp. 578-590, Oct. 2001.

[8] Bennett, J.C.R. and et al, "Delay jitter bounds and packet scale rate guarantee for expedited forwarding," IEEE/ACM Trans. Networking, Vol. 10 pp. 529-540, Aug. 2002.

[9] R. Stewart and et al "Stream Control Transmission Protocol," Internet Engineering Task Force (IETF), FRC 2960, Oct. 2000.

[10] K. Shuaib, S. Nawrot, H. Elsayed, M.Lee, and T. Saadawi,"Empirical evaluation of ATM CBR traffic transported under diverse operating conditions", Proc. of the ISCC conference, Athens, Greece June 1998, pp. 450-454.

[11] S. Banerjee, D. Tipper, B. Weiss and A. Khalil,"Traffic Experiments on the vBNS wide area ATM networks", IEEE communication magazine, August 1997, pp 126-133.

[12] R. Mandeville," Product evaluation, ATM switches", Data communication, March 1995.

[13] ATM Theory and Applications, D. McDysan and D. Spohn, McGraw-Hill Series, 1994.

[14] H. Elsayed and T. Saadawi "Jitter Equalization to Maintain QoS for Multimedia Traffic" IASTED'2003, Feb. 10 - 13, Innsbruck, Austria. pp.643 – 648.

[15] H. Elsayed, A. Abd El Al, et al "Synchronization Algorithm for SCTP Network" ICDCS'2003, May. 19 - 22, Rhode Island, USA. pp. 576 – 581.

[16] T. Saadawi, M.Ammar, A.Elhakeem, Fundamentals of Telecommunication Networks, New York, NY: John Wiley and Sons, 1994.

[17] Probabilty Random Variables, and Stochastic Processes, Athanasios Papoulis,McGraw-Hill, 1984.

[18] Y.Xie, C. Liu, M.Lee, and T.Saadawi, Multipoint multimedia teleconference system with adaptive synchronization, IEEE Journal on Selected Areas in Communications, September 1996.

[19] W. Matragi, K. Sohraby, and C. Bisdikia, "Jitter Calculus in ATM Networks: Multiple Node Case," Proc. IEEE INFOCOM'94, Toronto, Ont., Canada, June 12-16, 1994 pp. 2c.4.1-2d.4.10.

[20] A. Privalov and K. Sohraby, "Per-Stream Jitter Analysis in CBR ATM Multiplexors," IEE/ACM Transactions on Networking, Vol. 6, No. 2, April, 1998, pp. 141-149.

[21] J. Walker and A. Cantoni, "Determining Parameters to Minimize Jitter generation in the SRTS Methods," IEE Transaction on Networking, Vol. 46, No. 1, January, 1998, pp. 82-90.

[22] Randall R. Stewart, et al, "Stream Control Transmission Protocol (SCTP) : A Reference guide," Addison-Wesley Pub Co, Dec. 15, 2001.

[23] Probability Random Variables, and Stochastic Processes, Athanasios Papoulis,McGraw-Hill, 1984.

# Publications

## Selected Publications

[1] **H. Elsayed** and T. Saadawi "Jitter Equalization to Maintain QoS for Multimedia Traffic" IASTED'2003, Feb. 10 - 13, Innsbruck, Austria. pp.643 – 648.

[2] **H. Elsayed**, A. Abd El Al, et al "Synchronization Algorithm for SCTP Network" ICDCS'2003, May. 19 - 22, Rhode Island, USA. pp. 576 – 581.

[3] S. Nawrot, **H. Elsayed**, et al "Performance of a Hybrid Terrestrial/Satellite ATM Network – Experimental Results for CBR Traffic CTD and CDV QoS Parameters" Milcom,'98, 1998, USA.

[4] K. Shuaib, S. Nawrot, **H. Elsayed**, M.Lee, and T. Saadawi,"Empirical evaluation of ATM CBR traffic transported under diverse operating conditions", Proc. of the ISCC conference, Athens, Greece June 1998, pp. 450-454.

[5] K. Shuaib, **H. A. Elsayed**, et al "Experimental and Simulation Results in the Transport of MPEG-2 over ATM," SPIE'97, Nov. 3-5, 1997, Texas, USA.

[6] **H. A. Elsayed**, R. Iskander, et al "Increasing Noise Channel Utilization For Asymmetric Digital Communication Networks", ICM'96 Conf., Dec. 16-18, 1996, Cairo, Egypt.

[7] **H. A. Elsayed**, N. Elnady, and S. Elramly "A Proposal Amendment for Sliding Window Protocol Asymmetric Traffic Loading Communication Links"" June 14-16, 1995, APCC"5 Conf., Kobe Japan.

## Under submission

[1] **H. Elsayed**, T. Saadawi, and M. Lee , "Synchronization Algorithm for SCTP Non Periodic Traffic"

[2] **H. Elsayed**, T. Saadawi, and M. Lee , "Jitter Equalization Analysis for Periodic Multimedia Traffic"