Completeness of Certain Bimodal Logics for Subset Spaces

by

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Abstract

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Advisor: Professor Rohit Parikh

Subset Spaces were introduced by L. Moss and R. Parikh in [7]. These spaces model the reasoning about knowledge of changing states.

In [1] a kind of subset space called intersection space was considered and the question about the existence of a set of axioms that is complete for the logic of intersection spaces was addressed. In [6] we introduced two kinds of subset spaces, namely *quasi–intersection* and *directed spaces* and proved that any set of axioms for directed frames also characterizes intersection spaces.

We give here the solution to the question of a complete axiomatization for intersection spaces by giving a denumerable set of axioms that is complete for directed spaces. We also show that it not possible to reduce this set of axioms to a finite set.

For Ale, Gabriel and Daniela

I would like to thank Rohit Parikh, Larry Moss and Andy Dabrowski for introducing me to the problem solved here.

Rohit Parikh, Larry Moss and Konstantinos Georgatos inserted a lot of comments about how to interpret my own results. I took their works as mine in writing this thesis. Also I would thank them for listening to my seminars and sending suggestions by electronic mail.

The final form of this thesis was written following Melvin Fitting, Roman Kossak and Rohit Parikh's suggestions.

I would like to thank Marcelo Finger who has the credit for many of the pieces of notation and jargon used here.

It is necessary to say that many nice pieces of ideas and proofs are to be credited to Rohit Parikh – Thanks, the tea was nice and I will tell everyone!

Contents

Chapter 1

1.1 Introduction

Subset Spaces were introduced in the context of reasoning about knowledge as the semantics for a bimodal logic that would interpret the gain in knowledge versus the effort to obtain it. This logic has two modalities: K which represents the knowledge in some state, from an observer's view, and $\Box$ which represents an improvement of the observer's knowledge by refining its view (the applied effort).

Among the several kind of subset spaces introduced by *Dabrowski, Moss and Parikh*, DMP, in [1], we focus on one in particular in which the question of finding a complete axiomatization for them was open: The *intersection spaces.* In [6] we introduced two classes of subset spaces – *quasi–intersection* and *directed spaces* – and proved that their logic is the same as the logic of intersection spaces. DMP's question is answered here using the equivalence result obtained in [6].

Consider two modalities, K and $\Box$, and a denumerable set of atoms, $\mathcal{A}$. Define the set $\mathcal{F}$ of formulas as the smallest set closed under the rules:

- For every $A \in \mathcal{A}$, $A \in \mathcal{F}$
- If ϕ and ψ are in $\mathcal{F}$, so are $\phi \wedge \psi$ and $\neg\phi$
- If ϕ is in $\mathcal{F}$, so are $\Box\phi$ and $K\phi$.

Definition 1.1.1 *Given a formula ψ, S_ψ is the set of subformulas of ψ. The complexity of ψ is the cardinality of S_ψ.*

If ϕ and ψ are two formulas, the usual abbreviations $\phi \vee \psi$, $\phi \Rightarrow \psi$, $L\phi$ and $\Diamond\phi$ stand for $\neg(\neg\phi \wedge \neg\psi)$, $\neg(\phi \wedge \neg\psi)$, $\neg K \neg\phi$ and $\neg\Box\neg\phi$, respectively.

The semantics for $\mathcal{L}$ is given by **Subset frames**, frames of the form $\langle X, \mathcal{O} \rangle$, where X is a non–empty set of points and $\mathcal{O}$, the *set of opens*, is a set of subsets of X.

Definition 1.1.2 *Given an interpretation α, i.e. a mapping from $\mathcal{A}$ into the set of subsets of X, $\mathcal{P}(X)$, the* model $\langle X, \mathcal{O}, \alpha \rangle$ *is called a* **subset space**. *Given $(x, U) \in X \times \mathcal{O}$ and ψ a bimodal formula, the satisfaction relation, $x, U \models_{\mathcal{M}} \psi$, is defined recursively:*

If ψ is atomic, $x, U \models_{\mathcal{M}} \psi$ iff $x \in \alpha(\psi)$;

If ψ is $\neg\phi$, $x, U \models_{\mathcal{M}} \psi$ iff $x, U \not\models_{\mathcal{M}} \phi$;

If ψ is $\phi \wedge \chi$, $x, U \models_{\mathcal{M}} \psi$ iff $x, U \models_{\mathcal{M}} \phi$ and $x, U \models_{\mathcal{M}} \chi$;

If ψ is $K\phi$, $x, U \models_{\mathcal{M}} \psi$ iff for all $y \in U$, $y, U \models_{\mathcal{M}} \phi$;

If ψ is $\Box\phi$ $x, U \models_{\mathcal{M}} \psi$ iff for all $V \in \mathcal{O}$, if $x \in V \subseteq U$, then $x, V \models_{\mathcal{M}} \phi$

Definition 1.1.3 *Given a subset frame $\langle X, \mathcal{O} \rangle$ and $x \in X$, we call the set $\{U \in \mathcal{O} | x \in U\}$ the set of neighborhoods of x and we denote it by $\mathcal{N}x$.*

Definition 1.1.4 *Subset frames whose set of opens is closed under finite intersections are called* **Intersection Frames**. *If for all $x \in X$ and $U, V \in \mathcal{O}$, whenever $x \in U$ and $x \in V$, there exists $W \in \mathcal{O}$ so that $x \in W \subseteq (U \cap V)$, then the frame $\mathcal{X}$ is called a* **Directed Frame**.

We will call a subset space based on an intersection frame an **Intersection Space** *and a subset space based on a directed frame a* **Directed Space**.

Clearly any intersection frame is a directed frame. Now, given a directed space $\mathcal{M}$, if $Th(\mathcal{M}) = \{\psi | \models_{\mathcal{M}} \psi\}$, the following result is shown in [6]:

Theorem 1.1.5 *If $\mathcal{M}$ is a directed space, there exists an intersection space $\mathcal{M}'$ so that $Th(\mathcal{M}) = Th(\mathcal{M}')$.*

Consider the following axioms and rules of inference:

1) All instances of tautologies
2) $A \Rightarrow \Box A$, for all atoms A
3) $\Box(\psi \Rightarrow \phi) \Rightarrow (\Box\psi \Rightarrow \Box\phi)$
4) $\Box\psi \Rightarrow \psi$
5) $\Box\psi \Rightarrow \Box\Box\psi$
6) $K(\psi \Rightarrow \phi) \Rightarrow (K\psi \Rightarrow K\phi)$
7) $K\psi \Rightarrow \psi$
8) $\psi \Rightarrow KL\psi$
9) $K\psi \Rightarrow KK\psi$
10) $K\Box\psi \Rightarrow \Box K\psi$
WD) $\Diamond\Box\psi \Rightarrow \Box\Diamond\psi$
M_n) $(\Box L\Diamond\phi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n) \Rightarrow L(\Diamond\phi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n)$
Modus Ponens, $\dfrac{\psi,\ \psi \Rightarrow \chi}{\chi}$
$\Box$–necessitation $\dfrac{\Box\psi}{\psi}$
K–necessitation. $\dfrac{K\psi}{\psi}$

Observe that axioms 2 to 5 are axioms for $S4$ and axioms 6 to 9 are axioms for $S5$.

Axiom 10 is called the *Cross Axiom*. This axiom is linked with the fact that given two opens U and V, then $U \subseteq V$ is the same as $x \in U$ implies $x \in V$.

WD stands for the Weak Directed Axiom, a sound axiom over the class of *directed spaces*.

In [1], [7] and [3] completeness and decision procedures are shown for several classes of subset spaces. Besides, in [1] it is shown that axioms 1 to 10 plus WD are not a complete set of axioms for the logic of intersection spaces.

In this paper, we introduce the set of axioms M_n and show in Lemma 1.1.6 that they are sound for directed spaces. We also prove that axioms 1 to 10, WD and M_n are a complete set of axioms for Directed Spaces. Using Theorem 1.1.5, we have then solved the problem of completeness for Intersection Spaces.

After proving completeness, we show in Section 4 that the axioms M_n cannot be reduced to a finite number of axiom schemes. For each $n \in \mathbb{N}$, we build a series of models $\mathcal{M}_n$ and show that all axioms for intersection spaces are valid in $\mathcal{M}_n$, for $1 \leq j < n$, the axioms M_j are valid in $\mathcal{M}_n$ and that some instance ψ of the axiom M_n is not valid over $\mathcal{M}_n$.

The examples are reported in Section 5 and illustrate several sections of this paper.

Lemma 1.1.6 *For each $n \in \mathbb{N}$, the axiom M_n is sound for directed spaces.*

Proof: Let $\mathcal{M} = \langle X, \mathcal{O}, \alpha \rangle$ be a directed space. Let x, U be so that $x, U \models \Box L \Diamond \phi \wedge (\bigwedge_{1 \leq i \leq n} \Diamond K \psi_i)$. We have that $(\forall W \in \mathcal{N}x)(W \subseteq U \Rightarrow ((\exists y \in W)(y, W \models \Diamond \phi)))$ and $(\forall 1 \leq j \leq n)(\exists V_j \in \mathcal{N}x)(V_j \subseteq U \wedge \ x, V_j \models K\psi_j)$.

Pick $W \in \mathcal{N}x$ with $W \subseteq \cap\{V_i | 1 \leq i \leq n\}$. Let $y \in W$ be so that $y, W \models \Diamond\phi$. Then $y, W \models \Diamond\phi$. As $W \subseteq U$, we have $y, U \models \Diamond\phi$.

We claim that $y, U \models \Diamond\phi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n$: Because $x, V_i \models K\psi_i$ and $y \in \cap\{V_i | 1 \leq i \leq n\}$, then $y, V_i \models K\psi_i$. As $(\forall 1 \leq i \leq n)(y, U \models \Diamond K\psi_i)$, then $y, U \models \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n$. So,

$$y, U \models \Diamond\phi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n$$

Therefore,

$$x, U \models L(\Diamond\phi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n). \blacksquare$$

1.2 A Survey of Related Work

We present here a list of works about *subset spaces*. The papers are listed in chronological order and, after a short presentation of their contents, we relate this thesis with the other papers in the list.

Theoretical Aspects of Reasoning and The Logic of Knowledge, Lawrence S. Moss and Rohit Parikh (see [7]).

In their paper, the authors introduced a bimodal logic for reasoning about knowledge during knowledge acquisition. Their paper started a project of searching for a suitable language for describing topological reasoning.

Also, the authors gave a contribution to the development of logics of knowledge, introducing a logical model that interprets the effort versus the gain in knowledge acquisition. The several papers we list above make use of this interpretation.

In their paper, Moss and Parikh (MP) introduced the general idea of subset spaces. In MP's paper,

A) Axioms 1 to 10 are introduced,

B) Axioms 1 to 10 are are proven to be sound and complete for subset spaces.

C) Subset spaces whose set of opens is closed under intersection and union are introduced and discussed in MP paper,

D) The union axiom:

$$\Diamond(K\psi \land \phi) \land L\Diamond(K\psi \land \chi) \to \Diamond(K\Diamond\psi \land \Diamond\phi \land L\Diamond\chi)$$

is introduced and shown sound for spaces closed under union.

E) Axiom WD is introduced and shown sound for the classes spaces closed under intersection.

We note that the directed spaces were introduced in [6] and, since directed spaces are logically equivalent to intersection spaces, WD is sound for directed spaces.

Topological Reasoning and the Logic of Knowledge, A. Dabrowski, L.S, Moss and R. Parikh (see [1]).

DMP's paper is an extension of some results shown in MP's paper.

In DMP's paper,

A) The following classes of interpretations for the logic of subset spaces are discussed: Subset frames, intersection frames, lattice frames and complete lattice frames.

B) A completeness theorem for the logic of subset spaces is shown in a direct construction that uses the properties of maximal consistent sets.

C) In spite of the failure of the finite model property (shown by an example), DMP paper shows decidability for subset space logic.

D) An example is given in order to show that axioms 1 to 10 plus WD are sound but not a complete set of axioms for intersection spaces.

E) A complete discussion about the classes of intersection spaces is given and the question about the existence of an complete set of axioms for intersection spaces is conjectured.

In DMP paper, the system whose axioms are 1 to 10, WD and the Union axiom are called *topologic*. We have then,

F) It is shown that the topologic axioms give a complete axiomatization of topologies. Indeed, on all lattice spaces.

G) The finite model property, thus the decidability of topologic is shown using a different argument from that used in Georgatos' thesis (see [3]).

Modal Logic for Topological Spaces, Konstantinos Georgatos (see [3]).

Georgatos' thesis, KG, main subject is the topologic system. In his thesis,

A) The author obtains a completeness theorem for topologic, independently of the DMP paper result.

B) Finite model property for topologic was first shown there.

Georgatos extends the concept of subset spaces to the system MP^* – the systems MP^* are the class of subset spaces closed under intersection and union – and

C) Shows necessary and sufficient conditions in which a Kripke model can be turned into a set-theoretic model of MP^*.

D) Among the results is the finite model property and decidability for MP^*.

E) Several algebraic models for those models are presented.

Knowledge Theoretical Properties for Topological Spaces, Konstantinos Georgatos (see [4]).

In that paper, the author,

A) Describes the validity problem for topological spaces.

B) Studies the model based on the basis of a topological space closed under finite unions and prove it is equivalent to the topological space it generates.

C) Show finite satisfiability for the class of topological models.

Reasoning about Knowlewdge in Computational Trees, K. Georgatos (see [5]).

Unlike the above cited papers, Georgatos shows a new interpretation for the classes of directed spaces on introducing the Tree–like spaces, defined as: For all U, V opens, either $U \subseteq V$, $V \subseteq U$ or $U \cap V = \emptyset$.

The Tree like spaces are a suitable class of subset spaces for modeling reasoning about computation.

The author proposed the following set of axioms and rules:

Axioms 1 to 10,

11. $\Box(\Box\phi \rightarrow \psi) \vee \Box(\Box\psi \rightarrow \phi)$

12 $\Box K\phi \wedge K(\Box\phi \rightarrow \Box\psi) \rightarrow \Box K(\Box\phi \rightarrow \Box\psi)$.

The rules Modus Ponens, K–necessitation and $\Box$–necessitation.

Georgatos paper shows

A) The above set of axioms is a sound and complete axiomatization for tree like spaces.

B) The set of theorems is decidable: Tree like spaces have finite model property.

Subset Space Logics and Logics of Knowledge and Time, B. Heinemann, see [2]).

In his thesis, the author studies several logics of knowledge and effort. These logics are not indeed the original bimodal logic introduced by MP, but Heinemann's studies are based on the work of MP. Again, we see a different interpretation that goes in the direction of MP's original goals in introducing the logic of subset spaces.

Heinemann introduces the functional interpretation of the modal operator varying over subsets. In this way a "topological" *single step*–operator, the *nexttime*–operator, appears.

First the author studies the subset space logics of knowledge and time for the single–agent case. Subsequently, he generalizes on certain fragments of propositional temporal logic to the subset space logic.

The author studies both the linear and branching time structures. Afterwards the methods and results are extended to the multi–agent case.

Heinemann's thesis concerns the basic topics of logics in computer science: *completeness, decidability and complexity*, but in the multi–agent case, however, the author examines also some notions which only occur if more than one agent is present, such as *common knowledge*.

Chapter 2

2.1 Theoretical Preliminaries

Let $\mathcal{L}$ be the directed space language whose set of literals is $\mathcal{A}=\{A_1, A_2, \ldots\}$ and let X be the set of all maximal consistent sets of formulas in $\mathcal{L}$.

Define the canonical underlying set X by $x \in X$ iff x is a maximal consistent set in the logic of directed spaces. Let $\mathcal{M}_{\mathcal{L}}$ be the Kripke model, hereafter called the canonical model based on X endowed with the relations $\overset{L}{\longleftrightarrow}$, $\overset{\diamond}{\longrightarrow}$ and the canonical valuation α given by

$x \overset{L}{\longleftrightarrow} y$ iff for all formulas ψ, $K\psi \in x$ implies $\psi \in y$;

$x \overset{\diamond}{\longrightarrow} y$ iff for all formulas ψ, $\Box\psi \in x$ implies $\psi \in x$;

For all $A \in \mathcal{A}$, $\alpha(A) = \{x \in X | A \in x\}$.

Definition 2.1.1 *Let $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\diamond}{\longrightarrow}, \alpha\rangle$ be a Kripke model of $\mathcal{L}$. Denote the usual satisfaction relation over $Y \times \mathcal{L}$ by $\models_{\mathcal{N}}$, or, when the context is clear, $\models$.*

Observe that we used the same notation for the relations associated with the modalities $\Box$ and K for the two (Kripke) models: $\mathcal{N}$ (Definition 2.1.1) and the canonical model $\mathcal{M}_{\mathcal{L}}$. We proceed this way when it is clear where an arrow ($\overset{L}{\longleftrightarrow}$ or $\overset{\diamond}{\longrightarrow}$) is defined. We work here with Kripke models of $\mathcal{L}$, the canonical model (a special Kripke model of $\mathcal{L}$) and subset spaces for $\mathcal{L}$ (also Kripke models with the arrows $\overset{L}{\longleftrightarrow}$ and $\overset{\diamond}{\longrightarrow}$ given in Definition 1.1.2).

Definition 2.1.2 *Let $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\diamond}{\longrightarrow}, \alpha\rangle$ be a Kripke model of $\mathcal{L}$. $\mathcal{N}$ is a* directed model *if*

1 $\stackrel{L}{\longleftrightarrow}$ *is an equivalence relation;*

2 $\stackrel{\diamond}{\longrightarrow}$ *is transitive and reflexive;*

3 If $x \stackrel{\diamond}{\longrightarrow} y$ *and* $y \stackrel{L}{\longleftrightarrow} z$*, then there exists* $w \in Y$ *so that* $x \stackrel{L}{\longleftrightarrow} w$ *and* $w \stackrel{\diamond}{\longrightarrow} z$*;*

4 $\stackrel{\diamond}{\longrightarrow}$ *is weakly directed: If* $x \stackrel{\diamond}{\longrightarrow} y$ *and* $x \stackrel{\diamond}{\longrightarrow} z$*, there exists* w *so that* $y \stackrel{\diamond}{\longrightarrow} w$ *and* $z \stackrel{\diamond}{\longrightarrow} w$*.*

we also say that $\mathcal{N}$ *is a* directed and above directed model *if it satisfies 1 to 4 above plus*

5 For all $x, y, z \in Y$*, if* $y \stackrel{\diamond}{\longrightarrow} x$ *and* $z \stackrel{\diamond}{\longrightarrow} x$*, there exists* w*, such that* $w \stackrel{\diamond}{\longrightarrow} y$ *and* $w \stackrel{\diamond}{\longrightarrow} z$*.*

The following Lemma is proved in DMP paper, [1]:

Lemma 2.1.3 *The canonical model* $\mathcal{M}_{\mathcal{L}}$ *is a directed model and for all* x, ψ, $x \models_{\mathcal{M}_{\mathcal{L}}} \psi$ *iff* $\psi \in x$*.*

Definition 2.1.4 *Let* $\mathcal{N}_i = \langle Y_i, \stackrel{L}{\longleftrightarrow}_i, \stackrel{\diamond}{\longrightarrow}_i, \alpha_i \rangle$*, for* $i \in \{1, 2\}$ *be two Kripke models for* $\mathcal{L}$*. We say that* $\mathcal{N}_1$ *is a* strong expansion *(of* $\mathcal{N}_2$*) if there exists a mapping* F *- called a fiber mapping - from* Y_1 *onto* Y_2 *so that*

1. *For all literals* A, $F(\alpha_1(A)) = \alpha_2(A)$*;*
2. $(\forall y, y' \in Y_1)((y \stackrel{\diamond}{\longrightarrow}_1 y') \Rightarrow (F(y) \stackrel{\diamond}{\longrightarrow}_2 F(y')))$*;*
3. $(\forall y \in Y_1)(\forall x' \in Y_2)((F(y) \stackrel{\diamond}{\longrightarrow}_2 x') \Rightarrow (\exists y' \in Y_1)(y \stackrel{\diamond}{\longrightarrow}_1 y' \wedge F(y') = x'))$*;*
4. $(\forall y, y' \in Y_1)((y \stackrel{L}{\longleftrightarrow}_1 y') \Rightarrow (F(y) \stackrel{L}{\longleftrightarrow}_2 F(y')))$*;*
5. $(\forall y \in Y_1)(\forall x' \in Y_2)((F(y) \stackrel{L}{\longleftrightarrow}_2 x') \Rightarrow (\exists y' \in Y_1)(y \stackrel{L}{\longleftrightarrow}_1 y' \wedge F(y') = x'))$*.*

Lemma 2.1.5's proof is straightforward and we skip it. We note to the reader that is familiar with the idea of bisimulation that a strong expansion is a kind of bisimulation, where the fiber mapping plays the role of a relation between the points of two models.

Lemma 2.1.5 *Let $\mathcal{N}_i = \langle Y_i, \stackrel{L}{\longleftrightarrow}_i, \stackrel{\diamond}{\longrightarrow}_i, \alpha_i \rangle$, for $i \in \{1,2,3\}$ be Kripke models for $\mathcal{L}$ and suppose that $\mathcal{N}_1$ is a strong expansion of $\mathcal{N}_2$ and $\mathcal{N}_2$ is a strong expansion of $\mathcal{N}_3$, with fiber mapping given by F_1 and F_2 respectively. Then $\mathcal{N}_1$ is a strong expansion of $\mathcal{N}_3$, with fiber mapping $F_2 \circ F_1$.*

Lemma 2.1.6 (The Truth Lemma) *Given $\mathcal{N}_i = \langle Y_i, \stackrel{L}{\longleftrightarrow}_i, \stackrel{\diamond}{\longrightarrow}_i, \alpha_i \rangle$ for $i \in \{1,2\}$, two Kripke models for $\mathcal{L}$ and F a fiber mapping from Y_1 onto Y_2, then for all $y \in_1 Y$, for all $\psi \in \mathcal{L}$, $y \models_{\mathcal{N}_1} \psi$ iff $F(y) \models_{\mathcal{N}_2} \psi$.*

Proof: By induction on the complexity of ψ:

i If ψ is A, an atomic formula, then $y \models_{\mathcal{N}_1} \psi$ iff $y \in \alpha_1(A)$, iff $F(y) \in \alpha_2(A)$, iff $F(y) \models_{\mathcal{N}_2} \psi$;

ii The cases $\psi = \phi \wedge \chi$ and $\psi = \neg\phi$ are easy and left to the reader;

iii If $\psi = K\phi$, we have: If $y \models_{\mathcal{N}_1} K\phi$, for all $y' \in Y_1$, if $y \stackrel{L}{\longleftrightarrow}_1 y'$, then $y' \models_{\mathcal{N}_1} \phi$. Hence, by our inductive hypothesis, for all $y' \in Y_1$, if $y \stackrel{L}{\longleftrightarrow}_1 y'$ then $F(y') \models_{\mathcal{N}_2} \phi$. Now, by Definition 2.1.4 part 5, for all $x' \in Y_2$, if $F(y) \stackrel{L}{\longleftrightarrow}_2 x'$ then $x \models_{\mathcal{N}_2} \phi$, hence $F(y) \models_{\mathcal{N}_2} K\phi$.

Reciprocally, if $F(y) \models_{\mathcal{N}_2} K\phi$ then for all $x' \in Y_2$, if $F(y) \stackrel{L}{\longleftrightarrow}_2 x'$ then $x' \models_{\mathcal{N}_2} \phi$. By the inductive hypothesis, for all $y' \in Y_1$, if there exists $x' \in Y_2$ with $F(y) \stackrel{L}{\longleftrightarrow}_2 x'$ and $F(y') = x'$, then $y' \models_{\mathcal{N}_1} \phi$. Now by 2.1.4 part 4, for all $y' \in Y_1$, if $y \stackrel{L}{\longleftrightarrow}_1 y'$ then $y' \models_{\mathcal{N}_2} \phi$ so, $y \models_{\mathcal{N}_1} K\phi$;

iv For case $\psi = \Box\phi$, the procedure is analogous to iii.■

Chapter 3

In this Chapter, we prove that axioms 1)–10), W D and M_n are complete for the logic of directed spaces. In Section 3.1, we focus on certain subsets of Kripke Models of $\mathcal{L}$, the $\diamond$–classes and the Pp's, and study them in the canonical model $\mathcal{M}_{\mathcal{L}}$. In Section 3.2, using the M–axiom Lemma, we show that there exists a directed and above directed model $\mathcal{M}'$ that is a strong expansion of $\mathcal{M}_{\mathcal{L}}$. In Section 3.3, we enlarge the model $\mathcal{M}'$ we built and force it into an directed space for $\mathcal{L}$ via a bulldozer–like procedure (see [8]) applied over its points.

3.1 Some Properties of the Kripke Models of $\mathcal{L}$

Definition 3.1.1 *Let $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\diamond}{\longrightarrow} \alpha\rangle$ be a Kripke model for $\mathcal{L}$. Define $a \overset{\diamond}{\sim} b$ iff there exists $c \in Y$ so that $a \overset{\diamond}{\longrightarrow} c$ and $b \overset{\diamond}{\longrightarrow} c$.*

Lemma 3.1.2 *Given a weakly directed Kripke model $\mathcal{N}$,*

1. *$\overset{\diamond}{\sim}$ is an equivalence relation;*
2. *In the canonical model $\mathcal{M}_{\mathcal{L}}$, for all $a, a' \in X$, $a \overset{\diamond}{\sim} a'$ iff $a \cap F = a' \cap F$ where $F = \{\Diamond\Box\phi | \phi \text{ is a formula}\}$.*

Proof: 1. The relation $\overset{\diamond}{\sim}$ is clearly reflexive and symmetric. In order to show transitivity, if $a \overset{\diamond}{\sim} a'$ and $a' \overset{\diamond}{\sim} a''$, there exist c_1 and c_2 and $a, a' \overset{\diamond}{\longrightarrow} c_1$ and $a', a'' \overset{\diamond}{\longrightarrow} c_2$. As $\mathcal{N}$ is weakly directed and $a' \overset{\diamond}{\longrightarrow} c_1, c_2$, there exists d so that $a' \overset{\diamond}{\longrightarrow} c_1, c_2 \overset{\diamond}{\longrightarrow} d$. Hence $a, a'' \overset{\diamond}{\longrightarrow} d$, so, $a \overset{\diamond}{\sim} a''$. Since $\overset{\diamond}{\sim}$ is reflexive, symmetric and, as we just showed, transitive, $\overset{\diamond}{\sim}$ is an equivalence relation.

2. Suppose $a \overset{\diamond}{\sim} a'$ and $\Diamond\Box\phi \in a$, then there must exist c with $a \overset{\diamond}{\longrightarrow} c$ so that $\Box\phi \in c$. Also, there is d with $a, a' \overset{\diamond}{\longrightarrow} d$. As $a \overset{\diamond}{\longrightarrow} c, d$, there exists e so that $a \overset{\diamond}{\longrightarrow} c, d \overset{\diamond}{\longrightarrow} e$. So $\Box\phi \in e$ and $\Diamond\Box\phi \in a'$.

Suppose now $a \overset{\diamond}{\not\sim} a'$. Then there exist $\Box\phi \in a$, $\Box\psi \in a'$ with $\{\Box\phi, \Box\psi\}$ not consistent, so $\vdash \Box\phi \to \Diamond\neg\psi$, so $\vdash \Diamond\Box\phi \to \Diamond\neg\psi$. But $\Box\psi \in a'$, so $\Diamond\Box\phi \notin a'$. ■

We denote $\overline{a}$ as the equivalence class of an element a under $\overset{\diamond}{\sim}$ and call it a $\Diamond$-class.

Observation 3.1.3 *Note that in the canonical model a $\Diamond$-class is always downward directed under $\overset{\diamond}{\longrightarrow}$, but not necessarily upwards directed, as we can see in Example 5.1.1.*

Let F be as in Lemma 3.1.2. Then for all $a, b \in X$, $a \overset{\diamond}{\sim} b$ iff $a \cap F = b \cap F$.

Definition 3.1.4 *Let $char(\overline{a}) = a \cap F$. We note that $char(\overline{a})$ depends only on $\overline{a}$ and not on the particular representative a.*

Note also that if $\Diamond\Box\psi, \Diamond\Box\phi \in char(\overline{a})$, then $\Diamond\Box(\psi \wedge \phi) \in char(\overline{a})$: Let $a, b \in \overline{a}$ be such that $\Box\psi \in a$ and $\Box\phi \in b$. As there exists $c \in \overline{a}$ with $a, b \overset{\diamond}{\longrightarrow} c$, then $\Box(\psi \wedge \phi) \in c$. So, $\Diamond\Box(\psi \wedge \phi) \in char(\overline{a})$.

It immediately follows by compactness that there exists $c \in \bar{a}$ so that if $\Diamond\Box\psi \in char(\bar{a})$, then $\Box\psi \in c$. This c is a minimal element in $\bar{a}$ (not usually unique).

Let us now distinguish maximal upwards and downwards closed sets in X, the pseudo-points. Later, the pseudo-points of X will be identified with points in a directed space expansion of X.

Definition 3.1.5 *Given a Kripke model $\mathcal{N} = \langle Y, \xleftrightarrow{L}, \xrightarrow{\Diamond} \alpha\rangle$ for $\mathcal{L}$ and a $\Diamond$-class $\bar{a}$ in Y, we say:*

1. *$A \subseteq \bar{a}$ is* directed above *iff for any two elements a and b in A, there exists $c \in A$ such that $c \xrightarrow{\Diamond} a$ and $c \xrightarrow{\Diamond} b$. This element c is called an* upper bound *for a and b in A;*

2. *A* pseudo-point *(Pp) A of $\bar{a}$ is any directed above $A \subseteq \bar{a}$ which is maximal upwards closed, i.e., for every $w \in \bar{a} - A$, there exists $u \in A$ such that w and u have no upper bound. We denote pseudo-points of a $\Diamond$-class by capital Roman letters.*

Notation 3.1.6 *Given a $\Diamond$-class $\bar{a}$ and $b \xrightarrow{\Diamond} b' \in \bar{a}$, let*

1. *$min_{\Diamond}\bar{a}$ be the set of $\Diamond$-minimal elements of $\bar{a}$;*

2. *$max_{\Diamond}\bar{a}$ be the set of all $\Diamond$-maximal elements of $\bar{a}$;*

3. *The interval $[b', b]$ be the set $\{x | b \xrightarrow{\Diamond} x \xrightarrow{\Diamond} b'\}$.*

DMP's example A (in [1]) shows that the set of $\Diamond$-minimal and $\Diamond$-maximal elements are not necessarily singletons.

In the following, *[c]* will mean that the next result holds in the canonical model but not necessarily in arbitrary Kripke models.

Lemma 3.1.7 *[c] Given a $\Diamond$-class $\bar{a} \subseteq X$,*

1. *For all $b \in \bar{a}$, there exists a Pp $A \subseteq \bar{a}$ with $b \in A$;*

2. *There exists a $\Diamond$-minimal element $a_0 \in \bar{a}$;*

3. *For each Pp A in $\bar{a}$, there exists a $\Diamond$-maximal element $a_1 \in A$;*

4. *An interval* $[a_1, a_0]$ *is a Pp of* $\bar{a}$ *iff* a_1 *is a* $\Diamond$*-maximal element of* $\bar{a}$ *and* a_0 *is a* $\Diamond$*-minimal element of* $\bar{a}$*;*

5. *For each Pp A, there exists a* $\Diamond$*-maximal element* $a_1 \in A$ *and a* $\Diamond$*-minimal element* $a_0 \in A$, $A = [a_1, a_0]$.

Proof: 1) Let $\mathcal{A}$ be the set of all directed above subsets that contain a. Now, $\subseteq$ is a partial order over $\mathcal{A}$, and a union of a chain in $\mathcal{A}$ is again an element of $\mathcal{A}$. By Zorn's Lemma, there is a maximal chain and its union is a *Pp* A of $\bar{a}$ with $a \in A$.

2) and 3) are a straightforward application of compactness. We show 2):

Let $F = \{\Box\psi | (\exists a \in \bar{a})(\Box\psi \in a)\}$. Then F is a consistent set of formulas: Let $\{\Box\psi_1, \ldots, \Box\psi_n\} \subseteq F$. There exists $a_1, \ldots, a_n \in \bar{a}$ such that $\Box\psi_i \in a_i \in \bar{a}$ for all $1 \leq i \leq n$. By directeness there exists $b \in \bar{a}$ such that $(\forall 1 \leq i \leq n)(a_i \stackrel{\Diamond}{\longrightarrow} b)$. Hence $\Box\psi_i \in B$ for all $1 \leq i \leq n$ and $\{\Box\psi_1, \ldots, \Box\psi_n\}$ is consistent. By compactness, n is consistent, thus there exists $a_0 \in X$ so that $F \subseteq a_0$.

Because $char(\bar{a}) \subseteq a_0$, $a_0 \in \bar{a}$. Besides, for all $a \in \bar{a}$, as $\{\Box\psi | \Box\psi \in a\} \subseteq F$, then $a \stackrel{\Diamond}{\longrightarrow} a_0$ hence a_0 is a $\Diamond$–minimal element in $\bar{a}$.

4) and 5) follows from the fact that an interval is upwards and downwards closed set and a *Pp* is a maximal upwards and downwards closed set.■

Observe that a_0 is $\Diamond$ minimal in $\bar{a}$ iff for all formula ψ, $\Diamond\Box\psi \to \Box\psi \in a_0$.

Definition 3.1.8 *Given two sets* $Z_1, Z_2 \subseteq X$ *let* $Z_1 \leq_L Z_2$ *if for all* $a_1 \in Z_1$ *there exists* $a_2 \in Z_2$ *so that* $a_1 \stackrel{L}{\longleftrightarrow} a_2$ *(*$\leq_L$ *is a quasi-order). The comparability relation,* $comp_K$*, given by the quasi-order* $\leq_L$ *is called the relation of K-comparability, i.e.,* $comp_K(Z, W)$ *if* $Z \leq_L W$ *or* $W \leq_L Z$.

We say that two sets $Z_1, Z_2 \subseteq X$ *are K-linked (*$Z_1 \equiv_L Z_2$*) if* $Z_1 \leq_L Z_2$ *and* $Z_2 \leq_L Z_1$.

Lemma 3.1.9 *[c] Let* $\bar{a}$ *be a* $\Diamond$*-class in X. Let* $\mathcal{C}$ *be a chain of Pps of* $\bar{a}$ *totally ordered by* $\leq_L$*. Then there exists a Pp* A_0 *of* $\bar{a}$ *so that for all* $A \in \mathcal{C}$, $A \leq_L A_0$.

Proof: The set $F = char(\bar{a}) \cup \{\Diamond L\phi | (\exists a \in \cup\mathcal{C})(\phi \in a)\}$ is consistent. So, there exist a_0 and a *Pp* A of $\bar{a}$ so that $F \subseteq a_0$ and $a_0 \in A$. Now, for all $a_1 \in \cup\mathcal{C}$, there exists a_2 such that $char(\bar{a}) \cup \{L\phi | \phi \in a_1\} \subseteq a_2$. So $a_0 \stackrel{\Diamond}{\longrightarrow} a_2 \stackrel{L}{\longleftrightarrow} a_1$, hence A must be a maximal element for the chain $\mathcal{C}$.■

Definition 3.1.10 *A Pp A is a K*-superset *of* $\bar{a}$ *if for all Pp* A' *of* $\bar{a}$, $comp_K(A, A')$ *implies* $A' \leq_L A$.

Remark: Note that for every Pp B of $\bar{a}$, there exists a K–superset A of $\bar{a}$ such that $B \leq_L A$.

Definition 3.1.11 *Let* $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\diamond}{\longrightarrow}, \alpha\rangle$ *be a Kripke model for* $\mathcal{L}$. *Let* $Z \subseteq Y$. *Define*

- $K(Z) = \{y \in Y | (\exists z \in Z)(y \overset{L}{\longleftrightarrow} z)\}$, *the closure of* Z *under* K;
- Z *is* K*-closed if* $Z = K(Z)$;
- *If* $y \in X$, *let* $K(y)$ *be* $K(\{y\})$. *Call* $K(y)$ *the* K*-segment of* y;
- U *is a* K*-segment iff* $(\exists y \in U)(U = K(y))$.

Note that for any $Z \subseteq Y$, $K(Z) = K(K(Z))$. Also, note that U is a K–segment iff $(\forall y \in U)(U = K(y))$.

Denote the set of K–segments in Y by $\mathcal{K}$ and the elements of $\mathcal{K}$ by lowercase Greek letters $\{\zeta, \eta, \xi, \ldots\}$, with or without sub–index.

Definition 3.1.12 *Let* $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\diamond}{\longrightarrow}, \alpha\rangle$ *be a Kripke model for* $\mathcal{L}$, *let* $Z \subseteq Y$. *Define*

- $\preceq$, *a quasi-order over* $\mathcal{K}$, *by* $\zeta \preceq \eta$ *iff for all* $y \in \zeta$ *there exists* $x \in \eta$ *so that* $x \overset{\diamond}{\longrightarrow} y$;
- $\zeta \bigtriangledown \eta$ *iff* $\zeta \preceq \eta \wedge \eta \preceq \zeta$;
- $T \subseteq \mathcal{K}$ *is an* η*-chain if*
  - T *is linearly ordered with respect to* $\preceq$;
  - η *is a minimum (with respect to* $\preceq$*) in* T;
  - $\eta \in T$.

Note that if we were in a *directed space*, $\preceq$ would be a partial order and $\bigtriangledown$ would be equality. However, in a Kripke model, $\preceq$ is only a quasi–order in general and $\bigtriangledown$ is an equivalence relation. Moreover, in a directed space, a K–segment is an open set.

3.2 A Directed and Above Directed Model for $\mathcal{L}$

We show now that there exists a directed and above directed Kripke model $\mathcal{M}'$ which is a strong expansion of the canonical model $\mathcal{M}_{\mathcal{L}}$.

Lemma 3.2.1 *[c] Given two* $\Diamond$*-classes* $\bar{a}$ *and* $\bar{b}$*,* $\bar{a} \leq_L \bar{b}$ *iff* $K(min_\Diamond \bar{a}) \cap \bar{b} \neq \emptyset$.

Proof: Suppose $K(min_\Diamond \bar{a}) \cap \bar{b} \neq \emptyset$. Let $b \in K(min_\Diamond \bar{a}) \cap \bar{b}$ and $a_0 \in min_\Diamond \bar{a}$ with $a_0 \stackrel{L}{\longleftrightarrow} b$. As $a_0 \in min_\Diamond \bar{a}$, *for all* $a \in \bar{a}$, $a \stackrel{\Diamond}{\longrightarrow} a_0$. Now, as for all $a \in \bar{a}$, we have, $a \stackrel{\Diamond}{\longrightarrow} a_0 \stackrel{L}{\longleftrightarrow} b$. By applying Definition 2.1.2 part 3, for all $a \in \bar{a}$, there exists $b' \in \bar{b}$ such that $a \stackrel{L}{\longleftrightarrow} b' \stackrel{\Diamond}{\longrightarrow} b$. Thus, for all $a \in \bar{a}$, there exists $b' \in \bar{b}$ such that $a \stackrel{L}{\longleftrightarrow} b'$, i.e. $\bar{a} \leq_L \bar{b}$.

For the converse, if $\bar{a} \leq_L \bar{b}$, then for all $a \in \bar{a}$, there exists $b \in \bar{b}$ such that $a \stackrel{L}{\longleftrightarrow} b$. In particular, for all $a_0 \in min_\Diamond \bar{a}$, there exists $b \in \bar{b}$ such that $a_0 \stackrel{L}{\longleftrightarrow} b$. So $b \in K(min_\Diamond \bar{a})$, hence, $K(min_\Diamond \bar{a}) \cap \bar{b} \neq \emptyset$.■

Corollary 3.2.2 *[c] Given two* $\Diamond$*-classes* $\bar{a}$ *and* $\bar{b}$*,*

1. $\bar{a} \equiv_L \bar{b}$ *iff* $K(min_\Diamond \bar{a}) \cap \bar{b} \neq \emptyset$ *and* $K(min_\Diamond \bar{b}) \cap \bar{a} \neq \emptyset$*;*
2. $(\bar{a} <_L \bar{b})$ *iff* $K(min_\Diamond \bar{a}) \cap \bar{b} \neq \emptyset$ *and* $K(min_\Diamond \bar{b}) \cap \bar{a} = \emptyset$*;*
3. $\neg(comp_K(\bar{a}, \bar{b})$ *iff* $K(min_\Diamond \bar{a}) \cap \bar{b} = \emptyset$ *and* $K(min_\Diamond \bar{b}) \cap \bar{a} = \emptyset$.

Lemma 3.2.3 (M Axiom Lemma) *[c] Given two* $\Diamond$*-classes* $\bar{a}$ *and* $\bar{b}$ *in* X*. Given* $a_0, a_1 \in \bar{a}$ *and* $b_0 \in \bar{b}$*, if* $a_1 \stackrel{\Diamond}{\longrightarrow} a_0 \stackrel{L}{\longleftrightarrow} b_0$*, then there exists* $b_1 \in \bar{b}$ *such that* $[a_0, a_1] \leq_L [b_0, b_1]$.

Proof: Let $U = \{y \in \bar{b} | (\exists x \in [a_0, a_1])(x \stackrel{L}{\longleftrightarrow} y)\}$.

Since $a_1 \stackrel{\Diamond}{\longrightarrow} x$ for all $x \in [a_0, a_1]$, we have $a_1 \supseteq \{\Diamond L\psi | (\exists y \in U)(\psi \in y)\}$. As for all $x \in [a_0, a_1]$, $x \stackrel{\Diamond}{\longrightarrow} a_0 \stackrel{L}{\longleftrightarrow} b_0$, then, $a_1 \supseteq \{\Box L \Diamond \eta | \eta \in b_0\}$. So,

$$a_1 \supseteq \{\Diamond L\psi | (\exists y \in U)(\psi \in y)\} \cup \{\Box L \Diamond \eta | \eta \in b_0\}.$$

or, equivalently, since for all formulas ψ, $L\psi \equiv KL\psi$,

$$a_1 \supseteq \{\Diamond KL\psi | (\exists y \in U)(\psi \in y)\} \cup \{\Box L \Diamond \eta | \eta \in b_0\}.$$

Let $F = \{\Diamond KL\psi | (\exists y \in U)(\psi \in y)\} \cup \{\Diamond\eta | \eta \in b_0\}$. Let us show that F is consistent: For any $n \in \mathbb{N}$, for all $\{\Diamond KL\psi_1 \wedge \ldots \wedge \Diamond KL\psi_n\} \in \{\Diamond KL\psi | (\exists y \in U)(\psi \in y)\}$ and $\Diamond\eta \in \{\Diamond\eta | \eta \in b_0\}$, by axiom M_n,

$$\vdash (\Box L\Diamond\eta \wedge \Diamond KL\psi_1 \wedge \ldots \wedge \Diamond KL\psi_n) \Rightarrow L(\Diamond\eta \wedge \Diamond KL\psi_1 \wedge \ldots \wedge \Diamond KL\psi_n).$$

Now $\Box L\Diamond\eta \wedge \Diamond KL\psi_1 \wedge \ldots \wedge \Diamond KL\psi_n \in a$ then $L(\Diamond\eta \wedge \Diamond KL\psi_1 \wedge \ldots \wedge \Diamond KL\psi_n) \in a$. So $\Diamond\eta \wedge \Diamond KL\psi_1 \wedge \ldots \wedge \Diamond KL\psi_n$ is consistent. By compactness, it follows that F is consistent and therefore for some $b_1 \in X$, $b_1 \supseteq F$.

As $\{\Diamond\eta | \eta \in b_0)\} \subseteq b_1$, $b_1 \xrightarrow{\Diamond} b_0$. Now, for all $x \in [a_0, a_1]$, there exists $y \in U$ so that $y \xleftrightarrow{L} x$. Since $b_1 \supseteq F$, there exists z so that $b_1 \xrightarrow{\Diamond} z \xleftrightarrow{L} y$. Now, applying the transitivity and symmetry of $\xleftrightarrow{L}$, for all $x \in [a_0, a_1]$ there exists z so that $b_1 \xrightarrow{\Diamond} z \xleftrightarrow{L} y$. So, for $[b_0, b_1]$, we have $[a_0, a_1] \leq_L [b_0, b_1]$.■

Lemma 3.2.4 *Given two $\Diamond$-classes $\bar{a}$ and $\bar{b}$,*

1. *If $\bar{a} \leq_L \bar{b}$, then for all Pp A of $\bar{a}$ there exists a Pp B of $\bar{b}$ such that $A \leq_L B$*
2. *If $\bar{a} \equiv_L \bar{b}$, then for all K-supersets A of $\bar{a}$ there exists a K-superset B of $\bar{b}$ so that $A \equiv_L B$ (and vice versa).*

Proof: *Part 1:* Suppose that $A = [a_0, a_1]$. Since $\bar{a} \leq_L \bar{b}$, there exists $b_0 \in \bar{b}$ such that $a_0 \xleftrightarrow{L} b_0$. Also, by M Axiom Lemma, there exists $b_1 \in \bar{b}$ such that $[a_0, a_1] \leq_L [b_0, b_1]$. Let B be a Pp such that $[b_0, b_1] \subseteq B$. Then $A = [a_0, a_1] \leq_L [b_0, b_1] \leq_L B$.

Part 2: Given a superset A of $\bar{a}$, there exists B a Pp of $\bar{b}$ such that $A \leq_L B$. Also, there exists A' a Pp of $\bar{a}$ such that $B \leq_L A'$. Since $comp_K(A, A')$ and A is a K-superset, we get $A' \leq_L A$. Hence, $B \leq_L A$ and $A \equiv_L B$.■

Notation 3.2.5 *Given a $\Diamond$-class $\bar{a}$, $\mathrm{UC}(\bar{a})$, the upper core of $\bar{a}$, is the intersection of all Pp of $\bar{a}$.*

Definition 3.2.6 *Given a $\Diamond$-class $\bar{a}$, $a_0 \in min_\Diamond\bar{a}$, A a Pp of $\bar{a}$ and $b \in K(a_0)$, let $S_{b,A}$ be the set of all maximal (w.r.t. $\leq_L$) Pps B such that $b \in B$.*

Lemma 3.2.7 *$S_{b_0,A}$ as given in Definition 3.2.6. is non-empty.*

Proof: $S_{b_0,A}$ *is non-empty:* Let $[a_0, a_1] = A$. As $a_0 \overset{L}{\longleftrightarrow} b$, by M Axiom Lemma, there exists $b_1 \in \bar{b}$ so that $A \leq_L [b, b_1]$. As $[b, b_1]$ is contained in a Pp of C of $\bar{b}$, clearly $A \leq_L C$ and $b \in C$.

Pick U a maximal (w.r.t. $\leq_L$) chain of Pps such that for all $C \in U$, $b \in C$.

Consider the set $F = \{\Diamond\phi | \phi \in b\} \cup \{\Diamond L\psi | (\exists c \in \cup U)(\psi \in c)\}$. By compactness, F is consistent. So, the set $Cons = \{c_1 \in \bar{b} | F \subseteq c_1\}$ is non-empty.

As for all $c_1 \in$ $Cons$, $\{\Diamond\phi | \phi \in b\} \in c_1$, then for all $c_1 \in$ $Cons$, $c_1 \overset{\Diamond}{\longrightarrow} b$. Let $c_1 \in$ $Cons$ and let B be a Pp's such that $c_1 \in B$. As $\{\Diamond L\psi | (\exists c \in \cup U)(\psi \in c)\} \in c_1$, then for all $C \in U$, $C \leq_L B$. As U is a maximal (w.r.t. $\leq_L$) chain of Pps, B is a maximal (w.r.t. $\leq_L$) Pp.■

Observation 3.2.8 *Let $\bar{a}$ be a $\Diamond$-class, $a_0 \in min_\Diamond \bar{a}$ and A be a Pp of $\bar{a}$. Let $T_{a_0,A} = \cup\{S_{b_0,A} | b_0 \in K(a_0)\}$. Then for all $b \in K(a_0)$, there exists $B \in T_{a_0,A}$ such that $b \in B$.*

We show in Proposition 3.2.11 that there exists a directed and above directed Kripke model strong expansion of the canonical model. Example 5.1.2 illustrates Proposition 3.2.11.

Definition 3.2.9 *Given a $\Diamond$-class $\bar{a}$ and $a_0 \in \bar{a}$, we say that $C \subseteq \bar{a}$ is an a_0-chain if there exists $a_1 \in max_\Diamond(\bar{a})$ and C is a maximal totally ordered subset of $[a_0, a_1]$.*

Lemma 3.2.10 *[c] Let $\bar{a}$ and $\bar{b}$ be two $\Diamond$-classes. Let $a_1 \in max_\Diamond(\bar{a})$, $b_1 \in max_\Diamond(\bar{b})$. Suppose that the two intervals $[a_0, a_1], [b_0, b_1]$ are so that $[a_0, a_1] \leq_L [b_0, b_1]$. Then for all a_0-chains $C \subseteq [a_0, a_1]$, there exists a b_0-chain $C' \subseteq [b_0, b_1]$ such that $C \leq_L C'$.*

Proof: For all $x_1 \overset{\Diamond}{\longrightarrow} x_2 \overset{\Diamond}{\longrightarrow} \ldots \overset{\Diamond}{\longrightarrow} x_n$ in C, by repeatedly applying the Cross Axiom, we obtain a chain $y_1 \overset{\Diamond}{\longrightarrow} y_2 \overset{\Diamond}{\longrightarrow} \ldots \overset{\Diamond}{\longrightarrow} y_n$ in $[a_0, a_1]$ so that for all $1 \leq i \leq n$, $x_i \overset{L}{\longleftrightarrow} y_i$.

Now, for all $\psi_i \in x_i$, the formula $(L\psi_1 \wedge \Diamond(L\psi_2 \wedge \Diamond(L\psi_n)\ldots)) \in y_1$. By applying compactness, the set

$$F = \{(L\psi_1 \wedge \Diamond(L\psi_2 \wedge \Diamond(L\psi_n)\ldots)) | (\exists(y_1 \overset{\Diamond}{\longrightarrow} y_2 \overset{\Diamond}{\longrightarrow} \ldots \overset{\Diamond}{\longrightarrow} y_n \subseteq [a_0, a_1]) \\ (\exists(x_1 \overset{\Diamond}{\longrightarrow} x_2 \overset{\Diamond}{\longrightarrow} \ldots \overset{\Diamond}{\longrightarrow} x_n \subseteq C)(\forall 1 \leq i \leq n)(x_i \overset{L}{\longleftrightarrow} y_i) \wedge \psi_i \in x_i\}$$

is consistent. Therefore, there exists a b_0-chain C' contained in $[b_0, b_1]$ such that for all $x \in C$, there exists $y \in C'$ with $x \overset{L}{\longleftrightarrow} y$.■

Proposition 3.2.11 *Given the canonical model* $\mathcal{M}_{\mathcal{L}} = \langle X, \xrightarrow{\diamond}, \xleftrightarrow{L} \alpha\rangle$, *there exists a directed and above directed Kripke model* $\mathcal{M}' = \langle Y', \xrightarrow{\diamond} *, \xleftrightarrow{L} *, \alpha*\rangle$ *that is a strong expansion of* $\mathcal{M}_{\mathcal{L}}$.

Proof: Let Pp be the set of all *Pps* of X.

For $\eta \in \mathcal{K}$, we say that η is of type 1 ($\eta \in T1$) if for all $a \in \eta$, a is not a $\diamond$–minimal element of a $\diamond$–class in X, otherwise, η is of type 2 ($\eta \in T2$). Also, we say that $a \in X$ is of type 1 (type 2) iff $K(a)$ is of type 1 (type 2).

Note that if $a \xleftrightarrow{L} b$, then a and b are of the same type.

We will now define the set $Y' \subseteq X \times \text{Pp} \times \mathcal{P}(\text{Pp})$ so that the first coordinate projection F from Y' to X gives us an expansion of $\mathcal{M}_{\mathcal{L}}$.

For all $\eta \in \mathcal{K}$ if η is of type 1, let P_η be the set of all S such that there exists $a \in X$ and C an a–chain such that

$$S = \{B | C \leq_L B \wedge B \cap \zeta \neq \emptyset\}$$

Define $Y_1' = \{(a, A, S) | (\exists \eta \in T1)(S \in P_\eta \wedge A \in S \wedge a \in A \cap \eta)\}$.

Note that for all $S \in P_\eta$, we have $F[\{(a, A, S) | A \in S \wedge a \in A \cap \eta)\}] = \eta$: Let $a_0 \in \eta$ and C be an a_0–chain so that $T = \{K(x) | x \in C\}$. Let $C \subseteq [a_0, a_1]$. By M Axiom Lemma 3.2.3, for all $b_0 \in \eta$, there exists an interval $[b_0, b_1]$ with $[a_0, a_1] \leq_L [b_0, b_1]$. As $C \subseteq [a_0, a_1]$, then $C \leq_L [b_0, b_1]$. So, for all $a \in C$, there exists $b \in [b_0, b_1]$ with $a \xleftrightarrow{L} b$. As $T = \{K(x) | x \in C\}$, then for all $\zeta \in T$, there exists $b \in [b_0, b_1] \cap \zeta$. If $B \in \text{Pp}$ is such that $[b_0, b_1] \subseteq B$, then $B \cap \zeta \neq \emptyset$. So, for all $\zeta \in T$, $B \cap \zeta \neq \emptyset$. Hence, for all $b_0 \in \eta$, there exists $B \in S$ so that $(b, B, S) \in Y_1'$.

For all $\eta \in T2$,

- Given $a_0 \in min_\diamond(\bar{a}) \cap \eta$ and a *Pp* A such that $a_0 \in A$, let $S = T_{a_0,A}$ as defined in 3.2.8.
- For all $\eta \in T2$, let $S \in P_\eta$ if for some $a_0 \in \eta$, for some $A \in$Pp, $S = T_{a_0,A}$.

Define $Y_2' = \{(a, A, S) | (\exists \eta \in T2)(S \in P_\eta \wedge A \in S \wedge a \in A \cap \eta)\}$.

It follows from the definition of each $T_{a_0,A}$ (Observation 3.2.8) that for all $\eta \in T2$, for all $S \in P_\eta$, $F[\{(a, A, S) | A \in S \wedge a \in A \cap \eta)\}] = \eta$.

Let $Y' = Y_1' \cup Y_2'$.

Let us now define the relations $\overset{L}{\longleftrightarrow}*$ and $\overset{\Diamond}{\longrightarrow}*$ over Y' as

- $(a, A, S) \overset{L}{\longleftrightarrow} *(b, B, S)$;
- $(a, A, S) \overset{\Diamond}{\longrightarrow} *(b, A, S')$ iff $a \overset{\Diamond}{\longrightarrow} b$ and $S' \subseteq S$.

For all $A \in$Pp, let $A' = \{(a, A, S) \in Y' | a \in A\}$ Let $F((a, A, S)) = a$ and for all literal R, let $(a, A, S) \in \alpha * (R)$ iff $a \in \alpha(R)$.

Let us now show that $\mathcal{M}'$ is a directed and above directed Kripke model strong expansion of $\mathcal{M}_{\mathcal{L}}$ and F is a fiber mapping. We show:

1. For all $a \overset{L}{\longleftrightarrow} b$ and $(a, A, S) \in Y'$, there exists B such that $(a, A, S) \overset{L}{\longleftrightarrow} *(b, B, S)$;
2. For all $a_1 \overset{\Diamond}{\longrightarrow} a_0$ and $(a_1, A, S) \in Y'$, there exists S' such that $(a_1, A, S) \overset{\Diamond}{\longrightarrow} *(a_0, A, S')$;
3. For all $A \in$Pp, there exists a $\Diamond$–maximal element (a, A, S) in A' in which for all $x \in A'$, $(a, A, S) \overset{\Diamond}{\longrightarrow} *x$;
4. For all $A \in$Pp, there exists a $\Diamond$–minimal element (a_0, A, S) in A' in which for all $x \in A'$, $x \overset{\Diamond}{\longrightarrow} *(a_0, A, S)$;
5. $\overset{L}{\longleftrightarrow} *$ is an equivalence relation;
6. $\overset{\Diamond}{\longrightarrow} *$ is transitive and reflexive;
7. If $(a_1, A, S) \overset{\Diamond}{\longrightarrow} *(a_0, A, S') \overset{L}{\longleftrightarrow} *(b_0, B, S')$, there exists $(b_1, B, S) \in Y'$ such that $(a_1, A, S) \overset{L}{\longleftrightarrow} *(b_1, B, S) \overset{\Diamond}{\longrightarrow} *(b_0, B, S')$.

Items 1 and 2 above correspond to showing that Definition 2.1.4 part 3 and part 5 hold – the other clauses follow directly from the definition of $\alpha*$, $\overset{\Diamond}{\longrightarrow} *$ and $\overset{L}{\longleftrightarrow} *$. Items 3 to 7 are to show that $\mathcal{M}'$ is directed and above directed. Because items 5, 6 and 7 imply that $\mathcal{M}'$ is a Kripke model and each A' is a $\Diamond$–class, then items 3 and 4 imply that $(Y', \overset{\Diamond}{\longrightarrow} *)$ is directed and above directed. We have then:

1 and 5: Follows from the fact that the first coordinate projection of $K(((a, A, S))$ is $K(a)$.

2: *If* $\eta = K(a_1)$ *and* $\zeta = K(a_0)$ *are of type* $T1$, consider a *Pp* B, $b \in B$ and C_b a b–chain so that S is generated by the η–chain $T = \{K(x)|x \in C_b\}$. Let D be an a_1–chain so that $C_b \leq_L D$ (see Lemma 3.2.10). Let C be an a_0–chain such that $D \subseteq C$. Consider $S' \in P_\zeta$ given by the η-chain $T = \{K(y)|y \in C\}$. We will show now that $S' \subseteq S$: Let $B' \in S'$. Then there exists $[b'_1, b'_0] \subseteq B'$ with $C \leq_L [b'_1, b'_0]$. As $C_b \leq_L C$, then $C_b \leq_L [b'_1, b'_0]$. Thus for all $\zeta \in T$, there exists $b' \in B' \cap \zeta$. So, for all $\zeta \in T$, $B' \cap \zeta \neq \emptyset$, hence $B' \in S$.

If $\eta \in T1$ *and* $\zeta \in T2$, let b_0 be a $\diamond$–minimal element of $\overline{b_0}$ with $b_0 \in \zeta$. Let B be a *Pp* so that $b_0 \in B \in S$.

Suppose that $S = \cap\{A|(\exists\zeta \in T)(A \cap \zeta \neq \emptyset)\}$ and consider $S' = S_{b_0,A}$. We will show now that $S' \subseteq S$: If $C \in S'$, then $B \leq_L C$. Now, $B \in S$, so for all $\zeta' \in T$, there exists $b \in B \cap \zeta'$. Now, $B \leq_L C$ implies that there exists $c \in C$ with $c \overset{L}{\longleftrightarrow} b$. Thus, for all $\zeta' \in T$, there exists $c \in C$ with $c \in \zeta'$, so $C \in S$.

If η *and* ζ *are of type* $T2$, let $S = S_{b_1,A}$. Let $c_0 \in \zeta$ be a $\diamond$–minimal element for some $\diamond$–class $\bar{c}$. Let $C \in S$ be so that $c_0 \in C$ and consider $S' = S_{c_0,C}$. We will show now that $S' \subseteq S$: Suppose that $D \in S'$. Then $C \leq_L D$. Also $C \in S$ implies $B \leq_L C$. So, $B \leq_L D$. The maximality conditions over the elements of S' also follows over S. Thus $D \in S'$ implies $S \in S$.

3: For all $A \in$Pp, let $a_1 \in max_\diamond(A)$ and $S = \{A \in \text{Pp}|A \cap K(a_1 \neq \emptyset\}$. Let $(a, A, S') \in A'$. As $S' \supseteq S$, then $(a_1, A, S) \overset{\diamond}{\longrightarrow} *(a, A, S')$.

4: Let $a_0 \in min_\diamond(A)$ and let $S' = T_{a_0,A}$. We have that (a_0, A, S') is a $\diamond$–minimal element of A': For all $(a, A, S) \in A'$, as $a_0 \in min_\diamond(A)$, we have $a \overset{\diamond}{\longrightarrow} a_0$. Now, as for all $B \in S'$, $A \leq_L B$ and $B \in T_{a_0,A}$, then $B \in S$. So, $S' \subseteq S$ and therefore, $(a, A, S) \overset{\diamond}{\longrightarrow} *(a_0, A, S')$.

6: Follows because $\overset{\diamond}{\longrightarrow}$ and $\subseteq$ are transitive and reflexive relations.

7: Let b_1 be so that $a_1 \overset{L}{\longleftrightarrow} b_1 \overset{\diamond}{\longrightarrow} b_0$. As $b_1 \in K(a_1)$, $F(\{(c, C, S)\} = K(a_1)$ and $B \in S' \subseteq S$, then $(a_1, A, S) \overset{L}{\longleftrightarrow} *(b_1, B, S) \overset{\diamond}{\longrightarrow} *((b_0, B, S'))$.■

Let us call $\mathcal{M}'$ the canonical directed model, the DA–model and note that *we do not prove* that all $\diamond$–maximal elements in the distinct *Pp*'s of the canonical directed model are K–linked. This is not necessarily true in the canonical model and just reflects the fact we impose the set of all points is an open.

Observation 3.2.12 *In the canonical directed model any $\Diamond$-class is such that its image under the fiber mapping F is a Pp of a $\Diamond$-class in X. Conversely, for any Pp A of a $\Diamond$-class in X, there exists a $\Diamond$-class $\overline{A}$ in $\mathcal{M}'$ so that $F(\overline{A}) = A$.*

3.3 A Directed Space for $\mathcal{L}$

Now, in order to obtain a directed space strong expansion of the canonical model, we must focus the expansion in the set of points of Y', the underlying set of $\mathcal{M}'$, that satisfy $x \overset{\diamond}{\longleftrightarrow} *y$, and $x \neq y$ or $x \overset{L}{\longleftrightarrow} *y$ with x and y in the same $\diamond$-class. We will use a bulldozer-like procedure in order to force the relation $\overset{\diamond}{\longrightarrow}$ in our model to be a partial order and to make sure that any K-segment intersects any $\diamond$-class at most once.

Definition 3.3.1 *Given a partially ordered set $(X, \leq)$ and a quasi-ordered set $(Y, \preceq)$, we say that a mapping f from X to Y is an order bisimulation if*

1. *If $x \leq y$ then $f(x) \preceq f(y)$ and;*
2. *If $y' \preceq f(x)$, there exists $y \in X$ with $y \leq x$ and $f(y) = y'$*

We say that f is an embedding iff f is an order bisimulation and onto.

Definition 3.3.2 *Given a set Z, P an equivalence relation on Z, and F a mapping from a subset of Z onto a subset of Z, we say that F respects P if $f(x) = y$ always implies $(x, y) \in P$.*

Recall that $a \searrow = \{c \in \bar{a} | a \overset{\diamond}{\longrightarrow} c\}$.

Definition 3.3.3 *Let $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\diamond}{\longrightarrow}, \alpha\rangle$ be a Kripke model for $\mathcal{L}$, $\bar{b}$ a $\diamond$-class in $\mathcal{N}$ and $a, a' \in \bar{b}$. We say that a card a' if there exists an embedding G of $a \searrow$ onto $a' \searrow$ that respects $\overset{L}{\longleftrightarrow}$.*

Observe that if G is such an embedding from $a \searrow$ onto $a' \searrow$ and c, c' satisfy $G(c) = c'$, then c *card* c'.

Lemma 3.3.4 *Let $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\diamond}{\longrightarrow}, \alpha\rangle$ be a Kripke model for $\mathcal{L}$, let $\bar{b}$ be a $\diamond$-class in $\mathcal{N}$ and $a, a' \in \bar{b}$. Suppose that G is an embedding from $a \searrow$ onto $a' \searrow$ that respects $\overset{L}{\longleftrightarrow}$. Then for all $c \in a \searrow$, $c' \in a' \searrow$ and for all formula ψ, if $G(c) = c'$, then $c \models_{\mathcal{N}} \psi$ iff $c' \models_{\mathcal{N}} \psi$.*

Proof: Suppose ψ is a formula of smallest possible complexity such that $c \models_{\mathcal{N}} \psi$ and $c' \models_{\mathcal{N}} \neg\psi$. We prove a contradiction.

- If $\psi = A$ is an atomic formula: For all $x, y \in \bar{b}$, $x \models_{\mathcal{N}} A$, iff $y \models_{\mathcal{N}} A$. As $c, c' \in \bar{b}$, then $c \models_{\mathcal{N}} A$ iff $c' \models_{\mathcal{N}} A$.

- If $\psi = K\phi$ for some formula ϕ: Since c *card* c', $c \overset{L}{\longleftrightarrow} c'$. Hence $c \models_{\mathcal{N}} \psi$ iff $c' \models_{\mathcal{N}} \psi$;

- If $\psi = \Box\phi$ for some formula ϕ: Suppose $c \models_{\mathcal{N}} \Box\phi$ and $c' \models_{\mathcal{N}} \Diamond\neg\phi$. So for some $d' \in c' \searrow$, $d' \models_{\mathcal{N}} \neg\phi$. Because G is an embedding, G satisfies Definition 3.3.1 part 2. Hence, there exists $d \in c \searrow$ so that $G(d) = d'$. Now, $c \overset{\Diamond}{\longrightarrow} d$ implies $d \models_{\mathcal{N}} \phi$. Hence, $d \models_{\mathcal{N}} \phi$ and $d' \models_{\mathcal{N}} \neg\phi$.

  Suppose that $\Diamond\neg\phi \in c$, then there exists $d \in c \searrow$ such that $d \models_{\mathcal{N}} \neg\phi$. G is an embedding, so it satisfies Definition 3.3.1 part 1. So, $c' \overset{\Diamond}{\longrightarrow} G(d) = d'$. Now, $c' \overset{\Diamond}{\longrightarrow} d'$ implies $d' \models_{\mathcal{N}} \phi$. So, $d \models_{\mathcal{N}} \phi$ and $d' \models_{\mathcal{N}} \neg\phi$. Conclusion from the above that ψ is not of the form $\Box\phi$;

- If ψ or $\neg\psi$ is formed by a connective $\wedge$, the case follows straightforward.

Conclude then that for all formulas ψ, for all $c \in a \searrow$, $c' \in a' \searrow$, if c *card* c', then $c \models_{\mathcal{N}} \psi$ iff $c' \models_{\mathcal{N}} \psi$.■

Definition 3.3.5 *Let $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\Diamond}{\longrightarrow}, \alpha\rangle$ be a Kripke model for $\mathcal{L}$, let $Z \subseteq Y$. Define*

- *For each ζ in $\mathcal{K}$ $[\zeta] = \{\eta \in \mathcal{K} | \zeta \bigtriangledown \eta\}$;*

- *Let $[\mathcal{K}]$ be the set of equivalence classes under $\bigtriangledown$.*

Definition 3.3.6 *Given $[\eta]$ in $[\mathcal{K}]$ and a $\Diamond$-class $\bar{a}$, $[\eta] \otimes \bar{a}$ stands for $\cup\{\zeta \cap \bar{a} | \zeta \in [\eta]\}$.*

Definition 3.3.7 *Let $\mathcal{N} = \langle Y, \overset{L}{\longleftrightarrow}, \overset{\Diamond}{\longrightarrow}, \alpha\rangle$ be a Kripke model for $\mathcal{L}$ and $a, b \in Y$. Define:*

- b is predecessor of a *if $a \overset{\Diamond}{\longrightarrow} b$ and for all $c \in Y$, if $a \overset{\Diamond}{\longrightarrow} c \overset{\Diamond}{\longrightarrow} b$, then $b \overset{\Diamond}{\longrightarrow} c$.*

- b is predecessor of a in $K(b)$ *if $a \overset{\Diamond}{\longrightarrow} b$ and for all $c \in K(b)$, if $a \overset{\Diamond}{\longrightarrow} c \overset{\Diamond}{\longrightarrow} b$, then $b \overset{\Diamond}{\longrightarrow} c$.*

- b is minimum in $K(b)$ *if for all $c \in K(b)$, if $c \overset{\Diamond}{\longrightarrow} b$, then $b \overset{\Diamond}{\longrightarrow} c$.*

- b' is successor of a *if* $b' \xrightarrow{\Diamond} a$ *and for all* $c' \in Y$, *if* $b' \xrightarrow{\Diamond} c' \xrightarrow{\Diamond} a$, *then* $c' \xrightarrow{\Diamond} b'$.
- b' is successor of a in $K(b')$ *if* $b' \xrightarrow{\Diamond} a$ *and for all* $c' \in K(b')$, *if* $b' \xrightarrow{\Diamond} c' \xrightarrow{\Diamond} a$ *then* $c' \xrightarrow{\Diamond} b'$.
- b' is maximum in $K(b')$ *if for all* $c' \in K(b')$, *if* $b' \xrightarrow{\Diamond} c'$ *then* $c' \xrightarrow{\Diamond} b'$.

Note that we read $a \xrightarrow{\Diamond} b$ as b is less than a. This is contrary to the idea of quasi-order given by arrow $\xrightarrow{\Diamond}$, nevertheless, our intuition is that $x \xrightarrow{\Diamond} y$ represents two pairs *point-open*, $x = (a, U)$, $y = (a, V)$ and $x \xrightarrow{\Diamond} y$ iff $V \subseteq U$, i.e., V is less than U in the order $\subseteq$.

Lemma 3.3.8 *[c] Let $\bar{a}$ be a $\Diamond$-class in the canonical model $\mathcal{M}_{\mathcal{L}}$, $b \in \bar{a}$ with $K(b) \in [K(a)]$ and $K(b) \neq K(a)$. Then a has successors, predecessors, maximum and minimum in $K(b)$ w.r.t. the quasi-order $\xrightarrow{\Diamond}$.*

Proof: Let $A = \{\Box\psi | \psi \in a\}$, $B = \{\Diamond\phi | \phi \in a\}$. Let $C = \{L\chi | L\chi \in b\}$.

Given a maximal totally ordered (w.r.t. $\xrightarrow{\Diamond}$) set Γ with $a \in \Gamma$, then a successor, predecessor, maximum and minimum of a in $K(b)$ w.r.t. $\xrightarrow{\Diamond}$ that lie in Γ, respectively contain the sets

$$S = B \cup C \cup \{\Box\phi | (\exists b \in \Gamma \cap K(b))((b \xrightarrow{\Diamond} a) \wedge (\Box\phi \in b))\}$$

$$P = A \cup C \cup \{\Diamond\phi | (\exists b \in \Gamma \cap K(b))((a \xrightarrow{\Diamond} b) \wedge (\Diamond\phi \in b))\}$$

$$Max = B \cup C \cup \{\Box\phi | (\forall b \in \Gamma \cap K(b))((b \xrightarrow{\Diamond} a) \wedge (\Box\phi \in b))\}$$

$$Min = A \cup C \cup \{\Diamond\phi | (\forall b \in \Gamma \cap K(b))((a \xrightarrow{\Diamond} b) \wedge (\Diamond\phi \in b))\}$$

Let us see that S is consistent and for all x that contain S, $x \in \Gamma$ and x is a successor of a in $K(b)$. The other proofs follow a similar reasoning.

S is consistent: For all set of formulas $G = B \cup C \cup \{\Box\phi | (\exists b_1, \ldots, b_n \in \Gamma \cap K(b))(\forall b_i \in \{b_1, \ldots, b_n\})((b_i \xrightarrow{\Diamond} a) \wedge (\Box\phi \in b_i))\}$. Because $\{b_1, \ldots, b_n\}$ is finite, there exists $b_j \in \{b_1, \ldots, b_n\}$ such that $b_i \xrightarrow{\Diamond} b_j$ for all $b_i \in \{b_1, \ldots, b_n\}$. As $G \subseteq b_j$, then G is consistent and, by compactness, S is consistent.

If $S \subseteq x$, as Γ is a totally ordered set, then $x \in \Gamma$ and x is a successor of a in $K(b)$: As $B \subseteq x$, $x \xrightarrow{\Diamond} a$. As $C \subseteq x$, then $x \in K(b)$. Also, if $b \in \Gamma \cap K(b)$, then

$\{\Box\phi|\Box\phi \in b\} \subseteq x$, because $x \subseteq \cup\{\Box\phi|(b \in \Gamma \cap K(b))((b \stackrel{\Diamond}{\longrightarrow} a) \wedge (\Box\phi \in b))\}$. So, $b \stackrel{\Diamond}{\longrightarrow} x$, hence x is a successor of a in $K(b)$ iff $a \stackrel{L}{\longleftrightarrow} b$.■

Notation 3.3.9 *Given $\eta \in \mathcal{K}$ and $a \in \cup[\eta]$, denote the set of successors of a in $K(b)$, for all $K(b) \in [\eta] - K(a)$ by $Suc_{[\eta]}(a)$.*

Observation 3.3.10 *Given a Kripke model $\mathcal{M} = \langle Y, \stackrel{L}{\longleftrightarrow}', \stackrel{\Diamond}{\longrightarrow}', \alpha' \rangle$, for all $a, b \in Y$,*

1. *b is successor and predecessor of a in $K(b)$ iff $a \stackrel{\Diamond}{\longleftrightarrow} b$;*
2. *$a \stackrel{\Diamond}{\longleftrightarrow} b$ and $a \stackrel{L}{\longleftrightarrow} b$ implies $a \models \psi$ iff $b \models \psi$ for all formula ψ. Thus, if $\mathcal{M}$ is the canonical model, because each point in X is a set of maximal consistent formulas, $a = b$.*

Definition 3.3.11 *Given $\eta \in \mathcal{K}$, we say that $[\eta]$ is* regular *iff for all $\zeta \in [\eta]$, for all $a, b \in \zeta$, there exists an embedding (w.r.t. the quasi-order $\stackrel{\Diamond}{\longrightarrow}$) from $Suc_{[\eta]}(a)$ onto $Suc_{[\eta]}(b)$ that respects $\stackrel{L}{\longleftrightarrow}$.*

We must show, in order to obtain the expansion of the canonical model, that any $[\zeta] \in \mathcal{K}$ is regular.

Complete the set of atoms of $\mathcal{L}$ by adding new distinct atoms P_0, P_1, let $\mathcal{L}'$ be extension of $\mathcal{L}$ obtained by adjoining this new set of atoms. Let $\Psi \equiv \neg P_0 \vee \neg P_1$.

Lemma 3.3.12 *Given $a, b \in X$ with $a \stackrel{L}{\longleftrightarrow} b$, the following sets are consistent in $\mathcal{L}'$:*

$$F_{P_0} = \{\psi_0 \wedge P_0 | \psi_0 \in a\} \cup \{L(\psi_1 \wedge P_1) | \psi_1 \in b\} \cup \{\Psi\} \cup \{K(\phi_1 \vee \neg P_1) | \phi_1 \in b\}$$

$$F_{P_1} = \{\psi_1 \wedge P_1 | \psi_1 \in b\} \cup \{L(\psi_0 \wedge P_0) | \psi_1 \in a\} \cup \{\Psi\} \cup \{K(\phi_0 \vee \neg P_0) | \phi_0 \in a\}$$

Proof: Let us show F_{P_0} consistency, since F_{P_1} consistency has analogous proof. First let us show that the set of formulas $F_1 = \{\psi_0 \wedge P_0 | \psi_0 \in a\} \cup \{L(\psi_1 \wedge P_1) | \psi_1 \in b\} \cup \{\Psi\}$ is consistent:

As $\{\psi_0 | \psi_0 \in a\} \cup \{L(\psi_1 | \psi_1 \in b\}$ is consistent and P_0 and P_1 are new letters, then if $\{\psi_0 \wedge P_0 | \psi_0 \in a\} \cup \{L(\psi_1 \wedge P_1) | \psi_1 \in b\} \cup \{\Psi\}$ is not consistent, then we would have

$$\vdash \neg(P_0 \wedge LP_1 \wedge \Psi)$$

Now, as $\neg\Psi \equiv P_0 \wedge P_1$, we have $\vdash P_0 \wedge LP_1 \to P_0 \wedge P_1$. But $P_0 \wedge LP_1 \to P_0 \wedge P_1$ is not a theorem in our language.

Let $F_2 = F_1 \cup \{K(\phi_1 \vee \neg P_1) | \phi \in b\}$. We show now that F_2 is consistent: Suppose otherwise that

$$\vdash (\psi_0 \wedge P_0 \wedge L(\psi_1 \wedge P_1)) \wedge \Psi \Rightarrow L(\neg\phi_1 \wedge P_1)$$

for some $\psi_0 \in a$ and $\psi_1, \phi_1 \in b$. We have:

$$\vdash (\psi_0 \wedge P_0 \wedge \neg P_1) \wedge L(\psi_1 \wedge P_1)) \to L(\neg\phi_1 \wedge P_1)$$

As the first part of the above implication – $\Phi \equiv (\psi_0 \wedge P_0 \wedge \neg P_1) \wedge L(\psi_1 \wedge P_1))$ – is consistent, we don't have the second part of the implication – $L(\neg\phi_1 \wedge P_1)$ – trivially derivated from $\vdash \Phi \to \chi$ for all formula χ.

Now, $L(\neg\phi_1 \wedge P_1)$ has the subformula P_1, no suformula $\neg P_1$ and P_1 is linked to a connective $\wedge$. Also, $L\chi_1 \wedge L\chi_2 \to L(\chi_1 \wedge \chi_2)$ is not a theorem of our language, so our proof can be reduced to $\vdash L(\psi_1 \wedge P_1)) \to L(\neg\phi_1 \wedge P_1)$.

Now, a proof of $L(\psi_1 \wedge P_1) \to L(\neg\phi_1 \wedge P_1)$ is obtained just from K-necessitation over $\phi_1 \vee \neg P_1 \to \neg\psi_1 \vee \neg P_1$. Because $\psi_1 \wedge \phi_1 \in b$, we cannot prove $\psi_1 \wedge P_1 \to \neg\phi_1 \wedge P_1$.■

Notation 3.3.13 *Given the extension $\mathcal{L}'$ of $\mathcal{L}$ obtained by adding the new atoms P_0 and P_1, denote by $\mathcal{M}_{\mathcal{L}'} = \langle X', \xrightarrow{\diamond}', \xleftrightarrow{L}', \alpha' \rangle$ the canonical model for $\mathcal{L}'$.*

Corollary 3.3.14 *[c] Given $a_0 \xleftrightarrow{L} b_0$ in X and $\mathcal{L}'$ the extension of $\mathcal{L}$ obtained by adding the new atoms P_0 and P_1, there exists $a_{0,P_0} \xleftrightarrow{L}' b_{0,P_1}$ in X' so that $a_0 \subseteq a_{0,P_0}$, $b_0 \subseteq b_{0,P_1}$ and either*

1. *If $a_{0,P_0} \xleftrightarrow{L}' x$ and $P_1 \in x$, then $x = b_{0,P_1}$;*
2. *If $b_{0,P_1} \xleftrightarrow{L}' y$ and $P_0 \in y$, then $y = a_{0,P_0}$.*

Proof: Just let $a_{0,P_0}, b_{0,P_1} \in X'$ with $F_{P_0} \subseteq a_{0,P_0}$ and $b_{0,P_1} \in X$ such that $\{\psi_1 \wedge P_1) | \psi_1 \in b\} \subseteq b_{0,P_1}$ or $F_{P_1} \subseteq b_{0,P_{10}}$ and $a_{0,P_0} \in X$ such that $\{\psi_0 \wedge P_0) | \psi_0 \in a\} \subseteq a_{0,P_0}$.■

Observation 3.3.15 *[c] Let $\mathcal{M}_{\mathcal{L}} = \langle X, \xrightarrow{\diamond}, \xleftrightarrow{L} \alpha \rangle$ and $\mathcal{M}_{\mathcal{L}'} = \langle X', \xrightarrow{\diamond}', \xleftrightarrow{L}', \alpha' \rangle$, as in Notation 3.3.13. For all $x' \in X'$, the set x of all formulas $\psi \in x'$ such that ψ is a formula in $\mathcal{L}$ is a maximal consistent set of formulas of $\mathcal{L}$. So $x \in X$.*

Also, given $y \in X$, as y is consistent, there exists $y' \in X'$ such that $y \subseteq y'$.

Lemma 3.3.16 *[c] Let $\mathcal{M}_{\mathcal{L}} = \langle X, \xrightarrow{\Diamond}, \xleftrightarrow{L} \alpha\rangle$ and $\mathcal{M}_{\mathcal{L}'} = \langle X', \xrightarrow{\Diamond}', \xleftrightarrow{L}', \alpha'\rangle$, as in Notation 3.3.13. Let $x', y' \in X'$, $x, y \in X$ be such that $x \subseteq x'$ and $y \subseteq y'$. Then*

1. *If $x' \xrightarrow{\Diamond}' y'$, then $x \xrightarrow{\Diamond} y$;*
2. *If $x' \xleftrightarrow{L}' y'$, then $x \xleftrightarrow{L} y$;*
3. *If $x \xrightarrow{\Diamond} y$, there exists y'' so that $y \subseteq y''$ and $x' \xrightarrow{\Diamond}' y''$;*
4. *If $x \xrightarrow{\Diamond} y$, there exists x' so that $x \subseteq x''$ and $x'' \xrightarrow{\Diamond}' y'$;*
5. *If $x \xleftrightarrow{L} y$, there exists y'' so that $y \subseteq y''$ and $x' \xleftrightarrow{L}' y''$;*
6. *There exists $S \subseteq x' \searrow$, called a* homo-segment *for $x \searrow$ such that for all $x \xrightarrow{\Diamond} z \xrightarrow{\Diamond} y$, there exists z' such that $x' \xrightarrow{\Diamond}' z' \xrightarrow{\Diamond}' y'$.*

Proof: 1) For all formula $\Box\psi$, if $\Box\psi \in x$, then $\Box\psi \in x'$. As $x' \xrightarrow{\Diamond}' y'$, then $\psi \in y'$. Now, $\Box\psi \in \mathcal{L}$, so, $\psi \in y$.

2) Analogously to 1.

3) Let $F = \{\Box\psi | \Box\psi \in x'\} \cup \{\phi | \phi \in y\}$. We show that F is consistent:

If $\vdash \neg\{\Box\psi \wedge \phi\}$, for some $\Box\psi \in x'$ and $\phi \in y$, then $\vdash \Box\psi \rightarrow \neg\phi$. So, $\vdash \Box\psi \rightarrow \Box\neg\phi$. As $\Box\psi \in x'$, then $\Box\neg\phi \in x'$. As $\phi \in y$, then $\Diamond\phi \in x$. Now, $x \subseteq x'$, then $\Diamond\phi \in x'$. We obtain then $\Box\neg\phi \wedge \Diamond\phi \in x'$, a contradiction.

4) Let $G = \{\Diamond\psi | \psi \in y'\} \cup \{\phi | \phi \in x\}$. Then G is consistent:

If $\vdash \neg\{\Diamond\psi \wedge \phi\}$, for some $\psi \in y'$ and $\phi \in x$, then $\vdash \Diamond\psi \rightarrow \neg\phi$. As $\neg\phi \in \mathcal{L}$, there exists a formula $\chi \in \mathcal{L}$, such that $\psi \rightarrow \chi$ and $\vdash \Diamond\chi \rightarrow \neg\phi$. Now, $\psi \rightarrow \chi$ and $\psi \in y'$, so, $\chi \in y'$. Also, $\chi \in \mathcal{L}$, so, $\chi \in y$.

We have then: $\chi \in y$ and $x \xrightarrow{\Diamond} y$ implies, $\Diamond\chi \in x$. As $\vdash \Diamond\chi \rightarrow \neg\phi$ and $\phi \in x$, we obtain a contradiction.

5) Is analogous to 3.

6) Follows by compactness.■

Lemma 3.3.17 *[c] For all $\eta \in \mathcal{K}$, $[\eta]$ is regular.*

Proof: Let $a_0, b_0 \in X$ be such that $a_0 \overset{L}{\longleftrightarrow} b_0$ and c_0, d_0 be respectively successors of a_0 and b_0 in $K(c)$. Suppose that there exists f so that $d_0 \overset{\diamond}{\longrightarrow} f \overset{\diamond}{\longrightarrow} b_0$ and for all $e \in K(f)$, $\neg(c_0 \overset{\diamond}{\longrightarrow} e \overset{\diamond}{\longrightarrow} a_0)$.

By Corollary 3.3.14, there exists $c_{0,P_0}, d_{0,P_1} \in X'$ so that $c_0 \subseteq c_{0,P_0}$, $d_0 \subseteq d_{0,P_1}$ and if $d_{0,P_1} \overset{L}{\longleftrightarrow}{}' y$ and $P_0 \in y$, then $y = c_{0,P_0}$. Thus $K(c_{0,P_0}) \cap \overline{(c_{0,P_0})} = \{c_{0,P_0}\}$.

Now, it follows from the Cross–Axiom that for $a' \in X'$ with $c_{0,P_0} \overset{\diamond}{\longrightarrow}{}' a'$ and $a_0 \subseteq a'$, if C is a homo–segment that contains a', then there exists a homo–segment C' with $C \leq_L C'$ and $a' \overset{L}{\longleftrightarrow}{}' b'$ for some $b' \in X'$ with $b_0 \subseteq b'$. As c_0 and d_0 are successors of a_0 and b_0 in $K(c)$, then $c_{0,P_0} \overset{L}{\longleftrightarrow}{}' d'_0$, for some d'_0 such that $d_0 \subseteq d'_0$. Moreover c_{0,P_0} and d'_0 are successors of a' and b', respectively.

As C' is a homo–segment, there exists $f' \in X'$ with $d'_0 \overset{\diamond}{\longrightarrow}{}' f' \overset{\diamond}{\longrightarrow}{}' b'$ and $f \subseteq f'$.

Now, C is a homo–segment, and for all $e' \in K(f')$, $\neg(c_{0,P_0} \overset{\diamond}{\longrightarrow}{}' e' \overset{\diamond}{\longrightarrow}{}' a')$. We have then: $d' \overset{\diamond}{\longrightarrow}{}' f' \overset{L}{\longleftrightarrow}{}' e'$ and, by applying the Cross axiom, there exists c' with $d' \overset{L}{\longleftrightarrow}{}' c' \overset{\diamond}{\longrightarrow}{}' e'$. Besides, $(\forall e' \in K(f))(\neg(c_{0,P_0} \overset{\diamond}{\longrightarrow}{}' e' \overset{\diamond}{\longrightarrow}{}' a')$ implies $c' \neq c_{0,P_0}$. Now, $d' \overset{L}{\longleftrightarrow}{}' c'$ and $d' \overset{L}{\longleftrightarrow}{}' c_{0,P_0}$ implies $c' \overset{L}{\longleftrightarrow}{}' c_{0,P_0}$, a contradiction to $K(c_{0,P_0}) \cap \overline{(c_{0,P_0})} = \{c_{0,P_0}\}$.■

Lemma 3.3.18 *[c] Given $\zeta \in \mathcal{K}$, $a_1, a'_1, a_0 \in X$ and $\bar{a}$ a $\diamond$-class,*

1. *If $a_1, a'_1 \overset{\diamond}{\longrightarrow} a_0$, $a_1, a'_1 \overset{L}{\longleftrightarrow} a_0$ and a_1, a'_1 are successor of a_0 in $K(a_0)$, then $a_1 = a_0$;*
2. *If $F' \subseteq [\eta] \otimes \bar{a}$ is so that $F \equiv_L F'$ and $b \overset{\diamond}{\longleftrightarrow} c$ for all $b, c \in F'$. Then $F = F'$.*

Proof: 1) Let G from $[a_0, a'_1]$ onto $[a_0, a_1]$ be given by: For all $x \neq a_0$, $G(x') = x$ if $x \overset{L}{\longleftrightarrow} x'$ and $G(a_0) = a_0$.

As for all $\zeta \in [K(a_0)]$, there exists only one $x \in [a_0, a'_1] - \{a_0\}$, then G is well defined. As for all $\zeta \in [K(a_0)]$, there exists only one $x \in [a_0, a_1] - \{a_0\}$, then G is onto. Besides, $[\zeta]$ is regular, so G is an embedding. Extend G to $a_0 \searrow = \{x \in X | a_0 \overset{\diamond}{\longrightarrow} x\}$ onto $a_0 \searrow$ as $G(x) = x$, the identity mapping. Now, G respects $\overset{L}{\longleftrightarrow}$ and therefore, a_1 *card* a'_1. As $a_1, a'_1 \in X$, then $a_1 = a'_1$.

2) Let $F \searrow = \{x \in \Gamma | (\exists b \in F)(b \overset{\diamond}{\longrightarrow} x)\}$ and $F' \searrow = \{x \in \Gamma | (\exists b \in F')(b \overset{\diamond}{\longrightarrow} x)\}$.

Define G from $F \searrow$ onto $F' \searrow$ as G restricted to $F' \searrow$ is the identity mapping $F(x) = x$ and G restricted to $F \searrow - F' \searrow$ onto F' is so that for all $y \in F \searrow - F' \searrow$, for all $z \in F$, if $y \overset{L}{\longleftrightarrow} z$, then $F(y) = z$.

It follows from the definition of G that G respects $\overset{L}{\longleftrightarrow}$. We must verify that G in an embedding (w.r.t the quasi-order $\overset{\Diamond}{\longrightarrow}$).

Let $x, y \in F' \searrow$ be such that $x \overset{\Diamond}{\longrightarrow} y$. If $x \in F' \searrow$, then $G(x) \overset{\Diamond}{\longrightarrow} G(y)$ follows from the fact that G is the identity mapping. If $x, y \in F \searrow -F' \searrow$, then $G(x), G(y) \in F'$, so $G(x) \overset{\Diamond}{\longleftrightarrow} G(y)$. If $x \in F \searrow$ and $y \in F' \searrow$, then $G(y) = y$. As $(\forall y \in F' \searrow)(\forall x' \in F')(x \overset{\Diamond}{\longrightarrow} y)$, then $G(x) \overset{\Diamond}{\longrightarrow} G(y)$.

Now, let $G(x), y' \in F' \searrow$ be such that $G(x) \overset{\Diamond}{\longrightarrow} y'$. Let $y = y'$. It follows from 1) that for all $x \in F \searrow -F' \searrow$, for all $y \in F' \searrow$, $x \overset{\Diamond}{\longrightarrow} y$, so, either $x \in F' \searrow$ and $x \leq y$ or $G(x) \in F'$. So $x \in F \searrow -F' \searrow$ implies $x \overset{\Diamond}{\longrightarrow} y$.■

Observation 3.3.19 *We have shown that any class $[\eta]$ in the canonical model $\mathcal{M}_{\mathcal{L}}$ is regular and that any point $a_0 \in X$ has predecessors and successors in any $\zeta \in [\eta]$. Moreover, by Lemma 3.3.18, there exists only one element in $K(a)$, a_∞, such that a_∞ is the successor and the predecessor of a_∞ in $K(a_0)$. Thus, it follows that the set of* successors *of a_0 in $K(a_0)$ can be denoted by $a_\infty \overset{\Diamond}{\longrightarrow} a_n \overset{\Diamond}{\longrightarrow} a_{n-1} \overset{\Diamond}{\longrightarrow} \ldots \overset{\Diamond}{\longrightarrow} a_0$, where a_∞ is the maximum in $K(a_0)$.*

Lemma 3.3.20 *[c] Given $[\eta] \in [\mathcal{K}]$, $a_0 \in \cup[\eta]$, $\{a_\infty, a_n, a_{n-1}, \ldots a_0\} \subseteq K(a_0)$ with $a_\infty \overset{\Diamond}{\longrightarrow} a_n \overset{\Diamond}{\longrightarrow} a_{n-1} \overset{\Diamond}{\longrightarrow} \ldots \overset{\Diamond}{\longrightarrow} a_0$ and a_∞ is a successor of a_n in $K(a_0)$, then $a_0 = a_1 = \ldots a_{n-1} = a_n$.*

Proof: Let b_∞ be so that $\neg(a_n \overset{L}{\longleftrightarrow} b_\infty)$ and $a_\infty \overset{\Diamond}{\longleftrightarrow} b_\infty$. Let $a_{n,P_0} \overset{L}{\longleftrightarrow}{}' a_{\infty,P_1}$ be so that if $x \overset{L}{\longleftrightarrow}{}' a_{\infty,P_1}$ and $P_0 \in x$, then $x = a_{n,P_0}$ (see Corollary 3.3.14).

Let $a'_\infty, a'_n, a'_{n-1} \in X$ be such that $a_{n,P_0} \overset{L}{\longleftrightarrow}{}' a'_\infty$, $a'_n \overset{L}{\longleftrightarrow}{}' a'_{n-1}$, $a_{n,P_0} \overset{\Diamond}{\longrightarrow}{}' a'_{n-1}$ and $a'_\infty \overset{\Diamond}{\longrightarrow}{}' a'_n$ with $a_\infty \subseteq a'_\infty$, $a_n \subseteq a'_n$ and $a_{n-1} \subseteq a'_{n-1}$. Suppose also that $[a'_{n-1}, a_{n,P_0}]$ and $[a'_n, a'_\infty]$ are complete.

Let $\psi \in a_\infty$ with $\neg\psi \in b_\infty$. Then a'_∞ contains formulas of the form

$$\psi \wedge \Diamond(\neg\psi \wedge \Diamond(\psi \wedge \Diamond(\neg\psi \wedge \ldots)))$$

Now, we cannot obtain an embedding of $[a'_{n-1}, a_{n,P_0}]$ onto $[a'_n, a'_\infty]$. So $a'_{n-1} = a_{n,P_0}$. Thus $a_{n-1} = a_n = \ldots = a_0$.■

Corollary 3.3.21 *For any $\diamond$-class $\bar{b}$ and $[\eta] \in [\mathcal{K}]$, if $[\eta] \otimes \bar{b} \neq \emptyset$, then:*

1. *For all $\zeta \in [\eta]$, $\zeta \cap \bar{b}$ is a unitary set a^{ζ}_{∞};*

2. *For all $\zeta \in [\eta]$, either $\zeta \cap \bar{b} = \{a^{\zeta}_{\infty}\}$ or $\zeta \cap \bar{a} = \{a^{\zeta}_{\infty}, a^{\zeta}_{0}\}$ with $a^{\zeta}_{\infty} \xrightarrow{\diamond} a^{\zeta}_{0}$;*

3. *For all $\zeta \in [\eta]$, $(\zeta \cap \bar{a}, \xrightarrow{\diamond})$ has the same order as $\omega+1$, where ω is the first non-finite ordinal and $\omega+1$ denotes its successor. We denote $(\zeta \cap \bar{a}, \xrightarrow{\diamond})$ as $\{a^{\zeta}_{\infty} \xrightarrow{\diamond} \dots \xrightarrow{\diamond} a^{\zeta}_{n} \xrightarrow{\diamond} a^{\zeta}_{n-1} \xrightarrow{\diamond} \dots \xrightarrow{\diamond} a^{\zeta}_{0}\}$.*

Moreover, case ii holds at some $\bar{a} \in [\eta]$ iff there is no $\bar{b} \in [\eta]$ in which case iii holds.

Corollary 3.3.22 *All statements from 3.3.8 to 3.3.20 can be extended to the DA- model $\mathcal{M}' = \langle Y', \xleftrightarrow{L} *, \xrightarrow{\diamond} *, \alpha * \rangle$.*

Proof: Just note that any assertion about the canonical model holds in a *Pp* of X, so in a $\diamond$-class of Y'.■

We finalize Section 3.3 with the construction of a *Directed Space* for $\mathcal{L}$. First we will built a Directed Space strong expansion of the DA- model $\mathcal{M}' = \langle Y', \xleftrightarrow{L} *, \xrightarrow{\diamond} *, \alpha * \rangle$.

Proposition 3.3.23 *For each $[\eta] \in \mathcal{K}$ of $\mathcal{M}'$, there exists a digraph $G_{[\eta]} = (V_{[\eta]}, \xrightarrow{\diamond}_{[\eta]}, \xleftrightarrow{L}_{[\eta]})$, for $\xleftrightarrow{L}_{[\eta]}$ an equivalence relation and $\xrightarrow{\diamond}_{[\eta]}$ a partial order, and a mapping $H_{[\eta]}$ from $G_{[\eta]}$ onto $[\eta]$ so that Lemma 2.1.4 part 1 to Lemma 2.1.4 part 5 holds in $[\eta]$.*

Proof: *If cases i and ii hold for all $\diamond$-classes that intersect $[\eta]$:* Given a $\diamond$-class $\bar{a}$, then

- If there exists $\zeta \in [\eta]$ so that $|\zeta \cap \bar{a}| = 2$, consider $\{(a^{\zeta}_{\infty}, p/q, p'/q') | p/q < p'/q' \in \mathbb{Q}\} \cup \{(a^{\zeta}_{0}, p/q) | p/q \in \mathbb{Q}\}$;

- If for all $\zeta \in [\eta]$, $|\zeta \cap \bar{a}| = 1$, consider the set $\{(a^{\zeta}_{\infty}, p/q) | p/q \in \mathbb{Q}\}$.

Let $\leq_{[\eta]}$ be an order over $[\eta]$ and $\xrightarrow{\diamond}_{[\eta]}$ be the smallest transitive relation so that:

- If there exists $\zeta \in [\eta]$ so that $|\zeta \cap \bar{a}| = 2$,

  – If $\zeta \leq_{[\eta]} \zeta'$, $\forall p/q, p'/q'$, $(a^{\zeta'}_{\infty}, p/q, p'/q') \xrightarrow{\diamond}_{[\eta]} (a^{\zeta}_{\infty}, p/q, p'/q')$;

  – If $p'/q' < p''/q''$, $\forall p/q$, $\forall zeta, \zeta'$, $(a^{\zeta}_{\infty}, p/q, p''/q'') \xrightarrow{\diamond}_{[\eta]} (a^{\zeta'}_{\infty}, p/q, p'/q')$;

- $\forall p/q, p'/q'$, $(a^{\zeta}_{\infty}, p/q, p'/q') \xrightarrow{\diamond}_{[\eta]} (a^{\zeta}_{0}, p/q)$.

- If for all $\zeta \in [\eta]$, $|\zeta \cap \overline{a}| = 1$,
  - If $\zeta \leq_{[\eta]} \zeta'$, $\forall p/q$, $(a^{\zeta}_{\infty}, p/q) \xrightarrow{\diamond}_{[\eta]} (a^{\zeta'}_{\infty}, p/q)$;
  - If $p/q < p'/q'$, $(a^{\zeta}_{\infty}, p'/q') \xrightarrow{\diamond}_{[\eta]} (a^{\zeta'}_{\infty}, p/q)$.

For all $\zeta \in [\eta]$, for all $\diamond$–classes $\overline{a}$, $\overline{a'}$ and $\overline{a''}$, let $\xleftrightarrow{L}_{[\eta]}$ be the equivalence closure of:

- $(a^{\zeta}_{\infty}, p/q, p'/q') \xleftrightarrow{L}_{[\eta]} (a'^{\zeta}_{\infty}, r/s, p'/q') \xleftrightarrow{L}_{[\eta]} (a''^{\zeta}_{\infty}, p'/q')$;
- $(a^{\zeta}_{\infty}, p/q, p'/q') \xleftrightarrow{L}_{[\eta]} (a^{\zeta}_{0}, p'/q')$, if a'^{ζ}_{0} is defined.

If cases i and iii hold for all $\diamond$-class that intersect $[\eta]$: For all $\diamond$–class $\overline{a}$

- If $\zeta \cap \overline{a} = \{a^{\zeta}_{\infty} \xrightarrow{\diamond} \ldots \xrightarrow{\diamond} a^{\zeta}_{n} \xrightarrow{\diamond} a^{\zeta}_{n-1} \xrightarrow{\diamond} \ldots \xrightarrow{\diamond} a^{\zeta}_{0}\}$, consider the set $\{a^{\zeta}_{\infty}, r, s, t, u) | r, s, t, u \in \mathbf{Z} \wedge t > u\} \cup \{(a^{\zeta}_{j}, r, t, t+j) | r \in \mathbf{Z} \wedge t, j \in \mathbf{N}\}$;
- If $\zeta \cap \overline{a} = \{a^{\zeta}_{\infty}\}$, consider the set $\{a^{\zeta}_{\infty}, s, u) | s, u \in \mathbf{Z}\}$.

Let $\leq_{[\eta]}$ be the order over $min_{[\eta]}$ and $\xrightarrow{\diamond}_{[\eta]}$ be the transitive closure of

- If $\zeta \cap \overline{a} = \{a_{\infty} \xrightarrow{\diamond} \ldots \xrightarrow{\diamond} a_{n} \xrightarrow{\diamond} a_{n-1} \xrightarrow{\diamond} \ldots \xrightarrow{\diamond} a_{0}\}$, for all $\zeta, \zeta' \in [\eta]]$, consider
  - $(\forall r, s, t, u)$, $(a^{\eta'}_{\infty}, r, s, t, u) \xrightarrow{\diamond}_{[\eta]} (a\eta_{\infty}, r, s, t, u)$ if $\eta' \leq_{[\eta]} \eta$;
  - $(a^{\eta}_{\infty}, r, s, t, u) \xrightarrow{\diamond}_{[\eta]} (a^{\eta'}_{\infty}, r, s', t, u')$, if $s' < s$;
  - $(a^{\eta}_{\infty}, r, s, t, u) \xrightarrow{\diamond}_{[\eta]} (a^{\eta'}_{\infty}, r, s, t, u')$, if $u' < u$;
  - $(a^{\eta}_{\infty}, r, s, t, u) \xrightarrow{\diamond}_{[\eta]} (a^{\eta'}_{j}, r, t, t+j)$, for all s, s';
  - $(a_{j}, r, t, t+j) \xrightarrow{\diamond}_{[\eta]} (a'_{j}, r, t, t+j)$, for $a_{j} \xrightarrow{\diamond} a'_{j}$;
  - $(a_{j}, r, t, t+j) \xrightarrow{\diamond}_{[\eta]} (a'_{j'}, r, t, t+j')$, for all $j' < j$.
- If for all $\zeta \in [\eta]$, $\zeta \cap \overline{a} = \{a^{\zeta}_{\infty}\}$, for all $\zeta, \zeta' \in [\eta]]$, consider
  - $(a^{\eta}_{\infty}, s, u) \xrightarrow{\diamond}_{[\eta]} (a^{\eta'}_{\infty}, s, u')$ if $\eta' \leq_{[\eta]} \eta$
  - $(a^{\eta}_{\infty}, s, u) \xrightarrow{\diamond}_{[\eta]} (a^{\eta'}_{\infty}, s, u')$ for all $u' < u$.

Let $\overset{L}{\longleftrightarrow}_{[\eta]}$ be the equivalence closure of $(a_\infty, r, s, t, u) \overset{L}{\longleftrightarrow}_{[\eta]} (a_\infty, r', s, t', u) \overset{L}{\longleftrightarrow}_{[\eta]}$ $(a_{u-t''}, s, t'', u))$.

Denote the first coordinate projection onto $\zeta \cap \overline{a}$ by $H_{[\eta]}$ and let, for all $A \in \mathcal{A}$, $x \in \alpha_{[\eta]}(A)$ iff $H_{[\eta]}(x) \in \alpha(A)$.■

Theorem 3.3.24 *Given a language $\mathcal{L}$, there exists a directed model $\mathcal{D} = \langle X^+, \mathcal{O}^+, \alpha^+\rangle$ strong expansion of the DA- model.*

Proof: For all $\eta \in \mathcal{K}$, consider $\mathtt{D}_\eta$ the set of all K-segments of $G_{[\eta]}$. Consider for all $[\eta] \in [\mathcal{K}]$, for all $\eta \in [\eta]$, the set $\mathtt{N}_\eta$ given by: $N \in \mathtt{N}_\eta$ iff $(\forall[\zeta])(\eta \preceq \zeta \Rightarrow (\exists\zeta' \in [\zeta])(\forall\zeta'' \in [\zeta])(\forall D \in \mathtt{D}_\zeta)(D \in N) \Leftrightarrow \zeta'' \preceq \zeta')$.

Let D^+ be the union of all sets $N \in \mathtt{N}_\eta$ for all $\eta \in \mathcal{K}$.

For all $\diamond$–class $\overline{a}$ in Y', let $x_{\overline{a}} \in X^+$ iff exists a $\mathcal{D} \subseteq D^+$ so that for all $\eta \in \mathcal{K}$, if $\eta \cap \overline{a} \neq \emptyset$, then only one $D \in \mathtt{N}_\eta$ belongs to $\mathcal{D}$ and for all $\eta, \zeta \preceq \nu$, $D \in \mathtt{N}_\nu$, $D' \in \mathtt{N}_\eta$, $D'' \in \mathtt{N}_\zeta$ belongs to $\mathcal{D}$, implies $D', D'' \subseteq D$

For all $\zeta \in \mathcal{K}$, $\eta \in [\eta]$, define $V_D \in \mathcal{O}^+$ if $V_D = \{x \in X^+ | (\exists\zeta \in \mathcal{K})(\exists D \in \mathtt{N}_\zeta)(D \in x)\}$ and let, for all $x_{\overline{a}} \in X^+$, $F''(x_{\overline{a}}) = \overline{a}$. Define $x \in \alpha_{[\eta]}(A)$ iff $A \in char(\overline{a})$. Now, by making the composition of F' with F, the fiber mapping from Y' into X, we obtain a subset space $\mathcal{D} = \langle X^+, \mathcal{O}^+, \alpha^+\rangle$ strong expansion of $\mathcal{M}_{\mathcal{L}}$.

In order to show that $\mathcal{D}$ is an intersection space, note that for all $x_{\overline{a}} \in X^+$, the arrow $\overset{\diamond}{\longrightarrow}$ over $\mathcal{N}x_{\overline{a}}$, given in $\mathcal{O}^+$ by the order $\subseteq$ is given by the order $\subseteq$ over all $D \in D^+$ that belongs to $x_{\overline{a}}$, and this order is weakly directed.■

Corollary 3.3.25 *Given a language $\mathcal{L}$, there exists a directed model $\mathcal{D} = \langle X^+, \mathcal{O}^+, \alpha^+\rangle$ strong expansion of the canonical model.*

Corollary 3.3.26 *The axioms 1 to 10, WD and the axioms M_n are complete for directed spaces.*

Chapter 4

4.1 No Finite Axiomatization for Directed Spaces

In this section, we prove that axioms 1) - 10), WD and the axioms M_n cannot be reduced to a finite set of axiom schemas.

We use the compactness results obtained in DMP's paper and the denumerable set of counter-models $\mathcal{M}_n$ that are adapted from DMP's paper, [1] to obtain the results we claim.

Example 4.1.1 (DMP_n) *Let $\mathcal{L}$ be a language for subset spaces whose atomic symbols are $\{A, Z\} \cup \{P_i | 1 \leq i \leq n\}$. Let us consider, for every $n \in N$, a subset space $\mathcal{M}_n = \langle X_n, \mathcal{O}_n, \alpha_n \rangle$ given by:*

The set of points is $X_n = \{a, z_1, \ldots, z_n, p_1, \ldots, p_n\}$. The set of opens $\mathcal{O}_n$ is given by: X_n, $U_{j,t} = \{a, z_j, p_t\}$, $j \neq t$, $V_j = \{a, z_j\}$, $1 \leq j \leq n$, and $\{p_j\}$, for all $1 \leq j \leq n$

Let the valuation α_n is given by $A \mapsto \{a\}$, $Z \mapsto \{z_1, \ldots, z_n\}$, for all $i \leq n$, $P_i \mapsto \{p_i\}$.

Then for all n, for all $j < n$, the axioms M_j are valid in $\mathcal{M}_n$ but there exists an instance of axiom M_n that is not valid over $\mathcal{M}_n$. Moreover axioms 1) - 10) plus WD are valid in each $\mathcal{M}_n$.

Proof: Let $n \in N$ be fixed. First let us show that for all $j < n$ any instance of an axiom M_j is valid in $\mathcal{M}_n$.

Since axiom M_m implies axiom M_j, for all $j \leq m$, it is enough to show that for any formulas $\phi, \psi_1, \ldots, \psi_{n-1}$, if for some $x \in X_n$, $W \in \mathcal{O}_n$, we have

$$x, W \models \Box L \Diamond \phi \wedge \Diamond K \psi_1 \wedge \ldots \wedge \Diamond K \psi_{n-1}$$

then

$$x, W \models L(\Diamond \phi \wedge \Diamond K \psi_1 \wedge \ldots \wedge \Diamond K \psi_{n-1})$$

or, equivalently, there exists $y \in W$ so that

$$y, W \models \Diamond\phi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_{n-1}$$

Let us rename $\Diamond\phi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_{n-1}$ as Φ.

We can have that $x, W \models \Diamond\phi$ or $x, W \models \Box\neg\phi$. If $x, W \models \Diamond\phi$, then trivially, $x, W \models \Phi$ so $x, W \models L\Phi$.

So suppose $x, W \models \Box\neg\phi$.

if $x = p_j$, then since we have $\{p_j\} \in \mathcal{O}_n$, we get $p_j, \{p_j\} \models L\Diamond\phi$, and hence $p_j, \{p_j\} \models \Diamond\phi$. This yields $p_j, W \models \Diamond\phi$, i.e. $x, W \models \Diamond\phi$.

Suppose $x = z_j$, and $z_j, X_n \models \Box\neg\phi$. Then since $x, W \models \Box L\Diamond\phi$, we must have $z_j, V_j \models L\Diamond\phi$, and hence $a, V_j \models \Diamond\phi$, for $z_j, V_j \not\models \Diamond\phi$ and a is the only other element in V_j. Hence $a, W \models \Diamond\phi$. Now a belongs to all open sets to which z_j belongs. So $a, W \models \Diamond K\psi_i$ for all i and we get $a, W \models \Phi$.

If $W = U_{j,t}$ or $W = V_j$, then the restriction of X_n to W is a directed space. For if $X' = W$ and $\mathcal{O}' = \{V \in \mathcal{O} | V \subseteq W\}$ then for all atomic letters A, define $\alpha'(A) = \alpha_n(A) \cap W$. Then $\mathcal{M}' = \langle X', \mathcal{O}', \alpha' \rangle$ is a directed model that satisfies $(\forall V \in \mathcal{O}')(\forall x \in X')(\forall \psi \in \mathcal{L})\{(x, V \models_{\mathcal{M}'} \psi) \Leftrightarrow (x, V \models_{\mathcal{M}_n} \psi)\}$. Hence M_{n-1} is valid on W.

From the above it follows that we can assume that $x = a$ and that $W = X_n$. Since $a, X_n \models \Box L\Diamond\phi$, we also have $a, V_j \models L\Diamond\phi$ and hence, since $a, X_n \not\models \Diamond\phi$, we must have $z_j, V_j \models \Diamond\phi$ for all j.

Since $a, X_n \models \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_{n-1}$, for each $i \leq n-1$, there exists $W_i \subseteq X_n$ in $\mathcal{N}a$ such that $a, W_i \models K\psi_i$. Let $\mathcal{S}$ be the collection of all these W_i.

If $\mathcal{S} = \{X_n\}$ then pick $y \in X_n$ so that $y, X_n \models \Diamond\phi$. Then $y, X_n \models \Phi$.

Otherwise all W_i in $\mathcal{S}$ (other than X_n) are either of the form V_j or $U_{j,t}$. There are at most $n-1$ such W_i and hence at most $n-1$ subscripts t are involved. Now we note that for all j, j', $Th(z_j, U_{j,t}) = Th(z_{j'}, U_{j',t})$, and $Th(z_j, V_j) = Th(z_{j'}, V_{j'})$. Also, as we noted, $z_j, X_n \models \Diamond\phi$. So pick a single j different from all the t involved and replace (i) every $U_{j',t}$ in $\mathcal{S}$ by $U_{j,t}$ and (ii) every $V_{j'}$ by V_j, to get $\mathcal{S}'$. This z_j (with $\mathcal{S}'$) will satisfy Φ.

Now let us verify that an instance of axiom M_n is not valid in $\mathcal{M}_n$.

For each $i \leq n$, let ϕ_i be the formula $K\neg P_1 \wedge \ldots \wedge K\neg P_{i-1} \wedge LP_i \wedge K\neg P_{i+1} \wedge \ldots \wedge K\neg P_n$. By the $S5$ nature of the K-axioms, for all $i \leq n$, $\phi_i \equiv K\phi_i$.

Let us show that $a, X_n \models \Box L\Diamond Z \wedge \Diamond K\phi_1 \wedge \ldots \wedge \Diamond K\phi_n$, but no z_i satisfies $z_i, X_n \models \Diamond K\phi_1 \wedge \ldots \wedge \Diamond K\phi_n$. Therefore, $a, X_n \not\models L\{\Diamond Z \wedge \Diamond K\phi_1 \wedge \ldots \wedge \Diamond K\phi_n\}$.

We have that $a, X_n \models \Box L\Diamond Z \wedge \Diamond K\phi_1 \wedge \ldots \wedge \Diamond K\phi_n$, since

- For all $W \in \mathcal{N}a$, there exists $i \leq n$ so that $z_i \in W$, so, as $\alpha_n(Z) = \{z_1, \ldots, z_n\}$, then $a, X_n \models \Box L\Diamond Z$;
- If $m \neq j$ then $a, U_{m,j} \models K\phi_j$. Hence for all $j \leq n$, $a, X_n \models \Diamond K\phi_j$.

Also, let $m \leq n$. We have that $z_m, X_n \not\models \Diamond K\phi_1 \wedge \ldots \wedge \Diamond K\phi_n$, for:

- $z_m, X_n \models LP_1 \wedge \ldots \wedge LP_n$,
- For all $j \neq m$, $z_m, U_{m,j} \not\models LP_m$ and
- $z_m, V_m \models K\neg P_1 \wedge \ldots \wedge K\neg P_n$,

hence, for all $j \leq n$, we have $z_i, X_n \models \Box\neg\phi_i$, so, for all $i \leq n$, $z_i, X_n \models \Box L\neg\phi_1 \vee \ldots \vee \Box L\neg\phi_n$.

The axioms 1) - 10) are valid on each model $\mathcal{M}_n$ for they are examples of subset spaces and Weak Directed axiom follows from the fact that for all formulas χ, for all $i, j \leq n$,

$$a, V_i \models \chi \text{ iff } a, V_j \models \chi,$$
$$z_i, V_i \models \chi \text{ iff } z_j, V_j \models \chi$$

The set M that consists of Axioms 1) - 10), Weak Directed axiom and the axioms M_n are a complete set of axioms for directed spaces, see section 3, therefore, in view of the above example, if a finite set of axioms, say F is complete for directed spaces, we would have: $M \vdash F$, hence, using compactness, for some $D_1, \ldots D_n$, we would have $D_1, \ldots D_n \vdash F$ which is not possible in view of the series of models $\mathcal{M}_n$.

Chapter 5

5.1 Examples

Example 5.1.1 *Let* $\mathcal{M} = \langle X, \mathcal{O}, \alpha\rangle$, *where* $X = \{a, z_0, z_1, z_2, p, q\}$, $\mathcal{O} = \{X, U_1 = \{a, z_0, z_1, p\}, U_2 = \{a, z_0, z_2, q\}, V_1 = \{a, z_0, z_1\}, V_2 = \{a, z_0, z_2\}, V_0 = \{a, z_0\}\}$. *and* $\alpha(A) = \{a\}$, $\alpha(Z) = \{z_0, z_1, z_2\}$, $\alpha(P) = \{p\}$ *and* $\alpha(Q) = \{q\}$. *If* $\overline{z}$ *is the* $\diamond$*-class that contains* $Th((z_0, V_0))$, *we have that* $\overline{z}$ *is directed but not above directed.*

Proof: If $T(x, W)$ stands for $Th((x, W))$, we have

$$\begin{array}{l} T(a,X) \overset{L}{\longleftrightarrow} T(z_0,X) \overset{L}{\longleftrightarrow} T(z_1,X) \overset{L}{\longleftrightarrow} T(z_2,X) \overset{L}{\longleftrightarrow} T(p,X) \overset{L}{\longleftrightarrow} T(q,X) \\ T(a,U_1) \overset{L}{\longleftrightarrow} T(z_0,U_1) = T(z_1,U_1) \overset{L}{\longleftrightarrow} T(p,U_1) \\ T(a,U_2) \overset{L}{\longleftrightarrow} T(z_0,U_2) = T(z_2,U_2) \overset{L}{\longleftrightarrow} T(q,U_2) \\ T(a,V_0) = T(a,V_1) = T(a,V_2) \\ T(z_0,V_0) = T(z_0,V_1) = T(z_0,V_2) = T(z_1,V_1) = T(z_2,V_2) \\ T(a,V_0) \overset{L}{\longleftrightarrow} T(z_0,V_0) \end{array}$$

and

$$\begin{array}{l} T(a,X) \overset{\diamond}{\longrightarrow} T(a,U_1), T(a,U_2) \overset{\diamond}{\longrightarrow} T(a,V_0) \\ T(z_0,X) \overset{\diamond}{\longrightarrow} T(z_0,U_1), T(z_0,U_2) \overset{\diamond}{\longrightarrow} T(z_0,V_0) \\ T(z_1,X) \overset{\diamond}{\longrightarrow} T(z_0,U_1) \overset{\diamond}{\longrightarrow} T(z_0,V_0) \\ T(z_2,X) \overset{\diamond}{\longrightarrow} T(z_0,U_2) \overset{\diamond}{\longrightarrow} T(z_0,V_0) \\ T(p,X) \overset{\diamond}{\longrightarrow} T(p,U_1) \\ T(q,X) \overset{\diamond}{\longrightarrow} T(q,U_2) \end{array}$$

Now, $T(z_0, X)$, $T(z_1, X)$, $T(z_2, X)$ are two maximum for $T(z_0, V_0)$ in the order $\overset{\diamond}{\longrightarrow}$ that cannot be above directed by any $Th(x, W)$.

Example 5.1.2 *If* $X \subseteq \{\overline{a}, \overline{b}\}$, $\overline{a} = \{a_0, a_1, a_2, a_3\}$ *and* $\overline{b} = \{b_0, b_1, b_2, b_3, b_4, b_1', b_2', , b_3'\}$ *with* $\overline{a}$ *given by* $a_3 \overset{\diamond}{\longrightarrow} a_1, a_2 \overset{\diamond}{\longrightarrow} a_0$ *and* $\overline{b}$ *given by the Pps* $B_1 = \{b_3 \overset{\diamond}{\longrightarrow} b_1 \overset{\diamond}{\longrightarrow} b_0\}$, $B_2 = \{b_4 \overset{\diamond}{\longrightarrow} b_2 \overset{\diamond}{\longrightarrow} b_0\}$ *and* $B_3 = \{b_3' \overset{\diamond}{\longrightarrow} b_1', b_2' \overset{\diamond}{\longrightarrow} b_0\}$ *with:* $a_0 \overset{L}{\longleftrightarrow} b_0$, $a_1 \overset{L}{\longleftrightarrow}$

$b_1 \overset{L}{\longleftrightarrow} b'_1$, $a_2 \overset{L}{\longleftrightarrow} b_2, \overset{L}{\longleftrightarrow} b'_2$ and $a_3 \overset{L}{\longleftrightarrow} b_3 \overset{L}{\longleftrightarrow} b'_3 \overset{L}{\longleftrightarrow} b_4$. (M Axiom Lemma justify this third Pp $B_3 \in \bar{b}$), then Y' will be given as:

Call the Pp given by $\bar{a}$ as A_0 and let

$A'_0 = \{(a_3,0) \overset{\diamond}{\longrightarrow} *(a_1,0), (a_2,0) \overset{\diamond}{\longrightarrow} *(a_0,1,0), (a_0,2,0) \overset{\diamond}{\longrightarrow} *(a_0,0,0)\}$,

$B_3 = \{(b'_3,3) \overset{\diamond}{\longrightarrow} *(b'_1,3), (b'_2,3) \overset{\diamond}{\longrightarrow} *(b_0,1,3), (b_0,2,3) \overset{\diamond}{\longrightarrow} *(b_0,3,3)\}$,

$B_1 = \{(b_3,1) \overset{\diamond}{\longrightarrow} *(b_1,1) \overset{\diamond}{\longrightarrow} *(b_0,1,1)\}$

$B_2 = \{(b_4,2) \overset{\diamond}{\longrightarrow} *(b_2,2) \overset{\diamond}{\longrightarrow} *(b_0,2,2)\}$

and

$\{(a_3,0) \overset{L}{\longleftrightarrow} *(b'_3,3) \overset{L}{\longleftrightarrow} *(b_3,1) \overset{L}{\longleftrightarrow} *(b_4,2)\}$,

$\{(a_1,0) \overset{L}{\longleftrightarrow} *(b'_1,3) \overset{L}{\longleftrightarrow} *(b_1,1)\}$,

$\{(a_2,0) \overset{L}{\longleftrightarrow} *(b'_2,3) \overset{L}{\longleftrightarrow} *(b_2,2)\}$

$\{(a_0,1,0) \overset{L}{\longleftrightarrow} *(b_0,1,3) \overset{L}{\longleftrightarrow} *(b_0,1,1)\}$

$\{(a_0,2,0) \overset{L}{\longleftrightarrow} *(b_0,2,3) \overset{L}{\longleftrightarrow} *(b_0,2,2)\}$

$\{(a_0,0,0) \overset{L}{\longleftrightarrow} *(b_0,3,3)\}$

Example 5.1.3 *We use the fact that S is the set of all $U \subseteq W$ so that for some $1 \le i \le n$, $x, U \models \diamond K\psi_i$ in order to show that for at least one $U \in S$, $x, U \models \diamond K\psi_i$ for all $1 \le i \le n$. If for example $n = 3$, then $X_3 = \{a, z_1, z_2, z_3, p_1, p_2, p_3\}$,*

$$
\begin{aligned}
U_{1,2} &= \{a, z_1, p_2\}\\
U_{1,3} &= \{a, z_1, p_3\}\\
U_{2,1} &= \{a, z_2, p_1\}\\
U_{2,3} &= \{a, z_2, p_3\}\\
U_{3,1} &= \{a, z_3, p_1\}\\
U_{3,2} &= \{a, z_3, p_2\}\\
V_1 &= \{a, z_1\}\\
V_2 &= \{a, z_2\}\\
V_3 &= \{a, z_3\}
\end{aligned}
$$

and, for $W = X_3$, the possible sets S are:

$S = \{X_3\}$ or
$S = \{X_3, U_{1,2}, U_{3,2}\}$ or
$S = \{X_3, U_{1,3}, U_{2,3}\}$ or
$S = \{X_3, U_{2,1}, U_{3,1}\}$ or
$S = \{X_3, U_{2,1}, U_{3,1}, U_{1,2}, U_{3,2}\}$ or
$S = \{X_3, U_{1,3}, U_{2,1}, U_{2,3}, U_{3,1}\}$ or
$S = \{X_3, U_{1,2}, U_{1,3}, U_{2,3}, U_{3,2}\}$ or
$S = \{X_3, U_{1,2}, U_{1,3}, U_{2,1}, U_{2,3}, U_{3,1}, U_{3,2}\}$ or
$S = O_3$

for if some V_i *is so that* $x, V_i \models \Diamond K\psi_i$, *for all* $W \in \mathcal{O}_3$, *if* $V_i \subseteq U \subseteq X$ *then* $x, W \models \Diamond K\psi_i$, *and for all* $1 \leq i, j \leq 3$, *for all formula* η, $x, V_i \models \eta$ *iff* $x, V_j \models \eta$ *and for all* $(j, t) \in D$ *and* $(j', t) \in D$, *for all formula* η, $a, U_{j,t} \models \eta$ *iff* $a, U_{j',t} \models \eta$, $z_j, U_{j,t} \models \eta$ *iff* $z_{j'}, U_{j',t} \models \eta$ *and* $p_t, U_{j,t} \models \eta$ *iff* $p_t, U_{j',t} \models \eta$.

Chapter 6

6.1 Further Topics

The solution of completeness problem still lacks of some analysis about the existence of procedures in order to show if a given formula in the language of Intersection Spaces is valid or not, i.e., is the Logic for Intersection Spaces decidable?. Our next step in to write in that direction.

Even though Intersection Spaces have no finite model property, our efforts in solving completeness problem indicate that a finite Kripke model can be turned into a Directed Space. So, since Intersection Spaces are logically equivalent to Directed Spaces, if there is a decision procedure for the logic of Intersection Spaces can be obtained.

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IMAGE EVALUATION TEST TARGET (QA-3)

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