A

RULING EUCLIDEAN 3-SPACE

by

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ABSTRACT

RULING EUCLIDEAN 3-SPACE

by

Evan Siegel

Advisor: Professor John Velling

This essay studies $\mathbf{C}^r$ foliations of $\mathbf{R}^3$ by lines, or *rulings* of $\mathbf{R}^3$. In addition to rulings of $\mathbf{R}^3$, by 1) parallel lines and 2) lines which lie in parallel planes which themselves foliate $\mathbf{R}^3$ there are rulings by 3) a skew set of lines. Rulings of the third type have a "flexibiliy." The Jacobian of the Gauss map of the leaves in the foliation is studied. Flow along leaves between transverse planes preserves area if and only if the determinant of this Jacobian vanishes. Connected sets of parallel leaves must be convex. The set of leaves parallel to a given leaf can have no more than two connected components. These components can either be a pair of coplanar half-planes or they are on opposite sides of a pair of asymptotic planes.

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I also acknowledge the help of Professor Richard Sacksteder, who read major portions of the dissertation in its early and final stages.

Naturally, I alone take responsibility for any errors which survive into the present work.

DEDICATION

This dissertation is dedicated to

my parents, Eugene and June,

my wife and *ayshet khayel*, Mina Zand,

and

Mercutio and Ginger

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0 Introduction

0.1 Basic objects and a summary of results

Definition 0.1 *A* $\mathbf{C}^r$ ruling *of* $\mathbf{R}^3$ *consists of a family of lines such that*

1) $\mathbf{R}^3$ *is the disjoint union of this family of lines and*

2) *for some neighborhood of each line, there exists a cartesian coordinate system such that that line is parallel to* $(0, 0, 1)$ *and each line through any point* (x, y, z) *in that neighborhood is parallel to* $(p(x, y, z), q(x, y, z), 1)$, $p, q \in \mathbf{C}^r$ *and* $p, q, \to 0$ *as* $x^2 + y^2 \to 0$.

The lines of the ruling are also refered to as leaves.

A directed ruling *is a ruling in which the leaves are given a direction in a continuous fashion. We note that any ruling of* $\mathbf{R}^3$ *can be made a directed ruling since* $\mathbf{R}^3$ *is simply-connected.*

We have modified the usual continuity condition in the common definition of a k-dimensional $\mathbf{C}^r$ foliation of an n-manifold [1] in a manner permitted by the current problem: In Theorem 2.3, we will show that the degree of continuity is a constant of the leaf.

In this essay, we begin the process of developing a basic theory of $\mathbf{C}^r$ rulings of $\mathbf{R}^3$.

Before proceeding, we make a definition.

Definition 0.2 *A set of lines is called* skew *if it contains no parallel pair of lines.*

In §1.1, we begin by observing that in addition to the obvious rulings of $\mathbf{R}^3$, by 1) parallel lines and 2) lines which lie in parallel planes which themselves foliate $\mathbf{R}^3$ ("integrate into parallel planes"), there are also rulings by 3) skew sets of lines. We will refer to rulings which are locally of the first, second, or third type.

In §§1.2.1–1.2.3, we show that a foliation may include a neighborhood which includes neighborhoods of the first and third type ruling and neighborhoods of first and second type rulings. We also demonstrate that half-planes of parallel lines can be included in rulings which are otherwise of the third type.

§2 concerns local structure of foliations.

In §2.1, we introduce a coordinate system for lines which will be used in the course of the rest of the paper.

In §2.2, we generalize the discussion started in §1.1 of the flexibility of a ruling on a *local* level.

In §2.3, we consider the variation of the direction of the leaves close to a given leaf. Among other things, we prove that properties such as differentiability are constants of the leaf.

In §2.4, we examine under what circumstances the leaves of a ruling are perpendicular to a family of continuous surfaces which also foliates $\mathbf{R}^3$.

§2.5 considers analogies between the properties of a ruling of $\mathbf{R}^3$ and those of ruled surfaces. It concludes that analogous properties only obtain under limited circumstances.

In §2.6, we consider the ruling as a flow. We find that this flow preserves area if and only if the ruling is trivial.

§3 discusses global relations between leaves. In §3.1, we find that connected sets of leaves parallel to a given leaf are convex and therefore path-connected. We then prove that if a line intersects the set of leaves parallel to a given leaf in two intervals one of which is bounded, the ruling integrates into parallel planes. Using this as a lemma, we prove that for any ruling which does not integrate into parallel planes, if the set of leaves parallel to a given leaf is not connected, 1) each of its connected components includes a half-plane; 2) a plane containing a leaf in two connected components of this set intersects these connected components in a pair of disjoint closed half-planes; 3) if one of this set's connected components is not a half-plane, neither is the other; 4) this set can have no more than two connected components, and 5) these connected components are each asymptotic to a pair of half-planes.

1 A menagerie of rulings of $\mathbf{R}^3$

1.1 The three types of rulings of $\mathbf{R}^3$

We now introduce three types of rulings of $\mathbf{R}^3$; they will serve as basic examples of these rulings. They are distinguished as follows: Let N be the map from (x, y, z) to (p, q) defined implicitly in Definition 0.1. Then the dimension of the range of S will be different for each type.

The first type of ruling consists entirely of parallel leaves. The dimension of S is 0. We call such a ruling the *trivial* ruling.

In rulings of the second type, the leaves lie in parallel planes, but the directions of the leaves ruling each plane vary continuously as one moves through the planes. The dimension of S is 1.

In rulings of the third type, no two leaves are parallel. The dimension of S is 2. We call such a ruling a *skew ruling.* Rulings of this type may be constructed as follows: We fill $\mathbf{R}^3$ less a line ℓ with concentric circular one-sheeted hyperboloids (OSHs) whose intersections with a plane through ℓ are a family of hyperbolas which continuously increase in eccentricity as the distance between their vertices and ℓ increases. After reviewing the fact that OSHs are ruled surfaces, we will show that as the OSHs in this construction approach ℓ, the directions of their ruling lines approach that of ℓ, thus establishing continuity. We use these *rulings via OSHs* to generate interesting rulings of $\mathbf{R}^3$.

We first demonstrate that OSHs are ruled surfaces. Consider the OSH with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{1}$$

where $a, b, c > 0$. For any α, $0 \leq \alpha \leq 2\pi$, the coordinates of the points on the line parameterized by

$$\left(a \cos\alpha - \frac{a}{c} t \sin\alpha, b \sin\alpha + \frac{b}{c} t \cos\alpha, t\right), \tag{2}$$

$t \in \mathbf{R}$, satisfy the equation for the above OSH. We can see explicitly that this parameterizes the entire OSH. Letting $\cos\epsilon = c/\sqrt{c^2+t^2}$ and $\sin\epsilon = t/\sqrt{c^2+t^2}$, the cross-section of the OSH through the plane $z = t$ becomes $\frac{\sqrt{c^2+t^2}}{c}\left(a\cos(\alpha+\epsilon), b\sin(\alpha+\epsilon)\right)$, *i.e.*, each intersection of the OSH with the planes $z = t$, for each $t \in \mathbf{R}$, is produced by the parameterization.

The locus of points in the OSH with equation (1) for which the z term vanishes we call its *waist.* If an OSH is defined by the above equation with $a = b$, we call it *circular.* If $a \neq b$, we call it *elliptical.*

We now find a condition under which two circular OSHs with concentric waists do not intersect. Let

$$\frac{x^2+y^2}{{a_1}^2} - \frac{z^2}{{c_1}^2} = 1 \tag{3}$$

and

$$\frac{x^2+y^2}{{a_2}^2} - \frac{z^2}{{c_2}^2} = 1 \tag{4}$$

be two OSHs. Then they intersect only if

$$x^2+y^2 = {a_1}^2\left(1+\frac{z^2}{{c_1}^2}\right) = {a_2}^2\left(1+\frac{z^2}{{c_2}^2}\right), \tag{5}$$

and so

$$z^2 = \frac{{a_2}^2 - {a_1}^2}{{a_1}^2/{c_1}^2 - {a_2}^2/{c_2}^2}. \tag{6}$$

Then the OSHs will not intersect if $a_1 < a_2$ and $c_1/a_1 \geq c_2/a_2$. If c is a smooth function of a, $0 < a < \infty$, we get, passing to the limit as $|a_1 - a_2|$ approaches zero,

$$\frac{d}{da}\left(\frac{c}{a}\right) \leq 0, \tag{7}$$

i.e., as the radius of the waist decreases, the leaves of the OSHs do not become less vertical.

We next demonstrate that we can foliate $\mathbf{R}^3$ less a line with circular OSHs, and that the directions of the lines which foliate these OSHs approach that of the deleted line as the waists of the OSHs approach zero.

Example 1.1 (Rulings via OSHs) Consider the family of OSHs given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{(1/a)^2} = 1. \tag{8}$$

The last denominator is a continuous function of the first and satisfies inequality (7). Its intersection with the plane $z = t$, for each $t \in \mathbf{R}$, is a family of concentric circles which foliate the plane less $(0, 0, t)$. Therefore it foliates $\mathbf{R}^3$ with the z-axis deleted. Refering now to the expression in (2), we see that the leaf through $(a\cos\alpha, a\sin\alpha, 0)$ in the foliation of one member of this family of OSHs is parallel to the vector $(-a^2\sin\alpha, a^2\cos\alpha, 1)$, and so the leaves foliating the family of OSHs approach the z-axis as a approaches zero. Thus, the lines foliating the OSHs united with the line along the z-axis form a foliation of $\mathbf{R}^3$.

We now show that this is a skew ruling. We see that the projection onto the plane $z = 0$ of each leaf is tangent to the waist of the OSH in which it lies. Since a pair of leaves can only be parallel if their projection onto a common plane is parallel, a necessary condition for two leaves in this foliation to be parallel is that their points of tangency under this projection to their respective waists lie on a common diameter through $(0, 0, 0)$. On the other hand, since $c = 1/a$ in this case, the inequality in (7) is strict. Therefore, not even pairs of leaves perpendicular to a common diameter through the origin can be parallel. This set of foliating leaves is therefore skew.

This example enables us to state a theorem.

Theorem 1.1 $\mathbf{R}^3$ *may be ruled by a skew set of lines.*

We conclude this discussion by noting that we may derive sufficient conditions for a family of *elliptical* OSHs to not intersect. Consider a pair of

OSHs with equations

$$\frac{x^2}{a_i{}^2} + \frac{y^2}{b_i{}^2} - \frac{z^2}{c_i{}^2} = 1, \quad i = 1, 2. \tag{9}$$

Eliminating x between these equations, we obtain

$$y^2\left(\frac{a_1{}^2}{b_1{}^2} - \frac{a_2{}^2}{b_2{}^2}\right) + z^2\left(\frac{a_2{}^2}{c_2{}^2} - \frac{a_1{}^2}{c_1{}^2}\right) = a_1{}^2 - a_2{}^2. \tag{10}$$

There will be no intersection if $a_1 < a_2$, and $\frac{a_1}{b_1} \geq \frac{a_2}{b_2}$ and $\frac{a_1}{c_1} \leq \frac{a_2}{c_2}$. Then, if b and c are smooth functions of a, this condition can be written infinitesimally as

$$\frac{d}{da}\left(\frac{a}{c}\right) \geq 0 \quad \text{and} \quad \frac{d}{da}\left(\frac{a}{b}\right) \leq 0. \tag{11}$$

We can similarly show that the OSHs will also not intersect if one of the following two other conditions obtain: If c and a are smooth functions of b,

$$\frac{d}{db}\left(\frac{b}{c}\right) \geq 0 \quad \text{and} \quad \frac{d}{db}\left(\frac{b}{a}\right) \leq 0; \tag{12}$$

or, if a and b are smooth functions of c,

$$\frac{d}{dc}\left(\frac{c}{a}\right) \leq 0 \quad \text{and} \quad \frac{d}{dc}\left(\frac{c}{b}\right) \leq 0. \tag{13}$$

1.2 Variations and local constructions

1.2.1 A ruling of $\mathbf{R}^3$ via OSHs which includes a cylinder of parallel leaves.

In the following example, we exhibit a ruling of $\mathbf{R}^3$ in which the leaves foliating the circular OSHs converge to that of the natural foliation of a cylinder.

Example 1.2 (Rulings including a cylinder of parallel leaves) Refering to the notation of (1), set $b = a$ and parameterize $c = c(a)$ as

$$c(a) = \begin{cases} e^{a_0/(a-a_0)} & \text{if } a_0 < a; \\ \infty & 0 \leq a \leq a_0, \end{cases} , \tag{14}$$

where a_0 is a fixed positive number. We now check that this satisfies the condition in (7) for nonintersection:

$$\frac{d}{da}\left(\frac{a}{e^{a_0/(a-a_0)}}\right) = \left(1 + \frac{aa_0}{(a_0 - a)^2}\right) e^{a_0/(a_0-a)} > 0. \tag{15}$$

which is clearly positive for all values of a.

Finally, we note that since these leaves are parallel to the vector

$$\left(-e^{-a/(a-a_0)} \sin\alpha, e^{-a/(a-a_0)} \cos\alpha, 1\right),$$

which approaches $(0, 0, 1)$ as a approaches a_0, their directions approach that of the leaves of a natural foliation of the cylinder as they themselves approach this cylinder.

We note in passing that it is easy to have a ruling of the second type except for an open set which is a ruling of the first type.

1.2.2 A ruling of $\mathbf{R}^3$ via OSHs which includes a bounded sheet of parallel leaves

We begin with the following definitions:

Definition 1.1 *We call a set of parallel leaves* bounded *if its intersection with a plane perpendicular to its member leaves is bounded. Otherwise, we call* *it* unbounded. *A* connected component of parallel leaves (CCPL) *consisting of at least two elements and with no interior is called a* sheet.

We now demonstrate that we can foliate $\mathbf{R}^3$ less a convex bounded sheet with elliptical OSHs in order to form a ruling in which the direction of the leaves approach that of the leaves in the natural ruling of the bounded sheet.

Example 1.3 (Rulings including a strip of parallel leaves) The family of OSHs defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{(a+1)^2} - \frac{z^2}{(1/a)^2} = 1, \tag{16}$$

satisifies the inequalities in (12), and as a approach zero, the distance between the waist and the bounded sheet $x = 0$, $-1 < y < 1$, $z \in \mathbf{R}$ approaches zero, while the direction of the leaves through the waist approach that of the natural foliation of the sheet. This latter fact can be demonstrated by noting that the leaves through the waist of an OSH represented by the above equation are parallel to the vector $(-a^2 \sin\alpha, a(a+1)\cos\alpha, 1)$, which approaches $(0, 0, 1)$ as a approaches zero.

We now show that the set of sheets parallel to a given leaf need not be bounded or even connected, even if the ruling is not trivial or does not integrate into parallel planes.

Example 1.4 (Rulings including two coplanar half-planes) We may include in the ruling leaves which integrate into a pair of coplanar parallel half-planes. Let the half-planes lie in the (x, y)-plane, the one terminating at the line $x = 1$, the other at $x = -1$ in that plane. Foliate the remaining space with the OSHs

$$\frac{x^2}{\left(1 - \frac{1}{s+1}\right)^2} + \frac{y^2}{s^2} - z^2 = 1. \tag{17}$$

We now prove that this family of OSHs satisfies the conditions in (11), thus proving that no pair of them intersect. Let $r = 1 - 1/(s+1)$, so that $s = 1/(1-r) - 1$. Then, in the notation of (11), $\frac{a}{c} = r$ and $\frac{a}{b} = \frac{r}{\frac{1}{1-r} - 1} = 1 - r$. The respective derivatives of these functions with respect to r are 1 and -1. This proves that the conditions are satisfied.

Each OSH given by (17) may be ruled with leaves which we may parameterize as

$$\left(\left(1 - \frac{1}{s+1}\right)(\cos\theta + t\sin\theta), s(\sin\theta - t\cos\theta), t\right), \tag{18}$$

$0 \le \theta < 2\pi$, $t \in \mathbf{R}$. From this we see that as $s \to \infty$, $\theta \ne \pm\pi/2$, the direction of the leaves approaches the direction of the leaves which form the natural ruling of the half-planes, *i.e.*, parallel to the y-axis, while their intersections with the plane $z = 0$ approach the two sheets. Moreover, as $s \to 0$, their direction approaches $(0, 0, 1)$.

1.2.3 Rulings of $\mathbf{R}^3$ via hyperbolic paraboloids; rulings including a half-plane strip

We may use the fact that hyperbolic paraboloids (HPs), *i.e.* surfaces with equations of the form

$$x = \frac{y^2}{a^2} - \frac{z^2}{b^2}, \tag{19}$$

are ruled surfaces to construct still other interesting rulings of $\mathbf{R}^3$. That HPs are themselves ruled surfaces can be seen by parameterizing (x, y, z) in the above equation as $(2st + s^2, a(s+t), bt)$, where $s \in \mathbf{R}$ is a parameter for the set of leaves ruling the HP and $t \in \mathbf{R}$ is the parameter for the points of any given leaf ruling the HP. That s and t parameterize the entire HP can be seen by noting that as t varies along $\mathbf{R}$, the entire set of parabolas generated by the intersection of the HP with $z = t$, for each $t \in \mathbf{R}$ is produced. (We can also prove that HPsare ruled surfaces by notng that each one is projectively equivalent to an OSH.) We now show that $\mathbf{R}^3$ may be ruled with such HPs.

Consider the HP

$$x = \frac{y^2}{a^2} + c - \frac{z^2}{b^2}. \tag{20}$$

For $z = 0$, we get $x = \dfrac{y^2}{a^2} + c$, which has for its graph a parabola with vertex at $(c, 0)$, focus at $(\frac{1}{4}a^2 + c, 0)$, and directrix with equation $x = -\frac{1}{4}a^2 + c$. Consider the pair of HPs given by

$$x = \frac{y^2}{a_i^2} + c_i - \frac{z^2}{b_i^2}, \quad i = 1, 2. \tag{21}$$

The HPs represented by these equations will not intersect if the following equation does not have a solution:

$$y^2\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right) + z^2\left(\frac{1}{b_2^2} - \frac{1}{b_1^2}\right) = c_2 - c_1. \tag{22}$$

This will be the case if, when c increases, b does not decrease and a does not increase, or when

$$\frac{db}{dc} \geq 0 \quad \text{and} \quad \frac{da}{dc} \leq 0. \tag{23}$$

In more geometric terms, as the vertex moves up the x-axis, the focus and the directrix should approach each other.

Example 1.5 (Rulings including one half-plane) We may use HPs to develop a ruling of $\mathbf{R}^3$, less a half-plane, such that when we give the half-plane its natural ruling, we obtain a continuous ruling of $\mathbf{R}^3$. We rule $\mathbf{R}^3$ after deleting the half-plane $1 \leq x < \infty$, $y = 0$, with the following HPs:

$$x - \frac{1}{1+a} = \frac{y^2}{a^2} - a^2 z^2, \tag{24}$$

$0 < a < \infty$. Since the leaves may be parameterized as

$$\left(\frac{2st + s^2 + 1}{a+1}, a(t+s), \frac{t}{a} \right), \tag{25}$$

where $s, t \in \mathbf{R}$, as a increases, the leaves become more vertical and so we may complete a continuous ruling of all of $\mathbf{R}^3$ by ruling the half-plane with vertical lines.

2 Local Relations

2.1 An alternate parameterization of the lines

In defining rulings in Section 0.1, we introduced one way of parameterizing lines. We now present another.

Any two non-parallel lines may be joined by a unique mutual orthogonal which intersects these lines at their *points of proximity*. Refering to Figure 1 on the next page, fix a reference leaf. Any line not parallel to this reference

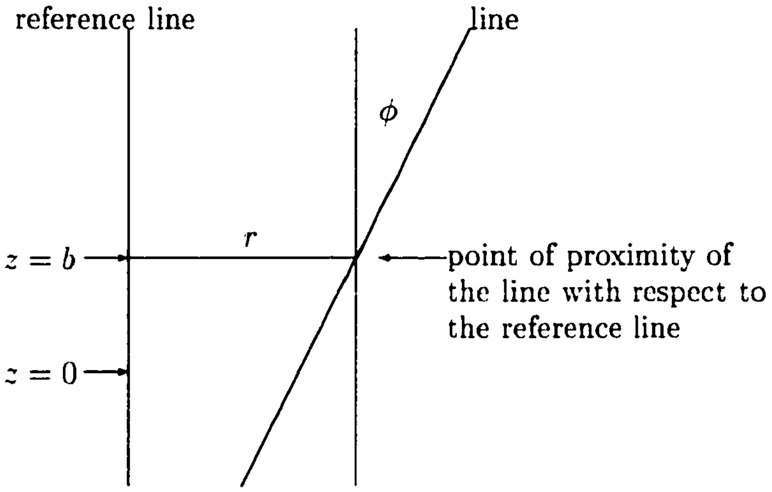

Figure 1: (b, r, θ, ϕ) coordinates

leaf has a unique point of proximity on it. Let b denote the position of this point on the fixed line relative to a fixed reference point on it; r the distance between the points of proximity; θ the angle of the mutual orthogonal with respect to some fixed plane in which the reference line lies; and ϕ the angle between the direction of the reference line and the other line (oriented so that the reference line, the oriented mutual perpendicular from the reference leaf to the line being described, and the positive direction in the plane perpendicular to the mutual orthogonal form a right-hand system) uniquely determine this other line. We will refer to these as the (b, r, θ, ϕ) coordinates.

2.2 More general conditions for a ruling of space

In §1.1, we found necessary conditions on OSHs which would make them integral surfaces of a ruling of $\mathbf{R}^3$. We now find a local condition which generalizes those calculations. We assume in this section that for each leaf, there exists a neighborhood such that no two leaves in it are parallel.

Fix a reference leaf and call it λ. Consider two leaves, λ_1 and λ_2. Label them according to the (b, r, θ, ϕ) system. Let Π_i be the plane containing leaf λ_i perpendicular to the mutual orthogonal between it and the fixed leaf. If $\Pi_1 \neq \Pi_2$, the leaves will intersect if and only if they intersect on ℓ, the intersection of Π_1 and Π_2, a line which will be parallel to λ. In the following calculation, we fix the plane through λ on which $\theta = 0$ as the plane which includes ℓ. We label the coordinates in the obvious way. Note that the distance between λ and ℓ is $r_1 \sec\theta_1 = r_2 \sec\theta_2 = r_2 \sec(\Delta\theta + \theta_1)$, where $\Delta\theta = \theta_2 - \theta_1$. We may assume that $0 < \Delta\theta < \pi$.

Then

$$r_2 \cos\theta_1 = r_1 \cos(\Delta\theta + \theta_1) = r_1 (\cos\Delta\theta \cos\theta_1 - \sin\Delta\theta \sin\theta_1), \tag{26}$$

so that

$$r_1 (\cos\Delta\theta - \sin\Delta\theta \tan\theta_1) = r_2 \tag{27}$$

and

$$r_1 \tan\theta_1 = \frac{-r_2 + r_1 \cos\Delta\theta}{\sin\Delta\theta}. \tag{28}$$

Similarly,

$$r_2 \tan\theta_2 = \frac{r_1 - r_2 \cos\Delta\theta}{\sin\Delta\theta}. \tag{29}$$

The difference between the z-coordinates of the points of intersection of ℓ with λ_1 and with λ_2,

$$b_1 + r_1 \tan\theta_1 \cot\phi_1 - (b_2 - (r_2 \tan\theta_2) \cot\phi_2), \tag{30}$$

has a fixed sign and may without loss of generality be assumed positive. Then

$$-\Delta b + r_2 \cot \phi_2 \tan \theta_2 + r_1 \cot \phi_1 \tan \theta_1 > 0, \tag{31}$$

or, using (28) and (29) and letting $r = r_1$, $\Delta r = r_2 - r_1$, $\phi = \phi_1$, and $\Delta\phi = \phi_2 - \phi_1$, we obtain

$$-\Delta b + \frac{1 - \tan\phi \tan\Delta\phi}{\tan\phi + \tan\Delta\phi} \frac{r - (r + \Delta r)\cos\Delta\theta}{\sin\Delta\theta} + \frac{1}{\tan\phi} \frac{r\cos\Delta\theta - (r + \Delta r)}{\sin\Delta\theta} > 0, \tag{32}$$

Simple trigonometry allows us to rewrite this as follows:

Proposition 2.1′ *In a ruling of* $\mathbf{R}^3$ *in which no two leaves are parallel, if the leaves are parameterized by the* (b, r, θ, ϕ) *parameters,*

$$-\Delta b + \frac{1 - \tan\phi \tan\Delta\phi}{\tan\phi + \tan\Delta\phi} \frac{2r\sin^2\frac{\Delta\theta}{2} - \Delta r\cos\Delta\theta}{\sin\Delta\theta} - \frac{1}{\tan\phi} \frac{2r\sin^2\frac{\Delta\theta}{2} + \Delta r}{\sin\Delta\theta} > 0$$

will have a fixed sign and never equal zero.

A second and closely related inequality is derived from considering the determinant $\left|\vec{\imath}_1, \vec{\imath}_2, \Delta\vec{r}\right|$, where $\vec{\imath}_1$ and $\vec{\imath}_2$ are unit vectors respectively parallel to λ_1 and λ_2, two leaves in the ruling, and $\Delta\vec{r}$ is a vector with its base on λ_1 and its terminus on λ_2. This value is independent of the choice of base and terminus because the difference between any two such choices is parallel to a linear combination of $\vec{\imath}_1$ and $\vec{\imath}_2$. We therefore choose as their initial and terminal points their respective points of proximity with the fixed leaf. Consider the vector with its base on the fixed leaf and its terminus on leaf k (with leaf $k = 0$ indexing the parameters defining our reference leaf). Any point on λ_k will have coordinates

$$r_k = (r_k \cos\theta_k, r_k \sin\theta_k, 0), \tag{33}$$

where r_k and θ_k are the polar coordinates of $\lambda_k \cap \Pi$, Π being a plane parallel to the (x, z)-plane, with origin at $\lambda \cap \Pi$. Also, recalling that $\phi_k \neq 0$, we may let

$$\vec{i}_k = (\sin\phi_k \sin\theta_k, -\sin\phi_k \cos\theta_k, \cos\phi_k)\,, \tag{34}$$

where ϕ is the vertical angle of λ_k with respect to λ, since $\vec{r}_k \cdot \vec{i}_k = 0$ and $\vec{i}_0 \cdot \vec{i}_k = \cos\phi_k$ and $\left\|\vec{i}_0\right\| = \left\|\vec{i}_k\right\| = 1$. Then

$$\Delta\vec{r} = (r_1 \cos\theta_1 - r_2\cos\theta_2, r_1\sin\theta_1 - r_2\sin\theta_2, b_1 - b_2)\,. \tag{35}$$

The above considerations show that the condition that no pair of leaves intersect is equivalent to

$$D = \begin{vmatrix} \sin\phi_1\sin\theta_1 & -\sin\phi_1\cos\theta_1 & \cos\phi_1 \\ \sin\phi_2\sin\theta_2 & -\sin\phi_2\cos\theta_2 & \cos\phi_2 \\ r_1\cos\theta_1 - r_2\cos\theta_2 & r_1\sin\theta_1 - r_2\sin\theta_2 & b_1 - b_2 \end{vmatrix} \tag{36}$$

having a fixed sign for all pairs of leaves in the ruling and never equaling zero; we may assume that it is positive. So

$$\begin{aligned} D &= \begin{vmatrix} \sin\theta_1 & -\cos\theta_1 & \cot\phi_1 \\ \sin\theta_2 & -\cos\theta_2 & \cot\phi_2 \\ r_1\cos\theta_1 - r_2\cos\theta_2 & r_1\sin\theta_1 - r_2\sin\theta_2 & b_1 - b_2 \end{vmatrix} \sin\phi_1\sin\phi_2 \\ &= \begin{vmatrix} \sin\theta_1 & -\cos\theta_1 & \cos\phi_1\sin\phi_2 \\ \sin\theta_2 & -\cos\theta_2 & \sin\phi_1\cos\phi_2 \\ r_1\cos\theta_1 - r_2\cos\theta_2 & r_1\sin\theta_1 - r_2\sin\theta_2 & (b_1 - b_2)\sin\phi_1\sin\phi_2 \end{vmatrix} \\ &= (r_1\cos\theta_1 - r_2\cos\theta_2) \begin{vmatrix} -\cos\theta_1 & \cos\phi_1\sin\phi_2 \\ -\cos\theta_2 & \sin\phi_1\cos\phi_2 \end{vmatrix} \\ &\quad - (r_1\sin\theta_1 - r_2\sin\theta_2) \begin{vmatrix} \sin\theta_1 & \cos\phi_1\sin\phi_2 \\ \sin\theta_2 & \sin\phi_1\cos\phi_2 \end{vmatrix} - \Delta b \begin{vmatrix} \sin\theta_1 & -\cos\theta_1 \\ \sin\theta_2 & -\cos\theta_2 \end{vmatrix} \sin\phi_1\sin\phi_2 \end{aligned}$$

$$
\begin{aligned}
&= -(r_1\cos\theta_1 - r_2\cos\theta_2)(\sin\phi_1\cos\phi_2\cos\theta_1 - \cos\phi_1\sin\phi_2\cos\theta_2) \\
&\quad -(r_1\sin\theta_1 - r_2\sin\theta_2)(\sin\phi_1\cos\phi_2\sin\theta_1 - \cos\phi_1\sin\phi_2\sin\theta_2) \\
&\quad -\Delta b(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)\sin\phi_1\sin\phi_2 \\
&= -r_1\sin\phi_1\cos\phi_2 - r_2\cos\phi_1\sin\phi_2 + r_1\cos\Delta\theta\cos\phi_1\sin\phi_2 \\
&\quad +r_2\cos\Delta\theta\sin\phi_1\cos\phi_2 - \Delta b\sin\Delta\theta\sin\phi_1\sin\phi_2 > 0. \qquad (37)
\end{aligned}
$$

Rewriting, we obtain the following:

Proposition 2.2 *In a ruling of* $\mathbf{R}^3$ *in which no pair of leaves are parallel, the leaves being parameterized by the* (b, r, θ, ϕ) *parameters,*

$$
\begin{aligned}
&-\Delta b(\tan\phi + \tan\Delta\phi)\tan\phi\sin\Delta\theta \qquad (38)\\
&\quad -(1 - \tan\phi\tan\Delta\phi)\left(2r\sin^2\frac{\Delta\theta}{2} - \Delta r\cos\Delta\theta\right) - \left(2r\sin^2\frac{\Delta\theta}{2} + \Delta r\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\Delta b\sin\Delta\theta\sin\phi_1\sin\phi_2 \qquad (39)\\
&\quad +(r_2\cos\Delta\theta - r_1)\sin\phi_1\cos\phi_2 + (r_1\cos\Delta\theta - r_2)\cos\phi_1\sin\phi_2
\end{aligned}
$$

will have a fixed sign and never equal zero.

The difference between the two inequalities lies in including the information in the inequalities (28) and (29).

The second statement of (39) implies the condition in (7): Without loss of generality, we may assume that $\phi = r = 0$. Then (39) reduces to $\Delta r\sin\Delta\phi$. We obtain our result by dividing by Δr^2 and taking the limit as $\Delta r \to 0$. We note that the resulting parameter $\lim_{\Delta r\to 0}\frac{\sin\Delta\phi}{\Delta r} = \frac{d\phi}{dr}$ will reappear later in this paper.

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2.3 The Jacobian of the mapping from the position of a leaf to its direction

We assume throughout this section that the ruling is 1) directed and 2) $\mathbf{C}^1$. Give $\mathbf{R}^3$ coordinates so that a particular leaf lies along the z-axis. In a sufficiently small neighborhood of it, each leaf will intersect the (x, y)-plane at a single point. Then to each point in the (x, y)-plane is associated the values (p, q) as defined in §2.1. We examine the Jacobian of *the Gauss map*

$$(x, y, 0) \xrightarrow{N_0^\infty} (p(x, y, 0), q(x, y, 0)), \tag{40}$$

i.e.

$$J = \begin{pmatrix} p_x & p_y \\ q_x & q_y \end{pmatrix}. \tag{41}$$

2.3.1 J in the case the OSH is a surface of revolution

We first examine J in the special case that the foliation integrates into coaxial, circular OSHs with coplanar waists as discussed in §1.1. In this case, in terms of (b, r, θ, ϕ) coordinates, b is constant and ϕ only depends on r. The points of proximity in $\mathbf{R}^3$ with respect to the leaf which forms the axis of these OSHs will be

$$(r\cos\theta, r\sin\theta, 0). \tag{42}$$

and

$$(p, q) = (-\sin\theta\tan\phi, \cos\theta\tan\phi). \tag{43}$$

The coordinates (x, y, z) of a point on a leaf will be $(r\cos\theta, r\sin\theta, 0) + z\,(p, q, 1)$, and, from (43), $x = (r\cot\phi)q + zp$ and $y = -(r\cot\phi)\,p + zq$.

Then, letting $\tilde{\rho} = r\cot\phi$, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z & \tilde{\rho} \\ -\tilde{\rho} & z \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \tag{44}$$

so that

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \dfrac{xz - y\tilde{\rho}}{z^2 + \tilde{\rho}^2} \\ \dfrac{x\tilde{\rho} + yz}{z^2 + \tilde{\rho}^2} \end{pmatrix}. \tag{45}$$

Let $R = z^2 + \tilde{\rho}^2$. Then, after some calculation, we obtain

$$\begin{aligned} \tilde{J} &= \frac{1}{R}\begin{pmatrix} z & -\tilde{\rho} \\ \tilde{\rho} & z \end{pmatrix} \\ &\quad + \frac{1}{R^2}\begin{pmatrix} y\tilde{\rho}^2\tilde{\rho}_x - 2xz\tilde{\rho}\tilde{\rho}_x - yz^2\tilde{\rho}_x & y\tilde{\rho}^2\tilde{\rho}_y - 2xz\tilde{\rho}\tilde{\rho}_y - yz^2\tilde{\rho}_y \\ -x\tilde{\rho}^2\tilde{\rho}_x - 2yz\tilde{\rho}\tilde{\rho}_x + xz^2\tilde{\rho}_x & -x\tilde{\rho}^2\tilde{\rho}_x - 2yz\tilde{\rho}\tilde{\rho}_y + xz^2\tilde{\rho}_y \end{pmatrix} \end{aligned} \tag{46}$$

$$= \frac{1}{R}\begin{pmatrix} z & -\tilde{\rho} \\ \tilde{\rho} & z \end{pmatrix} + \frac{1}{R^2}\begin{pmatrix} \tilde{\rho}^2 - z^2 & 2z\tilde{\rho} \\ -2z\tilde{\rho} & \tilde{\rho}^2 - z^2 \end{pmatrix}\begin{pmatrix} y\tilde{\rho}_x & y\tilde{\rho}_y \\ -x\tilde{\rho}_x & -x\tilde{\rho}_y \end{pmatrix} \tag{47}$$

.

Defining $J = \tilde{J}$ and $\rho = \tilde{\rho}$ for $x = y = 0$, we obtain

$$J = \frac{1}{R}\begin{pmatrix} z & -\rho \\ \rho & z \end{pmatrix} \tag{48}$$

and

$$J^{-1} = \begin{pmatrix} z & \rho \\ -\rho & z \end{pmatrix} = zI + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\rho, \tag{49}$$

where

$$\rho = \lim_{r\to 0} r\cot\phi = \frac{1}{\lim_{r\to 0}\left(\dfrac{\tan\phi}{r}\right)} = \left(\frac{d\phi}{dr}\right)^{-1}. \tag{50}$$

We will discuss ρ in §2.5. In the meantime, we note that the imaginary component of each eigenvalue of J (*i.e.*, $(z \pm i\rho)/R$) is the Gaussian curvature of a ruled surface consisting of leaves of the foliation which are tangent to the central leaf and the radial direction [5, p. 192]. Evaluated at $z = 0$, this gives ρ, the reciprocal of the rate at which the leaf "leans" in the direction perpendicular to the radial direction.

2.3.2 J in the general case

We now examine J in the general case, in which the b and ϕ in the (b, r, θ, ϕ) parameterization are allowed to vary with θ for fixed r.

As in §2.3.1, we will consider the Jacobian of the map in a neighborhood of the leaf through $(x, y, 0) = (0, 0, 0)$ to the direction of the leaf through it; call it $J(0)$, indicating that it is from the origin of the plane $z = 0$. If N_0^z is the map which takes a point from the (x, y)-plane through a vertical distance z along the leaf running through it, we can then factor the Gauss map N_0^∞ of (40) as

$$(x, y) \stackrel{N_0^z}{\longmapsto} (x + p(x, y, z)z, y + q(x, y, z)z) \stackrel{N_z^\infty}{\longmapsto} (p, q), \tag{51}$$

where z is considered as constant. The Jacobian of the second map for any fixed z we will call $J(z)$, indicating that it is from the origin of a plane on which z is constant; the J of §2.3.1 is then $J(0)$. The Jacobian of the first map is $I + J(0)z$. Since $N_z^\infty N_0^z = N_0^\infty$ we have, by the chain rule,

Lemma 2.1 $J(z)\,(I + zJ(0)) = J(0)$.

From this we prove

Corollary 2.1 *The eigenvectors of $J(z)$ are independent of z. If μ_0 is an eigenvalue associated with a given eigenvector of $J(0)$, the eigenvalues of $J(z)$ corresponding to this same eigenvector are $\frac{\mu_0}{1+z\mu_0}$.*

Proof: Let $\vec{v}$ be an eigenvector of $J(0)$ with eigenvalue μ_0. Then

$$\begin{aligned} J(z)\,(I + zJ(0))\,\vec{v} &= J(0)\vec{v} \\ J(z)\,(\vec{v} + z\mu_0\vec{v}) &= \mu_0\vec{v} \\ J(z)\,(1 + z\mu_0)\,\vec{v} &= \mu_0\vec{v} \\ J(z)\vec{v} &= \frac{\mu_0}{1+z\mu_0}\vec{v}. \end{aligned} \tag{52}$$

The following are immediate consequences of Corollary 2.1:

Corollary 2.2 *If one eigenvector has zero eigenvalue for any value of z, it has zero eigenvalue for all values of z.*

Corollary 2.3 *The rank of $J(z)$ is independent of z.*

Theorem 2.0 *If an eigenvalue of $J(z)$ along a given leaf is real, it is zero.*

Proof: Restrict to the invariant subspace of $\mathbf{R}^2$ corresponding to the non-vanishing eigenvalues; let $\bar{J}(z)$ be the matrix of the restriction of $J(z)$ to this subspace. From Lemma 2.1, $\det(\bar{J}(z))\det(I + z\bar{J}(0)) = \det(\bar{J}(0))$. By Corollaries 2.2 and 2.3, $\det(I + z\bar{J}(0))$ does not vanish for any real z and $\bar{J}(z)$, and so $J(z)$, has no real nonzero eigenvalues. □

Corollary 2.4 *$(I + zJ(0))$ is invertible for all z.*

Lemma 2.2 *$J(z_1)J(z_2) = J(z_2)J(z_1)$ for all z.*

Proof: By Corollary 2.4, we may write $J(z) = J(0)(I + zJ(0))^{-1}$. Then $J(z_1)J(z_2) = J(0)(I + z_1J(0))^{-1}J(0)(I + z_2J(0))^{-1}$
$= J(0)(I + z_2J(0))^{-1}J(0)(I + z_1J(0))^{-1} = J(z_2)J(z_1)$. □

Repeatedly differentiating both sides of the equation of Lemma 2.1 with respect to z and utilizing Lemma 2.2, we get $J^{(k)}(z)(I+zJ(0))+kJ^{(k-1)}(z)J(0) = 0$. An induction argument shows that this can be written in the following form:

Theorem 2.1 $J^{(k)}(z) = (-1)^{k-1}k!J(z)J(0)^k(I + zJ(0))^{-k}$

If $J(z)$ is invertible, Lemma 2.1 yields $J^{-1}(z) = J^{-1}(0)(I + zJ(0))$, which implies that $\dfrac{\partial J^{-1}(z)}{\partial z} = J^{-1}(0)J(0) = I$, from which we obtain

Theorem 2.2 *If $J(z)$ is invertible,* $\dfrac{\partial J^{-1}(z)}{\partial z} = I$.

Differentiating the equation of Lemma 2.1 by one of the independent variables, say x, we get, after some simple algebra,

$$J_x(z)(I + zJ(0)) = (I - zJ(z))J_x(0).$$

Repeated differentiation of this with respect to the independent variables, collecting the terms with the highest derivative on the left side of the equation, and the right-multiplication of each side by the inverse of their common factor, $I + zJ(0)$ (which will always exist for $z \neq 0$ by Corollary 2.4), displays the derivatives of $J(z)$ in terms of derivatives of $J(z)$ of lower order and derivatives of $J(0)$ of the same order in each of the independent variables. From this, we arrive at the following conclusion:

Theorem 2.3 *The degree of continuity of a ruling in the neighborhood of a leaf is a constant on each leaf of that ruling.*

2.4 A condition for a locally trivial ruling

Theorem 2.4 *The leaves of a ruling are locally normals to a common continuous surface if and only if the ruling consists of parallel lines.*

Proof: It is clear that if the ruling consists of parallel lines, its leaves are perpendicular to a common surface, *i.e.*, any plane perpendicular to them.

To prove the converse, consider a simple closed curve $\Sigma = \Sigma(s)$ in a plane Π perpendicular to a given leaf λ with $\lambda \cap \Pi$ in its interior with respect to Π, with $\Sigma(0) = \Sigma(1)$. Assume that Σ is small enough so that none of the leaves through it lie in Π. We choose a coordinate system in which $\vec{k} = (0, 0, 1)$ lies along λ. We may then chose direction vectors for the leaves such that, in the notation of §2.2, $[\vec{k}, \vec{l}(s), \vec{r}(s)]$ will have a fixed sign or be zero. If these leaves are perpendicular to a common continuous surface, as s increases, the z coordinate of the point of intersection of the leaves through Σ with that surface are either nonincreasing or nondecreasing. Therefore this z-coordinate must remain constant and the leaves through Σ must all be parallel to λ. Anticipating Proposition 3.2, all the leaves running through the interior of Σ with respect to Π must also be parallel to λ.

Now let ℓ be a line in Π intersecting λ. Again, anticipating Proposition 3.2, the CCCP including λ is convex, and so we can assume that ℓ intersects Σ at two points. We repeat this argument for any simple closed curve in Π centered on the intersections of Σ with ℓ. Continuing this process, we obtain an increasing cover of ℓ of closed sets with leaves parallel to λ running through them. Moreover, this process cannot end a finite distance from λ on either side of its intersection with ℓ since the closure of such a limit would also have leaves through it parallel to λ, since the set of leaves parallel to λ is closed. We have thus proven that the ruling includes a parallel plane of leaves. We now appeal to Proposition 3.1, which concludes the proof of the theorem. □

Definition 2.1 *If for a foliation of an n-dimensional space by k-dimensional leaves there exists a second foliation of the space by $(n-k)$-dimensional leaves such that the leaves of the first foliation are perpendicular to the leaves of the second foliation at their points of intersection, the foliation is said to be* orthogonal integrable.

We have thus proven:

Theorem 2.4′ *A ruling of $\mathbf{R}^3$ is orthogonal integrable precisely when it consists of parallel leaves.*

It is demonstrated in [2] that this is not the case in $\mathbf{H}^3$. The surfaces orthogonal to geodesic fields in $\mathbf{H}^3$ are generated as envelopes of horospheres. The analogous objects in $\mathbf{R}^3$ to horospheres in $\mathbf{H}^3$ are parallel planes, and a family of parallel planes in $\mathbf{R}^3$ has only the trivial envelope.

2.5 Rulings of $\mathbf{R}^3$ and properties of ruled surfaces.

For convenience's sake, we repeat a definition made in §2.1 and develop it:

Definition 2.2 *For any pair of non-parallel lines ℓ_1 and ℓ_2 in $\mathbf{R}^3$, there exist points $p_1 \in \ell_1$ and $p_2 \in \ell_2$ such that the distance between p_1 and p_2 is less than the distance between $q_1 \in \ell_1$ and $q_2 \in \ell_2$, where either $q_1 \neq p_1$ or $q_2 \neq p_2$. We call p_1 and p_2 their respective* points of proximity.

We now note that the triple product of §2.2, $\left|\vec{i}_1, \vec{i}_2, \Delta\vec{r}\right|$, is closely related to the *parameter of distribution* of a ruled surface, $\left|\vec{i}, \vec{i}', \vec{r}'\right|$, where $r(s)$ is a curve through the lines of the ruled surface parameterized by s, the arc length along the curve, and the derivative is taken with respect to s [3, p. 191]. Indeed,

$$\begin{aligned}\lim_{|\Delta s|\to 0}\left|\vec{i}(s), \vec{i}(s+\Delta s), \Delta\vec{r}\right| &= \lim_{|\Delta s|\to 0}\left|\vec{i}(s), \frac{\vec{i}(s+\Delta s)-\vec{i}(s)}{\Delta s}, \frac{\Delta\vec{r}}{\Delta s}\right| |\Delta\vec{s}|^2 \\ &= \left|\vec{i}(s), \vec{i}'(s), r'(\vec{s})\right| (ds)^2. \qquad (53)\end{aligned}$$

The parameter of distribution can be understood as the the maximum value of $\frac{d\phi}{dr}$ evaluated at any curve running through the leaf at its point of intersection with that leaf. These maximum values are achieved along the curve through the leaf defined as follows:

Definition 2.3 *Allow a family of lines in a ruled surface to approach a fixed line on which the parameter of distribution does not vanish. The limit of their points of proximity is called the* point of striction. *The locus of such points is called the* curve of striction [3, pp. 192-193].

The parameter of distribution of a ruled surface is a function of the leaf, *i.e.* it is independent of on which point of the leave it is measured: one of the vectors in the triple product which defines it lies along the leaf and so displacement in that direction will not change the value of this triple product. However, in a ruled space, the parameter of distribution is in general not defined for each leaf. Rather, the parameter of distribution on a leaf depends on and integral surface of the ruled space in which it is measured. (We also note here that the variable ρ of §2.3.1 is the reciprocal of the parameter of distribution [4, p. 245]).

Proposition 2.3 *If two ruled surfaces are tangent to each other along a leaf and the parameters of distribution on that leaf with respect to each of these surfaces are nonvanishing, the points of striction on the leaf with respect to each surface will be the same.*

Proof: The normal to the tangent plane of a ruled surface on a leaf in it has the property that if the parameter of distribution is non-vanishing, it will rotate through an angle π as the point it is measured on moves along that leaf. The point of striction has the property that this normal plane will have turned $\pm\frac{\pi}{2}$ as the point on which it is measured at comes in from $\pm\infty$ [3, p. 194]. If two ruled surfaces are tangent along a leaf, they will share tangent planes for each point on that line, and so will share the point of striction on that line. □

The question of the existence of a "surface of striction" analogous to the curve of striction naturally arises. A necessary condition for this is that b be defined and fixed for all θ (in the (b, r, θ, ϕ) coordinates) on each leaf. An example of a surface of striction is the plane passing through the common waists of the OSHs discussed in §1.1. Assuming that our surface of striction is smooth, we show that this is in a sense the only possibility locally:

Proposition 2.4 *A ruling of* $\mathbf{R}^3$ *will only have a surface of striction in a neighborhood if for each leaf in that neighborhood there exists a cartesian coordinate system such that* $J(z)$ *is skew symmetric for some* z.

We may assume that the surface intersects the leaf in question at $z = 0$. Fix a central leaf, λ. The square of the distance from λ to a point on leaf running through $(\Delta x, \Delta y, 0)$ along a line perpendicular to λ will be $(\Delta x + (p_x \Delta x + p_y \Delta y)\, t)^2 + (\Delta y + (q_x \Delta x + q_y \Delta y)\, t)^2$. Setting the derivative of this with respect to t equal to zero and solving for t, we obtain a fraction whose numerator is $(p_x \Delta x + p_y \Delta y)\, \Delta x + (q_x \Delta x + q_y \Delta y)\, \Delta y$ or, letting $\tan\theta = \Delta y / \Delta x$, $p_x + (p_y + q_x)\tan\theta + q_y \tan^2\theta$. By a suitable choice of coordinates, we may choose the surface of striction to be tangent to the plane $z = 0$ at λ. Then we need each term in the above expression to vanish for all θ, and the result follows. □

2.6 preserving orthogonal flows

Consider an area in a ruling of $\mathbf{R}^3$ in a plane bounded by a simply-connected curve. We assume that the area is small enough so that none of the leaves through it lie in the plane. Letting this plane be $z = 0$, we call the map N_0^t of §2.3.2 the *orthogonal flow* for time t. We now examine the circumstances under which area is preserved under orthogonal flow.

Theorem 2.5 *The area of a planar region* $\mathcal{R}$ *with a boundary of finite length is preserved under orthogonal flow only if there exists a leaf in its interior such that the set of leaves parallel to it intersects the boundary of* $\mathcal{R}$.

Proof: Let λ be a leaf which intersects the interior of $\mathcal{R}$ which lies in a plane, Π. Assume that none of the leaves on the boundary of $\mathcal{R}$ are parallel to λ. Then the vertical angle of the direction of the leaves on the boundary and that of λ is bounded away from zero since the boundary of $\mathcal{R}$ is compact. Hence the distance between the intersection of the boundary leaves with a plane a distance d from Π from the intersection of λ with that plane can be made arbitrarily large for *all* the points on the boundary as d increases. Therefore the area of the image under orthogonal flow will not be preserved. □

Another expression of this is

Theorem 2.5′ *The area of a planar region $\mathcal{R}$ is preserved under orthogonal flow only if there exists a simply-connected subset $\mathcal{R}'$ of $\mathcal{R}$ with boundary of finite length which intersects a leaf such that the set of leaves parallel to it intersects the boundary of $\mathcal{R}'$.*

A condition on the J of §2.3.2 measured on a leaf can be obtained as follows:

Theorem 2.6 *The induced measure on the tangent map of the area in the neighborhood of a leaf is preserved under orthogonal flow if and only if* $\operatorname{tr} J = \det J = 0$ *on that leaf.*

Proof: The Jacobian of orthogonal flow is, as we have seen in §2.3.2, $I + zJ$, where z is fixed. Area will be preserved under this mapping if and only if $|I + zJ| = 1$. For J a 2×2 matrix, this is equivalent to $\operatorname{tr} J = \det J = 0$. □

Corollary 2.5 *If the area of an open set of a plane is preserved under orthogonal flow, the leaves through that set integrate into parallel planar strips of leaves.*

3 Global Relations

In this chapter, we consider the distribution of parallel leaves and how it affects the ruling as a whole.

We begin this discussion with two lemmas which hold for foliations of $\mathbf{R}^n$ by k-planes. In them, let Λ be the set of leaves parallel to a fixed leaf, Π an $(n-k)$-plane perpendicular to Λ, and π the orthogonal projection onto Π.

Lemma 3.1 $\pi(\Lambda^c) = \pi(\Lambda)^c$.

Proof: We first demonstrate that $\pi(\Lambda^c) \subseteq \pi(\Lambda)^c$. If we assume, by way of contradiction, that $\pi(\Lambda^c) \cap \pi(\Lambda) \neq \emptyset$, then there would exist a point $p \in \Pi$ and leaves $\mu \notin \Lambda$ and $\lambda \in \Lambda$ such that $p \in \pi(\mu) \cap \pi(\lambda)$. We note that since λ is the k-plane perpendicular to Π and so is $\pi^{-1}(p)$, $\lambda = \pi^{-1}(p)$. Then the condition on p immediately implies that $\mu \cap \lambda \neq \emptyset$.

We now show that $\pi(\Lambda^c) \supseteq \pi(\Lambda)^c$. π is surjective because for every point in $p \in \Pi$, $\pi(p) = p$. If we assume, by way of contradiction, that $\pi(\Lambda^c)^c \cap \pi(\Lambda)^c \neq \emptyset$, then there would exist $p \in \Pi$ such that $p \notin \pi(\Lambda^c)$ and $p \notin \pi(\Lambda)$, contradicting the fact that π is surjective. □

The continuity of the ruling implies a second lemma.

Lemma 3.2 Λ *is a closed set.*

We now return to the discussion of rulings of $\mathbf{R}^3$.

3.1 The distribution of parallel leaves

In this section, unless otherwise stated, let λ be a leaf, $\lambda(s)$ be the leaf containing point s, λ_1, λ_2, and λ_3 be parallel leaves, Λ be the union of leaves parallel to λ_1, λ_2, and λ_3, Λ_1, Λ_2, and Λ_3 path components of Λ, Π be a plane perpendicular to Λ, and π be the orthogonal projection from $\mathbf{R}^3$ to Π.

Proposition 3.1 *If a ruling of $\mathbf{R}^3$ contains one plane of leaves, it integrates into parallel planes.*

Proof: Let Π be a plane which is itself ruled by lines. Every leaf in the ruling of $\mathbf{R}^3$ not contained in Π must be parallel to it. Consider any other plane Π' parallel to Π. Then a leaf through any point in Π' must be parallel to Π and hence altogether contained in Π'. □

Proposition 3.2 *CCPLs in a ruling of $\mathbf{R}^3$ are convex.*

Proof: Let λ_1 and λ_2 be members of the same CCPL. Then $\pi(\lambda_1)$ and $\pi(\lambda_2)$ are two points in Π. If on the line segment between $\pi(\lambda_1)$ and $\pi(\lambda_2)$ there existed a point the leaf λ through which was not in Λ, $\pi(\lambda)$ would be a line separating Π into two half-planes. By Lemma 3.1, $\pi(\lambda) \cap \pi(\Lambda) = \emptyset$. Then $\mathbf{R}^3 \setminus \pi^{-1}(\pi(\lambda))$ would be a union of two non-empty open disjoint sets, each containing one of λ_1 or λ_2, forming a disconnection of the CCPL. □

From the fact that convex sets are path-connected, we derive a corollary.

Corollary 3.1 *CCPLs in $\mathbf{R}^3$ are path-connected.*

Remark 3.1 The previous corollary allows us to write all statements about CCPLs in $\mathbf{R}^3$ as statements about *path components of parallel leaves* (PCPLs).

Lemma 3.3 *For any ruling which does not integrate into parallel planes, if there exists a line ℓ such that $\ell \setminus \Lambda$ includes a non-empty bounded component A, then $\ell \setminus \Lambda = A$*

Proof: We may assume without loss of generality that $\ell \subset \Pi$.

Suppose $\ell \setminus \Lambda$ contained a second component, B.

Claim: *For any $p_1 \in \pi^{-1}(A), p_2 \in \pi^{-1}(B), \pi(\lambda(p_1)) \parallel \pi(\lambda(p_2))$.*

If this claim is true, then the leaves through $\pi^{-1}(\pi(p_1))$ form a plane of parallel leaves, and by Proposition 3.1, the ruling integrates into parallel planes, thus proving the lemma.

If the claim were false, $\pi(\lambda(p_1))$ and $\pi(\lambda(p_2))$ (which must be lines by Lemma 3.1) would intersect at some point r. Then the leaves through the points in the interval I between $i_1 = \pi^{-1}(r) \cap \lambda(p_1)$ and $i_2 = \pi^{-1}(r) \cap \lambda(p_2)$ would all be in Λ^c by Lemma 3.1. Since $\pi^{-1}(\ell)$ is a plane, each leaf through $\pi^{-1}(r)$ either intersects it once or is parallel to it. Then the projection onto Π of each leaf through $\pi^{-1}(r)$ intersects ℓ once or is parallel to it.

Let $J = \{j | j \in I \text{ and } \lambda(j) \cap \pi^{-1}(A) \neq \emptyset\}$. J is clearly open in I because the ruling is continuous. Also, $i_1 \in J$, so $J \neq \emptyset$. However, J must also be closed in I: For any $k \in I$ and sequence $\{j_n\} \subset J$ with limit k, $\lambda(k) \cap \overline{\pi^{-1}(A)} \neq \emptyset$. Since $\lambda(k) \cap \Lambda = \emptyset$ by Lemma 3.1, $\lambda(j) \cap \pi^{-1}(A) \neq \emptyset$ and so $k \in J$. Then J is open and closed in I and so $J = I$. But $i_2 \notin J$. This contradiction proves the claim and so proves the lemma. □

Theorem 3.1 *For any ruling which does not integrate into parallel planes, if Λ is not connected, then*

1) Each of its connected components includes a half-plane.

2) A plane containing leaves in two connected components of Λ intersects these connected components in a pair of disjoint closed half-planes.

3) If one of Λ's connected components is a half-plane then Λ is the union of two coplanar half-planes.

4) Let Λ_1 and Λ_2 be connected components of Λ with non-empty interior and let C_j denote the intersection of Λ_j with Π. Then there are two intersecting lines P_1 and P_2 in Π such that C_1 and C_2 lie in opposing sectors of $\Pi \setminus (P_1 \cup P_2)$. P_1 and P_2 can be chosen in such a way that one end of each is asymptotic to C_1. If P_1' and P_2' denote the corresponding lines for C_2, then P_1' is parallel to or coincides with P_1 and P_2' is parallel to or coincides with P_2.

5) Λ has at most two connected components.

Proof: To prove 1), suppose the ruling does not integrate into parallel planes and that λ_1 and λ_2 are members of distinct PCPLs, Λ_1 and Λ_2 respectively. Let ℓ be a line intersecting the Λ_is. By Lemma 3.3, if the ruling does not integrate into parallel planes, for $i = 1, 2$, $\ell \cap \Lambda_i$ must be a half-line, and so $\pi^{-1}(\ell \cap \Lambda_i)$ (which is in Λ_i by Lemma 3.1) must include a half-plane.

The proof of 1) just given implies the stronger statement 2).

To prove 3), let ℓ be a line in Π which intersects a half-plane component Λ_1 of Λ at a single point p. Lemma 3.3 shows that $\ell \setminus \{p\}$ cannot intersect another connected component of Λ. Therefore, any other connected component of Λ

must lie completely in the plane containing Λ_1.

To prove 4), let ℓ be any supporting line to C_1 (*i.e.*, any line containing a point p_1 of C_1 and such that C_1 is contained entirely in one of the closed half-planes determined by ℓ). Now it will be shown that C_2 must be in the opposite closed half-plane.

Suppose that there were a point p_2 of C_2 in the interior of the half-plane containing C_1, and let ℓ^* be the line determined by p_1 and p_2. Then ℓ^* would intersect ℓ tranversally at p_1. The component of $\ell^* \setminus \{p_1\}$ containing p_2 clearly contains points not in Λ. Since the other component contains no points of C_1, it also contains points not in Λ. This contradicts Lemma 3.3 and shows that C_1 and C_2 lie in opposite half-planes with ℓ as their common boundary.

This result is true for any supporting line. Let m be a line connecting a point q_1 in ∂C_1 to q_2 in ∂C_2. The points can be chosen so that m contains interior points of both C_1 and C_2. Let S denote the segment from q_1 to q_2. We may assume that m is oriented so that one component, R_1, of $\partial C_1 \setminus \{q_1\}$ will be on the right of m and the other, L_1, will be on the left. Let ∂C_1 be parameterized by arclength with $s > 0$ corresponding to R_1 and $s < 0$ corresponding to L_1. Let $\alpha(s)$ denote the angle between m and the supporting line $\ell(s)$ and S at points s where the tangent line is unique (where, with a slight abuse of notation, we identify the points on ∂C_1 with the value of the parameter s corresponding to them). The line $l(s)$ is assumed to be oriented so that C_1 lies to the left of $\ell(s)$. Then $\alpha(s)$ is a non-decreasing function with discontinuities at points s where there is more than one supporting line. Clearly $\alpha_\infty = \lim_{s\to\infty} \alpha(s)$ exists and is non-positive.

Each supporting line must intersect S at some point $p(s)$ because C_1 and C_2 lie on opposite sides of it. As s increases, $p(s)$ moves toward q_2. It follows that there is a limiting line P_1 which intersects S at p_∞.

To see that P_1 is an asymptote of C_1, note first that P_1 cannot contain interior points of C_1, since no $\ell(s)$ does. But there are points of C_1 arbitrarily close to P_1. This is clear if $\alpha_\infty = \alpha(s)$ for large s. Otherwise, if s is large enough, $\ell(s)$ intersects S at a point arbitrarily close to p_∞. Since for such an s, $\alpha_\infty > \alpha(s)$, there will always be a point on ∂C_1 in the strip bounded by P_1 and the line parallel to P_1 through $p(s)$ for s arbitrarily high.

P_2, P_1', and P_2' are constructed analogously. It remains to be shown that for a suitable choice of indices, $P_1' || P_1$ and $P_2' || P_2$, where identical lines are considered parallel.

Choose the indices on P_1' and P_2' such that the end of P_1' that is asymptotic to C_2 is in the left half-plane determined by m with its orientation. To see that P_1 and P_1' are parallel or coincide, suppose that they intersected at a point. There are two cases to consider: a) if P_1' intersects S nearer to q_2 than P_1 does, P_1' would have to enter the interior of C_1, but that is impossible because P_1' separates C_1 and C_2; or b) if P_1' intersects S further from q_2 than P_1 does, P_1 would enter the interior of C_2, which is impossible for similar reasons. Analogous arguments show that P_2 and P_2' are either parallel or coincide.

To prove 5), note that in the notation of the proof of 4), a third component C_3 would have to lie in the sector of $\Pi \setminus (P_1 \cup P_2)$ opposing the sector containing C_1 and in the sector of $\Pi \setminus (P_1' \cup P_2')$ opposing the sector containing C_2. The intersection of the sectors in which C_3 would have to lie is therefore a point, a line segment, or a parallelgram. In particular, C_3 would have to be bounded. □

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