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INVESTIGATIONS IN HOMOGENEOUS COSMOLOGIES

by

CHRIST FTACLAS

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the degree of Doctor of Philosophy, The City University of New York.

1978

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

INVESTIGATIONS IN HOMOGENEOUS COSMOLOGIES

by

CHRIST FTACLAS

Advisers: (for the City University) Professor Daniel Greenberger, and Professor Jeffrey M. Cohen, University of Pennsylvania

Spatially homogeneous relativistic cosmological models are explored and a number of new properties are presented. It is demonstrated that motions associated with rotational symmetry need not preserve the homogeneity of physical fields. A model is presented in which physically defined preferred directions violate rotational symmetry. Some perturbed models are given which exhibit shear, expansion and non-vanishing vorticity. The unperturbed models are the Bianchi Type III spacetimes of Kantowski and Sachs. In addition, an introduction to the mathematical framework of Relativity is included with particular emphasis on exterior calculus and the theory of symmetric spaces.

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I thank all those who have taught me at the City University. In particular, I would like to express my appreciation to Professor Jeffrey M. Cohen whose guidance, patience and understanding have sustained me. In addition, for many valuable discussions, I would like to thank Doctor Nikos Batakis, Professor Alex Harvey and Professor Daniel Greenberger.

Lastly, I thank the foundation builders who have given us a Physics to delight in, to strive to understand and if fortune smiles, to advance a small step.

TO MY PARENTS FOR GIVING ME THE OPPORTUNITY

AND TO SHERRE FOR GIVING ME THE REASON.

Preface - Some Notational and Other Conventions

1. Throughout the text, except where indicated, the summation convention is in effect and repeated indices are an implicit sum.

2. In general, Latin indices assume the values 1, 2 and 3, and Greek indices the values 0, 1, 2 and 3. Where spaces of arbitrary dimensionality N are discussed in Chapter 2, all indices range from 1 to N.

3. In the system of units used, $c = 8\pi G = 1$ so that the field equations are: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha}{}_{\alpha} = T_{\mu\nu}$. Where it is intuitively useful, quantities in more common units are repeated parenthetically.

4. Chapters are numbered sequentially as are sections within them. For instance, 3.2 is the second section of Chapter 3. Equations are numbered sequentially in the section in which they appear so that: (3.2.7) is the seventh numbered equation in (3.2).

5. Figures and tables are numbered sequentially in the chapters which they appear.

6. References are indicated by a numbered superscript and listed by number at the end of the text.

7. Round brackets on indices denote symmetrization and square brackets anti-symmetrization. Thus:

$$u_{(ab)} = \frac{1}{2} (u_{ab} + u_{ba})$$
$$u_{[ab]} = \frac{1}{2} (u_{ab} - u_{ba})$$

8. No attempt has been made in typestyle (i.e. boldface) to differentiate abstracted tensor valued quantities from their components, as the context will always make the distinction clear.

9. Occasionally, partial derivative operators are denoted by the letter p with a subscript so that p_i is the partial derivative with respect to x^i. Otherwise, as usual, a comma will denote partial differentiation and a semi-colon will denote covariant differentiation.

10. Footnotes will be designated in the text by a Latin superscript and are listed following the references.

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1. Introduction

The following chapters include an exposition of the basics of relativistic cosmology, and a series of researches into the nature of homogeneous spacetimes and some of the consequences of the assumption of homogeneity. Chapter 2 is a non-rigorous exposition of the mathematical framework of the General Theory of Relativity which has been included primarily to establish notation and point of view. Chapter 3 applies Relativity to cosmology and develops the theory of symmetric spaces and spatially homogeneous world models.

New results are presented in Chapters 4, 5 and 6. In Chapter 4, some properties of homogeneous spacetimes are explored and it is demonstrated that rotational symmetries need not preserve the homogeneity of physical fields associated with spacetime. In Chapter 5, a rotationally symmetric spacetime is presented in which physically defined preferred directions violate rotational symmetry. In Chapter 6, some perturbed cosmological models are given which are shearing and expanding with non-zero vorticity.

Specific results are discussed in the chapters in which they are obtained and broadly related to other work in the field in Chapter 7.

2. - An Introduction to General Relativity

This section contains a brief introduction to the mathematical framework of General Relativity. It is by no means complete or entirely rigorous, and is intended primarily to establish a consistent framework from which further results can be developed. For more complete treatments of the full theory the reader is referred to references (1-7). In addition, the mathematical tools introduced in this section are developed in references (8-11) and will not be cited individually.

Spacetime will be modeled as a differentiable manifold with pseudo-Riemannian metric. Tensor operation, defined in the tangent spaces at each point in the manifold, and derivative operations requiring connections between the tangent spaces of adjacent points are defined. The techniques of exterior calculus and differential forms are developed as a natural framework in which to treat homogeneous spacetimes.

2.1 - Differentiable Manifolds

A differentiable manifold of dimension n is a point set which is locally indistinguishable from R^n the n-fold product of the real line with itself. More explicitly, a point set M is a differentiable manifold if there exist pairs (U_α, φ_α) where U_α is an open set in M and φ_α is a homeomorphism of U_α onto W_α an open set in R^n such that:

(1) $U_1 \cup U_2 \cup \ldots\ldots\ldots\ldots = M$

(2) for any α, β if $U_\alpha \cap U_\beta$ is not empty then $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a differentiable mapping in R^n.

For p any point in U_α of M the $\varphi_\alpha(p) = (x^1(p),\ x^2(p) \ldots\ldots\ldots x^n(p))$ are called the coordinate mappings of p. Where $U_\alpha \cap U_\beta$ is not empty, points in the intersection have two coordinate descriptions $\varphi_\alpha(p), \varphi_\beta(p)$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$ (the composition of φ_α with φ_β^{-1}) is a coordinate transformation in R^n. Condition (1) above guarantees that every point in M is mapped into R^n and (2) assures the differentiability of these mappings. A pair $(U_\alpha, \varphi_\alpha)$ is called a chart and an atlas is a collection of charts such that the union of their open sets is M.

Functions may be defined on M by their action on the points of R^n;

$$f(p) = f(\varphi_\alpha(p)) = f(x^i(p))$$

Likewise differentiability may be defined on M but it is clear that functions are differentiable to the extent that the φ_α are differentiable. We may define a C^ℓ atlas (or a C^ℓ manifold) as one in which the functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are C^ℓ (possess continuous partial derivatives to all orders $\leq \ell$). A homeomorphism τ

of M onto itself is called a diffeomorphism if τ and τ^{-1} are differentiable. Diffeomorphisms preserve the class of differentiable structures on M.

In addition to the mathematical definition of M the following physical restrictions are imposed as assumptions about the nature of spacetime:

(1) M is connected - Any two points in M can be joined by a continuous curve lying entirely in M.

(2) M is Hausdorff - For any two points p, q in M there exist neighborhoods P of p and Q of q such that $P \cap Q$ is empty.

(3) M is oriented - The Jacobians of the transformations are all chosen to have the same sign. Orientability is an intrinsic property of M not the fashion in which coordinates are chosen.

(4) M is paracompact - There exists a covering of M such that for every p in M there is a neighborhood A which intersects only a finite number of covering sets (i.e. - every covering of M has a locally finite refinement).

This definition of a differentiable manifold permits the extension of concepts familiar in R^n such as continuity and differentiability of scalars to suitably defined open sets in M.

2.2 - Vectors and Forms on the Manifold

For M a differentiable manifold, a mapping γ of the real line into M is a parametized curve in M. If t is the curve parameter on the real line, then $\gamma(t)$ is assumed to be a differentiable function of t. At the point x_0 in M where $\gamma(0)=x_0$ the following product can be defined for any function f which is differentiable in a neighborhood of x_0:

$$\langle\gamma, f\rangle = \left[\frac{d}{dt}(f\circ\gamma)\right]_{t=0} \tag{2.2.1}$$

Under this product, the following equivalence relations arise:

(a) An equivalence class of curves such that $\langle\gamma_1, f\rangle=\langle\gamma_2, f\rangle$ defines a tangent vector at x_0. The set of all possible tangent vectors at x_0 defines the tangent space T_{x_0} at x_0.

(b) An equivalence class of functions such that $\langle\gamma, f_1\rangle=\langle\gamma, f_2\rangle$ defines a differential df, or one-form at x_0. The set of all one-forms at x_0 defines the cotangent or dual space $T^*_{x_0}$ at x_0.

If α is the tangent vector to the curve γ then df is defined by :

$$\langle\alpha, df\rangle = \langle\gamma, f\rangle \tag{2.2.2}$$

Both the tangent and dual spaces may be given the structure of a vector space.

If $(x^1, x^2, \ldots\ldots x^n)$ are coordinates in a region surrounding x_0 define a family of curves $\gamma^j(t)$ by :

$$\gamma^j(t) = x_o^j + x^j(t) = x_o^j + t$$

The tangent vector to γ^j at x_o is p_j. Defining functions f^i by $f^i = x^i$ the differential df^i at x_o is then dx^i and the product (2.2.1) is :

$$<p_j, dx^i> = \frac{d}{dt}(x_o^j + t)\delta^i{}_j = \delta^i{}_j \tag{2.2.3}$$

The duality relation (2.2.3) implies the linear independence of the p_i and the dx^i since if $h = c_i dx^i = 0$ then $<p_j, h> = c_j = 0$. The p_i (or dx^i) are a basis in the tangent (or cotangent) space. Writing out the product (2.2.1) gives :

$$<\gamma, f> = \left[\frac{d}{dt} f(\gamma(t))\right]_{t=0} = \left[f_{,i}\frac{dx^i}{dt}\right]_{t=0} = <\alpha, df>$$

or using (2.2.3) and the linearity of the product (2.2.1) :

$$df = f_{,i}dx^i = u_i dx^i \qquad \alpha = \frac{dx^i}{dt}p_i = v^i p_i$$

where :

$$<\gamma, f> = <\alpha, df> = (u_i v^i)_{t=0}$$

The tangent space is the space of contravariant vectors with generic element:

$$\alpha = u^i p_i \tag{2.2.4}$$

Its elements are the mappings on functions $f \to u^i f_{,i} \in R^1$. The dual space is the space of covariant vectors with generic element:

$$\beta = u_i dx^i \tag{2.2.5}$$

Its elements are the linear functionals on vectors defined by $df(\alpha) = \alpha f$. The forms (2.2.4, 5) make manifest the behavior of elements of T_{x_o} and $T^*_{x_o}$ under coordinate transformations $\bar{x}^i = \bar{x}^i(x)$ (where x represents the set x^i);

$$\alpha = u^i p_i = u^i \bar{x}^j{}_{,i} \bar{p}_j = u^i a^j{}_i \bar{p}_j = \bar{u}^j \bar{p}_j \tag{2.2.6}$$

$$\beta = u_i dx^i = u_i (\bar{p}_j x^i) d\bar{x}^j = u_i b^i{}_j d\bar{x}^j = \bar{u}_j d\bar{x}^j \tag{2.2.7}$$

$$a^i{}_j b^j{}_k = \delta^i{}_k$$

Spaces of higher order can be constructed by taking the Cartesian products of T and T*. Thus, the space TxT* (it is understood that these spaces carry subscripts assigning them to one point in M) has a basis $p_i \otimes dx^j$ and generic elements:

$$A = A^i{}_j p_i \otimes dx^j$$

The product (2.2.1) is not as yet an inner product since no norm has been assigned to vectors, nor has any other metric function been defined. The dual and tangent spaces remain distinct. A metric may be assigned to M by the physical observation that Special Relativity is essentially a correct theory over small distance scales. Accordingly, we can require that for all p in M there exist local coordinates x^α (specializing to four dimensions) such that the metric tensor g has the form:

$$g = \eta_{\alpha\beta} dx^\alpha \otimes dx^\beta$$

where $\eta_{\alpha\beta}$ = diag (-1, +1, +1, +1) is Minkowskian. In any other coordinate system $(\bar{x}^\alpha = \bar{x}^\alpha (x^\beta))$:

$$g = \eta_{\alpha\beta} b^\alpha{}_\gamma b^\beta{}_\delta d\bar{x}^\gamma \otimes d\bar{x}^\delta = g_{\gamma\delta} d\bar{x}^\gamma \otimes d\bar{x}^\delta \qquad (2.2.8)$$

where the $b^\gamma{}_\delta$ are defined by (2.2.7). The norm of an element of T, (2.2.4) is then :

$$(u,u) = g_{\gamma\delta} u^\gamma u^\delta = u_\gamma u^\gamma = |u|^2 \qquad (2.2.9)$$

In addition to defining lengths and angles it is evident from (2.2.9) that the metric tensor makes identifications between elements of T and T* and renders (2.2.1) an inner product. Tensor valued quantities may therefore be regarded as abstractions having representations in either dual or tangent spaces. The vector (tensor of rank 1) U, for instance, may be written $U = U_\alpha dx^\alpha$, or $U = U^\alpha p_\alpha$ where $U_\alpha = g_{\alpha\beta} U^\beta$. More generally any nonsingular linear combination of the dx^α (p_α) may be chosen as a basis in T*(T), even where the basis elements are not exact differentials (cannot be written as p_α). A basis for which the metric takes the form (2.2.8) is a coordinated basis.

2.3 - Directed Derivatives

In keeping with the coordinate free ideas of the previous section it is suitable to redefine the metric operation as a symmetric, bilinear mapping of ordered pairs of vectors into R^1. (This definition can be naturally extended to inner products on tensors of higher rank.) Thus, if Y and Z are vectors, and (Y,Z) represents their inner product, we have:

(a)	$(Y,Z) = (Z,Y)$	(symmetric)
(b)	$(Y+W,Z) = (Y,Z)+(W,Z)$	(bilinear)
	$(aY,Z) = (Y,aZ) = a(Y,Z)$	$(a \in R)$
(c)	$(Y,Z) = 0$ for all $Z \rightarrow Y = 0$	(nonsingular)

Condition (c) above is weaker than the positive-definite condition $(Y,Y)=0 \rightarrow Y=0$. It permits the existence of null vectors and labels the manifold as pseudo-Riemannian. For Y_α any basis in T the corresponding components of the metric tensor are given by :

$$g_{\alpha\beta} = (Y_\alpha, Y_\beta)$$

From the definition of vectors as differential operators it follows that for X a vector and f a scalar field :

$$Xf = X^\alpha f_{,\alpha} = (X, df)$$

is the change in f in the direction X. It is logical to extend this concept to the change in a vector field in the direction of another. Although such an extension is straightforward in Euclidean space difficulties are encountered in the general Riemannian space. The problem is essentially that differentiation implies the comparison of vectors at two different points in space but the considerations of the previous sections indicate that such vectors are actually in different spaces (T_p, $T_{p'}$). Comparison requires a "connection" between these spaces giving rise to a new derivative operation. If we designate this operation by D_Y the derivative in the direction Y then D_Y can be defined by :

$$\begin{aligned}
&(a)\quad D_Y(Z+W)=D_YZ+D_YW \\
&(b)\quad D_{Y+W}(Z)=D_YZ+D_WZ \\
&(c)\quad D_{fY}Z=fD_YZ \\
&(d)\quad D_Y(fZ)=(Yf)Z+fD_YZ
\end{aligned} \qquad (2.3.1)$$

where Y, Z and W are vectors and f is a function.

For a given basis X_α taking $Y=Y^\alpha X_\alpha$ and $Z=Z^\alpha X_\alpha$, from the rules (2.3.1) D_YZ may be written :

$$\begin{aligned}
D_YZ &= Y^\alpha D_\alpha(Z^\beta X_\beta) \\
&= Y^\alpha[(X_\alpha Z^\beta)X_\beta + Z^\beta D_\alpha X_\beta]
\end{aligned} \qquad (2.3.2)$$

where $D_\alpha = D_{X_\alpha}$. It is evident that the action of D is completely specified by (2.3.1) and its action on basis vectors. Since the change in a vector is itself a vector $D_\alpha X_\beta$

may be expanded on the X_α to give:

$$D_\alpha X_\beta = \Gamma^\mu{}_{\alpha\beta} X_\mu \tag{2.3.3}$$

where the $\Gamma^\mu{}_{\alpha\beta}$ are the coefficients of connection (if the basis is holonomic i.e. $[X_\alpha, X_\beta] = 0$, then the $\Gamma^\mu{}_{\alpha\beta}$ are called Christoffel symbols). Using (2.3.3), (2.3.2) can be written:

$$D_Y Z = Y^\alpha (X_\alpha Z^\beta + \Gamma^\beta{}_{\mu\alpha} Z^\mu) X_\beta = Y^\alpha Z^\beta{}_{;\alpha} X_\beta$$

Strictly speaking the requirement of the previous section that there exist locally a Minkowskian coordinate system should include the requirement that in such a system the $\Gamma^\mu{}_{\alpha\beta}$ vanish since this is also a characteristic of Minkowski space. This constrains the $\Gamma^\mu{}_{\alpha\beta}$ by requiring that the torsion T defined by:

$$T = D_Y Z - D_Z Y - [Y, Z] \tag{2.3.4}$$

vanish, or equivalently that $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$ in a holonomic basis. The $\Gamma^\mu{}_{\alpha\beta}$ are then completely specified by the condition $T = 0$ and $Dg = 0$, that is the covariant derivative of the metric tensor vanish.

2.4 - Differential Forms and Exterior Calculus

Tensor fields of rank 2 can be formed by the product T^*xT^*. A basis in the product space will be $dx^\alpha \otimes dx^\beta$ with the properties :

$$(dx^\alpha + dx^\beta) \otimes dx^\gamma = dx^\alpha \otimes dx^\gamma + dx^\beta \otimes dx^\gamma$$
$$dx^\gamma \otimes (dx^\alpha + dx^\beta) = dx^\gamma \otimes dx^\alpha + dx^\gamma \otimes dx^\beta$$
$$mdx^\alpha \otimes dx^\beta = dx^\alpha \otimes mdx^\beta = m(dx^\alpha \otimes dx^\beta) \tag{2.4.1}$$

Since the tensor product is not in general symmetric, we may define the exterior product to be the anti-symmetric part of the tensor product as :

$$dx^\alpha \wedge dx^\beta = \frac{1}{2}\,(dx^\alpha \otimes dx^\beta - dx^\beta \otimes dx^\alpha) \tag{2.4.2}$$

where " $\wedge$" denotes the exterior product and (2.4.2) defines a 2-form. From (2.4.1, 2) it follows that the exterior product is linear in each of its arguments and antisymmetric on their exchange. The space of two-forms is thus a vector space with dimensionality equal to the number of distinct pairs of n objects where n=dimM. Calling functions zero-forms with only one basis element (which can be taken to be 1) and with T^* the space of one-forms with dim $T^*=n=\binom{n}{1}$, then (2.4.2) can be generalized to a space of p-forms with basis :

$$\omega^p = dx^{a_1} \wedge dx^{a_2} \wedge \ldots. \wedge dx^{a_p}$$

and dimensionality $\binom{n}{p}$. In the above, $1 \leq a_1, a_2 \ldots a_p \leq n$.
For $p>n$, ω^p contains a repeating index and must vanish. From the properties of $\binom{n}{p}$ it is clear that $\dim \omega^p = \dim \omega^{n-p}$. A mapping from ω^p to ω^{n-p} is provided by the Hodge or duality operator. For λ^p an element in the space of ω^p where :

$$\lambda^p = \frac{1}{p!} \lambda_{a_1, a_2, \ldots, a_p} dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p}$$

and $\lambda_{a_1, \ldots, a_p}$ is antisymmetric in all of its indices, the element $*\lambda^p$ an element in ω^{n-p} is defined by:

$$\lambda^p \wedge *\lambda^p = (*\lambda^p, *\lambda^p) dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \qquad (2.4.3)$$

where the inner product on p-forms is induced by the inner product in T^* as follows: If $\sigma^p = \sigma^{a_1} \wedge \ldots \wedge \sigma^{a_p}$ then :

$$(\sigma^p, \sigma^p) = \det(\sigma^{a_i}, \sigma^{a_j})$$

In the general case the metric $g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$ can be diagonalized (g is a symmetric, non-degenerate matrix) to yield an orthonormal basis of one-forms ω^α (Cartan basis) where :

$$\begin{aligned} g &= \eta_{\alpha\beta} \omega^\alpha \otimes \omega^\beta \\ \eta_{\alpha\beta} &= \mathrm{diag}(-1,+1,+1,+1) \end{aligned} \qquad (2.4.4)$$

The action of the duality operator on the elements of basis p-forms associated with

(2.4.4) based on (2.4.3) are summarized in Table (2.1).

For any scalar field f , df is given by:

$$df = f_{,\alpha} dx^{\alpha}$$

where the operator "d" has mapped the 0-form f into the 1-form df . Generalizing this operation, we can define the exterior derivative "d" to be the operation which maps p-forms into p +1 - forms such that :

$$d(\lambda_1 + \lambda_2) = d\lambda_1 + d\lambda_2 \tag{2.4.5}$$

$$d(\lambda^p \wedge \theta^q) = d\lambda^p \wedge \theta^q + (-1)^p \lambda^p \wedge d\theta^q \tag{2.4.6}$$

$$d(d\lambda^p) = 0 \tag{2.4.7}$$

Condition (2.4.5) is just the linearity associated with derivative operators. (2.4.6) follows from $d(dx^{\alpha})=0$ and the antisymmetry of the exterior product and (2.4.7) is the statement of the equality of mixed partial derivatives.

The action of d on 1-forms as generalized from (2.4.5) is:

$$d(v_{\alpha} dx^{\alpha}) = v_{\alpha,\beta} dx^{\beta} \wedge dx^{\alpha}$$

and so on for other p-forms. On vectors (elements of T), d operates on components as scalar fields to yield a vector with 1-form valued components[a]:

$$d(v^{\alpha} p_{\alpha}) = d(v^{\alpha}) p_{\alpha} + v^{\alpha} d(p_{\alpha})$$

Table 2.1 - THE DUALITY OPERATOR

For a p-form λ, $*\lambda$ is defined from:

$$\lambda \wedge *\lambda = (*\lambda, *\lambda)\ \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$$

In a four dimensional space with signature (-,+,+,+) the double dual obeys:

$$*(*\lambda^p) = (-1)^{p+1}\ \lambda^p$$

On basis p-forms:

$$*1 = -\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$$
$$*\omega^0 = \omega^1 \wedge \omega^2 \wedge \omega^3$$
$$*\omega^i = \omega^0 \wedge \omega^j \wedge \omega^k \qquad \text{(i, j, k cyclic)}$$
$$*(\omega^0 \wedge \omega^i) = \omega^j \wedge \omega^k \qquad \text{(i, j, k cyclic)}$$

Thus for:

(a) $J = J_\mu \omega^\mu$,

$$*J = \frac{1}{3!}\ J^\mu \eta_{\alpha\beta\nu\mu} \omega^\alpha \wedge \omega^\beta \wedge \omega^\nu$$

(b) $F = \frac{1}{2}\ F_{\mu\nu} \omega^\mu \wedge \omega^\nu$,

$$*F = -\ \frac{1}{4}\ F^{\mu\nu} \eta_{\mu\nu\alpha\beta} \omega^\alpha \wedge \omega^\beta$$

(c) $A = \frac{1}{3!}\ A_{\lambda\mu\nu} \omega^\lambda \wedge \omega^\mu \wedge \omega^\nu$,

$$*A = \frac{1}{3!}\ A^{\lambda\mu\nu} \eta_{\alpha\lambda\mu\nu} \omega^\alpha$$

Where for orthonormal frames:

$$\eta_{\alpha\beta\gamma\delta} = \eta_{[\alpha\beta\gamma\delta]} = \epsilon_{\alpha\beta\gamma\delta}$$
$$\eta_{0123} = 1$$

and $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor

2.5 - Relativity in the Language of Exterior Calculus

For the orthonormal basis constructed in section (2.4) the basis of 1-forms can be written :

$$\omega^\alpha = a^\alpha{}_\beta dx^\beta \tag{2.5.1}$$

where the $a^\alpha{}_\beta$ are in general functions of all the coordinates. Dual to the ω^α is a set of basis vectors ω_α such that :

$$(\omega_\alpha, \omega^\beta) = \delta_\alpha{}^\beta \tag{2.5.2}$$

which together with (2.5.1) and the duality relation (2.2.3) gives:

$$\omega_\alpha = b^\beta{}_\alpha p_\beta$$

where $b^\beta{}_\alpha a^\alpha{}_\gamma = \delta^\alpha{}_\gamma$. Applying d on ω_α and writing the p_α in terms of the ω_α gives :

$$d\omega_\alpha = \omega^\beta{}_\alpha \omega_\beta \tag{2.5.3}$$

where $\omega^\beta{}_\alpha$ is a connection 1-form in analogy with the action of D_α on basis vectors. The connection forms can be expanded on the ω^α to give :

$$\omega^\alpha{}_\beta = \gamma^\alpha{}_{\beta\mu}\omega^\mu$$

where the $\gamma^\alpha{}_{\beta\mu}$ are the Ricci rotation coefficients[b]. The exterior derivative of $(\omega_\gamma, \omega_\delta) = \eta_{\gamma\delta}$ gives :

$$(\omega^{\mu}{}_{\gamma}\omega_{\mu}, \omega_{\delta}) + (\omega_{\gamma}, \omega^{\mu}{}_{\delta}\omega_{\mu}) = 0$$
$$\omega^{\mu}{}_{\gamma}\eta_{\mu\delta} + \omega^{\mu}{}_{\delta}\eta_{\gamma\mu} = 0$$
$$\omega_{\gamma\delta} + \omega_{\delta\gamma} = 0$$

(2.5.4)

which together with (2.5.3) completely specifies the $\omega^{\gamma}{}_{\delta}$.

Defining the displacement of a point dP by :

$$dP = dx^{\alpha}p_{\alpha} = \omega^{\alpha}\omega_{\alpha}$$

(dP is not an exterior derivative) then :

$$\begin{aligned} d(dP) &= d\omega^{\alpha}\omega_{\alpha} - \omega^{\alpha}d\omega_{\alpha} \\ &= (d\omega^{\alpha} - \omega^{\beta} \wedge \omega^{\alpha}{}_{\beta})\omega_{\alpha} \end{aligned}$$

The 2-form $d\omega^{\alpha} - \omega^{\beta} \wedge \omega^{\alpha}{}_{\beta}$ is the torsion T defined in (2.3.4) and is required to vanish giving the structure equation :

$$d\omega^{\alpha} = -\omega^{\alpha}{}_{\beta} \wedge \omega^{\beta} \qquad (2.5.6)$$

Since $d\omega_{\alpha}$ is not the exterior derivative of a differential form $d(d\omega_{\alpha})$ does not vanish and :

$$\begin{aligned} d(d\omega_{\alpha}) &= d(\omega^{\beta}{}_{\alpha}\omega_{\beta}) \\ &= (d\omega^{\beta}{}_{\alpha} + \omega^{\beta}{}_{\mu} \wedge \omega^{\mu}{}_{\alpha})\omega_{\beta} \qquad (2.5.7) \\ &= \theta^{\beta}{}_{\alpha}\omega_{\beta} \end{aligned}$$

The curvature 2-form $\theta^{\alpha}{}_{\beta}$ defined in (2.5.7) has components in the basis $\omega^{\alpha} \wedge \omega^{\beta}$ given by :

$$\begin{aligned} \theta^{\alpha}{}_{\beta} &= d\omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\mu} \wedge \omega^{\mu}{}_{\beta} \\ &= \frac{1}{2} R^{\alpha}{}_{\beta\mu\nu} \omega^{\mu} \wedge \omega^{\nu} \end{aligned} \tag{2.5.8}$$

where $R^{\alpha}{}_{\beta\mu\nu}$ is the Riemann curvature tensor and the definition of curvature as a second derivative ($d^2\omega_{\mu} = \theta^{\nu}{}_{\mu}\omega_{\nu}$) is consistent with Euclidean concepts.

From $d(d\omega^{\alpha}) = 0$ and (2.5.6) we get :

$$\begin{aligned} d(d\omega^{\alpha}) &= -d\omega^{\alpha}{}_{\beta} \wedge \omega^{\beta} + \omega^{\alpha}{}_{\beta} \wedge d\omega^{\beta} \\ &= -(d\omega^{\alpha}{}_{\lambda} + \omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}{}_{\lambda}) \wedge \omega^{\lambda} \\ 0 &= \theta^{\alpha}{}_{\lambda} \wedge \omega^{\lambda} \end{aligned}$$

or :

$$R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\gamma\delta\beta} + R^{\alpha}{}_{\delta\beta\gamma} = 0$$

From (2.5.9)

$$\begin{aligned} d\theta^{\alpha}{}_{\beta} &= d(d\omega^{\alpha}{}_{\beta}) + d(\omega^{\alpha}{}_{\gamma} \wedge \omega^{\gamma}{}_{\beta}) \\ &= \theta^{\alpha}{}_{\gamma} \wedge \omega^{\gamma}{}_{\beta} - \omega^{\alpha}{}_{\gamma} \wedge \theta^{\gamma}{}_{\beta} \end{aligned}$$

which are the Bianchi identities,

$$R_{\alpha\beta[\gamma\delta;\nu]} = 0$$

In summary, the diagonalization of the metric tensor yields the one-form basis ω^μ and the duality relations (2.5.2) and (2.2.3) determine the dual vector basis ω_μ. The structure equations (2.5.4,6) uniquely fix the connection 1-forms $\omega^\mu{}_\nu$ and the curvature 2-forms $\theta^\mu{}_\nu$ are calculated from (2.5.8). The components of the Riemann tensor are the coefficients of the curvature 2-forms in the basis $\omega^\mu \wedge \omega^\nu$ and the Ricci tensor is as usual the "trace" of the Riemann tensor :

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$$

The Einstein tensor $G_{\mu\nu}$ is the divergence free structure $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu}$ where $R = R^\mu{}_\mu$ and the field equations are :

$$G_{\mu\nu} = T_{\mu\nu} - \lambda\eta_{\mu\nu}$$

where λ is the cosmological constant and $T_{\mu\nu}$ is the stress-energy tensor of whatever fields inhabit spacetime.

For u a vector field :

$$u = u^\mu \omega_\mu$$

and the exterior derivative of u is

$$
\begin{aligned}
du &= u^{\mu}{}_{,\nu}\,\omega^{\nu}\omega_{\mu} + u^{\mu} d\omega_{\mu} \\
&= (u^{\nu}{}_{,\mu} + \gamma^{\nu}{}_{\delta\mu} u^{\delta})\,\omega^{\mu}\omega_{\nu} \\
&= u^{\nu}{}_{;\mu}\,\omega^{\mu}\omega_{\nu}
\end{aligned}
$$

where $u^{\mu}{}_{;\nu}$ is the covariant derivative in orthonormal frames and $u^{\mu}{}_{,\nu}$ is shorthand for $\omega_{\nu} u^{\mu}$ the basis vectors applied as differential operators on the components of u.

More applications of exterior calculus will be persued in the following chapters. The advantages of orthonormal Cartan bases is of course that the metric is always Minkowskian and together with exterior calculus, results may be expressed in invariant (coordinate free) form rather than covariant form. A straightforward calculation reveals, for instance, that the exterior derivative is independent of the frame in which it is calculated. The disadvantage, if any, is that the basis 1-forms in general are not hypersurface forming, that is the ω^{μ} are usually not exact differentials of any functions so that global coordinate surfaces cannot be found. Where the differential structure of the ω^{μ} is simple, however, as in the case of homogeneous spacetimes the method is extremely powerful.

2.6 - Symmetries of Spacetime

We consider a manifold M and assume for the moment global coordinates x^ν. A homeomorphism, τ, of M onto itself is then a coordinate transformation in M. We assume further that τ is of the form :

$$\tau : x^\nu \rightarrow \bar{x}^\nu(x,s) \tag{2.6.1}$$

where $x = \{x^\nu\}$ and s is a real parameter such that $\bar{x}^\nu$ is a continuous function of s and $\bar{x}^\nu(x,0) = x^\nu$ (τ is connected to the identity). From (2.2.6,7) it follows that each such τ induces a mapping τ_* (τ^*) on the elements of T(T*). For τ near the identity we can consider τ (p) as moving p in M as s is varied away from zero. At each point of the orbit a unique tangent vector is defined which is the generator of infinitesimal transformations:

$$Z = Z^\nu p_\nu = \left(\frac{d}{ds}\bar{x}^\nu\right)_{s=0} p_\nu \tag{2.6.2}$$

If τ is effective on M then (2.6.2) defines a vector field on M. The existence of global coordinates and to some extent τ itself are artificial in that any vector field on M with suitable coninuity can be regarded as generating infinitesimal transformations on M. Finite transformations, as always, can be built up from successive infinitesimal ones.

Assuming that the field Z is defined on M, then for p in M, an infinitesimal transformation to p' can be written :

$$x^{\nu}{}_{p'} = x^{\nu}{}_{p} + Z^{\nu}{}_{p}\delta s \tag{2.6.3}$$

Let F be any tensor field on the manifold (continuous, differentiable, etc.) then $L_Z F$, the Lie derivative of F with respect to Z can be defined as :

$$L_Z F = \lim_{\delta s \to 0} \frac{F(p') - F'(p)}{\delta s} \tag{2.6.4}$$

where $F(p')$ is the value of F at p' and $F'(p)$ is the value of F at p' induced by the mappings τ_*, τ^* of T, T^*. For example, if F is a vector field, $X = X^{\alpha} p_{\alpha}$, since X is assumed analytic:

$$X^{\alpha}(p') = X^{\alpha}(p\ ) + X^{\alpha}{}_{,\beta} Z^{\beta} \delta s \tag{2.6.5}$$

Similarly, under the transformation (2.6.3), the components of $X'(p)$ can be written

$$\begin{aligned} X'^{\alpha}(p) &= X^{\beta}(p)(\bar{x}^{\alpha}{}_{,\beta})_{p'=p} \\ &= X^{\beta}(p)(\delta^{\alpha}{}_{\beta} + Z^{\alpha}{}_{,\beta}\delta s)_{p'=p} \\ &= X^{\alpha}(p) + Z^{\alpha}{}_{,\beta} X^{\beta} \delta s \end{aligned} \tag{2.6.6}$$

Thus, from (2.6.5,6)

$$L_Z X = \lim_{\delta s \to 0} \frac{(X^\alpha{}_{,\beta} Z^\beta - Z^\alpha{}_{,\beta} X^\beta)\delta s}{\delta s} p_\alpha$$

$$= (X^\alpha{}_{,\beta} Z^\beta - Z^\alpha{}_{,\beta} X^\beta) p_\alpha$$

$$L_Z X = [Z, X] \qquad (2.6.7)$$

Similarly, for covariant vectors

$$L_Z(Y_\alpha dx^\alpha) = (Y_{\alpha;\beta} Z^\beta + Z^\beta{}_{;\alpha} Y_\beta) dx^\alpha \qquad (2.6.8)$$

The action of the Lie derivative on any tensor field is defined by its action in (2.6.7,8). The vanishing of $L_Z F$ implies F is invariant under the transformations generated by **Z** (i.e. F'(x) is the same function in the $\bar{x}$ that F is in the x). It is evident that any F can be defined at a point and $L_Z F = 0$ can be used to construct an invariant F on M.

In Relativity, of course, the metric tensor plays a special role. Those transformations which leave the metric invariant are called motions or isometries of M. From (2.6.8) ;

$$L_Z g = (g_{\alpha\beta;\nu} Z^\nu + g_{\alpha\nu} Z^\nu{}_{;\beta} + g_{\nu\beta} Z^\nu{}_{;\alpha}) dx^\alpha \otimes dx^\beta$$

$$= (Z_{\alpha;\beta} + Z_{\beta;\alpha}) dx^\alpha \otimes dx^\beta$$

If Z is to generate an isometry then $L_Z g = 0$ and Z must satisfy Killing's equation:

$$Z_{\alpha;\beta} + Z_{\beta;\alpha} = 0 \tag{2.6.9}$$

The set of all transformations which leave g invariant form a group (two successive transformations leave g invariant as does the identity transformation $x' = x$) which for analytic transformations is a Lie group. The space spanned by the Z_a ("a" labels different isometries) is a Lie algebra which is isomorphic to the Lie algebra defined on the group manifold G of transformations leaving g invariant[12].

Together with the constraint equation (2.6.9) the n components Z^α and n^2 components $Z_{\alpha;\beta}$ define $\frac{1}{2}n(n+1)$ independent quantities implying that in a space of dimension n, at most $\frac{1}{2}n(n+1)$ independent isometries can exist.

3 - Relativistic Cosmology

The fundamental program of relativistic cosmology (assuming the correctness of Einstein's equations) is the generation of a solution to the field equations which agrees in detail with all observations. In this sense, every observation of nature is, in fact, "cosmological". To say that this program is at the very least optimistic is an understatement. It cannot be executed at the present time for two basic reasons:

(a) There is no general agreement as to the nature and significance of the observations.

(b) The fine structure of the Universe (stars, galaxies, etc.) render the mathematical problem intractable with current techniques.

Point (a) can be accomodated by adopting a modified program in which models are constructed to conform to some limited data set which is considered "cosmological" in that it pertains to the large scale structure of the Universe.

Point (b) requires the same approach used in many fields of physics such as defining gas dynamics or electromagnetic theory in the light of atomic structure. The object is to, in some sense, divide the Universe into cells large compared to any fine structure and small compared to characteristic cosmological dimensions. Average values of density, velocity, etc., are ascribed to these cells leading to a continuum formulation of cosmology. Assuming that the length scale of quantum mechanics exists independent of the macrocosm, then in any singularity of spacetime (taken here to mean a point of infinite density) the cell model must break down even if all other physics remains valid. Since it is the four-dimensional manifold which is being subdivided there are temporal constraints as well on cell size which also must be violated at a singularity.

Away from any singularity, current observations[13, 14] on the distribution of galaxies indicate a superclustering effect on the length scale of

$\ell_{sc} = 9\text{-}18 \times 10^{25}$cm (30-60 Mpc). If the age of the Universe is taken as $t_u = 2 \times 10^{28}$cm (2×10^{10}yrs) then ℓ_{sc} is only $\approx .5\text{-}1\%$ of t_u. Thus, for the current Universe there may not be a sufficiently large gap between large and small scale structures to permit the cell approximation. This point will be discussed further in section (2.2).

These problems notwithstanding, the cell approximation serves to define those measurements pertinent to the modified program discussed above. Clearly, those observations are to be taken as cosmological which are meaningful on the scale of the cells themselves. Thus, a 10% Helium density in one cell is a local property but a similar density in every cell is a cosmological property.

Throughout this work the cell approximation will be used. The Universe will be modeled as a continuum with local properties defined as continuous tensor fields on the manifold. Although this approach is certainly subject to criticism it seems at present to be the only route to achievable solutions to the field equations.

In this chapter the motion of a cosmological fluid will be discussed and applied to a discussion of the cosmological principle. The metric structure of homogeneous spacetimes will be developed as a prelude to the research results reported in the following chapters.

3.1 - The Motion of a Cosmological Fluid[c]

We assume the Universe filled with a fluid whose world lines define a congruence on M. The tangent vectors to these world lines form a vector field U on M (the four-velocity) taken to be normalized so that :

$$(U,U) = -1 \tag{3.1.1}$$

For any observer with four-velocity u a change of basis (boost plus rotation) such that :

$$X_0 = U \qquad (X_i, X_0) = 0 \qquad (X_i, X_j) = \delta_{ij}$$

defines a frame in which the observer is at rest. The spacelike hypersurface spanned by the X_i is orthogonal to u and is (locally) the locus of points simultaneous with the observer. Direct knowledge of the local hypersurface of simultaneity is forbidden any observer by causality. Observations are restricted to the surface and interior of the past light cone resulting in a set of null or timelike connection vectors joining the observer to points in his neighborhood. The position of any particle in the hyperplane of simultaneity (rest space of the observer) can be approximated by the component of the connection vector orthogonal to the four velocity of the observer. This effectively takes the current position of a particle to be its currently observed position. The connection vector X may be written:

$$X = X_{\parallel} + X_{\perp} \tag{3.1.2}$$

where the subscripts $//$ and $\perp$ refer to the components of X parallel and perpendicular to the observer four velocity U. From (3.1.2) it follows that:

$$X_\perp = X - X_{//} \qquad (3.1.3)$$
$$= X + (X,U)U$$

where the plus sign in (3.1.3) is a consequence of $(U,U) = -1$. In component form (3.1.3) can be written:

$$(X_\perp)^\alpha = X^\alpha + X^\beta U_\beta U^\alpha$$
$$= (\delta^\alpha{}_\beta + U^\alpha U_\beta)X^\beta$$
$$= h^\alpha{}_\beta X^\beta \qquad (3.1.4)$$

where $h^\alpha{}_\beta$ is idempotent $(h^\alpha{}_\beta h^\beta{}_\gamma = h^\alpha{}_\gamma)$ and projects onto the hyperplane orthogonal to U. The time rate of change of $X_\perp$ is given by:

$$(X_\perp)^\bullet = X^\bullet + (X,U)^\bullet U + (X,U)U^\bullet \qquad (3.1.5)$$

where $(\ )^\bullet$ denotes $_{;\gamma}U^\gamma$ $(= D_U)$. The observed part of this velocity is that portion which is orthogonal to U. The second term on the right hand side of (3.1.5) is parallel to U and can be ignored. The third term is already orthogonal to U((U,U)= -1) and the first term can be rewritten as follows:

Since X is assumed to join two nearby points write its components as δx^γ and expand X as follows:

$$
\begin{aligned}
(X^\alpha)^\bullet = \delta x^\alpha{}_{;\beta} U^\beta &= (\delta x^\alpha)_{,\beta} U^\beta + \Gamma^\alpha{}_{\beta\gamma} \delta x^\beta U^\gamma \\
&= \frac{d}{ds}(\delta x^\alpha) + \Gamma^\alpha{}_{\beta\gamma} \delta x^\beta U^\gamma \\
&= \delta U^\alpha + \Gamma^\alpha{}_{\beta\gamma} \delta x^\beta U^\gamma \\
&= U^\alpha{}_{,\beta} \delta x^\beta + \Gamma^\alpha{}_{\beta\gamma} \delta x^\beta U^\gamma \\
&= U^\alpha{}_{;\beta} \delta x^\beta = U^\alpha{}_{;\beta} X^\beta
\end{aligned}
\tag{3.1.6}
$$

Using (3.1.5, 6), $(X^\alpha)^\bullet$ becomes:

$$
\begin{aligned}
(X^\alpha_\perp)^\bullet_\perp &= U^\alpha{}_{;\beta} X^\beta + U_\beta (U^\alpha)^\bullet X^\beta \\
&= (U^\alpha{}_{;\beta} + U_\beta U^{\bullet\alpha}) X^\beta \\
&= v^\alpha{}_\beta X^\beta
\end{aligned}
\tag{3.1.7}
$$

From $v^\alpha{}_\beta \, h^\beta{}_\gamma = v^\alpha{}_\gamma$ it follows that (3.1.7) can also be written:

$$
(X^\alpha_\perp)^\bullet_\perp = v^\alpha{}_\beta (X^\beta)_\perp \tag{3.1.8}
$$

The matrix $v^\alpha{}_\beta$ is orthogonal to U and can be separated into its trace, symmetric traceless and anti-symmetric components which characterize respectively isotropic expansion, shear and rotation:

$$\text{Expansion: } \theta = v^{\lambda}{}_{\lambda} = U^{\lambda}{}_{;\lambda}$$
$$\text{Shear: } \sigma_{\alpha\beta} = v_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta}$$
$$\text{Rotation: } \Omega_{\alpha\beta} = v_{[\alpha\beta]} \tag{3.1.9}$$

The reason for this choice of names follows from the action of each of these quantities on an initially spherical distribution of particles:

Expansion - preserves the shape and orientation of the initial distribution but alters the volume. A scale length can be defined from $\frac{\dot{\ell}}{\ell} = \frac{1}{3}\theta$

Shear - alters the shape, but leaves the volume constant. The directions of the principle axes of shear (eigenvectors of $\sigma_{\alpha\beta}$) remain unchanged. From $\sigma_{\alpha\beta}$ the shear scalar $\sigma^2 = \sigma_{\alpha\beta}\sigma^{\alpha\beta}$ can be defined where: $\sigma^2=0 \leftrightarrow \sigma_{\alpha\beta}=0$.

Vorticity - preserves shape and volume but alters orientation leaving only one direction fixed. From $\Omega_{\alpha\beta}$ the vector

$$\Omega^{\alpha} = \frac{1}{2}\epsilon^{\beta\gamma\delta\alpha}U_{\beta}\Omega_{\gamma\delta} \qquad (g_{\gamma\delta} = \eta_{\gamma\delta}) \tag{3.1.10}$$

can be defined where Ω^{α} is the direction of the axis of rotation. From (3.1.9,10) the following relations can be deduced:

$$\Omega^{\alpha}U_{\alpha} = \Omega^{\alpha\beta}U_{\beta} = \Omega^{\alpha}\Omega_{\alpha\beta} = 0$$

and the magnitude of the vorticity vector is:

$$|\Omega| = (\Omega^{\alpha}\Omega_{\alpha})^{\frac{1}{2}} = (\tfrac{1}{2}\Omega_{\alpha\beta}\Omega^{\alpha\beta})^{\frac{1}{2}}$$

where $|\Omega| = 0 \leftrightarrow \Omega^{\alpha} = 0 \leftrightarrow \Omega^{\alpha\beta} = 0$.

The vanishing of the vorticity is the condition that the local hypersurfaces of simultaneity mesh to form global hypersurfaces[15] allowing the existence of a cosmic time.[16]

The above defined quantities together with the "acceleration"

$$(U^{\alpha})^{\bullet} X_{\alpha} = U^{\alpha}{}_{;\beta} U^{\beta} X_{\alpha} = D_U U \tag{3.1.11}$$

serve to describe completely the motion of a cosmological fluid. (In (3.1.11) X_{α} is a basis vector.)

3.2 - The Cosmological Principle

Simply stated, the Cosmological Principle imposes on the Universe homogeneity and isotropy. If it is to be interpreted as an observational statement (i.e. the Universe looks homogeneous and isotropic) evolution is prohibited since observations along the past light cone must indicate a constant density. This strict interpretation leads to Einstein's Static Universe[d]. More generally it is assumed that the Principle implies the existence of homogeneous and isotropic spacelike sections through the Universe. From the work of the previous section it follows that isotropy requires that the shear, vorticity and acceleration vanish which permits the existence of global hypersurfaces of simultaneity to which the matter four-velocity is orthogonal[15]. Orthogonality of the four-velocity implies the matter is at rest on the spacelike sections. A complete analysis[53] leads to the Friedman models whose properties are well documented.

Although elevated in status by being given a name, the Cosmological Principle is in fact a set of simplifying assumptions which permit an easy solution of the field equations. It is perhaps unfortunate that the enunciation of the Principle preceded any data to support it. The very language of observational Astronomy is permeated with concepts such as "Hubble constant" and "deceleration parameter" which are inherent to Friedman models. Astronomical anisotropies are usually reported as velocities required in an isotropic system to produce the observed anisotropy. In what seems like a circular argument the generally accepted test that a flux or portion thereof be cosmological in origin is that it be isotropic. A careful evaluation of the data seems to indicate that the observational evidence[54, 55] is equally good (or bad) for both homogeneity and isotropy.

From the point of view of the work in section 2.6 the Cosmological Principle would imply the existence of a six parameter group of motions acting on spacelike hypersurfaces. The surfaces are maximally symmetric and the solutions (Friedman models) are well explored. From the cell point of view discussed in the introduction to this chapter, however, the interpretation of the Cosmological Principle is slightly more complicated. Homogeneity would imply that the cell structure can be translated along any spacelike direction without affecting the average values assigned to cells. Isotropy implies the existence of a timelike direction at every point such that the cell average values are invariant under a rotation of the cell structure about any spacelike direction orthogonal to both the timelike and any spacelike direction. A further requirement is that no invariant direction be associated with the cells themselves. This requires for instance, that the average values of electric or magnetic field, angular momentum and pressure gradient (spacelike component) vanish when averaged over the cell volume. Here again the local supercluster may cause problems. Recent measurements[39] are consistent with a rotation of the supercluster with a period of approximately $10t_u$ where t_u is the age of the Universe. For a typical galaxy in the supercluster, the ratio of spin to orbital angular momentum is given by:

$$L_s/L_o = (M_g R_g{}^2 t_s)/(M_g R_s{}^2 t_g) \qquad (3.2.1)$$

where M_g, R_g and t_g are the mass, radius and rotation period for the galaxy and R_s, t_s are the corresponding values for the supercluster. For a typical galaxy $t_g \approx t_u/50$ and as was already stated $t_s \approx 10t_u$ giving for (3.2.1)

$$L_s/L_o \approx 500\,(R_g/R_s)^2 \doteq 5\times10^{-4} \tag{3.2.2}$$

where $R_g \approx 15$ Kpc and $R_s \approx 15$ Mpc. As (3.2.2) indicates the total angular momentum of the supercluster is predominantly orbital since a similar calculation for galaxies indicates that spin angular momentum of stars is negligible compared to their orbital angular momentum. It has already been demonstrated that the supercluster is certainly a significant part of any cosmological volume element and it is difficult to understand how the total angular momentum of such a volume element could vanish. This result combined with assumed homogeneity, has lent further significance to the understanding of cosmological vorticity.

Taking the Cosmological Principle as a set a simplifying assumptions, a reasonable course of action is to determine how much these assumptions can be weakened while still permitting a solution of the field equations. This leads naturally to the investigation of homogeneous spacetimes pursued in the following sections.

3.3 - Homogeneous Spacetimes

Throughout the remainder of this work, a spacetime which is homogeneous on spacelike sections will be referred to as a homogeneous spacetime. The assumption of spatial homogeneity is sometimes called the narrow Cosmological Principle or the strong Copernican Principle[18].

By assumption, a homogeneous spacetime can be modeled as a family S of homogeneous spacelike hypersurfaces such that for p and q, any two points on an element of S, there exist isometries of the space such that $p' = q$: the isometry group acts transitively on surfaces $t =$ constant. Since all points in a three dimensional neighborhood N containing p (such that N lies entirely in an element of S) are equivalent to p, it follows that a minimum dimensionality for the group of motions acting on S is three. Maximal symmetry for a three dimensional hypersurface requires six isometries so that the dimensionality of the group of motions of a homogeneous spacetime must be such that $3 \leqslant n \leqslant 6$. When the mapping from p to q is unique on elements of S, the group action is said to be simply transitive on S. In this case, homogeneity is generated by a three parameter group, G_3, whose generators span elements of S. Where the mapping from p to q is not unique, the group action is said to be multiply transitive on elements of S. In this case the group of motions is either G_4 or G_6 since G_5 cannot act transitively on three dimensional spaces[19]. (Simply stated, three dimensions permit three translations and one rotation (G_4, rotational symmetry) or three translations and three rotations (G_6, spherical symmetry) but not three translations and two rotations.) Since Lie algebras of order four and six have three dimensional sub-algebras, we can always consider homogeneity as arising from the action of a simply transitive G_3. There is one possible exception, when the group of motions is G_4 but the subgroup G_3 is effective on maximally symmetric two dimensional surfaces. This case has been examined by Kantowski and Sachs[20]. The surfaces must have constant curvature which is either positive or negative. If it is negative, the group of motions admits a simply transitive G_3. In the positive case, the lone exception arises in which the sub-group G_3 is multiply transitive on two dimensional surfaces rather than simply transitive on three dimensional surfaces. With this exception noted, we will consider homogeneous spacetimes as invariant under a group of motions G_3 whose action is simply transitive on spacelike hypersurfaces. If x^i $(i = 1, 2, 3)$ are coordinates on elements of S, and T_a $(a = 1, 2, 3)$ are the generators of G_3 then:

$$T_a = T_a{}^i(x) p_i \tag{3.3.1}$$

where x denotes the set x^i. If the family S is parametized by time t, then the vector field p_t is geodesic and the elements of S are geodesically parallel (ref. 19, chapt. 5). A suitable metric form is then[21]:

$$g = -dt \otimes dt + h \tag{3.3.2}$$

where:

$$h = h_{ij} dx^i \otimes dx^j \tag{3.3.3}$$

The h_{ij} are in general functions of the x^i and t and are to be determined by the condition:

$$(h_{ij}T_a{}^j)_{;k} + (h_{kj}T_a{}^j)_{;i} = 0 \tag{3.3.4}$$

That is the T_a satisfy Killings equations. A form for h can always be found by constructing a basis $X_a = X_a{}^i p_i$ on elements of S such that:

$$L_{T_b} X_a = [T_b, X_a] = 0 \tag{3.3.5}$$

The X_a, by (3.3.5) are invariant under the translations generated by the T_a (ref. 19, chapt. 3). Similarly, $X_a \otimes X_b$ is also invariant and if the components of h in this basis are constants with respect to the T_a, then g will be invariant under the T_a. The T_a only operate on three dimensional surfaces, so the most general constants with respect to their action are functions of time, and an explicit form for g is:

$$g = -p_t \otimes p_t + \alpha^{ab} X_a \otimes X_b \tag{3.3.6}$$

or in the dual space:

$$g = -dt \otimes dt + \beta_{ab} \sigma^a \otimes \sigma^b$$

where the invariant one-forms σ^a are dual to the X_a:

$$(\sigma^a, X_b) = \delta^a{}_b \tag{3.3.7}$$

and $\alpha^a{}_b = \alpha^\alpha{}_b(t)$, $\beta^a{}_b = \beta^a{}_b(t)$, $\alpha^\alpha{}_b \beta^b{}_c = \delta^a{}_c$.

When the group of motions is G_4, more than one G_3 can exist [f]. For instance, a rotationally symmetric Bianchi Type I spacetime has the same group structure for its motions as a rotationally symmetric Type VII_o ($h = o$) spacetime[27]. In the case of G_4 there is rotational symmetry, but homogeneity requires a symmetry axis at every point. This symmetry class is called Locally Rotationally Symmetric (LRS)[31] spacetimes. (See 3.4 for a definition of the Bianchi types.)

3.4 - Construction of the Invariant Basis

We assume a three parameter translation group with elements T_a whose action is simply transitive on spacelike hypersurfaces. A basis for structures of the T_a was given by Lie and Scheffers[22] in 1893 and applied to the motions of spaces by Bianchi[23, 24]. Consequently, the classifications of homogeneous spacetimes have come to be called Bianchi types. The possible structures of G_3 are presented in Table 3.1 and numbered according to a composite classification scheme due to Bianchi, Taub[21] and Estabrook, Wahlquist and Behr[25]. A spacetime will be designated as a particular Bianchi type if it is invariant under a translation group of that type.

Assuming any structure from Table 3.1 we have:

$$[T_a, T_b] = C^c{}_{ab} T_c \tag{3.4.1}$$

where the $C^c{}_{ab}$ are constants such that :

$$C^c{}_{ab} = C^c{}_{[ab]} \tag{3.4.2}$$

and the T_a satisfy the Jacobi identity:

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0 \tag{3.4.3}$$

Once a structure is chosen, the equations (3.4.1) can be integrated to construct a realization of the T_a as vectors on the three-spaces t =constant. The T_a are then of the form $T_a = T_a{}^i p_i$ where the $T_a{}^i$ are only functions of x^1, x^2 and x^3. That the group action is simply transitive implies that the T_a span these three-spaces.

Table 3.1 - The Bianchi Classification[g]

TYPE	STRUCTURE		
	$[X_1, X_2]$	$[X_1, X_3]$	$[X_2, X_3]$
I	0	0	0
II	0	0	X_1
III	0	X_1	0
IV	0	X_1	$X_1 + X_2$
V	0	X_1	X_2
VI_h ($h \neq 0, 1$)	0	X_1	hX_2
VII_h ($h^2 \neq 4$)	0	X_2	$-X_1 + hX_2$
VIII	X_3	X_2	X_1
IX	X_3	$-X_2$	X_1

Writing (3.4.1) in component form gives the differential equations which must be solved for the $T_a{}^i$:

$$T_a{}^i T_b{}^j{}_{,i} - T_b{}^i T_a{}^j{}_{,i} = C^c{}_{ab} T_c{}^j \tag{3.4.4}$$

In general, the equations (3.4.4) underdetermine the $T_a{}^i$ so that the coordinate realization of the T_a is not unique. It will be shown that this indeterminacy does not affect the curvature calculation.

If $A^b = A^b{}_i dx^i$ are the one forms dual to the T_a then:

$$(A^b, T_a) = \delta^b{}_a \tag{3.4.5}$$

or in component form:

$$A^b{}_i T_a{}^i = \delta^b{}_a \tag{3.4.6}$$

Similarly:

$$A^b{}_i T_b{}^j = \delta_i{}^j \tag{3.4.7}$$

Multiplying (3.4.4) by $A^a{}_k A^b{}_m$ gives:

$$A^b{}_m T_b{}^j{}_{,k} - A^a{}_k T_a{}^j{}_{,m} = C^c{}_{ab} T_c{}^j A^a{}_k A^b{}_m \tag{3.4.8}$$

or:

$$\lambda_{mk}{}^j - \lambda_{km}{}^j = C^c{}_{ab} T_c{}^j A^a{}_k A^b{}_m \tag{3.4.9}$$

where:

$$\lambda_{mk}{}^{j} = A^{b}{}_{m}T_{b}{}^{j}{}_{,k} \tag{3.4.9a}$$

The invariant basis with elements X_a is defined from:

$$[X_a, T_b] = 0 \tag{3.4.10}$$

We first show that the X_a are the elements of a group. Writing the Jacobi identity (3.4.3) for the vectors X_a, X_b and T_c, the only remaining term, in the light of (3.4.10) will be:

$$[T_c, [X_a, X_b]] = 0 \tag{3.4.11}$$

The commutator $[X_a, X_b]$ is a vector and can be expanded on the X_a to give:

$$[X_a, X_b] = D^{c}{}_{ab}X_c \tag{3.4.12}$$

which on substitution into (3.4.11), using (3.4.10) results in:

$$(T_c D^{d}{}_{ab})X_d = 0 \tag{3.4.13}$$

Since both the T_c and X_d span the space, it follows that the D^{a}_{bc} are constants and the X_a form a group. Groups related by (3.4.10) are called reciprocal. Writing (3.4.10) in component form:

$$X_a{}^{i}T_b{}^{j}{}_{,i} - T_b{}^{i}X_a{}^{j}{}_{,i} = 0 \tag{3.4.14}$$

and multiplying by A^b_k gives:

$$X_a{}^j{}_{,k} - \lambda_{ki}{}^j X_a{}^i = 0 \tag{3.4.15}$$

which is a differential equation for the components X_a^i. The equations (3.4.15) admit three linearly independent solutions (ref. 19, p.114) which define the X_a. Writing (3.4.12) in component form:

$$X_a{}^i X_b{}^j{}_{,i} - X_b{}^i X_a{}^j{}_{,i} = D^c{}_{ab} X_c{}^j \tag{3.4.16}$$

and using (3.4.15) yields:

$$X_a{}^i X_b{}^k \lambda_{ik}{}^j - X_b{}^i X_a{}^k \lambda_{ik}{}^j = D^c{}_{ab} X_c{}^j \tag{3.4.17}$$

If the one forms $\sigma^a = \sigma^a{}_i dx^i$ are dual to the X_a so that:

$$\sigma^a{}_i X^i{}_b = \delta^a{}_b \tag{3.4.18}$$

then multiplying (3.4.17) by $\sigma^a{}_m \sigma^b{}_n$ gives:

$$\lambda_{mn}{}^j - \lambda_{nm}{}^j = D^c{}_{ab} X_c{}^j \sigma^a{}_m \sigma^b{}_n \tag{3.4.19}$$

or, using (3.4.9)

$$D^c{}_{ab} X_c{}^j \sigma^a{}_i \sigma^b{}_k = C^c{}_{ab} T_c{}^j A^a{}_k A^b{}_i \tag{3.4.20}$$

Coordinates can always be normalized on each hypersurface $t =$ constant such that at a given point p; $X_c^j(p) = T_c^j(p)$ in which case from (3.4.20):

$$D^c{}_{ab} = -C^c{}_{ab}$$

and the constants of structure of the reciprocal group can be chosen to be the negatives of those of the translation group. Other normalizations are, of course, possible.

For the one forms $\sigma^a = \sigma^a{}_i dx^i$, the exterior derivative $d\sigma^a$ is given by:

$$d\sigma^a = \frac{1}{2} (\sigma^a{}_{i,j} - \sigma^a{}_{j,i}) \; dx^j \wedge dx^i \tag{3 4.21}$$

Differentiating (3.4.18) gives:

$$\sigma^a{}_{i,j} X_b{}^i + \sigma^a{}_i X_b{}^i{}_{,j} = 0 \tag{3 4.22}$$

Multiplying (3.4.22) by A^b_k and using (3.4.10) yields:

$$\sigma^a{}_{k,j} + \sigma^a{}_i \lambda_{kj}{}^i = 0 \tag{3.4.23}$$

which, on substitution in (3.4, 21) gives:

$$d\sigma^a = \frac{1}{2} (\lambda_{ij}{}^k - \lambda_{ji}{}^k) \sigma^a{}_k dx^j \wedge dx^i \tag{3.4.24}$$

which from (3.4.19) can be written:

$$d\sigma^a = \frac{1}{2} D^c{}_{db} X_c{}^k \sigma^d{}_j \sigma^b{}_i \sigma^a{}_k dx^j \wedge dx^i$$

or:

$$\begin{aligned} d\sigma^a &= \frac{1}{2} D^a{}_{bc} \sigma^c \wedge \sigma^b \\ &= \frac{1}{2} C^a{}_{bc} \sigma^b \wedge \sigma^c \end{aligned}$$

It remains only to demonstrate that the form for h (3.3.3) is such that the T_a

satisfy Killing's equation when:

$$h = \gamma_{ab}(t)\sigma^a{}_i\sigma^b{}_j dx^i \otimes dx^j \tag{3.4.26}$$

Expanding Killing's equations (3.3.4) and using the symmetry of the Christoffel symbols and (3.4.9a) gives:

$$h_{ij,k} = h_{im}\lambda_{jk}{}^m + h_{mj}\lambda_{ik}{}^m \tag{3.4.27}$$

in which case inserting $h_{ij} = \gamma_{ab}(t)\sigma^a{}_i\sigma^b{}_j$ and using (3.4.23) reduces (3.4.27) to an identity and Killing's equations are satisfied.

Thus, the general metric form of a homogeneous spacetime may be written:

$$g = -dt \otimes dt + \gamma_{ab}(t)\sigma^a \otimes \sigma^b \tag{3.4.28}$$

where the σ^a satisfy (3.4.25). Following the methods discussed in Section 2.5, the metric (3.4.26) can be diagonalized to yield the orthonormal Cartan basis ω^α, such that:

$$g = \eta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta \tag{3.4.29}$$

where $\eta_{\alpha\beta}$ is Minkowskian and the ω^α can be written:

$$\omega^\alpha = a^\alpha{}_\beta\sigma^\beta \tag{3.4.30}$$

with $\sigma^0 = dt$. It is evident from the form of (3.4.2) that the $a^\alpha{}_\beta$ will be only functions of time. Hence, the exterior derivatives of the ω^α and the Ricci rotation coefficients defined in 2.5 will be only functions of time, and the structure constants C^c_{ab}. Since the curvature depends only on the rotation coefficients and their first derivatives (in complete analogy with the construction of the Ricci tensor from the Christoffel symbols), the Ricci tensor will depend only on the a^α_β and the C^a_{bc}. It follows, then, that the construction of coordinate realizations of the T_a and X_a is somewhat artificial, since it is only the group structure and the matrix $a^\alpha{}_\beta$ which determine the curvature. That such a formulation should exist is logical since presumably in a homogeneous spacetime the curvature should depend only on which hypersurface is examined, not on any particular location on the hypersurface.

3.5 - Bianchi Type VII_h, An Example

As an example of the process described in Sections 3.3, 4, we consider the group type VII_h whose structure from Table 3.1 is:

$$[T_1, T_2] = 0 \qquad [T_1, T_3] = T_2 \qquad [T_2, T_3] = -T_1 + hT_2 \tag{3.5.1}$$

In (3.5.1), h is a constant and a different group structure is defined for each value of h. Integration of (3.5.1) leads to the generators:

$$T_1 = p_2 \quad T_2 = p_3 \quad T_3 = p_1 - x^3 p_2 + (x^2 + hx^3) p_3 \tag{3.5.2}$$

which do not vanish for any x^1, x^2, x^3. From (3.5.2) the one-forms A^b satisfying (3.4.5) can be found and from (3.4.9a) the non-zero $\lambda_{ab}{}^c$ are:

$$\lambda_{31}{}^2 = 1 \qquad \lambda_{21}{}^3 = -1 \qquad \lambda_{31}{}^3 = -h \tag{3.5.3}$$

which from (3.4.15) give as the differential equations for the X_a:

$$X_a{}^2{}_{,1} + X_a{}^3 = 0 \qquad X_a{}^3{}_{,1} = X_a{}^2 + hX_a{}^3$$
$$X_a{}^i{}_{,j} = 0 \quad (i \neq 2,3 \;,\; j \neq 1) \tag{3.5.4}$$

The equations (3.5.4) are equivalent to the second order equation:

$$X_a{}^3{}_{,1,1} + X_a{}^3 - hX_a{}^3{}_{,1} = 0 \tag{3.5.5}$$

Equation (3.5.5) is to be integrated such that the X_a have the structure (3.5.1) with the structure constants reversed in sign. These solutions are:

$$
\begin{aligned}
X_1 &= Ap_3 + (B - kA)p_2 \\
X_2 &= (kA + B)p_3 - Ap_2 \\
X_3 &= p_1
\end{aligned}
\tag{3.5.6}
$$

where:

$$A = e^{kx^1} \cosh ax^1 \qquad B = ae^{kx^1} \sinh ax^1 \tag{3.5.7}$$

and:

$$k = \frac{1}{2} h \qquad a^2 + 1 = \frac{1}{4} h^2 \tag{3.5.8}$$

For $h^2 < 4$ the equations (3.5.6) are well behaved under the transformation $a \rightarrow ia$ but at $h^2 = 4$ equation (3.5.5) has a single root and $X_1 = X_2$. However, at $h = \pm 2$, the transformation:

$$
\begin{aligned}
Y_1 &= \pm T_2 - T_1 \\
Y_2 &= T_1 \\
Y_3 &= \pm T_3
\end{aligned}
\tag{3.5.9}
$$

bring the relations (3.5.1) to the form:

$$[Y_1, Y_2] = 0 \qquad [Y_1, Y_3] = Y_1 \qquad [Y_2, Y_3] = Y_1 + Y_2 \tag{3.5.10}$$

which comparison with Table 3.1 shows to be the structure of Bianchi Type IV.

Bianchi Type VII_h is an infinite set of groups with the parameter h taking on all real values except $h = \pm 2$ at which point the structure is Bianchi Type IV.

4.- Some Properties of Homogeneous Spacetimes

We have shown that if the homogeneity of a spacetime is generated by a simply transitive group of motions G_3 then its geometry is determined by two independent factors: first, the structure of G_3 which allows an invariant basis in the surfaces of transitivity of the group; and second, a linear mapping of this basis to an orthonormal set which defines a metric on these surfaces. The group structure can be chosen from the nine, transitive, three parameter groups of Bianchi (Table 3.1) but for each such group or "Bianchi Type" a wide variety of metric forms is possible. The majority of work done to date investigating particular Bianchi types has been for the case where the invariant basis is an orthogonal set[26]. More general investigations into the relative roles of group and metric structure have been undertaken[27,28,29] particularly with respect to curvature characteristics. This chapter is a collection of new results concerning calculations which usually attend the construction of cosmological models. The geodesic and continuity equations, as well as vorticity and Maxwell's equations are investigated and re-expressed in forms which make manifest the role of group structure. Conclusions for individual as well as classes of Bianchi Types are discussed, and invariance of homogeneous fields under additional motions of the spacetime is investigated.

4.1 - Homogeneous Spacetimes

In Section (3.4, 5) it was shown that the metric form:

$$g = -dt \otimes dt + h_{ij}(t)\sigma^i \otimes \sigma^j \tag{4.1.1}$$

will admit as Killing vectors the generators of a Bianchi Group if the one-forms σ^i of (4.1.1) obey:

$$d\sigma^i = \frac{1}{2} C^i{}_{jk}\sigma^j \wedge \sigma^k \tag{4.1.2}$$

where the $C^i{}_{jk}$ are the group structure constants, "d" denotes the exterior derivative and "$\wedge$" the exterior or "wedge" product. The surfaces t = constant in (4.1.1) are orthogonal to the geodesic world-lines with tangent vectors p_t . The vectors σ_i dual to the σ^i are the generators of the group reciprocal to the Bianchi group with commutators:

$$[\sigma_j, \sigma_k] = -C^i{}_{jk}\sigma_i \tag{4.1.3}$$

In general, the metric form (4.1.1) is skew but can be diagonalized to give a non-holonomic Cartan basis ω^α where;

$$g = \eta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta \tag{4.1.4}$$

and $\eta_{\alpha\beta}$ = diag (-1,+1,+1,+1).

In (4.1.4) ω^o = dt and the ω^i are linear combinations of the σ^i with time dependent coefficients:

$$\omega^\alpha = a^\alpha{}_\beta \sigma^\beta \tag{4.1.5}$$

where (4.1.1) requires $a^o_o = 1$ and $a^o_i = a^i_o = 0$. The invertability of (4.1.5) implies the non-singularity of a_{ij}. From the duality relation $(\sigma^\alpha, \sigma_\beta) = \delta^\alpha{}_\beta$ the basis vectors ω_γ may be found:

$$\omega_\alpha = b^\beta{}_\alpha \sigma_\beta \qquad \sigma_\beta = a^\alpha{}_\beta \omega_\alpha \tag{4.1.6}$$

where $a^\alpha{}_\beta b^\beta{}_\gamma = \delta^\alpha{}_\gamma$

The one-forms σ^i together with $\sigma^o = dt$ form a time-independent basis (with time-dependent metric) which connects to the Cartan and coordinate frames. Orthonormal frame results can be translated to the σ^α basis both through (4.1.5, 6) and the invariance properties of the exterior derivative and inner product. The plan of the following sections is to use these various ways of expressing the same idea to obtain alternative expressions for familiar quantities.

4.2 - The Four-Velocity and Geodesic Motion

The matter content of spacetime is represented as a congruence (non-intersecting, space filling curves) of world lines whose normalized tangent vectors constitute a field u on the manifold. The exterior derivative of u is

$$du = u_{\alpha,\beta}\omega^{\beta} \wedge \omega^{\alpha} + u_{\alpha}\gamma^{\alpha}{}_{\mu\beta}\omega^{\mu} \wedge \omega^{\beta} \tag{4.2.1}$$

where $u_{\alpha,\beta} = \omega_{\beta}u_{\alpha}$ and the $\gamma^{\alpha}{}_{\delta\beta}$ are the Ricci rotation coefficients. Using the anti-symmetry of the exterior product (4.2.1) can be written:

$$du = \frac{1}{2}(du)_{\alpha\beta}\omega^{\alpha} \wedge \omega^{\beta} = \frac{1}{2}(u_{\beta;\alpha} - u_{\alpha;\beta})\omega^{\alpha} \wedge \omega^{\beta} \tag{4.2.2}$$

where $(du)_{\alpha\beta} = (du)_{[\alpha\beta]}$. From (4.2.2) it follows that the equations:

$$(du)_{\alpha\beta}u^{\beta} = 0 \tag{4.2.3}$$

together with the normalization of u are the geodesic equations of motion. More generally, defining the "acceleration" one-form a as:

$$a = a_{\beta}\omega^{\beta} \tag{4.2.4}$$

then an invariant definition of a is:

$$a = -*(u \wedge *du) \tag{4.2.5*}$$

in which case the geodesic equations of motion are a = 0 or from (4.2.5)

$$u \wedge *du = 0 \tag{4.2.6}$$

All equations marked with an asterisk will be derived in Section 4.7. For homogeneous geodesic motion, the components of u in the Cartan basis are only functions of time and from (4.1.5) u can be written:

$$u = u_\alpha(t)\omega^\alpha = u_\alpha a^\alpha{}_\beta(t)\sigma^\beta = v_\beta\sigma^\beta \tag{4.2.7}$$

where the v^β are also only functions of the time. In a similar fashion, the

$$v^\beta = u^\alpha b^\beta{}_\alpha \tag{4.2.8}$$

can be obtained from (4.1.6). Denoting by v the one form $v_\beta\sigma^\beta$ then:

$$dv = v_i{}^\bullet\sigma^o \wedge \sigma^i + \tfrac{1}{2}v_i C^i{}_{jk}\sigma^j \wedge \sigma^k \tag{4.2.9}$$

where the dot denotes differentiation with respect to time. From (4.2.9) and (4.2.3) or (4.2.6) the geodesic equations can be written:

$$v_i{}^\bullet v^i = 0 \qquad \text{(i)}$$

$$v_i{}^\bullet v^o - v_j C^j{}_{ik}v^k = 0 \qquad \text{(ii)} \tag{4.2.10}$$

together with $v_\alpha v^\alpha =$ constant.

Multiplying (4.2.10) ii by v^i indicates that (4.2.10)i is a consequence of the anti-symmetry of the $C^i{}_{jk}$. From (4.2.10)i and $v_\alpha v^\alpha =$ constant there follows:

$$2v_o v_o{}^\bullet - v_i v^{i\bullet} = 0 \tag{4.2.11}$$

For the special case when $a^{\alpha}{}_{\beta}$ is diagonal, $\omega^i = A_i \sigma^i$ (no sum)

and the geodesic equations (4.2.10)ii can be written :

$$(A_i u_i)^{\bullet} = \sum_{k,j} \frac{A_k}{A_j} \frac{u_k u_j}{u_o} C^k{}_{ij} \qquad \text{(i not summed over)}$$

Equations (4.2.10) are obtained directly from the group and metric structure without the need for intermediate calculation of connection forms. Metric dependence is suppressed and general analysis of geodesic motion is considerably simplified. For example, in Bianchi Type IX spacetimes ($C^i{}_{jk}$ =1, i, j, k cyclic) equation (4.2.10)ii is:

$$v_i{}^{\bullet} v^o + v_j v^k - v_k v^j = 0 \qquad \text{(i, j, k cyclic)}$$

and it follows that $\sum_i v_i{}^{\bullet} v_i = 0$ or that $\sum_i (v_i)^2$ is a constant of the motion. Similarly:

(a) In Type VIII spacetimes ($C^1{}_{23} = C^2{}_{31} = -C^3{}_{12} = 1$), it follows that $v_1{}^2 + v_2{}^2 - v_3{}^2$ is a constant of the motion.

(b) In Type VII_o spacetimes ($C^1{}_{22} = C^2{}_{31} = 1$), $v_1{}^2 + v_2{}^2$ is a constant of the motion.

(c) In Type V spacetimes ($C^1{}_{13} = C^2{}_{23} = 1$), v_1/v_2 is a constant of the motion.

4.3 - Vorticity

For a fluid characterized by a four velocity u there is an associated vorticity tensor $\Omega_{\alpha\beta}$ given by (3.1.9) as:

$$\Omega_{\alpha\beta} = u_{[\alpha;\beta]} + a_{[\alpha}u_{\beta]} \tag{4.3.1}$$

and a vorticity vector Ω^{α} given for orthonormal frames by (3.1.10) as:

$$\Omega^{\alpha} = \frac{1}{2}\epsilon^{\beta\gamma\delta\alpha}u_{\beta}\,\Omega_{\gamma\delta} \tag{4.3.2}$$

which measures the rotation of the matter with respect to a gyroscopic basis. Defining the vorticity 2-form Ω' by $\Omega' = \frac{1}{2}\Omega_{\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}$, then by (4.2.2) and (4.3.1), Ω' is given by:

$$\Omega' = \frac{-1}{2}(du + u\wedge a) \tag{4.3.3}$$

and the vorticity one-form with components given by (4.3.2) can be written:

$$\Omega = -{*}(u\wedge\Omega') = \frac{1}{2}{*}(u\wedge du) \tag{4.3.4*}$$

which can be inverted to give:

$$\Omega' = -{*}(u\wedge\Omega) \tag{4.3.5*}$$

From (4.3.4,5) the magnitude of the vorticity is:

$$\Omega_{\alpha}\Omega^{\alpha} = -{*}(\Omega\wedge{*}\Omega) = -{*}(\Omega'\wedge{*}\Omega') = \frac{1}{8}(du_{\alpha\beta}du^{\alpha\beta} + 2a_{\beta}a^{\beta}) \tag{4.3.6*}$$

For geodesics of the form (4.2.7) in homogeneous spacetimes, equations (4.2.10) can be used to eliminate the v_i from (4.2.9) giving the generic geodesic vorticity vector as a function of only the v_i and $C^i{}_{jk}$. In Type V, for example, ($C^1{}_{13} = C^2{}_{23} = 1$) from (4.3.3):

$$*\Omega = \frac{1}{2}\, u \wedge du = \frac{1}{2}\left(\frac{v_1}{v_o}\, \sigma^o \wedge \sigma^1 \wedge \sigma^3 + \frac{v_2}{v_o}\, \sigma^o \wedge \sigma^2 \wedge \sigma^3 \right)$$

The vanishing of Ω is of interest since the condition $\Omega = 0$ implies local simultaneity is globally extendable. From (4.3.4, 5) $\Omega = 0 \Leftrightarrow \Omega' = 0$ and from (4.3.4):

$$du = a \wedge u \tag{4.3.7}$$

For geodesics, $a = 0$ and (4.3.7) requires the vanishing of du. Equations (4.3.5) and (4.3.6) show that $du = 0$ is certainly a sufficient condition for $\Omega = a = 0$, but (4.3.7) implies it is a necessary condition as well, and all vorticity-free geodesics are the solutions of $du = 0$. Equivalently:

$$u \wedge du = 0 = u \wedge *du \rightleftarrows du = 0 \quad \text{if } (u,u) \neq 0 \tag{4.3.8}$$

(reminiscent of the vector statement $A \times B = 0 = A \cdot B \rightleftarrows A = 0$ if $B \cdot B \neq 0$).

That (4.3.8) is generally true follows from the Frobenius Integration Theorem (ref. 9,, p. 92) which states that the vanishing of $u \wedge du$ implies u is of the form:

$$u = f dg \tag{4.3.9}$$

or:

$$du = f^{-1} df \wedge u \tag{4.3.10}$$

Eq. (4.2.6) then implies:

$$(f_{,\alpha}u_\beta - f_{,\beta}u_\alpha)u^\beta = 0 \tag{4.3.11}$$

or:

$$df = w^{-1}(f_{,\beta}u^\beta)u \tag{4.3.12}$$

where w is the norm of u.

If $f_{,\beta} = 0$ or $f_{,\beta}u^\beta \doteq 0$ the vanishing of du follows immediately; otherwise it follows from substitution of (4.3.12) into (4.3.10).

Homogeneity was not assumed in deriving (4.3.8) and it follows that in any spacetime all the vorticity free geodesics are the solutions of du = 0 subject to the normalization condition $(u,u) = \text{constant} \neq 0$.

For the Bianchi types in particular, the two forms $\sigma^\alpha \wedge \sigma^\beta$ are linearly independent, and from (4.2.9) du = 0 requires:

$$\begin{aligned} &v_i{}^{\bullet} = 0 && \text{(i)} \\ &v_i C^i{}_{jk} = 0 && \text{(ii)} \end{aligned} \tag{4.3.13}$$

which for the group structures of Table 3.1 requires $v_i = 0$ if $d\sigma^i \neq 0$. Consideration of (4.3.13) leads to the following observations:

(a) In Bianchi Type I $(d\sigma^i = 0)$ the vorticity vanishes for all homogeneous geodesic motion.

(b) In Bianchi Type IX the only homogeneous geodesic motion with vanishing vorticity is the co-moving case: $u^0 = 1$

(c) In Bianchi Type V

the solutions of du= 0 are $v_1 = v_2 = 0$, $v_3 =$ constant. Furthermore, all solutions of the geodesic equations of the form $v_i =$ constant require $\Omega = 0$. This follows from the geodesic equations which from (4.2.10) with $\dot{v}_i = 0$ are:

$$v_1 v^3 = v_2 v^3 = v_1 v^1 + v_2 v^2 = 0$$

If $\Omega \neq 0$ then at least one of v_1, v_2 must not vanish and v_3 must be zero. The vanishing of v^3, however, requires $v_o v^o = -1$ and the matter is co-moving implying $\Omega = 0$.

4.4 - Class A and Class B Spacetimes

The groups G_3 and the spacetimes derived from them may be divided into two classes based on the nature of the group structure constants. Following Ellis and MacCallum[27] we label those groups for which the sum $C^j{}_{ij}$ vanishes as Class A, and those for which $C^j{}_{ij} \neq 0$ as Class B. If i, j, k is any even permutation of 1, 2, 3 then:

$$d(\sigma^i \wedge \sigma^j) = \frac{1}{2} C^i{}_{mn}\sigma^m \wedge \sigma^n \wedge \sigma^j - \frac{1}{2} C^j{}_{mn}\sigma^i \wedge \sigma^m \wedge \sigma^n \tag{4.4.1}$$

Since $j \neq 1$ the sums on the right in (4.4.1) vanish unless n, m take on the values i, k or j, k in the first and second terms respectively, in which case (4.4.1) becomes[30]:

$$d(\sigma^i \wedge \sigma^j) = C^m{}_{km}\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \tag{4.4.2}$$

Thus (4.4.2) vanishes or not, according to the structure of the group of motions. Two short applications of this property follow:

i) The Equation of Continuity

For any one form $j = j_\alpha \omega^\alpha$ the operation:

$$\delta j = *d*j \tag{4.4.3}$$

is the divergence $j^\gamma{}_{;\gamma}$. For:

$$j = j_o\omega^o + j_i\omega^i \tag{4.4.4}$$

and the metric form (4.1.1), $*j$ is given by:

$$*j = j_o \omega^1 \wedge \omega^2 \wedge \omega^3 + j_1 \omega^o \wedge \omega^2 \wedge \omega^3 + \text{cyclic} \tag{4.4.5}$$

in the σ^α basis (4.4.5) can be written:

$$*j = J_o{}^* \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + J_i{}^* \sigma^o \wedge \sigma^2 \wedge \sigma^3 + \text{cyclic} \tag{4.4.6}$$

where $J_0^* = j_o |a|$ $\quad$ ($|a| = \det(a^i{}_j)$) and:

$$J_i{}^* = j_p (a^m{}_j a^n{}_k - a^m{}_k a^n{}_j) \quad \left(\begin{smallmatrix} i,j,k \\ p,m,n \end{smallmatrix} \text{ cyclic}\right) \tag{4.4.7}$$

Exterior differentiation of (4.4.7) gives:

$$d*j = (J_o{}^*)^{\cdot} \sigma^o \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 - (J_1{}^*) \sigma^o \wedge d(\sigma^2 \wedge \sigma^3) + \text{cyclic} \tag{4.4.8}$$

which by virtue of (4.4.2) gives for the continuity equation $(\delta j = 0)$:

$$(J_o{}^*)^{\cdot} - \sum_i J_i{}^* C^k{}_{ik} = 0 \tag{4.4.9}$$

Thus the continuity equation is integrable in all Class A spacetimes with solution:

$$J_o{}^* = j_o |a| = \text{constant} \tag{4.4.10}$$

ii) Maxwell's Equations and Charged Homogeneous Spacetimes.

For the Maxwell two-form given by:

$$f = \frac{1}{2} f_{\alpha\beta} \omega^\alpha \wedge \omega^\beta = e_i \omega^i \wedge \omega^o + b_1 \omega^2 \wedge \omega^3 + \text{cyclic} \tag{4.4.11}$$

Maxwell's equations are given by:

$$df = 0 \qquad \text{(i)}$$
$$d{*}f = {*}j \qquad \text{(ii)} \qquad (4.4.12)$$

where j is the current one-form. In the σ^α basis:

$$f = E_i \sigma^i \wedge \sigma^o + B_1 \sigma^2 \wedge \sigma^3 + \text{cyclic} \qquad (4.4.13)$$

where $E_i = e_j a^j{}_i$ and:

$$B_i = b_p(a^m{}_j a^n{}_k - a^m{}_k a^n{}_j) \quad (\begin{smallmatrix} i,j,k \\ p,m,n \end{smallmatrix} \text{ cyclic})$$

Similarly:

$${*}f = B_i{*}\sigma^i \wedge \sigma^o - E_i{*}\sigma^2 \wedge \sigma^3 + \text{cyclic} \qquad (4.4.14)$$

where $B_i^* = b_j a^j{}_i$ and

$$E{*}_i = e_p(a^m{}_j a^n{}_k - a^m{}_k a^n{}_j) \quad (\begin{smallmatrix} i,j,k \\ p,m,n \end{smallmatrix} \text{ cyclic})$$

Using (4.4.13), equation (4.4.12)i becomes:

$$df = \tfrac{1}{2} E_i C^i{}_{jk} \sigma^j \wedge \sigma^k \wedge \sigma^o + B_1{}^{\bullet} \sigma^o \wedge \sigma^2 \wedge \sigma^3 + B_1 d(\sigma^2 \wedge \sigma^3) + \text{cyclic}$$

which using (4.4.2) gives the equations

$$(B_i)^{\bullet} + E_m C^m{}_{jk} = 0 \qquad \text{(i, j, k cyclic)}$$
$$\sum_i B_i C^k{}_{ik} = 0 \qquad (4.4.15)$$

Using equation (4.4.6) for $*j$ gives for (4.4.12)ii

$$(E_i{*})^{\cdot} - B_m C^m{}_{jk} = -J_i{*} \qquad \text{(i, j, k cyclic)}$$

$$\sum_i E_i{*} C^k{}_{ik} = -J_o{*} \tag{4.4.16}$$

Equations (4.4.16) indicate that $J^*_o = j_o |a|$, and consequently j_o, must vanish if $C^k{}_{ik}=0$, thus prohibiting charged homogeneous solutions in all Class A spacetimes.

Equations (4.4.15) and (4.4.16) may be used to analyze general solutions to Maxwell's equations. In Type I, for instance, (4.4.15,16) give in the source free case

$$B_i = \text{constant} \qquad E^*_i = \text{constant}$$

which are linear equations for the e_i and b_i.

In Type IX spacetimes eq. (4.4.15,16) give :

$$B_i{}^{\cdot} + E_i = 0 \tag{4.4.17}$$

$$(E_i{*})^{\cdot} - B_i{*} = -J_i{*}$$

which demonstrate that for $A = B_i\, \sigma^i$

$$dA = B_i{}^{\cdot} \sigma^o \wedge \sigma^i + B_1 \sigma^2 \wedge \sigma^3 + \text{cyclic}$$

and if (4.4.17) is satisfied, $f = dA$ and a global vector potential has been constructed.

4.5 - Fields on Homogeneous Spacetimes

We consider a homogeneous spacetime, and two observers 0 and 0' related by an infinitesimal coordinate transformation generated by the vector field R. If x and $\bar{x}$ are coordinates in 0 and 0' respectively and R is assumed effective on surfaces t constant and spacelike everywhere then:

$$\bar{x}^i = x^i + R^i \delta s \tag{4.5.1}$$

We assume further that R generates isometries of the space so that if g(x), $T_a(x)$ and $X_a(x)$ are respectively, the metric tensor, generators of translations and invariant basis ($[T_a, X_b] = 0$) in 0, then $g(\bar{x})$, $T_a(\bar{x})$ and $X_a(\bar{x})$ are the corresponding quantities in 0'. A homogeneous vector field in 0 can be written:

$$V = V^o(t)p_t + V^a(t)X_a{}^i p_i \tag{4.5.2}$$

. Under transformation from 0 to 0', V can be written:

$$V = V^o p_t + V^a X_a{}^i \bar{x}^j{}_{,i} \bar{p}_j \tag{4.5.3}$$

If V is to be homogeneous in 0', that is with respect to the $X_a(\bar{x})$, we must have:

$$X_a{}^i(x)\bar{x}^j{}_{,i} = c^b{}_a(s) X_b{}^j(\bar{x}) \tag{4.5.4}$$

where the $c^b{}_a$ are functions of only the parameter s defined in (4.5.1). Using the explicit relations (4.5.1) we have:

$$X_b{}^j(\bar{x}) = X_b{}^j(x) + X_b{}^j{}_{,i}R^i\delta s \quad \text{(a)}$$

$$\bar{x}^j{}_{,i} = \delta^j{}_i + R^j{}_{,i}\delta s \quad \text{(b)} \qquad (4.5.5)$$

$$c^b{}_a(s) = \delta^b{}_a + d^b{}_a\delta s \quad \text{(c)}$$

where in (4.5.5 c) since $s = 0$ is the identity, $c^b{}_a(0) = \delta^b{}_a$ and the constants $d^b{}_a$ are defined by:

$$d^b{}_a = \left[\frac{d}{ds}\, c^b{}_a\right]_{s=0}$$

Using (4.5.5a, b, c) in (4.5.3) gives to first order in δs:

$$X_a{}^iR^j{}_{,i} - R^iX_a{}^j{}_{,i} = d^b{}_aX_b{}^j \qquad (4.5.6)$$

or

$$[X_a, R] = d^b{}_aX_b \qquad (4.5.7)$$

Equation (4.5.7) will be recognized as the requirement that a coordinate transformation map the X_a into the same group in the new coordinates (ref. 19, p. 109). From the reciprocal relation between the X_a and T_a, or by writing (4.5.3) for the T_a and continuing as before, it follows that the T_a must obey a relation of the form (4.5.7):

$$[T_a, R] = w^b{}_aT_b \qquad (4.5.8)$$

where the w^b_a are also constants. If R is itself an element of the translation group, (4.5.8) is clearly satisfied and it follows that translations preserve homogeneity (in fact (4.5.8) may be taken as proof that the requirement that the translations preserve homogeneity is that they form a group). If (4.5.8) is satisfied for every element R of the group of motions then the translations are an invariant subgroup of the group of motions. Where the translations are not an invariant subgroup (and the group of motions is larger than G_3) an isometry associated with rotation will map homogeneous tensor fields of rank greater than 0 into inhomogeneous ones.

In Euclidian space with Cartesian coordinates x^i, the Abelian group with elements p_i generates translations and the elements of 0(3) are generated by:

$$R_i = x^j p_k - x^k p_j \qquad \text{(i, j, k cyclic)}$$

It is easily verified that the R_i satisfy (4.5.8) and preserve homogeneity. It is this independence of observer orientation which has led to the conceptualization of homogeneity as a physical property of fields. If (4.5.8) is violated in the spacetimes of General Relativity, it is apparent that such a conceptualization cannot be extended.

An example of a spacetime in which (4.5.8) cannot be satisfied is the open Friedman model with metric:

$$g = -dt \otimes dt + A(t)^2[e^{2z}(dx \otimes dx + dy \otimes dy) + dz \otimes dz] \qquad (4.5.9)$$

and motions (ref. 11, p. 236):

$$T_1 = p_x \qquad R_1 = yp_x - xp_y$$

$$T_2 = p_y \qquad R_2 = -xp_z + \tfrac{1}{2}(x^2 - y^2 - e^{-2z})p_x + xyp_y$$

$$T_3 = -p_z + xp_x + yp_y \qquad R_3 = -yp_z + \tfrac{1}{2}(y^2 - x^2 - e^{-2z})p_y + xyp_x \tag{4.5.10}$$

The structure of the group of motions is:

$$[T_1,T_2] = 0 \qquad [R_1,R_2] = R_3$$

$$[T_1,T_3] = T_1 \qquad [R_1,R_3] = -R_2$$

$$[T_2,T_3] = T_2 \qquad [R_2,R_3] = 0 \tag{4.5.11}$$

and

$$[R_1,T_1] = T_2 \qquad [R_2,T_1] = T_3 \qquad [R_3,T_1] = -R_1$$

$$[R_1,T_2] = -T_1 \qquad [R_2,T_2] = R_1 \qquad [R_3,T_2] = -T_3$$

$$[R_1,T_3] = 0 \qquad [R_2,T_3] = -R_2 \qquad [R_3,T_3] = -R_3 \tag{4.5.12}$$

Strictly speaking the R_a as defined by (4.5.10) have no fixed points (the equations $(R_a)^i = 0$ have no solution), but the R_a can be redefined as follows:

$$R_1' = R_1 \qquad R_2' = R_2 + \tfrac{1}{2}T_1 \qquad R_3' = R_3 + \tfrac{1}{2}T_2 \tag{4.5.13}$$

It is easily verified that the R_a' leave the origin fixed and from (4.5.11, 12) it follows that they have the structure 0(3) (Type IX). This redefinition does not affect subsequent arguments since:

i) If there exists any isometry which does not satisfy (4.5.8) a problem arises.

ii) The R_a' do not satisfy (4.5.8). If the T_a are not an invariant subgroup, redefinitions of the form (4.5.13) will not alter that fact.

The basis of one-forms :

$$\sigma^0 = dt \qquad \sigma^1 = e^z dx \qquad \sigma^2 = e^z dy \qquad \sigma^3 = dz \tag{4.5.14}$$

are invariant under the (Type V) T_a suggesting the form :

$$V = V_0(t)\sigma^0 + V_a(t)\sigma^a \tag{4.5.15}$$

will represent a homogeneous vector field. As the structure (4.5.12) shows, R_1, which generates the LRS case satisfies (4.5.8) whereas R_2 and R_3 do not. A more physical picture of the situation is afforded by the following argument:

Imagine that a group of non-interacting test particles is to be propagated in the spacetime with metric (4.5.9). The four-velocity of the particles $U = U_\alpha \sigma^\alpha$ is constrained by the conditions:

i) all the U_i do not vanish.

ii) U is homogeneous; $U_\alpha = U_\alpha(t)$

iii) U is geodesic; $U \wedge {*dU} = 0$ (4.5.16)

iv) U is vorticity free; $U \wedge dU = 0$

According to the results of Section 4.3 the only form for U satisfying (4.5.16) is:

$$U = U_0\sigma^0 + U_3\sigma^3 \qquad (U_3 = c,\ U_0{}^2 = 1 + (c/A)^2)$$

where c is a constant and A is defined in (4.5.9). Under coordinate transformations generated by R_2 or R_3, U will remain geodesic and vorticity free but not homogeneous, since the only four-velocity satisfying (4.5.16) in the transformed system is along the new σ_3 axis and inspection of (4.5. 10) shows that both R_2 and R_3 alter the z axis hence change σ_3.

In the flat Friedman model where the translation group is taken to be Abelian, the rotation group is 0(3) and, as has been discussed, (4.5.8) is satisfied. Similarly, in the closed model, the invariant basis itself generates rotations and the R_a and T_a are both Type IX and commute satisfying (4.5.8). For the open Friedman models, the translation group can also be taken[27] as Type VII_h. Since like Type V, the structure of VII_h can be chosen so that only one of the σ^i obeys $d\sigma^i = 0$, the above argument beginning with equation (4.5.15) can be applied to it as well. Thus only in the open Friedman models is it impossible to choose the translation group so that (4.5.8) is satisfied. It follows that in these models it is impossible to have an observer independent concept of homogeneity[h].

4.6 - Homogeneity and the Field Equations

It has been shown that the isometries of a homogeneous spacetime do not necessarily preserve homogeneity. Although it may be felt that a unique prescription for the construction and recognition of homogenous fields is a requirement in any homogeneous spacetime, a stricter view would hold that homogeneity need only be preserved for those observable fields which are consistent with the field equations (i.e. included in the stress-energy tensor). Here, two cases are of interest: where the group of motions is G_4 or G_6 (again assumed acting only on spacelike hypersurfaces). Where the group of motions is G_6 the hypersurfaces t= constant are maximally symmetric and presumably no physically defined preferred direction is consistent with the field equations. Where the group of motions is G_4, standard symmetry considerations suggest that only preferred directions parallel to the axis of symmetry are allowed. Since the axis of symmetry is, by definition, preserved by rotation, it would seem that the problem under discussion would never arise if the field equations are used as the criterion of acceptable homogeneous fields.

The above symmetry argument for spacetimes invariant under a G_4 of motions is not correct, however, in that it assumes that the symmetries of the fields from which $T_{\mu\nu}$ is constructed are constrained by the symmetries of $T_{\mu\nu}$ (i.e. geometrodynamics is always valid). A Type V, LRS cosmological model will be presented in Chapter 5 as a counter example, in which physically defined preferred directions violate geometrical symmetries. Although for this model the rotation (R_1 of 6.9) preserves homogeneity, there does not appear to be a general argument which would prevent the existence of a solution of the field equations which is LRS and for which the additional motion does not preserve the homogeneity of fields included in the stress-energy tensor.

4.7 - Some More Exterior Calculus

In this section, equations marked with an asterisk in Sections 4.2, 3 will be derived, and some applications will be presented.

i) The "acceleration" 1-form - For a two-form

$$F = \frac{1}{2} F_{\mu\nu} \omega^{\mu} \wedge \omega^{\nu} \qquad \text{where} \qquad F_{\mu\nu} = F_{[\mu\nu]}$$

$$*F = -\frac{1}{4} F^{\alpha\beta} \epsilon_{\alpha\beta\gamma\delta} \omega^{\gamma} \wedge \omega^{\delta}$$

If $g = g_{\mu} \omega^{\mu}$ then:

$$g \wedge *F = -\frac{1}{4} g_{\mu} F^{\alpha\beta} \epsilon_{\alpha\beta\gamma\delta} \omega^{\mu} \wedge \omega^{\gamma} \wedge \omega^{\delta}$$

and:

$$*(g \wedge *F) = -\frac{1}{4} g^{\mu} F_{\alpha\beta} \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\nu\mu\gamma\delta} \omega^{\nu}$$

$$= F_{\alpha\beta} g^{\beta} \omega^{\alpha} \tag{4.7.1}$$

Thus for $F = du$ and $g = u$:

$$*(u \wedge *du) = du_{\alpha\beta} u^{\beta} \omega^{\alpha}$$

or from (4.2.2)

$$*(u \wedge *du) = -a \tag{4.7.2}$$

As an application of (4.7.2) consider the Lorentz force on a particle of charge q and mass m. With $f_{\alpha\beta}$ defined by (4.4.11) we have for the equation of motion:

$$a_\alpha = \frac{q}{m} f_{\alpha\beta} u^\beta$$

which from (4.7.1,2) is:

$$-*(u \wedge *du) = \frac{q}{m} *(u \wedge *f)$$

or:

$$u \wedge *(du + \frac{q}{m} f) = 0$$

which is the equation of motion in invariant form.

Similarly, for a perfect fluid with pressure P and density ρ the stress energy tensor can be written:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P\eta_{\mu\nu}$$

Using (4.7.2) and (4.4.3) the divergence of $T_{\mu\nu}$ can be written as the one-form:

$$T_\mu{}^\nu{}_{;\nu}\omega^\mu = \delta[(\rho + P)u]u + (\rho + P)a + dP$$

ii) <u>The Vorticity</u> - For the vorticity 2-form $\Omega' = \frac{1}{2}\Omega_{\alpha\beta}\omega^\alpha \wedge \omega^\beta$:

$$u \wedge \Omega' = \frac{1}{2}\Omega_{\alpha\beta}u_\gamma \omega^\gamma \wedge \omega^\alpha \wedge \omega^\beta$$

and :

$$*(u \wedge \Omega') = \frac{1}{2} \Omega^{\alpha\beta} u^{\gamma} \epsilon_{\delta\gamma\alpha\beta} \omega^{\delta}$$

which from (3.1.10) can be written:

$$*(u \wedge \Omega') = -\Omega_{\alpha} \omega^{\alpha} = -\Omega \qquad (4.7.3)$$

From (4.7.3)

$$u \wedge \Omega = u \wedge *(u \wedge \Omega') = \frac{-1}{2} \Omega^{\alpha\beta} u^{\gamma} u_{\delta} \epsilon_{\lambda\gamma\alpha\beta} \omega^{\delta} \wedge \omega^{\lambda}$$

and:

$$*(u \wedge \Omega) = \frac{-1}{4} \Omega_{\alpha\beta} u_{\gamma} u^{\delta} \epsilon^{\gamma\alpha\beta\lambda} \epsilon_{\delta\mu\nu\lambda} \; \omega^{\mu} \wedge \omega^{\nu}$$

$$= \frac{-1}{2} \Omega_{\mu\nu} \omega^{\mu} \wedge \omega^{\nu} = -\Omega'$$

The magnitude of the vorticity can be obtained using Table 2.1 and (4.3.4) as:

$$\Omega \wedge *\Omega = \frac{1}{4} *(u \wedge du) \wedge (u \wedge du)$$

$$= \frac{-1}{16} u^{\alpha} du^{\beta\gamma} \epsilon_{\delta\alpha\beta\gamma} u_{\lambda} du_{\mu\nu} \omega^{\lambda} \wedge \omega^{\mu} \wedge \omega^{\nu} \wedge \omega^{\delta}$$

$$= \frac{-1}{8} (du_{\alpha\beta} du^{\alpha\beta} + 2a^{\beta} a_{\beta}) \; \omega^{o} \wedge \omega^{1} \wedge \omega^{2} \wedge \omega^{3}$$

a similar calculation shows that:

$$\Omega \wedge {}^*\Omega = \Omega' \wedge {}^*\Omega'$$

5. - An Anisotropic Bianchi Type V Spacetime with Null Electromagnetic Field

In relativistic cosmology, it is generally assumed though unproven that symmetries of the geometry of spacetime imply symmetries of the physically measurable quantities associated with spacetime. Equivalently, the observables from which the stress energy tensor is constructed are assumed invariant under the motions admitted by the metric structure.

This chapter provides a counter example to the above assumption. A rotationally symmetric cosmological model is presented in which the electric and magnetic field components orthogonal to the axis of symmetry do not vanish. (A non-rotationally symmetric electromagnetic field gives rise to a rotationally symmetric stress energy tensor.) The spacetime is of Bianchi Type V and admits a multiply transitive G_4 of isometries acting on three dimensional spacelike hypersurfaces. It contains a source-free, null electromagnetic field with components in the symmetry plane. Since null fields are normally associated with radiation solutions, the model can be viewed as containing a wave (moving along the axis of symmetry) whose polarization vector defines a preferred direction in the symmetry plane. The Einstein-Maxwell equations are also integrated for the null electromagnetic field in combination with an ultra-relativistic gas assumed uncoupled from the field. In both cases the relationship of the present solutions to the known Friedman solutions is indicated and the relative contributions of matter and electromagnetic energy to singularity structure are examined.

5.1 - Type V Geometry

From the results of Chapter 3 and Table 3.1 it follows that Bianchi Type V spacetimes can be characterized by a basis of invariant one-forms σ^i with differential structure:

$$d\sigma^1 = \sigma^3 \wedge \sigma^1 \qquad d\sigma^2 = \sigma^3 \wedge \sigma^2 \qquad d\sigma^3 = 0 \tag{5.1.1}$$

A general metric form admitting a simply transitive G3 of Type V structure as a group of motions is:

$$g = -dt \otimes dt + \gamma_{ab}(t)\sigma^a \otimes \sigma^b \tag{5.1.2}$$

where the σ^i have the structure (5.1.1).

When $\gamma_{ab}(t)$ is given by $A^2(t)\ \delta_{ab}$, (5.1.2) describes the open Friedman spacetimes. For the LRS case with σ^3 the preferred direction (5.1.2) becomes:

$$g = -dt \otimes dt + A^2(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) + C^2\sigma^3 \otimes \sigma^3 \tag{5.1.3}$$

Defining the orthonormal Cartan basis ω^α by :

$$\omega^0 = dt \qquad \omega^1 = A\sigma^1 \qquad \omega^2 = A\sigma^2 \qquad \omega^3 = C\sigma^3 \tag{5.1.4}$$

(5.1.3) can be written:

$$g = \eta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta \tag{5.1.5}$$

where $\eta_{\alpha\beta}$ is Minkowskian with components

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1) \tag{5.1.6}$$

From (5.1.1,4) the differential structure of the ω^{α} is:

$$\begin{aligned}
d\omega^{o} &= 0 \\
d\omega^{1} &= (\ln A)^{\bullet}\omega^{o} \wedge \omega^{1} + C^{-1}\omega^{3} \wedge \omega^{1} \\
d\omega^{2} &= (\ln A)^{\bullet}\omega^{o} \wedge \omega^{2} + C^{-1}\omega^{3} \wedge \omega^{2} \\
d\omega^{3} &= (\ln C)^{\bullet}\omega^{o} \wedge \omega^{3}
\end{aligned} \tag{5.1.7}$$

where $(\ )^{\bullet}$ denotes differentiation with respect to time. The non-zero Ricci rotation coefficients are then:

$$\gamma^{1}{}_{01} = (\ln A)^{\bullet} = \gamma^{2}{}_{02} \qquad \gamma^{3}{}_{03} = (\ln C)^{\bullet} \qquad \gamma^{1}{}_{31} = \gamma^{2}{}_{32} = C^{-1} \tag{5.1.8}$$

giving for the non-zero components of the Ricci tensor:

$$\begin{aligned}
R_{00} &= -\left(2\,\frac{A^{\bullet\bullet}}{A} + \frac{C^{\bullet\bullet}}{C}\right) && \text{(a)} \\
R_{11} = R_{22} &= \frac{A^{\bullet\bullet}}{A} + \left(\frac{A^{\bullet}}{A}\right)^{2} + \frac{A^{\bullet}C^{\bullet}}{A\,C} - \frac{2}{C^{2}} && \text{(b)} \\
R_{33} &= \frac{C^{\bullet\bullet}}{C} + 2\,\frac{A^{\bullet}C^{\bullet}}{A\,C} - \frac{2}{C^{2}} && \text{(c)} \\
R_{03} &= -\frac{2}{C}\left(\frac{A^{\bullet}}{A} - \frac{C^{\bullet}}{C}\right) && \text{(d)}
\end{aligned} \tag{5.1.9}$$

with trace :

$$R = R^{\gamma}{}_{\gamma} = 2\left[\, 2\,\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + 2\,\frac{\dot{A}}{A}\frac{\dot{C}}{C} + \left(\frac{\dot{A}}{A}\right)^2 - \frac{3}{C^2}\right] \qquad (5.1.10)$$

5.2 - Maxwell's Equations

In the basis $\omega^\alpha \wedge \omega^\beta$ the Maxwell 2-form f is given by

$$f = \frac{1}{2} f_{\alpha\beta}\omega^\alpha \wedge \omega^\beta = e_i\omega^i \wedge \omega^o + b_1\omega^2 \wedge \omega^3 + \text{cyclic} \tag{5.2.1}$$

and Maxwell's equations in the absence of sources are

$$df = 0 \tag{5.2.2}$$

and

$$d *f = 0 \tag{5.2.3}$$

Null electromagneitc fields obey the additional restrictions :

$$f \wedge f = f \wedge *f = 0 \tag{5.2.4}$$

which require that the invariants $e \cdot b$ and $e^2 - b^2$ vanish. Rather than assuming the null field condition, it will be demonstrated that the only solutions of (5.2.2, 3) consistent with homogeneity and the LRS assumption require (5.2.4). If the e_i and b_i are assumed to be functions of t alone, (5.2.2) gives (using (5.2.1) and (5.1.7)):

$$\begin{aligned} e_1A + (b_2CA)^{\bullet} &= 0 \\ (b_1AC)^{\bullet} - e_2A &= 0 \\ b_3 &= 0 \end{aligned} \tag{5.2.5}$$

and (5.2.3) gives

$$(e_2AC)^{\bullet} - b_1A = 0$$

$$(e_1AC)^{\bullet} + b_2A = 0 \tag{5.2.6}$$

$$e_3 = 0$$

Defining new field variables E_i and B_i by

$$E_i = ACe_i \qquad \text{and} \qquad B_i = ACb_i \tag{5.2.7}$$

and a new time variable τ by

$$dt = Cd\tau \tag{5.2.8}$$

equations (5.2.4,5) become:

$$B_2' + E_1 = 0 \qquad\qquad E_2' - B_1 = 0 \tag{5.2.9}$$

$$B_1' - E_2 = 0 \qquad\qquad E_1' + B_2 = 0$$

where the prime denotes differentiation with respect to τ. Equations (5.2.9) are equivalent to the set of second order equations:

$$B_i'' = B_i \qquad\qquad E_i'' = E_i \tag{5.2.10}$$

with general solutions:

$$B_1 = a\text{Cosh }\tau + b\text{Sinh }\tau$$

$$B_2 = c\text{Cosh }\tau + d\text{Sinh }\tau \tag{5.2.11}$$

where a, b, c, d are constants and E_1, E_2 are given by (5.2.9). In order for the stress-energy tensor of the electromagnetic field:

$$(T_{e-m})^{\mu\nu} = f^{\mu}{}_{\alpha} f^{\nu\alpha} - \frac{1}{4} \eta^{\mu\nu} f_{\alpha\beta} f^{\alpha\beta} \tag{5.2.12}$$

to be consistent with the Ricci tensor (5.1.9) it is necessary that $T^{11}{}_{e-m} = T^{22}{}_{e-m}$ which from (5.2.11, 12) gives

$$c^2 - d^2 = a^2 - b^2 \tag{5.2.13}$$

similarly $T^{12}{}_{e-m} = 0$ gives $bd = ac$ which together with (5.2.13) implies:

$$a = kb \qquad c = kd \qquad (k^2 = 1)$$

and the solutions of (5.2.9) consistent with (5.1.9) must be of the form:

$$B_1 = kE_2 = \alpha e^{k\tau} \qquad B_2 = -kE_1 = \beta e^{k\tau} \tag{5.2.14}$$

where α, β are constants and $k^2 = 1$. From (5.2.14) it follows that $E \cdot B = E^2 - B^2 = 0$.

The non-vanishing components of the stress energy tensor (5.2.12) are

$$\begin{aligned} T^{00}{}_{e-m} &= T^{33}{}_{e-m} = \gamma^2 e^{2k\tau} (AC)^{-2} \\ T^{03}{}_{e-m} &= -kT^{00}{}_{e-m} \end{aligned} \tag{5.2.15}$$

where $\gamma^2 = \alpha^2 + \beta^2$ and (5.2.15) is the canonical form for the stress energy tensor of the null electromagnetic field[32]. The connection with radiative solutions of Maxwell's equations can be made by noting that if f is null, $T^{\mu\nu}{}_{e-m}$ can be written[32]

$$T^{\mu\nu}{}_{e-m} = \rho\lambda^{\mu}\lambda^{\nu} \tag{5.2.16}$$

where λ^{μ} is the null eigenvector of $T^{\mu\nu}$. The zero divergence of (5.2.16) with $\lambda = \lambda_0\omega^0 + \lambda_3\omega^3$ yields (5.2.15) and $k = \pm 1$ corresponds to the choice $\lambda_3 = \pm \lambda_0$. The field configuration is that of a wave moving along the three axis $[(E \times B)_3 \neq 0]$ with polarization fixed by the constants α, β. The field equations depend only on $\gamma^2 = \alpha^2 + \beta^2$ which is invariant under rotations in the 1-2 plane, and the polarization may be independently specified.

5.3 - Solution of the Einstein-Maxwell Equations

The stress-energy tensor (5.2.12) is traceless and the Maxwell-Einstein equations (with zero cosmological constant) are:

$$R_{\mu\nu} = (T_{e-m})_{\mu\nu} \tag{5.3.1}$$

with T^{e-m} given by (5.2.15). Writing the Ricci tensor (5.1.9) in terms of τ and using (5.2.15) gives for (5.3.1):

$$R_{00} = \frac{-1}{C^2}\left[2\,\frac{A''}{A} \quad 2\,\frac{A'C'}{A\,C} + \frac{C''}{C} - \left(\frac{C'}{C}\right)^2\right] = \gamma^2\,\frac{e^{2k\tau}}{(AC)^2} \qquad \text{(a)}$$

$$R_{11} = R_{22} = \frac{1}{C^2}\left[\frac{A''}{A} + \left(\frac{A'}{A}\right)^2 - 2\right] = 0 \qquad \text{(b)}$$

$$R_{33} = \frac{1}{C^2}\left[\frac{C''}{C} - \left(\frac{C'}{C}\right) + 2\,\frac{A'C'}{A\,C} - 2\right] = \gamma^2\,\frac{e^{2k\tau}}{(AC)^2} \qquad \text{(c)}$$

$$R_{03} = \frac{-2}{C^2}\left(\frac{A'}{A} - \frac{C'}{C}\right) = k\gamma^2\,\frac{e^{2k\tau}}{(AC)^2} \qquad \text{(d)}$$

(5.3.2)

Equation (5.3.2b) can be rewritten as:

$$\left(\frac{A'}{A}\right)' = 2\left[1 - \left(\frac{A'}{A}\right)^2\right] \tag{5.3.3}$$

with solutions:

$$A = A_o(\text{Sinh } 2\tau)^{\frac{1}{2}} \qquad \left(\frac{A'}{A}\right)^2 > 1 \qquad \text{(a)}$$

$$A = A_o(\text{Cosh } 2\tau)^{\frac{1}{2}} \qquad \left(\frac{A'}{A}\right)^2 < 1 \qquad \text{(b)} \qquad (5.3.4)$$

$$A = A_o e^{k^*\tau} \qquad \frac{A'}{A} = k^*,\ k^{*2} = 1 \qquad \text{(c)}$$

In equations (5.3.4), A_0 is a constant of integration and the origin of has been fixed at zero. Since the mapping $\sigma^{1(2)\prime} = A_0 \sigma^{1(2)}$ is an automorphism of the Type V structure (5.1.1), A_0 may be chosen to be 1 without loss of generality. Similarly in view of (5.2.8) any constant multiplying C may be removed from the metric as a conformal factor although such constants will be retained throughout.

Using (5.3.2b, d) in (5.3.2a) gives the constraint equation

$$3\left[1 - \left(\frac{A'}{A}\right)^2\right] = \gamma^2 \frac{e^{2k\tau}}{A^2}\left(k\frac{A'}{A} - 1\right) \tag{5.3.5}$$

which on substitution of (5.3.4a, b, c) with $A_0 = 1$ gives:

	$k = +1$	$k = -1$
$\left\lvert\frac{A'}{A}\right\rvert > 1$	$\gamma^2 = -3$	$\gamma^2 = +3$
$\left\lvert\frac{A'}{A}\right\rvert < 1$	$\gamma^2 = -3$	$\gamma^2 = -3$
$\left\lvert\frac{A'}{A}\right\rvert = 1$	$k = k^*$	

(5.3.6)

and there are two solutions corresponding to (5.3.4a, c). Equation 5.3.2d) may now be integrated for C to give

$$C = C_0 e^{3\tau/2}(\operatorname{Sinh} 2\tau)^{-1/4} \qquad \left\lvert\frac{A'}{A}\right\rvert > 1 \tag{5.3.7a}$$

$$C = C_0 \exp\left[\left(1 + \frac{1}{2}\gamma^2\right)k\tau\right] \qquad \frac{A'}{A} = k,\ k^2 = 1 \tag{5.3.7b}$$

Substitution in equations (5.3.2) of solutions (5.3.4a, 5.3.7a) and (5.3.4c, 5.3.7b) verifies that all field equations are satisfied. For C given by (5.3.7a) the metric can be written :

$$g = C^2(\sigma^3 \otimes \sigma^3 - dt \otimes dt) + A^2(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) \tag{5.3.8}$$
$$A = (\mathrm{Sinh}\ 2\tau)^{\frac{1}{2}} \qquad C = C_o e^{3\tau/2} A^{-1/2}$$

and the time dependence of the field amplitudes is given by

$$b_1 = -e_2 = \alpha F_1$$

$$b_2 = e_1 = \beta F_1$$

where $F_1 = e^{-\tau}(AC)^{-1} = e^{-5\tau/2}(\mathrm{Sinh}\ 2\tau)^{-\frac{1}{4}}$. The singularity is of the line type with the transverse directions collapsing to zero accompanied by a longitudinal expansion to infinity. The field amplitudes are infinite at the singularity and approach zero as τ approaches infinity. For large τ, A and C go as e^{τ}.

For C given by (5.3.7b), the solutions defined by $k = \pm 1$ are simply time reversals of each other. Choosing $k = +1$, (5.2.8) is easily integrated, and the metric may be written:

$$g = -dt \otimes dt + (t)^{2/h}(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) + (ht)^2(\sigma^3 \otimes \sigma^3)$$
$$h = 1 + \frac{1}{2}\gamma^2 \tag{5.3.9}$$

with the time dependence of the field variables given by

$$b_1 = e_2 = \alpha F_2$$

$$b_2 = -e_1 = \beta F_2$$

where $F_2 = e^{\tau}(AC)^{-1} = (ht)^{-1}$. The singularity is a point type and the solution (5.3.9) approaches the empty Friedman model as γ^2 approaches zero.

Defining the longitudinal and transverse expansion rates as $H_L = C^{-1}C^{\cdot}$ and $H_T = A^{-1}A^{\cdot}$ respectively, then the fractional anisotropy in the expansion rate (independent of time coordinate) is given by

$$H_T^{-1}(H_L - H_T) = 3(1 - e^{4\tau})^{-1} \qquad \text{(solution 5.3.8)}$$

$$H_T^{-1}(H_L - H_T) = \frac{1}{2}\gamma^2 \qquad \text{(solution 5.3.9)} \qquad (5.3.10)$$

As (5.3.10) shows, the solution (5.3.8) tends to isotropic expansion for large τ and (5.3.9) maintains a constant anisotropy independent of τ.

5.4 - Solutions with Ultra-Relativistic Matter

The incorporation of matter into the solutions of Section 5.3 is complicated by the electric field components which must couple with ionized matter. The relative contributions of matter and electromagnetic energy to singularity structure are of formal interest however. Accordingly we consider an additional contribution to the stress energy tensor (5.2.16) describing an ultra-relativistic gas with equation of state $P = \rho$, not coupled to the electromagnetic field. The matter stress-energy is described by:

$$T^M_{\mu\nu} = (\rho + P)\delta^o_\mu \delta^o_\nu + P\eta_{\mu\nu} \tag{5.4.1}$$

whose zero divergence gives:

$$(\rho + P)^{-1} d\rho + d(\ln A^2 C) = 0 \tag{5.4.2}$$

with solution:

$$\rho = \theta^2 (A^2 C)^{-2} \tag{5.4.3}$$

where $\theta^2 (>0)$ is a constant. The non-zero components of T^M are:

$$T^M_{\mu\nu} = \theta^2 (A^2 C)^{-2} \delta_{\mu\nu} \tag{5.4.4}$$

with:

$$T = T^M_{\mu\nu} \eta^{\mu\nu} = \theta^2 (A^2 C)^{-2} \tag{5.4.5}$$

From the field equations :

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T\eta_{\mu\nu} \tag{5.4.6}$$

it follows that the solutions (5.3.4a,b,c) remain unchanged. The constraint equation (5.3.5) can now be written:

$$3[(\frac{A'}{A})^2 - 1)] = \gamma^2 \frac{e^{2k\tau}}{A^2}(1 - k\frac{A'}{A}) + \theta^2 A^{-4} \tag{5.4.7}$$

which on substitution of (5.3.4a,b,c), gives the results :

	$k = +1$	$k = -1$
$\lvert\frac{A'}{A}\rvert > 1$	$3 = \theta^2 + \gamma^2$	$3 = \theta^2 + \gamma^2$
$\lvert\frac{A'}{A}\rvert < 1$	$-3 = \theta^2 + \gamma^2$	$-3 = \theta^2 + \gamma^2$
$\lvert\frac{A'}{A}\rvert = 1$	$\theta^2 = 0(k=k^*)$	$2\gamma^2 + \theta^2 = 0(k=-k^*)$

(5.4.8)

Inspection of (5.4.8) shows that the case $\left|\frac{A'}{A}\right| \leq 1$ is now excluded if θ^2 is assumed positive definite. From (5.4.8) it follows that:

$$\gamma^2 = k(\theta^2 - 3) \qquad (k^2 = 1) \tag{5.4.9}$$

Equation (4.5) can be integrated for C to give:

$$C = C_o A^{(1 + k\gamma^2/2)} e^{\gamma^2\tau/2} \qquad (A = \text{Sinh}^{\frac{1}{2}} 2\tau) \tag{5.4.10}$$

where again, direct substitution verifies that all field equations are satisfied.

For k= +1 the solution (5.4.10) has a point singularity ($A, C \to 0$ as $\tau \to 0$) for all values of γ^2 and does not exist for $\theta^2 < 3(\gamma^2 < 0)$. Since θ^2 cannot vanish, the model is not connected with those of Section 5.3. For

$k = -1$, C is given by:

$$C = C_o A^{(1 - \gamma^2/2)} e^{\gamma^2 \tau/2} \qquad (5.4.11)$$

In this case for $0 \leq \gamma^2 < 2$ $(1 < \theta^2 \leq 3)$ the singularity is a point, for $2 < \gamma^2 \leq 3$ $(0 \leq \theta^2 < 1)$ C is infinite at $\tau = 0$ ($A = 0$). At $\gamma^2 = 2$ $(\theta^2 = 1)$, $C = C_o e^{\tau}$ is finite at $\tau = 0$. For $k = -1$, setting $\theta^2 = 0$ recovers the solutions (5.3.8).

All solutions contained in (5.4.10) approach the isotropic $P = \rho$ Friedman models:

$$A = (\mathrm{Sinh}\ 2\tau)^{\frac{1}{2}}$$

$$\theta^2 = 3$$

$$dt = A d\tau$$

as γ^2 approaches zero.

From (4.5.10) the fractional anisotropy in the expansion rate is:

$$H_T^{-1}(H_L - H_T) = k\gamma^2(1 + e^{-4k\tau})^{-1}$$

which approaches zero for large τ independent of γ^2 if $k = -1$ and approaches $+\gamma^2$ for $k = +1$.

5.5 - Symmetry and Singularity Structure of the Models

The Einstein-Maxwell equations have been integrated for an LRS Bianchi Type V spacetime containing a source free, null electromagnetic field and an added (non-interacting) gas with the equation of state $P = \rho$. The most striking feature of the models is the existence of a preferred direction in the symmetry plane of the geometry. There is, however, no contradiction in this result. Associating the symmetries of the geometry of spacetime with the symmetries of the physics of spacetime is a concept inherent to the geometrodynamic[33] view of relativity. For the null electromagnetic field, the geometrodynamic view fails in that the field tensor $f_{\alpha\beta}$ cannot be uniquely constructed[32] (to within a constant duality rotation) from $T_{\alpha\beta}$. Thus in the null case, the geometry of $T_{\alpha\beta}$ is not necessarily communicated to the physics of $f_{\alpha\beta}$.

The singularity behavior of the models is also interesting. For the solution (5.11), which includes the models (4.8) without gas, the singularity is characterized by transverse collapse and longitudinal expansion to infinity as long as $2\theta^2 < \gamma^2$, and for mixtures such that $2\theta^2 > \gamma^2$ the singularity is always a point. This is a somewhat surprising result in that it would be expected that the singularity structure would depend on the relative contributions of the matter and electromagnetic mass densities given by:

$$\rho_{e-m} = e^2 + b^2 = 2\gamma^2 \frac{e^{2k\tau}}{(AC)^2} \quad \text{and} \quad \rho_M = \theta^2 (A^2 C)^{-2} \tag{5.5.1}$$

whose ratio $\frac{\rho_{e-m}}{\rho_M} = 2\frac{\gamma^2}{\theta^2}A^2 e^{2k\tau}$ approaches zero as $\tau \rightarrow 0$. In investigating a group of cosmological models with magnetic fields, Thorne[7] reported that (for his models) when the ratio $\frac{\rho_{e-m}}{\rho_M}$ approached zero as the singularity was approached, the magnetic field had a negligible effect on the structure and dynamics of the singularity. In the present case the ratio $\frac{\rho_{e-m}}{\rho_M}$ approaches zero as $\tau \rightarrow 0$ yet the structure of the singularity depends on the parameter $\frac{\gamma^2}{\theta^2}$.

6. - Some Perturbed Cosmological Models

In investigating general classes of "tilted" homogeneous cosmological models, King and Ellis[28] have suggested that some simple models of this type might be found among the Bianchi Type III spacetimes. The "tilted" homogeneous models are those in which the matter has a non-zero spatial velocity component in the frame in which the Universe is homogeneous, thus "tilting" the co-moving frame with respect to the hypersurfaces of homogeneity. This suggestion is pursued in this chapter with the object of generating Type III spacetimes with non-zero vorticity. For the metric forms considered, it was found that the field equations were amenable to perturbation techniques, valid for small values of "tilt". Such methods have inherent limitations in that the ultimate effects of vorticity on singularity structure are not generally available, but much information about the structure of spacetime at later epochs can be gained.

Three solutions will be discussed in 6.3,4 , all of which represent perturbations on a space of Bianchi Type III geometry. In Section 6.1 the unperturbed metric is discussed. It is the open model (case 2) of Kantowski and Sachs[20]. The geometry is spatially homogeneous, locally rotationally symmetric (LRS) and shearing, with vanishing vorticity. In 6.2 the metric is modified to admit the possibility of vorticity as well as additional shear components arising from a motion of the matter not associated with rotation. The metric is analyzed and its possible solutions classified. In 6.3, a solution is found which, to lowest order, preserves the diagonal field equations, and for which the vorticity does not vanish. A totally anisotropic solution is presented in 6.4 which can be either rotating, or "tilted" and non-rotating. The models are discussed in 6.5 and the domain of validity of the perturbation assumptions is analyzed.

6.1 - The Unperturbed Metric - A Bianchi Type III Spacetime

It has been shown that spatially homogeneous world models can be viewed as a family S of homogeneous spacelike hypersurfaces parametized by time. If elements of S are the hypersurfaces of transitivity of a three-parameter isometry group G_3, the generators of G_3 can be used to establish an invariant basis on elements of S. If the basis one-forms σ^i are invariant under G_3, the σ^i can be chosen such that:

$$d\sigma^i = \frac{1}{2} C^i{}_{jk}\sigma^j \wedge \sigma^k \tag{6.1.1}$$

where the C^i_{jk} are the constants of structure of G_3. For Bianchi Type III spacetimes, the structure (6.1.1) is:

$$d\sigma^1 = \sigma^1 \wedge \sigma^2 \qquad d\sigma^2 = d\sigma^3 = 0 \tag{6.1.2}$$

and a metric form admitting an additional motion associated with rotational symmetry is:

$$g = -dt \otimes dt + A^2(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) + C^2\sigma^3 \otimes \sigma^3 \tag{6.1.3}$$

where the functions A and C depend only on t, and the σ^i have the structure (6.1.2). The metric (6.1.3) can be written:

$$g = \eta_{\alpha\beta}\,\omega^\alpha \otimes \omega^\beta \tag{6.1.4}$$

where $\eta_{\alpha\beta}$ is Minkowskian:

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1) \tag{6.1.5}$$

and the Cartan basis ω^α is given by:

$$\omega^0 = dt \quad \omega^1 = A\sigma^1 \quad \omega^2 = A\sigma^2 \quad \omega^3 = C\sigma^3 \tag{6.1.6}$$

The non-vanishing components of the Einstein tensor $G_{\mu\nu}$, where:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^\alpha{}_\alpha$$

are:

$$G_{00} = 2\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\frac{\dot{C}}{C} - \frac{1}{A^2} \qquad \text{(a)}$$

$$G_{11} = G_{22} = -\left(\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C}\right) \qquad \text{(b)} \qquad (6.1.7)$$

$$G_{33} = -\left(2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 - \frac{1}{A^2}\right) \qquad \text{(c)}$$

where the dot indicates differentiation with respect to time. These equations have been integrated for a pressureless perfect fluid (dust) by Kantowski and Sachs[20], for various equations of state by Kantowski[35], and for dust and an axial magnetic field by Thorne[34]. In all cases the models are open, with the world lines of matter orthogonal to the hypersurfaces of homogeneity. In the present case it will be useful to integrate the field equations directly for co-moving dust. With the cosmological constant taken to be zero, the field equations are just:

$$G_{\alpha\beta} = T_{\alpha\beta} \tag{6.1.8}$$

where $G_{\alpha\beta}$ is given by (6.1.7) and $T_{\alpha\beta}$ has only one non-vanishing component:

$$T_{00} = \rho = M(A^2C)^{-1} \tag{6.1.9}$$

where M is a constant of integration. The equation $G_{33} = 0$ can be written (with A as the dependent variable) as:

$$\frac{d}{dA}(A\dot{A}^2) = 1$$

Integration yields:

$$\dot{A}^2 = 1 + k_1A^{-1}$$

and:

$$t - t_o = xf - k_1\ln(x+f) \tag{6.1.10}$$

where k_1 and t_o are constants of integration and:

$$\begin{aligned} x^2 &= A \\ f &= A^{\frac{1}{2}}\dot{A} = (x^2 + k_1)^{\frac{1}{2}} \end{aligned} \tag{6.1.11}$$

The G_{00} or G_{11} equations can be integrated to give:

$$C = x^{-1}f[k_3 + \ln(x + f)] - 1 \tag{6.1.12}$$

where k_3 is a constant of integration, and C has been scaled so that the constant M, defined in (6.1.9), has been chosen as 1. The general solution is characterized by zero vorticity, volume expansion:

$$\theta = 2H_T + H_L$$

and shear:

$$(\sigma^*)^2 = \frac{1}{3}(H_T - H_L)^2$$

where $H_T = (\ln A)^{\cdot}$ and $H_L = (\ln C)^{\cdot}$ are the transverse and longitudinal expansion rates respectively. The notation (σ^*) will be used to differentiate the shear from elements of the invariant basis. The metric (6.1.3) may be written:

$$g = x^4[-4f^{-2}dx \otimes dx + \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2] + C^2(x)\sigma^3 \otimes \sigma^3 \quad (6.1.13)$$

where f(x) and C(x) are given by (6.1.11) and (6.1.12). As (6.1.10,11) indicate, if $k_1 = 0$ then A becomes the time. Similarly for any solution in which A grows without bound $A^{\cdot} \to 1$ as $A \to \infty$. For $k_1=0$, from (6.1.12), C can be written:

$$C = \frac{1}{2}\ln\left(\frac{A}{A^*}\right) \quad (6.1.14)$$

where:

$$t = A - A^* \qquad \text{and} \qquad A^* = \frac{1}{4}\exp 2(1 - k_3) \quad (6.1.15)$$

The metric is:

$$g = -dA \otimes dA + A^2(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) + \frac{1}{4}\left(\ln \frac{A}{A^*}\right)^2 \sigma^3 \otimes \sigma^3 \quad (6.1.16)$$

where the solution (6.1.16) is exact if $k_1 = 0$, and the asymptotic limit of all solutions for which $A \to \infty$. Together with its evident simplicity, it is an ideal starting point for a perturbation calculation.

The solution (6.1.16) is new and does not occur in references 3, 7 or 8. The solutions presented in these references are given parametrically and (6.1.16) is outside the range of models permitted by the parametrization.

6.2 - A "Tilted" Type III Spacetime

In order to generalize the metric of 6.1 and introduce the possibility of vorticity, the metric form (6.1.4) was considered where the orthonormal Cartan basis ω^α is given by:

$$\omega^0 = dt \qquad \omega^1 = A(\sigma^1 + h\sigma^2)$$
$$\omega^2 = B\sigma^2 \qquad \omega^3 = C\sigma^3 \tag{6.2.1}$$

where, again, the σ^i obey (6.1.2), and the functions A, B, C, and h depend only on t. The non-zero components of the Einstein tensor are now:

$$G_{00} = \frac{A^\cdot}{A}\frac{B^\cdot}{B} + \frac{A^\cdot}{A}\frac{C^\cdot}{C} + \frac{B^\cdot}{B}\frac{C^\cdot}{C} - \frac{1}{B^2} - \frac{1}{4}Q^2$$

$$G_{11} = -\left(\frac{B^{\cdot\cdot}}{B} + \frac{C^{\cdot\cdot}}{C} + \frac{B^\cdot}{B}\frac{C^\cdot}{C} - \frac{3}{4}Q^2\right)$$

$$G_{22} = -\left(\frac{A^{\cdot\cdot}}{A} + \frac{C^{\cdot\cdot}}{C} + \frac{A^\cdot}{A}\frac{C^\cdot}{C} + \frac{1}{4}Q^2\right)$$

$$G_{33} = -\left(\frac{A^{\cdot\cdot}}{A} + \frac{B^{\cdot\cdot}}{B} + \frac{A^\cdot}{A}\frac{B^\cdot}{B} + \frac{1}{4}Q^2 - \frac{1}{B^2}\right)$$

$$G_{01} = -\frac{Q}{B} \qquad G_{02} = \frac{1}{B}\left(\frac{A^\cdot}{A} - \frac{B^\cdot}{B}\right) \qquad G_{12} = \frac{1}{2}Q(\ln QA^2C)^\cdot \tag{6.2.2}$$

where $Q = h^\cdot \frac{A}{B}$.

For the stress-energy tensor:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta \tag{6.2.3}$$

the equations (6.2.2) indicate the four-velocity u should be of the form:

$$u = u_o \omega^o + u_1 \omega^1 + u_2 \omega^2 \tag{6.2.4}$$

The zero divergence of $T_{\alpha\beta}$ gives:

$$\begin{aligned} &(\ln \rho u^o ABC)^\bullet = (\ln u^1 A)^\bullet = u^2 (u^o B)^{-1} \\ &(\ln u^2 B)^\bullet = -\frac{(u^1)^2}{u^2 u^o B} + Q\,\frac{u^1}{u^2} \\ &u^o (u^o)^\bullet + \frac{A^\bullet}{A}(u^1)^2 + \frac{B^\bullet}{B}(u^2)^2 + Q u^1 u^2 = 0 \end{aligned} \tag{6.2.5}$$

For $T_{\alpha\beta}$ given by (6.2.3), the possible models associated with the structure (6.2.1) can be classified in the following way:

Case 1. h =constant. This is equivalent to h= 0 since the mapping:

$$(\sigma^1)^\bullet = \sigma^1 + h\sigma^2 \qquad (\sigma^2)^\bullet = \sigma^2 \quad (\sigma^3)^\bullet = \sigma^3$$

restores (6.2.1) to its form for h =0, and is an automorphism of the Type III structure (6.1.2). Alternatively, the field equations (6.2.2) depend only on derivatives of h. An examination of (6.2.1, 2) with h =0 shows that two situations are possible: (a) the LRS case with A = B discussed in 6.1 and; (b) the totally anisotropic, "tilted" case $A \neq B \neq C$ with $u = u_0\, \omega^0 + u_2\, \omega^2$ and $(u^2 B)_{,0} = 0$. This model is non-rotating, and the field equations have never been integrated.

Case 2. $h = h(t)$. For this case, a solution requires that both u^1 and u^2 be different from zero. This follows from (6.2.2) in that if $u^2 = 0$, $G_{02} = 0$ requires $A = B$, and $G_{11} - G_{22}$ yields a negative value for the square of u^1. Conversely, if $u^1 = 0$, $G_{01} = 0$ requires h = constant, and we return to the solutions of case 1. For the four velocity (6.2.4) there is an associated vorticity vector given by:

$$\Omega = \frac{1}{2} *(u \wedge du) = \frac{1}{2} u_1 (Bu_o)^{-1} \omega^3 \qquad (6.2.6)$$

where the equations of motion (6.2.5) have been used to eliminate time derivatives. The vorticity is manifestly orthogonal to u, as expected, and is non-zero whenever u_1 does not vanish. Although it does not depend explicitly on Q(h), the vorticity vanishes for $Q = 0$ since this implies $G_{01} = 0$, and consequently $u_1 = 0$. Any solution of case 2 contains a solution of case 1b.

6.3 - A Solution which Preserves the Diagonal Field Equations

The work of (6.2) suggests that a perturbed rotating solution might be obtained by the introduction of a small component of velocity u^1. This follows from the vorticity (6.2.6) which depends only on u^1, and from the field equations (6.2.2, 3) in which all terms preventing the vanishing of u^2 are quadratic in Q. We assume:

$$u^o = 1, \qquad u^1 = \lambda \underline{u}^1 \qquad \text{and} \quad h = \lambda \underline{h} \tag{6.3.1}$$

where $\lambda << 1$. The component u^2 is assumed of order λ^2 or smaller. To lowest order in λ, the equations (6.2.5) give:

$$\underline{u}^1 A = \alpha \qquad \text{and} \quad \rho ABC = 1 \tag{6.3.2}$$

where α is a constant. The G_{01} or G_{12} equations give:

$$\underline{h}^{\bullet} = \alpha\rho(\frac{B}{A})^2 \tag{6.3.3}$$

and the G_{02} equation requires A = B, and to lowest order in λ the diagonal field equations are those of Section II with solutions (6.1.12, 13) or (6.1.16). In terms of (6.1.16), $\underline{h}$ can be evaluated as:

$$\underline{h} = \alpha \int \rho dA = -2\alpha(A*)^{-1}Ei(2C) \tag{6.3.4}$$

where, again, $2C = \ln A/A^*$ and Ei(x) is the exponential integral:

$$Ei(x) = \int_x^{\infty} (xe^x)^{-1}dx$$

Thus $\underline{h}$ vanishes when u^1 vanishes ($\alpha = 0$) and approaches zero as $A \to \infty$.

The vorticity vector (6.2.6) is:

$$\Omega = \frac{1}{2} \lambda \alpha A^{-2} \omega^3 \tag{6.3.5}$$

which is the value expected from conservation of angular momentum, or vorticity propagation equations[53]. The model resembles a test field solution in that the metric functions A and C remain unchanged. The metric, however, describes a spacetime in which every co-moving observer sees the matter in his neighborhood rotating with respect to a Fermi-Walker propagated (gyroscopic) basis. The metric can be written:

$$g = g_o + \lambda g_1$$

where g_o is given by (6.1.16) and:

$$g_1 = -2A^2\alpha(A*)^{-1}\mathrm{Ei}(\ln \frac{A}{A*})(\sigma^1 \otimes \sigma^2 + \sigma^2 \otimes \sigma^1)$$

6.4 - A Totally Anisotropic Solution

Since a complete solution of case 2 requires two spatial velocity components, a more general set of perturbation assumptions is:

$$u^o = 1 \qquad u^1 = \lambda\tilde{u}^1 \qquad u^2 = \lambda\tilde{u}^2 \qquad h = \lambda\tilde{h} \tag{6.4.1}$$

where, again, $0 < \lambda << 1$. Under the assumptions (6.4.1), $A = C$ can no longer be maintained if $u^2 \neq 0$ requiring:

$$A = A_o + \lambda A_1 = A_o(1 + \lambda a_1) \qquad C = C_o + \lambda C_1 = C_o(1 + \lambda c_1)$$
$$B = A_o + \lambda B_1 = A_o(1 + \lambda b_1) \qquad \rho = \rho_o + \lambda\rho_1 = \rho_o(1 + \lambda\tilde{\rho}_1) \tag{6.4.2}$$

where the subscripts $(_o)$ refer to the unperturbed models of Section II. Using (6.4.1,2) in the Einstein tensor (6.2.2) and the stress-energy tensor (6.2.3) gives for $G_{\alpha\beta} = T_{\alpha\beta}$:

$$(\ln A_o)^\bullet(a_1 + b_1 + 2c_1)^\bullet + (\ln C_o)^\bullet(a_1 + b_1)^\bullet + 2b_1A_o^{-2} = \rho_o\tilde{\rho}_1$$
$$C_o(b_1^\bullet A_o^2C_o)^\bullet + A_o(c_1^\bullet C_o^2A_o)^\bullet = 0$$
$$C_o(a_1^\bullet A_o^2C_o)^\bullet + A_o(c_1^\bullet C_o^2A_o)^\bullet = 0$$
$$[(a_1 + b_1)^\bullet A_o^3]^\bullet + 2b_1A_o = 0 \tag{6.4.3}$$
$$\tilde{h}^\bullet = \rho_o\tilde{u}^1A_o \qquad (\tilde{h}^\bullet A_o^2C_o)^\bullet = 0 \qquad (b_1 - a_1)^\bullet = \rho_o\tilde{u}^2A_o$$

where the unperturbed field equations (6.1.7, 8, 9) have been used to eliminate zero order terms. Using (6.4.1, 2) in equations (6.2.5) gives:

$$(\tilde{\rho}_1 + a_1 + b_1 + c_1)^{\bullet} = \beta A_o^{-2}$$

$$\tilde{u}^1 A_o = \alpha \qquad \tilde{u}^2 A_o = \beta \tag{6.4.4}$$

The equations (6.4.3, 4) show that to first order in λ , the rotating, and tilted non-rotating models (case 2 and case 1b respectively) decouple, with h completely determined by the off-diagonal equations.

Using for A_o and C_o the solution (6.1, 16), the system (6.4.3, 4) can be integrated to give:

$$b_1 = \beta A_o^{-1}[(1 + x)\ln x - x]$$

$$a_1 = b_1 + 2\beta A*^{-1}Ei(x)$$

$$\begin{aligned} c_1 = -b_1(2 + \tfrac{1}{x}) + \beta(A*xe^x)^{-1}[(2x^2+5x+5)\ln x - 2x^2-2x+4] \\ - \beta A*^{-1}(2 - \tfrac{4}{x})Ei(x) \end{aligned} \tag{6.4.5}$$

$$\tilde{\rho}_1 = -2\beta(A*xe^x)^{-1}[(x^2+2x+2)\ln x + 2-x^2] - 4\beta(xA*)^{-1}Ei(x)$$

$$\tilde{h} = 2\alpha A*^{-1}Ei(x)$$

where $x = 2C_o = \ln A/A^*$ and all constants of integration have been set equal to zero to insure the perturbation vanishes whenever u^1, u^2 vanish. The rotation vector is that of(6.3.5) and the shear scalar is:

$$(\sigma^*)^2 = (\sigma^*)_o{}^2 + \frac{1}{3}\lambda(H_T - H_L)_o(a_1 + b_1 + 2c_1 + \tilde{u}^2)^{\bullet}$$

where:

$$(a_1 + b_1 + 2c_1 + \tilde{u}^2)^{\bullet} = \rho_o[A\tilde{\rho}_1 - \beta \ln x + b_1 A_o(1-x) - \frac{1}{2}\beta x]$$

The volume expansion is given by:

$$\theta = \theta_o + \lambda\theta_1 = \theta_o + \lambda(a_1 + b_1 + c_1 + \tilde{u}^2)^{\bullet} \qquad (6.3.6)$$

where $(\sigma^*)_o$ and θ_o are the unperturbed quantities given in (6.1) and:

$$(a_1 + b_1 + c_1 + \tilde{u}^2)^{\bullet} = -\beta\rho_o[(x^2 + x + 2 + \frac{2}{x})\ln x - x^2 + \frac{2}{x} + \frac{2A_o}{A^*x} Ei(x)]$$

Using (6.4.1,2) and (6.2.1) the metric may be written:

$$g = g_o + \lambda g_1 \qquad (6.3.7)$$

where g_o is given by (6.1.3) and g_1 is:

$$g_1 = A_o{}^2[a_1\sigma^1 \otimes \sigma^1 + b_1\sigma^2 \otimes \sigma^2 + 2\tilde{h}\sigma^{(1} \otimes \sigma^{2)}] + c_1 C_o{}^2\sigma^3 \otimes \sigma^3 \qquad (6.3.8)$$

6.5 - Model Evaluation

The unperturbed models of (6.1) were not compared in detail to observations in references 20, 34 and 35 since observational data at the time precluded open world models, and the high early anisotropy exhibited by the solutions seemed inconsistent with micro-wave background observations. For later times, however, as pointed out in reference 20, the solutions are not unlike the Friedman models, and consequently bear some analysis. Using the solutions (6.1.16) and taking:

$$\gamma = \frac{H_L}{H_T} = \left(\ln \frac{A_o}{A^*}\right)^{-1} \tag{6.5.1}$$

where, again, $H_L = (\ln C_o)_{,0}$ and $H_T = (\ln A_o)_{,0} = A_o^{-1}$, and defining:

$$H_o = \frac{1}{3}\,(2H_T + H_L) = \frac{1}{3}\,\frac{(2+\gamma)}{A_o} \tag{6.5.2}$$

gives a model with the properties:

$$\begin{aligned}
&\text{Age:} \quad t = A_o - A^* = \tfrac{1}{2}\, t_c(1 - e^{-1/\gamma})(2+\gamma) \\
&\text{Density:} \quad \rho = (A_o^2 C)^{-1} = 6\gamma(2+\gamma)^{-2}\rho_c \\
&\text{Shear:} \quad (\sigma^*)^2 = 3H_o{}^2(1-\gamma^2)(2+\gamma)^{-2}
\end{aligned} \tag{6.5.3}$$

where ρ_c and t_c are the density and age respectively of an Einstein-DeSitter model with expansion rate H_o. As (6.5.1) indicates as the models evolve from $A = A^*$ (singularity) to A approaching infinity, γ evolves from an infinite value to zero. At the shear-free epoch ($\gamma = 1$) the models are slightly younger ($t = .95t_c$) and slightly less dense ($\rho = 2/3\,\rho_c$) than the corresponding Einstein-DeSitter model. An interesting feature of the shear-free epoch is that the deceleration parameters:

$$(q_o)_T = A(A^{\cdot -1})^{\cdot} = 0 \quad \text{and} \quad (q_o)_L = C(C^{\cdot -1})^{\cdot} = 1$$

are consistent with both open and closed models depending on the direction of observation.

The temperature anisotropy in the microwave background may be calculated by assuming either:

a) The surface of last scattering is coincident with the surface of recombination (deep origin) in which case:

$$\frac{T_a}{T_e} = (\frac{\rho_e}{\rho_a})^{1/3} = 10^3 \tag{6.5.4}$$

where T is the temperature of the microwave background and the subscripts a and e refer to photon absorption and emission times respectively. Or

b) The surface of last scattering has been displaced relative to the surface of recombination by an intervening ionized inter-galactic medium (local origin)[36] in which case:

$$\tau = \int_{t_e}^{t_a} \sigma_T n_e dt \sim 1 \tag{6.5.5}$$

where τ is the optical depth, σ_T is the Thompson scattering cross-section and n_e is the density of free electrons. Equation (6.5.5) assumes that the thermal energy of both the microwave photons and scattering electrons is small compared to the electron rest mass.

Under assumption (a), using (6.5.1), (6.5.4) becomes:

$$\left(\frac{A_e}{A^*}\right)^2 \ln \frac{A_e}{A^*} = \frac{e^{2/\gamma}}{\gamma} \times 10^9$$

which for γ of order of magnitude 1 gives:

$$\frac{T_T}{T_L} = \left(\frac{A_e}{A_a}\right)\left(\frac{C_a}{C_e}\right) = e^{-3/\gamma} \times 10^9 \qquad (6.5.6)$$

Under assumption (b), taking $\tau = 1$ and assuming the entire mass content of the Universe is in the form of ionized hydrogen gives:

$$1 = (4.26 \times 10^{26} \text{ cm}) A^{*-1}[Ei(2C_e) - Ei(2C_a)] \qquad (6.5.7)$$

From (6.5.1,2):

$$A^* = 3e^{-1/\gamma}(2 + \gamma)h_o^{-1} \times 10^{28} \text{ cm} \qquad (6.5.8)$$

where h_o is the Hubble constant in units of 10 km sec^{-1} Mpc^{-1}. Using (6.5.8) in (6.5.7) gives:

$$Ei(2C_e) = Ei(\gamma^{-1}) + 70.4\ h_o^{-1}e^{-1/\gamma}(2 + \gamma) \qquad (6.5.9)$$

Again, assuming γ to be of order 1, and $1 < h_o < 10$, the second term on the right in (6.5.9) is much larger than the first, and the exponential integral can be expanded to give:

$$2C_e = e^{-k} \qquad\qquad k = 70.4 h_o^{-1} e^{-1/\gamma}(2+\gamma)$$

and:

$$\frac{T_T}{T_L} = (\gamma e^{1/\gamma})^{-1}(e^k + 1) \tag{6.5.10}$$

Inspection of (6.5.6, 10) shows that observational constraints on the magnitude and angular variation of the Hubble constant make it impossible to satisfy limits on the anisotropy in the microwave background. Although the particular solution (6.1.16) was used to obtain these results, they accurately reflect difficulties encountered in fitting the more general models of Section II.

Turning to the perturbed models (6.4.7,8), Fig. 6.1 is a plot of the first order quantities a_1, b_1, c_1 and $\tilde{\rho}_1$ multiplied by the constant A^*/β. The perturbation assumptions (6.4.1,2) require these quantities, together with h, u_1 and u_2, to be, in absolute value, much less than λ^{-1}. As Fig. 6.1 indicates, although the quantities a_1 and b_1 are not monotonic in absolute value, they are bounded from above and below by $\tilde{\rho}_1$ and c_1 which tend monotonically to zero. Thus, if the perturbation assumptions are satisfied at some t_o, they are satisfied for all $t > t_o$. At the epoch $\gamma = 1$:

$$\left|\lambda c_1\right| \ll 1 \rightarrow \left|u_2\right| \ll .24 \qquad\qquad \left|\lambda \tilde{\rho}_1\right| \ll 1 \rightarrow \left|u_2\right| \ll .23 \tag{6.5.11}$$

Similarly:

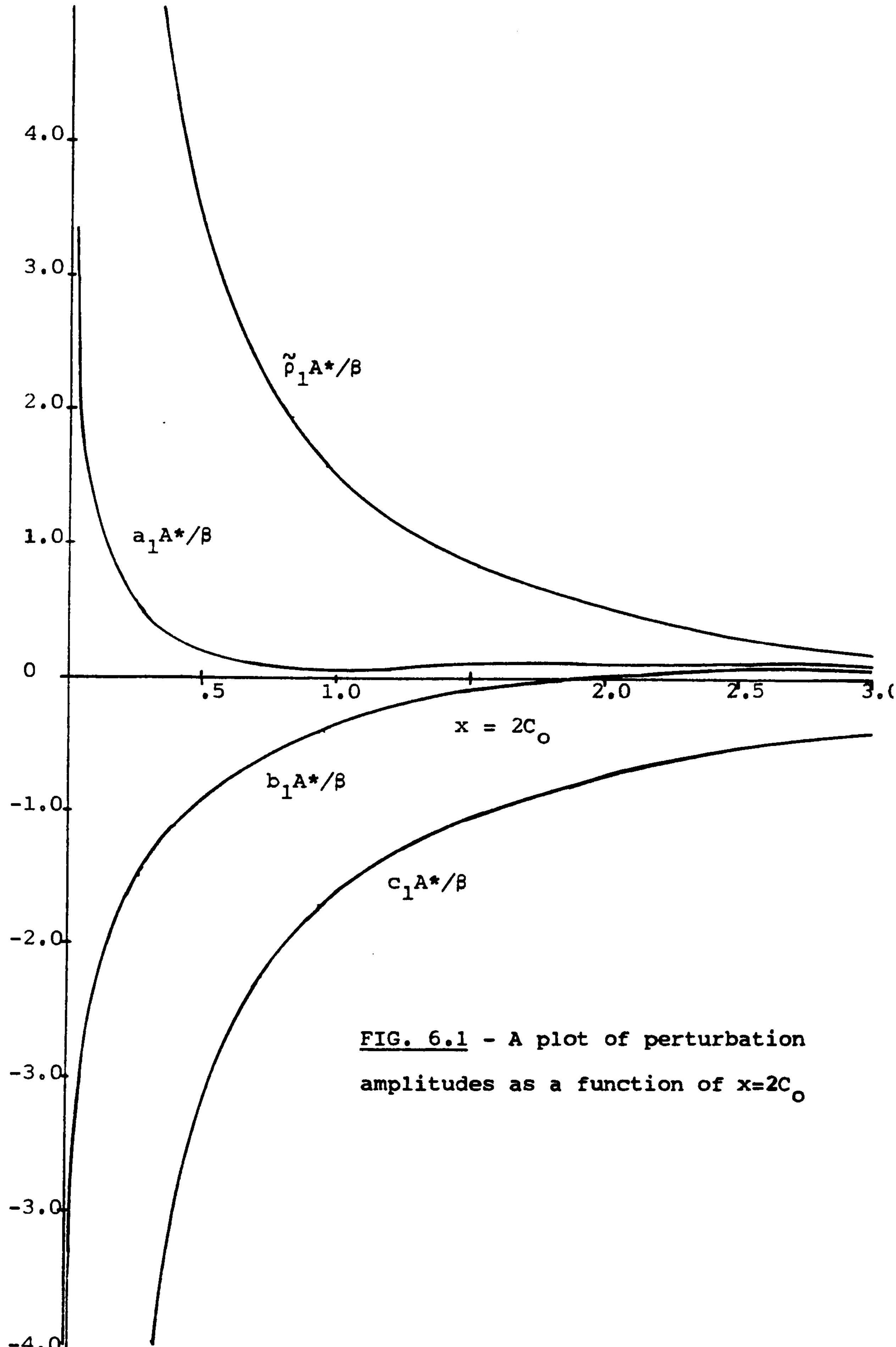

FIG. 6.1 - A plot of perturbation amplitudes as a function of $x=2C_o$

$$|h| \ll 1 \rightarrow |u^1| \ll .84 \tag{6.5.12}$$

which bounds the magnitude of the vorticity vector at $\gamma=1$ by:

$$|\Omega| \ll .42\ H_o$$

where $H_o = A_o^{-1}$.

A surprising feature of the curves in Fig. 6.1 is the relative magnitudes of the perturbation amplitudes. For $u_1 = 0$, the perturbation introduces a small matter velocity component along the σ_2 axis, yet a_1 and b_1 the resulting changes in the 1-2 plane (which was originally isotropic) are smaller than c_1 the perturbed amplitude in the σ_3 direction. That this is an observable effect may be verified by going to a co-moving orthonormal basis π^α with elements:

$$\begin{aligned} \pi^o &= u = -\omega^o + u_2\omega^2 \\ \pi^1 &= \omega^1 \\ \pi^2 &= \omega^2 - u_2\omega^o \\ \pi^3 &= \omega^3 \end{aligned} \tag{6.5.13}$$

In terms of the π^α, the shear tensor is diagonal with only three non-vanishing components. At $\gamma=1$:

$$\begin{aligned} \sigma^* = {} & \lambda(a_1{}^\bullet + u_2{}^\bullet - \tfrac{1}{3}\,\theta_1)\ \pi^1 \otimes \pi^1 + \lambda(b_1{}^\bullet - \tfrac{1}{3}\,\theta_1)\ \pi^2 \otimes \pi^2 \\ & + \lambda(c_1{}^\bullet - \tfrac{1}{3}\,\theta_1)\ \pi^3 \otimes \pi^3 \end{aligned} \tag{6.5.14}$$

where θ_1 is given by (5.6). Defining H_i as the expansion rate in the π_i direction, then[53]:

$$H_i = (\sigma^*)_{ii} + \frac{1}{3}(\theta_o + \lambda\theta_1) \qquad \text{(no sum)} \tag{6.5.15}$$

The fractional difference between the expansion rates in the i and j directions is then:

$$\alpha_{ij} = 3(H_i - H_j)(\theta)^{-1} = 3(\sigma^*_{ii} - \sigma^*_{jj})(\theta_o + \lambda\theta_1)^{-1} \tag{6.5.16}$$

From (6.4.5, 6) and (6.5.14), the α_{ij} can be evaluated at $\gamma=1$:

$$\alpha_{21} = 3u_2 \qquad \alpha_{23} = 7.38u_2 \qquad \alpha_{13} = 4.38u_2 \tag{6.5.17}$$

The α_{ij} can be measured as anisotropies in the Hubble flow and (6.5.17) indicates that the observable effects of the perturbation are larger outside the plane of initial symmetry than in the plane. Observed values of the α_{ij} are uncertain due to effects of the Local Supercluster[37,38,39] but assuming $\alpha_{ij} < .1$ the upper bound of u_2 as determined from (6.17) is about 20 times smaller than that determined from (6.11).

This work has illustrated some of the difficulties in solving and evaluating anisotropic cosmological models. We have demonstrated the existence of stable perturbations of the models of Sections II but not disproved the existence of unstable modes since the solutions (6.4.5, 8) were obtained by choosing all constants of integration as zero.

7. - Discussion

The results presented in Chapters 4 - 6 fall into two categories:

(a) The relationship of geometrical to physical symmetries (Chapters 4, 5)

(b) The inconsistency of typical anisotropic models with current observational data.

Although these areas are broadly related as efforts to understand cosmology beyond the standard model, they are most easily discussed separately.

7.1 - Geometrical and Physical Symmetries

It was shown in (4.5) that rotational Killing transformations need not preserve the homogeneity of physical fields, and in Chapter 5 that the null field need not be rotationally symmetric if the geometry is. The problem of invariance of the electromagnetic field under geometrical symmetries has been investigated for the null case[40,41] and the non-null case[42,43] where the stress-energy tensor is purely electromagnetic. The results of these investigations can be summarized[44] as:

$$L_X F = k \, {*F} \tag{7.1.1}$$

where $L_X F$ is the Lie derivative of the electromagnetic field tensor F with respect to any Killing vector field X, and k is given by:

$$k = (d\alpha,\ X) \qquad \text{if F is non-null}$$

and dk is parallel to λ if F is null. Where λ is the propagation vector of the null field ($T^{e-m}_{\alpha\beta} = \rho\lambda_\alpha\lambda_\beta$), and α is the complexion[32,33] of the non-null field. For the non-null case, if F is assumed homogeneous:

$$L_T F = 0 = (da,\ T) \ {*F} \tag{7.1.2}$$

implies $(d\alpha, T) = 0$ where T is the generator of any translation. Since the translational Killing vectors span the hypersurfaces t = constant, (7.1.2) requires $d\alpha$ to be of the form:

$$d\alpha = \alpha_{,0}\,\omega^0$$

and it follows that $(d\alpha\ , R) = 0$ if R is the generator of any isometry such that $(R, \omega^{o}) = 0$. Thus for the non-null field if F is assumed homogeneous, it must also be rotationally symmetric and solutions like that of Chapter 5 can only occur for the null field[i]. However, since the complexion is completely determined by the Ricci tensor, it may be inconsistent in the non-null case to assume F is homogeneous, even in a homogeneous spacetime[45].

Further work in this area is needed to determine the exact degree to which isometries constrain physical fields, and in particular the degree to which homogeneous spacetimes require homogeneous physics.

7.2 - Anisotropic Spacetimes

Current interest in anisotropic spacetimes stems from several considerations:

(a) The only observational data consistent with the assumption of strict isotropy is the lack of angular variation in the microwave background[54].

(b) Strict isotropy requires very special initial conditions for which no cause can, at present, be discerned[46].

(c) In a truly isotropic spacetime, inhomogeneities such as galaxies are difficult to understand.

(d) Particle and event horizons[47] characteristic of Friedman cosmologies seem to contradict the maintenance of isotropy[55].

However, as the work in Chapter 6 illustrates, although anisotropic models can be made consistent with "local" observations such as the magnitude and angular variation of the expansion rate, they fail, in general, to satisfy constraints on the anisotropy of the microwave background. Even where models satisfy all observations[34], the required initial conditions are generally as special as those leading to Friedman cosmologies. An alternative program is to assume arbitrary initial conditions together with a mechanism, or set of mechanisms characteristic of the physics of hot dense matter which rapidly damps anisotropy. Among the mechanisms considered are: neutrino viscosity[48,49], particle creation by the curvature of spacetime[50,51] and anisotropic radiation pressure[52]. Here also, much further work is needed. For the model (6.1.16) it may be possible to calculate the exact evolution of the anisotropic radiation pressure and thus estimate its effects.

The assumption of dissipative viscous forces, however, cannot remove all problems. It can be shown[29] that the asymptotic dynamics of certain Bianchi Types precludes isotropy in the limit of large time. This would seem again to limit possible models since at some large time the stress energy is presumably ignorable and subsequent evolution is anisotropic. It may well be that the ultimate cosmological issue will again be "initial conditions".

Footnotes

(a) In a space with zero torsion, $d(p_\alpha) = \Gamma^{\beta}{}_{\alpha\mu} dx^\mu p_\beta$ where the $\Gamma^{\beta}{}_{\alpha\mu}$ are the Christoffel symbols.

(b) The Ricci rotation coefficients are the coefficients of connection in an orthonormal basis

(c) Material in this section is based on the work in reference 53.

(d) Or steady state models which will not be discussed here.

(e) From $(p_t, T_a) = 0$ and $(T_a)_{(\alpha;\beta)} = 0$ it follows that $D_t p_t = 0$(p_t is geodesic).

(f) If the group action is multiply transitive, then for points p and q, there exist transformations a and b such that: $ap = bp = q$, or ab^{-1} leaves p fixed[57]. Isometries with fixed points are rotations. For G_4 (or G_6) the group of motions may admit more than one simply transitive subgroup G_3. In this case the identification of the translation will depend on the particular coordinate realization given the group(see 4.5)

(g) Coordinate realizations for the generators of translations and the invariant basis elements can be found in references 21 and 56.

(h) That rotations can intermix elements of the isometry goup has been pointed out in the literature[27,57], but the consequences of such mixing and the criterion (4.5.8) have not been discussed.

(i) Since $k=(d\alpha, R)$ is a constant for the non-null field, and R must vanish somewhere if it is to generate rotations, it follows[43] that k=0 for any rotation independant of assumed homogeneity.

(j) In (6.5.6) and subsequently T_T and T_L are the microwave background temperatures as measured in the transverse and longitudinal directions respectively.

References

1. Adler, R., Bazin, M. and Schiffer, M.: Introduction to General Relativity. McGraw-Hill, New York (1965)

2. Weinberg, S.: Gravitation and Cosmology. Wiley, New York (1972)

3. Landau, L.D. and Lifshitz, E.M., The Classical Theory of Fields. Pergamon Press, Oxford (1962)

4. Misner, C.W., Thorne, K.S. and Wheeler, J.A.: Gravitation, W.H. Freeman, San Francisco (1970)

5. Hawking, S.W. and Ellis, G.F.R.: The Large Scale Structure of Spacetime. Cambridge University Press (1973)

6. Witten, L. (editor): Gravitation, An Introduction to Current Research. Wiley, New York (1962)

7. C. and B. deWitt (editors): Relativity, Groups and Topology. Blackie and Son, London (1964)

8. Auslander, L. and Mackenzie, R.E.: Introduction to Differentiable Manifolds. McGraw-Hill, New York (1963)

9. Flanders, H.: Differential Forms. Academic Press, New York (1963)

10. Hicks, N.J.: Notes on Differential Geometry. D. Van Nostrand, Princeton (1965)

11. Eisenhart, L.P.: Riemannian Geometry. Princeton University Press, Princeton (1964)

12. Cohn, P.M.: Lie Groups. Cambridge University Press, Cambridge, G.B. (1957)

13. deVaucouleurs, G.: Nature <u>182</u>, 1478 (1958)

14. deVaucouleurs, G.: Nature <u>220</u>, 868 (1968)

15. Gödel, K.: Rev. Mod. Phys. <u>21</u>, 447 (1949)

16. Gödel, K.: Proc. Int. Cong. Math. <u>1</u>, 175 (1950)

17. Sachs, B.K. (editor): Proceedings of the International School of Physics - Course 47. Academic Press, New York (1971)

18. Bondi, H.: Cosmology. Cambridge University Press, Cambridge (1952)

19. Eisenhart, L.P.: Continuous Groups of Transformation. Princeton University Press (1933)

20. Kantowski, R. and Sachs, R.K.: J. Math. Phys. 7, 443 (1966)

21. Taub, A.H.: Ann. of Math. 53, 472 (1951)

22. Lie, S. and Scheffers, G.: Vorlesungen über Continuierliche Gruppen mit Geometrischen und Anderen Anwendungen. Teubner, Liepzig (1893)

23. Bianchi, L.: Lezioni sulla Teoria dei Gruppi Continuifiniti di Transformazioni. Spoerri, Pisa (1918)

24. Bianchi, L.: Soc. Italiana delle Science, Mem. di Mat., 11, ser. 3, 267 (1897)

25. Estabrook, F.B., Wahlquist, H.D. and Behr, C.G.: J. Math. Phys. 9, 497 (1968)

26. cf. Heckman, O. and Schucking, E.: Chapter 11 of Ref. 6

27. Ellis, G.F.R. and MacCallum, M.A.H.: Comm. Math. Phys. 12, 108 (1969)

28. King, A.R. and Ellis, G.F.R.: Comm. Math. Phys. 31, 209 (1973)

29. MacCallum, M.A.H.: Comm. Math. Phys. 20, 57 (1971)

30. Hughston, L.P. and Jacobs, K.C.: Ap. J. 160, 147 (1970)

31. Ellis, G.F.R.: J. Math. Phys. 8, 1171 (1967)

32. Witten, L.: Chapter 9 in Ref. 6.

33. Misner, C.W. and Wheeler, J.A.: Ann. Phys. 2, 525 (1957)

34. Thorne, K.S.: Ap. J. 148, 51 (1967)

35. Kantowski, R.: Ph.D. thesis, University of Texas (unpublished).

36. Bahcall, J.N. and Salpeter, E.E.: Ap. J. 142, 1677 (1965)

37. Sandage, A.R. and Tammann, G.A.: Ap. J. 196, 313 (1975)

38. DeVaucouleurs, G.: Ap. J. 205, 13 (1976)

39. Rubin, V.C., Thonnard, N., Ford, W.K. and Roberts, M.S.: Astron. J. 81, 719 (1976)

40. Wainwright, J. and Yaremowicz, P.E.A.: Gen. Rel. & Grav. 7, 595 (1976)

41. Coll, B.: C. R. Acad. Sci., Paris 280; A1773 (1975)

42. Ray, J.R. and Thompson, E.L.: J. Math. Phys. 16, 345 (1975)

43. Michalski, H. and Wainwright, J.: Gen. Rel. and Grav. 6, 289 (1975)

44. Coll, B.: J. Math. Phys. 18, 1919 (1977)

45. John Ray - Private communication

46. Collins, C.B. and Hawking, S.W.: Ap. J. 180, 317 (1973)

47. Rindler, W.: Mon. Not. Roy. Ast. Soc. 116, 663 (1956)

48. Matzner, R.A. and Misner, C.W.: Ap. J. 171, 415 (1973)

49. Matzner, R.A.: Ap. J. 171, 433 (1972)

50. Zel'dovich, Ya.B.: Soviet Physics - J.E.T.P. 34, 1159 (1972)

51. Parker, L. and Fulling, S.A.: Phys. Rev. D7, 2357 (1973)

52. Press, W.H.: Ap. J., 205, 311 (1976)

53. Ellis, G.F.R.: article in Ref. 17 - pp 104-182

54. Peebles, P.J.E.: Physical Cosmology. Princeton University Press (1971)

55. Sciama, D.W.: article Ref. 17, pp. 183-228.

56. Ryan, M.P. and Shepley, L.C.: Homogeneous Relativistic Cosmologies. Princeton University Press, Princeton, (1975)

57. MacCallum, M.A.H.: article in Cargese Lectures in Physics, Vol. 6. Gordon and Breach, New York (1973)