Can Fuzzy Decision-Making Be Another Piece of the Puzzle?

by

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Abstract

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This dissertation examines whether the widespread human tendency of forming fuzzy subjective beliefs can be accountable for two prominent theoretical conundrums: the excess consumption growth puzzle and the equity premium puzzle.

The first chapter of this dissertation is devoted to the excess consumption growth puzzle. Angus Deaton discovered that there was a prolonged period during post- World War II U.S. history when the aggregate consumption continued growing, while the ex-post real interest rate remained negative. This troubling coexistence is not consistent with the Euler equation derived from a family of conventional intertemporal consumption models and was coined by Deaton as "the excess of consumption growth puzzle." This chapter establishes that under stochastic wealth constraint neither precautionary motives nor subjective pessimism and doubt can entirely resolve the puzzle. Using a simple two-period life-cycle consumption model, we demonstrate that, when the representative consumer forms fuzzy beliefs, then the resulting optimal saving behavior is consistent with the excess consumption growth. Hence, the explicit treatment of fuzzy decision-making in intertemporal consumption models can go a long way in settling the puzzle of excess consumption growth.

In the second chapter we examine whether the equity premium puzzle can be uncovered if the representative agent is endowed with a mechanism of combining robust forecasts with subjective fuzzy beliefs about forthcoming consumption growth. The puzzle is introduced by Mehra and Prescott, who show that the U.S. historical average excess return on risky equity over that of the riskless bonds is much higher than what could be explained by the family of conventional intertemporal stochastic consumption models. We use an infinite-horizon dynamic model to demonstrate in general equilibrium settings that, when the growth rate of consumption in every random state of the world is perceived by a pessimistic representative agent as a fuzzy rather than crisp magnitude, then the resulting Euler equation can successfully be calibrated to account for on average low risk-free rates as well as large equity premiums. We conclude that introduction of fuzzy uncertainty as an additional layer of uncertainty in intertemporal representative agent models can be very fruitful in settling the equity premium puzzle.

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Needless to say, any mistakes and inaccuracies that this dissertation may contain are my sole responsibility.

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Can Fuzzy Decision-Making Explain Growing Consumption During Eras of Negative Real Interest Rates?

1.1 Introduction

One of the manifests offered by a large family of life-cycle representative rational agent models is that personal consumption is expected to decline whenever an individual's time preference exceeds the real rate of interest. Deaton [9],[10], however, reports some puzzling evidence that there was a prolonged period in the U.S. post-war history when the aggregate consumption continued to grow, while the prevailing average real rates of interest remained negative. Responding to this conundrum, some authors[1] propose that the existence of precautionary motives can, to a large degree, be accountable for excessive saving and therefore excessively growing consumption. Nevertheless, as it is noted by Deaton [10] and demonstrated in this paper, the explicit treatment of precautionary saving in a representative agent framework does not entirely uncover this discrepancy between the average positive consumption growth and the negative real rate of return on risk-free assets. The coefficients of risk aversion and time preference that calibrate the model to match the post-World War II data

1 See [6] for example

are of unreasonable magnitudes, even when the precautionary motives are properly accounted for.

In order to uncover puzzles of this sort, theories often deviated from the rationality assumption. One notable advance in this respect is presented by Abel [1], who shows that pessimism and doubt in the subjective probability distribution of the consumption growth may significantly reduce the equity premium puzzle[2]. Subjectivity in the consumer's intertemporal optimization problem is also justified by an important finding that the consumer confidence index, a data proxy for consumers' subjective optimism-pessimism, is an unmistakable predictor power of consumption growth [2].

Even though the models with subjective probability distribution are favored by the above-mentioned characteristics, there are some general issues that ought to be brought up. First, as we show below, the incorporation of subjectivity in the form of optimism-pessimism and doubt into the otherwise conventional stochastic model by itself does not entirely resolve the "puzzle of excess consumption growth." Even with pessimism and doubt present in the model, the parameter space that reconciles growing consumption with the negative real rate of interest includes only unreasonable ranges. The second remark is philosophical in nature. It is related to one unsettled inquiry in regard to why consumers should remain pessimistic or optimistic for a prolonged period of time. Perhaps the following example will motivate the validity of

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[2] Also see [7].

this inquiry. Consider a consumer whose real endowment in each period depends on a repeated gamble that consists of a simple flip of a coin. In every period she is awarded by a monetary prize if she wins and nothing otherwise. The award or a part of it can be consumed or saved for the forthcoming periods when the gambling outcomes turn unfavorable. The size of the award does not change from one period to another. The same gamble is repeated in every period of the consumer's life. Within a few trials the consumer will be able to learn the true stochastic properties of the endowment process, and her subjective optimism-pessimism or doubt regarding future outcomes shall no longer affect her saving/consumption decisions. The same holds even if the coin is not fairly weighted. Naturally, the amount of precautionary savings built by more risk averse consumers must be in excess of what their less risk averse counterparts save. The description of this gamble resembles general underlying settings assumed by many rational expectation models, and intuitively the neutrality of the consumer's subjectivity should apply to them as well. So what is missing in stochastic rational expectation models that may justify the effect of consumer's optimism-pessimism on their consumption decisions? In order to answer this question, we consider another gamble, which is more realistic in its nature. This time the consumer's endowment in each period depends on a lottery that is similar to many conducted by municipalities or states. Typically, the probability of winning depends on a scheme that is not subject to straight-forward calculations. Moreover, the amount of the win depends on the number of participants and winners in the lottery, as well as complicated provisions of the tax law. The utility loss from going through the trouble of finding out about all the exact details can be tremendous. At the same time, the utility loss from setting consumption according to some reasonable rule of thumb rather than to the optimal permanent-income decision rule can be microscopic [8]. As a result, the consumers who participate in this lottery commonly have a vague or fuzzy idea about the odds of winning and the amount of the win. Most commonly they remain in the state of belief that the probability of winning is *small* and the amount of the win is *large.* In these settings gamblers are likely to draw heavily on subjective factors. From the consumer's point of view, the uncertainty of this lottery stems not only from the fact that the outcome is subject to a chance, but also that the probability of winning and the amount of the gain are not known exactly and will never be. We, therefore, recognize that there are two layers of uncertainty that the participants of these sorts of lotteries face. Purely stochastic uncertainty exists because the outcome may or may not happen with certain probability. The second layer of uncertainty stems from the fact that the exact win and the probability structure of the gamble are not precisely known and perhaps will never be. The degree of blurriness of the linguistically defined values *large* win and *small* probability constitutes what we define as fuzzy uncertainty or fuzziness. Consumers seldom attempt to resolve this blurriness since the failure to clarify it results in only a minimal utility loss. It is because of this ever-present blurriness that the consumer's pessimism or optimism impacts their consumption-saving choices. Hence, by drawing parallels between the first and second gambles described above, we attempt to highlight the hypothesis that the consumer's optimism-pessimism affects her optimal consumption plans, as long as fuzziness is an attribute of the decision process. As a consequence, we conjecture that only incorporation of fuzziness into the model, rather than modification of its stochastic properties, is a fruitful venue for resolution of the puzzle.

In reality, fuzziness may be present even if stochastic uncertainty is not. Salaried workers, for example, who receive their paychecks with probability close to one often fail to quote the exact figure for their disposable annual income. Most commonly, this figure is quoted by respondents as an approximation *around* some benchmark, round number. Clearly, what constitutes uncertainty in this case is the word *"around"* rather than the stochastic nature of the income.

This paper addresses the "puzzle of excess consumption growth" by explicitly treating fuzziness of future wealth as the only source of uncertainty in a simple, two-period life-cycle consumption model. We extend the methodology proposed by Stahlecker et al. [28] and Hauenschild and Stahlecker [14] to the case of a constant relative risk aversion felicity function and show that when future wealth is uncertain in a fuzzy sense, then the puzzle of excess consumption growth can successfully be resolved.

This paper is organized as follows. Section 1.2 reviews the baseline and purely stochastic model of the consumer's intertemporal choice with objective expectations.

This section introduces the puzzle of excess consumption growth and demonstrates that the baseline stochastic model fails to simultaneously match the post- World War II moments of consumption growth and the interest rates. Section 1.3 incorporates subjective uncertainty into the baseline stochastic model introduced in Section 1.2. We show in this section that, even though the subjectivity in the form of pessimism and doubt improves the model's performance, it does not completely uncover the puzzle. Section 1.4 develops a simple model of consumer's intertemporal choice under fuzzy wealth constraint. This section demonstrates that the model can be reasonably calibrated to match the post-war moments of consumption growth with those of the real interest rate once fuzziness is accounted for. Section 1.5 concludes.

1.2 Two-Period Model with Stochastic Uncertainty and Objective Expectations

[3]A representative agent maximizes the following two-period, time-separable utility function

$$\max_{c_1} E\{U(c_1, c_2)\} = u(c_1) + \frac{1}{1+\delta} Eu(c_2) \tag{1.1}$$

Subject to a constraint

$$A_2 = y_2 + (1 + r_2)(y_1 - c_1) \tag{1.2}$$

and a transversality condition

$$A_2 = c_2 \tag{1.3}$$

[3] This section closely follows Deaton [10], pp. 63-66.

Here c_1 and c_2 are the present and future consumptions respectively, y_1 and y_2 are the present and future incomes, r_2 is the net real rate of interest, and $\delta > 0$ is the rate of time preference. Positivity of the rate of time preference is a widespread assumption made in the related literature[4]. The second-period labor income and the real return on savings are unknown at the time when the consumption decision is made. These are unknown in a stochastic sense. Hence, there is an objective joint probability distribution function that is associated with the random second-period income and the real interest. In order to evaluate the model, we first eliminate the wealth constraint by replacing (1.3) into (1.2) and then the resulting expression into (1.1). The optimal level of consumption should satisfy the following first order necessary condition:

$$u'(c_1) = E\left\{\frac{1+r_2}{1+\delta}u'(c_2)\right\} \tag{1.4}$$

We further modify (1.4) by assuming that the consumer's preferences are described by a constant relative risk aversion (CRRA) power utility function of the form $u(c) = \frac{c^{1-\lambda}}{1-\lambda}$, so that $u'(c) = c^{-\lambda}$, where $\lambda > 0$ is the coefficient of risk aversion[5]. Hence, (1.4) can be rewritten in the form:

$$1 = E\left\{\frac{1+r_2}{1+\delta}\left(\frac{c_2}{c_1}\right)^{-\lambda}\right\} \tag{1.5}$$

4 Kocherlakota [17] shows that the assumption of positivity of the discount factor is not necessary for achieving a well defined solution in infinite horizon growth economies.

5 λ is required to be greater than zero in order to ensure concavity of the felicity function. Estimates of λ vary in empirical literature. Altug [3] finds it to be near zero. Mankiw, Rotemberg, and Summers [23] establish that it is near 0.5. Kehoe [16], and Hansen and Singleton [13] estimate λ to be around 1. Tobin and Dolde [29] 1.5. Friend and Blume [11] found it to be around 2. Zeldes's [30] estimates of λ are in the region of 2.3. Mankiw [21], [22] 4, and 3 respectively. Levy et al. [19] survey the related empirical and experimental literature and conclude that relative risk aversion parameter, λ, is in the range of 0.6 and 2.

It is frequently assumed in the literature that the consumption growth and the real interest rate are jointly normally distributed. We, therefore, can take the expectation in (1.5) explicitly if the same assumption is made here [6]

$$E \exp[(r_2 - \delta) - \lambda \ln \Delta c_2] = 1 \tag{1.6}$$

$$(Er_2 - \delta) - \lambda E \ln \Delta c_2 + \frac{\lambda^2}{2} Var\{\ln \Delta c_2 - \lambda^{-1} r_2\} = 0 \tag{1.7}$$

where $\Delta \ln c_2 = \ln c_2 - \ln c_1$, and we assume that r_2 and δ are small enough to allow for the approximation. We further evaluate (1.7):

$$E\Delta \ln c_2 = \lambda^{-1}(Er_2 - \delta) + \frac{\lambda}{2}\omega(\lambda) \tag{1.8}$$

Where

$$\begin{aligned} \omega(\lambda) &= Var(\Delta \ln c_2) + \frac{1}{\lambda^2} Var(r_2) \\ &\quad - \frac{2}{\lambda} Cor(\Delta \ln c_2, r_2)\sqrt{Var(\Delta \ln c_2)}\sqrt{Var(r_2)} \end{aligned} \tag{1.9}$$

Here $\frac{\lambda}{2}\omega(\lambda)$ constitutes the contribution of precautionary motives to the consumption growth. Intuitively, greater individual variances of the consumption growth and the interest rate, and smaller covariance between consumption growth and the interest rate induce consumers to save in excess of what they would have if they had perfect foresight about their future consumption. It is fairly straight forward to check whether the historical estimates of the ex-post real interest rate and the consumption growth can be reconciled with (1.8).

---

[6] If a random variable $V\tilde{\ }N(m, s^2)$, then the non-negative random variable $Q = e^V$ has a log-normal distribution with n-th moment given by $E(Q^n) = \exp[nm + \frac{1}{2}n^2 s^2]$.

Statistics (3Q1953-4Q1984 N=126)	Point Estimates (as reported by Deaton)	99% Confidence Interval Estimates	
		Lower Limit	Upper Limit
$E\Delta \ln c_2$	0.005	0.003668	0.006332
Er_2	-0.0006	-0.002069	0.000869
$Var(\Delta \ln c_2)$	0.000023	0.000017	0.000033
$Var(r_2)$	0.000028	0.000021	0.000040
$Cor(\Delta \ln c_2, r_2)$	0.197028	-0.032605	0.406902

Table 1.1: Historical moments of the consumption growth and the real rate of interest between 3Q1953 and 4Q1984. The confidence interval estimates of the mean consumption growht and that of the real interest rate are obtained using the upper variance limits to represent extremely doubtful views

The second column of Table 1.1 reproduces estimates of the first, second, and cross moments of the consumption growth and the real interest rate reported by Deaton [9], [10] for the period between 3Q1953 and 4Q1984 (126 observations). This author reports that the estimate of the covariance between consumption growth and the real interest rate was 0.000005, which, given point estimates of the variance, amounts to the correlation coefficient equal to 0.197. To calibrate the model to these values, we first substitute them into (1.9), then the result into (1.8). We then search for values of the coefficient of risk aversion, λ, and the coefficient of time preference, δ, which turn the resulting expression into an exact equality. The graph in Figure 1.1 represents pairs of λ and δ that accomplish this objective. A quick examination of the figure reveals that, in order to calibrate (1.8) to the long run empirical moments while maintaining the assumptions of positivity of both λ and δ, the coefficient of risk aversion should be allowed to exceed 435. These values are out of any consideration since any range of λ that exceeds 20 would imply almost zero marginal utility of virtually any positive level of consumption. Hence, there are no acceptable values of the behavioral parameters in these conventional settings that would accommodate for growing consumption while the real rate of return is negative. This argument highlights the puzzle of excess consumption growth.

1.3 Two-Period Model with Stochastic Uncertainty and Subjective Expectations

In this section we introduce subjectivity into the consumer's inter-temporal optimization problem (1.1)-(1.3). Following Abel [1], we model subjectivity by allowing the representative agent to form subjectively pessimistic or optimistic as well as doubtful expectations about the stochastically uncertain variables in the model. Subjectivity is modeled by shifting the relevant shape parameters of the joint and marginal probability distributions of the consumption growth and the real rate of interest. The outcome of pessimistic/optimistic views is achieved by shifting the subjective mean below/above its objective counterpart. Doubtful views, on the other hand, are achieved by increasing the objective spread, and therefore the variance, of the probability distribution around its mean. If the representative agent's subjective choices are the driving force of the actual aggregate consumption process, then the subjective moments can be substituted into (1.8).

$$E^{*}\Delta \ln c_2 = \lambda^{-1}(E^{*}r_2 - \delta) + \frac{\lambda}{2}\omega^{*}(\lambda) \tag{1.10}$$

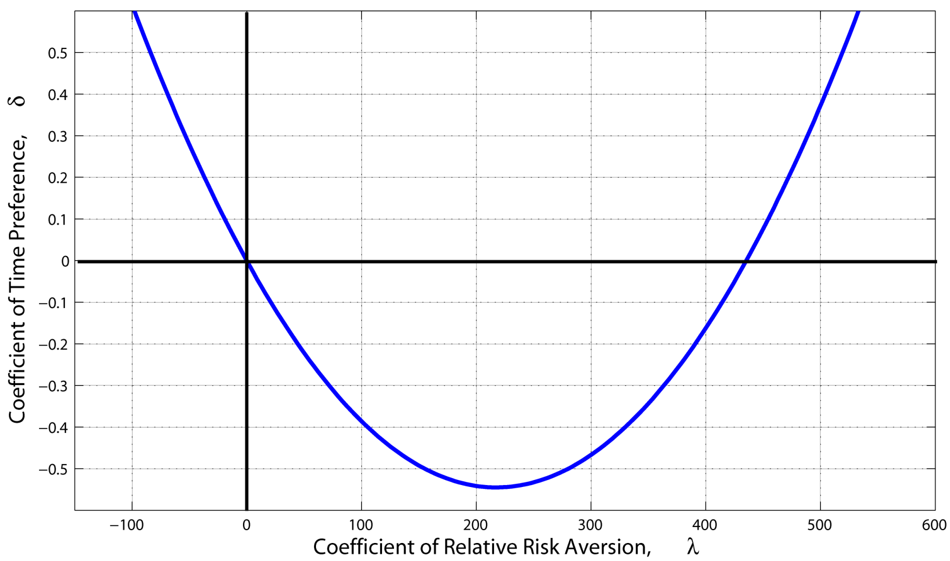

Figure 1.1: Values of the coefficient of risk aversion, λ, and the coefficient of time preference, δ, that calibrate the baseline model (1.1)-(1.3) and (1.8) to the historical moments of consumption growth and the real rate of interest exhibited in the second column of Table 1.1

where, as before

$$\begin{aligned}\omega^*(\lambda) &= Var^*(\Delta \ln c_2) + \frac{1}{\lambda^2} Var^*(r_2) \\ &\quad - \frac{2}{\lambda} Cor^*(\Delta \ln c_2, r_2) \sqrt{Var^*(\Delta \ln c_2)} \sqrt{Var^*(r_2)}\end{aligned} \tag{1.11}$$

Here and in any consequent occurrence superscripted stars denote moments computed with respect to the subjective probability distribution. Similar to Abel ([1]), we describe subjectivity by two specific characteristics: optimism/pessimism and doubt in regard to the uncertain variables of interest. Uniform optimism or pessimism is a rightward or leftward shift respectively of the marginal probability distribution. Doubt is a mean preserving increase in the spread of the objective probability distribution. More specifically, let

$$E^* \Delta \ln c_2 = E \Delta \ln c_2 + \xi \tag{1.12}$$

$$E^* r_2 = E r_2 + \eta \tag{1.13}$$

$$\omega^*(\lambda) = \omega(\lambda) + \phi(\lambda) \tag{1.14}$$

Here ξ and η are subjectivity shift parameters that can take either positive or negative values to represent optimism and pessimism respectively. Similarly, ϕ is a doubt shift parameter which, when positive, denotes doubt and overconfidence when negative. Substituting (1.12), (1.13), and (1.14) into (1.11) then the result into (1.10) and slightly modifying the resulting expression, we obtain:

$$E\Delta \ln c_2 = \lambda^{-1}(Er_2 - \delta) + \frac{\lambda}{2}\omega(\lambda) + \frac{1}{\lambda}\eta + \frac{\lambda}{2}\phi(\lambda) - \xi \tag{1.15}$$

It is clear from (1.15) that for any given λ the expression

$$\frac{1}{\lambda}\eta + \frac{\lambda}{2}\phi(\lambda) - \xi \tag{1.16}$$

can be chosen to compensate for the puzzling discrepancy that exists between the left and right hand sides of (1.8). It should be noted that in these settings the success of calibration is not crucially dependent on negativity of ξ neither on positivity of ϕ. Figure 1.2 exhibits a subset of the parameter space $\{\lambda, \xi, \eta, \delta, \phi\}$ that calibrates (1.15) to the post-World War II moments of the consumption growth and the real rate of interest that are reported in the second column of Table 1.1. There are clearly some reasonable values of the coefficient of risk aversion, λ (between 0 and 3 on the graph) and time preference, $\delta > 0$ that calibrate the model to conform to the post-World War II data. For instance, if $\lambda = 2.6$, and quarterly discount factor $\delta = 0.005$, then $\xi = -0.0013$, $\eta = 0.0038$ together with $\phi = 0.0087$ would predict 0.5% quarterly consumption growth at -0.06% quarterly real rate of interest.

It is natural, however, to question how large expression (1.16) can reasonably get. Is it possible to settle the puzzle if this difference is reasonably large? In order to answer these questions, we build a case for a representative consumer for whom this difference is very large under reasonable assumptions. We conjecture that, if the calibration is not satisfactory for this hypothetical extreme case, then it

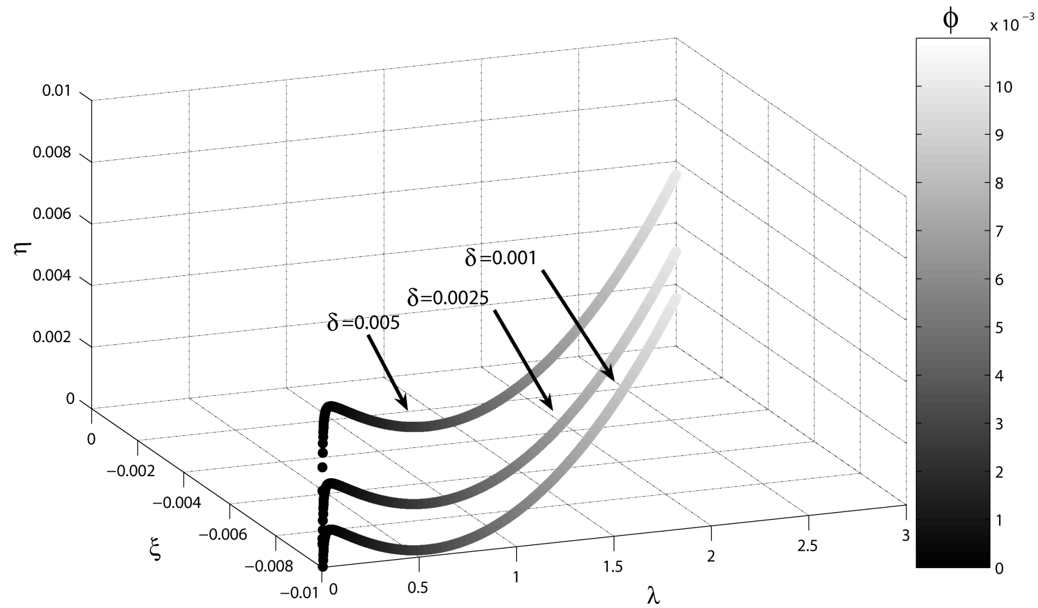

Figure 1.2: Values of the behavioral parameters in (1.15) that calibrate the model with stochastic uncertainty and subjective expectations to the historical moments of consumption growth and the real rate of interest exhibited in the second column of Table 1.1

may not possibly be for others. The hypothetical consumer for whom the difference in (1.16) is extremely large must be remarkably optimistic about the real rate of interest ($\max\{\eta\}$)[7], extremely pessimistic about the consumption growth ($\min\{\xi\}$), and extremely doubtful ($\max\{\phi(\lambda)\}$). Hence

$$\max\{\frac{1}{\lambda}\eta + \frac{\lambda}{2}\phi(\lambda) - \xi\} = \frac{1}{\lambda}\max\{\eta\} + \frac{\lambda}{2}\max\{\phi(\lambda)\} - \min\{\xi\} \tag{1.17}$$

If subjectivity arises only because of possible measurement error[8], then the sample estimates of the confidence intervals of the mean, the variance, and the covariance may suggest the order of extremes that these statistics may take with certain significance level. The 99 percent confidence interval estimates of the marginal and cross moments of the consumption growth and the real rate of interest between 3Q1953 and 4Q1984 are reported in the last two columns of Table 1.1. We use the upper confidence limits of the variances of the consumption growth and the interest rate jointly with the lower limit of the correlation between these two series in order to account for extreme doubt. Consequently, we use the upper variance bounds of the consumption growth and of the real rate to obtain the interval estimates of the corresponding means. The lower limit of the mean consumption growth is used as a subjective estimate of the extreme pessimism about this variable. Similarly, the upper interval estimate of the mean real interest rate is used to represent extreme subjective opti-

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[7] Optimism about the real rate is equivalent to an understatement of the forthcoming inflation rate.

[8] Abel [1] notes that standard errors may not necessarily be the sole cause of pessimism and doubt. Other relevant factors are not directly quantifiable.

mism about future real rate. Hence, we obtain:

$$\begin{aligned}
\max\{\phi(\lambda)\} &= \overline{\omega^*(\lambda)} - \omega(\lambda) \\
&= \overline{Var^*(\Delta \ln c_2)} + \frac{1}{\lambda^2}\overline{Var^*(r_2)} \\
&\quad -\frac{2}{\lambda}\underline{Cor^*(\Delta \ln c_2, r_2)}\sqrt{\overline{Var^*(\Delta \ln c_2)}}\sqrt{\overline{Var^*(r_2)}} \\
&\quad -Var(\Delta \ln c_2) - \frac{1}{\lambda^2}Var(r_2) \\
&\quad +\frac{2}{\lambda}Cor(\Delta \ln c_2, r_2)\sqrt{Var(\Delta \ln c_2)}\sqrt{Var(r_2)}
\end{aligned} \tag{1.18}$$

$$\min\{\xi\} = \underline{E^*\Delta \ln c_2} - E\Delta \ln c_2 = 0.0037 - 0.005 = -0.0013 \tag{1.19}$$

$$\max\{\eta\} = \overline{E^* r_2} - Er_2 = 0.00087 + 0.0006 = 0.00147 \tag{1.20}$$

where the upper and lower bars denote the upper and lower interval limits respectively. We do not report individual values of the doubt parameter $\max\{\phi(\lambda)\}$ explicitly since there is one estimate for each given value of the coefficient of risk aversion λ. In order to obtain an Euler representation for an extremely doubtful consumer with extreme optimistic expectations about the interest rate and extreme pessimistic views about the consumption growth, we substitute (1.18),(1.19), and (1.20) into (1.17) and the resulting expression into (1.15). The coefficients of risk aversion and time preference that calibrate the resulting Euler equation to the historical moments reported by Deaton are displayed in Figure 1.3. Even though subjectivity of the representative agent in the stochastic environment improves the overall performance it does not eliminate the puzzle. In order to calibrate the stochastic model with extreme doubt and subjectivity to the sample moments displayed in the second column of Table 1.1 either δ or λ should be negative , or λ should exceed 220. Once again, these values can not be accepted as normal.

Hence, we find that subjectivity and doubt in a representative agent stochastic inter-temporal model do offer a tentative solution to the excess consumption growth puzzle. A more careful examination reveals, however, that this solution is not feasible under normal circumstances.

1.4 Two-Period Model with Fuzzy Uncertainty and Subjectivity

Consider once again a representative agent who maximizes a two-period time-separable utility function

$$\max_{c_1}\{U(c_1, c_2)\} = u(c_1) + \frac{1}{1+\delta}u(c_2) \tag{1.21}$$

Subject to a constraint:

$$A_2 = \text{around}[y_2 + (1 + r_2)(y_1 - c_1)] \tag{1.22}$$

and a transversality condition:

$$c_2 = A_2 \tag{1.23}$$

What makes this model different from the ones presented in Sections 1.2 and 1.3 is the type of uncertainty that the representative consumer faces. Unconventionally, the future wealth is uncertain only in a fuzzy sense. Hence, the representative consumer

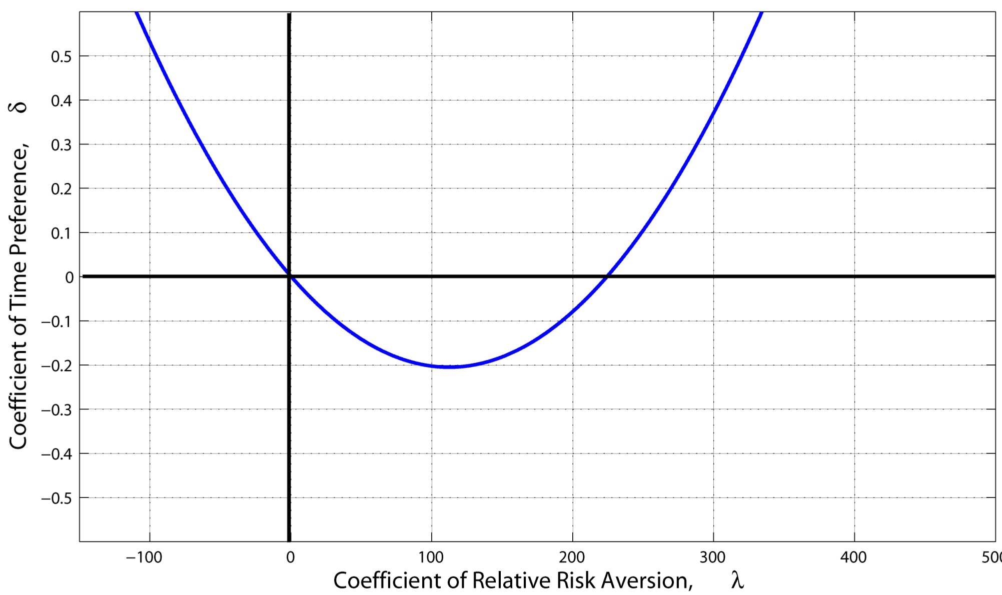

Figure 1.3: Values of the coefficient of risk aversion, λ, and the coefficient of time preference, δ, that calibrate the model with extreme subjectivity and extreme doubt, hence equation (1.15), to the historical moments of consumption growth and the real rate of interest exhibited in the second column of Table 1.1

knows for certain, or with probability one, that her future wealth is going to be *around* some perfectly known value $\widetilde{A}_2$. What constitutes uncertainty in these settings is the word "around," which potentially may arise due to complexity of the tax system that the consumer does not attempt to predict. One question that we adress in this paper is how much "around" will make the model fit to the long run consumption and real interest rate statistics. The degree of membership of the future wealth A_2 to the fuzzy set described by the sentence "wealth around $\widetilde{A}_2$" is modeled by a commonly used triangular membership function[9].

$$\mu(A_2) = \begin{cases} \dfrac{A_2 - (1-z)\widetilde{A}_2}{\widetilde{A}_2 z} \text{ if } (1-z)\widetilde{A}_2 \leq A_2 \leq \widetilde{A}_2 \\ \dfrac{(1+z)\widetilde{A}_2 - A_2}{z\widetilde{A}_2} \text{ if } \widetilde{A}_2 \leq A_2 \leq (1+z)\widetilde{A}_2 \\ 0 \text{ otherwise} \end{cases} \tag{1.24}$$

where A_2 is the actual second-period wealth. Generally, membership functions are a mathematical way of functionally assigning a degree between zero and one to which any given element belongs to a given fuzzy set. More specifically, membership function (1.24) prescribes an exact degree to which any given level of the second period wealth A_2 can be described as "around $\widetilde{A}_2$." Clearly, the further away A_2 is from $\widetilde{A}_2$, the less should the membership of A_2 be to the fuzzy expression "around $\widetilde{A}_2$." The parameter z alter the degree of fuzziness carried by the membership function: greater z represent more fuzziness in the model and conversely the fuzziness vanishes as z approaches to zero. This parameter characterizes what exactly consumers

[9] Our consequent analysis does not crucially depend on the coice of the mebership function. Virtually any unimodal membership function would lead us to the same results.

have in mind by vague "around $\widetilde{A}_2$." If fuzzy set "around $\widetilde{A}_2$" includes only values of A_2 that are closely clustered around $\widetilde{A}_2$ then z should be small. Greater values of z are more appropriate if one considers even distant values of A_2 as being "around $\widetilde{A}_2$." Since the degree of fuzzy uncertainty carried by z is one of the four parameters that are used in the consequent calibration process it is important to further specify which values of z we regard as reasonable. It clearly depends on which values of A_2 in particular consumers regard as "around $\widetilde{A}_2$." Consider, for example, a case of a representative consumer who believes that her forthcoming income is going to be around $\widetilde{A}_2 = \$1,000$. Let us further assume that for her the range $\$1,000 \pm 75$ ($\pm 7.5\%$) is deemed more than 75% as being "around $\widetilde{A}_2$." Moreover, any second period wealth that falls beyond the range $\$1,000 \pm 300$ ($\pm 30\%$) is not considered as being "around $\widetilde{A}_2$." This example describes a sort of belief that may be encountered among people. As Figure 1.4 indicates this degree of fuzziness can be achieved by setting $z = 0.3$. Whether $z = 0.3$ is an acceptable value for percent fuzziness or not is a question that must be tested in a separate study. In our consequent analysis we classify any value of fractional fuzziness attained by setting $z < 0.75$ as tolerable.

The α-cut of the membership function (1.24) is given by:

$$A_2(\alpha^*) = \widetilde{A}_2 \pm \widetilde{A}_2 z(1-\alpha) \tag{1.25}$$

where $\alpha \in [0,1]$. Expression $z(1-\alpha)$ in (1.25) represents some relative or percent measure of fuzziness that distorts the benchmark value of the second-period wealth $\widetilde{A}_2$ from its actual value A_2. Given the α-cut at $\alpha = \alpha^*$, an extremely optimistic

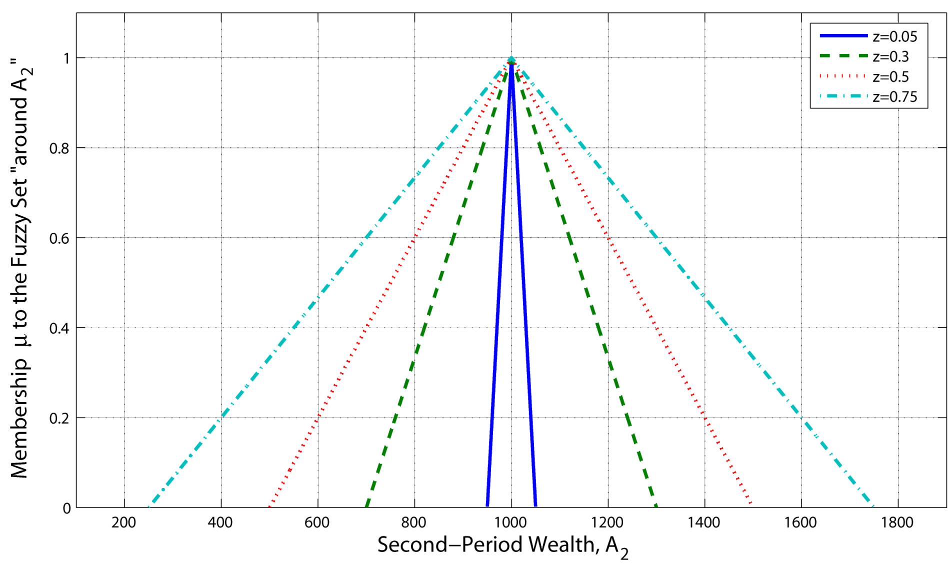

Figure 1.4: The degree of membership of given second-period wealth A_2 to the linguistically defined fuzzy set "around $\widetilde{A}_2$" when the membership function is defined as in (1.24), and $\widetilde{A}_2 = \$1,000$

consumer believes that her wealth in period two is going to be:

$$A_2(\alpha^*) = \widetilde{A}_2 + \widetilde{A}_2 z(1-\alpha^*) \tag{1.26}$$

Perceived wealth of a remarkably pessimistic consumer can be written as:

$$A_2(\alpha^*) = \widetilde{A}_2 - \widetilde{A}_2 z(1-\alpha^*) \tag{1.27}$$

We rewrite the utility function of an optimistic consumer by first substituting (1.26) into (1.23) and then the resulting expression into (1.21):

$$U(c_1, c_2) = u(c_1) + \frac{1}{1+\delta} u(\widetilde{A}_2 + \widetilde{A}_2 z(1-\alpha^*)) \tag{1.28}$$

Similarly, given an α-cut at $\alpha = \alpha^*$, we rewrite the utility of a pessimistic individual

$$U(c_1) = u(c_1) + \frac{1}{1+\delta} u(\widetilde{A}_2 - \widetilde{A}_2 z(1-\alpha^*)) \tag{1.29}$$

Utility of an individual whose optimism falls between these two extremes is

$$\begin{aligned} U(c_1, c_2) &= u(c_1) + \frac{1}{1+\delta}\{qu((1 - z(1-\alpha^*))\widetilde{A}_2) \\ &\quad +(1-q)u((1+z(1-\alpha^*))\widetilde{A}_2)\}, \end{aligned} \tag{1.30}$$

where q is an optimism-pessimism index that may take values between 0 and 1 and represents extreme optimism and extreme pessimism respectively at these extremes. In these partial equilibrium settings we assume that the optimism-pessimism index, q, is an exogenous parameter that depends on socioeconomic events that are not captured by the model. Aggregating (1.30) over all possible α-cuts, we get a fuzzy version of the utility function in (1.21):

$$U(c_1, c_2) = u(c_1) + \frac{1}{1+\delta}\{q \int_0^1 u((1 - z(1-\alpha))\widetilde{A}_2)\, d\alpha \qquad (1.31)$$
$$+(1-q)\int_0^1 u((1 + z(1-\alpha))\widetilde{A}_2)\, d\alpha\}$$

The optimal first-period consumption is subject to a first order necessary condition that can be obtained by substituting transition equation (1.22) into (1.31) and differentiating the resulting expression with respect to first-period consumption.

$$u'(c_1) = \frac{1+r_2}{1+\delta}\{q \int_0^1 (1 - z(1-\alpha))u'((1 - z(1-\alpha))\widetilde{A}_2)\, d\alpha \qquad (1.32)$$
$$+(1-q)\int_0^1 (1 + z(1-\alpha))u'((1 + z(1-\alpha))\widetilde{A}_2)\, d\alpha\}$$

We further modify (1.32) by again assuming that the consumer's preferences can be described by a conventional constant relative risk aversion utility function of the form $u(c) = \dfrac{c^{1-\lambda}}{1-\lambda}$:

$$c_1^{-\lambda} = \frac{1+r_2}{1+\delta}(c_2)^{-\lambda}\{q \int_0^1 (1 - z(1-\alpha))^{1-\lambda}\, d\alpha \qquad (1.33)$$
$$+(1-q)\int_0^1 (1 + z(1-\alpha))^{1-\lambda}\, d\alpha\}$$

where c_2 is what the second-period consumption would have been had the consumer had a non-fuzzy perfect foresight that her future wealth was going to be equal to the benchmark amount $\widetilde{A}_2$. After evaluating the integrals explicitly, (1.33) can be simplified to:

$$c_1^{-\lambda} = \frac{1+r_2}{1+\delta}(c_2)^{-\lambda}\left[\frac{(2q-1)-q(1-z)^{2-\lambda}+(1-q)(1+z)^{2-\lambda}}{z(2-\lambda)}\right]$$

which after taking logs on both sides, can be rewritten as:

$$\Delta \ln c_2 = \frac{1}{\lambda}(r_2-\delta)+\frac{1}{\lambda}\ln\left(\frac{(2q-1)-q(1-z)^{2-\lambda}+(1-q)(1+z)^{2-\lambda}}{z(2-\lambda)}\right) \quad (1.34)$$

where $\Delta \ln c_2 = \ln c_2 - \ln c_1$. Here again we make an assumption that the interest rate and the discount factor are reasonably small to allow for the approximation. It can be shown (see [4]) that Euler equation (1.33) and its collapsed version (1.34) carry a few very interesting features. For example, it can be shown that, when future labor income is uncertain in a fuzzy sense, the difference in consumption patterns of two identically optimistic or pessimistic but differently risk-averse consumers is the largest when both are neither excessively optimistic nor pessimistic. Hence, as extreme optimism or pessimism becomes widespread, the heterogeneity in consumption patterns between more risk-averse and less risk-averse individuals disappears. It also can be noted from (1.33) that, as z converges to zero, thus as fuzzy uncertainty disappears, fuzzy Euler equation (1.33) converges to a conventional, perfect foresight, intertemporal Euler equation of the form:

$$(c_1)^{-\lambda} = \frac{1+r_2}{1+\delta}(c_2)^{-\lambda} \quad (1.35)$$

Again we attempt to calibrate Euler equation (1.34) to the post-World War II moments of the consumption growth and the real rates of interest. We search for a space of optimism-pessimism index, q, coefficient of risk-aversion, λ, and relative fuzziness, z, that solve Euler equation (1.34) by allowing δ to take four discreet values: $\delta = 1\%$, $\delta = 3\%$, $\delta = 5\%$, and $\delta = 10\%$ per quarter. The graphical results that correspond to each of these cases are presented in Figures 1.5-1.8 respectively. The solution curves characterize all the combinations of optimism-pessimism and risk-aversion that, given the historical consumption growth, the ex-post real rate of interest, relative fuzziness, and the rate of time preference, conform with (1.34). It can easily be observed from all the graphs that there are some fairly sensible degrees of optimism-pessimism and coefficients of risk aversion for which the predicted moments of the consumption growth match with their empirical counterparts. For instance, if consumers discount the future consumption at a quarterly rate of $\delta = 3\%$, (Figure 1.8), and percent fuzziness $z = 29\%$, then the coefficient of relative risk aversion $\lambda = 1.5$ coupled with an optimism-pessimism index $q = 0.7$ (moderately pessimistic) would predict 0.5% quarterly consumption growth at the quarterly real rate of interest $r_2 = -0.06\%$. Hence, the puzzle of the excess consumption growth disappears once the fuzzy uncertainty is properly accounted for.

It is interesting to see that the excessive consumption growth and therefore excessive first-period saving, can be observed for pessimistic as well as optimistic individuals (Ranges of optimism-pessimism index and coefficient of risk aversion that are located in the southwest corners of Figures 1.5-1.8).

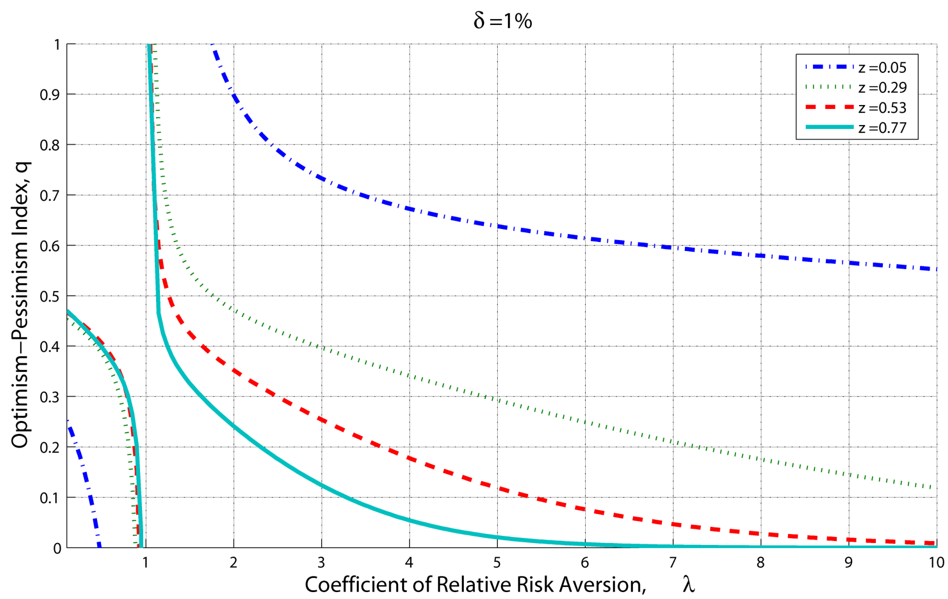

Figure 1.5: Optimism-pessimism index, q, vs. positive coefficients of risk aversion, λ, that predict quarterly consumption growth equal to its historical post-World War II average of 0.5% given negative quarterly real interest rate -0.06% and positive quarterly discount factor of 1%

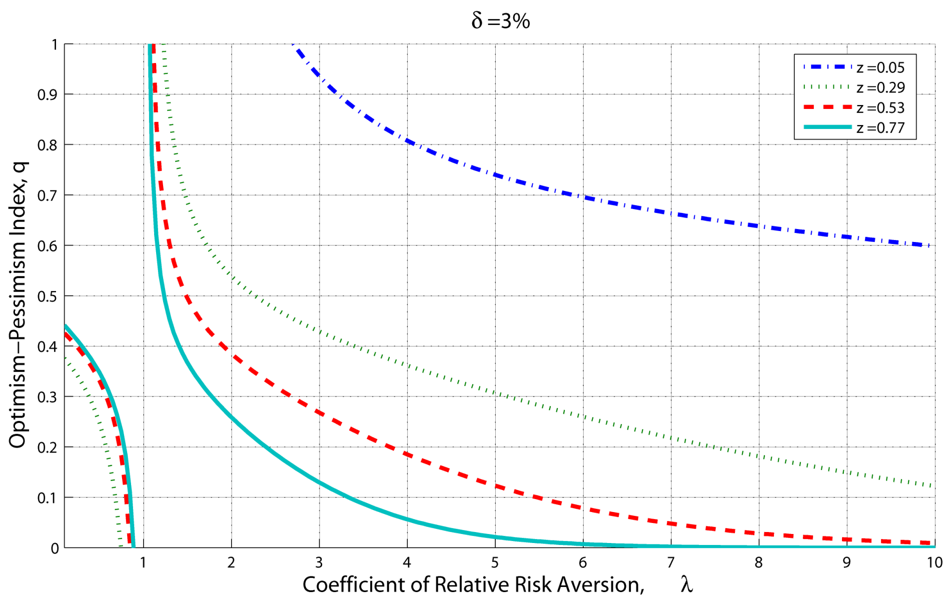

Figure 1.6: Optimism-pessimism index, q, vs. positive coefficients of risk aversion, λ, that predict quarterly consumption growth equal to its historical post-World War II average of 0.5% given negative quarterly real interest rate -0.06% and positive quarterly discount factor of 3%

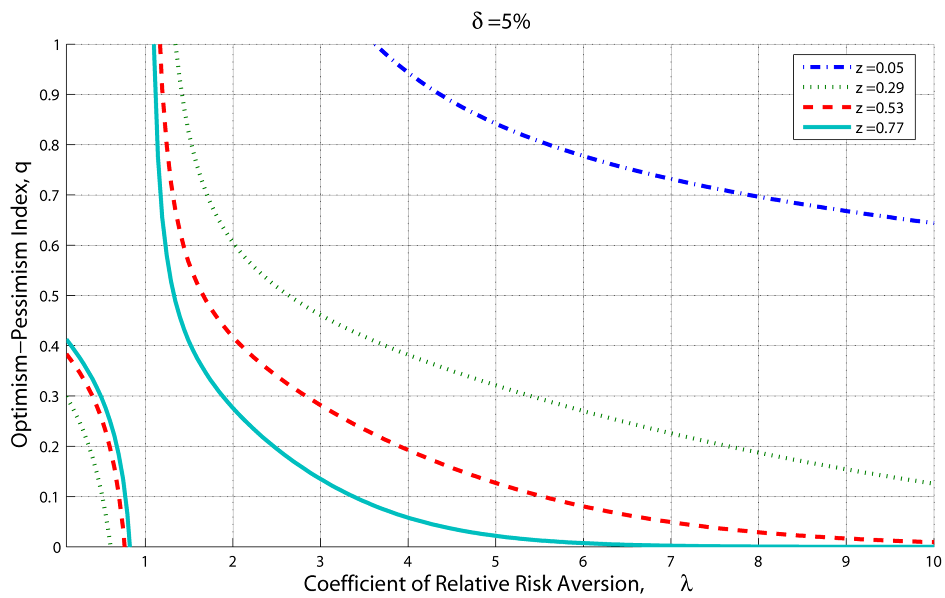

Figure 1.7: Optimism-pessimism index, q, vs. positive coefficients of risk aversion, λ, that predict quarterly consumption growth equal to its historical post-World War II average of 0.5% given negative quarterly real interest rate -0.06% and positive quarterly discount factor of 5%

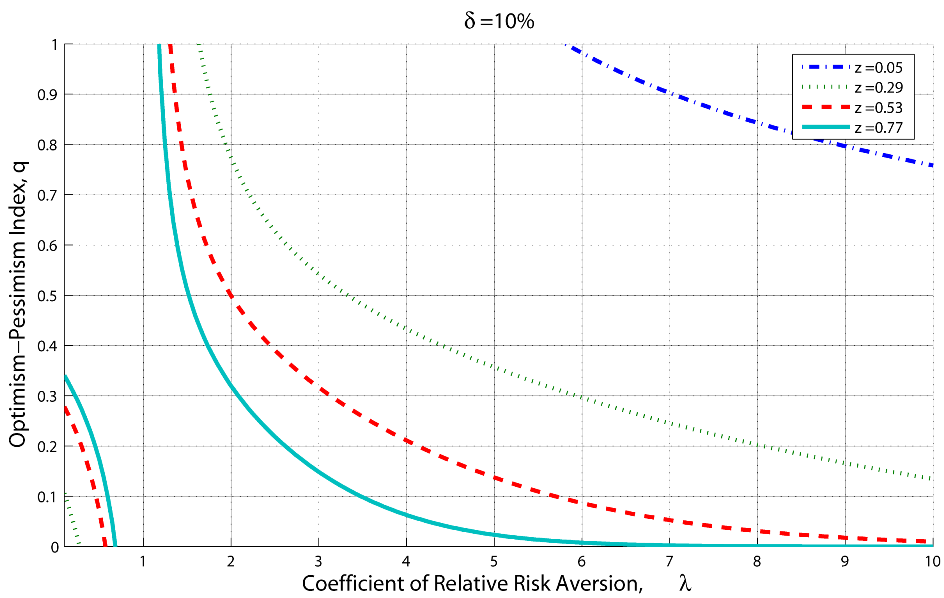

Figure 1.8: Optimism-pessimism index, q, vs. positive coefficients of risk aversion, λ, that predict quarterly consumption growth equal to its historical post-World War II average of 0.5% given negative quarterly real interest rate -0.06% and positive quarterly discount factor of 10%

1.5 Conclusion

Deaton [9],[10] reports that the U.S. post-World War II history includes more than 30 years when the aggregate consumption grew on average despite the average real interest remaining below zero. This fact goes against the predictions offered by a widely used family of rational expectation representative agent models. Responding to this puzzle, few theories suggest that precautionary motives and persistent subjectivity of the consumers with stochastic wealth constraints may account for this "excess consumption growth puzzle." This paper demonstrates that when the wealth constraint is purely stochastic, neither precautionary motives nor subjective optimism/pessimism and doubt can offer a satisfactory resolution to the puzzle. We propose a simple two-period model with fuzzy wealth constraint that, under reasonable parameterization, is consistent with simultaneously growing consumption and a negative real interest rate and therefore explains the puzzle.

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Chapter 2
Can Fuzzy Decision-Making Explain the Equity Premium Puzzle?

2.1 Introduction

Difficulties arise when the traditional intertemporal stochastic model with a rational representative agent is employed to explain the patterns of yields on various assets. One of the most prominent problems, the equity premium puzzle, is discovered by Mehra and Prescott [24], who show that, in order to reproduce the historical average excess return on equity over that of the risk-free asset by employing a conventional stochastic consumption model, one should allow the rates of risk aversion and time preference to take values beyond the conventionally assumed ranges.

While numerous attempts were made to resolve the equity premium puzzle, many were proven unsatisfactory along critical dimensions. The poor merit of the modern financial theory in explaining the statistics that describe the recent evolution of the U.S. financial markets is highlighted in [25], [18]. One promising direction is suggested by Abel [1], who shows, using a version of Lucas's fruit-tree model [20], that simultaneous presence of pessimism and doubt in the subjective probability distribution of the consumption growth may eliminate the equity premium puzzle[10].

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[10] Also see Cecchetti [7]

Unfortunately, subjectivity characterized by both pessimism and doubt is not found in data. Giordani and Sőderlind [12] analyze U.S. survey data between 1972 and 2003 and find that, while individual consumption forecasts are indeed pessimistic, they are rather overconfident than doubtful.

The observed trace of long-run subjective consumer pessimism in the data by itself goes against the rationality hypothesis and requires some further conceptual and theoretical motivation. In order to exhibit the nature of the conflict between persistent pessimism (or for that matter optimism) and rationality, consider the following simple example. A gambler, whose real endowment depends on a flip of a coin, is awarded by a monetary prize if won and nothing otherwise. The same gamble is repeated in every period of the individual's life. The entire award or a part of it can be consumed or saved for the forthcoming periods when the gambling outcomes turn unfavorable. The size of the award does not change from one period to another. It is obvious that within few trials the gambler will be able to learn the true stochastic properties of the endowment process, and her subjective pessimism or doubt regarding future outcomes shall no longer affect her saving/consumption decisions. Even if the coin is not fairly weighted, the gambler should rationally choose the current saving level to smooth out her lifetime consumption. Naturally, the amount of precautionary saving built by more risk averse individuals must be in excess of what their less risk averse counterparts save. This simple gamble exhibits the way that many rational expectation models generally treat uncertainty, and the neutrality of consumers' subjectivity applies to them as well.

The question then is what is missing in the stochastic rational expectations models that may justify the effect of investors' optimism-pessimism on their equilibrium choices? In our view, the answer to this question should be sought in the structure of uncertainty that consumers and investors face and in the way that the flow of information is processed by them. While in reality information necessary for objective forecasts is available, some elements of it are in scattered rather than concentrated form [15]. Hence, current information, I_t, at the decision maker's disposal can be presented as a sum of two elements:

$$I_t = S_t + C_t$$

where C_t and S_t are the portions of available information in concentrated and scattered form respectively. C_t can readily be used for objective statistical forecasts. S_t by itself, on the other hand, is quite useless for the purposes of statistical inference, but can be converted to C_t at some cost. The costly process of concentration of S_t into C_t may entail sizeable utility losses. At the same time, the utility loss from acting according to some reasonable rule of thumb rather than according to the optimal decision rule can be microscopic [8]. As a result, complete concentration of scattered information, S_t, may not be cost efficient and only obvious elements, C_t, are used to make objective statistical forecasts. Hence, continuous presence of S_t may be taken as an equilibrium outcome[11]. Traditional rational expectations models (such as [20]), however, disregard the presence of S_t and accommodate for C_t only. Nevertheless, in reality individuals do not discard elements of S_t that were too costly to convert into C_t. The bits of scattered information, S_t, are used to form fuzzy, often qualitative and linguistic, beliefs. It is these fuzzy beliefs due to S_t that may even in the long run be subjectively optimistic or pessimistic. Moreover, since in reality processing of S_t and therefore formation of fuzzy beliefs is a natural way of cultivating subjectivity, we conjecture that explicit incorporation of fuzzy decision making mechanism into the traditional settings is also a natural methodology of capturing observed pessimism.

There are many real life examples that support our reasoning. Participants of municipal or state lotteries, for instance, rarely know precisely what the probability is of winning and the exact amount of the disposable prize[12]. While this sort of fundamental information may usually be obtained from publicly available sources in its most precise form, many lottery participants rarely go through the trouble of obtaining it. Typically, they believe that the probability of winning is "small" or "close to zero," and the amount of the win is "large" or "around" some number. Statements such as "large win" and "small probability" are the type of beliefs being formed under scattered information, S_t. Quite interestingly, the same fuzzy belief about the lottery, such as "small probability" of winning, can be shared by both lottery participants (optimists) as well as non-participants (pessimists).

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11 The question of determination of the optimal mix of C_t and S_t by itself is an interesting question but is not in the scope of this paper.

12 The complicated provisions of the tax law may be one reason why the lottery participants may not be aware of the exact take home win.

The degree of blurriness of the linguistically defined values, such as "large win," "around zero," or "small probability," is what we define as fuzzy uncertainty or fuzziness. The notion of fuzzy uncertainty should not be confused with randomness or stochastic uncertainty, which arises when the occurrence of some outcome is subject to a chance. In reality, fuzzy uncertainty may be present even if the outcome is not random. Salaried workers, for example, who receive their paychecks with probability close to one often fail to quote the exact figure for their disposable annual income. Most commonly this figure is quoted by respondents as an approximation "around" some benchmark, round number. Clearly, what constitutes uncertainty in this case is the word "around" rather than the stochastic nature of the salary itself.

This paper addresses the equity premium puzzle by explicitly treating fuzzy uncertainty with respect to future income as an additional type of uncertainty in an infinite-horizon representative agent model. We extend the defuzzyfication methodology proposed by Stahlecker et al. [28] and Hauenschild and Stahlecker [14] to the case of a constant relative risk aversion felicity function and show that, when the future wealth is uncertain not only in stochastic but also in a fuzzy sense, then the equity premium puzzle can successfully be resolved.

The paper is organized as follows. Section 2.2 describes the data used in this study. Section 2.3 presents a version of Lucas's classical fruit-tree model [20], which serves as a baseline for the consequent modifications. Here we also introduce the equity premium puzzle. Section 2.4 incorporates the notion of fuzzy uncertainty into the baseline Lucas's model and demonstrates that the equity premium can be resolved if fuzzy uncertainty is properly accounted for. Subsection 2.4.1 introduces the notion of fuzzy uncertainty using a membership function and its alpha cuts, two important concepts in the mathematics of fuzzy sets. Subsection 2.4.2 presents the methodology of combining fuzzy uncertainty first with the notion of mathematical expectations (1) and second with the concept of expected utility (2). Subsection 2.4.3 presents the infinite-horizon utility maximization problem and its general equilibrium solution for asset prices in an economy with fuzzy beliefs and random consumption growth. Here we demonstrate analytically that the presence of fuzzy uncertainty and the prevailence of pessimistic views will reduce the return on the risk-free asset and increase the equity premium relative to the prediction of the baseline model. Subsection 2.4.4 develops the methodology and presents the results of the model's calibration to historical moments of consumption growth, the risk-free rate, and equity premium. This section demonstrates that the model with fuzzy uncertainty is capable of reproducing the long-run financial statistics without pushing deep parameters beyond the conventionally accepted ranges. Section 2.5 concludes.

2.2 Data

This study is based on three annual data series for the period between 1929 and 2007:

1. Nominal S&P 500 index and dividend yield on S&P 500. The monthly data series are produced and maintained by Robert Shiller as an online data

supplement for [27]. We annualize this monthly data by taking averages of monthly indexes.

2. Nominal risk-free interest rate. This is the annual average of monthly series of Fama risk-free rate compiled by the Center of Research in Security Prices (CRSP), which in its turn is based on average of bid-ask prices of three month U.S. treasuries. CRSP reports continuously compounded yields on U.S. treasuries. We convert continuously compounded yields to simple yields to ensure conformity with other data.

3. Nominal and real per capita annual consumption expenditures were obtained from National Income and Product Account (NIPA) tables published by the Bureau of Economic Analysis (BEA).

We obtain consumption deflator by dividing nominal consumption by real consumption. Real variables are consequently obtained by dividing the corresponding nominal magnitudes by this deflator. The real return on equity is obtained by summing the real capital gain and real dividend yield. We obtain the real gross ex-post risk-free rate by dividing the gross nominal rate by the gross forthcoming percent change in the consumption deflator. The real equity premia are obtained by subtracting the risk-free rate from the real return on equity. Table 2.1 presents the first and second moments of the annual data used in this paper and, for comparison purposes, data used by Mehra and Prescott [24]. Regardless of the sample period and the data

		Data Used by Mehra and Prescott		Refreshed Data	
		1889-1978	1929-1978	1929-1978	1929-2007
Growth of per capita real consumption (%)	Mean	1.83	1.70	2.05	1.99
	STD	3.59	2.84	2.61	2.16
Real return on relatively riskless security (%)	Mean	0.99	-0.58	-0.32	0.72
	STD	5.75	4.65	4.42	4.06
Real return on S&P 500 (%)	Mean	7.03	6.40	6.04	7.29
	STD	16.63	17.95	17.48	16.05
Risk Premium (%)	Mean	6.04	6.98	6.35	6.58
	STD	18.59	19.68	19.25	17.15

Table 2.1: Historical moments of real consumption growth, the real risk-free rate of interest, real return on equity, and risk premium between 1928 and 2008

used, the historical average real return on risk-free bonds is observed to be less than one percent, and the average annual real risk premium is between six and seven percent. The annual growth rate of real per capita consumption was around two percent. Calibration in this paper is made to the moments of the refreshed data set for the period between 1929 and 2007.

2.3 The Equity Premium Puzzle. The Model with Purely Stochastic Uncertainty and Objective Expectations.

In order to demonstrate the equity premium puzzle we start by considering a discreet version of Lucas's [20] fruit-tree economy, where a representative agent maximizes an endless sum of discounted utility by choosing consumption of fruits and holdings of shares of fruit bearing trees:

$$\max E[U_t] = \max_{c} E_t \left\{ \sum_{j=0}^{\infty} \beta^j u(C_{t-j}) \right\} \tag{2.1}$$

Here operator E_t represents period t expectations of the stochastic variables in the model, β is the discount factor or the representative agent's time preference, and C_t is the period t consumption. The felicity $u()$ is a concave and bounded function. In the conventional Lucas's setting, the trees produce fruits that serve as dividends for the owners of the tree shares. While the provision of future dividends is unknown, the representative agent is presumed to know the stochastic properties of the growth rate of random dividends. The maximization problem in (2.1) holds a solution that must obey the stream of Euler equations:

$$u'(C_t) = \beta E_t\{R_{t+1} u'(C_{t+1})\} \tag{2.2}$$

where R_{t+1} is the forthcoming gross return on any available asset. We, following convention, assume that the representative agent's felicity function can be represented by a constant relative risk aversion (CRRA) utility function

$$u(C_t) = (1-\gamma)^{-1} C_t^{1-\gamma} \tag{2.3}$$

where γ is the coefficient of relative risk aversion. We rewrite (2.2) by using (2.3) in the following more convenient form:

$$1 = \beta E_t\{R_{t+1}(\frac{C_{t+1}}{C_t})^{-\gamma}\} \tag{2.4}$$

Due to its generality, relation (2.4) can be applied to any asset. In order to apply Euler equation (2.11) to the risk-free bond, we use the fact that the risk-free rate is known ahead of time and rewrite (2.4) in the following form [13]:

$$R^f_{t+1} = \frac{1}{\beta E\{(\frac{C_{t+1}}{C_t})^{-\gamma}\}} \tag{2.5}$$

Gross equilibrium return on equity, R^e, can be obtained by first recognizing that:

$$R^e_{t+1} = \frac{P^e_{t+1} + D_t}{P^e_t} \tag{2.6}$$

where P^e_{t+1} and P^e_t are time $t+1$ and time t post-dividend prices of risky equity respectively, while D_t is the time t dividend payment made to the holders of equity. We assume that the economy is in equilibrium and therefore the entire output or income is paid out in dividends to shareholders ($Y_t = D_t$), and all the dividend income is consumed by the shareholders ($C_t = D_t$). One of the virtues that follow from this assumption is that the growth rate of dividends and the growth rate of consumption inherit the stochastic properties of the output growth. We assume that the growth rates of output, X_t, dividends, D_{t+1}/D_t, and consumption, C_{t+1}/C_t, are i.i.d. random variables and in equilibrium are equal to each other:

$$X_{t+1} = \frac{D_{t+1}}{D_t} = \frac{C_{t+1}}{C_t} \tag{2.7}$$

Because of this assumption and because the CRRA utility function is homogeneous, the post-dividend price of equity can be expressed as a linear function of the current dividend payment:

$$P^e_t = wD_t \tag{2.8}$$

[13] The time subscripts can be dropped from the expectations operator without loss of generality.

Substituting (2.8) into (2.6), we rewrite it in the following format:

$$R^e_{t+1} = \frac{w+1}{w} X_{t+1} \tag{2.9}$$

We further substitute (2.9) into (2.4) and keep in mind (2.7):

$$1 = \beta E_t \{ \frac{w+1}{w} X^{1-\gamma}_{t+1} \} \tag{2.10}$$

which implies that

$$\frac{w+1}{w} = \frac{1}{\beta E_t \{ X^{1-\gamma}_{t+1} \}} \tag{2.11}$$

We finally substitute (2.11) into (2.9) and take expectations to obtain:

$$E\{R^e\} = \frac{E\{X_{t+1}\}}{\beta E\{X^{1-\gamma}_{t+1}\}} \tag{2.12}$$

The expected premium on equity over the return on the risk-free assets then can be written in a fractional form:

$$\frac{E\{R^e\}}{R^f} = \frac{E\{X_{t+1}\} E\{X^{-\gamma}_{t+1}\}}{E\{X^{1-\gamma}_{t+1}\}} \tag{2.13}$$

In order to demonstrate that the model and the results given by (2.13) cannot be used to generate accurate predictions about the long-run average equity premium and the risk free rate, we make another simplifying assumption that the gross rate of growth of consumption, X_t, is a log-normally distributed random variable[14]. The log-normality assumption is frequently encountered in the related literature and implies that the k^{th} moment of the log-normally distributed random variable X can be written as:

$$E\{X^k_{t+1}\} = \exp[k\mu + \frac{1}{2} k^2 \sigma^2]$$

[14] If the random variable $V = \ln(X)$ is normally distributed then the random variable X itself is log-normally distributed.

where $\mu = E\{\ln(X_t)\}$ and $\sigma^2 = Var\{\ln(X_t)\}$. Using this result as well as (2.13), the log of the gross equity premium, $\ln(\frac{E\{R^e\}}{R^f}) = \eta$, can be written as:

$$\eta = \gamma\sigma^2 \tag{2.14}$$

The above equation represents the model's prediction about the equilibrium log equity premium, η, for given variance of the log consumption growth, σ^2, and the coefficient of risk aversion, γ. In reality, however, individual risk preferences measured by γ cannot easily be quantified from non-experimental and aggregate data[15]. A useful way of evaluating (2.14) is to ask the question that was first raised by Mehra and Prescott [24] twenty-five years ago: what is the coefficient of risk aversion, γ^*, implied by the model and therefore by (2.14), if one assumes that the long-run sample based estimate of the variance of consumption growth, $\overline{\sigma}^2$, is an unbiased estimator of the true variance σ^2, and that the long-run sample average equity premium, $\overline{\eta}$, is representative of the underlying true equilibrium value of η. The answer to that question can be obtained by solving (2.14) for η and replacing all the parameters with their respective sample based estimates.

$$\gamma^* = \frac{\overline{\eta}}{\overline{\sigma}^2}$$

[15] γ is required to be greater than zero in order to ensure concavity of the felicity function. Estimates of γ vary in literature. Altug [3] finds it to be near zero. Mankiw, Rotemberg, and Summers [23] establish that it is near 0.5. Kehoe [16], and Hansen and Singleton [13] estimate γ to be around 1. Tobin and Dolde's [29] estimate is 1.5. Friend and Blume [11] found it to be around 2. Zeldes's [30] estimates of γ are in the region of 2.3. Mankiw [21], [22] 4, and 3 respectively. Barski et al. [5] explore the results of a survey and find that the median coefficient of relative risk aversion is approximately 7. Levy et al. [19] survey the related empirical and experimental literature and conclude that relative risk aversion parameter, γ, is in the range between 0.6 and 2.

Annual data from 1929 to 2007 imply that $\overline{\eta} = 0.0632$ and $\overline{\sigma}^2 = 0.000448$[16]. It follows that $\gamma^* = 140.92$. This value cannot be treated as anything acceptable since any value of the coefficient of risk aversion exceeding 20 would imply virtually zero marginal utility of any additional consumption and, therefore, cannot be perceived as plausible.

Similarly, we may be interested in what the value is of the discount factor, β^*, that reconciles the model in this section with sample moments implied by the data. The answer to this question follows from the expression for the equilibrium log risk-free rate, r^f, which can be obtained from (2.5).

$$r^f = -\ln(\beta) + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 \tag{2.15}$$

Solving (2.15) for β, and replacing the right-hand side parameters with their sample based estimates would result in the following expression:

$$\beta^* = \exp(\gamma^*\overline{\mu} - \frac{1}{2}(\gamma^*)^2\overline{\sigma}^2 - \overline{r}^f) \tag{2.16}$$

which in its turn signifies that the implied coefficient of time preference $\beta^* = 0.1865$ [17]. Intuitively, under the logarithmic utility function ($\gamma = 1$) and zero interest rate, the coefficient of time preference can be interpreted as the maximum amount of current consumption that in equilibrium one would be willing to sacrifice for one extra

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16 From Table 2.1, $\overline{r}^f = \ln(1.0072) = 0.0072$, therefore $\overline{\eta} = \ln(1.0729/1.0072) = 0.0632$. Also due to log-normality, $\frac{Var(X)}{(E(X))^2} = (\exp(\overline{\sigma}^2) - 1)$
$\Longrightarrow \overline{\sigma}^2 = \ln(\frac{Var(X)}{(E(X))^2} + 1) = \ln(\frac{(0.0216)^2}{(1.0199)^2} + 1) = 0.000448$

17 Due to log-normality $E(X) = \exp(\mu + \frac{1}{2}\sigma^2), \mu = \ln(E(X)) - \frac{1}{2}\sigma^2 \Longrightarrow \overline{\mu} = \ln(1.0199) - \frac{1}{2}0.000448 = 0.0197$.

unit of next period's consumption. Hence, $\beta^* = 0.1865$ implies that only if offered more than 5.36 units of consumption[18] at the beginning of the next year would the consumer be willing to sacrifice one unit of present consumption. While nothing in theory precludes us from using such a low discount factor as acceptable, it is rarely encountered in reality.

This overall inability of the model to replicate the long-run first moments of the equity premium and the risk free rate within reasonable parameter ranges constitutes what Mehra and Prescott [24] infamously coined as the equity premium puzzle.

2.4 The Model with Fuzzy and Stochastic Uncertainty

2.4.1 Modeling Fuzzy Uncertainty. The Membership Function

We assume that only concentrated and obvious elements of the information at the agents' disposal are used for objective statistical forecast of the future. The remaining portion is overly scattered and complex for statistical forecasts. Investors do not, however, dispose of these scattered pieces of information. They rely on their possibly irrational senses and instincts to process it into fuzzy subjective beliefs. The objective statistical forecasts are consequently combined with these fuzzy subjective beliefs into somewhat subjective projections about the variables of interest.

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[18] $\frac{1}{0.1865} = 5.36$

It is worthwhile to note that the formation of fuzzy beliefs may occur even if the outcomes are not random. It is easy to imagine a situation, for instance, when the payoff of some investment project is not random, yet some imperfections in the flow of information prevent the investor from having complete knowledge about it. In this case the investor may believe that the return on this investment is bound to be around some Y percent. This kind of belief has two noteworthy characteristics: first, the word "around" prevents it from being classified as certain; and second, the investor may not be attempting to clarify this uncertainty any further if the cost of concentrating or crispening this knowledge exceeds the potential improvement in her utility. We model the type of uncertainty that exists about the linguistically defined outcomes such as "around Y percent" by means of fuzzy sets.

Definition 1 *A fuzzy set on a classical set X is defined as follows:*

$$\widetilde{A} = \{(x, \mu(x))|x \in X\}$$

where $0 \leq \mu(x) \leq 1$ is the membership function that quantifies the grade of membership of the elements x on X to the fuzzy set $\widetilde{A}$ by mapping possible values of X into the interval $[0, 1]$.

Any particular return on investment is partially or entirely classified as a member of the fuzzy set linguistically defined as "around Y percent" when the degree of membership μ to the fuzzy set "around Y percent" is non-zero. For instance, when the actual return is 9.95 percent, then the degree of membership of this outcome to the fuzzy set "around 10 percent" must be close to one, hence $\mu(9.95) \approx 1$. When the return on investment is 0.5 percent, on the other hand, it can hardly be described as belonging to the fuzzy set "around 10 percent." It follows that the degree of membership of 0.5 percent return to the fuzzy set "around 10 percent," $\mu(0.5\%)$, is equal or very close to zero. We assume that the membership function that defines the degree to which the payoff from investment Y belongs to the fuzzy set linguistically described as "around $\widehat{Y}$"[19] has the following form:

$$\mu(Y) = \begin{cases} \dfrac{Y-(1-z)\widehat{Y}}{z\widehat{Y}} & \text{if } (1-z)\widehat{Y} \leq Y \leq \widehat{Y} \\ \dfrac{(1+z)\widehat{Y}-Y}{z\widehat{Y}} & \text{if } \widehat{Y} \leq Y \leq (1+z)\widehat{Y} \\ 0 & \text{otherwise} \end{cases} \tag{2.17}$$

This type of membership functions, frequently referred to as triangular, is fairly general and commonly used in the related literature[20]. Figure 2.1 exhibits a family of membership functions given by (2.17) for few different values of z and $\widehat{Y} = 100\%$. The choice of this particular membership function is due to its simplicity and of computational convenience. Any related membership function would ultimately imply the same set of results that we obtain in this paper. The term $z \in [0, 1]$ in (2.17) is a parameter that describes the fractional degree of fuzziness and reflects the amount of diluted and unconcentrated information at an agent's disposal. If $z = z^*$, then any particular level of income, Y, that falls within the range $\widehat{Y} \pm z^*\widehat{Y}$ can be considered as belonging to the fuzzy set "around $\widehat{Y}$" and the degree of membership of Y

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19 Very often we refer to variables marked by hats, such as $\widehat{Y}$, as benchmark values. Note that there is nothing uncertain about the benchmark payoff, $\widehat{Y}$ when fuzzines is the only source of uncertainty .

20 For an introductory exposure on fuzzy-set theory and the related literature refer to [26]

to this fuzzy set is given by the membership $\mu(Y)$ in (2.17). Naturally, the smaller z, the smaller is the range that contains the members of the fuzzy set "around $\widehat{Y}$." In a limiting case as z approaches to 0, fuzzy uncertainty disappears, and the fuzzy set "around $\widehat{Y}$" contains only one element, $\widehat{Y}$ itself. Stated in more general terms, this convenient property implies that non-fuzzy sets can be regarded as a particular limiting case of fuzzy sets at $z \to 0$. As we demonstrate below, any result in this paper that reflects the presence of fuzzy uncertainty holds the corresponding result obtained under conventional, non-fuzzy settings as a particular case at z approaching to 0.

Definition 2 *An $\alpha - cut$ of a fuzzy set $\widetilde{A}$ with a membership function μ for given $\alpha \in [0, 1]$ is defined by the following crisp set:*

$$\{x \in \mathbb{R} | \mu(x) \geq \alpha\} \tag{2.18}$$

Intuitively, an $\alpha - cut$ of a given membership function at given α is a set of all points along the horizontal axis under the membership function for which the value of the membership function is greater than α. It can be shown that the upper and lower bounds of an $\alpha - cut$ of membership function (2.17) are given by:

$$\widehat{Y} \pm \widehat{Y} z(1 - \alpha), \;\; \forall \alpha \in [0, 1] \tag{2.19}$$

2.4.2 Combining Fuzzy and Stochastic Uncertainty

Before we proceed with the exhibit of the formal model, it is necessary to lay down what the fuzzy equivalents of the two important notions are: the expected value and

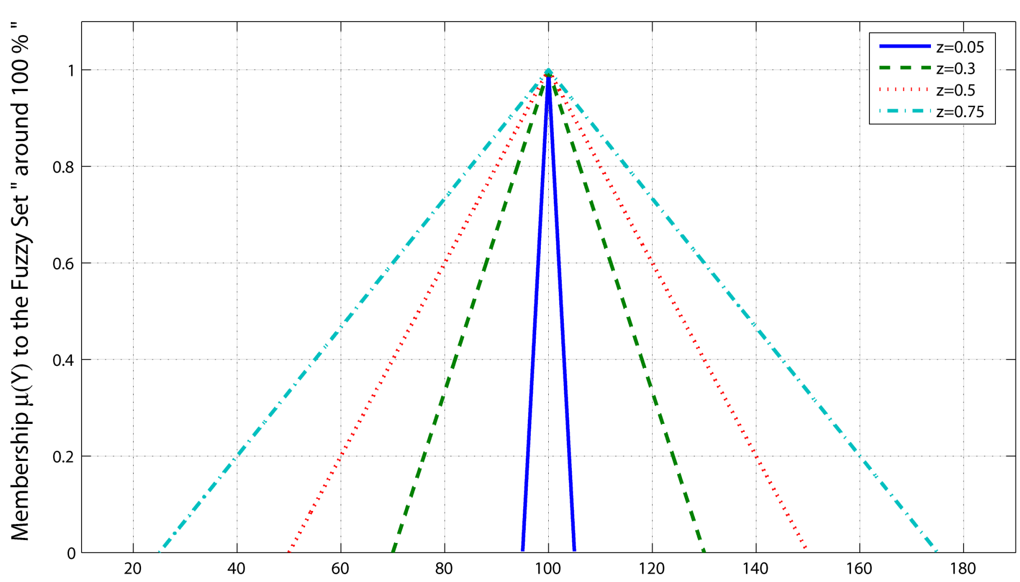

Figure 2.1: The degree of membership of given return on investment, $Y\%$, to the linguistically defined fuzzy set "around $\widehat{Y}$ percent" when the membership function is defined as in (2.17), and $\widehat{Y} = 100\%$

the expected utility. We would like to define mathematically what the expected value is of the payoff "around $\widehat{Y}_{t+k}$" when $\widehat{Y}_{t+k}$ itself is a random variable. Moreover, we need to formalize the notion of the expected utility of the payoff "around (again random) $\widehat{Y}_{t+k}$." In developing these two fundamental concepts in the following two subsections respectively, we follow the strategy of first treating $\widehat{Y}_{t+k}$ as a known, non-random magnitude while the notion of fuzziness around $\widehat{Y}_{t+k}$ is introduced. Consequently, when the fuzzy uncertainty is formalized, we relax the assumption of non-randomness of $\widehat{Y}_{t+k}$. As an outcome of this technique, we hope to capture the conjectured above human tendency of combining objective statistical forecasts with subjective fuzzy beliefs.

Expectations of Fuzzy as Well as Random Income

In order to introduce fuzziness in its pure form[21] we first conjecture that the forthcoming payoff from investment is not random. Yet investors may form fuzzy beliefs about the return on investment when it is impossible or excessively costly to evaluate some of the diluted and scattered information. For the same fuzzy belief about the forthcoming return on investment "around $\widehat{Y}_{t+1}$" optimistic individuals would lean towards the payoffs above $\widehat{Y}_{t+1}$, while pessimistic individuals towards payoffs bellow $\widehat{Y}_{t+1}$. Given the upper and lower bounds of an α-cut in (2.19), extremely optimistic individuals will believe that the forthcoming payoff is going to be

[21] The methodology developed in this and the following subsections was introduced by Stahlecker et al. [28] and Hauenschild and Stahlecker [14] for the case of a quadratic utility function.

equal to the upper bound in (2.19)[22]:

$$\widetilde{Y}^{o}_{t+1} = \widehat{Y}_{t+1} + \widehat{Y}_{t+1} z(1-\alpha)$$

Conversely, the lower bound in (2.19) is what extremely pessimistic individuals will forecast their $t+1$ period's return on investment to be:

$$\widetilde{Y}^{p}_{t+1} = \widehat{Y}_{t+1} - \widehat{Y}_{t+1} z(1-\alpha)$$

Fuzzy beliefs about forthcoming return on investment formed by more centric individuals given an α-cut can be written as a weighted average of extremely optimistic and pessimistic views:

$$\begin{aligned}
\widetilde{Y}_{t+1} &= q\widetilde{Y}^{p}_{t+1} + (1-q)\widetilde{Y}^{o}_{t+1} \\
&= q(\widehat{Y}_{t+1} - \widehat{Y}_{t+1} z(1-\alpha)) + (1-q)(\widehat{Y}_{t+1} + \widehat{Y}_{t+1} z(1-\alpha)) \\
&= \widehat{Y}_{t+1} + (1-2q)(1-\alpha) z \widehat{Y}_{t+1} \qquad (2.20)
\end{aligned}$$

Here $q \in [0,1]$ is the consumers' or investors' optimism index, with $q = 0$ representing extremely optimistic and $q = 1$ extremely pessimistic views. By aggregating (2.20) over all possible $\alpha \in [0,1]$, we obtain an expression for fuzzy income antici-

22 In this paper tildes over symbols denote presence of fuzzy uncertainty

pated by an individual with optimism-pessimism measured by q:

$$\begin{aligned}
\widetilde{Y}_{t+1} &= \int_0^1 (\widehat{Y}_{t+1} + (1-2q)(1-\alpha)z\widehat{Y}_{t+1})\, d\alpha & (2.21)\\
&= \widehat{Y}_{t+1} + (1-2q)z\widehat{Y}_{t+1}\int_0^1 (1-\alpha)d\alpha \\
&= \widehat{Y}_{t+1} + (1-2q)\frac{z}{2}\widehat{Y}_{t+1} \\
&= (1+(1-2q)\frac{z}{2})\widehat{Y}_{t+1} & (2.22)\\
&= \delta\widehat{Y}_{t+1} & (2.23)
\end{aligned}$$

Note that the fuzzy income is perceived to be less (more) than the benchmark value $\widehat{Y}_{t+1}$ when the decision maker is pessimistic (optimistic), hence when $0.5 < q \leq 1$ ($0 \leq q < 0.5$).

Expression (2.23) is the crisp or defuzzified version of the fuzzy belief about forthcoming income. Interestingly, (2.23) is somewhat equivalent to mathematical expectations when the variable of interest is random rather than fuzzy. What distinguishes (2.22) and (2.23) from conventional measures of expected value, however, is the presence of the exogenously determined optimism-pessimism index q that may bias decision makers perception about her future income.

We now introduce randomness into the analysis by conjecturing that the benchmark income $\widehat{Y}_{t+1}$ itself may be a random variable with a given c.d.f. $F(\widehat{Y}_{t+1})$. It follows that that:

$$\begin{aligned}
E[\widetilde{Y}_{t+1}] &= (1+(1-2q)\frac{z}{2})E[\widehat{Y}_{t+1}] \\
&= \delta E[\widehat{Y}_{t+1}]
\end{aligned}$$

where

$$\delta = 1 + (1 - 2q)\frac{z}{2} \tag{2.24}$$

Obviously, $\lim_{z \longrightarrow 0} \delta = 1$, hence the conventional, non-fuzzy alternative of hitherto analysis can be modeled as a limiting case of the setup with fuzziness present. It is also worth mentioning that δ is less than one in the presence of fuzziness ($z \neq 0$) whenever the representative agent is pessimistic about future payoffs ($q > 0.5$).

Expected Utility of Fuzzy as Well as Random Income

This subsection is devoted to the mathematical foundation of the concept of expected utility from fuzzy payoffs. Similar to the purely stochastic representation, utility of defuzzified income is not equal to the defuzzified utility of fuzzy income, hence $u(\widetilde{Y}) \neq \widetilde{u}(Y)$ [23]. Similar to our analysis in the previous section, we will first assume that the benchmark payoff $\widehat{Y}_{t+1}$ is a known constant magnitude. Consequently, we will relax this assumption and allow it to be a random magnitude with a given probability distribution.

For an optimistic individual the utility of some payoff "around (known) $\widehat{Y}_{t+1}$" for a given $\alpha - cut$ at any $\alpha \in [0, 1]$ can be presented as:

$$\widetilde{u}^o = u((1 + (1 - \alpha)z)\widehat{Y}_{t+1}) \tag{2.25}$$

[23] In a purely stochastic case under non-linear utility function, utility of expected income is not equal to expected utility, hence $u(E(y)) \neq E(u(y))$

where as above $u(\ldots)$ is a strictly concave and bounded function. Similarly, anticipated utility as perceived by a pessimistic individual is written as:

$$\widetilde{u}^p = u((1-(1-\alpha)z)\widehat{Y}_{t+1}) \tag{2.26}$$

Anticipated utility of an individual whose optimism is somewhere in between of these two extremes can be presented as a weighted average of these two extremes with the weight again being the optimism-pessimism index:

$$\begin{aligned}
\widetilde{u}(\widehat{Y}_{t+1}) &= q\widetilde{u}^p + (1-q)\widetilde{u}^o \\
&= qu((1-(1-\alpha)z)\widehat{Y}_{t+1}) \\
&\quad +(1-q)u((1+(1-\alpha)z)\widehat{Y}_{t+1})
\end{aligned}$$

We aggregate the above over all possible alpha cuts at $\alpha \in [0,1]$:

$$\begin{aligned}
\widetilde{u}(\widehat{Y}_{t+1}) &= q\int_0^1 u((1-(1-\alpha)z)\widehat{Y}_{t+1})d\alpha \\
&\quad +(1-q)\int_0^1 u((1+(1-\alpha)z)\widehat{Y}_{t+1})d\alpha
\end{aligned} \tag{2.27}$$

If the utility function is homogeneous of degree n, then (2.27) can be written as

$$\begin{aligned}
\widetilde{u}(\widehat{Y}_{t+1}) &= u(\widehat{Y}_{t+1})[q\int_0^1 (1-(1-\alpha)z)^n d\alpha \\
&\quad +(1-q)\int_0^1 (1+(1-\alpha)z)^n d\alpha]
\end{aligned} \tag{2.28}$$

If, for example, the utility function is of CRRA form

$$u(Y) = \frac{Y^{1-\gamma}}{1-\gamma} \tag{2.29}$$

and, therefore, is homogeneous of degree $1-\gamma$, with γ being the coefficient of risk aversion, then the utility of income "around $\widehat{Y}_{t+1}$" can be written as:

$$\begin{aligned}
\widetilde{u}(\widehat{Y}_{t+1}) &= q\int_0^1 \frac{((1-z(1-\alpha))\widehat{Y}_{t+1})^{1-\gamma}}{1-\gamma}d\alpha \\
&\quad +(1-q)\int_0^1 \frac{((1+z(1-\alpha))\widehat{Y}_{t+1})^{1-\gamma}}{1-\gamma}d\alpha \\
&= \frac{(\widehat{Y}_{t+1})^{1-\gamma}}{1-\gamma}[q\int_0^1 (1-z(1-\alpha))^{1-\gamma}d\alpha \\
&\quad +(1-q)\int_0^1 (1+z(1-\alpha))^{1-\gamma}d\alpha]
\end{aligned}$$

The above can be further simplified:

$$\widetilde{u}(\widehat{Y}_{t+1}) = \theta\frac{(\widehat{Y}_{t+1})^{1-\gamma}}{1-\gamma} = \theta u(\widehat{Y}_{t+1}) \tag{2.30}$$

where

$$\theta = \frac{(2q-1) - q(1-z)^{2-\gamma} + (1-q)(1+z)^{2-\gamma}}{(2-\gamma)z} \tag{2.31}$$

Parameter θ in (2.31) conveniently summarizes the differential effect of optimism-pessimism, q, fuzzy uncertainty, z, and risk aversion, γ, on individual utility. Utility of a payoff "around $\widehat{Y}_{t+1}$" is different from utility of a payoff precisely $\widehat{Y}_{t+1}$ only by a multiplicative parameter, θ. Figure 2.2 exhibits a contour plot of the functional dependence of θ on the coefficient of optimism-pessimism, q, the coefficient of relative risk aversion, γ, and the fuzziness parameter, z. It is apparent from the graph that

θ increases above unity for sufficiently pessimistic and risk averse investors. This becomes more pronounced when fuzziness expressed in terms of z becomes larger (graph on the right for $z = 0.2$). Since $\theta > 1$ for pessimistic and risk averse individuals, it may be tempting to conclude from (2.30) that, for a risk averse and pessimist investor, utility of a payoff "around $\widehat{Y}_{t+1}$" is greater than utility of $\widehat{Y}_{t+1}$. It is not, however, the case since the CRRA utility function (2.29) spans over the negative range for the coefficients of risk aversion greater than one. Hence, for any $\gamma > 1$, utility of a payoff "around $\widehat{Y}_{t+1}$" is less rather than greater than non-fuzzy utility of $\widehat{Y}_{t+1}$, $\widetilde{u}(\widehat{Y}_{t+1}) < u(\widehat{Y}_{t+1})$.

Again, it is important to notice that the $\lim_{z \to 0} \theta = 1$; hence, the non-fuzzy alternative of the model can be build as a limiting case of the fuzzy representation here

One important question that should be considered is how fuzziness propagates as the projection horizon expands. We assume that the information about outcomes more distant in the future must be more scattered than that of the events on the immediate horizon. This assumption implies that fuzzy uncertainty must expand as the projection horizon increases. We assume that fuzzy uncertainty, summarized by coefficient θ, expands exponentially with the horizon. Hence

$$\widetilde{u}(\widehat{Y}_{t+k}) = \theta^k u(\widehat{Y}_{t+k}) \tag{2.32}$$

In order to introduce random uncertainty, we again assume that the benchmark level of income $\widehat{Y}_{t+k}$ by itself is stochastic with a given probability distribution

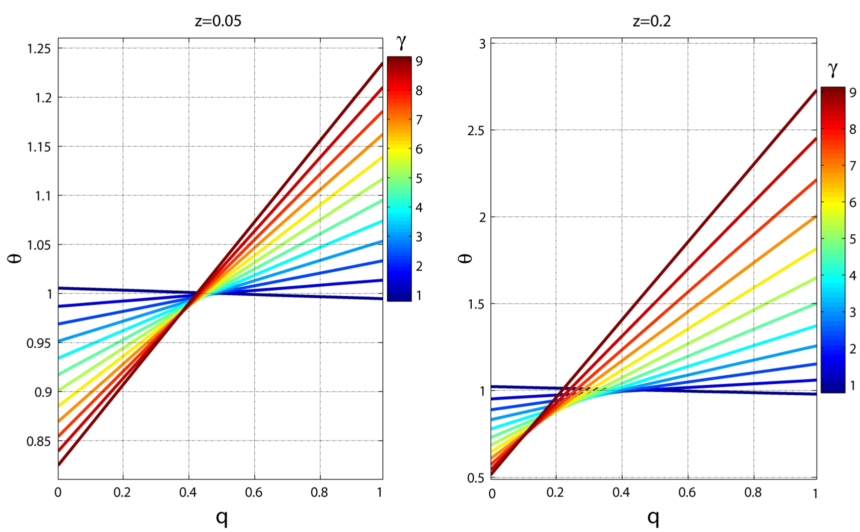

Figure 2.2: Graphical representation of the functional dependence of θ on q, γ, and z given by (2.31)

$F(\widehat{Y}_{t+k})$. Expression (2.32) implies that, for any homogeneous utility function, the expected utility of a payoff "around random $\widehat{Y}_{t+k}$" is given by

$$E[\widetilde{u}(\widehat{Y}_{t+k})] = \theta^k E[u(\widehat{Y}_{t+k})] \tag{2.33}$$

2.4.3 The Model with CRRA Utility Function

At any given time an investor with a CRRA utility function solves the following dynamic stochastic optimization problem:

$$\max E[\widetilde{u}_t] = \max_c E\left\{\frac{1}{1-\gamma}\sum_{k=0}^{\infty}(\beta\theta)^k \widehat{C}_{t+k}^{1-\gamma}\right\} \tag{2.34}$$

where β is the rate of time preference, θ is the fuzzy factor given by (2.31), and γ is the coefficient of risk aversion.

One reason for concern may be the fact that for pessimistic and risk averse investors who do not discount the future heavily the product of β and θ in (2.34) may be greater than unity. As it is shown by Kocherlakota [17], however, well-defined competitive equilibria with positive interest rates may exist in infinite horizon maximization problems such as (2.34), even with a discount factor (in our case $\beta\theta$) greater than one.

In general equilibrium settings, the growth rate of consumption inherits the stochastic properties of the output and dividend process, hence $\widehat{X}_{t+1} = \frac{\widehat{D}_{t+1}}{D_t} = \frac{\widehat{C}_{t+1}}{C_t}$[24]. The consumer's/investor's maximization problem in (2.34) possesses a set of

[24] There is no contextual but only notational difference between $\widehat{X}$ in this section and X in Section 2.3.

solutions that obey the following Euler condition:

$$1 = E_t[R_t \beta \theta \widehat{X}_{t+1}^{-\gamma}] \tag{2.35}$$

where R_t again is the gross return on the asset under consideration.

We first apply pricing equation (2.35) to the risk-free asset. Using the fact that the return on the risk-free asset is known a priory, we can write:

$$\widetilde{R}^f = \frac{1}{\beta \theta E[\widehat{X}_{t+1}^{-\gamma}]} \tag{2.36}$$

where $\widetilde{R}^f$ is the return on the risk-free asset in the presence of fuzzy uncertainty. By taking logs on both sides of (2.36), we obtain the expression for the net risk-free rate in the presence of fuzzy uncertainty

$$\widetilde{r}^f = r^f - \ln(\theta) \tag{2.37}$$

where lower-case r^f denotes the net risk-free rate without fuzziness and can be obtained by taking logs on both sides of (2.5). It is clear from (2.37) that the risk free-rate in the presence of fuzzy uncertainty ($z \neq 0$) will be less than its non-fuzzy counterpart whenever θ is greater than one. As it was noted above, $\theta > 1$ whenever the representative consumer is pessimistic and risk-averse.

The Euler equation for risky equity in the presence of fuzzy uncertainty is given by:

$$1 = E_t[\widetilde{R}_{t+1}^e \beta \theta \widehat{X}_{t+1}^{-\gamma}] \tag{2.38}$$

where $\widetilde{R}^e$ is the gross one-period return on risky equity. We obtain the expression for the equilibrium gross rate of return on risky equity by again acknowledging that the ex-dividend return on a risky asset is given by the sum of its capital gain and dividend yields.

$$R^e_{t+1} = \frac{P^e_{t+1} + D_{t+1}}{P^e_t} \tag{2.39}$$

It can be shown, that when the dividend series are independent and the utility function is homogeneous, then the equilibrium price of the equity security can be presented in the form:

$$P^e_t = \omega D_t \tag{2.40}$$

Substituting (2.40) into (2.39), we get:

$$R^e_{t+1} = \frac{\omega + 1}{\omega} \frac{D_{t+1}}{D_t} \tag{2.41}$$

We have to recognize that the forthcoming dividend is fuzzy in order to convert expression (2.41) to an equivalent with fuzzy uncertainty present. Hence,

$$\widetilde{R}^e_{t+1} = \frac{\omega + 1}{\omega} \frac{\widetilde{D}_{t+1}}{D_t} \tag{2.42}$$

By using (2.23), we can rewrite (2.42) as:

$$\widetilde{R}^e_{t+1} = \frac{\omega + 1}{\omega} \delta \frac{\widehat{D}_{t+1}}{D_t} \tag{2.43}$$

where δ is given by (2.24).We substitute (2.43) into (2.38) and solve for $\dfrac{\omega + 1}{\omega}$:

$$\frac{\omega + 1}{\omega} = \frac{1}{\beta \delta \theta E_t[\widehat{X}^{1-\gamma}_{t+1}]} \tag{2.44}$$

Substitution of (2.44) into (2.41) produces the expression for the return on the equity asset:

$$\widetilde{R}^e = \frac{\widehat{X}_{t+1}}{\beta \delta \theta E_t[\widehat{X}^{1-\gamma}_{t+1}]} \tag{2.45}$$

Expected return on equity is given by:

$$E[\widetilde{R}^e] = \frac{E[\widehat{X}_{t+1}]}{\beta\delta\theta E[\widehat{X}_{t+1}^{1-\gamma}]} \tag{2.46}$$

Finally the equity premium in the ratio form is obtained by dividing (2.46) by (2.36):

$$\frac{E[\widetilde{R}^e]}{\widetilde{R}^f} = \frac{E[\widehat{X}_{t+1}]E[\widehat{X}_{t+1}^{-\gamma}]}{\delta E[\widehat{X}_{t+1}^{1-\gamma}]} \tag{2.47}$$

By taking logs on both sides of (2.47), we rewrite the net equity premium formula in the following form:

$$\widetilde{\eta} = \eta - \ln(\delta) \tag{2.48}$$

where $\widetilde{\eta}$ and η are the net equity premiums in the presence and absence of fuzzy uncertainty respectively, and δ is given by (2.24). We reiterate that $\delta < 1$; hence $\ln(\delta) < 0$, whenever the representative agent is pessimistic ($q > 0.5$). In that case the predicted equity premium with fuzziness will be greater than without.

In sum, for a pessimistic representative investor introduction of fuzzy uncertainty will imply larger predicted equity premium and lower predicted risk-free rate relative to what the baseline model predicts.

2.4.4 Calibrating the Model

In order to calibrate the model developed in Subsection 2.4.3 to the long-run sample averages, we again make a simplifying assumption that the gross rate of consumption growth is log-normally distributed. This assumption is very common in the related literature. The expected equity premium in (2.47) can further be evaluated as:

$$\begin{aligned} \frac{E[\widetilde{R}^e]}{\widetilde{R}^f} &= \frac{e^{\mu+\frac{1}{2}\sigma^2}e^{-\gamma\mu+\frac{1}{2}\gamma^2\sigma^2}}{\delta e^{(1-\gamma)\mu+\frac{1}{2}(1-\gamma)^2\sigma^2}} \\ &= \frac{1}{\delta}e^{\mu(1-\gamma)+\frac{1}{2}(1+\gamma^2)\sigma^2-\mu(1-\gamma)-\frac{1}{2}(1-\gamma)^2\sigma^2} \\ &= \frac{e^{\sigma^2\gamma}}{\delta} \end{aligned} \tag{2.49}$$

which amounts to the net equity premium:

$$\widetilde{\eta} = \sigma^2\gamma - \ln(\delta) \tag{2.50}$$

or

$$\widetilde{\eta} = \sigma^2\gamma - \ln(1 + (1-2q)\frac{z}{2})$$
$$\ln(1 + (1-2q)\frac{z}{2}) = \sigma^2\gamma - \widetilde{\eta}$$

The above can be solved for the optimism-pessimism index, q, which would calibrate the model for given γ^*, z^*, and historical average equity premium $\overline{\eta}$ and the variance of log consumption growth $\overline{\sigma}^2$:

$$q^* = \frac{1}{2} - \frac{(e^{\overline{\sigma}^2\gamma^*-\overline{\eta}} - 1)}{z^*}$$

As a result, we obtain the optimism-pessimism index implied by the model, and $\overline{\sigma}$, $\overline{\eta}$, γ^*, and z^* Once the implied coefficient of optimism-pessimism, q^*, is obtained, we use it along with γ^* and z^* to generate θ^* from (2.31):

$$\theta^* = \frac{(2q^*-1) - q^*(1-z^*)^{2-\gamma^*} + (1-q^*)(1+z^*)^{2-\gamma^*}}{(2-\gamma^*)z^*} \tag{2.51}$$

Lastly, to calibrate the coefficient of time preference, β, we use the assumption of log-normality of consumption growth to rewrite the equilibrium risk-free rate in (2.36) in the following form:

$$\widetilde{R}^f = \frac{1}{\beta\theta e^{-\gamma\mu+\frac{1}{2}\gamma^2\sigma^2}} \tag{2.52}$$

By taking logs, on both sides of (2.52) we get to the net risk-free rate:

$$\widetilde{r}^f = \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 - \ln(\beta) - \ln(\theta) \tag{2.53}$$

which can be solved for β

$$\beta = e^{\gamma\mu-\frac{1}{2}\gamma^2\sigma^2-\overline{r}^f-\ln(\theta)} \tag{2.54}$$

Substituting $\gamma = \gamma^*$and $\theta = \theta^*$ from (2.51), and the sample based moments of log consumption growth, $\overline{\mu}$, and, $\overline{\sigma}^2$, as well as the sample based average risk-free rate, $\overline{r}^f$, into the right-hand side of (2.54) will result in the implied value of β^*.

$$\beta^* = e^{\gamma^*\overline{\mu}-\frac{1}{2}(\gamma^*)^2\overline{\sigma}^2-\overline{r}^f-\ln(\theta^*)}$$

The isosurface of the calibration results is presented in Figure 2.3. As it can be quickly inferred from the graph, there are some acceptable ranges of the deep parameters, γ, q, and β, as well as non-significant values of fractional fuzziness, z, that successfully calibrate the model to the historical average financial statistics presented in Table 2.1. For instance, the combination of the coefficient of risk aversion $\gamma = 1.37$, optimism-pessimism index $q = 0.96$, discount factor $\beta = 0.99$, and fractional fuzziness, $z = 0.13$ would result in 0.72% risk-free rate and risk premium 6.32%.

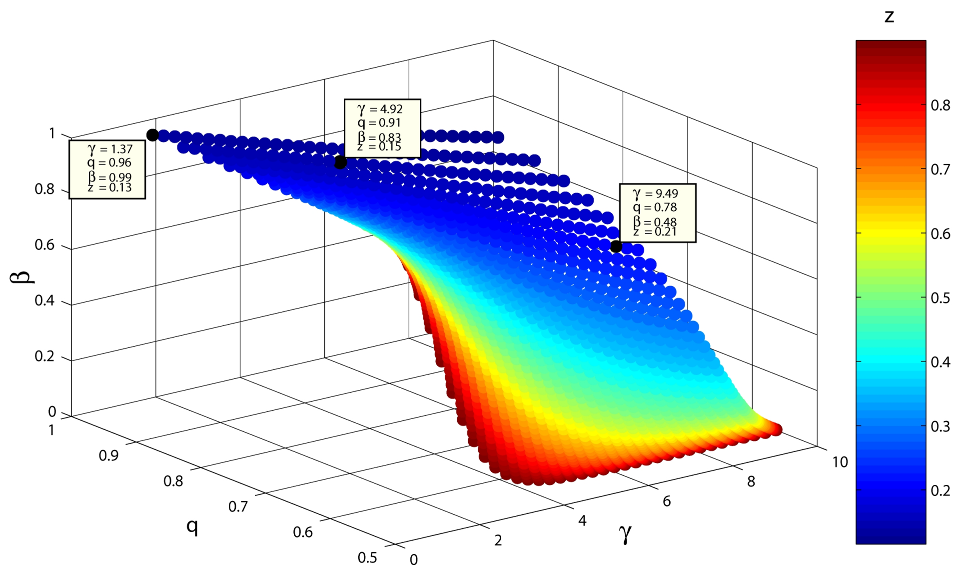

Figure 2.3: Values of the coefficient of risk aversion, γ, optimism-pessimism, q, time preference, β, and fuzziness, z, that result in 0.72% risk-free rate and 6.32% risk premium

2.5 Conclusion

We use an infinite-horizon dynamic model to demonstrate that, when the growth rate of consumption in every random state of the world is perceived by the pessimistic representative agent as a fuzzy rather than crisp magnitude, then the resulting Euler equation can successfully be calibrated to account for on average low historical risk-free rates as well as large observed average equity premiums. Hence, explicit treatment of fuzziness as an additional layer of uncertainty in intertemporal representative agent models can be very fruitful in settling the equity premium puzzle, even with no doubt present in the model .

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