ON THE BAUMSLAG-SOLITAR GROUPS AND CERTAIN GENERALIZED FREE PRODUCTS

by

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Abstract

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The class of the Baumslag groups *G(m,n)* yields examples of finitely generated 1-relator groups that fail to be residually finite [7]. With the utilization of "the Magnus breakdown" of 1-relator groups, in conjunction with the Reidemeister-Schreier method, our analysis of the structure of the groups *G(m,n)* exhibits, as subgroups, the class of the Baumslag-Solitar groups *B(m,n)*.

In 1991 D.I. Moldavanskii [24] gave a complete solution to the isomorphism problem for the class of the Baumslag-Solitar groups. This thesis takes a different approach to the problem of pairwise distinguishing the members of the class of the Baumslag-Solitar groups.

In 1966, S. Lipschutz [17] solved the conjugacy problem for the generalized free product of free groups with cyclic amalgam. In 1962, G. Baumslag [2] proved that a certain generalized free product G of free group F and free abelian group A with cyclic amalgam is residually free. Motivated by the desire to extend this result, we derived an algorithm for solving the conjugacy problem in a special case of this generalized free product G.

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Besides my parents Martin and Margaret Clement, this thesis is dedicated to the loving memories of my grandmother Faith Gladys Clement as well as my mentor Br. Leonard Dennehy—all of whom have been inspiring examples of impeccable character and great sources for my learning.

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Introduction

The Baumslag-Solitar group *B(2,3)* $=< a, b;\ a^{-1}b^2a = b^3 >$ was the simplest example of a finitely generated 1-relator non-Hopfian group (i.e., a group that is isomorphic to a proper subgroup of itself)[8]. The class of groups *B(m,n)* $=< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0,\ n \neq 0,\ m,\ n \in \mathbb{Z} >$ were introduced by Gilbert Baumslag and Donald Solitar in 1962 [12]; they gave a counter-example (i.e., *B(2,3)*) to the commonly believed notion that every finitely generated 1-relator group is Hopfian (i.e., is not isomorphic to a proper subgroup of itself). The isomorphism problem for the groups *B(m,n)* was not solved up until 1991, when D.I. Moldavanskii [24] gave a complete solution to it. In this thesis we give an alternative to Moldavanskii's solution.

In **Chapter 1** we discuss free groups and presentations and we describe the Reidemeister-Schreier method. This method of Kurt Reidemeister and Otto Schreier introduced in 1927 enables one to find presentation for a subgroup of a group given by generators and defining relations. The

Reidemeister-Schreier method plays a useful role in understanding the nature of finitely presented groups. We then include some general facts used in this thesis and give examples of presentations for certain types of groups.

Chapter 2 deals with the intriguing class of the Baumslag-Solitar groups. These groups, which have deceptively simple presentations *B(m,n)* $=< a, b;\, a^{-1}b^m a = b^n \,|\, m \neq 0,\, n \neq 0,\; m,\, n \in \mathbb{Z} >$, were introduced by Gilbert Baumslag and Donald Solitar in 1962 and have been a rich source of examples and counter-examples since then. In our solution to the isomorphism problem for the class of groups *B(m,n)*, three principal building blocks play a central role.

(I) Repeated use of **Lemma 2.2.18** (and variations of it—see **Corollary 2.2.33 and Lemma 2.2.35**): Let G and H be groups and φ be an isomorphism between G and H. Let $G^{(1)}$, $G^{(2)}$ and $H^{(1)}$, $H^{(2)}$ be the first and second derived groups of G and H, respectively. Then φ **induces isomorphisms** between their corresponding **factor groups**, $G/G^{(1)} \cong_{\widetilde{\varphi}} H/H^{(1)}$, $G/G^{(2)} \cong_{\widehat{\varphi}} H/H^{(2)}$, $G^{(1)}/G^{(2)} \cong_{\check{\varphi}} H^{(1)}/H^{(2)}$, etc.

(II) We associate certain torsion-free abelian factor groups of *B(m,n)* $=< a, b;\, a^{-1}b^m a = b^n \,|\, m \neq 0,\, n \neq 0,\; m,\, n \in \mathbb{Z} >$ with the subgroups $\Lambda_{mn} = \{ \frac{\ell}{(mn)^k} |\;\; m \neq 0, n \neq 0, m, n, \ell, k \in \mathbb{Z}\}$ of **the additive**

group of the rational numbers.

(III) We capitalize on the inherent **semi-direct product** nature of *B(m,n)* and analyze inherited actions of the infinite cyclic group on certain subgroups of *B(m,n)* and their factor groups.

Our **main theorem** appears in **Chapter 2** as **Theorem 2.2.26** states:

Let $G = B(m,n) =< a, b;\, a^{-1}b^m a = b^n \,|\, m \neq 0,\, n \neq 0,\ m, n \in \mathbb{Z} >$ and $H = B(m', n') =< x, y;\, x^{-1}y^{m'}x = y^{n'} \,|\, m' \neq 0,\, n' \neq 0,\ m', n' \in \mathbb{Z} >$. Then $G \cong_{\varphi} H$ if and only if $m = m'$ and $n = n'$.

This amounts to a solution to the isomorphism problem for this class of 1-relator groups. The isomorphism problem is the most challenging of the three fundamental problems formulated in Max Dehn's 1911 paper [14] about finitely presented groups. These are as follows:

The word problem: Let G be a group given by a finite presentation $G =< X; Y >$. Is there an algorithm to decide whether or not any given word represents the identity in G?

The conjugacy problem: Let G be a group given by a finite presentation $G =< X; Y >$. Is there an algorithm which decides whether or not any pair of words u and v represent conjugate elements in G?

The isomorphism problem: Let $G =< X; R >$ and $H =< Y; S >$ be any pair of finite presentations. Is there an algorithm to decide whether or not G is isomorphic to H?

The solution to the word problem for 1-relator groups is known [19]. A sufficient amount of work have gone into the conjugacy problem [11](O5), and even still very little progress have been made in solving the isomorphism problem [11](O4) for 1-relator groups in general.

Chapter 3 deals with a certain generalized free product. The idea here is that we take a free group $F =< x, y >$ of rank 2, with an element u that generates its own centralizer in F, and a free abelian group A of rank 2, with an independent set $\{t, v\}$ of generators, and form the generalized free product $G = F \underset{\langle u\rangle = \langle v\rangle}{*} A$ with cyclic amalgam. With the help of a proposition by Gilbert Baumslag [2], we were able to show that G is residually free by constructing, for an arbitrarily picked non-trivial element in G, a homomorphism from G onto the free group component F that keeps the image of the element non-trivial. Also, we derived an algorithm for solving the conjugacy problem for G.

Chapter 4 is an outgrowth of **Chapter 2** in the sense it deals with the Baumslag groups $G(m,n) =< a, b;\ a^{m^{a^b}} = a^n \,|\, m \neq 0,\ n \neq 0,\ m, n \in \mathbb{Z} >$, each of which contains an embedded copy of $B(m,n)$ by virtue of possessing a subgroup N that is an ascending union of generalized free products of isomorphic copies of the Baumslag-Solitar group *B(m,n)* with cyclic amalgams. Based on the evidence encountered so far, **Chapter 4** ends with the conjecture that the isomorphism problem can be solved for the class of the Baumslag groups along lines similar to our solution given for the class of the Baumslag-Solitar groups.

Notations

$X \subseteq G$	X a subset of the group G
$gp(X)$	the group generated by X
$gp_G(X)$	the normal closure of X in G
$< X; Y >$	the group presented by generators X and relators Y
1	the identity element or the trivial group
${}_H 1$	the identity homomorphism on the group H
$\dot{\cup}$	the disjoint union
$\bigcup \uparrow$	the ascending union
$f\,!$	$f \in F$ written uniquely as a X-word in the free group F
$\overline{tx}$	the representatives of tx of E in F
$\phi_*\|_E$	the restriction of ϕ_* to E
$w(\widetilde{x})$	the reduced X-word.
$\delta(t, x)$	the reduced word $tx(\overline{tx})^{-1} \neq 1$
$f \vee g$	the last letter of f cancels with the first letter of g

$f \vartriangle g$	the last letter of f does not cancels with the first letter of g
$[x, y]$	the commutator $x^{-1}y^{-1}xy$ of x and y
$[X, Y]$	$gp([x, y] \mid x \in X, y \in Y)$
$< a >$	the infinite cyclic group generated by a
G/H	the factor group (quotient) of G by H
$\twoheadrightarrow$	surjection
$G * H$	the free product of G and H
$G \cong H$	G is isomorphic to H
$G \cong_{\varphi} H$	G is isomorphic to H via φ
x^{y}	the conjugate $y^{-1}xy$ of x by y
$G^{(1)}$	$[G, G]$, the first derived group of G
$G^{(2)}$	$[G^{(1)}, G^{(1)}]$, the second derived group of G
$\mathbb{Z}$, $\mathbb{Q}$	the set of integers and rational numbers respectively
$ker(\psi)$	the kernel of the homomorphism ψ
$E = A \rtimes Q$	E written as a semi-direct product of A by Q
$B(m, n)$	the Baumslag-Solitar groups with relation $a^{-1}b^{m}a = b^{n}$
$G(m, n)$	the Baumslag groups with relation $a^{m^{a^{b}}} = a^{n}$
$G \times H$	the direct product of G and H
$\xrightarrow{\sim}$	isomorphism

$Aut(G)$	the automorphism group of G
$Inn(G)$	the inner automorphism group of G
$H \leq_c G$	H is a characteristic subgroup of G
$im(\psi)$	the image of the homomorphism ψ
$\Longrightarrow$	this implies
$\Longleftrightarrow$	if and only if
C_n	the cyclic group of order n
C_∞	the infinite cyclic group
$\mathbb{Q}^+$	the additive group of the rational numbers
$\|G\|$	the order of the group G
$\tau(A)$	the torsion subgroup of the group A
$\zeta(G)$	the center of the group G
$\varrho(r)$	the Reidemeister-Schreier rewrite of the relator r
$g \sim h$	g is conjugate to h
Λ_{mn}	$\{\frac{\ell}{(mn)^k} \mid m \neq 0,\, n \neq 0,\, \ell, m,\, n \in \mathbb{Z}\} \subseteq \mathbb{Q}^+$
$G \underset{K}{*} H$	the generalized free product of G and H with amalgam K
$\{G * H;\, \varphi(K) = K\}$	the generalized free product of G and H with amalgam K and identifying isomorphism φ

Chapter 1

Free Groups And Presentations

We begin with a collection of facts, concepts, and notational conventions used throughout this thesis.

1.1 Preliminaries

Definition 1.1.1. *Let X be a set, and G be a group. Let $X \subseteq G$, then $gp(X)$ denotes the subgroup of G* ***generated by*** *X—i.e., the smallest subgroup of G containing X. It follows that $gp(X) = \{x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} \mid x_i \in X, \epsilon_i = \pm 1\}$. We call an expression of the form $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ $(x_i \in X, \epsilon_i = \pm 1)$ an* ***X-word****.*

Definition 1.1.2. *An X-word $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ $(x_i \in X, \epsilon_i = \pm 1)$ is called* ***reduced*** *if $x_i = x_{i+1}$ implies $\epsilon_i \neq -\epsilon_{i+1}$, $(i = 1, ..., n-1)$.*

Definition 1.1.3. *Let* $X \subseteq G$*, then* $gp_G(X)$ *denotes the smallest normal subgroup of* G *containing* X*, or it is often refer to as the* ***normal closure*** *of* X *in* G*. So* $gp_G(X) = gp(gxg^{-1} \mid g \in G, x \in X)$*.*

Definition 1.1.4. *Let* X *be a set. Then a group* F *is said to be* ***free*** *on* X *if* $F = gp(X)$ *and for all* $f \in F$*,* $f \neq 1$*, where* $f \,! = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ *is a non-empty reduced X-word, and* ! *indicates that the right hand side is the unique expression of* f*.*

Definition 1.1.5. *Let* F *be a free group on a set* X*. Let* E *be a subgroup of* F*. A complete set of representatives* T *of (right) cosets of* E *in* F *is a (right)* ***transversal*** *if* $1 \in T$*.*

Definition 1.1.6. *Let* F *be a free group on a set* X *and let* E *be a subgroup of* F*. A (right) transversal of* E *in* F *is called a (right)* ***Schreier transversal*** *if* $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_{n-1}^{\epsilon_{n-1}} \in T$ *whenever* $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} \in T$*.*

Definition 1.1.7. *Suppose* G *is a group and* $G \cong F/N$*, where* F *is a free group on a set* X *and* $N = gp_F(R)$ *with* $R \subseteq F$*, and with a map* $\phi : X \longrightarrow G$*, which extends to a homomorphism* $\phi_* : F \longrightarrow G$ *with kernel* N*, then we write* $G =< X; R >$ *and we term* $< X; R >$ *a* ***presentation*** *of* G*. The elements of* X *are called the* ***generators*** *and those of* R *are called* ***defining***

relators. *Sometimes we use the notation* $G =< X; \{r = 1 \mid r \in R\} >$ *in place of* $< X; R >$ *and we refer to the expression* $\{r = 1 \mid r \in R\}$ *as a set of* ***defining relations*** *for* G.

Definition 1.1.8. *Given a presentation of the form* $G =< X; R \cup S >$, *we say that* S *is a* ***set of consequences*** *of* R *if in the ambient free group,* S *is contained in the normal closure of* R. *We may think of this presentation as a result of added redundant relations.*

Definition 1.1.9. *Given a particular presentation of a group, other presentations for the same group can be obtained by certain moves called* ***Tietze transformations***. *They are as follows:*

T1- *add a new set of generators together with a relation for each new generator which defines the new generator in terms of the previously existing generators. That is, if* $G =< X\dot{\cup}Y; R >$, *then* $G =< X\dot{\cup}Y; R \cup yw(\widetilde{x})^{-1} \mid y \in Y >$ *where* $w(\widetilde{x})$ *is a reduced* X*-word corresponding to* y *in* Y.

T1' *(reverse of* ***T1****)- delete a subset of the generators, each of which appears only in a relation which defines it in terms of a disjoint subset of the other generators, and at the same time delete those relations. That is, if* $G =< X\dot{\cup}Y; R \cup yw(\widetilde{x})^{-1} \mid y \in Y >$ *and* R *does not contain any words involving the elements* y *in* Y, *then* $G =< X; R >$.

T2 - *add a set of consequences of the already existing relations. That is, if* $G =< X; R >$*, and* S *is a set of consequence of* R*, then* $G =< X; R \cup S >$*.*

T2′ *(reverse of **T2**)- delete a set of relations which is a set of consequences of the other relations. That is, if* $G =< X; R \cup S >$*, and* S *is a set of consequence of* R*, then* $G =< X; R >$*.*

Note: As we have done already here, for simplification we use " $=$ " instead of " $\cong$ " whenever we are referring to equivalent (isomorphic) presentations under Tietze tranformations.

Theorem 1.1.1. *(O. Schreier) Every subgroup of a free group is free.*

We will prove this theorem by proving a series of lemmas.

Lemma 1.1.2. *Let* F *be a free group on a set* X*. Let* E *be a subgroup of* F*, then there exist a right Schreier transversal* T *of* E *in* F*.*

Proof. Let us define $\ell(Ef) = min\{\ell_x(a)|\, a \in Ef\}$, where $\ell_x(a)$ represent the length of a as an X-word and $\ell(E) = 0$.
We construct T inductively on $\ell(Ef)$. Choose 1 to be representative of E. Let $T = \bigcup T_n$, T_n consist of the representatives of all the cosets of length less than or equal to n, where $T_0 = \{1\}$. Suppose T_{n-1} is constructed and is a Schreier transversal for $n \geqslant 1$. Consider a coset of length n, say, Ef,

then there exists $a_1 \cdots a_n \in Ef$, $(a_i \in X \cup X^{-1})$. So $Ef = Ea_1 \cdots a_n$. Observe that $\ell(Ea_1 \cdots a_{n-1}) \leq n-1$. Let $a'_1 \cdots a'_m$ be the representatives of $Ea_1 \cdots a_{n-1}$. Thus, $a_1 \cdots a_{n-1} = ea'_1 \cdots a'_m$, where $e \in E$. This implies that $Ef = Ea'_1 \cdots a'_m a_n$, therefore $m = n-1$. So we choose $a'_1 \cdots a'_{n-1}a_n$ to be the representatives of Ef. Finally, $T_n = T_{n-1} \cup$ {all the representatives of the cosets of length n chosen in the form $a'_1 \cdots a'_{n-1}a_n$}. □

Lemma 1.1.3. *Let F be a free group on the set X and let $E \leq F$. If T is a right Schreier transversal of E in F, then E is generated by the set $Y = \{tx(\overline{tx})^{-1} \neq 1 \mid t \in T, x \in X\}$, where $\overline{tx}$ denotes the representatives of the right cosets tx of E in F, i.e., $Etx = E\overline{tx}$.*

Proof. Notice that since $Etx = E\overline{tx}$, it follows that multiplying by $(\overline{tx})^{-1}$, we get $Etx(\overline{tx})^{-1} = E$. So the elements $tx(\overline{tx})^{-1} \in E$, and therefore $Y \subseteq E$. Let $e \in E$, where $e = x_1x_2 \cdots x_n$, $x_i \in X^{\pm 1}$, $i = 1, ..., n$, be a reduced word in X. Let us define a sequence of elements of T inductively as follows: $t_1 = 1$, $t_{i+1} = \overline{t_ix_i}$, $i = 1, ..., n$. Put $y_i = t_ix_it_{i+1}^{-1} = t_ix_i(\overline{t_ix_i})^{-1} \in Y$, $i = 1, ..., n$.
So that $y_1y_2 \cdots y_n = t_1x_1t_2^{-1}t_2x_2t_3^{-1} \cdots x_nt_{n+1}^{-1} = 1x_1 \cdots x_nt_{n+1}^{-1} = 1et_{n+1}^{-1}$. $(\star)$
Since the left-hand side of the equation $(\star)$ belongs to E, so does t_{n+1}^{-1}. But we know that $t_{n+1}^{-1} \in T$ and that $E \cap T = 1$, this implies that $t_{n+1}^{-1} = 1$. So $(\star)$ expresses e as a Y-word, proving that Y generates E. □

Lemma 1.1.4. *Let F be a free group on the set X and let $E \leq F$. If T is a right Schreier transversal of E in F, then E is* ***freely*** *generated by the set $Y = \{tx(\overline{tx})^{-1} \neq 1 \mid t \in T, x \in X\}$, where $\overline{tx}$ denotes the representatives of the right cosets tx of E in F, i.e., $Etx = E\overline{tx}$.*

Proof. By **Lemma 1.1.3**, we showed that Y generates E, now we need to show that every reduced Y-word is not the trivial word ($\neq 1$). Recall that F being free on X, this implies for all $f \in F$, $f \neq 1$, where $f\ ! = x_1 \cdots x_n$ reduced, $x_i \in X \cup X^{-1}$, and the length of f, defined $\ell_x(f)$ is n, i.e., $\ell(f) = n$, and $\ell(1) = 0$. If $f, g \in F$, we define $f \vartriangle g \Longleftrightarrow \ell_x(fg) = \ell(f) + \ell(g)$ and $f \vee g \Longleftrightarrow \ell_x(fg) < \ell(f) + \ell(g)$. Let $\delta(t,x) = tx(\overline{tx})^{-1} \neq 1$. Now

(i) If $\delta(t,x) \neq 1$, then $\delta(t,x) = t \vartriangle x \vartriangle (\overline{tx})^{-1}$.

To show (i) above, suppose $t \vee x$, then $t = s \vartriangle x^{-1}$, $t \in T$, and since T is a Schreier transversal, $s \in T$, i.e., $\bar{s} = s$. So we have $tx(\overline{tx})^{-1} = sx^{-1}x(\overline{sx^{-1}x})^{-1} = ss^{-1} = 1$. However this contradicts the assumption that $\delta(t,x) \neq 1$. The case $x \vee (\overline{tx})^{-1}$, or equivalently $\overline{tx} \vee x^{-1}$ can be done in a similar way as (i) above.

(ii) If $\delta(t,x) \neq 1$ then, $\delta(t,x) = \delta(s,y) \Longleftrightarrow t = s$ and $x = y$.

To show (ii) above, we see by (i) that $t \vartriangle x \vartriangle (\overline{tx})^{-1} = s \vartriangle y \vartriangle (\overline{sy})^{-1}$. If $\ell(t) = \ell(s)$, $t = s$ and $x = y$. If $\ell(t) < \ell(s)$, tx is an initial segment of s. So

$\overline{tx} = tx$ and therefore $tx(\overline{tx})^{-1} = 1$, which is a contradiction. Similarly, the other case cannot occur.

(iii) If $w = (\delta(t_1, x_1))^{\epsilon_1} \cdots (\delta(t_n, x_n))^{\epsilon_n}$ is a reduce Y-word in the symbols $\delta(t, x)$(which by (ii) are distinct elements if the symbols are distinct), then $w = \cdots \triangle x_1^{\epsilon_1} \triangle \cdots \triangle x_n^{\epsilon_n} \triangle \cdots$, i.e., the x and x^{-1} in the middle of $\delta(t, x)$ and $(\delta(t, x))^{-1}$ respectively do not cancel in the computation of the reduced X-word representing w.

To show (iii) above, observe that on computing any reduced words

$(\delta(t, x))^{\pm}(\delta(t, y))^{\pm}$, we have only four possibilities

$\cdots \triangle\ x \ \triangle \cdots \triangle\ y\ \triangle \cdots$,

$\cdots \triangle\ x\ \triangle \cdots \triangle\ y^{-1}\ \triangle \cdots$,

$\cdots \triangle\ x^{-1}\ \triangle \cdots \triangle\ y\ \triangle \cdots$,

$\cdots \triangle\ x^{-1}\ \triangle \cdots \triangle\ y^{-1}\ \triangle \cdots$.

This means that the letters shown never cancels, so we have established the form of w. This completes the proof of **Theorem 1.1.1.** □

Definition 1.1.10. *Let $G =< X; R >$ be a presentation of a group G. Then G comes equipped with a* ***presentation map*** *$\phi : X \longrightarrow G$. If $r = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ $(x_i \in X, \epsilon_i = \pm 1)$, then we use the notation $r = r(\widetilde{x}) \in R$ as a shorthand for r expressed as a reduced X-word in R. Now suppose H is* *another group and* $\psi : x\phi \longrightarrow H$*, then we define* $r(\widetilde{(x\phi)\psi}) =$ $((x_1\phi)\psi)^{\epsilon_1} \cdots ((x_n\phi)\psi)^{\epsilon_1} \in H.$

Theorem 1.1.5. *(W. Von Dyck) Let* $G =< X; R >$ *be a presentation of a group* G *with presentation map* ϕ*. Furthermore, suppose that* H *is any group such that* $\psi : x\phi \longrightarrow H$*. If for each* $r = r(\widetilde{x}) \in R$*,* $r(\widetilde{(x\phi)\psi}) =_H 1$*, then the map* ψ *defines a homomorphism from* G *into* H*.*

1.2 The Reidemeister-Schreier method

Given a presentation of a group G, ***the Reidemeister-Schreier method*** shows one how to compute a presentation for a subgroup H of the given group G. First, we remind ourselves that the presentation $G =< X; R >$

(i) comes equipped with a presentation map $\phi : X \longrightarrow G$;

(ii) if F denotes the free group on X, then ϕ extends to a unique homomorphism $\phi_* : F \twoheadrightarrow G$;

(iii) if K stands for the kernel of ϕ_*, then $K = gp_F(R) = gp(frf^{-1} \mid f \in F, r \in R)$ and by the first isomorphism theorem $\phi_* : F/K \xrightarrow{\sim} G$.

Suppose now that H is a subgroup of G, then H can be presented by generators and defining relations as follows: Observe that the pre-image of H in F/K is isomorphic to E/K, for some subgroup E in F. We seek generators for the free subgroup E. To this end, let T be a right Schreier transversal of E in F. Then E is free on Y, where $Y = \{tx(\overline{tx})^{-1} \neq 1 \mid t \in T, x \in X\}$ according to **Lemma 1.1.4**. Notice that due to the isomorphism between E/K and H, $\phi|_Y$ is a presentation map of H, so $H =< Y; W >$, with W still to be determined (in terms of Y-words). We know that $K = gp(frf^{-1} \mid f \in F, r \in R) = gp_E(\{trt^{-1} \mid t \in T, r \in R\})$. To verify the equality above we observe that since T is a right Schreier transversal of E in F, the cosets of E partition F, so $F = \dot{\bigcup}\{et \mid e \in E, t \in T\}$. Every element of F can be written in the form $f = et$, $e \in E$, $t \in T$. Thus $K = gp(frf^{-1} \mid f \in F, r \in R)$

$= gp(\{(et)r(et)^{-1} \mid e \in E, t \in T, r \in R\})$

$= gp(\{e(trt^{-1})e^{-1} \mid e \in E, t \in T, r \in R\})$

$= gp_E(\{trt^{-1} \mid t \in T, r \in R\})$.

We still face the problem of re-expressing the relators in W, a subset of F, in terms of the generators of E above. The method provides for methodical replacement of the X-words in relation specifying elements of H by Y-words. Let $\varrho(trt^{-1})$ be the rewrite of trt^{-1} as a reduced Y-word. So W above can be written as $W = \{\varrho(trt^{-1}) \mid t \in T, r \in R\}$. Now H can be presented: $H =< \{tx(\overline{tx})^{-1} \neq 1 \mid t \in T, x \in X\}; \varrho(trt^{-1}) \mid t \in T, r \in R\} >$.

1.3 Some general facts : Presentations for certain types of groups

Definition 1.3.1. *Let G be a group and let A and B be subgroups. Then we say that G is a* ***product*** *of A and B if $gp(A \cup B) = G$.*

Note that if G is a product of A and B and $g \in G$, then $g = a_1 b_1 \cdots a_n b_n$

$(a_i \in A,\ b_i \in B,\ 1 \leq i \leq n)$.

Definition 1.3.2. *Suppose that G is a product of A and B, if*

(i) $A \cap B = 1$;

(ii) ab=ba for all $a \in A$, $b \in B$;

then we call G the (internal) ***direct product*** *of A and B and we write $G = A \times B$. If $g \in G$, then $g = ab$.*

Definition 1.3.3. *Let G be a product of A and B, then we say that G is the* ***free product*** *of A and B which we express by writing $G = A * B$ if*

(i) $a_1 b_1 \cdots a_n b_n$ represents a non-trivial element of G whenever

$a_i \in A - \{1\};\ b_i \in B - \{1\}$.

(ii) $b_1 a_1 \cdots b_n a_n$ represents a non-trivial element of G whenever

$a_i \in A - \{1\},\ b_i \in B - \{1\}$.

It follows that $A \cap B = 1$.

Definition 1.3.4. *A group G is the* ***generalized free product*** *of its subgroups A and B, amalgamating $H = A \cap B$, if*

i) $G = gp(A \cup B)$;

ii) every "strictly alternating" $A \cup B$-product $x_1 x_2 \cdots x_n$ $(n > 0)$ represents a non-trivial element of G. In such a case we write $G = A \underset{H}{*} B$ or $\{A * B; H\}$. The product above is termed ***strictly alternating*** *if each $x_i \in (A-H) \cup (B-H)$ and consecutive x's comes from different factors, i.e., if $x_i \in (A-H)$, then $x_{i+1} \in B-H$, and if $x_i \in (B-H)$, then $x_{i+1} \in A-H$ $(i = i, ..., n-1)$. The group G has the following universal mapping property: for every group G^* and every pair of homomorphisms $\alpha : A \longrightarrow G^*$, $\beta : B \longrightarrow G^*$ such that $\alpha|_H = \beta|_H$, there exists a unique homomorphism $\gamma : G \longrightarrow G^*$ which agrees with α on A and β on B.*

In the case H is cyclic, we call $A \underset{H}{*} B$ the generalized free product of A and B with **cyclic amalgam** or simply the ***cyclically pinched*** generalized free product of A and B.

In the case $H = A \cap B = \{1\}$, i.e., $G = A \underset{\{1\}}{*} B$ or $\{A * B; \{1\}\}$ is referred to as the ***free product*** of A and B as mentioned before.

We may recall, in short, that by the group G presented as $< X ; R >$, we mean the quotient group of a free group on X by the the normal closure of the words in R.

Given a presentation $< X\,;\, R >$, we can always find a group G such that $G =< X\,;\, R >$, for example $G = F/gp_{_F}(R)$ when we choose $\phi : X \longrightarrow G$ to be the natural homomorphism $\phi : x \mapsto xgp_{_F}(R)$.

Below we give examples of certain types of presentations of groups.

It is perhaps interesting to note that $G =< a, b\,;\, a^{-1}ba = b^2,\, b^{-1}ab = a^2 >$ is a presentation for the ***trivial group***. For taking the first relation $a^{-1}ba = b^2$ and multiplying by b^{-1}, we get $\underbrace{b^{-1}a^{-1}b}a = b$. Now taking the inverse of the second relation $b^{-1}ab = a^2$, we get $\underbrace{b^{-1}a^{-1}b} = a^{-2}$. Substituting a^{-2} for $\underbrace{b^{-1}a^{-1}b}$ into the first relation $\underbrace{b^{-1}a^{-1}b}a = b$ we get $a^{-2}a = b$, so $a^{-1} = b$. But this simply means that the relation $a^{-1}ba = b^2$ becomes $b^2a = b^2$, so $a = 1$ and therefore $b = 1$. Hence $G =< 1 >$, the trivial group.

Despite the success in deciding that the presentation above stood for the trivial group, there is no algorithm to decide in general whether or not a given presentation represents the trivial group. Therefore knowledge of how certain presentation type correspond to certain types of group constructions is desirable.

Example 1.3.1. (i) *The presentation* $C_n =< x; x^n >$ *stands for* ***cyclic group of order n***.

(ii) *The presentation* $C_\infty = < x >$ *stands for the* ***infinite cyclic group****.*

Example 1.3.2. *Given a group* G *with presentation* $< X; R >$*, then a presentation for the* ***abelianization*** $G/[G, G]$ *of* G *is expressed as* $< X\, ;\, R \cup \{[x_1, x_2],\, x_1, x_2 \in X\} >$*.*

Example 1.3.3. *Given groups* G *and* H *with presentations* $< X; R >$ *and* $< Y; S >$ *respectively, then a presentation for the* ***direct product*** $G \times H$ *of* G *and* H *is* $< X \dot{\cup} Y\, ;\, R \cup S \cup \{[x, y],\, x \in X, y \in Y\} >$*.*

Example 1.3.4. *Given groups* G *and* H *with presentations* $< X; R >$ *and* $< Y; S >$ *respectively, then a presentation for the* ***free product*** $G * H$ *of* G *and* H *is* $< X \dot{\cup} Y\, ;\, R \cup S >$*.*

Example 1.3.5. *Given groups* G *and* H *with presentations* $< X; R >$ *and* $< Y; S >$ *respectively, if* K *is a group equipped with two monomorphisms* $\varphi : K \longrightarrow G,\quad \theta : K \longrightarrow H$*, then the* ***generalized free product*** $G \underset{K}{*} H$ *of* G *and* H *with amalgam* K *has presentation* $< X \dot{\cup} Y\, ;\, R \cup S \cup \{\varphi(k)\theta(k^{-1}) \mid k \in K\} >$*.*

Example 1.3.6. *A group* G *is said to be a finitely generated* ***1-relator group*** *with presentation* $< x_1, x_2, ..., x_n\, ;\, r >$ *if it has finite number of generators* $x_1, x_2, ..., x_n$ *and a single defining relator* r*.*

In this work we will focus our attention mainly to the special classes of 1-relator groups:

(i) *The **Baumslag-Solitar groups** having presentation*

B(m,n) $=< a, b;\ a^{-1}b^{m}a = b^{n}\,|\, m \neq 0,\ n \neq 0,\ m,\ n \in \mathbb{Z} >$.

(ii) *The **Baumslag groups** having presentation*

G(m,n) $=< a, b;\ a^{m^{a^{b}}} = a^{n}\,|\, m \neq 0,\ n \neq 0,\ m,\ n \in \mathbb{Z} >$.

Example 1.3.7. *In our investigation of these two classes, presentation for another construct will be utilized. Suppose a family of groups with known presentation* $G_i =< X_i\,;\, R_i >$ *is given, indexed by a set* $I : \{G_i\,(i \in I\}$ *and suppose that* $G_i \leq G_{i+1}\,(i \in I)$, *a compatible family of connecting morphisms* $\varphi_i : G_i \longrightarrow G_{i+1}\,(i \in I)$ *is also given in terms of the generating sets* X_i *of the known presentations. Then the presentation for the "direct limit" or "ascending union" of these groups can be obtained by augmenting the union of the generating sets, the union of the relators by the inclusion identifications (see **Chapter 4**).*

Chapter 2

The Baumslag-Solitar Groups

This chapter describes our solution to the isomorphism problem for the groups in the title above. The main technical tools are **Lemma 2.2.18** and its variations, together with quotients respecting semi-direct product structure, in context of some associations with subgroups of the additive group of rational numbers.

2.1 Historical background and motivation

We focus our attention on the special class of 1-relator groups with presentations $B(m,n) =< a, b;\, a^{-1}b^m a = b^n \,|\, m \neq 0,\, n \neq 0,\ m,\, n \in \mathbb{Z} >$ known as ***the Baumslag-Solitar groups***, introduced by Gilbert Baumslag and Donald Solitar in 1962 [12]. Their successfully achieved aim was to show the existence of a finitely generated 1-relator non-Hopfian group—i.e., a finitely generated 1-relator group which is isomorphic to one of its proper factor groups. Thus, the belief that every finitely generated 1-relator group is Hopfian was refuted [8] when the Baumslag-Solitar group $B(2,3) =< a, b\,;\, a^{-1}b^2a = b^3 >$ was shown to be a finitely generated 1-relator non-Hopfian group. This proceeds as follows: Consider the group

$$G = B(2,3) =< a, b\,;\, a^{-1}b^2a = b^3 > \qquad (1)$$

We apply a series of the Tietze transformations $T1, T1', T2, T2'$ (see **Chapter 1**) and together "the Magnus' breakdown" of 1-relator groups.

Consider the factor group $H = G/C$ (where $C = gp_G(b(b^{-1}a^{-1}ba)^{-2})$ of G obtained by adding the relation $b = (b^{-1}a^{-1}ba)^2$ to the relation in (1), we obtain,

$$H =< a, b\,;\, a^{-1}b^2a = b^3,\ b = (b^{-1}a^{-1}ba)^2 > \qquad T2$$

$$=< a, b, c;\, a^{-1}b^2a = b^3,\ b = (b^{-1}a^{-1}ba)^2,\ c = (b^{-1}a^{-1}ba) > \qquad T1$$

$$=< a, b, c;\, a^{-1}b^2a = b^3,\ b = c^2,\ c = (b^{-1}a^{-1}ba) > \qquad T2/T2'$$

$$=< a, c\,;\, a^{-1}c^4a = c^6,\ c = c^{-2}a^{-1}c^2a > \qquad T1'/T2/T2'$$

$$=< a, c\,;\, a^{-1}c^4a = c^6,\ a^{-1}c^2a = c^3 > \qquad T2/T2'$$

$$=< a, c\,;\, a^{-1}c^2a = c^3 >. \qquad T2'$$

Therefore, H is isomorphic to G via simple symbol replacement. Note that $b(b^{-1}a^{-1}ba)^{-2}$ defines a non-trivial element in $B = gp_G(b)$. To see this, applying "Magnus' breakdown" of 1-relator groups to B, we obtain $B =< ..., b_{-1}, b_0, b_1, ...; ..., b_0^2 = b_{-1}^3,\ b_1^2 = b_0^3, ... >= \bigcup_{\substack{i\geq 0\\ j\geq 0}}^{\infty} \uparrow B_{-i,j}$ which is an ascending union of generalized free products. The relator $b(b^{-1}a^{-1}ba)^{-2}$ can be rewritten as $b_0(b_0^{-1}b_1)^{-2} = b_0(b_0^{-1}b_1)^{-1}(b_0^{-1}b_1)^{-1} = b_0b_1^{-1}b_0b_1^{-1}b_0$. Now in the generalized free product, $B_{0,1} =< b_0, b_1 ;\ b_1^2 = b_0^3 >$. By the uniqueness of the normal form $b_0b_1^{-1}b_0b_1^{-1}b_0 \neq_{B_{0,1}} 1$. Thus, G given in (1) is isomorphic to the proper factor group H of itself, i.e., G is non-Hopfian.

A.I. Malcev's classic theorem [21] tells us that finitely generated residually finite groups are Hopfian. However, the converse is not true, and a counter-example will be seen below.

Hopficity (i.e., not to be isomorphic to a proper factor group of itself) is an instance of what is often referred to as ***finiteness conditions***, i.e., conditions satisfied by finite groups which may or may not hold for some infinite groups; a few manifestations of them in the class of the groups *B(m,n)* are noted in the list below.

(1) *B(m,n)* is ***residually finite*** (i.e., the intersection of all of its subgroups of finite index is trivial) if and only if $|m| = |n|$ or $|m| = 1$ or $|n| = 1$.

(2) *B(m,n)* is ***Hopfian*** if and only if it is residually finite or $\pi(m) = \pi(n)$, where $\pi(m)$ and $\pi(n)$ denotes the set of prime divisors of m and n respec-

tively. For example, *B(2,4)* is Hopfian but not residually finite.

In this regard the Baumslag-Solitar groups fall into three distinct classes:

(i) those which are residually finite;

(ii) those which are Hopfian but not residually finite;

(iii) and those that are non-Hopfian.

However, the way in which we pairwise distinguish these groups in **Section 2.2** do not involve any deliberate use of this foregoing classification.

2.2 A solution to the isomorphism problem for the Baumslag-Solitar groups *B(m,n)*

In this section we will focus our attention to a solution to isomorphism problem for the Baumslag-Solitar groups.

2.2.1 The groups *B(1,n)*

Theorem 2.2.1. *Let E be the semi-direct product of Λ_n and Q, where $\Lambda_n = \{\frac{\ell}{n^k} | \, n \neq 0, \, n, \, \ell, \, k \in \mathbb{Z}\}$ thought as subgroups of the additive group $\mathbb{Q}^+$ of rational numbers, and $Q =< q >$ the infinite cyclic group, where q acts on Λ_n by multiplication by n. For each $n \neq 0$, $n \in \mathbb{Z}$, E is isomorphic to the Baumslag-Solitar group B(1,n) $=< a, b; a^{-1}ba = b^n >$.*

Definition 2.2.1. *Let E be a group and let Λ and Q be subgroups of E. Then E is said to be a* ***semi-direct product*** *of Λ and Q, written $E = \Lambda \rtimes Q$ if*

i) $\Lambda \trianglelefteq E$;

ii) $\Lambda \cap Q = 1$;

iii) $E = Q\Lambda$.

Definition 2.2.2. *Given a (multiplicatively-written) group Q, a (right)* ***Q-module*** *is an (additively-written) abelian group Λ together with an action of Q on Λ (on the right) such that the following axioms hold:*

i) $(b + b')q = bq + b'q, \quad b, b' \in \Lambda,\ q \in Q$;

ii) $b(qq') = (bq)q', \quad b \in \Lambda,\ q, q' \in Q$;

iii) $b \cdot 1 = b, \quad b \in \Lambda$.

We will use a series of lemmas to prove **Theorem 2.2.1** above.

Lemma 2.2.2. *The group E is generated by q and b_0, $q \in Q$, $b_0 \in \Lambda_n$, where $q = (n, 0)$ and $b_0 = (1, 1)$.*

Proof. Notice that since Λ_n is Q-module, and E is a semi-direct product, multiplication in E has the form $(n^i, \frac{\ell}{n^k})(n^j, \frac{\ell'}{n^{k'}})= \left(n^{i+j}, \frac{\ell}{n^k}n^j + \frac{\ell'}{n^{k'}}\right)$. Also $Q = \{q^i = (n^i, 0) | i \in \mathbb{Z}\}$ and $\{b_0^j = (1, j) | j \in \mathbb{Z}\}$. We need to show that if we take a typical element of E, say, $\left(n^i, \frac{\ell}{n^k}\right)$, where $n \neq 0$, n, ℓ, i, $k \in \mathbb{Z}$, we can express it in terms of q and b_0. Observe that $(n^i, \frac{\ell}{n^k})$

$= (n^i, 0)(1, \frac{\ell}{n^k}) = (n^i, 0)(1, n^{-k})^\ell = q^i(1,\, n^{-k})^\ell.$ $(\star)$

Put $b_{-k} = (1, n^{-k})$ and we notice that

$b_{-1} = (1, n^{-1}) = (n, 1)(n^{-1}, 0) = (n, 0)(1, 1)(n^{-1}, 0) = qb_0q^{-1}$

$b_{-2} = (1, n^{-2}) = (n^2, 1)(n^{-2}, 0) = (n^2, 0)(1, 1)(n^{-2}, 0) = q^2b_0q^{-2}$

$\vdots$

$b_{-k} = (1, n^{-k}) = (n^k, 1)(n^{-k}, 0) = (n^k, 0)(1, 1)(n^{-k}, 0) = q^kb_0q^{-k},$ $(\star')$

we substitute $q^kb_0q^{-k}$ for $(1, n^{-k})$ in $(\star)$,

so $(n^i,\, \frac{\ell}{n^k}) = q^i(1, n^{-k})^\ell = q^i(q^kb_0q^{-k})^\ell = q^iq^{k\ell}b_0^\ell q^{-k\ell} = q^{i+kl}b_0^\ell q^{-k\ell}.$

Thus q and b_0 generates E.

Observe that $q^{-1}b_0q = (n^{-1}, 0)(1, 1)(n, 0)$

$= (n^{-1}, 1)(n, 0)$

$= (1, n) = \underbrace{(1, 1)(1, 1)\cdots(1, 1)}_{n} = b_0^n.$ $(\star'')$

Thus, $E =< q, b_0\ ;\ q^{-1}b_0q = b_0^n, ... >$. □

Lemma 2.2.3. *The mapping* $\phi : B(1,n) \longrightarrow E$ *defined by*

$$a \longmapsto q$$

$$b \longmapsto b_0$$

is a homomorphism.

Proof. By Von Dyck's theorem. □

Lemma 2.2.4. *Every element of the group E can be written as (normal form) $q^i b_{-k}^{\ell}$, where b_{-k} is defined to be $q^k b_0 q^{-k}$.*

Proof. By **Lemma 2.2.2** this holds. However, if we take an arbitrary element of E, say, $q^r b_0^s q^t b_0^u q^v$, where $r, s, t, u, v \in \mathbb{Z}$, then this element can be rewritten as (normal form) $q^i b_{-k}^{\ell}$. Latter on we will see why this is a key step in showing that E and *B(1,n)* are isomorphic.

Note that in the proof of **Lemma 2.2.2**, the relation

$b_{-k} = (1, n^{-k}) = q^k b_0 q^{-k}$ $(\star')$

and the relation $q^{-1} b_0 q = b_0^n$ $(\star'')$

both hold.

Observe that $b_{-k}^{\ell} = (1, n^{-k})^{\ell} = q^k b_0^{\ell} q^{-k}$ $(\star''')$

Thus, by using equations $(\star')$, $(\star'')$, $(\star''')$ and inserting appropriate pinches, i.e., (a pinch is a word in the form $q^{-1}q$ or qq^{-1}), we can rewrite an arbitrary element, say, $q^r b_0^s q^t b_0^u q^v$, (where $r, s, t, u, v \in \mathbb{Z}$) in E as $q^i b_{-k}^{\ell}$ (normal form). Note that if $k < 0$ in $q^i b_{-k}^{\ell}$, then $q^i b_{-k}^{\ell} = q^i b_k^{\ell} = q^i b_0^{nk\ell}$, i.e., $b_k = b_0^{nk}$, for $k > 0$. □

Lemma 2.2.5. *The words $q^i b_{-k}^{\ell} = q^r b_{-k}^{s}$ hold in E if and only if $i = r$ and*

$\ell = s$.

Proof. Since E is a semi-direct product, E has a unique form. Recall that $E = Q\Lambda_n$.

$q^i b_{-k}^{\ell} = q^r b_{-k}^{s}$.

Multiplying by q^{-i}, we get $b_{-k}^{\ell} = q^{-i}q^r b_{-k}^{s}$. Multiplying by b_{-k}^{-s}, we get $b_{-k}^{\ell} b_{-k}^{-s} = q^{-i}q^r$. But since $Q \cap \Lambda_n = 1$, then $b_{-k}^{\ell} b_{-k}^{-s} = q^{-i}q^r$ if and only if $b_{-k}^{\ell} b_{-k}^{-s} = 1$, which implies that $\ell = s$ and $q^{-i}q^r = 1$, and so $i = r$. □

Lemma 2.2.6. *The group E may be presented as follows:*

$E =< q, b_0, b_{-1}, b_{-2}, ...; \; q^{-1}b_0 q = b_0^n, \; q^{-1}b_{-1}q = b_0, \; q^{-1}b_{-2}q = b_{-1}, ...,$ $b_{-1}^n = b_0, \; b_{-2}^n = b_{-1}, \; b_{-3}^n = b_{-2}, ..., b_{-1} = qb_0q^{-1}, \; b_{-2} = qb_{-1}q^{-1}, \; b_{-3} = qb_{-2}q^{-1} \;, ... >.$

Proof. All the relations in the equations $(\star')$, $(\star'')$ and $(\star''')$ hold for the group E. Since all the relations in the presentation above are a set of consequence of these three relations. For instance observe that

$$b_{-1}^n = \underbrace{(1, n^{-1})(1, n^{-1}) \cdots (1, n^{-1})}_{n} = (1, \, nn^{-1}) = (1, 1) = b_0.$$

$$b_{-2}^n = \underbrace{(1, n^{-2})(1, n^{-2}) \cdots (1, n^{-2})}_{n} = (1, \, nn^{-2}) = (1, n^{-1}) = b_{-1}$$

$\vdots$

$$b_{-k}^{n} = \underbrace{(1, n^{-k})(1, n^{-k} \cdots (1, n^{-k})}_{n} = (1, nn^{-k}) = (1, n^{-(k-1)}) = b_{-(k-1)}. \quad (\star^{iv})$$

Thus E may be presented as claimed. □

Lemma 2.2.7. *Every element of the group B(1,n) can be written uniquely in the form* $a^i b_{-k}^{\ell}$*, using the notation* $b_{-k} = a^k b_0 a^{-k}$*, similarly to that of the group* E*.*

Proof. $B(1,n) =< a, b; a^{-1}ba = b^n >$. Put $b = b_0$ in the relation $a^{-1}ba = b^n$ and put $b_{-k} = a^k b_0 a^{-k},\ k \in \mathbb{Z}$ (†)

then $a^{-1}b_0 a = b_0^n$. (†′)

holds from the relation in *B(1,n)*. We see that $b_{-1} = ab_0a^{-1}$.

$b_{-2} = a^2 b_0 a^{-2} = a(ab_0a^{-1})a^{-1} = a(b_{-1})a^{-1}$.

$\vdots$

$b_{-k} = ab_{-(k-1)}a^{-1}$.

Also since $b_{-1} = ab_0a^{-1}$.

$b_{-1}^n = ab_0^n a^{-1} = a(a^{-1}b_0a)a^{-1} = b_0$.

$b_{-2}^n = a^2 b_0^n a^{-2} = a(ab_0^n a^{-1})a^{-1} = a(a(a^{-1}b_0a)a^{-1})a^{-1} = ab_0a^{-1} = b_{-1}$.

Continuing this this manner, we get $b_{-k}^n = b_{-(k-1)}$. (†″)

Thus, by using equations (†), (†′), (†″), and using appropriate inserted pinches we can rewrite an arbitrary element $a^r b_0^s a^t b_0^u a^v$, (where $r, s, t, u, v \in \mathbb{Z}$) in *B(1,n)* as $a^i b_{-k}^{\ell}$ (normal form). Note that if $k < 0$ in $a^i b_{-k}^{\ell}$, then

$a^i b_{-k}^{\ell} = a^i b_k^{\ell} = a^i b_0^{nk\ell}$, i.e., $b_k = b_0^{nk}$, for $k > 0$.

The elements in *B(1,n)* corresponds similarly to that of the elements of E.

Notice that under the homomorphism ϕ : *B(1,2)* $\longrightarrow E$,

$\phi(a^i b_{-k}^{\ell}) = q^i b_{-k}^{\ell}$.

Observe that $q^i b_{-k}^{\ell} = 1$, if and only if $i = \ell = 0$.

That is, by looking at $(n^i, \frac{\ell}{n^k})$,

$(n^i, \frac{\ell}{n^k}) = (1, 0)$ if and only if $i = \ell = 0$.

Since ϕ is a homomorphism, $\phi(1) =_E 1$,

so $a^i b_{-k}^{\ell} = 1$ in *B(1,n)* if and only if $i = \ell = 0$. Thus elements of *B(1,n)* has the corresponding form $a^i b_{-k}^{\ell}$. □

Lemma 2.2.8. *The group B(1,n) may be presented as follows:*

B(1,n) $=< a, b_0, b_{-1}, b_{-2}, ...;\ a^{-1}b_0 a = b_0^n,\ a^{-1}b_{-1}a = b_0,$

$a^{-1}b_{-2}a = b_{-1}, ..., b_{-1}^n = b_0,\ b_{-2}^n = b_{-1},\ b_{-3}^n = b_{-2}, ..., b_{-1} = ab_0a^{-1},\ b_{-2} =$

$ab_{-1}a^{-1},\ b_{-3} = ab_{-2}a^{-1},\ ... >$.

Proof. All the relations in the equations (†), (†′) and (†″) hold for the group E. Since all the relations in the presentation above are a set of consequence of these three relations. Thus *B(1,n)* may be presented as claimed. □

Lemma 2.2.9. *The map ϕ is an isomorphism.*

Proof. We know from **Lemma 2.2.3** that ϕ is a homomorphism.

ϕ : *B(1,n)* $\longrightarrow$ *E*

$\phi(a^i b_{-k}^{\ell}) = \phi(a^j b_{-k}^{m})$

$\Longleftrightarrow q^i b_{-k}^{\ell} = q^j b_{-k}^{m}$

$\Longleftrightarrow i = j$, and $\ell = m$, so ϕ is injective.

Since E is generated by q and b_0 and *B(1,2)* is generated by

a and $b = b_0$ and $\phi(ab_0) = qb_0$.

Hence ϕ is surjective.

Thus, ϕ is an isomorphism. □

Thus, we have a good understanding of the nature of *B(1,n)*.

2.2.2 Pairwise distinctions amongst the groups *B(m,n)*

Definition 2.2.10. *Let G be a group. The* ***derived series (commutator series)*** $G \geq G' \geq G'' \geq G''' \geq \cdots \geq G^k \geq \cdots$ *of G is defined by*

$G^{(0)} = G, \quad G^{(k+1)} = [G^{(k)}, G^{(k)}]$. *Recall that $[G,G] = gp([x,y] \mid x, y \in G)$. The* ***first derived group*** *or* ***commutator subgroup*** *of G is defined by $G^{^{(1)}} = [G,G]$. The* ***second derived group*** *of G is defined by $G^{^{(2)}} = [G^{^{(1)}}, G^{^{(1)}}]$.*

Definition 2.2.11. *A group whose second derived group is trivial is called* ***metabelian****.*

Thus a group G is metabelian if and only if it has a normal abelian subgroup A such that G/A is abelian. This follows from the lemma below:

Lemma 2.2.12. *Let G be a group and let N be a normal subgroup of G, then*

$$[G/N, G/N] = [G, G]N/N.$$

Proof. $[G/N, G/N] = gp([gN, hN] \mid g, h \in G)$

$\iff \{[g_1N, h_1N][g_2N, h_2N] \cdots [g_kN, h_kN] \mid g_k, h_k \in G\}$

$\iff \{[g_1, h_1]N[g_2, h_2]N \cdots [g_k, h_k]N \mid g_k, h_k \in G, \}$

$\iff \{[g_1, h_1][g_2, h_2] \cdots [g_k, h_k]N \mid g_k, h_k \in G\}$

$\iff \{[g_1, h_1][g_2, h_2] \cdots [g_k, h_k]nN \mid \forall n \in N,\ g_k, h_k \in G\}$

$\iff \{[g_1, h_1][g_2, h_2] \cdots [g_k, h_k]NN \mid g_k, h_k \in G\}$

$\iff \{[g_1, h_1][g_2, h_2] \cdots [g_k, h_k]N/N \mid g_k, h_k \in G\}$

$\iff gp([g, h]N \mid g, h \in G)/N$

$\iff [G, G]N/N.$ □

Definition 2.2.3. *A subgroup H of a group G is* ***normal*** *in G if it is invariant under all inner automorphisms of G, i.e., if $\iota_g(H) = H, \forall\, \iota_g \in$*

$Inn(G)$; equivalently, if $g^{-1}Hg = H, \forall\, g \in G$.

Definition 2.2.4. *A subgroup H of a group G is **characteristic** in G if it is invariant under all automorphisms of G, i.e., if* $\alpha(H) = H,\ \ \forall\, \alpha \in Aut(G)$.

Hence every characteristic subgroup of G is normal in G.

Lemma 2.2.13. *The relation "characteristic" is transitive, i.e., if $K \leq H \leq G$ with K characteristic in H, and H characteristic in G, then K is characteristic in G; in short, if $K \leq_c H \leq_c G$, then $K \leq_c G$.*

Proof. Let α be an automorphism of G ($\alpha \in Aut(G)$); and let $\breve{\alpha}$ the restriction of α to H. Then since H is characteristic in G, $\alpha(H) = H$, so $\alpha \in Aut(H)$. But $\breve{\alpha}(H) = H$. So $\alpha, \breve{\alpha} \in Aut(H)$. Since K is characteristic in H, $\alpha(K) = \breve{\alpha}(K) = K$. Hence K is characteristic in G. □

Lemma 2.2.14. *If $K \leq H \leq G$ with K characteristic in H, and H normal in G, then K is normal in G; in short, if $K \leq_c H \trianglelefteq G$, then $K \trianglelefteq G$.*

Proof. Let $g \in G$. Since $H \trianglelefteq G$, the conjugation map ($\iota_g \in Inn(G)$, i.e., $g^{-1}Hg = H$) induced by g, $\forall g \in G$, maps H to itself, and defines an automorphism of H ($\iota_g \in Aut(H), \forall g \in G$). Since $K \leq_c H$, this automorphism maps K to itself ($\iota_g(K) = K, \forall g \in G$) and so $g^{-1}Kg = K$, $\forall g \in G$, therefore, $K \trianglelefteq G$. □

Since we will be dealing with isomorphisms, it is then appropriate to state the isomorphism theorems.

Theorem 2.2.15. ***(First Isomorphism Theorem)*** *Let G and H be groups and let $\varphi : G \longrightarrow H$ be a homomorphism. Then*

$$G/ker(\varphi) \cong Im(\varphi).$$

Theorem 2.2.16. ***(Second Isomorphism Theorem)*** *Let H be a subgroup of G and let N be normal subgroup of G. Then N is a normal subgroup of HN, HN is a subgroup of G and*

$$HN/N \cong H/H \cap N.$$

Theorem 2.2.17. ***(Third Isomorphism Theorem)*** *Let G be a group and let H and N be normal subgroups of G with N a subgroup of H. Then H/N is normal in G/N and*

$$G/N/H/N \cong G/H.$$

Lemma 2.2.18. *Let G and H be arbitrary groups. Let $\varphi : G \xrightarrow{\sim} H$ be an isomorphism. Then*

(i) $\varphi([G, G]) = [H, H]$,

(ii) $\varphi([G^{(1)}, G^{(1)}]) = [H^{(1)}, H^{(1)}]$,

(iii) *φ induces an isomorphism* $\widetilde{\varphi} : G/[G,G] \xrightarrow{\sim} H/[H,H]$,

(iv) *φ induces an isomorphism* $\widehat{\varphi} : G/G^{(2)} \xrightarrow{\sim} H/H^{(2)}$,

(v) *φ induces an isomorphism* $\check{\varphi} : G^{(1)}/G^{(2)} \xrightarrow{\sim} H^{(1)}/H^{(2)}$.

Proof. (i) Let $a, b \in G$. Notice

$\varphi([a,b]) = \varphi(a^{-1}b^{-1}ab) = \varphi(a^{-1})\varphi(b^{-1})\varphi(a)\varphi(b) = \varphi(a)^{-1}\varphi(b)^{-1}\varphi(a)\varphi(b)$ $= [\varphi(a), \varphi(b)]$ because φ is also a homomorphism. Hence products of commutators goes to products of commutators. Hence $\varphi([G,G]) \subseteq [H,H]$. Since $\varphi : G \longrightarrow H$ is an isomorphism, there is the corresponding map $\varphi^{-1} : H \longrightarrow G$. Hence similarly $\varphi^{-1}([H,H]) \subseteq [G,G]$. Now if we compose these two maps, then $\varphi \circ \varphi^{-1} =_{[H,H]} 1$ and $\varphi^{-1} \circ \varphi =_{[G,G]} 1$. This implies that φ induces an isomorphism from $[G,G]$ into $[H,H]$.

(ii) Similar to the proof of (i) by just relabeling.

(iii) We consider the map $\widetilde{\varphi} : G/[G,G] \longrightarrow H/[H,H]$ defined by

$\widetilde{\varphi}(g[G,G]) = \varphi(g)[H,H]$, $g \in G$.

(a) Then $\widetilde{\varphi}$ is a well defined map, for if g_1, $g_2 \in G$,

and $\widetilde{\varphi}(g_1[G,G]) = \widetilde{\varphi}(g_2[G,G])$, that is, $\varphi(g_1)[H,H] = \varphi(g_2)[H,H]$,

i.e., $\varphi(g_1)^{-1}\varphi(g_2) \in [H,H]$ which implies that $\varphi(g_1^{-1}g_2) \in [H,H]$ since φ is a homomorphism. Applying φ^{-1} to the above we get

$g_1^{-1}g_2 \in \varphi^{-1}([H,H]) = [G,G]$ as needed.

(b) An easy check shows that $\widetilde{\varphi}$ a homomorphism.

$\widetilde{\varphi}(g_1[G,G]g_2[G,G]) = \widetilde{\varphi}(g_1g_2[G,G]) = \varphi(g_1g_2)[H,H]$

$= \varphi(g_1)\varphi(g_2)[H,H] = \varphi(g_1)[H,H]\varphi(g_2)[H,H] = \widetilde{\varphi}(g_1[G,G])\widetilde{\varphi}(g_2[G,G])$.

(c) $\widetilde{\varphi}$ is surjective since φ was such: given an element $h[H,H] \in H/[H,H]$, the choice $g = \varphi^{-1}(h)$ assures that $\widetilde{\varphi}(g[G,G]) = h[H,H]$, i.e.,

$\varphi(\varphi^{-1}(h))[H,H] = h[H,H]$.

(d) $\widetilde{\varphi}$ injective as well for suppose $\widetilde{\varphi}(g[G,G]) =_{H/[H,H]} 1$,

then $\varphi(g)[H,H] = [H,H]$, so $\varphi(g) \in [H,H]$. Applying φ^{-1}, we get

$g \in \varphi^{-1}([H,H]) = [G,G]$, so $(g[G,G]) =_{G/[G,G]} 1$.

So indeed $\widetilde{\varphi}$ has trivial kernel.

Hence, φ induces an isomorphism $\widetilde{\varphi} : G/[G,G] \xrightarrow{\sim} H/[H,H]$.

The argument produced for (iii) can be replicated with the obvious notational adjustments to get (iv) and also (v) once we appeal to the fact that $G^{(2)}$ is noted to be a characteristic subgroup of $G^{(1)}$ and G. $\square$

Note: We can make some distinctions amongst the Baumslag-Solitar groups *B(m,n)* using Tietze transformations.

Lemma 2.2.19. *The Baumslag-Solitar groups*

B(–m,–n) $=< a, b;\ a^{-1}b^{-m}a = b^{-n} \,|\, m \neq 0,\ n \neq 0,\ m,\ n \in \mathbb{Z} >$

can be written as B(m,n) $=< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0,\ n \neq 0,\ m,\ n \in \mathbb{Z} >$.

Proof. We can achieve this result via Tietze transformations, i.e., in this case, multiplying the relation by appropriate powers of the generators.

$B(-m,-n) =< a, b\,;\ a^{-1}b^{-m}a = b^{-n} >$

$=< a, b\,;\ b^{-m}a = ab^{-n} >$

$=< a, b\,;\ a = b^m ab^{-n} >$

$=< a, b\,;\ ab^n = b^m a >$

$=< a, b\,;\ b^n = a^{-1}b^m a >= B(m,n)$ □

Theorem 2.2.20. *Let* $G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m,\ n \in \mathbb{Z} >$ *and let* $H = B(m', n') =< x, y;\ x^{-1}y^{m'}x = y^{n'} \,|\, m' \neq 0,\ n' \neq 0,\ m',\ n' \in \mathbb{Z} >$. *If* $n = m + 1$ *and* $G \cong_\varphi H$, *then* $n' = m' + 1$, *moreover,* $m = m'$ *and* $n = n'$ *must hold.*

We will prove the theorem above by proving series of lemmas. But first observe that $G = gp(a, b)$. Let $\bar{a}$ be the coset representative of a in $G/[G, G]$ and $\bar{b}$ be the coset representative of b in $G/[G, G]$.

Then $G/[G, G] = gp(a[G, G], b[G, G]) = gp(\bar{a}, \bar{b})$.

Lemma 2.2.21. *Let* $G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m,\ n \in \mathbb{Z} >$, *if* $n = m + 1$, *then* $G/[G, G] = gp(a[G, G]) =< \bar{a} >$.

Proof. $G/[G,G]$ can be presented as follows:

$G/[G,G] =< a, b;\ a^{-1}b^m a = b^{m+1}, ab = ba >$

$=< a, b;\ b^m a = ab^{m+1},\ ab = ba > \quad T2/T2'$

$=< a, b;\ b^m a = ab^{m+1},\ b^{m-1}ab = b^m a > \quad T2/T2'$

$=< a, b;\ ab^{m+1} = b^{m-1}ab,\ b^m a = ab^{m+1},\ b^{m-1}ab = b^m a > \quad T2$

$=< a, b;\ ab^{m+1} = b^{m-1}ab,\ b^{m-1}ab = b^m a > \quad T2'$

$=< a, b;\ ab^m = b^{m-1}a,\ ab = ba > \quad T2/T2'$

$=< a, b;\ b^{m-1}a = ab^m,\ ab = ba >$ rearranging the relation above

$\vdots$

$=< a, b;\ b^{m-m}a = ab^{m-(m-1)},\ ab = ba >$

$=< a, b;\ a = ab,\ ab = ab >$

$=< a, b;\ 1 = b,\ ab = ab > \quad T2/T2'$

$=< a, b;\ 1 = b > \quad T2/T2'$

$=< a > \quad T1'$

$= gp(a[G,G]) =< \bar{a} > .$ □

So $G/[G,G]$ is infinite cyclic on $a[G,G] = \bar{a}$. Now let $B = gp_G(b)$, then it follows that

$G/B =< a, b;\ a^{-1}b^m a = b^{m+1}, b = 1 >$

$=< a, b;\ b = 1 >$

$=< a >$

$= gp(aB)$.

So G/B is infinite cyclic on aB.

Furthermore, by Magnus' method or by the method of Reidemeister and Schreier, one can present B. Indeed if we use Magnus' method and set $b_i = a^{-i}ba^i$ $(i \in \mathbb{Z})$, then we find that B has the following presentation:

$$B =< ..., b_i, ...; ..., b_i^m = b_{i-1}^{m+1}, ... > .$$

Lemma 2.2.22. *The group $B = gp_{_G}(b) = [G, G]$.*

Proof. We know that $[G, G]$ is the smallest normal subgroup of G such that the factor group $G/[G, G]$ is abelian. We also know that G/B is infinite cyclic on aB, where $B = gp_{_G}(b)$, in particular, G/B is abelian. So $B \supseteq [G, G]$. Therefore, we have (i) $[G, G] \leq B$.

We need to show that (ii) $B \leq [G, G]$.

Now $G/[G, G] = gp(a[G, G], b[G, G])$. We focus on the relation $a^{-1}b^m a = b^{m+1}$ in $G/[G, G]$. Looking at $G/[G, G]$ in terms coset representatives, we have $(b[G, G])^{m+1} = b^{m+1}[G, G] = (a^{-1}b^m a)[G, G] = b^m[G, G]$.

So $b^{m+1}[G, G] = b^m[G, G]$. Multiplying b^{-m}, it follows that $b[G, G] = [G, G]$ and therefore that $b \in [G, G]$. Observe that B is generated by the conjugates of b. If $b \in [G, G]$, then the conjugates of b are in $[G, G]$, and so all the products of the conjugates are in $[G, G]$. Therefore all of B is in $[G, G]$. So $B \leq [G, G]$. Thus, $B = [G, G]$. $\square$

Now $H = B(m', n') =< x, y;\, x^{-1} y^{m'} x = y^{n'} >$. But by **Lemma 2.2.18** and **Lemma 2.2.21**, if $n = m + 1$ and $G \cong_{\varphi} H$, then $G/G^{(1)} \cong_{\widetilde{\varphi}} H/H^{(1)}$ and since $G/[G, G]$ is infinite cyclic on $a[G, G] = \bar{a}$, then $H/[H, H]$ must be infinite cyclic on $x[H, H] = \bar{x}$, hence $n' = m' + 1$.

To prove the second part of the theorem, we then need to look at $G/G^{(2)}$ and $H/H^{(2)}$. But since the structure of $G/G^{(2)}$ is difficult to unravel at the moment, first we will look at its subgroup $B/[B, B] = [G, G]/G^{(2)}$ and make some associations with a subgroup of the rational numbers. To simplify our notation, let $B^\sharp = B/[B, B] = [G, G]/G^{(2)} = gp(b_i G^{(2)}) = gp(b'_i)$, where b'_i is the coset representative of b_i in $[G, G]/G^{(2)}$.

So

$$B^\sharp =< ..., b'_i, ...; ..., b'^{m}_i = b'^{m+1}_{i-1}, ..., ..., [b'_i, b'_j] = 1, ...(i, j \in \mathbb{Z}) > .$$

Lemma 2.2.23. *Let*

$$\Lambda_{m(m+1)} = \{ \frac{\ell}{(m(m+1))^k} \mid m > 0,\, \ell,\, k \in \mathbb{Z} \}$$

be an additive subgroup of $\mathbb{Q}$*. Then*

$$\Lambda_{m(m+1)} \cong B^{\sharp} = [G,G]/G^{(2)}.$$

Proof. We want to compare $B^{\sharp}$ to $\Lambda_{m(m+1)}$. Note that $B^{\sharp}$ is a multiplicative group and $\Lambda_{m(m+1)}$ is an additive group. We define a map μ from the set of generators of $B^{\sharp}$ to $\Lambda_{m(m+1)}$ as follows: $\mu : b'_i \mapsto \left(\frac{m+1}{m}\right)^i$. Observe that $\left(\frac{m+1}{m}\right)^i$, $i > 0$ is in $\Lambda_{m(m+1)}$ because $\left(\frac{m+1}{m}\right)^i\left(\frac{m+1}{m+1}\right)^i = \frac{(m+1)^{2i}}{m^i(m+1)^i}$ is of the form $\frac{\ell}{(m(m+1))^k}$, where $(m+1)^{2i} = \ell$ and $i = k$.
Also $\left(\frac{m}{m+1}\right)^i$, $i > 0$ is in $\Lambda_{m(m+1)}$ because $\left(\frac{m}{m+1}\right)^i\left(\frac{m}{m}\right)^i = \frac{m^{2i}}{m^i(m+1)^i}$ is of the form $\frac{\ell}{(m(m+1))^k}$, where $m^{2i} = \ell$ and $i = k$.
We claim that the defining relations of $B^{\sharp}$ go into relations in $\Lambda_{m(m+1)}$. To see this, observe that the images of b'_i and b'_{i-1} under μ are $\left(\frac{m+1}{m}\right)^i$ and $\left(\frac{m+1}{m}\right)^{i-1}$ respectively and that of $b'^m_i = b'^{m+1}_{i-1}$ is $m\left(\frac{m+1}{m}\right)^i =$ (m+1)$\left(\frac{m+1}{m}\right)^{i-1}$, which means that the images of the defining relations for $B^{\sharp}$ are relations in $\Lambda_{m(m+1)}$. Observe that the commuting relations in $\Lambda_{m(m+1)}$ are automatically satisfied because $\Lambda_{m(m+1)} = \left\{\frac{\ell}{(m(m+1))^k} \mid m > 0,\ \ell,\ k \in \mathbb{Z}\right\}$ is an abelian group. So μ extends to a homomorphism from $B^{\sharp}$ to $\Lambda_{m(m+1)}$ by Von Dyck's theorem.
Notice that if we take the element $(b'_0 b'^{-1}_1)^{m+1} = b'^{m+1}_0 b'^{-m}_1 b'^{-1}_1 = b'^m_1 b'^{-m}_1 b'^{-1}_1 =$

$b_1'^{-1}$ and the element $(b_0'b_1'^{-1})^m = b_0'^m b_1'^{-m} = b_0'^m b_0'^{-m} b_0'^{-1} = b_0'^{-1}$. So the subgroup generated by b_0' and b_1' is infinite cyclic on $b_0'b_1'^{-1}$. Let $c_0' = b_0'b_1'^{-1}$ and $c_1' = b_1'b_2'^{-1}$, then we see that $c_1'^m = c_0'^{m+1}$ relation holds. Now continuing in this manner, the relation $c_i'^m = c_{i-1}'^{m+1}$ holds altogether. This implies that the subgroup generated by finite number of the b_i''s is infinite cyclic. Thus $B^\sharp$ is a union of infinite cyclic groups. Now we show μ is injective as follows:

In summary, consider a non-trivial element in $B^\sharp$; it lies in one of those infinite cyclic groups. If we take the generator of that infinite cyclic group, then some power of it give b_0. Then its image under μ will be non-trivial because b_0 is non-trivial.

In particular, suppose the element $b' \neq 1 \in B^\sharp$, and $B^\sharp$ is the union of infinite cycles generated by, say v_i'. So $gp(b_0') = gp(v_0') < gp(v_1') < gp(v_2') < ....$ So $b' \in v_i'$ for some choice of i, say $b' = v_i'^k$, $k \neq 0$, since $b' \neq 1$. Notice that $b_0' \in gp(v_i')$ implies that $b_0' = v_i^r$, $r \neq 0$. Then $\mu(b_0) = 1 \neq 0$, implies that $\mu(v_i^k) = k\mu(v_i) \neq 0$, which implies that $\mu(v_i) \neq 0$. Therefore, $\mu(b') = \mu(v_i) \neq 0$. Hence, μ is injective. Now we show μ is surjective, that is, given and ℓ and i in $\frac{\ell}{m^i(m+1)^i}$, can we find an element in $B^\sharp$ that maps to it? Notice that the $gcd(m^i, (m+1)^i) = 1$. So there exists integers α and β such that $\alpha(m^i) + \beta(m+1)^i = 1$, that is, $\ell\alpha(m^i) + \ell\beta(m+1)^i = \ell$. Observe that

$$\begin{aligned}\mu(b'^{\ell\alpha}_{-i}b'^{\ell\alpha}_{i}) &= \ell\alpha(\frac{m}{m+1})^i + \ell\beta(\frac{m+1}{m})^i \\ &= \ell\alpha(\frac{m^i}{(m+1)^i}) + \ell\beta(\frac{(m+1)^i}{m^i}) \\ &= \frac{\ell\alpha(m^i)(m^i) + \ell\beta(m+1)^i(m+1)^i}{(m+1)^i(m^i)} \\ &= \frac{\ell\alpha(m^{2i}) + \ell\beta(m+1)^{2i}}{(m(m+1))^i} \\ &= \frac{\ell}{(m(m+1))^i}.\end{aligned}$$

Hence μ is an isomorphism. □

The exact same argument works for H. Now

$$Y =< ..., y_i, ...; ..., y_i^{m'} = y_{i-1}^{m'+1}, ... > .$$

Next we want to look at $Y/[Y, Y]$, i.e., $[H, H]/H^{(2)}$. Let $Y^\sharp = Y/[Y, Y] = [H, H]/H^{(2)}$.

So

$$Y^\sharp =< ..., y'_i, ...; ..., y_i'^{m'} = b_{i-1}'^{m'+1}, ..., ..., [y'_i, y'_j] = 1, ...(i, j \in \mathbb{Z}) > .$$

Therefore, the following lemma also holds:

Lemma 2.2.24. *Let*

$$\Lambda_{m'(m'+1)} = \{\frac{\ell}{(m'(m'+1))^k} \mid m' > 0,\ \ell,\ k \in \mathbb{Z}\}$$

be an additive subgroup of $\mathbb{Q}$*. Then*

$$\Lambda_{m'(m'+1)} \cong Y^{\sharp} = [H,H]/H^{(2)}.$$

Proof. Similar to the proof of the lemma above. □

Now let $L = G/G^{(2)}$. Let Let a'' be the coset representative of a in $G/G^{(2)}$ and b'' be the coset representative of b in $G/G^{(2)}$. Then

$$L = G/G^{(2)} = gp(aG^{(2)}, bG^{(2)}) = gp(a'', b'').$$

We can to identify L with $G/[G,G]$. By the third isomorphism theorem,

$$L/B^{\sharp} = G/G^{(2)}/[G,G]/G^{(2)} \cong G/[G,G].$$

So $L/B^{\sharp}$ is infinite cyclic since $G/[G,G]$ is infinite cyclic. The fact is that since a is of infinite order modulo $[G,G] = B$, then we see that a'' is of infinite order modulo $B^{\sharp}$.

Similarly let

$$M = H/H^{(2)} = gp(xH^{(2)}, yH^{(2)}).$$

And let $x'' = xH^{(2)}$ and $y'' = yH^{(2)}$ be the coset representatives, then $M = gp(x'', y'')$. By the third isomorphism theorem,

$$M/Y^{\sharp} \cong H/H^{(2)}/[H,H]/H^{(2)} \cong H/[H,H].$$

So $M/Y^\sharp$ is infinite cyclic since $H/[H,H]$ is infinite cyclic. And since x is of infinite order modulo $[H,H]=Y$, then we see that x'' is of infinite order modulo $Y^\sharp$.

In general, any Baumslag-Solitar group can be written as semi-direct product B(m,n) $= B \rtimes < a >$, where $B = gp_G(b)$.

Observe that in this case,

$$G = G^{(1)} \rtimes < a >$$

and the relation $a^{-1}b^m a = b^{m+1}$ holds in G, and the action of a on $G^{(1)}$ which shifts the b_i's in $G^{(1)}$ induces a similar action of a'' on $G^{(1)}/G^{(2)}$. Hence

$$G/G^{(2)} = G^{(1)}/G^{(2)} \rtimes < aG^{(2)} >$$

which is

$$L = B^\sharp \rtimes < a'' > .$$

Thus, a typical element of L has the form $\{a''^k b' | \, k \in \mathbb{Z}, \, b' \in B^\sharp\}$ and the relation $a''^{-1} b'^m a'' = b'^{m+1}$ holds in L. Similarly observe that

$$H = H^{(1)} \rtimes < x >$$

and the relation $x^{-1}y^{m'}x = y^{m'+1}$ holds in H, and the action of x on $H^{(1)}$

which shifts the y_i's induces a similar action of x'' on $H^{(1)}/H^{(2)}$. Hence

$$H/H^{(2)} = H^{(1)}/H^{(2)} \rtimes < xH^{(2)} >$$

which is

$$M = Y^{\sharp} \rtimes < x'' > .$$

So a typical element of M has the form $\{x''^{k} y' |\, k \in \mathbb{Z},\, y' \in Y^{\sharp}\}$ and the relation $x''^{-1} y'^{m'} x'' = y'^{m'+1}$ holds in M.

Now L is an extension of an abelian group by an abelian group, i.e., L is metabelian. Similarly, M is an extension of an abelian group by an abelian group, i.e., M is metabelian. Observe that $L/[L, L]$ is generated by $a''[L, L]$ is infinite cyclic. It is isomorphic to the factor group $G/[G, G]$ by the third isomorphism theorem.

Lemma 2.2.25.

$$[L, L] = B^{\sharp}$$

and

$$[M, M] = Y^{\sharp}.$$

Proof. Clearly it is sufficient to just prove the first. We make a direct application of the **Lemma 2.2.12**, that is,

$[L, L] = [G/G^{(2)}, G/G^{(2)}] = [G, G]G^{(2)}/G^{(2)} =_{G^{(2)} \trianglelefteq G^{(1)}} [G, G]/G^{(2)}$

$= B/[B, B] = B^{\sharp}$. □

Recall if $\widehat{\varphi} : L \xrightarrow{\sim} M$ is an isomorphism, then

(i) $\widehat{\varphi}([L, L]) = [M, M]$;

(ii) $\widehat{\varphi}$ induces an isomorphism $\overline{\varphi} : L/[L, L] \xrightarrow{\sim} M/[M, M]$.

Also recall that $L/[L, L]$ is infinite cyclic and so $M/[M, M]$ must be infinite cyclic by the induced isomorphism above. We look at the induce isomorphism $\overline{\varphi} : L/[L, L] \xrightarrow{\sim} M/[M, M]$ defined by $\overline{\varphi}(a''[L, L]) = \widehat{\varphi}(a'')\widehat{\varphi}([L, L)]) = \widehat{\varphi}(a'')[M, M]$. The generators of $M/[M, M]$ is $x''[M, M]$ or $x''^{-1}[M, M]$ since $M/[M, M]$ is infinite cyclic. An isomorphism takes generators to generators. So we have either

(i) $\widehat{\varphi}(a'')[M, M] = x''[M, M]$ or (ii) $\widehat{\varphi}(a'')[M, M] = x''^{-1}[M, M]$.

Now for (i) $\widehat{\varphi}(a'')[M, M] = x''[M, M]$,

$\Longrightarrow x''^{-1}\widehat{\varphi}(a'')[M, M] = [M, M]$,

$\Longrightarrow x''^{-1}\widehat{\varphi}(a'') \in [M, M]$,

$\Longrightarrow x''^{-1}\widehat{\varphi}(a'') = s$, where $s \in [M, M]$,

$\Longrightarrow \widehat{\varphi}(a'') = x''s$, where s is uniquely determined by $\widehat{\varphi}(a'')$.

Now we look at the relation $a''^{-1}b'^{m}a'' = b'^{m+1}$ in L,

$a'' \neq 1$, $b' \neq 1$, under $\widehat{\varphi}$.

$\widehat{\varphi} : L \xrightarrow{\sim} M$. So $\widehat{\varphi}(a''^{-1} b'^m a'') = \widehat{\varphi}(b'^{m+1})$,

$\Longrightarrow \widehat{\varphi}(a'')^{-1} \widehat{\varphi}(b'^m) \widehat{\varphi}(a'') = \widehat{\varphi}(b'^{m+1}),\ b' \in B^\sharp = [L, L]$,

$\Longrightarrow (x''s)^{-1} y'^m (x''s) = y'^{m+1}$, where and $y' \in [M, M]\ \ y' \neq 1$,

$\Longrightarrow s^{-1} x''^{-1} y'^m x'' s = y'^{m+1}$,

$\Longrightarrow x''^{-1} y'^m x'' = s y'^{m+1} s^{-1}$, and since elements in $[M, M]$ commutes,

$\Longrightarrow x''^{-1} y'^m x'' = y'^{m+1}$.

By taking the m'^{th} power of $x''^{-1} y'^m x'' = y'^{m+1}$,

$\Longrightarrow (x''^{-1} y'^m x'')^{m'} = (y'^{m+1})^{m'}$,

$\Longrightarrow x''^{-1} y'^{mm'} x'' = (y'^{m+1})^{m'}$,

$\Longrightarrow (x''^{-1} y'^{m'} x'')^m = (y'^{m+1})^{m'}$. But we know that $x''^{-1} y'^{m'} x'' = y'^{m'+1}$, holds in M. So by taking the m^{th} power of $x''^{-1} y'^{m'} x'' = y'^{m'+1}$,

$\Longrightarrow (x''^{-1} y'^{m'} x'')^m = (y'^{m'+1})^m$,

$\Longrightarrow (y'^{m'+1})^m = (y'^{m+1})^{m'}$,

$\Longrightarrow y'^{m'm+m} = y'^{m'm+m'}$,

$\Longrightarrow m'm + m = m'm + m'$.

So that $m = m'$, $y' \neq 1$, y' is of infinite order since $y' \in [M, M] = Y^\sharp \cong [H, H]/H^{(2)} \cong \Lambda_{m'(m'+1)} = \{\frac{\ell}{(m'(m'+1))^k} |\ m' > 0,\ \ell,\ k \in \mathbb{Z}\}$, and the subgroup $\Lambda_{m'(m'+1)}$ of the rational numbers has no elements of finite order. Now $m = m'$ and from our previous conditions $n = m + 1$ and $n' = m' + 1$ hold, hence $n = n'$. In case (ii) the result is similar. This completes the proof of **Theorem 2.2.20.**

Thus, if $m \neq m'$, then L and M are not isomorphic and therefore

$G = B(m,\ m{+}1)$ is not isomorphic to $H = B(m', n')$.

We can now strengthen this theorem to an even more general form which would constituent our **main theorem**:

Theorem 2.2.26. *Let* $G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0,\ n \neq 0,\ m,\ n \in \mathbb{Z} >$ *and let* $H = B(m', n') =< x, y;\ x^{-1}y^{m'}x = y^{n'} \,|\, m' \neq 0,\ n' \neq 0,\ m',\ n' \in \mathbb{Z} >$*. Then* $G \cong_\varphi H$*, if and only if* $m = m'$ *and* $n = n'$*.*

We will prove this theorem above by proving a series of lemmas:

Lemma 2.2.27. *Let* $G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$ *and let* $H = B(m', n') =< x, y;\ x^{-1}y^{m'}x = y^{n'} \,|\, m' \neq 0, n' \neq 0,\ m', n' \in \mathbb{Z} >$*. If* $m = m' = 1$ *and* $G \cong_\varphi H$*, then* $n = n'$*.*

Proof. We make use of **Lemma 2.2.18**. If $G \cong_\varphi H$, then $G/G^{(1)} \cong_{\tilde{\varphi}} H/H^{(1)}$.

$G/G^{(1)} =< a, b;\ a^{-1}ba = b^n, ab = ba >$

$=< a, b;\ ba = ab^n, ab = ba > \qquad T2/T2'$

$=< a, b;\ b = b^n, ab = ba > \qquad T2/T2'$

$=< a, b;\ 1 = b^{n-1}, ab = ba > \qquad T2/T2'$

$=< a > \times < b;\ 1 = b^{n-1} > = C_\infty \times C_{n-1}.$

Then we look at $H/H^{(1)} =< x, y;\ x^{-1}yx = y^{n'}, xy = yx >$

$=< x, y;\ yx = xy^{n'}, xy = yx >$ $\quad T2/T2'$

$=< x, y;\ y = y^{n'}, xy = yx >$ $\quad T2/T2'$

$=< x, y;\ 1 = y^{n'-1}, xy = yx >$ $\quad T2/T2'$

$=< x > \times < y;\ 1 = y^{n'-1} >$

$= C_\infty \times C_{n'-1}$. So

$\Longrightarrow C_\infty \times C_{n-1} \cong C_\infty \times C_{n'-1}$, (if A, B abelian and $A \cong B$, then $\tau(A) \cong \tau(B)$)

$\Longrightarrow C_{n-1} \cong C_{n'-1}$, (if A, B finite abelian and $A \cong B$, then $|A| = |B|$)

$\Longrightarrow n - 1 = n' - 1,$

$\Longrightarrow n = n'.$ □

So if $n \neq n'$, then $G/G^{(1)} \ncong_{\widetilde{\varphi}} H/H^{(1)}$

$\Longrightarrow G \ncong_{\varphi} H.$

Corollary 2.2.28. *Let* G = *B(m,n)* $=< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$. *Then* $G/G^{(1)} = C_\infty \times C_{n-m}$.

Proof. $G/G^{(1)} =< a, b;\ a^{-1}b^m a = b^n,\ ab = ba >$

$=< a, b;\ b^m a = ab^n,\ ab = ba >$

$=< a, b;\ b^m a = ab^n,\ b^{m-1}ab = b^m a >$

$=< a, b;\ ab^n = b^{m-1}ab,\ b^{m-1}ab = b^m a >$

$=< a, b;\ ab^{n-1} = b^{m-1}a,\ ab = ba >$

$=< a, b;\ b^{m-1}a = ab^{n-1},\ ab = ba >$

$\vdots$

$=< a, b;\ b^{m-m}a = a^{n-m},\ ab = ba >$

$=< a, b;\ a = ab^{n-m},\ ab = ba >$

$=< a, b;\ 1 = b^{n-m},\ ab = ba >$

$=< a > \times < b;\ 1 = b^{n-m} >$

$= C_\infty \times C_{n-m}.$ □

Corollary 2.2.29. *Let* $G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$. *If* $m = n$*, then* $G/G^{(1)} = C_\infty \times C_\infty$.

Corollary 2.2.30. *Let* $G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$. *If* $n = m + 1$*, then* $G/G^{(1)} = C_\infty$.

Lemma 2.2.31. *Let* $G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$. *Let* $H = B(m', n') =< x, y;\ x^{-1}y^{m'}x = y^{n'} \,|\, m' \neq 0, n' \neq 0,\ m', n' \in \mathbb{Z} >$. *If* $G \cong_\varphi H$*, then* $C_\infty \times C_{n-m} \cong G/G^{(1)} \cong_{\tilde{\varphi}} H/H^{(1)} \cong C_\infty \times C_{n'-m'}$ *and* $n - m = n' - m'$.

Proof. We know that if $G \cong_\varphi H$, then $G/G^{(1)} = C_\infty \times C_{n-m} \cong_{\tilde{\varphi}} H/H^{(1)} =$

$C_\infty \times C_{n'-m'}$. The groups $C_\infty \times C_{n-m}$ and $C_\infty \times C_{n'-m'}$ are finitely generated abelian groups. If two abelian groups are isomorphic, then their torsion subgroups are isomorphic, so $C_{n-m} \cong C_{n'-m'}$. If two finite cyclic groups are isomorphic then they both have the same order. So $n - m = n' - m'$. □

Let us further examine the group

$G = B(m,n) =< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$ when $m = n$.

Let's call that group $B(m) =< a, b;\ a^{-1}b^m a = b^m \,|\, m \neq 0,\ m \in \mathbb{Z} >$

$=< a, b;\ b^m a = ab^m \,|\, m \neq 0,\ m \in \mathbb{Z} >$.

Lemma 2.2.32. *The center of the group* $B(m) =< a, b;\ a^{-1}b^m a = b^m \,|\, m \neq 0,\ m \in \mathbb{Z} >$ *is* $gp(b^m)$.

Proof. Now b^m commutes with a and b^m commutes with b. Thus b^m commutes with generators of $B(m)$, so b^m commutes all elements of $B(m)$. So $gp(b^m) \subseteq \zeta(B(m))$, where we denote the center of $B(m)$ to be $\zeta(B(m))$. Let us look at $B(m)/gp(b^m)$. The subgroup $gp(b^m)$ is normal because it is in the center, so we can present the group

$B(m)/gp(b^m) =< a, b;\ a^{-1}b^m a = b^m,\ b^m = 1 >$

$=< a, b;\ b^m = 1 >$

$= C_\infty * C_m$.

We now understand this factor group, because it is a free product. Free products have trivial centers. So the center of $B(m)/gp(b^m) = C_\infty * C_m$ is trivial. Hence $gp(b^m)$ must be the entire center. $\square$

Thus, notice that if two groups are isomorphic, their centers are isomorphic, and the factor groups by their centers are isomorphic. Hence, this leads to the following:

Corollary 2.2.33. *If* $B(m) \cong_\varphi B(m')$*, then* $B(m)/gp(b^m) = C_\infty * C_m \cong_{\dot{\varphi}} C_\infty * C_{m'} = B(m')/gp(b^{m'})$ *and* $m = m'$.

We have the following as a result of **Lemma 2.2.31** and **Corollary 2.2.33** above.

Lemma 2.2.34. *Let* $G = B(m,n) =< a, b;\, a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$ *and* $H = B(m', n') =< x, y;\, x^{-1}y^{m'}x = y^{n'} \,|\, m' \neq 0, n' \neq 0,\ m', n' \in \mathbb{Z} >$*. If* $m = n$ *and* $G \cong_\varphi H$*, then* $m = m'$ *and* $n = n'$.

In the foregoing cases we relied on the fact that any isomorphism between two groups possessing torsion-free abelian quotients induces an isomorphism between such corresponding quotients. For the remaining cases that involve potential torsion in the abeilan quotients we proceed as follows:

Lemma 2.2.35. *Let G and H be any pair of groups and suppose φ is an isomorphism from G to H. Then the following holds:*

(i) If $S = \{x \in G \mid x^n \in G^{(1)} for\, some\; n > 0\}$ ($S/G^{(1)}$ is the torsion subgroup of $G/G^{(1)}$) and

$T = \{y \in H \mid y^n \in H^{(1)} for\, some\; n > 0\}$ ($T/H^{(1)}$ is the torsion subgroup of $H/H^{(1)}$), then $\varphi(S) = T$;

(ii) φ induces an isomorphism φ^* from G/S to H/T;

(iii) $\varphi(S^{(1)}) = T^{(1)}$;

(iv) φ induces an isomorphism $\widetilde{\varphi}^*$ from $G/S^{(1)}$ to $H/T^{(1)}$;

(v) φ induces an isomorphism $\widehat{\varphi}^*$ from $S/S^{(1)}$ to $T/T^{(1)}$.

Proof. Clearly S and T are subgroups of G and H respectively, in particular, for $u, v \in S$, with $u^m \in G^{(1)}$ and $v^k \in G^{(1)}$, computing in $G/G^{(1)}$ we have $(uv)^{mk}G^{(1)} = u^{mk}G^{(1)}v^{mk}G^{(1)} = (u^m \in G^{(1)})^k(v^k \in G^{(1)})^m = G^{(1)}$, so $(uv)^{mk} \in G^{(1)}$, hence S is multiplicatively closed by definition.

Moreover, we have closure under conjugation for S (and T).

For, by the definition of S, if $u \in S$, then there exists $m > 0$, such that $u^m \in G^{(1)}$, a normal subgroup of G. For $g \in G$, $(g^{-1}ug)^m = g^{-1}u^m g \in G^{(1)}$, hence $g^{-1}ug \in S$. (A similar argument applies for T).

Now for (i) recall if $u \in S$, then for some integer $n > 0$, $u^n \in G^{(1)}$. Since φ is a homomorphism, $\varphi(u^n) = (\varphi(u))^n \in H^{(1)}$ (**Lemma 2.2.18**). So $\varphi(u) \in T$, for all $u \in S$, therefore $\varphi(S) \subseteq T$.

However, φ^{-1} an isomorphism from H to G, so consequently $\varphi(S) = T$.

For (ii) we consider G/S and H/T, and the induced map φ^* :G/S$\longrightarrow H/T$ defined by $\varphi^*(gS) = \varphi(g)T$.

(1) Then φ^* is well defined, for if $g_1, g_2 \in G$, and $\varphi^*(g_1S) = \varphi^*(g_2S)$, that is, $\varphi(g_1)T = \varphi(g_2)T$, i.e., $\varphi(g_1)^{-1}\varphi(g_2) \in T$ which implies that $(g_1^{-1}g_2) \in S$.

(2) An easy check shows φ^* to be a homomorphism. That is, $\varphi^*(g_1Sg_2S)$ $= \varphi^*(g_1g_2S) = \varphi(g_1g_2)T = \varphi(g_1)\varphi(g_2)T = \varphi(g_1)T\varphi(g_2)T = \varphi^*(g_1S)\varphi^*(g_2S)$.

(3) Moreover, φ^* is surjective since φ was such: given an element $hT \in H/T$, the choice $g = \varphi^{-1}(h)$ assures that $\varphi^*(gS) = hT$, i.e., $\varphi(\varphi^{-1}(h))T = hT$.

(4) φ^* is injective as well for suppose $\varphi^*(gS) = 1_{H/T}$, i.e., $\varphi(g)T = T$, so $\varphi(g) \in T$ and so $g \in \varphi^{-1}(T) = S$, whence $gS = 1_{G/S}$ and so φ^* has trivial kernel and indeed φ induces an isomorphism $\varphi^* : G/S \xrightarrow{\sim} H/T$.

The argument produced for (i) and (ii) can be replicated with the obvious notational adjustments to get (iii) and also (iv) once we appeal to **Lemma 2.2.14** and observe that $S^{(1)} \trianglelefteq G$. □

Corollary 2.2.36. *Let* $S = \{g \in G \mid g^{\ell} \in G^{(1)} for\ some\ \ell > 0\}$ *and let*

$T = \{h \in H \mid h^\ell \in H^{(1)} for\ some\ \ell > 0\}$ *where* $G = B(\mathrm{m,n})$ *and* $H = B(m', n')$. *Then* $S = gp_G(b)$ *and* $T = gp_H(y)$ *respectively.*

Proof. Clearly it is sufficient to only to prove one of these claims.

Let $B = gp_G(b)$. We want to show $B = S$. First we must show $S \supseteq B$. Observe that $G/G^{(1)} \cong < a > \times < b; 1 = b^{n-m} > \cong C_\infty \times C_{n-m}$.

Now S contains b because $b^r \in G^{(1)}$ for some $r > 0$. So S contains the conjugates of b, and so it contains all of $B = gp_G(b)$. Second we must show $B \supseteq S$, ie., anything in S is contained in B. Notice that G/B is infinite cyclic on aB. In particular, G/B is abelian, so $B \supseteq G^{(1)}$. Let $g \in S$, so by definition, $g^\ell \in G^{(1)} \subseteq B$. Hence $g^\ell \in B$. Now $(gB)^\ell = g^\ell B = B$. So the element gB is of finite order. But we know that the elements gB of G/B are of infinite order. So this implies that $gB = B$, and so $g \in B$ and thus $B \supseteq S$. Hence, $S = gp_G(b)$. □

Corollary 2.2.37. *Let* $\varphi : G \xrightarrow{\sim} H$ *be an isomorphism. Put* P *for the torsion subgroup* $\tau(S/S^{(1)})$ *of* $S/S^{(1)}$ *and put* Q *for the torsion subgroup* $\tau(T/T^{(1)})$ *of* $T/T^{(1)}$. *Then,*

(i) φ *induces an isomorphism* $\varphi^\star : G/P \xrightarrow{\sim} H/Q$.

(ii) φ *induces an isomorphism from* $\varphi^\sharp : S/P \xrightarrow{\sim} T/Q$.

Lemma 2.2.38.

$$S/P \cong \Lambda_{mn} \subseteq \mathbb{Q}^{+},$$

and

$$T/Q \cong \Lambda_{m'n'} \subseteq \mathbb{Q}^{+}.$$

Proof. It is sufficient to prove the first. Observe that S/P is a torsion-free abelian subgroup of G/P and that the subgroup Λ_{mn} of $\mathbb{Q}$ is also torsion-free abelian. Let $B^{\sharp} = S/P$ and let b'_i be the representative of b_i in S/P. So $B^{\sharp} = S/P = gp(b_iP) = gp(b'_i)$. We define a map γ from the set of generators of S/P to Λ_{mn} as follows: $\gamma : b'_i \mapsto \frac{n^i}{m^i} = \left(\frac{n}{m}\right)^i$.

Observe that $\left(\frac{n}{m}\right)^i$, $i > 0$ is in Λ_{mn} because $\left(\frac{n}{m}\right)^i\left(\frac{n}{n}\right)^i = \frac{n^{2i}}{m^i n^i}$ is of the form $\frac{\ell}{mn^k}$, where $n^{2i} = \ell$ and $i = k$.

Also $\left(\frac{m}{n}\right)^i$, $i > 0$ is in Λ_{mn} because $\left(\frac{m}{n}\right)^i\left(\frac{m}{m}\right)^i = \frac{m^{2i}}{m^i n^i}$ is of the form $\frac{\ell}{mn^k}$, where $m^{2i} = \ell$ and $i = k$.

We claim that the defining relations of $B^{\sharp}$ go into relations in Λ_{mn}. To see this, observe that the images of b'_i and b'_{i-1} under μ are $\left(\frac{n}{m}\right)^i$ and $\left(\frac{n}{m}\right)^{i-1}$ respectively and that of $b'^m_i = b'^n_{i-1}$ is $m\left(\frac{n}{m}\right)^i = \mathrm{n}\left(\frac{n}{m}\right)^{i-1}$, which means that the images of the defining relations for $B^{\sharp}$ are relations in Λ_{mn}. Observe that the commuting relations in Λ_{mn} are automatically satisfied because

Λ_{mn}=$\{\frac{\ell}{(mn)^k}|\ \ m > 0, \ell, k \in \mathbb{Z}\}$ is an abelian group. So γ extends to a homomorphism from $B^\sharp$ to Λ_{mn} by Von Dyck's theorem.

But notice that since the b_i''s are the torsion elements in $G^{(1)}$ were factored out, then we are back to a similar situation as we had with $\Lambda_{m(m+1)}$ when $G/G^(1)$ was infinite cyclic. And so the rest of the proof is now reduce to that similarly to the rest of the proof of **Lemma 2.2.23**.

Hence γ is an isomorphism. □

Corollary 2.2.39. *Let $G = B(m,n)$ and $H = B(m', n')$. If $G \cong_\varphi H$, then,*

$$G/G^{(1)} \cong_{\widetilde{\varphi}} H/H^{(1)};$$

$$G/S^{(1)} \cong_{\widetilde{\varphi}^*} H/T^{(1)};$$

$$S/S^{(1)} \cong_{\widehat{\varphi}^*} T/T^{(1)};$$

$$G/P \cong_{\varphi^\star} H/Q;$$

$$S/P \cong_{\varphi^\sharp} T/Q;$$

and $n = n'$ and $m = m'$.

Proof. Recall that $S = gp_G(b)$. So that

$$G = S \rtimes < a >$$

(observe $G/S \cong< aS >\cong< a >$ is infinite cyclic) and the relation $a^{-1}b^m a = b^n$ holds in G and the action in G induces a similar action on $G/S^{(1)}$, hence

$$G/S^{(1)} = S/S^{(1)} \rtimes < aS^{(1)} > .$$

We can present $S/S^{(1)}$.

$S/S^{(1)} =< ..., b_i, ...; ..., b_i^m = b_{i-1}^n, ..., ..., [b_i, b_j] = 1, ...(i, j \in \mathbb{Z}) >$. Observe that S is an ascending union of generalized free products of infinite cyclic groups with cyclic amalgams. In other words $S/S^{(1)}$ is an abelian generalized free product. We see that if $m = 1$, then $S/S^{(1)}$ is isomorphic to the dyadic fractions of the form $\{1/n^k \mid \ell, k, n \in \mathbb{Z}\}$, where n is our n in the relation $b_i = b_{i-1}^n$. Now

$$G/S^{(1)} = S/S^{(1)} \rtimes < aS^{(1)} >,$$

so

$$G/S^{(1)}/P/S^{(1)} = S/S^{(1)}/P/S^{(1)} \rtimes < aS^{(1)}PS^{(1)} >,$$

$$\Longrightarrow G/P = S/P \rtimes < aPS^{(1)} > .$$

Let $L = G/P$, $B^\sharp = S/P$, $a'' =< aPS^{(1)} >$, so

$$L = B^\sharp \rtimes < a'' >,$$

where $B^\sharp =< ..., b'_i, ...; ..., b_i'^m = b_{i-1}'^n, ..., ..., [b'_i, b'_j] = 1, ...(i, j \in \mathbb{Z}) >$ is a torsion free abelian subgroup of L. And a typical element of L has the form

$\{a''^k b' | \, k \in \mathbb{Z}, \, b' \in B^\sharp \cong \Lambda_{m,n}\}$ and the relation $a''^{-1} b'^m a'' = b'^n$ holds in L.

Similarly,

$T = gp_G(y)$. So that

$$H = T \rtimes < x >$$

(observe $H/T \cong < xT >$ is infinite cyclic) and the relation $x^{-1} y^{m'} x = y^{n'}$ holds in H, and the action in H induces a similar action on $H/T^{(1)}$, hence

$$H/T^{(1)} = T/T^{(1)} \rtimes < xT^{(1)} > .$$

We can present $T/T^{(1)}$.

$T/T^{(1)} =< ..., y_i, ...; ..., y_i^{m'} = y_{i-1}^{n'}, ..., ..., [y_i, y_j] = 1, ...(i, j \in \mathbb{Z}) >$. Now

$$H/T^{(1)} = T/T^{(1)} \rtimes < xT^{(1)} >,$$

so

$$H/T^{(1)}/Q/T^{(1)} = T/T^{(1)}/Q/T^{(1)} \rtimes < xT^{(1)}QT^{(1)} >,$$

$$\Longrightarrow H/Q = T/Q \rtimes < xQT^{(1)} > .$$

Let $M = H/Q$, $Y^\sharp = T/Q$, $x'' =< xQT^{(1)} >$, so

$$M = Y^\sharp \rtimes < x'' >,$$

where $Y^\sharp =< ..., y'_i, ...; ..., y'^{m'}_i = y'^{n'}_{i-1}, ..., ..., [y'_i, y'_j] = 1, ...(i, j \in \mathbb{Z}) >$ is a torsion free abelian subgroup of M. And a typical element of M has the form $\{x''^k y' | \, k \in \mathbb{Z}, \, y' \in Y^\sharp \cong \Lambda_{m',n'}\}$ and the relation $x''^{-1} y'^{m'} x'' = y'^{n'}$ holds in M.

Observe $\varphi^\star : L \xrightarrow{\sim} M$.

Let $\vec{\varphi} : L/B^\sharp \xrightarrow{\sim} M/Y^\sharp$ defined by $\vec{\varphi}(a''B^\sharp) = \varphi^\star(a'')Y^\sharp$. Since both $L/B^\sharp$ and $M/Y^\sharp$ are isomorphic infinite cyclic, $\vec{\varphi}$ is guaranteed to be an isomorphism by **Corollary 2.2.39**. This implies that either (i) $\varphi^\star(a'')Y^\sharp = x''Y^\sharp$ or (ii) $\varphi^\star(a'')Y^\sharp = x''^{-1}Y^\sharp$. (Note the result of (ii) is similar to that of (i)).

Let take (i) $\varphi^\star(a'')Y^\sharp = x''Y^\sharp$,

$\Longrightarrow x''^{-1}\varphi^\star(a'')Y^\sharp = Y^\sharp$,

$\Longrightarrow x''^{-1}\varphi^\star(a'') \in Y^\sharp$,

$\Longrightarrow x''^{-1}\varphi^\star(a'') = y^\sharp$, for some $y^\sharp \in Y^\sharp$,

$\Longrightarrow \varphi^\star(a'') = x''y^\sharp$, where $y^\sharp$ is uniquely determined by $\varphi^\star(a'')$.

Now we look at the relation $a''^{-1}b'^m a'' = b'^n$ in L, $a'' \neq 1$, $b' \neq 1$, under $\varphi^\star$.

$\varphi^\star : L \xrightarrow{\sim} M$.

So $\varphi^\star(a''^{-1}b'^m a'') = \varphi^\star(b'^n)$,

$\Longrightarrow \varphi^\star(a'')^{-1}\varphi^\star(b'^m)\varphi^\star(a'') = \varphi^\star(b'^n)$, where $\varphi^\star(b') = y' \neq 1 \in Y^\sharp$,

$\Longrightarrow (x''y^\sharp)^{-1}y'^m(x''y^\sharp) = y'^n$, (note $\varphi^\star \mid B^\sharp \cong \varphi^\sharp$ and $\varphi^\sharp : B^\sharp \xrightarrow{\sim} Y^\sharp$),

$\Longrightarrow {y^\sharp}^{-1}x''^{-1}y'^m x''y^\sharp = y'^n$,

$\Longrightarrow x''^{-1}y'^m x'' = y^\sharp y'^n {y^\sharp}^{-1}$, $Y^\sharp \cong \Lambda_{m',n'} \subseteq \mathbb{Q}^+$ is torsion-free abelian,

$\Longrightarrow x''^{-1}y'^{m}x'' = y'^{n}$.

By taking the m'^{th} power of $x''^{-1}y'^{m}x'' = y'^{n}$,

$\Longrightarrow (x''^{-1}y'^{m}x'')^{m'} = (y'^{n})^{m'}$,

$\Longrightarrow x''^{-1}y'^{mm'}x'' = (y'^{n})^{m'}$,

$\Longrightarrow (x''^{-1}y'^{m'}x'')^{m} = (y'^{n})^{m'}$..........$(*)$, since elements $y' \in Y^{\sharp}$ commutes.

But we know that $x''^{-1}y'^{m'}x'' = y'^{n'}$, holds in M, so by taking the m^{th} power of $x''^{-1}y'^{m'}x'' = y'^{n'}$,

$\Longrightarrow (x''^{-1}y'^{m'}x'')^{m} = (y'^{n'})^{m}$, but by $(*)$

$\Longrightarrow (y'^{n})^{m'} = (y'^{n'})^{m}$,

$\Longrightarrow y'^{nm'} = y'^{n'm}$,

$\Longrightarrow nm' = n'm$.

And we know that $n - m = n' - m'$ holds.

So we have (1) $n - m = n' - m'$ and (2) $nm' = n'm$.

Multiplying (1) by n,

$\Longrightarrow n^2 - nm = nn' - nm'$,

$\Longrightarrow n^2 - nm = nn' - n'm$, since $nm' = n'm$

$\Longrightarrow n(n-m) = n'(n-m)$. Now

(i) if $n - m \neq 0$, we can divide by $n - m \neq 0$,

$\Longrightarrow n = n'$.

By (1) $nm' = nm$, dividing by $n \neq 0$,

we have $m = m'$.

(ii) if $n - m = 0$, i.e., $n = m$, then we are back to **Lemma 2.2.34:**

Let G = *B(m,n)* $=< a, b;\ a^{-1}b^m a = b^n \,|\, m \neq 0, n \neq 0,\ m, n \in \mathbb{Z} >$ and H = *B*(m', n') $=< x, y;\ x^{-1}y^{m'}x = y^{n'} \,|\, m' \neq 0, n' \neq 0,\ m', n' \in \mathbb{Z} >$, if $m = n$ and $G \cong_{\varphi} H$, then $m = m'$ and $n = n'$. □

So this completes the proof of our main theorem, **Theorem 2.2.26** in the forward direction. Now in the reverse direction the proof is obvious by just the relabeling of the generators.

Hence, this completes the entire proof of **Theorem 2.2.26**.

2.3 A construction involving the group *B(1,2)*

As a result of the study of G. Baumslag and D. Solitar, the group *B(1,2)* is Hopfian. We will show that if we form the generalized free product of two isomorphic copies of *B(1,2)* with cyclic amalgams, the resulting new group is non-Hopfian.

Theorem 2.3.1. *The generalized free product of two isomorphic copies of the Baumslag-Solitar group B(1,2)* $=< x, y; x^{-1}yx = x^2 >$ *is non-Hopfian.*

Proof. Let $A =< c, a; c^{-1}ac = a^2 >$, and let $B =< d, b; d^{-1}bd = b^2 >$. We may recall that A and B are the semi-direct products of the additive group of dyadic fractions, i.e., the subgroup of $\mathbb{Q}$ consisting of all rational numbers of the form $\frac{\ell}{2^m}$ by an infinite cyclic group where the infinite cyclic group acts by multiplication by 2. In particular, a and b are of infinite order. So we can form the generalized free product $G = A_{\langle a\rangle \overset{*}{=} \langle b\rangle} B$ (in the usual notation). The amalgamated subgroups are $H = gp(a)$ and $K = gp(b)$. Let $\alpha : A \longrightarrow G$ be defined by $\alpha : a \longmapsto a^2, c \longmapsto c$ (i.e., α is conjugation by c followed by the inclusion of A into G). Similarly, let $\beta : B \longrightarrow G$ be defined by $\beta : b \longmapsto b^2, d \longmapsto d$. Observe that α and β agree on H: $\alpha : a \longmapsto a^2$, $\beta : a(= b) \longmapsto a^2$. So both can be simultaneously be extended (by Von Dyck's theorem) to a homomorphism $\gamma : G \longrightarrow G$. Since $a^2, c, d \in \gamma(G)$, it follows that $a \in \gamma(G)$. Thus, $\gamma(G) = G$. Let us now consider the element $g = \underbrace{c^{-1}ac}\underbrace{d^{-1}b^{-1}d}$. If we gather $c^{-1}ac$ together and $d^{-1}b^{-1}d$ together, g becomes a strictly alternating $A \cup B-$product, so $g \neq 1$. But now observe that $\gamma(g) = c^{-1}a^2cd^{-1}b^{-2}d = a^4b^{-4} = a^4a^{-4} = 1$. So $G/ker(\gamma) \cong G$, with $ker(\gamma) \neq 1$. Thus, G is non-Hopfian. $\square$

Chapter 3

On A Certain Generalized Free Product

In **Chapter 2** we observe that the cyclically pinched construction does not necessarily preserve Hopficity. In this chapter we consider a certain cyclically pinched generalized free product of a free group and free abelian group and show the result to be residually free. Inspired by G. Baumslag's paper *"On generalized free products"* [2], we also devise an algorithm to solve the conjugacy problem for a certain cyclically pinched generalized free product of the special kind alluded to in the introduction.

To start with, we take a free group $F =< x, y >$ of rank 2, a non-trivial element u in F that generates its own centralizer in F, and a free abelian group A of rank 2, with an independent set $\{t, v\}$ of generators, then form their generalized free product $G = F \underset{\langle u \rangle = \langle v \rangle}{*} A$ (with cyclic amalgam). Through a constructive process, we show that G is residually free. We devise an algorithm for recognizing whether or not two elements are conjugate in G.

3.1 A theorem inspired by G. Baumslag's paper *"On generalized free products"*

In 1962, Gilbert Baumslag proved the following theorem and proposition:

Theorem 3.1.1. *([2]) Let F be a free group and let $u \in F$, $(u \neq 1)$. Furthermore, let A be a free abelian group, of countably infinite rank, with an independent set $X \cup \{v\}$ of generators. If u generates its own centralizer in F, then the generalized free product $G = F \underset{\langle u \rangle = \langle v \rangle}{*} A$ is a subgroup of the cartesian product of isomorphic copies of F.*

Proposition 3.1.2. *([2]) Let k be a positive integer and let F be a given free group. Suppose that $u, b_1, b_2, ..., b_k \in F$. Furthermore, suppose $b_1 u^{n_1} b_2 u^{n_2} \cdots b_k u^{n_k} = 1$ for infinitely many integral values of n_1, infinitely many integral values of $n_2, ...,$ and infinitely many integral values of n_k. Then there exist an i $(1 \leq i \leq k)$ such that $b_i u = u b_i$.*

We can rewrite the conclusion of **Theorem 3.1.1** in alternative terminology.

Definition 3.1.1. *A group G is said to be* ***residually free*** *if for every $g \in G$, $g \neq 1$, there exists a free group F and a homomorphism $\varphi : G \twoheadrightarrow F$ such that $\varphi(g) {\neq}_F 1$.*

So **Theorem 3.1.1** states that G is residually free. Hence with a restriction to two generators on the free subgroup F and the abelian subgroup A, we will provide a constructive proof to the following theorem:

Theorem 3.1.3. *Let $F =< x, y >$ be a free group of rank 2 and let $u \in F$, ($u \neq 1$). Furthermore, let A be a free abelian group of rank 2, with an independent set $\{t, v\}$ of generators. If u generates its own centralizer in F, then the generalized free product $G = F \underset{\langle u \rangle = \langle v \rangle}{*} A$ is residually free.*

Proof. As a start we view the data at hand. F is a free group, $u \in F$, $u \neq 1$, is a particular element of F generating its own centralizer, A is a free abelian group on $\{t, v\}$, and $G = F \underset{\langle u \rangle = \langle v \rangle}{*} A$ or alternatively written $G = \{F * A; u = v\}$. We pick a large positive integer n (in a manner later described) and then construct the map $\varphi : G \twoheadrightarrow F$ as follows:

$f \mapsto f,\ f \in F,$ and for assuring commuting images for commuting generators of A, $\quad u(= v) \mapsto u(= v),$ and $\quad t \mapsto u^n,$

where n is the chosen large positive integer (see below for the appropriate choices of n that will keep the image $\varphi(g)$ outside the kernel of φ).

Clearly the map specified above extends to a homomorphism φ from all of G since all the relators are arranged to be preserved; moreover the image is contained in F.

Let $gp(u)$ denote the amalgam, i.e., the subgroup generated by u in G. For $g \in G$, we have four cases to consider:

(i) $1 \neq g \in F - gp(u)$ (the choice of n immaterial);

(ii) $1 \neq g \in gp(u),\ g = u^i = v^i$ $(i \neq 0$, so choose $n > |i|$ to get $\varphi(g) \neq 1)$;

(iii) $1 \neq g \in A,\ g = v^i t^j$, $(j \neq 0$, choose $n > |i| + |j|$ for $\varphi(g) \neq 1)$;

(iv) $1 \neq g = f_1 a_1 \cdots f_k a_k,\ f_i \in F - gp(u),\ a_i \in A - gp(v),\ 1 \leq i \leq k.$

$= f_1(v^{i_1}t^{j_1}) \cdots f_k(v^{i_k}t^{j_k}),\ f_i \in F - gp(u),\ v^{i_\ell}t^{j_\ell} \in A - gp(v),\ i_\ell, j_\ell \in \mathbb{Z}$, $1 \leq \ell \leq k$. Here choose $n > |i_1| + |i_2| + \cdots + |i_k|$, for such a choice for n will assure $\varphi(g)= f_1 u^{i_1+nj_1} f_2 u^{i_2+nj_2} \cdots f_k u^{i_k+nj_k} \neq 1$. For otherwise we'd have the conditions of **Proposition 3.1.2** [2] satisfied, therefore some of the f_i's would have to commute with the u's. Since u was self-centralizing in F, this is impossible. □

3.2 A solution of the conjugacy problem for a certain generalized free product

In this section we devise an algorithm which decides whether or not pairs of element in the generalized free product given in **Theorem 3.1.3.** are conjugate even when the amalgamated generator in the free group F is possibly not self-centralizing.

We will now make use of the following theorem and propositions:

Theorem 3.2.1. *([20]) Let $G = A \underset{H=K}{*} B$. Then every element of G is conjugate to a cyclically reduced element of G. Moreover, suppose that g is a cyclically reduced element of G. Then:*

(i) If g is conjugate to an element h in H, then g is in some factor and there is a sequence $h, h_1, h_2, ..., h_t, g$ where h_i is in H and consecutive terms of the sequence are conjugate in a factor.

(ii) If g is conjugate to an element g' in some factor, but not in a conjugate of H, then g and g' are in the same factor and are conjugate in that factor.

(iii) If g is conjugate to an element $p_1 \cdots p_r$, where $r \geq 2$, and p_i, p_{i+1} as well as p_1,p_r are in distinct factors, then g can be obtained by cyclically permuting $p_1 \cdots p_r$ and then conjugating by an element of H.

Proposition 3.2.2. *[18](The generalized word problem) Given a finite subset U of a finitely generated free group F, there is an algorithm which decides whether or not elements in F are in $gp(U)$.*

Proposition 3.2.3. *[18] Given a finite subset $\{u, p_1, q_1\}$ in a free group F, there is an algorithm to decide whether or not there exist integers i and j such that $u^j p_1 u^i = q_1$ holds in F and if they do, finds them.*

So let $H = gp(u, p_1)$, if u and p_1 do not commute, then H is free on u and p_1. By **Proposition 3.2.2** we can decide if an element lies in H and if it does find an expression for it in terms of the given basis for the given subgroup H. Thus one can check whether or not q_1 lies in H. If q_1 does not lies in H, we are done. If q_1 lies in H, we can find a unique expression for it. Now using the group G in **Theorem 3.1.3** together with all of its properties (but not necessarily self-centralizing), **Theorem 3.2.1**, and **Proposition 3.2.3** above, we will prove the following:

Proposition 3.2.4. *There is an algorithm to decide whether or not two elements in $G = F \underset{\langle u \rangle = \langle v \rangle}{*} A$ are conjugate.*

Proof. We have $G = F \underset{\langle u \rangle = \langle v \rangle}{*} A$, where $A =< v, t; vt = tv >$ is a free abelian group, $F =< x, y >$ is a free group, and the cyclic amalgam $H = g(u) =$

$g(v) = K$. Our objective is: Given two elements in G, can we decide whether or not these two elements are conjugate in G? We would use (i), (ii), and (iii) in **Theorem 3.2.1** as a blueprint to derive this algorithm. Let g be a cyclically reduced element of G, then by (i) if g is conjugate to an element h in H, then g is in some factor and there is a sequence $h, h_1, h_2, ..., h_t, g$ where h_i is in H and consecutive terms of the sequence are conjugate in a factor.

So by (i) if $g \sim h$, then g is either in F or in A. Now h is in both F and A. Thus we know when two elements are conjugate in F and we know when two elements are conjugate in A. Two elements are conjugate in F if and only if one is a cyclic permutation of the other. This method is achieved just by writing the elements out in words and comparing them. Two elements are conjugate in A if and only if they are equal.

By (ii) if g is conjugate to an element g' in some factor, but not in a conjugate of H, then g and g' are in the same factor and are conjugate in that factor. This part is reduced to (i) again because we know whether or not two elements are conjugate in F and we know whether or not two elements are conjugate in A.

(iii) If g is conjugate to an element $p_1 \cdots p_r$, where $r \geq 2$, and p_i, p_{i+1} as well as p_1,p_r are in distinct factors, then g can be obtained by cyclically permuting

$p_1 \cdots p_r$ and then conjugating by an element of H.

For each cyclic permutation of $p = p_1 \cdots p_r$, we argue as follows: if $q = q_1 \cdots q_r \sim p = p_1 \cdots p_r$, and suppose that q_1 belongs to F, q_2 belong to A, then due to the uniqueness of the normal form for generalized free products there exists $h_1 \in H$ such that $q_1 = p_1 h_1$. So $p_1 h_1 q_2 \cdots q_r \sim p_1 \cdots p_r$, thus $h_1 q_2 \cdots q_r \sim p_2 p_3 \cdots p_r$. Similarly, for some h_2, $h_1 q_2 = p_2 h_2$, thus $q_2 = h_1^{-1} p_2 h_2$. Continuing in this manner , we get $q_r = h_{r-1}^{-1} p_r h_r$, for $h_r, h_{r-1}^{-1} \in H$. So by part (iii) if $q = q_1 \cdots q_r \sim p = p_1 \cdots p_r$, there exists an $h \in H$ such that either

1) $h^{-1} p_1 p_2 \cdots p_r h = q_1 q_2 \cdots q_r$

or

2) $h^{-1} p_3 p_4 \cdots p_r p_1 p_2 h = q_1 q_2 \cdots q_r$

$\vdots$

r) $h^{-1} p_{r-1} p_r p_1 \cdots p_{r-2} h = q_1 q_2 \cdots q_r$.

We see for example, in case 1, that $\underbrace{h^{-1} p_1 h}\underbrace{h^{-1} p_2 h} \cdots \underbrace{h^{-1} p_r h} = q_1 q_2 \cdots q_r$, where $h \in gp(u)$.

Thus we can rely on solving the generalized word problem for the subgroup of F and A respectively, generated by $\{u, p_1\}$, $\{u, p_2\}$,...,$\{u, p_r\}$ utilizing it to detect whether or not the tested element $q_1, q_2, ..., q_r$ falls into their questioned respective subgroups (see **Proposition 3.2.2**). For there to be a chance for some h to have this property above, we would have $h = u^i$ and also $u^{-i}p_1u^i = q_1$ $(*)$ in F for some i by the normal form for generalized free products and by **Proposition 3.2.3** referred to above. We would also have $q_1 \in gp(u, p_1)$. We can check to see if $q_1 \in gp(u, p_1)$ in case 1, the first permutation. If it does, we check to see if $q_2 \in gp(u, p_2)$, keeping this same permutation. If we continue this and $q_3 \in gp(u, p_3) \cdots q_r \in gp(u, p_r)$ in the first permutation and $u^{-i}p_ku^i = q_k$, $1 \leq k \leq r$ for the same i, then $q = q_1 \cdots q_r$ is conjugate to $p = p_1 \cdots p_r$. We can repeat this process and so if this procedure holds for <u>any</u> of the given r permutation, then q is <u>conjugate</u> to p. Now if in case 1 $q_1 \notin gp(u, p_1)$, then we go to case 2, the second permutation and repeat the process. Note that $q_1 \in gp(u, p_1)$ looks like $q_1 ! = u^{\alpha_1}p_1^{\beta_1}u^{\alpha_2}p_1^{\beta_2}\cdots$. So we can compare this word with $(*)$ and thus can tell whether or not $q_1 \in gp(u, p_1)$. If we continue with this pattern and for <u>any</u> given permutation $q_k \notin gp(u, p_k)$, $1 \leq k \leq r$, then we go to the next permutation down the series. If there exist a single k, $1 \leq k \leq r$, such that $q_k \notin gp(u, p_k)$ for every permutation going from case 1 to case r, then q is <u>not conjugate</u> to p.

What is said above can be stated in a more algorithmic approach as follows: For each cyclic permutation of $q_1q_2 \cdots q_r$ do the routine that we show for the first case $q_1 q_2 \cdots q_r$; first decide for q_1 if $q_1 \in gp(u, p_1)$. If "no", go to case 2, ie., $q_3 q_4 \cdots q_r q_1 q_2$. If "yes", remember the value of i and decide if $q_2 \in gp(u, p_2)$ and $u^{-i} p_2 u^i = q_2$. If "no", go to case 2, the next permutation. Repeat with each factor in the given permutation of $q_1 q_2 \cdots q_r$. Anytime there is a "no" for $q_k \notin gp(u, p_k)$, $1 \leq k \leq r$ or value of i not same as before, go to the next permutation. If there is a "no" for $q_k \notin gp(u, p_k)$, $1 \leq k \leq r$ or value of i not same as before for each of the r permutation, conclude "no q is not conjugate to p". Conclude "yes q is conjugate to p" if all answers to $q_k \in gp(u, p_k)$ end up with "yes" and $u^{-i} p_k u^i = q_k$, with the same value for i for any given permutation. The final decision will be reached by the r^{th} running of the process. □

Chapter 4

The Baumslag Groups

4.1 Some observations involving the groups *G(m,n)*

We recall that the Baumslag groups have presentation of the form

G(m,n) $=< a, b;\, a^{m^{a^b}} = a^n \,|\, m \neq 0,\, n \neq 0,\;\, m,\, n \in \mathbb{Z} >$. This chapter examines the structure of *G(m,n)* and confirms that it contains an embedded copy of the Baumslag-Solitar group *B(m,n)* $=< a, b;\, a^{-1}b^m a = b^n \,|\, m \neq 0,\, n \neq 0,\;\, m,\, n \in \mathbb{Z} >$ by virtue of possessing a normal subgroup N that is an ascending union of generalized free products of isomorphic copies of *B(m,n)* with cyclic amalgam at each stage.

With some inspection, alternative presentations can also be discerned for the Baumslag groups as follows:

G(m,n) $=< a, b;\, a^{m^{a^b}} = a^n \;|\; m \neq 0,\;\, n \neq 0,\, n,\, m \in \mathbb{Z} >$

$$=< a, b;\ [a^m, a^b] = a^{n-m} |\, m \neq 0,\ n \neq 0,\ m, n \in \mathbb{Z}) >$$

$$=< a, b;\ b^{-1}a^{-1}ba^m b^{-1}ab = a^n \mid m \neq 0,\ n \neq 0, n, m \in \mathbb{Z} >.$$

The Baumslag group *G(1,2)* first appeared in [7] as $G =< a, b;\ a = [a, a^b] >$ in which G. Baumslag showed that G fails to be residually finite.

In order to address some properties about *G(1,2)*, we need to state an important theorem that plays a central role in 1-relator group theory.

Theorem 4.1.1. *(**W. Magnus' Freiheitssatz**) Let G be a group with a single defining relator, i.e., $G =< x_1, ..., x_q;\ r >$. Suppose that r is cyclically reduced, i.e., the first and last letters in r are not inverses of each other. If each of $x_1, ..., x_q$ actually appears in r, then any proper subset of $x_1, ..., x_q$ freely generate a free group.*

The lemma below is an immediate consequence of W. Magnus' breakdown in the proof of the theorem above.

Lemma 4.1.2. *[4] Let $G =< b, x, ..., c;\ r = 1 >$ be a group with a single defining relation. Suppose that b occurs in r with exponent sum zero and that μ and ν are respectively the minimum and maximum subscripts of x occurring in r_0. If $\mu < \nu$ and if μ and ν occur only once in r_0 then $N = gp_G(x, ..., c)$ is free. Moreover if G is a 2-generator group with generators b and x, then*

N is free of rank $\nu - \mu + 1$.

Lemma 4.1.3. *The Baumslag group G(1,1) $=< a, b;\ a^{a^b} = a >$*

$=< a, b;\ a^{-1}b^{-1}a^{-1}bab^{-1}ab >$ contains a free group rank 3.

Proof. This is done by direct application of **Lemma 4.1.2** above. Observe that the exponent sum of a in r, denoted, $exp_r(a)$ is zero. Also, $exp_r(b) = 0$. Let us use the fact that the first is true, that is, $exp_r(a) = 0$. Put $b_i = a^{-i}ba^i$. Let $N = gp_G(b)$. Observe that $r_0 = b_1^{-1}b_2b_1^{-1}b_0$. Notice that in r_0, $\mu = 0$ and $\nu = 2$, so $\mu < \nu$, and both $\mu = 0$ and $\nu = 2$ occurs only once in r_0. Therefore $N = gp_G(b)$ is free of rank $\nu - \mu + 1 = 2 - 0 + 1 = 3$. □

The following is as a result of **Lemma 4.1.3** above:

Given a 1-relator group on two generators say a and b, if a commutes with a^b, then the normal subgroup generated by b is free.

4.1.1 A subgroup of the group *G(1,2)* as a certain generalized free product

With the use of the previous section's method, our analysis of the structure of *G(m,n)* recognizes *B(m,n)* as a subgroup. More precisely, we can state the result for *G(1,2)* as follows:

Proposition 4.1.4. *The Baumslag group G(1,2)* $=< a, b;\ a^{a^b} = a^2 >$ $=< a, b\ ; b^{-1}a^{-1}bab^{-1}ab = a^2 >$ *contains a normal subgroup* N *that is an ascending union of generalized free products of isomorphic copies of the Baumslag-Solitar B(1,2)* $=< x, y;\ x^{-1}yx = y^2 >$ *with cyclic amalgams.*

Proof. We will use the "Magnus breakdown" to find a presentation of the subgroup N of *G(1,2)*. We will invoke the results of Magnus' proof of the *Freiheitssatz.* Since b occurs with sum exponent zero, put $a_i = b^{-i}ab^i$, $(i \in \mathbb{Z})$. Let $N = gp_{G(1,2)}(a) = gp(..., a_{-1}, a_0, a_1, a_2, ...)$. Observe that *G(1,2)*/$N$ is infinite cyclic. Magnus incorporated the Reidemeister-Schreier method into his method. We first rewrite the relation of *G(1,2)* in terms of the generators of N and "bump up" each term in the relation by 1 going from left to right according to the elements in the relation and then repeat the process. Notice that $\varrho(b^{-1}a^{-1}bab^{-1}ab = a^2)$ is $a_1^{-1}a_0a_1 = a_0^2$, where $\varrho(b^{-1}a^{-1}bab^{-1}ab = a^2)$ is the Reidemeister-Schreier rewrite of the relator $b^{-1}a^{-1}bab^{-1}ab = a^2$. So N has the following presentation: $N =< ..., a_{-1}, a_0, a_1, ...;\ a_0^{-1}a_{-1}a_0 = a_{-1}^2,\ a_1^{-1}a_0a_1 = a_0^2,\ a_2^{-1}a_1a_2 = a_1^2, ... >$. It follows from Magnus' work that $N_{-1,0} =< a_{-1}, a_0\ ; a_0^{-1}a_{-1}a_0 = a_{-1}^2 >$, $N_{0,1} =< a_0, a_1\ ; a_1^{-1}a_0a_1 = a_0^2 >$,.., $N_{i,i+1} =< a_i, a_{i+1}\ ; a_{i+1}^{-1}a_ia_{i+1} = a_i^2 >$ and note that $N_{i,i+1} =< a_i, a_{i+1}\ ; a_{i+1}^{-1}a_ia_{i+1} = a_i^2 >$ for each $i \in \mathbb{Z}$ is isomorphic to the Baumslag-

Solitar group *B(1,2)*$=< x, y\,;\, x^{-1}yx = y^2 >$. Now, by Magnus' *Freiheitssatz*, a_{-1} and a_0 are both of infinite order in $N_{-1,0}$. Similarly, by the *Freiheitssatz* a_0 and a_1 are of infinite order in $N_{0,1}$. Thus, the generalized free product $N_{-1,1} = \{N_{-1,0} * N_{0,1};\, a_0\}$ is well-defined: elements of finite order if any, in $N_{-1,1}$ must be conjugate to such elements in either $N_{-1,0}$ or $N_{0,1}$ [25] but neither of them have elements of finite order. Following a copy of this argument we also have a well-defined torsion-free generalized free product $N_{-1,2} = \{N_{-1,1} * N_{1,2};\, a_1\}$. Clearly we can repeat the foregoing construction for all positive integers i, j, k and to obtain $N_{-i,j} = \{N_{-i,k} * N_{k,j};\, a_k\}$. Now we note that the generalized free product $N_{-1,1}$ has the presentation $< a_{-1}, a_0, a_1\ ;\ a_0^{-1}a_{-1}a_0 = a_{-1}^2,\ a_1^{-1}a_0a_1 = a_0^2 >$. Similarly, $N_{-1,2} =< a_{-1},\, a_0,\, a_1,\, a_2\,;\, a_0^{-1}a_{-1}a_0 = a_{-1}^2,\ a_1^{-1}a_0a_1 = a_0^2,\ a_2^{-1}a_1a_2 = a_1^2 >$. Thus, the ascending union $N = \bigcup_{\substack{i\geq 0\\ j\geq 0}}^{\infty} \uparrow N_{-i,j}$ has the presentation $< ..., a_{-1}, a_0, a_1, ...;$ $a_0^{-1}a_{-1}a_0 = a_{-1}^2,\ a_1^{-1}a_0a_1 = a_0^2,\ a_2^{-1}a_1a_2 = a_1^2, ... >$ which coincides with that obtained through the Magnus breakdown and the Reidemeister-Schreier rewriting process. Following through with the details of the construction, it is clear that the copy of *B(1,2)* in $N_{0,1}$, (etc) is a subgroup of N. $\square$

From the particulars of the proof for **Proposition 4.1.4**, it is also utterly clear that with the obvious adjustments with (m, n) in place of $(1, 2)$—the previous result generalizes for all pair of non-zero integers.

Theorem 4.1.5. *For any pair of non-zero integers m, n the Baumslag group B(m,n) is (isomorphic to) a subgroup of G(m,n).*

With some hesitation, however, based on the evidence in **Chapter 3** provided by the solution of the isomorphism problem for the class of the Baumslag-Solitar groups, we end this chapter with the following conjecture.

Conjecture 4.1.6. *Let* $G = G(\text{m,n}) =< a, b\ \ a^{m^{a^b}} = a^n | m \neq 0,\, n \neq 0,\ m,\, n \in \mathbb{Z}) >$ *and let* $H = G(\text{m}',\text{n}') =< x, y\,; x^{m'^{x^y}} = x^{n'} | m' \neq 0,\, n' \neq 0,\ m',\, n' \in \mathbb{Z}) >$*. Then* $G \cong_\varphi H$ *if and only if* $m = m'$ *and* $n = n'$*.*

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