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Studies on the amplitude-squared squeezed states of light

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STUDIES ON THE AMPLITUDE-SQUARED SQUEEZED STATES OF LIGHT

by

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Abstract

STUDIES ON THE AMPLITUDE-SQUARED SQUEEZED STATES OF LIGHT

by

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Amplitude-squared squeezed states are nonclassical states of light with reduced quantum noise in one quadrature of the square amplitude at the expense of enlarged fluctuation in another quadrature. Focusing on the investigation on these amplitude-squared squeezed states, this dissertation contains four different parts. The first and last parts are concentrated on the minimum uncertainty amplitude-squared squeezed states and their quantum statistical properties. In the first part, a particularly simple subset of the states are found and investigated. These states are constructed by applying a squeeze operator to a state that consists of a Hermite polynomial, whose argument is the mode creation operator multiplied by a constant, acting on the vacuum. Investigations reveal that these states may or may not be squeezed in the normal sense, and may or may not have sub-Poissonian photon statistics. In the second part, it is shown that large amount of amplitude-squared squeezing can be obtained outside the cavity while the maximum reduction of the noise of the square of the field amplitude inside the cavity is only by 50%. In the third part, the phase-insensitive amplification of amplitude-squared squeezing as well as two other higher-order squeezing, fourth-order squeezing and intrinsic fourth-order squeezing, is examined. It is found that for any

input state, both fourth-order squeezing and intrinsic fourth-order squeezing disappear at the output if the intensity gain is greater than 2. Amplitude-squared squeezing, on the other hand, can survive amplification at gains slightly greater than 2. Therefore, the photon-cloning limit (intensity gain equal 2) is not a fundamental limit to nonclassical behavior. In the last part, general solutions to the eigenvalue equation for amplitude-squared squeezed minimum uncertainty states have been found and investigated. The average photon number, quasi-probability Q-function, and photocount distribution of these states have been derived and explored. It is shown that symmetry of Q-function of any quantum state is related to the suppression of certain terms of its photocount probability. Finally, oscillation of the photocount probability in highly amplitude-squared squeezed minimum uncertainty states has been found and interpreted in terms of "interference in phase space".

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CHAPTER 1
INTRODUCTION

The electric field for a nearly monochromatic plane wave can be described by a complex amplitude which represents both the magnitude and the phase of the field. In the quantum-mechanical description, the real and imaginary parts of this amplitude obey the Heisenberg's uncertainty relation. If the uncertainties in the real and imaginary parts are equal, then the minimum uncertainty state is a coherent state, the closest quantum counterpart to a classical field with fluctuations in the two quadrature equal and minimizing the uncertainty product. The quantum fluctuations in such a coherent state are equal to the zero-point fluctuations and are randomly distributed in phase. These zero-point fluctuations represent the standard quantum limit to the reduction of noise in a signal. Even an ideal laser operating in a pure coherent state would still possess quantum noise due to zero-point fluctuations.

Other minimum uncertainty states are possible which have less fluctuations in one quadrature phase than a coherent state at the expense of increased fluctuations in the other quadrature phase. Such states, which have been called [1-5] squeezed states (other names include two photon coherent states, generalized coherent states), no longer have their quantum noise randomly distributed in phase. Such states offer intriguing possibilities. In the present optical communication systems which use coherent beams of laser light propagating in optical fibers, the ultimate limit to the noise is given by the quantum noise or zero-point fluctuations. If, instead, beams of squeezed light were used to transmit information in the quadrature phase that had reduced fluctuations, the quantum noise level could be reduced below the zero-point fluctuations. Optical communication systems based on light signals with phase sensitive quantum noise have been proposed by Yuen and Shapiro [6, 7].

The concept of squeezed states applies to other quantum mechanical systems. For example, they may have a role increasing the sensitivity of a gravitational wave detector. A standard bar detector for gravitational radiation may be treated as a harmonic oscillator. The effect of the gravitational radiation is so weak that the expected displacement of the bar is of the order of 10^{-19} cm. This is the same order of magnitude as the quantum mechanical uncertainty of the bar's position in its ground state. Thus the signal from the gravitational wave detector may be obscured by the zero-point fluctuations of the detector. This is a striking example of the influence of quantum fluctuation on a macroscopic system. In principle, a way of beating this problem is clear. In stead of the ground state of the oscillator with its quantum noise randomly distributed in phase, one prepares the oscillator in a squeezed state. One then measures the displacement due to the gravitational radiation in the quadrature with reduced fluctuations. In this way it should be possible to detect displacements less than the quantum mechanical uncertainty in the bar's position. Of course, this leaves a lot of technical questions unanswered. The problems and suggested solution are discussed elsewhere [8,9] in treatments of quantum non-demolition measurements.

In general, a state is squeezed when the uncertainty in one of its field quadrature components is less than it would be in a coherent state. Such a state will be called the "normal squeezed state" hereafter throughout the text in order to distinguish it from other squeezed states. It is possible to generalize the idea of squeezing by looking at fluctuations in variables more complicated than the mode amplitude, which leads to the definition of higher-order squeezing effects. Hong and Mandel defined a state to be squeezed to 2Nth order if the expectation value of the 2Nth power of the difference between a field quadrature component and its average value is less than it would be in a coherent state [10,11]. They found this type of squeezing in a number of nonlinear optical processes. Another approach has been explored by Braunstein and McLachlan [12]. They considered higher-order analogs of the squeeze operator in order to define what they called generalized squeezed states. A straightforward generalization of the normal squeezing is to consider squeezing in variables that are the real and imaginary parts of the square of the field amplitude, namely, squeezing associated with the square of the field amplitude, or the amplitude-squared squeezing. The amplitude-squared squeezed states, first proposed by Hillery around 1987 [15,16], are the main topics of this dissertation. Next, the mathematical definitions of the normal squeezing and the amplitude-squared squeezing will be given and followed by the goals of this research.

Normal Squeezed States

The amplitude of a single mode electric field (more generally the electric field of a mode of the electromagnetic field) is not a fixed quantity, there are always quantum mechanical fluctuations. The amplitude, having both a magnitude and a phase, is a complex number and is described by the mode annihilation operator a. It is also possible to characterize the amplitude by its real and imaginary parts which correspond to the Hermitian and anti-Hermitian parts of a,

$$X_1 = \frac{1}{2}(a^+ + a) \qquad\qquad X_2 = \frac{i}{2}(a^+ - a)\ , \tag{1.1}$$

respectively. These operators do not commute and, as a result, obey the uncertainty relation

$$\Delta X_1\, \Delta X_2 \geq \frac{1}{4}. \tag{1.2}$$

From this relation we see that the amplitude fluctuates within an "error box" in the complex plane whose area is at least 1/4. Coherent states, among them the vacuum state, are minimum uncertainty states with $\Delta X_1 = \Delta X_2 = 1/2$. A normal squeezed state, squeezed in the X_1 direction, has the property that $\Delta X_1 < 1/2$. A normal squeezed state need not to be a minimum uncertainty state, but those that are can be obtained by applying the squeeze operator

$$S(\zeta) = \mathrm{Exp}(\zeta^* a^2 - \zeta a^{+2}), \tag{1.3}$$

to a coherent state. The phase of the complex parameter ζ determines the direction of squeezing and its magnitude determines the extent of the squeezing.

Normal squeezed states are examples of nonclassical states, that is they cannot be described in terms of a nonnegative definite P representation [13]. This means that a field in a normal squeezed state cannot be modeled as a classical stochastic field. It should be noted that even though a normal squeezed state is nonclassical it can have a large number of photons. In fact, a highly normal squeezed state must have a large number of photons [14]. Thus we see that the usual association of large photon number with classical behavior is not correct.

Amplitude-Squared Squeezed States

The simplest generalization of forgoing normal squeezed states is to consider the fluctuations in the square of the field amplitude, a^2 [15, 16]. Following the example of normal squeezing one can break this variable into its real and imaginary parts

$$Y_1 = \frac{1}{2}(a^{+2} + a^2) \qquad Y_2 = \frac{i}{2}(a^{+2} - a^2). \tag{1.4}$$

The commutator of these operators is $[Y_1, Y_2] = i(2N + 1)$, where $N = a^+a$, and this leads to the uncertainty relation

$$\Delta Y_1 \Delta Y_2 \geq \langle N + 1/2 \rangle. \tag{1.5}$$

A state is amplitude-squared squeezed in the Y_1 direction if $(\Delta Y_1)^2 < \langle N + 1/2 \rangle$.

States with this property are nonclassical. This follows from the fact that $(\Delta Y_1)^2$ can be written as

$$\langle : (Y_1 - \langle Y_1 \rangle)^2 : \rangle = (\Delta Y_1)^2 - \langle N + 1/2 \rangle \ , \tag{1.6}$$

where the double dots indicate normal ordering. For a classical state the normally ordered term is always nonnegative so one can see that the onset of amplitude-squared squeezing corresponds to the onset of nonclassical behavior.

Amplitude-squared squeezed light can be generated in principle by second harmonic generation, degenerate parametric amplification and fourth-order down conversion [15,16]. A method of detecting amplitude-squared squeezing which employs second harmonic generation has also been proposed [16]. It makes use of the fact that amplitude-squared squeezing of the fundamental mode can be converted into normal squeezing of the second harmonic mode. The normal squeezing in the second harmonic can then be measured by means of homodyne detection.

Amplitude-squared squeezing was first discussed under the name SU(1,1) squeezing in a paper by Wodkiewicz and Eberly [17]. The reason for this name is that commutation relations of the operators Y_1, Y_2, and N are closely related to those of the Lie algebra SU(1,1). In particular this Lie algebra is described by three operators K_1, K_2, and K_3, whose commutation relations are given by

$$[K_1, K_2] = -iK_3 \qquad [K_2, K_3] = iK_1 \qquad [K_3, K_1] = iK_2 \ . \tag{1.7}$$

If one makes the identification $K_1 = Y_1/2$, $K_2 = -Y_2/2$ and $K_3 = (N+1/2)/2$, the above commutation relations are satisfied. This means that the representations of SU(1,1) can be used to study higher-order squeezing and this has been done by a number of authors [18-20].

Amplitude-squared squeezing can also be considered as a member of a family of higher order squeezing effects, which have been collectively designated quadratic squeezing [21], in which one studies the fluctuation behavior of variables quadratic, rather than linear, in the mode operators. In the case of a singe mode the square of the amplitude, which corresponds to a^2, is one such observable. If one considers two modes with annihilation operators a and b, then products such as ab and a^+b can be considered which leads to the sum squeezing and difference squeezing, proposed by Hillery [22]. These two modes higher-order squeezing effects, sum and difference squeezing, just like the single mode amplitude-squared squeezing effect, can be detected by first converting fluctuations in these quadratic quantities into fluctuations of a single mode amplitude through certain nonlinear optical processes after which they can be measured by standard techniques [22].

Goals of the Study

A normal squeezed state of light, as mentioned before, has the property that the quantum fluctuation in one quadrature component is reduced below that of a vacuum state at the expense of increased noise in the other component. It has been argued that the squeezed light has some potential applications in extremely high resolution measurement (such as gravitational wave detection) and in optical communication. Amplitude-squared squeezing is different from the normal squeezing since it is associated with the square of the field amplitude while normal squeezing is related to the field amplitude itself. In this dissertation, it is found that amplitude-squared squeezed states behave differently in some aspects from normal squeezing. For example, some amplitude-squared squeezed states can survive the amplification process with gain larger than 2, while all other nonclassical effects including normal squeezing disappear after amplified by an amplifier with gain of 2 [27,28]. The Q-functions of the amplitude-squared squeezed states indicate that there exits dramatic distinction between a normal squeezed state and a amplitude-squared squeezed state. Thus, it is worthwhile to investigate amplitude-squared squeezing thoroughly from a fundamental point of view as well as with regard to their potential applications.

In next chapter, I will begin with the studies of the eigenvalue equation of the minimum uncertainty states for amplitude-squared squeezing and find a particular simple subset of the states that satisfy the uncertainty relation as an equality. These states are constructed by applying a squeeze operator to a state that consists of a Hermite polynomial, whose argument is the mode creation operator multiplied by a constant, acting on the vacuum. An investigation of these states shows that amplitude-squared squeezed states may or may not be squeezed in the normal sense and also may or may not be states with sub-Poissonian photon statistics [29]. This subset of minimum uncertainty states of amplitude-squared squeezing is obtained by assuming that the expectation values of the two quadrature components Y_1 and Y_2 are related to their fluctuations in a special manner, which is not true for a general amplitude-squared squeezed minimum uncertainty state. The general solution will be discussed in last chapter.

Amplitude-squared squeezing is similar to normal squeezing in several respects. Both are nonclassical effects and can also be generated by some of the same systems. For example, a degenerate parametric oscillator will produce light in which both kinds of squeezing are present. In this device both types of squeezing can attain a maximum level of 50% inside the cavity [23-25], and an arbitrarily high level of squeezing outside the cavity [25,26]. This similarity between the normal squeezing and amplitude-squared squeezing is discussed in chapter 3.

In chapter 4, the phase-insensitive amplification of the amplitude-squared squeezing as well as other kinds of higher-order squeezing (fourth-order squeezing and intrinsic fourth-order squeezing) is examined. It is found that for any input state both fourth-order squeezing and intrinsic fourth-order squeezing disappear at the output if the intensity gain is greater than 2. Amplitude-squared squeezing, on the other hand, can survive amplification at gains slightly greater than 2.

In the last chapter, general solutions to the eigenvalue equation for amplitude-squared squeezed minimum uncertainty states have been found and investigated. An expression for the average photon number in these states is derived. The Q-functions as well as the photocount distribution of these states have been investigated. Plots of Q-functions indicate that the "manner" of amplitude-squared squeezing is different from that of normal squeezing. It is shown that 2-fold and 4-fold symmetry of Q-function in phase space is related to the suppression of certain terms of the photocount probability for a quantum state. Finally, oscillation of the photocount probability in highly squeezed minimum uncertainty states for amplitude-squared squeezing has been found and interpreted in terms of "interference in phase space".

CHAPTER 2

MINIMUM UNCERTAINTY STATES FOR AMPLITUDE-SQUARED SQUEEZING: HERMITE POLYNOMIAL STATES

The object in this chapter is to find and investigate a subset of the minimum uncertainty states for amplitude-squared squeezing. This subset has been chosen for its simplicity, and the more complicated general case will be treated in the final chapter. What is meant by a minimum uncertainty state is a state for which Eq. (1.5) is satisfied as an equality. These states have been called "intelligent states" by some authors [30,31], but we shall retain what we believe to be the more conventional nomenclature.

This chapter is organized as follows. In the first section the eigenvalue equation is discussed which allows one to find the minimum uncertainty states and some of the implications it has for their properties. In the second section we solve the equation and present the subset of its solutions which have a particularly simple form. In the third section the properties of these states are discussed. Results of this chapter are summarized in the last section.

Eigenvalue Equation

The minimum uncertainty states for amplitude-squared squeezing are the solutions to the eigenvalue problem

$$(Y_1 + i\,\lambda Y_2)\,|\psi\rangle = \beta\,|\psi\rangle, \tag{2.1}$$

where λ is real and β is complex [32]. This follows from examining the difference between the two sides in Eq. (1.5) and demanding that it be zero. The general procedure for accomplishing this is discussed in detail in Ref. [32], and it implies that only the states that satisfy Eq. (2.1) will satisfy Eq. (1.5) as an equality. We shall prove, in a somewhat different fashion, that the states that satisfy Eq. (2.1) are minimum uncertainty states. In doing so we gain insight into the meaning of the parameters λ and β. In the next section we shall solve this equation in order to find the minimum uncertainty states.

As a first step it is useful to take the expectation of the operator $Y_1 + i\,\lambda Y_2$ in the state $|\psi\rangle$ which is a solution of Eq. (2.1). This gives

$$\langle\psi| Y_1 |\psi\rangle + i\,\lambda \,\langle\psi| Y_2 |\psi\rangle = \beta\,. \tag{2.2}$$

Because Y_1 and Y_2 are hermitian their expectation values are real so that Eq. (2.2) implies that

$$\langle\psi| Y_1 |\psi\rangle = \beta_r\,, \qquad \langle\psi| Y_2 |\psi\rangle = \beta_i/\lambda\ , \tag{2.3}$$

where $\beta_r = \mathrm{Re}\,\beta$ and $\beta_i = \mathrm{Im}\,\beta$. Therefore, β is directly related to the expectation values of Y_1 and Y_2 in minimum uncertainty states.

Now let us multiply Eq. (2.1) by the operator $Y_1 - i\,\lambda Y_2$ and then take the inner product with $|\psi\rangle$. The result, after making use of Eq. (2.3) is

$$\langle\psi| Y_1^2 + \lambda^2\ Y_2^2 |\psi\rangle + i\,\lambda\,\langle\psi| [Y_1, Y_2] |\psi\rangle = |\beta|^2,$$

or

$$(\Delta Y_1)^2 + \lambda^2\ (\Delta Y_2)^2 = \lambda\,\langle\psi| 2N + 1 |\psi\rangle\ . \tag{2.4}$$

Note that this equation implies that λ must be greater than or equal to zero.

Multiplying Eq. (2.1) by $Y_1 + i\,\lambda Y_2$ and taking the inner product with $|\psi\rangle$ also provides useful information. After doing so and then taking real and imaginary parts one finds that

$$\langle\psi| Y_1^2 - \lambda^2 Y_2^2 |\psi\rangle = \mathrm{Re}(\beta^2), \quad \lambda\langle\psi| Y_1 Y_2 + Y_2 Y_1 |\psi\rangle = \mathrm{Im}(\beta^2). \tag{2.5}$$

If one now combines the first of these equations with the results in Eq. (2.2), one has

$$(\Delta Y_1)^2 - \lambda^2 (\Delta Y_2)^2 = \mathrm{Re}(\beta^2) - \beta_r^2 + \beta_i^2 = 0,$$

or

$$\Delta Y_1 = \lambda \Delta Y_2. \tag{2.6}$$

From this equation it is clear that λ plays the role of a squeezing parameter. If $\lambda = 1$, then the uncertainties are equal and there is no amplitude-squared squeezing. If $\lambda > 1$, then ΔY_2 is squeezed, and if $0 < \lambda < 1$, then ΔY_1 is squeezed.

Finally, if Eq. (2.6) is used to eliminate either ΔY_1 or ΔY_2 in Eq. (2.4), then one finds that

$$(\Delta Y_1)^2 = \lambda\langle\psi| N + 1/2 |\psi\rangle, \qquad (\Delta Y_2)^2 = (1/\lambda)\langle\psi| N + 1/2 |\psi\rangle. \tag{2.7}$$

These equations show that $\Delta Y_1 \Delta Y_2 = \langle\psi| N + 1/2 |\psi\rangle$ and prove that a solution of Eq. (2.1) is a minimum uncertainty state. They confirm as well the comments made in the preceding paragraph about the role of λ as a squeezing parameter. Note also that if we know that $|\psi\rangle$ is a solution of Eq. (2.1) then the photon number determines ΔY_1 and ΔY_2. That is, if $\langle\psi| N |\psi\rangle$ is known, then Eqs. (2.7) can be used to find ΔY_1 and ΔY_2 immediately, thus providing a certain economy of computation. This fact will prove useful later.

Solution of Eigenvalue Equation

We now want to solve Eq. (2.1) for $|\psi\rangle$. It is first useful to express the equation in terms of creation and annihilation operators:

$$[\,(1-\lambda)\,(a^+)^2/2+(1+\lambda)\,a^2/2\,]\,|\psi\rangle=\beta\,|\psi\rangle\,. \tag{2.8}$$

A natural approach at this point is to expand $|\psi\rangle$ in terms of number states. This, however, leads to a three term recurrence relation for the expansion coefficients which is difficult to solve. It is instead simpler to introduce the state

$$|\psi'\rangle=S(z)^{-1}\,|\psi\rangle\,, \tag{2.9}$$

which is related to $|\psi\rangle$ by the squeezing transformation $S(z)$. The parameter z will be chosen later. The state $|\psi'\rangle$ can then be expanded in photon number states. For the proper choice of z the recurrence relation which determines the expansion coefficients contains only two terms and can be easily solved.

If we let $z = r\,e^{i\theta}$, then we find that $|\psi'\rangle$ satisfies the equation

$$S(z)^{-1}[\,(1-\lambda)\,(a^+)^2/2+(1+\lambda)\,a^2/2\,]\,S(z)\,|\psi'\rangle=\beta\,|\psi'\rangle,$$

or

$$\{\,[\,(1-\lambda)\,(\cosh^2 r)/2+(1+\lambda)\,(e^{2i\theta}\sinh^2 r)/2\,]\,(a^+)^2$$

$$+\,[\,(1-\lambda)\,e^{-i\theta}/2+(1+\lambda)\,e^{i\theta}/2]\,\cosh r\,\sinh r\,(a^+a+a\,a^+)$$

$$+\,[\,(1-\lambda)\,(e^{-2i\theta}\sinh^2 r)/2+(1+\lambda)\,(\cosh^2 r)/2\,]\,a^2\,\}\,|\psi'\rangle=\beta\,|\psi'\rangle\,. \tag{2.10}$$

We now want to choose z so that the coefficient of $(a^+)^2$ vanishes. The condition for this is

$$\tanh^2 r = e^{-2i\theta}(\lambda-1)/(\lambda+1). \tag{2.11}$$

For $\lambda \geq 1$ we choose $\theta = 0$ and for $0 < \lambda < 1$ we take $\theta = \pi/2$. The parameter r is then chosen so than $\tanh^2 r$ is equal to the absolute value of the right hand side. Note that this is always possible as $|(\lambda-1)/(\lambda+1)| < 1$ for $\lambda > 0$. With these choices we find that for $0 < \lambda < 1$

$$\cosh r = [\,(1+\lambda)/2\lambda\,]^{1/2} \qquad \sinh r = [\,(1-\lambda)/2\lambda\,]^{1/2}\,, \tag{2.12}$$

and for $\lambda \geq 1$

$$\cosh r = [\,(1+\lambda)/2\,]^{1/2} \qquad \sinh r = [\,(\lambda-1)/2\,]^{1/2}\,. \tag{2.13}$$

These expressions can now be substituted into Eq. (2.10). The result for $0 < \lambda < 1$ is

$$[\,a^2 + i\,(1-\lambda^2)^{1/2}(a^+ a + 1/2)\,]\,|\psi'\rangle = \beta\,|\psi'\rangle\,, \tag{2.14}$$

and for $\lambda \geq 1$

$$[\,\lambda\, a^2 + (\lambda^2 - 1)^{1/2}\,(a^+ a + 1/2)\,]\,|\psi'\rangle = \beta\,|\psi'\rangle\,. \tag{2.15}$$

We now expand $|\psi'\rangle$ in terms of number states

$$|\psi'\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle\,, \tag{2.16}$$

and substitute this expression into Eqs. (2.14) and (2.15). This leads to the recurrence relations

$$C_{n+2} = [\,\frac{\beta - i\sqrt{1-\lambda^2}\,(n+1/2)}{\sqrt{(n+2)(n+1)}}\,]\,C_n\,, \tag{2.17}$$

for $0 < \lambda < 1$, and

$$C_{n+2} = [\,\frac{\beta - \sqrt{\lambda^2-1}\,(n+1/2)}{\lambda\sqrt{(n+2)(n+1)}}\,]\,C_n\,, \tag{2.18}$$

for $\lambda \geq 1$. Both C_0 and C_1 are arbitrary.

These recurrence relations are easily solved for any value of β and for $\lambda > 0$. The properties of the states which result from the general solution will be discussed in the last chapter. Here we wish to examine a particularly simple subset of solutions. This subset is found by noting that if β and λ are related in the proper way only a finite number of the coefficients C_n will be different from zero. In particular, if $0 < \lambda < 1$ and $\beta = i\,(1-\lambda^2)^{1/2}\,(m + 1/2)$, where m is a nonnegative integer, then the series for $|\psi'\rangle$ can be chosen to terminate with $C_m\,|m\rangle$ being the last term. Similarly, if $\lambda \geq 1$ and $\beta = (\lambda^2 - 1)^{1/2}\,(m + 1/2)$, again with m as a nonnegative integer, then C_0 and C_1 can be chosen so that $C_n = 0$ for $n > m$.

Let us find explicit expressions for the solutions. If $0 < \lambda < 1$, then for even m we have

$$|\psi'(m,\lambda)\rangle = \sum_{n=0}^{m/2} [\,i\,(1-\lambda^2)^{1/2}]^n \frac{2^n}{\sqrt{(2n)!}} \frac{(m/2)!}{(m/2-n)!}\, C_0\,|2n\rangle, \tag{2.19}$$

and for odd m

$$|\psi'(m,\lambda)\rangle = \sum_{n=0}^{(m-1)/2} [\,i\,(1-\lambda^2)^{1/2}]^n \frac{2^n}{\sqrt{(2n+1)!}} \frac{[(m-1)/2]!}{[(m-1)/2-n]!}\, C_1\,|2n+1\rangle\,. \tag{2.20}$$

If $\lambda \geq 1$, then for even m we have

$$|\psi'(m, \lambda)\rangle = \sum_{n=0}^{m/2} [\,(\lambda^2 - 1)^{1/2}/\lambda]^n \frac{2^n}{\sqrt{(2n)!}} \frac{(m/2)!}{(m/2 - n)!} C_0 \,|2n\rangle, \tag{2.21}$$

and for odd m we have

$$|\psi'(m, \lambda)\rangle = \sum_{n=0}^{(m-1)/2} [\,(\lambda^2 - 1)^{1/2}/\lambda]^n \frac{2^n}{\sqrt{(2n+1)!}} \frac{[(m-1)/2]!}{[(m-1)/2 - n]!} C_1 \,|2n+1\rangle. \tag{2.22}$$

The constants C_0 and C_1 are chosen so as to normalize the states.

It is possible to express these states in a relatively compact form in terms of Hermite polynomials. The n^{th} Hermite polynomial is given by

$$H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k (2x)^{n-2k} n! \,/[(n - 2k)!\, k!]\,, \tag{2.23}$$

where [n/2] denotes the greatest integer less than or equal to n/2. The state $|\psi'(m, \lambda)\rangle$ can be expressed as

$$|\psi'(m, \lambda)\rangle = C_m(\lambda)\, H_m(i\, \gamma(\lambda)\, a^+)\, |0\rangle, \tag{2.24}$$

where $C_m(\lambda)$ is a normalization constant and

$$\gamma(\lambda) = e^{\,i\pi/4}[(1-\lambda^2)^{1/2}/2]^{1/2}, \tag{2.25}$$

for $0 < \lambda < 1$, and

$$\gamma(\lambda) = [\,(\lambda^2 - 1)^{1/2}/2\lambda]^{1/2}\,, \tag{2.26}$$

for $\lambda \geq 1$. Finally, we can combine this with the squeezing transformation to give the minimum uncertainty states

$$|\psi(m, \lambda)\rangle = S(z)\, |\psi'(m, \lambda)\rangle = C_m(\lambda) S(z)\, H_m(i\, \gamma(\lambda)\, a^+)\, |0\rangle, \tag{2.27}$$

where z is defined by Eq. (2.11). It is these states which we now wish to examine further. It should again be noted that the states $|\psi(m, \lambda)\rangle$ are only a subset of the full set of minimum uncertainty states.

Properties of States

We now want to discuss some of the properties of the states $|\psi(m, \lambda)\rangle$. First we shall obtain more explicit expressions for the photon number and for the uncertainties in Y_1 and Y_2. Then it will be determined whether these states are squeezed in the normal sense and whether their photon statistics are sub-Poissonian.

Eq. (2.7) shows that if we know the average photon number for these states, then we know ΔY_1 and ΔY_2. Therefore, we need to find $\langle\psi(m, \lambda)|N|\psi(m, \lambda)\rangle$. This quantity can, in fact, be expressed rather simply in terms of the functions $|C_m(\lambda)|^2$. In order to see this first note that

$$\begin{aligned} \langle\psi(m, \lambda)|N|\psi(m, \lambda)\rangle = |C_m(\lambda)|^2 \{ & (\cosh^2 r + \sinh^2 r)\, \langle 0|H_m(-\, i\, \gamma^*\, a)\, a^+ a\, H_m(i\, \gamma\, a^+)|0\rangle \\ & + \cosh r \sinh r\, [\, e^{-i\theta} \langle 0|H_m(-\, i\, \gamma^*\, a)\, a^2\, H_m(i\, \gamma\, a^+)|0\rangle \\ & + e^{i\theta} \langle 0|H_m(-\, i\, \gamma^*\, a)(a^+)^2\, H_m(i\, \gamma\, a^+)|0\rangle\,]\, \} + \sinh^2 r\, . \end{aligned} \tag{2.28}$$

The equation

$$\langle\psi(m,\lambda)|\, a^2\, |\psi(m,\lambda)\rangle = \beta_r + i\,\beta_i/\lambda \ , \tag{2.29}$$

which follows from Eq. (2.3), and its complex conjugate allow us to solve for $\langle 0|H_m(-i\gamma^* a)\, a^2 H_m(i\gamma a^+)|0\rangle$ and $\langle 0|H_m(-i\gamma^* a)\ (a^+)^2 H_m(i\gamma a^+)|0\rangle$ in terms of $\langle 0|H_m(-i\gamma^* a)\, a^+a\, H_m(i\gamma a^+)|0\rangle$, β_r, and β_i/λ. The result for the expectation value of a^2 is

$$(\cosh^2 r + \sinh^2 r)\,\langle 0|H_m(-i\gamma^* a)\, a^2 H_m(i\gamma a^+)|0\rangle = [(\cosh^2 r - e^{2i\theta}\sinh^2 r)\,\beta_r$$

$$+\, i\,(\cosh^2 r + e^{2i\theta}\sinh^2 r)\beta_i/\lambda\,]\,/\,|C_m(\lambda)|^2$$

$$-\, e^{i\theta}\cosh r\,\sinh r\,\langle 0|H_m(-i\gamma^* a)\,(2a^+a + 1)\,H_m(i\gamma a^+)|0\rangle\ . \tag{2.30}$$

This result and its complex conjugate can then be substituted into Eq. (2.28). It remains to simplify the expectation value of a^+a in the Hermite polynomial state. Here the identity

$$a\, H_m(i\gamma a^+)|0\rangle = 2i\gamma m\, H_{m-1}(i\gamma a^+)|0\rangle\ , \tag{2.31}$$

leads to

$$\langle 0|H_m(-i\gamma^* a)\, a^+a\, H_m(i\gamma a^+)|0\rangle = 4|\gamma|^2 m^2\,\langle 0|H_{m-1}(-i\gamma^* a)H_{m-1}(i\gamma a^+)|0\rangle$$

$$= 4|\gamma|^2 m^2\,/\,|C_{m-1}(\lambda)|^2\ . \tag{2.32}$$

Finally, expressing the quantities β, γ, $\cosh r$, and $\sinh r$ in terms of λ and m gives for $0 < \lambda < 1$

$$\langle\psi(m,\lambda)|N|\psi(m,\lambda)\rangle = 2m^2\lambda(1-\lambda^2)^{1/2}\,|C_m(\lambda)|^2\,/\,|C_{m-1}(\lambda)|^2$$

$$+ (1-\lambda^2)(m + 1/2)/\lambda - (1-\lambda)/2, \tag{2.33}$$

and for $\lambda \geq 1$

$$\langle\psi(m, \lambda)|N|\psi(m, \lambda)\rangle = 2m^2\lambda(\lambda^2-1)^{1/2} |C_m(\lambda)|^2 / (\lambda^2|C_{m-1}(\lambda)|^2)$$

$$+ (\lambda^2-1)(m + 1/2)/\lambda - (\lambda-1)/(2\lambda) . \tag{2.34}$$

Let us now use these expressions to find $\langle N\rangle$, ΔY_1, and ΔY_2 for $|\psi(m, \lambda)\rangle$ for the cases $m = 0$, 1, and 2. A straightforward calculation gives

$$|C_0(\lambda)|^2 = 1, \qquad |C_1(\lambda)|^2 = 1/4|\gamma|^2, \qquad |C_2(\lambda)|^2 = 1/(32|\gamma|^2 + 4) . \tag{2.35}$$

Turning first to $|\psi(1, \lambda)\rangle$, which is a squeezed one-photon state, we find for $0 < \lambda < 1$

$$\langle N\rangle = (3 - \lambda)/2\lambda, \qquad (\Delta Y_1)^2 = 3/2 , \qquad (\Delta Y_2)^2 = 3/(2\lambda^2), \tag{2.36}$$

and for $\lambda \geq 1$

$$\langle N\rangle = (3\lambda - 1)/2, \qquad (\Delta Y_1)^2 = 3\lambda^2/2 , \qquad (\Delta Y_2)^2 = 3/2 . \tag{2.37}$$

Note that $\langle N\rangle \to \infty$ as λ goes to either zero or infinity. As $\lambda \to 0$, Y_1 becomes increasingly squeezed and $(\Delta Y_1)^2$ is equal to 3/2. Similarly, as $\lambda \to \infty$, Y_2 becomes more and more squeezed and $(\Delta Y_2)^2$ is equal to 3/2.

The state $|\psi(2, \lambda)\rangle$ is obtained by squeezing a linear combination of the vacuum and a two-photon state. For this state the photon number is

$$\langle N\rangle = [(15 - 14\lambda^2)/(6 - 4\lambda^2)\lambda] - 1/2 , \tag{2.38}$$

for $0 < \lambda < 1$ and

$$\langle N \rangle = [(15\lambda^3 - 14\lambda)/(6\lambda^2 - 4)] - 1/2 , \tag{2.39}$$

for $\lambda \geq 1$. This again diverges as λ goes to either zero or infinity. In this case, however, as λ goes to zero $(\Delta Y_1)^2$ goes to 5/2, and as λ goes to infinity $(\Delta Y_2)^2$ goes to 5/2.

It is also possible to derive an asymptotic expression for $|C_m(\lambda)|^2$ which is valid for large m. This is done in the Appendix A. If the result is then used to find the mean photon number for $|\psi(m, \lambda)\rangle$, then we find for m large

$$\langle N \rangle = (m + 1/2)/\lambda - m\lambda/[1+ (1- \lambda^2)^{1/2}] , \tag{2.40}$$

for $0 < \lambda < 1$ and

$$\langle N \rangle = \lambda(m + 1/2) - m/[\lambda+ (\lambda^2- 1)^{1/2}] , \tag{2.41}$$

for $\lambda \geq 1$. Therefore, for large m we have a situation similar to that for $m = 1$ and 2. That is as $\lambda \rightarrow 0$, the photon number diverges and $(\Delta Y_1)^2 \rightarrow (m + 1/2)$. As $\lambda \rightarrow \infty$, the photon number also diverges and $(\Delta Y_2)^2 \rightarrow (m + 1/2)$.

The simplest case is $m = 0$. The state $|\psi(0, \lambda)\rangle$ is just a squeezed vacuum so that this state is a minimum uncertainty state for both normal squeezing and amplitude-squared squeezing. The photon number in this case is

$$\langle N \rangle = 1/(2\lambda) - 1/2 , \tag{2.42}$$

for $0 < \lambda < 1$ and

$$\langle N\rangle = (\lambda - 1)/2\,, \tag{2.43}$$

for $\lambda \geq 1$. We can see that this state follows the pattern set by those which we have discussed previously.

It is of interest to determine whether the states $|\psi(m, \lambda)\rangle$ are squeezed in the normal sense. Clearly $|\psi(0, \lambda)\rangle$ being a squeezed vacuum is squeezed for all $\lambda \neq 1$. This is, however, not the case for $m \neq 0$. A short calculation shows that the state $|\psi(1, \lambda)\rangle$ is not squeezed if $3/5 < \lambda < 5/3$, but is squeezed otherwise. Therefore, amplitude-squared squeezed minimum uncertainty states may or may not be squeezed in the normal sense.

A similar conclusion is reached if we inquire whether these states have sub-Poissonian photon statistics. Here we start by deriving a simple expression for $(\Delta N)^2$ for $|\psi(m, \lambda)\rangle$. Noting that $Y_1{}^2 + Y_2{}^2 = N^2 + N + 1$, we obtain

$$(\Delta Y_1)^2 + (\Delta Y_2)^2 = \langle\, N^2 + N + 1\,\rangle - \beta_r{}^2 - (\beta_i/\lambda)^2\,. \tag{2.44}$$

Using Eq. (2.7) and solving the resulting expression for $(\Delta N)^2 - \langle\, N\,\rangle$ gives

$$(\Delta N)^2 - \langle\, N\,\rangle = [(\lambda^2 + 1)/\lambda]\,\langle\, N + 1/2\,\rangle + \beta_r{}^2 + (\beta_i/\lambda)^2 - \langle\, N + 1/2\,\rangle^2\,. \tag{2.45}$$

If the right-hand side of Eq. (2.45) is negative the photon statistics of $|\psi(m, \lambda)\rangle$ are sub-Poissonian. In order to illustrate what can happen, let us consider the case $m = 1$. Because $|\psi(1, 1)\rangle$ is just the one-photon number state, it is clear that for some range of λ the photon statistics are sub-Poissonian. A more detailed analysis using Eqs. (2.36), (2.37), and (2.45) shows that they are sub-Poissonian if

$$2/[1 + (11/3)^{1/2}] < \lambda < [1 + (11/3)^{1/2}]/2\,, \tag{2.46}$$

and they are not if λ is outside this range. One expects that this behavior is typical for odd m. The state $|\psi(m, 1)\rangle$, for odd m, is just the one photon number state and is, therefore, sub-Poissonian. As λ deviates from 1 the amount of squeezing in the state increases and eventually makes the state super-Poissonian.

Conclusion

A subset of the minimum uncertainty states, $|\psi(m, \lambda)\rangle$, has been presented for the variables which describe amplitude-squared squeezing. For these states the expectation values of Y_1 and Y_2 are related to their fluctuations. This is not true for a general amplitude-squared minimum uncertainty state where the expectation values of Y_1 and Y_2 can be specified independently of the squeezing parameter λ. The complete set of minimum uncertainty states will be discussed in the final chapter.

All of the states $|\psi(m, \lambda)\rangle$ have the property that they have zero mean amplitude, i. e. the expectation of the annihilation operator is zero. If one displaces one of these states with a displacement operator $D(\alpha)$, the minimum uncertainty character of the state is destroyed. This is because the amount of amplitude-squared squeezing, unlike the amount of normal squeezing, is not invariant under displacements.

Perhaps the most interesting result is that the squeezed vacuum is a minimum uncertainty state for amplitude-squared squeezing as well as for normal squeezing. Because such a state can be produced by a degenerate parametric amplifier it is the most likely of the states$|\psi(m, \lambda)\rangle$ to be seen in the laboratory. The other candidate for a state which is, perhaps, not too hard to observe is $|\psi(1, \lambda)\rangle$. This is a squeezed one-photon state and can be produced by using a one-photon number state as the input for a degenerate parametric amplifier.

Finally we note that as the amount of amplitude-squared squeezing in these states increases so does the photon number. A similar situation occurs with normal squeezing

[33]. Thus amplitude-squared squeezed minimum uncertainty states provide another example of states whose nonclassical behavior becomes most pronounced at large photon number.

CHAPTER 3

AMPLITUDE-SQUARED SQUEEZING INSIDE AND OUTSIDE A CAVITY

The degenerate parametric oscillator (DPO) has proven to be one of the best devices for producing normal squeezed states. Noise reduction 60% below the vacuum level has been achieved [35]. Here we want to determine whether this system is also a good source of amplitude-squared squeezed light.

Milburn and Walls [36] and Lugiato and Strihi [37] studied the squeezing of amplitude quadrature inside a cavity in a degenerate parametric oscillator at steady state. They showed that the maximum reduction of the quadrature fluctuations that can be obtained is by a factor of 2. This made the prospects of obtaining large amounts of squeezing seem rather bleak. The situation improved considerably when Yurke [38] investigated the squeezing of the transmitted field outside a cavity and found that a large amount of squeezing can be obtained near threshold. This result was confirmed and generalized by others [39-42] and made squeezing of practical as well as academic interest. In this paper we present the results of a study of the amplitude-squared squeezing of both the intracavity and output signal either above or below threshold of DPO. We find that a large amount of amplitude-squared squeezing can be obtained outside the cavity while the maximum squeezing inside the cavity is only by 50%, similar to the result for normal squeezing. Different methods have been employed to treat the fields since the behaviors of the intracavity and output fields are quite different above or below threshold of DPO.

Amplitude-Squared Squeezing Above Threshold Of DPO

A. Inside the Cavity

A degenerate parametric oscillator consists of 2 cavity modes, one at twice the frequency of the other, interacting via a $\chi^{(2)}$ nonlinearity. The master equation for the density operator ρ of the two mode system in the interaction picture is [37]

$$\frac{d\rho}{dt} = -i\,[H_I + H_{ext}, \rho] + (\Lambda_1 + \Lambda_2)\,\rho\,,$$

$$H_I = i\,(\chi/2)\,(a_1^{+2} a_2 - a_1^2 a_2^+)\,, \tag{3.1}$$

$$H_{ext} = i\,\epsilon\,(a_2^+ - a_2)\,,$$

$$\Lambda_i\,\rho = \gamma_i\,(2a_i \rho a_i^+ - a_i^+ a_i \rho - \rho a_i^+ a_i)\,, \qquad (i = 1, 2)$$

where H_I describes the interaction between mode 1, the signal, with frequency ω and mode 2, the pump, with frequency 2ω. H_{ext} describes the external pump and Λ_i the losses of the ith mode (i = 1, 2). a_i and γ_i are the annihilation operator and the damping constant of the ith mode. ∈, which is proportional to the amplitude of the driving field, is taken to be real and positive.

Above threshold ($\epsilon > \gamma_1\gamma_2/\chi$), the stable steady-state semiclassical solution is

$$\langle a_1 \rangle = [(2\gamma_1\gamma_2)^{1/2}/\chi]\,\eta^{1/2}\,, \tag{3.2}$$

where $\eta = \chi\epsilon/\gamma_1\gamma_2 - 1 > 0$.

Now, let us introduce the fluctuation operator of the signal mode

$$\Delta a_1 = a_1 - \langle a_1 \rangle\,. \tag{3.3}$$

Well above threshold, $\Delta a_1/\langle a_1 \rangle$ is proportional to $(\chi^2/\gamma_1\gamma_2)^{1/2}$ [37]. Thus for $(\chi^2/\gamma_1\gamma_2)^{1/2} << 1$ the fluctuations are small and one can neglect higher order fluctuation terms with respect to second order ones. For example,

$$\langle a_1^4\rangle = \langle\, (\, \langle a_1\rangle + \Delta a_1\,)^4\,\rangle$$

$$= \langle a_1\rangle^4 + 6\langle a_1\rangle^2 \langle(\Delta a_1)^2\rangle + O(\, \langle(\Delta a_1\,)^3\,\rangle\,)\,. \tag{3.4}$$

Using this approximation, we can write

$$\langle(\Delta Y_2)^2\rangle = -\frac{1}{4}\,[\, \langle(a_1^{\dagger 2} - a_1^2)^2\rangle - \langle(a_1^{\dagger 2} - a_1^2)\rangle^2\,]$$

$$= -\,2\frac{\gamma_1\gamma_2}{\chi^2}\,\eta[\langle(\Delta a_1^\dagger)^2\rangle + \langle(\Delta a_1)^2\rangle - \langle(\Delta a_1)(\Delta a_1^\dagger)\rangle - \langle(\Delta a_1^\dagger)(\Delta a_1)\rangle]$$

$$= 2\,\frac{\gamma_1\gamma_2}{\chi^2}\,4\eta\langle(\Delta X_2)^2\rangle\,, \tag{3.5}$$

where $X_2 = \frac{i}{2}\,(a_1^\dagger - a_1)$ and

$$\langle\, N_1\,\rangle = \langle\; a_1^\dagger\, a_1\,\rangle = 2\,\frac{\gamma_1\gamma_2}{\chi^2}\,\eta + \langle(\Delta a_1^\dagger)(\Delta a_1)\rangle\,. \tag{3.6}$$

By making use of the results of reference [37] for the expectation values of products of two fluctuation operators we obtain

$$\frac{\langle\,(\,\Delta\, Y_2\,)^2\rangle}{\langle\, N_1 + 1/2\,\rangle} = 4\,\frac{\eta}{\eta+1/2}\,\langle\,(\,\Delta\, X_2\,)^2\,\rangle = \frac{\eta}{\eta+1/2}\,[\,1 - \frac{1}{2}\,\frac{2+f+2\eta}{\eta f+2\eta+f+2}]\,, \tag{3.7}$$

where $f = \gamma_1/\gamma_2 > 0$. It is obvious that

$$\langle(\Delta\, Y_2)^2\rangle < \langle\, N_1 + 1/2\,\rangle, \tag{3.8}$$

so that the field is amplitude-squared squeezed.

For large η, i.e., very far above threshold,

$$\frac{\langle (\Delta Y_2)^2 \rangle}{\langle N_1 + 1/2 \rangle} = 1 - \frac{1}{f+2} , \tag{3.9}$$

and in the limit $f \to 0$, $\langle (\Delta Y_2)^2 \rangle$ is maximally squeezed and

$$\frac{\langle (\Delta Y_2)^2 \rangle}{\langle N_1 + 1/2 \rangle} = \frac{1}{2} . \tag{3.10}$$

We see, therefore, that inside the cavity above threshold the highest level of amplitude-squared squeezing that can be obtained is 50%.

B. Outside the Cavity

In previous calculation the amplitude-squared squeezing inside the cavity is associated with normal squeezing in ΔX_2. This is also true for the output signal if the amplitude of the output field is large. In this case, the fluctuation $\Delta a_1 = a_1 - \langle a_1 \rangle$ is small when compared with large amplitudes, we can again neglect higher order fluctuation terms with respect to the second order ones. We then find that

$$\langle (\Delta Y_1)^2 \rangle = \langle a_1^\dagger \rangle^2 \langle (\Delta a_1^\dagger)^2 \rangle + \langle a_1^\dagger \rangle \langle a_1 \rangle \langle \Delta a_1^\dagger \Delta a_1 + \Delta a_1 \Delta a_1^\dagger \rangle + \langle a_1 \rangle^2 \langle (\Delta a_1)^2 \rangle , \tag{3.11}$$

$$\langle N_1 \rangle = \langle a_1^\dagger \rangle \langle a_1 \rangle + \langle \Delta a_1^\dagger \Delta a_1 \rangle . \tag{3.12}$$

Let $\langle a_1 \rangle = |\langle a_1 \rangle| e^{i\theta}$ and if we define a general quadrature component X_ϕ to be $X_\phi = \frac{1}{2} (e^{i\phi} a_1^\dagger + e^{-i\phi} a_1)$, then

$$\langle(\Delta Y_1)^2\rangle - \langle N_1+1/2\rangle \approx |\langle a_1\rangle|^2 [\, e^{2i\theta}\langle(\Delta a_1^\dagger)^2\rangle + 2\langle\Delta a_1^\dagger \Delta a_1\rangle + e^{-2i\theta}\langle(\Delta a_1)^2\rangle \,]$$

$$\approx |\langle a_1\rangle|^2\, 4\, [\langle(\Delta X_\theta)^2\rangle - 1/4\,]. \tag{3.13}$$

This implies that the normal squeezing in the X_θ direction will bring on the amplitude-squared squeezing in Y_1 direction. According to Collett and Walls [42], nearly perfect squeezing in X_θ can be achieved when the high-frequency losses from the cavity are insignificant ($\gamma_2 << \gamma_1$) , thus, a large amount of squeezing in ΔY_1 can also be obtained under the same condition. Now we see that the squeezing behaviors above threshold of DPO of the output field are quite different from that of the intracavity signal. An arbitrary amount of amplitude-squared squeezing is attainable theoretically in the output port while the maximum squeezing in the intracavity mode is only by a factor of 2.

Amplitude-Squared Squeezing Blow Threshold Of DPO

A. Inside the Cavity

In this case, we can treat approximately the pumping field as a constant since its depletion is small, and then H_I can be written in the interaction picture as

$$H_I = i\,(\varepsilon\, a_1^{\dagger 2} - \varepsilon^*\, a_1^2), \tag{3.14}$$

where $\varepsilon = |\varepsilon|\, e^{i\vartheta}$ is a complex number. By making use of the simplified Master equation

$$\frac{d\rho}{dt} = -\,i\,[\, H_I\ , \rho\,]\ + \Lambda_1\, \rho\,, \tag{3.15}$$

we can calculate directly the steady-state solution of the values of $\langle a_1^{\dagger m}\, a_1^n\rangle$ (m, n= 0, 1, 2, 3, 4, m+n≤4) and then find $\langle(\Delta Y_i)^2\rangle$ (i=1, 2). It is found that the maximum squeezing in $\langle(\Delta Y_i)^2\rangle$ can be reached when $\vartheta = 0$ and

$$\frac{\langle(\Delta Y_2)^2\rangle}{\langle N_1 + 1/2\rangle} = 1 - \frac{2|\varepsilon|^2}{\gamma_1^2} \ . \tag{3.16}$$

Below threshold we have that $\gamma_1^2 > 4|\varepsilon|^2$, so the squeezing factor $\frac{\langle(\Delta Y_2)^2\rangle}{\langle N_1 + 1/2\rangle}$ is again always smaller than 1/2, similar to the result for above threshold inside the cavity of Eq. (3.10). So the amount of amplitude-squared squeezing for the intracavity signal mode of a DPO is always below 50%.

B. Outside the Cavity

Yurke [38] considered a ring configuration for a DPO shown in Fig.1. The cavity consists of one partly reflecting mirror M1, two totally reflecting mirrors M2 and M3. The input optical mode is a_2 and the output mode is a_1, both are of frequency ω. According to Yurke, the mode transformation for a no-loss parametric gain medium (PGM) is

$$c_2 = G\,c_1 + e^{i\Delta}(G^2 - 1)^{1/2}\,c_1^\dagger \ , \tag{3.17}$$

where the gain, G, is taken to be real, and $G > 1$. Δ is a parameter which depends on the phase of the field pumping the PGM.

For the partly reflecting mirror M1,

$$\begin{aligned} b_1 &= T\,a_2 - R\,b_2 \ , \\ a_1 &= R\,a_2 + T\,b_2 \ , \end{aligned} \tag{3.18}$$

where T and R are taken to be real and positive, and satisfy

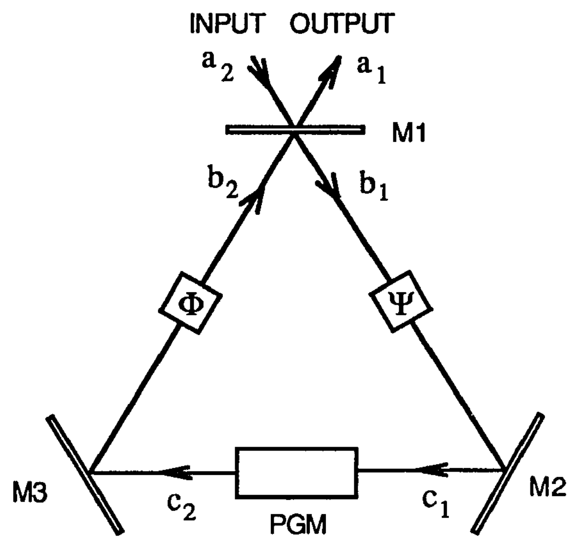

Fig. 1 Ring configuration for a degenerate parametric oscillator studied by Yurke. M1 is a partly reflective mirror, M2 and M3 are two totally reflective mirrors, PGM is the no-loss parametric gain medium, Ψ and Φ are phase shifts: Ψ -- from M1 to input port of PGM and Φ -- from output port of PGM to M1. a_2 and a_1 are input and output modes respectively.

$$T^2 + R^2 = 1. \tag{3.19}$$

In propagation from M1 to the input port of the PGM, the light accumulates a phase Ψ

$$c_1 = e^{i\Psi}\, b_1 \,. \tag{3.20}$$

Similarly,

$$b_2 = e^{i\Phi}\, c_2 \,, \tag{3.21}$$

where Φ is the phase shift from the output port of the PGM to M1.

Combining Eqs. (3.17) - (3.21), we find that a_1 is related to a_2 and a_2^+ as

$$a_1 = \mu\, a_2 + \nu\, a_2^\dagger , \tag{3.22}$$

where

$$\mu = |\mu|\, e^{i\phi_\mu} = \frac{2R + e^{i(\Psi+\Phi)}\, G + e^{-i(\Psi+\Phi)}\, G\, R^2}{1 + R^2 + 2\, G\, R \cos(\Psi+\Phi)} , \tag{3.23}$$

$$\nu = |\nu|\, e^{i\phi_\nu} = \frac{e^{i(\Delta+\Phi-\Psi)}\, (G^2-1)^{1/2}(1-R^2)}{1 + R^2 + 2\, G\, R \cos(\Psi+\Phi)} . \tag{3.24}$$

It is not difficult to verify that

$$|\mu|^2 - |\nu|^2 = 1, \tag{3.25}$$

which is necessary in maintaining the commutation relation $[\, a_1 , a_1^\dagger\,] = 1$.

Let $|0\rangle_2$ denote the vacuum state of the a_2 mode, then

$$a_2\, |0\rangle_2 = 0 , \tag{3.26}$$

By employing Eqs. (3.22), (3.25) and (3.26), we can calculate the fluctuations in the two quadrature of the square of the field amplitude at the output port when vacuum is fed into the input port

$$\langle(\Delta Y_1)^2\rangle = \frac{1}{2}\, [\, |\mu|^4 + |\nu|^4 + 2\, |\mu|^2\, |\nu|^2 \cos 2(\phi_\mu + \phi_\nu)\,] , \tag{3.27}$$

$$\langle(\Delta Y_2)^2\rangle = \frac{1}{2}\,[\,|\mu|^4 + |\nu|^4 - 2\,|\mu|^2\,|\nu|^2 \cos 2(\phi_\mu + \phi_\nu)\,]\,. \tag{3.28}$$

Since $N_1 = a_1^\dagger a_1$, we have

$$\langle\, N_1 + 1/2 \,\rangle = |\mu|^2 - 1/2\,. \tag{3.29}$$

The fluctuations in Y_1 are maximized and those in Y_2 are minimized when

$$\phi_\mu + \phi_\nu = 0\,. \tag{3.30}$$

Imposing this condition Eq. (3.27) and Eq. (3.28) reduce to:

$$\langle(\Delta Y_1)^2\rangle = \frac{1}{2}\,[\,|\mu|^2 + |\nu|^2\,]^2 = 2\,(\,|\mu|^2 - \frac{1}{2}\,)^2\,, \tag{3.31}$$

$$\langle(\Delta Y_2)^2\rangle = \frac{1}{2}\,. \tag{3.32}$$

Now it is clear that ΔY_2 is squeezed under the condition in Eq. (3.30) because $|\mu|^2 > 1$. Furthermore, for large values of $|\mu|$,

$$\frac{\langle(\Delta Y_2)^2\rangle}{\langle\, N_1 + 1/2 \,\rangle} = \frac{1}{2\,|\mu|^2 - 1} = \frac{1}{2\,|\mu|^2} + O\,[\,\frac{1}{|\mu|^4}\,]\,. \tag{3.33}$$

This means that large values of $|\mu|$ are favorable for obtaining large amounts of amplitude-squared squeezing.

From Eq. (3.23), large values of $|\mu|$ can be obtained if the system is just below threshold, i.e., if $2RG \cos(\,\Phi + \Psi\,)$ is close to $-\,(\,1 + R^2)$. This choice makes the denominator of Eq. (3.23) close to zero while the numerator remains finite. In order to satisfy this condition, one needs

$$G > \frac{1 + R^2}{2R}. \tag{3.34}$$

Thus, by choosing appropriately the value of G and R, and adjusting the distance between M1, M2 and M3, large amounts of amplitude-squared squeezing can occur at the output of DPO.

Conclusion

Amplitude-squared squeezing and normal squeezing are independent effects in general [16]. Here we present results which trace an analogy between the two. The maximum amplitude-squared squeezing which can be obtained inside a DPO cavity is 50%, exactly the same as that for normal squeezing. We also find that a large amount of amplitude-squared squeezing can be achieved outside the cavity, again similar to the situation for normal squeezing.

CHAPTER 4
AMPLIFICATION OF AMPLITUDE-SQUARED SQUEEZING AND OTHER HIGHER-ORDER SQUEEZING

When a signal passes through an amplifier, it is modified in two ways. The first is that the amplitude of the signal is amplified. The second is that noise is added to the signal. The noise can be either phase-dependent or phase-independent. Here we shall consider only phase-independent noise. The ultimate lower limit on the amount of noise which is added is provided by quantum mechanics. This has led to a considerable amount of work on the quantum properties of linear amplifiers [43].

The added noise is of particular concern when one wants to amplify a nonclassical state of the electromagnetic field. If too much phase-independent noise is added to such a state it loses its nonclassical character. Because the amount of noise added increases with the gain, one finds that for sufficiently high gains all nonclassical input states will become classical at the output. This was examined in detail for a phase-insensitive amplifier by Hong, Friberg, and Mandel [44]. They found that both squeezing and sub-Poissonian photon statistics, two of the most studied nonclassical properties, disappear at the output, no matter what the input state, if the gain is greater than two. They also found a gain above which all nonclassical properties of the output state vanish. This gain can exceed two. There is, therefore, a range of the gain below which the most well known nonclassical properties disappear, but which itself lies below the gain at which the output must be classical. It is an open question whether there are any nonclassical properties which survive into this range.

In order to investigate this point further we shall examine the effect of amplification on two additional nonclassical properties. Both are forms of higher-order squeezing.

The first is fourth-order squeezing as defined by Hong and Mandel [11]. The second is amplitude-squared squeezing.

A state is said to be squeezed to fourth order in the X_1 direction if [11]

$$\langle(\Delta X_1)^4\rangle < 3/16\,. \tag{4.1}$$

The right-hand side of the above inequality is the value that the expectation value on the left-hand side assumes in the vacuum state.

States with either fourth order squeezing or amplitude-squared squeezing can be produced by nonlinear optical devices. In fact, both types of squeezing can be present in the output of a degenerate parametric amplifier and in the fundamental mode in second harmonic generation [11, 15, 16]. Second harmonic generation also plays an important role in the detection of amplitude-squared squeezing. If the fundamental mode is amplitude-squared squeezed, then the second harmonic will be squeezed in the normal sense [15]. Normal squeezing can be measured by using a homodyne detector. Therefore, a frequency doubling crystal in conjunction with a homodyne detector can be used to measure amplitude-squared squeezing.

In the next section we shall discuss the model for the amplifier and how to extract results from it. In two subsequent sections the effects of amplification on fourth-order squeezing and amplitude-squared squeezing will be considered.

Amplifier

The density matrix of a single-mode field, $\rho(t)$, interacting linearly with a collection of two-level atoms, N_2 of which are in their upper states and N_1 of which are in their lower states, is governed by the master equation

$$\frac{d\rho}{dt} = \beta N_2(2a^{+}\rho\, a - a\, a^{+}\rho - \rho a\, a^{+}) + \beta N_1(2a\rho\, a^{+} - a^{+}\, a\rho - \rho a^{+}\, a)\,, \tag{4.2}$$

where β is a coupling constant. We have assumed that the field is resonant with the atomic transition frequency. From this equation Carusotto [45] was able to find the normally ordered characteristic function of $\rho(t)$, $C(\xi,t)$, which is defined as

$$C(\xi,t) = \mathrm{Tr}[\, e^{\xi a^+} e^{-\xi^* a}\, \rho(t)] \,. \tag{4.3}$$

He found that

$$C(\xi,t) = C_1(\xi,t)\, C_2(\xi,t) \,, \tag{4.4}$$

where

$$C_1(\xi,t) = \mathrm{Tr}[\, \exp(e^{i\omega t}\, e^{\beta(N_2-N_1)\,t}\, \xi a^+) \exp(-\, e^{-\,i\omega t}\, e^{\beta(N_2-N_1)\,t}\, \xi^* a)\, \rho(0)], \tag{4.5}$$

and

$$C_2(\xi,t) = \exp\{-\, [N_2/(N_2-N_1)]\, (e^{2\beta(N_2-N_1)\,t} - 1)\xi^*\xi\}. \tag{4.6}$$

Expectation values of creation and annihilation operators can be obtained by differentiating $C(\xi,t)$, i. e.

$$\langle a^{+n} a^m \rangle = \left(\frac{\partial}{\partial \xi}\right)^{n} \left(-\frac{\partial}{\partial \xi^*}\right)^{m} C(\xi,t)|_{\xi=0} \,. \tag{4.7}$$

In particular, for $n=0$ and $m=1$ we see that

$$\langle\, a(t)\, \rangle = e^{-\,i\omega t}\, e^{\beta(N_2-N_1)\,t} \langle\, a(0)\, \rangle \equiv G(t)\, \langle\, a(0)\, \rangle \,, \tag{4.8}$$

so that the system is an amplifier if $N_2 > N_1$. The quantity $|G(t)|^2$ is the gain of the amplifier. We can now use these results to study the amplification of higher-order squeezing.

Fourth-Order Squeezing

Let us first consider fourth-order squeezing as defined by Hong and Mandel. Suppose that a field state initially has values of $\langle(\Delta X_1)^2\rangle$ and $\langle(\Delta X_1)^4\rangle$ given by $\langle(\Delta X_1)^2\rangle_0$ and $\langle(\Delta X_1)^4\rangle_0$, respectively. Using the results of the previous section we find that at the output of the amplifier (after an interaction time t) $\langle(\Delta X_1)^4\rangle$ is given by

$$\langle(\Delta X_1)^4\rangle = |G|^4 \langle(\Delta X_1)^4\rangle_0 + (3/2)\,|G|^2\,(\,|G|^2 -1)\,[(N_1 + N_2)/(N_2 - N_1)]\,\langle(\Delta X_1)^2\rangle_0$$

$$+ (3/16)[(N_1 + N_2)/(N_2 - N_1)]^2\,(\,|G|^2 -1)^2\,. \tag{4.9}$$

Two immediate conclusions can be drawn from this expression. The first is that for $|G|^2 > 1$, if $\langle(\Delta X_1)^4\rangle < 3/16$, then it must be the case that $\langle(\Delta X_1)^4\rangle_0 < 3/16$. This follows from the fact that the second and third terms on the right-hand side of Eq. (4.9) are greater than or equal to zero. This means that if a state is fourth-order squeezed at the output, it must have been fourth-order squeezed at the input. The second conclusion is that if $|G|^2 > 2$, then $\langle(\Delta X_1)^4\rangle > 3/16$. This can be seen by noting that the third term on the right-hand side of Eq. (4.9) is greater than 3/16 for $|G|^2 > 2$, and that the first two terms are positive. Therefore, as for squeezing and sub-Poissonian statistics, fourth-order squeezing disappears for gains greater than two.

It is also possible to look at the amplification properties of what Hong and Mandel call intrinsic fourth-order squeezing. A state is intrinsically fourth-order squeezed if

$$\langle:(\Delta X_1)^4:\rangle < 0\,, \tag{4.10}$$

where the double dots denote normal ordering. Similar definitions hold for intrinsic squeezing of any even order. Fourth-order squeezing can be a result of intrinsic second-order squeezing, intrinsic-fourth order squeezing, or both. Application of the results for the amplifier gives

$$\langle : (\Delta X_1)^4 : \rangle = |G|^4 \langle (\Delta X_1)^4 \rangle_0 + (3/2) [\, |G|^2/(N_2 - N_1)] [(N_1 + N_2)|G|^2 - 2N_2] \langle (\Delta X_1)^2 \rangle_0$$

$$+ (3/16)[1/(N_2 - N_1)^2] [(N_1 + N_2) |G|^2 - 2N_2]^2 . \tag{4.11}$$

All three of the terms on the right-hand side are positive if

$$|G|^2 > 2N_2/(N_1 + N_2) . \tag{4.12}$$

For an amplifier $N_2 > N_1$ so that $2N_2/(N_1 + N_2) < 2$. Therefore, intrinsic fourth-order squeezing also disappears for gains greater than two.

Amplitude-Squared Squeezing

Let us now consider amplification of amplitude-squared squeezing. As we shall see the situation here is more complicated than that for fourth-order squeezing. Applying the results for the amplifier we find that the amplitude-squared squeezing at the output is related to that at the input by

$$(\Delta Y_1)^2 - \langle N + 1/2 \rangle = |G|^4 [(\Delta Y_1)_0{}^2 - \langle N + 1/2 \rangle_0] + [2N_2/(N_2 - N_1)]\, |G|^2 (|G|^2 - 1)\langle N \rangle_0$$

$$+ [N_2/(N_2 - N_1)]^2 (|G|^2 - 1)^2 . \tag{4.13}$$

An examination of this expression shows that in order to have amplitude-squared squeezing at the output, one must have it at the input.

We now want to find the maximum gain, $|G|^2_{max}$, for which amplitude-squared squeezing can occur at the output for given values of $(\Delta Y_1)_0$ and $\langle N \rangle_0$ at the input. In order to find it we set the left-hand side of Eq. (4.13) equal to zero and solve the resulting quadratic equation for $|G|^2$. The quantity $|G|^2_{max}$ is the larger of the two roots and is given by

$$|G|^2_{max} = [\kappa/(2\kappa\langle N\rangle_0 + \kappa^2 - \eta)]\,[\,(\langle N\rangle_0 + \kappa) + (\langle N\rangle_0^2 + \eta)^{1/2}]\,, \tag{4.14}$$

where $\kappa = N_2/(N_2 - N_1)$ and $\eta = \langle N + 1/2\rangle_0 - (\Delta Y_1)_0^2$. Note that $\kappa > 1$, and we shall assume that $\eta > 0$, i. e. the input state is amplitude-squared squeezed. We shall first derive a simple inequality satisfied by $|G|^2_{max}$, and then investigate whether $|G|^2_{max}$ can exceed 2.

We begin deriving an upper bound for $|G|^2_{max}$ by observing that

$$\langle N\rangle_0^2 + \eta = \langle N\rangle_0^2 + \langle N\rangle_0 + 1/2 - (\Delta Y_1)_0^2 < (\langle N\rangle_0 + 1/\sqrt{2})^2\,. \tag{4.15}$$

Substituting this result into Eq. (4.14) gives

$$|G|^2_{max} < 1 + (\eta + \kappa/\sqrt{2}\,)/(2\kappa\langle N\rangle_0 + \kappa^2 - \eta)\,. \tag{4.16}$$

We also have that

$$\eta + \kappa/\sqrt{2} \le \kappa/\sqrt{2} + 1/2 + \langle N\rangle_0\,,$$

and

$$2\kappa\langle N\rangle_0 + \kappa^2 - \eta \ge (2\kappa - 1)\langle N\rangle_0 + \kappa^2 - 1/2\,, \tag{4.17}$$

which imply that

$$|G|^2_{max} \leq 1 + (\kappa/\sqrt{2} + 1/2 + \langle N\rangle_0) \,/\, [(2\kappa - 1)\langle N\rangle_0 + \kappa^2 - 1/2] \,. \tag{4.18}$$

The second term on the right-hand side of Eq. (4.18) is a decreasing function of κ for $\kappa > 0$ as can be verified by taking its derivative. Therefore, it assumes its maximum value for $\kappa \geq 1$ when $\kappa = 1$, and

$$|G|^2_{max} \leq 2 + 1/[\sqrt{2}\,(1/2 + \langle N\rangle_0)] \,. \tag{4.19}$$

We see, then, that as $\langle N\rangle_0$ increases, the maximum gain which will produce amplitude-squared squeezing at the output goes to two from above.

The result in Eq. (4.19) suggests that amplitude-squared squeezing can persist at gains greater than the photon-cloning limit of two. What we now wish to determine is whether this is an artifact of the approximations made (i. e. the terms which were dropped) in deriving Eq. (4.19) or whether this nonclassical effect really can exist at gains greater than two. We begin by deriving a lower bound for $|G|^2_{max}$.

In order to derive the lower bound we shall make use of the inequalities

$$\begin{aligned} &(1 + x)^{1/2} \geq 1 + x/2 - x^2/8 \quad , \\ &(1 + x)^{-1} \geq 1 - x \,, \end{aligned} \tag{4.20}$$

which hold for $1 > x \geq 0$. Applying these to the expressions appearing in Eq. (4.14) gives

$$[\langle N\rangle_0^2 + \langle N\rangle_0 + 1/2 - (\Delta Y_1)_0^2\,]^{1/2} \geq \langle N\rangle_0[1 + (\nu/2) - \nu^2/8] \,, \tag{4.21}$$

where $\nu = 1/\langle N\rangle_0 + [1/2 - (\Delta Y_1)_0^2]/\langle N\rangle_0^2$, and

$$[(2\kappa - 1)\langle N\rangle_0 + \kappa^2 - 1/2 + (\Delta Y_1)_0^2]^{-1} \geq [1/\langle N\rangle_0(2\kappa - 1)]$$

$$\times\{1 - [\kappa^2 - 1/2 + (\Delta Y_1)_0{}^2]/\langle N\rangle_0(2\kappa - 1)\}. \tag{4.22}$$

Combining these we find that $|G|^2_{max}$ is greater than or equal to

$$|G|^2_{max} \geq [\kappa/(2\kappa-1)]\ \{1 - [\kappa^2 - 1/2 + (\Delta Y_1)_0{}^2]/\langle N\rangle_0(2\kappa-1)\}$$

$$\times[2 + \kappa/\langle N\rangle_0 + (\nu/2) - \nu^2/8]\ . \tag{4.23}$$

If we arrange the right-hand side in increasing powers of $1/\langle N\rangle_0$ we find

$$|G|^2_{max} \geq [\kappa/(2\kappa-1)]\ (2 + \{\kappa + 1/2 - 2[\kappa^2 - 1/2 + (\Delta Y_1)_0{}^2]/(2\kappa-1)\}\ (1/\langle N\rangle_0)$$

$$+ O(1/\langle N\rangle_0{}^2)\ )\ . \tag{4.24}$$

If κ and $(\Delta Y_1)_0{}^2$ are of order one so are the coefficients of the terms of higher order in $1/\langle N\rangle_0$. Therefore, by choosing an initial state with $\langle N\rangle_0$ large and $(\Delta Y_1)_0{}^2$ of order one these terms can be made as small as we wish, and they can be neglected. Turning our attention to the $1/\langle N\rangle_0$ term we find that

$$\kappa + 1/2 - 2[\kappa^2 - 1/2 + (\Delta Y_1)_0{}^2]/(2\kappa-1) \geq 0\ , \tag{4.25}$$

if $(\Delta Y_1)_0{}^2 \leq 1/4$.

Our conclusion is the following. If we want to achieve $|G|^2_{max} > 2$, then first we must choose κ very close to one. This will make the factor $\kappa/(2\kappa-1)$ close to its maximum value of 1. For this condition to be met we need $N_2 >> N_1$, i. e. a large inversion. It is then necessary to find an input state which has a large photon number and a value of $(\Delta Y_1)_0{}^2$ less than 1/4. This state will remain nonclassical after passing through an amplifier with a gain greater than two.

The amplitude-squared squeezed vacuum is a state which can meet these requirements. The state is a minimum uncertainty state for Y_1 and Y_2, i. e. $\Delta Y_1 \Delta Y_2 = \langle N+1/2 \rangle$. It is given in terms of photon number states by

$$|0; \lambda\rangle = c_0 \sum_{n=0}^{\infty} r^n \sqrt{a_n}\ |4n\rangle , \tag{4.26}$$

where $r = (\lambda-1)/(\lambda+1)$, c_0 is a normalization constant and

$$a_0 = 1 , \qquad a_n = \prod_{k=1}^{n} [(4k-2)(4k-3)]/[4k(4k-1)] , \qquad \text{for } n \geq 1. \tag{4.27}$$

Here λ is a parameter which describes the amount of amplitude-squared squeezing in the state. From Eq. (2.7) of chapter 2,

$$(\Delta Y_1)^2 = \lambda \langle N+1/2 \rangle , \tag{4.28}$$

so that for $0<\lambda<1$ the state is amplitude-squared squeezed in Y_1 direction. Further properties of this state are discussed in the Appendix B. It is shown there that the photon number for this state obeys

$$[\, \mu_1 + \mu_2 |\ln(1-r^2)|\,]^{-1} (1-\lambda)^2/4\lambda \leq \langle N \rangle \leq [\, \mu_3 + \mu_4 |\ln(1-r^2)|\,]^{-1} (1-\lambda)^2/4\lambda , \tag{4.29}$$

where the μ_j, $j=1$ to 4, are numerical constants whose values are given in the Appendix B. From this inequality it is clear first that $\langle N \rangle \to \infty$ as $\lambda \to 0$. One also has that $\lambda\langle N \rangle \to 0$ as $\lambda \to 0$ which, in combination with Eq. (4.28), implies that $\Delta Y_1 \to 0$ as $\lambda \to 0$. Therefore, the state $|0; \lambda\rangle$ for $\lambda<<1$ meets the requirements for a state to remain amplitude-squared squeezed for $|G|^2_{max} > 2$. One must choose λ large enough so that $\langle N \rangle >> 1$ and $(\Delta Y_1)^2 < 1/4$.

Conclusion

The amplification of higher-order squeezing by a phase insensitive amplifier has been studied. It is found that for any form of higher-order squeezing to be present at the the output of the amplifier it must have been present at the input. There is also a limit on how large the gain can be if there is still to be higher-order squeezing in the output field. This limit is independent of how much higher-order squeezing is present in the input field. For fourth -order squeezing of the Hong-Mandel type and for fourth-order intrinsic squeezing the maximum gain is two. For amplitude-squared squeezing the maximum gain can exceed two by a small amount.

Amplitude-squared squeezing is the first known nonclassical field property which can survive amplification at gains greater than two. Even though the allowed gain exceeds two by only a small amount, this example shows that not all nonclassical properties disappear at the photon cloning limit. It suggests that it would be worthwhile to see if there are other nonclassical properties which can persist at even higher gains.

CHAPTER 5

MINIMUM UNCERTAINTY STATES FOR AMPLITUDE-SQUARED SQUEEZED STATES: GENERAL SOLUTION

In chapter 2, the eigenvalue equation which gives the minimum uncertainty states for amplitude-squared squeezing has been studied. A particularly simple subset of these minimum uncertainty states was found. These are constructed by applying a squeeze operator to states called Hermite Polynomial States, so called because they are defined as the vacuum state acted on by operators in the form of Hermite polynomials whose argument is proportional to the mode creation operator. This subset of minimum uncertainty states of amplitude-squared squeezing is obtained by assuming that the expectation values of the two quadrature components Y_1 and Y_2 are related to their fluctuations in a special manner. This is not true for a general amplitude-squared squeezed minimum uncertainty state. It is the purpose of this chapter to provide a general solution to the eigenvalue equation which gives the minimum uncertainty states for amplitude-squared squeezing. The solution obtained here is both more general and, unfortunately, more complicated than the one obtained in the second chapter.

In the first section, we start again from the eigenvalue equation for amplitude-squared squeezing and find the general solution for the minimum uncertainty states. These can be expressed in terms of hypergeometric functions. A general expression of the average photon number in these minimum uncertainty states has been derived in next section. In sections thereafter, we studied the quasiprobability Q-function, photocount probability distribution, and the relation between the symmetry of Q-function plots and photocount probability distribution for these states. Results of this chapter are summarized in last section.

General Solution for Eigenvalue Equation

The eigenvalue equation Eq. (2.1) of the minimum uncertainty states for amplitude-squared squeezing has been studied in the second chapter. In order to get rid of the difficulty of three-term recurrence relation, we introduced the squeezing operator (transformation) S(z) in Eq. (2.9)

$$|\psi\rangle = S(z)\,|\psi'\rangle \tag{5.1}$$

where z is a function of λ and β (see chapter 1). If we now expand $|\psi'\rangle$ in terms of number states

$$|\psi'\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle\,, \tag{5.2}$$

with z given by Eqs. (2.11) though (2.13), we find a relatively simple two-term recurrence relation for C_n

$$C_{n+2} = \left[\frac{\beta - i\sqrt{1-\lambda^2}\,(n+1/2)}{\sqrt{(n+2)(n+1)}}\right] C_n\,, \tag{5.3}$$

for $0 < \lambda < 1$, and

$$C_{n+2} = \left[\frac{\beta - \sqrt{\lambda^2-1}\,(n+1/2)}{\lambda\sqrt{(n+2)(n+1)}}\right] C_n\,, \tag{5.4}$$

for $\lambda \geq 1$. Both C_0 and C_1 are arbitrary.

From now on, we will focus our study on the situation $\lambda \geq 1$ (squeezing in Y_2

direction), and all results obtained hereafter can be transferred directly to the situation $0 < \lambda < 1$ (squeezing in Y_1 direction). There are two independent solution to Eq. (5.4). One, denoted by $|\psi'\rangle_e$, is made up of even number states and the other, denoted by $|\psi'\rangle_o$, is a sum of odd number states.

Let us first look at the even number states $|\psi'\rangle_e$. With $n=2k$, Eq. (2.9) becomes

$$|\psi'\rangle_e = \sum_{k=0}^{\infty} C_{2k} |2k\rangle , \tag{5.5}$$

and

$$C_{2k} = \frac{\beta - \sqrt{\lambda^2-1}\,[2(k-1)+1/2]}{\lambda\sqrt{(2k)(2k-1)}}\; C_{2k-2}$$

$$= [-2\sqrt{1-\lambda^{-2}}]\; \frac{[1/4 - \beta/(2\sqrt{\lambda^2-1})] + (k-1)}{\sqrt{(2k)!}}\; C_{2k-2}$$

$$= [-2\sqrt{1-\lambda^{-2}}]^k\; \frac{[1/4 - \beta/(2\sqrt{\lambda^2-1})]_k}{\sqrt{(2k)!}}\; C_0 \tag{5.6}$$

where the notation $(\alpha)_k$ has been used which denotes $(\alpha)_k = \alpha\,(\alpha+1)\,...\,(\alpha+k-1)$ for any α. If we substitute Eq. (5.6) back to Eq. (5.5), we can write

$$|\psi'\rangle_e = C_0 \sum_{k=0}^{\infty} \frac{\left(1/4 - \beta/(2\sqrt{\lambda^2-1})\right)_k}{(2k)!}\; [-2\sqrt{1-\lambda^{-2}}\; a^{+2}]^k\; |0\rangle . \tag{5.7}$$

Note that we have used identity $|2k\rangle = \frac{a^{+2k}}{\sqrt{(2k)!}} |0\rangle$ in this equation.

The Kummer function ${}_1F_1(\xi, \zeta, Z)$ is defined as [46]

$${}_1F_1(\xi, \zeta, Z) = \sum_{k=0}^{\infty} \frac{(\xi)_k}{(\zeta)_k} \frac{Z^k}{k!} . \tag{5.8}$$

When $\zeta = 1/2$, it becomes

$$ {}_1F_1(\xi\,,1/2\,,Z) = \sum_{k=0}^{\infty} \frac{(\xi)_k}{(2\,k)\,!}\,(4\,Z)^k\,. \tag{5.9}$$

With the help of Eq. (5.9), we can express Eq. (5.7) as

$$ |\psi'\rangle_e = C_0\,{}_1F_1(\xi\,,1/2\,,\gamma\,a^{+2})\;|0\rangle\,, \tag{5.10}$$

where $\xi = 1/4 - \beta/(2\sqrt{\lambda^2 - 1})$ and $\gamma = -\frac{\sqrt{1-\lambda^{-2}}}{2}$. Thus the minimum uncertainty states for amplitude-squared squeezing can be expressed as

$$ |\psi\rangle_e = C_0\,S(z)\,\{{}_1F_1(\xi\,,1/2\,,\gamma\,a^{+2})\}\;|0\rangle\,. \tag{5.11}$$

Note that this is the solution associated with the even-number-state solution $|\psi'\rangle_e$. Following the same procedure, one can find the second solution

$$ |\psi\rangle_o = C_1\;S(z)\,\{{}_1F_1(\xi + 1/2, 3/2\,,\gamma\,a^{+2})\}\;|1\rangle\,. \tag{5.12}$$

It is clear at this point that both C_0 and C_1 play the role of normalization constants. They can be determined by $\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = 1$. As an example, let us derive an expression for C_0. By employing Eq. (5.7), we find that

$$ 1 = {}_e\langle\psi'|\psi'\rangle_e = |C_0|^2 \sum_{k=0}^{\infty} \frac{(\xi^*)_k(\xi)_k}{(2\,k)\,!}\left[4\,(1-\lambda^{-2})\right]^k $$

$$ = |C_0|^2\;{}_2F_1[\xi\,,\xi^*;\;1/2\,;\;(1-\lambda^{-2})]\,, \tag{5.13}$$

where ${}_2F_1[\xi , \xi^*;\ 1/2\ ;\ (1-\lambda^{-2})]$ is the hypergeometric function defined by [46]

$$ {}_2F_1[\xi_1 , \xi_2^*;\ \zeta\ ;\ Z] = \sum_{k=0}^{\infty} \frac{(\xi_1)_k (\xi_2)_k}{(\zeta)_k} \frac{Z^k}{k\,!} . \tag{5.14} $$

Thus one can write C_0 as

$$ C_0 = \{ {}_2F_1[\xi , \xi^*;\ 1/2\ ;\ (1-\lambda^{-2})] \}^{-1/2} , \tag{5.15} $$

where an arbitrary phase factor has been neglected.

The states $|\psi\rangle_e$ and $|\psi\rangle_o$ of Eqs. (5.11) and (5.12) are two independent solutions to the eigenvalue equation for minimum uncertainty states for amplitude-squared squeezing. Furthermore, any linear combination of these two states, where the values of λ and β are the same in both, is also a minimum uncertainty state for amplitude-squared squeezing.

Average Photon Number

We now want to discuss some of the properties of the minimum uncertainty states for amplitude-squared squeezing obtained in previous section. The first thing we are interested in is the average photon number in these states. If we know $\langle N \rangle$, then, since it is directly related to the uncertainties in Y_1 and Y_2 (see Eq.(2.7)), we know ΔY_1 and ΔY_2. We will again consider only the even-number-state solution $|\psi\rangle_e$. The second solution $|\psi\rangle_o$ can be treated quite similarly.

For this even-number-state solution $|\psi\rangle_e = S(z)\,|\psi'\rangle_e$ the average photon number can be written as

$$ \langle N \rangle_e = {}_e\langle \psi' |\ S^{-1}(z)\ a^+ a\ \ S(z) | \psi' \rangle_e $$

$$= {}_e\langle\psi'| \left[\frac{\sqrt{\lambda^2-1}}{2}\,(a^{+2}+a^2)+\lambda\, a^+ a\ +\frac{\lambda-1}{2}\right] |\psi'\rangle_e \tag{5.16}$$

By making use of Eq. (2.3), one finds

$$\langle a^2\rangle_e = {}_e\langle\psi'|\, S^{-1}(z)\, a^2\ S(z)|\psi'\rangle_e = \beta_r + i\,\beta_i/\lambda\,. \tag{5.17}$$

This equation enables us to derive the result

$${}_e\langle\psi'|\, a^{+2} + a^2\, |\psi'\rangle_e = \frac{2\beta_r}{\lambda} - 2\sqrt{1-\lambda^{-2}}\ ({}_e\langle\psi'|\, a^+a\, |\psi'\rangle_e + 1/2\,)\,, \tag{5.18}$$

which allows us to evaluate the first term on the right-hand side of Eq. (5.16). If we substitute Eq. (5.18) into Eq. (5.16), then we obtain

$$\langle N+1/2\rangle_e = {}_e\langle\psi'|\, N+1/2|\psi'\rangle_e\,/\lambda + \beta_r(1-\lambda^{-2})\,. \tag{5.19}$$

By employing Eq. (5.7), one finds

$$\begin{aligned}
{}_e\langle\psi'|\, a^+a\, |\psi'\rangle_e &= |C_0|^2 \sum_{k=1}^{\infty} \frac{(\xi^*)_k(\xi)_k}{(2k-1)!}\left[4\,(1-\lambda^{-2})\right]^k \\
&= |C_0|^2\left[4\,(1-\lambda^{-2})\,|\xi|^2\right]\sum_{k=0}^{\infty}\frac{(\xi^*+1)_k(\xi+1)_k}{(2k+1)!}\left[4\,(1-\lambda^{-2})\right]^k \\
&= |C_0|^2\left[4\,(1-\lambda^{-2})\,|\xi|^2\right]{}_2F_1[\xi+1\,,\,\xi^*+1;\ 3/2\,;\ (1-\lambda^{-2})]\ .
\end{aligned} \tag{5.20}$$

Substituting this equation back into Eq. (5.19), and with $|C_0|^2$ determined by Eq. (5.15), we find that

$$\langle N+1/2\rangle_e = \left[4\lambda^{-1}(1-\lambda^{-2})|\xi|^2\right] \frac{{}_2F_1(\xi^*+1, \xi+1; 3/2; 1-\lambda^{-2})}{}$$

$$\langle N+1/2\rangle_e = \left[4\lambda^{-1}(1-\lambda^{-2})|\xi|^2\right] \frac{{}_2F_1(\xi^*+1, \xi+1; 3/2; 1-\lambda^{-2})}{{}_2F_1(\xi^*, \xi; 1/2; 1-\lambda^{-2})}$$

$$+\frac{1}{2\lambda} + \beta_r\sqrt{(1-\lambda^{-2})}\,. \tag{5.21}$$

Employing the following identity for the hypergeometric function

$${}_2F_1(\xi_1+1, \xi_2+1; \zeta+1; z) = \frac{\zeta}{\xi_1\,\xi_2}\,\frac{d}{dz}\Big({}_2F_1(\xi_1, \xi_2;\ \zeta;\ z)\Big), \tag{5.22}$$

Eq. (5.21) can be written approximately as

$$\langle N+1/2\rangle_e \approx (\lambda^2-1)\,\frac{d}{d\lambda}\Big(\ln\big({}_2F_1(\xi^*, \xi;\ 1/2;\ 1-\lambda^{-2})\big)\Big) + \frac{1}{2\lambda} + \beta_r\sqrt{(1-\lambda^{-2})}\,, \tag{5.23}$$

where ξ^* and ξ have been treated as constants in the differentiation. Eq. (5.23) enables us to find the asymptotic behavior of $\langle N+1/2\rangle_e$ for highly squeezed states, namely $\lambda >> 1$. For the sake of simplicity, we assume that $\beta \approx 1$, which corresponds to states with small expectation values of $\langle Y_1\rangle$ and $\langle Y_2\rangle$, Eq. (5.23) then becomes (see Appendix C)

$$\langle N+1/2\rangle_e \approx \frac{\lambda}{\ln\lambda}\,, \tag{5.24}$$

which implies $\Delta Y_2 \approx \sqrt{1/\ln\lambda}$. Note that $\langle N+1/2\rangle_e \to \infty$ as λ goes to infinity, which means that a highly squeezed state contains a large number of photon. This behavior has already been noted in chapter 2.

Q-Functions

Next, we want to get an expression for the quasiprobability Q-function of the minimum uncertainty states for amplitude-squared squeezing. The Q-function for any state $|\psi\rangle$ is defined as

$$Q(\alpha) = \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2 , \tag{5.25}$$

where $|\alpha\rangle$ is a coherent state. With the help of Eq. (2.8), we can start to derive the Q-function of the minimum uncertainty states for amplitude-squared squeezing by noting that

$$\langle\alpha|\psi\rangle = \frac{1}{\beta}\ \langle\alpha|\left[\frac{1-\lambda}{2}a^{+2} + \frac{1+\lambda}{2}a^2\right]|\psi\rangle. \tag{5.26}$$

For a coherent state, we have

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle, \qquad a^+\,|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \alpha^*/2\right)|\alpha\rangle , \tag{5.27}$$

so that Eq. (5.26) can then be expressed as

$$\left[\frac{\partial^2}{\partial\alpha^{*2}} + \alpha\frac{\partial}{\partial\alpha^*} + \frac{\alpha^2}{4} - \frac{\lambda-1}{\lambda+1}\alpha^{*2} - \frac{2\beta}{\lambda+1}\right]\langle\alpha|\psi\rangle = 0 . \tag{5.28}$$

If we write $\langle\alpha|\psi\rangle$ in the following form

$$\langle\alpha|\psi\rangle = \mathrm{Exp}\left[-\frac{|\alpha|^2}{2} + \frac{1}{2}\sqrt{(\lambda-1)/(\lambda+1)}\ \ \alpha^{*2}\right] F(\alpha^*) , \tag{5.29}$$

then, after substituting Eq. (5.29) into Eq. (5.28) , we find that $F(\alpha^*)$ satisfies

$$\left[\frac{\partial^2}{\partial\alpha^{*2}} + 2\sqrt{\frac{\lambda -1}{\lambda +1}}\, \alpha^* \frac{\partial}{\partial\alpha^*} + \left(\sqrt{\frac{\lambda -1}{\lambda +1}} - \frac{2\beta}{\lambda +1}\right)\right] F(\alpha^*) = 0 . \tag{5.30}$$

This equation can be solved by means of a series expansion. If we assume

$$F(\alpha^*) = \sum_{n=0}^{\infty} B_n \, \alpha^{*n} , \tag{5.31}$$

then from Eq. (5.30) we obtain the recurrence relation for the expansion coefficients B_n

$$B_{n+2} = - \frac{2\sqrt{\frac{\lambda -1}{\lambda +1}}\; n + \left(\sqrt{\frac{\lambda -1}{\lambda +1}} - \frac{2\beta}{\lambda +1}\right)}{(n+2)(n+1)} \; B_n . \tag{5.32}$$

Comparing it with the previous recurrence relation for C_n of Eqs. (5.3) and (5.4), one finds that they are quite similar. Thus our discussions of B_n and also that of Eq. (5.30) can be made along the same line as that in the first section on C_n. For instance, there are again two independent solutions for $F(\alpha^*)$, one is made up of even powers of α^* and the other, is the sum of odd power terms. Now let us look at the even power series solution of $F(\alpha^*)$

$$F(\alpha^*) = \sum_{k=0}^{\infty} B_{2k} \, \alpha^{*2k} , \tag{5.33}$$

which, after making use of the recurrence relation Eq. (5.32), can be expressed in terms of Kummer function as

$$F(\alpha^*) = B_0 \cdot {}_1F_1\left[\xi ; 1/2; -\sqrt{\frac{\lambda -1}{\lambda +1}}\, \alpha^{*2}\right] , \tag{5.34}$$

where B_0 is a constant. A similar result can be derived for the odd solution of $F(\alpha^*)$. Again we will concentrate our investigation on Eq. (5.34) only. The Q-functions of the minimum uncertainty states for amplitude-squared squeezing are given by

$$Q(\alpha) = \frac{|B_0|^2}{\pi} \mathrm{Exp}\left[-|\alpha|^2 + \frac{1}{2}\sqrt{\frac{\lambda-1}{\lambda+1}}\ (\alpha^{*2} + \alpha^2)\right] \left|{}_1F_1\left[\xi\ ;\ 1/2;\ -\sqrt{\frac{\lambda-1}{\lambda+1}}\ \alpha^{*2}\right]\right|^2, \quad (5.35)$$

where we have employed Eqs. (5.25), (5.29), and (5.34). Letting $\alpha = x_1 + i\, x_2$, where x_1 and x_2 are real, one is now ready to plot the Q-function in the x_1-x_2 phase space. In Fig. 2 the Q-function of this special amplitude-squared squeezed state (Eq. (5.35)) with $\beta = 0$ (Fig. 2(c)) has been plotted, together with the Q-functions of a normal squeezed vacuum state $|0\rangle_{N.S.}$ (2(b)) and a vacuum state $|0\rangle$ (2(a)). The state with $\beta = 0$ can be regarded as the amplitude-squared squeezed vacuum state because it has the property that $\langle Y_1 \rangle = \langle Y_2 \rangle = 0$, and we use $|0\rangle_{A.S.S.}$ to denote it, where the subscript A.S.S. refers to amplitude-squared squeezing. In the same way, $|0\rangle_{N.S.}$ refers to a normal squeezed vacuum state. It is obvious that there is a dramatic difference between the Q-function of the state $|0\rangle_{A.S.S.}$ and that of $|0\rangle_{N.S.}$. The Q-function of an amplitude-squared squeezed vacuum state possesses a 4-fold symmetry while that of the normal squeezed vacuum state is a 2-fold symmetric one. The comparison between the Q-function of a vacuum state $|0\rangle$ (Fig. 2(a)) and that of the state $|0\rangle_{A.S.S.}$ suggests that the "manner" of squeezing for state $|0\rangle_{A.S.S.}$ is like a vacuum state "squeezed" along two perpendicular directions in the x_1-x_2 phase space, which is quite different from a normal squeezed states that is squeezed along a single direction (x_1). Note that a normal squeezed vacuum state is also an amplitude-squared squeezed minimum uncertainty state (but corresponding $\beta \neq 0$, actually $\beta = (\lambda^2 - 1)^{1/2}/2$), as has been discussed in chapter 2. Fig. 3 shows that when β is displaced away from 0, the corresponding Q-function expands along the x_1 direction and contracts in the x_2 direction, and the 4-fold symmetry is immediately destroyed and degraded into a 2-fold one. The Q-function gradually changes into that of a normal squeezed vacuum state when β reaches the designated value. All of the Q-functions in Fig. 2 are single-peaked

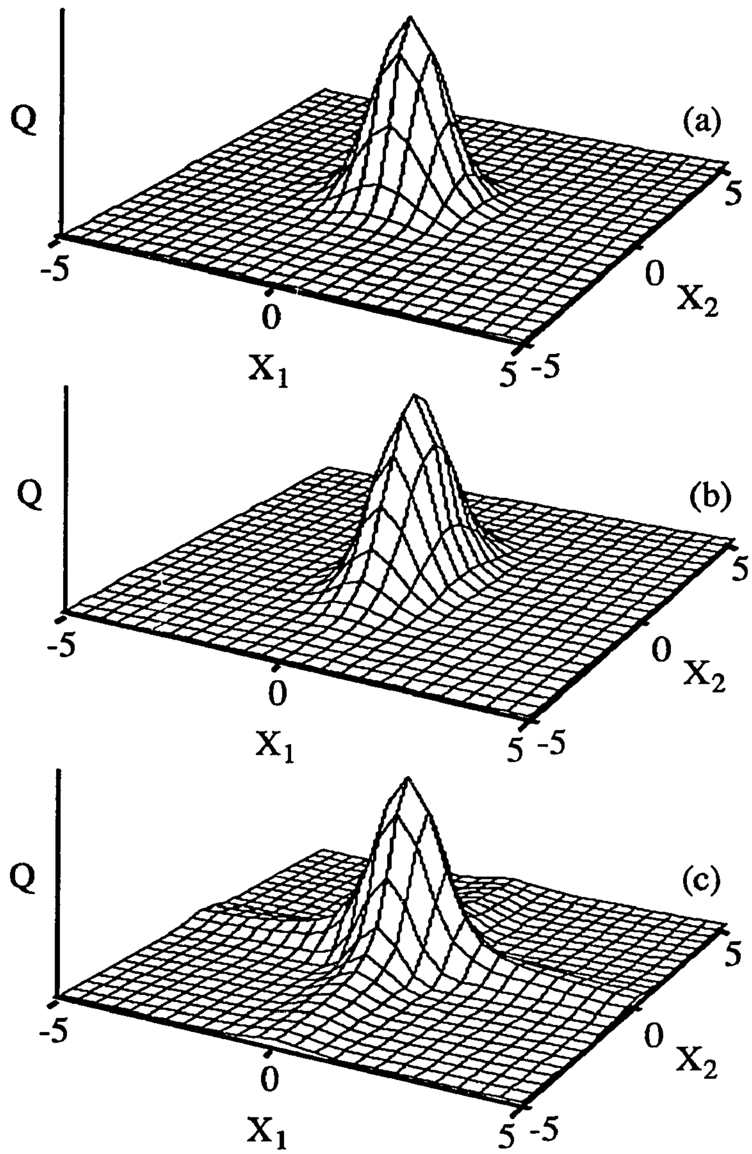

Fig. 2 Plots of the quasiprobability Q-function of: (a) a vacuum state $|0\rangle$; (b) a normal squeezed vacuum state $|0\rangle_{N.S.}$, which can be regarded as a vacuum state (a) which has been squeezed along x_1 direction; (c) an amplitude-squared squeezed vacuum state $|0\rangle_{A.S.S.}$, which can be regarded as a vacuum state (a) which has been squeezed along two perpendicular directions in x_1-x_2 phase space.

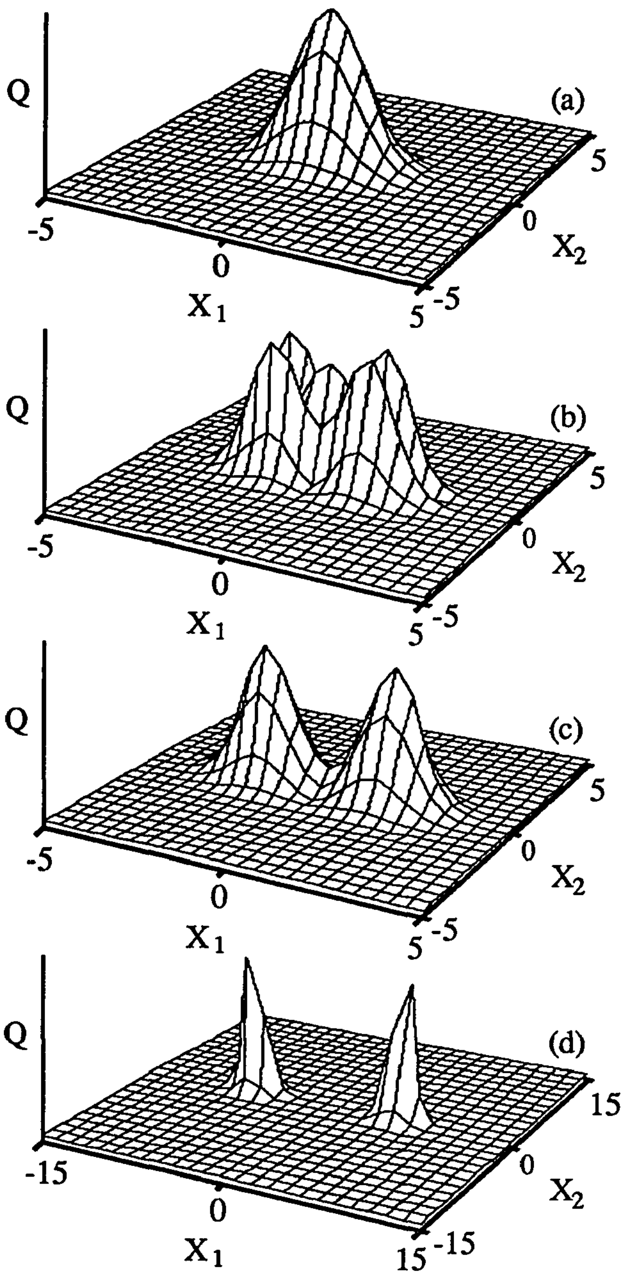

Fig. 3 Plots of the quasiprobability Q-function of the even-number-state solution (Eq.(5.35)) for amplitude-squared squeezed states with the same squeezing parameter $\lambda=2$, but different values of β: (a) $\beta=(\lambda^2-1)^{1/2}/2 = 0.87$; (b) $\beta=2.55$; (c) $\beta=4.30$; (d) $\beta=35.5$.

ones. But the single-peaked Q-function of the amplitude-squared squeezed vacuum state first splits in a complex manner as indicated by Fig. 3(b) and then evolve into a "twin-peak" ones when β becomes large compared to $(\lambda^2 - 1)^{1/2}/2$ (Fig. 3(c) and 3(d)). This suggests that the minimum uncertainty states for amplitude-squared squeezing with large β can be regarded as a superposition state of two states which are 180° out of phase (known as "Schrödinger Cats") and have equal displacement from the origin of the x_1-x_2 phase space. The displacement becomes larger when β increases (Fig. 3(d)).

In order to investigate this point further, we shall derive an asymptotic expression for the minimum uncertainty states of amplitude-squared squeezing with a large β but comparably small λ. For the sake of simplicity, we assume that $\lambda=1+\delta$ (where $0< \delta <<1$), with $\beta >> \lambda$. Then we will have

$$\sqrt{\lambda^2 - 1} \approx \sqrt{2\delta} << 1. \tag{5.36}$$

Let us denote $\sqrt{2\delta} = \Delta << 1$, Eq. (5.4) can be written into two parts

$$C_{n+2} \approx \frac{\beta}{\sqrt{(n+2)(n+1)}} C_n - \Delta \frac{n + 1/2}{\sqrt{(n+2)(n+1)}} C_n \tag{5.37}$$

Note both the fact $\lambda \approx 1$ and Eq. (5.36) have been used to derive the above equation. As a 0th order approximation, we can drop the second term on the right hand side of Eq. (5.37) which contributes much less than the first term does. Then we will get

$$C^0_{n+2} \approx \frac{\beta}{\sqrt{(n+2)(n+1)}} C^0_n \tag{5.38}$$

where the superscript 0 denotes the 0th order approximation. Eq. (5.38) is just the recurrence relation of the number state expansion coefficients of the superposition state of two coherent states, which is given by

$$|\alpha, \pm\rangle = |\alpha\rangle \pm |-\alpha\rangle , \tag{5.39}$$

with the value $\alpha = \sqrt{\beta}$. Thus Eqs. (5.11) and(5.12) can now be expressed as

$$|\psi\rangle_e \approx C_0\, S(\Delta/\sqrt{2})\left(|\sqrt{\beta}\rangle + |-\sqrt{\beta}\rangle \right), \tag{5.40}$$

and

$$|\psi\rangle_o \approx C_1\, S(\Delta/\sqrt{2})\left(|\sqrt{\beta}\rangle - |-\sqrt{\beta}\rangle \right), \tag{5.41}$$

both of them are superposition states of two normal squeezed states.

With the result of this 0th order approximation, one is ready to find the first order solution, which can be written as

$$C_{n+2} \approx C^0_{n+2} + C^1_{n+2} , \tag{5.42}$$

where C^1_n, which corresponds to the second term on the right hand side of Eq. (5.37), satisfies

$$C^1_{n+2} \approx -\Delta \frac{n+1/2}{\sqrt{(n+2)(n+1)}} C^0_n \approx -\frac{\Delta}{\beta}\left(\sqrt{\beta}\right)^{n+2} \frac{n+2}{\sqrt{(n+2)!}} C_0 . \tag{5.43}$$

By employing the identity $a^+|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(n+1)!}} (n+1)\,|n+1\rangle$, we can write the first order approximation of $|\psi\rangle$ as

$$|\psi\rangle_{e,o} \approx C_{0,1}\, S(\Delta/\sqrt{2}) \left\{ \left(|\sqrt{\beta}\rangle \pm |-\sqrt{\beta}\rangle \right) + \frac{\Delta}{\sqrt{\beta}} a^{+} \left(-|\sqrt{\beta}\rangle \pm |-\sqrt{\beta}\rangle \right) \right\}$$

$$|\psi\rangle_{e,o} \approx C_{0,1}\, S(\Delta/\sqrt{2}) \left\{ \left(|\sqrt{\beta}\rangle \pm |-\sqrt{\beta}\rangle \right) + \frac{\Delta}{\sqrt{\beta}} a^{+} \left(-|\sqrt{\beta}\rangle \pm |-\sqrt{\beta}\rangle \right) \right\} \tag{5.44}$$

where $C'_{0,1} = C_{0,1}\, e^{|d|^2/2 + d^* \sqrt{\beta}}$ and $d = -\frac{\Delta}{\sqrt{\beta}} << 1$. Note that we have neglect an arbitrary phase factor in $C'_{0,1}$. It is now evident that the effect of the first order correction is only a small shift of the two superposition states towards each other.

Photon Statistics

Next we want to study the photocount distribution of the amplitude-squared squeezed minimum uncertainty states, which, by definition, can be written as

$$P_n = |\langle n | \psi \rangle|^2 = |C'_n|^2 , \tag{5.45}$$

where $|n\rangle$ is a number state and C'_n is the coefficient of the number state expansion for $|\psi\rangle$

$$|\psi\rangle = \sum_{n=0}^{\infty} C'_n \, |n\rangle . \tag{5.46}$$

The recurrence relation for C'_n can be derived by making use of the Eq. (2.8) as

$$C'_{n+4} = \frac{2\beta}{(\lambda+1)\sqrt{(n+4)(n+3)}}\, C'_{n+2} + \frac{\lambda - 1}{\lambda + 1} \sqrt{\frac{(n+2)(n+1)}{(n+4)(n+3)}}\, C'_n , \tag{5.47}$$

which implies that there are two possible expansions, one is made up of all even number states, and the other of all odd number states with

$$C'_2 = \sqrt{2}\,\frac{\beta}{\lambda+1}\,C'_0\,, \qquad C'_3 = \sqrt{\frac{2}{3}}\,\frac{\beta}{\lambda+1}\,C'_1\,, \tag{5.48}$$

for even-number-state and odd-number-state expansion respectively, where C'_0 and C'_1 are determined in such a way that the following identity holds

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} |\, C'_n |^2 = 1\,. \tag{5.49}$$

An alternate way of finding an explicit expression for C'_n is by using the identity $1 = \frac{1}{\pi}\int d^2\alpha\ | \alpha \rangle\langle \alpha |$, then

$$C'_n = \frac{1}{\pi}\int d^2\alpha\ \langle n | \alpha \rangle\langle \alpha | \psi \rangle = \frac{1}{\pi}\int d^2\alpha\ \frac{\alpha^n}{\sqrt{n!}}\ \langle \alpha | \psi \rangle\, e^{-|\alpha|^2/2}\,, \tag{5.50}$$

where $\langle \alpha | \psi \rangle$, which is related to the Q-function for the state $| \psi \rangle$, has been already derived as Eqs. (5.29) and (5.34) for a even-number-state $| \psi \rangle_e$. Thus, we can write C'_n for $|\psi\rangle_e = C_0\, S(z)\, |\psi'\rangle_e$

$$C'_n = \frac{B_0}{\pi}\int d^2\alpha\ \frac{\alpha^n}{\sqrt{n!}}\, \mathrm{Exp}\left[-|\alpha|^2 + \frac{1}{2}\sqrt{\frac{\lambda-1}{\lambda+1}}\ \alpha^{*2}\right]$$

$$\cdot\ {}_1F_1\left[\xi\,;\,1/2;\,-\sqrt{\frac{\lambda-1}{\lambda+1}}\ \alpha^{*2}\right]. \tag{5.51}$$

It is not hard to recognize that the state with even-number-state expansion $\sum_{k=0}^{\infty} C'_{2k}\, | 2k \rangle$, where C'_{2k} satisfies Eqs. (5.47) and (5.48), is just the state given by Eq. (5.11)

$$|\psi\rangle_e = C_0\, S(z)\, |\psi'\rangle_e = \sum_{k=0}^{\infty} C'_{2k}\, |2k\rangle\,, \tag{5.52}$$

and the state with odd-number-state expansion is the state given by Eq. (5.12)

$$|\psi\rangle_o = C_1\, S(z)\, |\psi'\rangle_o = \sum_{k=0}^{\infty} C'_{2k+1}\, |2k+1\rangle\,. \tag{5.53}$$

This is due to the fact that the squeeze operator is quadratic in the mode operators.

We see, for instance, from Eq. (5.52) that all photocount amplitude C'_{2k+1} will be zero for even-number-state $|\psi\rangle_e$. Furthermore, if we take $\beta = 0$ in Eq. (5.47), we will get the photocount amplitude recurrence relation for the amplitude-squared squeezed vacuum state $|0\rangle_{A.S.S.}$

$$C'_{4k+4} = \frac{\lambda - 1}{\lambda + 1}\sqrt{\frac{(4k+2)(4k+1)}{(4k+4)(4k+3)}}\; C'_{4k}\,, \tag{5.54}$$

which implies that the nontrivial photocount amplitude are only those of C'_{4k} terms, where we have made the assumption $n = 4k,\ k = 0,\ 1,\ 2,\ \cdots$

One of the striking feature of normal squeezed states is the oscillation in photocount probability which was interpreted by Schleich and Wheeler [47,48] in terms of "interference in phase space". Fig. 4 shows that the same type of oscillation behavior exists in the photocount probability for highly squeezed amplitude-squared squeezed states. Furthermore, this oscillation behavior has a very strong dependence on the direction of amplitude-squared squeezing. In Fig. 4, dramatic oscillation appears when the direction of amplitude-squared squeezing is along the same direction as that of displacement, which is determined by the value of $\langle Y_1\rangle = \beta_r$, and $\langle Y_2\rangle = \beta_i / \lambda$. For example, if ΔY_1 is squeezed, and the displacement of the state is also along Y_1, which means $\langle Y_1\rangle = \beta_r \neq 0$ but $\langle Y_2\rangle = \beta_i / \lambda = 0$, then the oscillations occur (solid curve in Fig.

4). But if we choose the direction of squeezing along Y_2 instead of Y_1, while keeping the same value of β, then no oscillation appears at all (Fig. 4, dotted curve). This dependence of the oscillations on the direction of amplitude-squared squeezing is similar to that of normal squeezing, which can be explained through the x_1-x_2 phase space representation of a quantum state, where x_1 and x_2 are the real and imaginary

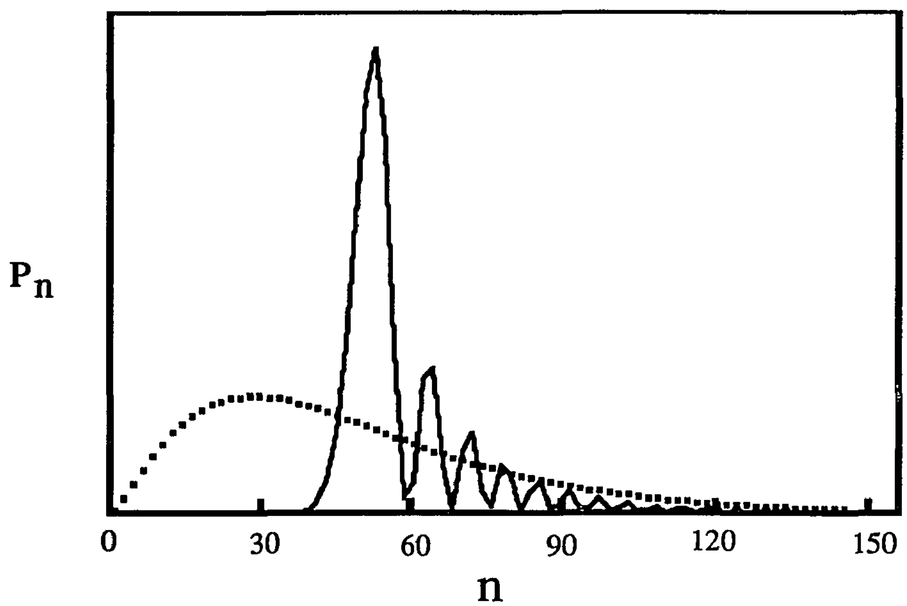

Fig. 4 Plots of photocount probability distributions P_n of the minimum uncertainty states for amplitude-squared squeezing with same β=50, but different λ, with λ=1/21 for solid curve and λ'=1/λ=21 for dotted curve, which means that they are both amplitude-squared squeezed with the same amount of squeezing but squeezed along different directions.

parts of a field amplitude, i.e., x_1 corresponds to the operator $X_1 = (a^+ + a)/2$ and x_2 to $X_2 = i(a^+ - a)/2$. A quantum state can be described by this x_1-x_2 phase space representation through the amplitude of the state of $\langle a \rangle = x_1 + i\,x_2$ which is centered on its mean value but fluctuates within an error box [49]. Different quantum states will have different error boxes. A coherent state corresponds to a circular error box of radius

1/2 while a normal squeezed state will have an elliptical error box, with the short axis of ellipse indicating the direction of squeezing. In the x_1-x_2 representation, a number state $|n\rangle$ is represented by a circular band centered at origin with radius $\sqrt{n + 1/2}$ and a width of $\sqrt{n}/2$. This follows from the fact that in terms of X_1 and X_2, $N = a^+a$ is given by

$$N = X_1^{\,2} + X_2^{\,2} - 1/2\,. \tag{5.55}$$

Therefore, a number state $|n\rangle$ corresponds to a curve $x_1^{\,2} + x_2^{\,2} = n + 1/2$, which is a circle of radius $\sqrt{n + 1/2}$. Since the value of n is discrete, each number state corresponds to a circular band which has an area of π and all of the bands taken together fill phase space. The oscillation behavior as well as its dependence on the direction of squeezing of normal squeezed states has been well interpreted in terms of "interference in phase space" by means of this phase space representation [48]. But it is rather difficult, although not impossible, to interpret the similar oscillation of highly amplitude-squared squeezed states by using the same x_1-x_2 phase space, since in x_1-x_2 phase space representation, amplitude-squared squeezed states become more complex than normal squeezed states, in other words, the shapes of their error boxes are irregular and varying with state's parameters λ and β, which can be seen from the plots of their Q-functions (Figs. 2&3). So it is quite natural at this point to introduce a new kind of phase space in terms of y_1 and y_2 which can be regarded as the real and imaginary parts of $\langle a^2\rangle$, since $Y_1 = (a^{+2} + a^2)/2$, $Y_2 = i(a^{+2} - a^2)/2$. This new phase space is unique to amplitude-squared squeezed states since in y_1-y_2 phase space, they are represented by an elliptical error box with the short axis indicating the direction of amplitude-squared squeezing. Now, however, the area of the ellipse is $\pi\langle N+1/2\rangle$ which is determined by the uncertainty relation of Eq. (1.5). Moreover, the center of the elliptical error box is the expectation value of y_1 and y_2, or the displacement of the state. The reason for this is as follows. The variable whose "eigenstate" corresponds to lines making an angle of $\theta+\pi/2$ with the y_1 axis is

$$Y(\theta) = (e^{i\theta} a^{+2} + e^{-i\theta} a^2)/2 = Y_1 \cos\theta + Y_2 \sin\theta\,, \tag{5.56}$$

so that the fluctuations of a state $|\psi\rangle$ in the direction at an angle θ to the y_1 axis is given by

$$\Delta Y(\theta) = [\,\langle\psi|\, Y(\theta)^2\,|\psi\rangle - \langle\psi|\, Y(\theta)\,|\psi\rangle^2\,]^{1/2}$$

$$= \Delta Y_1 \cos^2\theta + \Delta Y_2 \sin^2\theta + [\,\langle Y_1Y_2\rangle + \langle Y_2Y_1\rangle - 2\,\langle Y_1\rangle\langle Y_2\rangle\,]\,\sin\theta\cos\theta. \tag{5.57}$$

For our minimum uncertainty amplitude-squared squeezed states, the last term on the right hand side is zero, which follows from the fact that according to Eq. (2.1)

$$\langle\psi|\,(Y_1 + i\,\lambda Y_2)^2\,|\psi\rangle = \beta^2, \tag{5.58}$$

but $\beta = \langle\psi|\, Y_1 + i\,\lambda Y_2\,|\psi\rangle$, so Eq. (5.58) becomes

$$\langle\psi|\, Y_1^2 + i\,\lambda(Y_1Y_2 + Y_2Y_1) - \lambda^2 Y_2^2\,|\psi\rangle = (\,\langle\psi|\, Y_1 + i\,\lambda Y_2\,|\psi\rangle\,)^2. \tag{5.59}$$

Taking the imaginary part of both sides gives

$$\langle\psi|\, Y_1Y_2 + Y_2Y_1|\psi\rangle = 2\,\langle Y_1\rangle\langle Y_2\rangle, \tag{5.60}$$

thus Eq. (5.57) becomes

$$\Delta Y(\theta) = \Delta Y_1 \cos^2\theta + \Delta Y_2 \sin^2\theta, \tag{5.61}$$

which shows that the error box is elliptical for our minimum uncertainty amplitude-squared squeezed states in the y_1-y_2 phase space.

Now let us consider the representation of a number state $|n\rangle$ in this new phase space. It is still represented by a circular band, which follows from the relationship between Y_1, Y_2 and N

$$Y_1^2 + Y_2^2 = N^2 + N + 1, \tag{5.62}$$

therefore, the eigenstate of N with eigenvalue n in y_1-y_2 phase space corresponds to a curve $y_1^2 + y_2^2 = n^2 + n + 1$, which is a circle of radius $\sqrt{n^2 + n + 1}$. Since the number states are complete, i.e., $I = \sum_{n=0}^{\infty} | n \rangle \langle n |$, where I is the identity operator, this means that the phase space representation of each number state must have a finite area, which implies that each circle must have a width. The space between circles is $\delta R_n = \sqrt{(n+1)^2+(n+1)+1} - \sqrt{n^2+n+1} \approx 1$ (for $n >>1$) and we associate to each circle this width. Thus, the number state $|n\rangle$ is represented by the circular band with inner radius $\sqrt{n^2+n+1} - 1/2 \approx n$ and outer radius $\sqrt{n^2+n+1} + 1/2 \approx n + 1$. The circular band corresponding to state $|n\rangle$ has an area of $2\pi n$ and all of the bands taken together fill phase space.

In Fig.5, under this new y_1-y_2 phase space, two highly amplitude-squared squeezed states are represented by two long, thin cigar shaped areas, one is vertical indicating ΔY_1 is squeezed, and the second is horizontal, corresponding to a ΔY_2 squeezed state, and both have the same value of β (with $\beta_r \neq 0$ and $\beta_i = 0$, which means these two states are both displaced from the origin by the same amount along the same direction of y_1). The photocount distribution of a state is determined by the overlap between that state and the photon number state $|n\rangle$ in phase space. Fig.5 shows two different cases for this overlap. One, corresponding to the vertical cigar in which ΔY_1 is squeezed, has two overlap areas with a number state and both of them will contribute to the photocount probability P_n. These two areas have a fixed "phase" relation and thus the total P_n is not just the direct sum of the two overlap areas, but they interfere with each other and thus result in oscillation in P_n. The second one, corresponding to the horizontal cigar in which ΔY_2 is squeezed, has only one overlap area with $|n\rangle$, so no interference will occur in P_n. Thus it is obvious that although the two amplitude-squared squeezed states shown in Fig.5 are both amplitude-squared

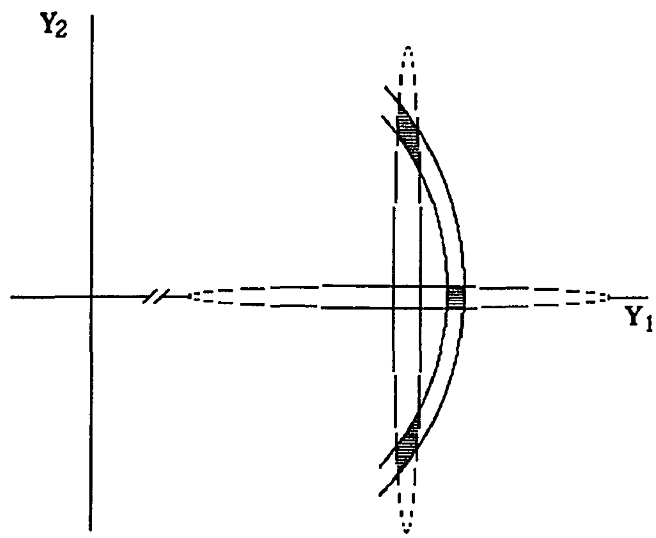

Fig. 5 In the y_1-y_2 phase space representation, a number state $|n\rangle$ is represented by a circular band with inner radius n and outer radius n+1(with n>>1), and the minimum uncertainty states for amplitude-squared squeezing correspond to long, thin "cigar shaped" regions. For the vertical cigar, $\lambda=1/21$, and $\beta=50$; and for the horizontal one, $\lambda'=1/\lambda=21$, $\beta=50$. Photocount probability distributions of these states are determined by the overlapping areas between the cigar and the circular band.

squeezed by the same amount and have the same displacement from the origin (note that the displacement of the amplitude-squared squeezed vacuum state $|0\rangle_{A.S.S}$ from the origin is equal to zero), one has dramatic oscillations in P_n, but P_n for the second state doesn't oscillate at all. The difference is due to the fact that the states are squeezed in different directions.

Symmetry of Amplitude-Squared Squeezed States

In previous section, we noticed that certain terms of the photocount probability distribution P_n vanish for $|\psi\rangle_e$, $|\psi\rangle_o$, and $|0\rangle_{A.S.S}$. Furthermore, this behavior will lead to certain symmetry of their corresponding Q-functions in x_1-x_2 phase space. We can make the connection between the photon number distribution and phase-space behavior more explicit by introducing the operator

$$U(\theta) = e^{i\theta a^{+} a} \ , \tag{5.63}$$

which when applied to a state rotates it counterclockwise by an angle θ in phase space. That is if the Q-function of the state $|\psi\rangle$ is $Q_\psi(\alpha)$, then the Q-function of $U(\theta)\,|\psi\rangle$ is $Q_\psi(\alpha e^{-i\theta})$. By a state with 2-fold symmetry we mean a state whose Q-function is invariant under a rotation by π. Therefore, if $|\psi\rangle$ is a state with a 2-fold symmetry, then $|\psi\rangle$ and $U(\pi)\,|\psi\rangle$ have the same Q-function.

States which are eigenstates of $U(\pi)$ satisfy this condition. Because $U(\pi)^2 = I$, if λ is an eigenvalue of $U(\pi)$, then it must satisfy $\lambda^2 = 1$. Therefore, the eigenvalues of $U(\pi)$ are 1 and -1. A state which is a superposition of number states all of whose numbers are even ($|\psi\rangle_e$) is an eigenstate corresponding to 1 and if a state is a superposition of odd number states ($|\psi\rangle_o$) it is an eigenstate with eigenvalue -1. An eigenstate of $U(\pi)$, therefore, satisfies

$$|\langle\alpha|\psi\rangle|^2 = |\langle\alpha|\, U(\pi)\, |\psi\rangle|^2 \ , \tag{5.64}$$

which implies that the Q-functions of $|\psi\rangle$ and $U(\pi)\,|\psi\rangle$ are the same. From this we can conclude that states which are superposition of only odd number states ($|\psi\rangle_o$) or even number states ($|\psi\rangle_e$) will have a 2-fold symmetry in phase space.

Similar reasoning can be used for states whose Q-function have a 4-fold symmetry. This means that the Q-function of $|\psi\rangle$ and $U(n\frac{\pi}{4})|\psi\rangle$, where $n=1, 2,$ or 3, are identical. In this case states which are of the form

$$|\psi\rangle = \sum_{k=0}^{\infty} C_{4k}\,|\,4k+q\,\rangle\,, \tag{5.65}$$

where q can be 0, 1, 2, or 3, satisfy this condition. For such states only every fourth photon number probability is nonzero and state $|0\rangle_{A.S.S}$ is such a state.

Q-functions shown in Figs.2 and 3 are of these special solutions with 2-fold ($|\psi\rangle_e$ and $|\psi\rangle_o$) and 4-fold ($|0\rangle_{A.S.S}$) symmetry. But they do not exhaust all possibilities for any minimum uncertainty states for amplitude-squared squeezing. As we mentioned in Sec.II, a more general solution can be built up by a linear combination of states $|\psi\rangle_e$ and $|\psi\rangle_o$ under the condition that both of them have the same value of λ and β, which is also a minimum uncertainty state for amplitude-squared squeezing. In order to make this point more clear, let us first rewrite Eq. (5.11) and Eq. (5.12) into a slightly different form

$$|\psi'\rangle_e = C_0 \sum_{k=0}^{\infty} \left(\frac{1}{\lambda}\right)^n \frac{1}{\sqrt{(2n)!}} \prod_{k=1}^{n} \left(\beta - \sqrt{\lambda^2-1}\,(2k-\frac{3}{2})\right) |2n\rangle\,, \tag{5.66}$$

$$|\psi'\rangle_o = C_1 \sum_{k=0}^{\infty} \left(\frac{1}{\lambda}\right)^n \frac{1}{\sqrt{(2n+1)!}} \prod_{k=1}^{n} \left(\beta - \sqrt{\lambda^2-1}\,(2k-\frac{1}{2})\right) |2n+1\rangle\,, \tag{5.67}$$

with $\lambda \geq 1$. Let $r = \dfrac{\beta}{2\sqrt{\lambda^2-1}}$, by using the notation $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)} = \prod_{k=0}^{n-1}(a+k)$, we can simplify the following expressions

$$\prod_{k=1}^{n} \left(\beta - \sqrt{\lambda^2-1}\,(2k-\frac{3}{2})\right) = \left(-2\sqrt{\lambda^2-1}\right)^n \left(\frac{1}{4}-r\right)_n\,, \tag{5.68}$$

$$\prod_{k=1}^{n}\left(\beta-\sqrt{\lambda^2-1}\,(2k-\tfrac{1}{2})\right)=\left(-2\sqrt{\lambda^2-1}\right)^n\left(\tfrac{3}{4}-r\right)_n . \tag{5.69}$$

Now with the help of Eqs. (5.68) and (5.69), Eqs. (5.66) and (5.67) can be written as

$$|\psi'\rangle_e = C_0 \sum_{\substack{n=0 \\ n\ \text{even}}}^{\infty} (-2)^{n/2}\frac{1}{\sqrt{n!}}\left(1-\lambda^{-2}\right)^{n/4}\frac{\Gamma(\frac{1}{4}-r+n)}{\Gamma(\frac{1}{4}-r)}\,|n\rangle , \tag{5.70}$$

$$|\psi'\rangle_o = C_1 \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} (-2)^{(n-1)/2}\frac{1}{\sqrt{n!}}\left(1-\lambda^{-2}\right)^{(n-1)/4}\frac{\Gamma(\frac{1}{4}-r+n)}{\Gamma(\frac{3}{4}-r)}\,|n\rangle . \tag{5.71}$$

If we choose $C_1 = i\sqrt{2}\,\left(1-\lambda^{-2}\right)^{1/4}\dfrac{\Gamma(\frac{3}{4}-r)}{\Gamma(\frac{1}{4}-r)}\,C_0$, then we can build a superposition state of the form

$$|\psi'\rangle_e + |\psi'\rangle_o = C_0\sum_{n=0}^{\infty}\left(-2\sqrt{1-\lambda^{-2}}\right)^{n/2}\frac{1}{\sqrt{n!}}\,\frac{\Gamma(\frac{1}{4}-r+n)}{\Gamma(\frac{1}{4}-r)}\,|n\rangle, \tag{5.72}$$

and thus another example of an amplitude-squared squeezed minimum uncertainty state could be

$$\begin{aligned}|\psi\rangle &= S(z)\left(|\psi'\rangle_e + |\psi'\rangle_o\right)\\ &= C_0\, S(z)\sum_{n=0}^{\infty}\left(-2\sqrt{1-\lambda^{-2}}\right)^{n/2}\frac{1}{\sqrt{n!}}\,\frac{\Gamma(\frac{1}{4}-r+n)}{\Gamma(\frac{1}{4}-r)}\,|n\rangle,\end{aligned} \tag{5.73}$$

where C_0 will be determined by the normalization of $|\psi\rangle$. It can be seen that the above state will have nonzero even and odd number photocount probabilities and thus it is not necessary for the Q-function of a general minimum uncertainty state for amplitude-

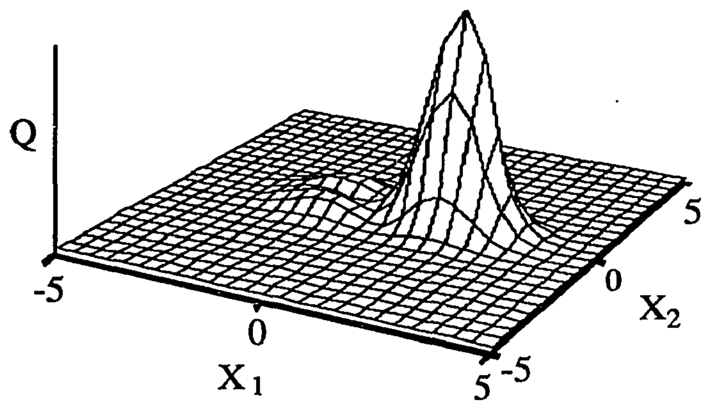

Fig. 6 Plot of quasiprobability Q-function of a general amplitude-squared squeezed state with λ=2 and β=4.30. It is neither 2-fold nor 4-fold symmetric.

squared squeezing to be either 2-fold or 4-fold symmetric in x_1-x_2 phase space. One example of the Q-function of this example is plotted in Fig. 6, which shows no 2-fold or 4-fold symmetry at all.

Conclusion

We have investigated the eigenvalue problem for the minimum uncertainty states for amplitude-squared squeezing. A general minimum uncertainty state can be expressed in terms of a squeeze operator acting on a state which can be written in the form of a Kummer function, whose argument is proportional to the creation operator, acting on the vacuum state. We have derived an expression for the average photon number in such states. We found that when the amount of amplitude-squared squeezing in these states increases, so does the photon number. This is quite similar to the situation for normal squeezed states.

The Q-function of one ($|\psi\rangle_e$) of the two independent eigenstates $|\psi\rangle_e$ and $|\psi\rangle_o$ of the eigenvalue equation for the amplitude-squared squeezed minimum uncertainty state has been derived and investigated. We found that they change both in shape and symmetry dramatically with state parameters λ and β. When β is zero, which corresponds to amplitude-squared squeezed vacuum states, the corresponding Q-function looks like that of a vacuum state squeezed along two perpendicular directions of the x_1-x_2 phase space and possesses a 4-fold symmetry. Immediately after β is increased from zero, the shape of the Q-function changes and the 4-fold symmetry degrades into a 2-fold one. The Q-function gradually changes into that of a normal squeezed vacuum state when β reaches a value of $\sqrt{\lambda^2 - 1} / 2$. Further increasing the value of β splits the single-peaked Q-function into a twin-peaked one. This suggests that the amplitude-squared squeezed state becomes a superposition states of two states which are 180° out of phase in phase space. As we previously mentioned, there exist two independent solutions to the equation for minimum uncertainty states for amplitude-squared squeezing, and their corresponding Q-functions have a 2-fold symmetry when plotted in x_1-x_2 phase space. Furthermore, this 2-fold symmetry is related to their photocount probability which stimulated us to investigate the general relationship between the symmetry of any state in phase space and its photocount probability. We found that any state with nonzero photocount probability P_n for $n=2k$ (or $n=2k+1$)only will have 2-fold symmetry in phase space (of its quasiprobability Q-function), while any state with nonzero P_n for $n=4k$ (or $n=4k+q$ where k is an integer and $q=0, 1, 2, 3$) only, will have a Q-function which is 4-fold symmetric. Finally, we introduced y_1-y_2 phase space to explain the occurrence of the oscillations in the photocount probability of highly amplitude-squared squeezed states, where y_1 and y_2 are related to $\alpha^2 = \langle a^2 \rangle$ and are the intrinsic variables for amplitude-squared squeezing. We found that, like the situation of a normal squeezed states, these oscillation are most pronounced when the displacement of these states occurs along the amplitude-squared squeezed quadrature, which can be considered as a "squeezed amplitude; diverse-phase" state in the y_1-y_2 phase space.

Appendix A: Asymptotic Expression For $|C_n(\lambda)|^2$

Here we want to find an asymptotic expression for $|C_n(\lambda)|^2$ which is valid as $n \to \infty$. We start by noting the expression for the generating function for the Hermite polynomials

$$e^{2zx - z^2} = \sum_{n=0}^{\infty} (z^n/n!)\, H_n(x)\,. \tag{A.1}$$

Application of this result twice gives

$$e^{-[z^2 + (z^*)^2]} \langle 0|e^{-2i\gamma^* a}\, e^{2i\gamma a^+}|0\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (z^*)^n z^m/(n!\, m!) \\ \times\langle 0|H_m(-i\gamma^* a)H_m(i\gamma a^+)|0\rangle\,. \tag{A.2}$$

The left-hand side of this equation can be simplified by noting that

$$e^{4|z|^2|\gamma|^2} = \langle 0|e^{-2i\gamma^* a}\, e^{2i\gamma a^+}|0\rangle\,. \tag{A.3}$$

Now set $z = r\, e^{i\theta}$ and integrate both sides of Eq. (A.2) with respect to θ from 0 to 2π. The result is

$$1/(2\pi)\, e^{4 r^2 |\gamma|^2} \int_0^{2\pi} d\theta\, e^{-2 r^2 \cos(2\theta)} = \sum_{n=0}^{\infty} [(r^2)^n/(n!)^2] \\ \times\langle 0|H_m(-i\gamma^* a)H_m(i\gamma a^+)|0\rangle\,. \tag{A.4}$$

Equating powers of r^2 on both sides, we get

$$\langle 0|H_m(-i\gamma^* a)H_m(i\gamma a^+)|0\rangle = [(2^{n-1}n!)/\pi] \int_0^{2\pi} d\theta\, [2|\gamma|^2 - \cos(2\theta)]^n$$

$$= [(2^{n-1}n!)/\pi] \int_0^{2\pi} d\theta\, (2|\gamma|^2 - \cos\theta)^n \, . \qquad \text{(A.5)}$$

In order to find an asymptotic form for the integral as $n \to \infty$, we first express the integrand as $\exp[\, n \ln(2|\gamma|^2 - \cos\theta)]$ and apply Laplace's method [34]. This involves finding the maximum of $\ln(2|\gamma|^2 - \cos\theta)$ (which occurs at $\theta = \pi$) and then expanding about it keeping only lowest-order terms. The resulting integral is Gaussian and can be performed. The result is

$$\int_0^{2\pi} d\theta\, (2|\gamma|^2 - \cos\theta)^n \approx (2|\gamma|^2 + 1)^n\, [\, \pi(4|\gamma|^2 + 2)/n\,]^{1/2} , \qquad \text{(A.6)}$$

so that

$$\langle 0|H_m(- i\, \gamma^* a)H_m(i\, \gamma\, a^+)|0\rangle \approx 2^{n-1}n!\, (2|\gamma|^2 + 1)^n\, [(4|\gamma|^2 + 2)/(n\pi)]^{1/2} \, , \qquad \text{(A.7)}$$

for large n. The function $|C_n(\lambda)|^2$ is just the reciprocal of this expression.

Appendix B: Properties of Amplitude-Squared Squeezed Vacuum State

Here we wish to discuss a number of the properties of the amplitude-squared squeezed vacuum, $|0;\lambda\rangle$. Its number state expansion is given in Eq. (4.26). This expansion is derived by solving the equation

$$(Y_1 + i\,\lambda Y_2)\,|\psi\rangle = 0\,. \tag{B.1}$$

The solution of this equation is guaranteed to be a minimum uncertainty state for Y_1 and Y_2 with the additional property that $\langle Y_1\rangle = \langle Y_2\rangle = 0$. It also satisfies the relations [6]

$$(\Delta Y_1)^2 = \lambda\,\langle N + 1/2\rangle\ , \qquad (\Delta Y_2)^2 = (1/\lambda)\,\langle N + 1/2\rangle\ . \tag{B.2}$$

We now want to prove the inequality in Eq. (4.29). We start by deriving inequalities for the coefficients a_n. We have that

$$\ln a_n = \sum_{m=1}^{n} \{\ \ln[1- 1/(2m)] + \ln[1- 2/(4m-1)]\ \}\ . \tag{B.3}$$

If we now use the fact that $\ln(1\text{-}x) \le -x$ for $0 \le x < 1$ we have that

$$\begin{aligned}\ln a_n \le \sum_{m=1}^{n} [\ (1/2m) + 2/(4m-1)\ ] &\le - \int_1^{n+1} ds\ (1/2s) - \int_1^{n+1} ds\ [2/(4s-1)] \\ &\le -(1/2)\ [\ \ln(n+1) + \ln(n+3/4) + \ln(4/3)\ ] \\ &\le -\ln(n+3/4) - (1/2)\ln(3/4).\end{aligned} \tag{B.4}$$

Exponentiating both sides gives

$$a_n \le (3/4)^{1/2}/(n + 3/4) \,. \tag{B.5}$$

In order to find a lower bound we note that $\ln(1\text{-}x) \ge -x/(1\text{-}x)$ for $0 \le x < 1$ so that

$$\ln a_n \ge -\sum_{k=1}^{n} [1/(2k-1) + 2/(4k-3)] \ge -\{1 + \int_1^n ds\ 1/(2s-1) + 2 + \int_1^n ds\ 2/(4s-3)\}$$

$$\ge -\{3 + \ln(n-1/2) + (3/2)\ln 2\} \,. \tag{B.6}$$

Exponentiating both sides gives, for $n \ge 1$,

$$a_n \ge (1/8)^{1/2}\, e^{-3}/(n -1/2) \,. \tag{B.7}$$

These expressions can now be used to find estimates for c_0^2 and $\langle N \rangle$. We have that

$$c_0^2 \sum_{n=0}^{\infty} a_n\, r^{2n} = 1,$$

$$\langle N \rangle = c_0^2 \sum_{n=0}^{\infty} 4n\, a_n\, r^{2n} \,. \tag{B.8}$$

Our results for a_n imply that

$$(4/3)^{1/2} [\sum_{n=0}^{\infty} r^{2n}/(n+3/4)]^{-1} \le c_0^2 \le \{1+\sum_{n=1}^{\infty} (1/8)^{1/2}\, e^{-3}\, r^{2n}/(n-1/2)\,\}^{-1} \,, \tag{B.9}$$

and

$$c_0^2 \sum_{n=1}^{\infty} (1/8)^{1/2} e^{-3} r^{2n} 4n/(n-1/2) \leq \langle N \rangle \leq c_0^2\, 2(3)^{1/2} \sum_{n=1}^{\infty} r^{2n} n/(n+3/4)\,. \qquad \text{(B.10)}$$

In order to obtain analytic bounds for c_0^2 we note that for $-1< x <1$

$$\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n!\,, \qquad \text{(B.11)}$$

so that

$$\begin{aligned} &\sum_{n=0}^{\infty} r^{2n}/(n+3/4) \leq 4/3 + \sum_{n=1}^{\infty} r^{2n}/n = 4/3 + |\ln(1-r^2)|\,, \\ &\sum_{n=1}^{\infty} r^{2n}(n-1/2) \geq \sum_{n=1}^{\infty} r^{2n}/n = |\ln(1-r^2)|\,. \end{aligned} \qquad \text{(B.12)}$$

Substituting these into the expression for c_0^2 gives

$$[(4/3)^{1/2} + (3/4)^{1/2} |\ln(1-r^2)|\,]^{-1} \leq c_0^2 \leq \{1 + (1/8)^{1/2} e^{-3} |\ln(1-r^2)|\,]^{-1}\,. \qquad \text{(B.13)}$$

In order to find more tractable bounds for $\langle N \rangle$ we use

$$\begin{aligned} &\sum_{n=1}^{\infty} r^{2n} n/(n+3/4) \leq \sum_{n=1}^{\infty} r^{2n} = r^2/(1-r^2)\,, \\ &\sum_{n=1}^{\infty} r^{2n} n/(n-1/2) \geq \sum_{n=1}^{\infty} r^{2n} = r^2/(1-r^2)\,, \end{aligned} \qquad \text{(B.14)}$$

so that

$$c_0^{2}(1/8)^{1/2}\, e^{-3}\, 4r^{2}/(1-r^{2}) \leq \langle N \rangle \leq c_0^{2}\, 2(3)^{1/2}\, r^{2}/(1-r^{2})\ . \tag{B.15}$$

The final step is to insert the lower bound for c_0^2 into the left-hand inequality and the upper bound for c_0^2 into the right-hand inequality in Eq. (B.15). The result is Eq. (4.29).

Appendix C: Derivation of Eq. (5.23)

According to Ref. [46], there exists an identity

$$_2F_1(\xi_1, \xi_2;\ \zeta;\ x) = \pi^{-1/2}\, 2^{\xi_1+\xi_2-3/2}\, \Gamma(1/2+\xi_1)\, \Gamma(1/2+\xi_2)\, (1-x)^{1/2\,(1/2-\xi_1-\xi_2)}$$

$$[P^{1/2-\xi_1-\xi_2}_{\xi_1-\xi_2-1/2}(x^{1/2}) + P^{1/2-\xi_1-\xi_2}_{\xi_1-\xi_2-1/2}(-x^{1/2})]\,, \qquad \text{(C.1)}$$

where P^a_b is the Associate Legendre Function. Thus, we have

$$_2F_1(\frac{1}{4}, \frac{1}{4};\ \frac{1}{2};\ 1-\lambda^{-2}) = \frac{\Gamma^2(3/4)}{2\pi^{1/2}}\ [P_{-\frac{1}{2}}(1-\frac{1}{2\lambda^2}) + P_{-\frac{1}{2}}(-1+\frac{1}{2\lambda^2})]\,, \qquad \text{(C.2)}$$

where we have used the approximation $x^{1/2}=\sqrt{1-\lambda^{-2}} \cong 1 - 1/(2\lambda^2)$ for $\lambda >> 1$. According to another identity of Ref. [46], we have

$$P_{-\frac{1}{2}}(1-\frac{1}{2\lambda^2}) + P_{-\frac{1}{2}}(-1+\frac{1}{2\lambda^2}) = \frac{2}{\pi}\ [K(\frac{1}{2\lambda}) + K(1-\frac{1}{8\lambda^2})]\,, \qquad \text{(C.3)}$$

where $K(x)$ is the Elliptic Integral. For $\lambda >> 1$,

$$K(\frac{1}{2\lambda}) \cong K(0) = 1.57\,, \qquad \text{(C.4)}$$

and $K(1 - 1/(8\lambda^2))$ can be found from

$$K(1-m) = [a_0 + a_1 m + a_2 m^2] + [b_0 + b_1 m + b_2 m^2] \ln(1/m) + \varepsilon(m)\,, \qquad \text{(C.5)}$$

for any $0< m <1$, and a_i, b_i (i=0, 1, 2) are constants, $\varepsilon(m) < 3 \times 10^{-5}$. Thus, we can write

$$K\left(1-\frac{1}{8\lambda^2}\right)=2\,b_0\ln\lambda\,, \tag{C.6}$$

where $b_0 = 1/2$. Substituting Eqs. (C.4) and (C.6) back into Eq.(C.3), then we find

$$P_{-\frac{1}{2}}\left(1-\frac{1}{2\lambda^2}\right)+P_{-\frac{1}{2}}\left(-1+\frac{1}{2\lambda^2}\right)\approx\frac{2}{\pi}\ln\lambda\,. \tag{C.7}$$

Now Eq. (C.2) can be written as

$${}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};1-\lambda^{-2}\right)=\frac{\Gamma^2(3/4)}{\pi^{3/2}}\ln\lambda\,. \tag{C.8}$$

Finally, Eq. (5.23) can be expressed approximately as

$$\langle N+1/2\rangle_e\approx\lambda^2\frac{d}{d\lambda}\ln\left[{}_2F_1\left(\frac{1}{4},\frac{1}{4};\frac{1}{2};1-\lambda^{-2}\right)\right]\approx\frac{\lambda}{\ln\lambda}\,. \tag{C.9}$$

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