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A CELLULAR CONSTRUCTION OF $\Omega^2\Sigma^2(X)$

by

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TABLE OF CONTENTS

INTRODUCTION

It has long been of interest to algebraic topologists to compute the homology and cohomology of the spaces $\Omega^n S^n(X)$, where X is a pointed CW-complex and Ω and S are loop space and suspension functors respectively. J.C. Moore [7], Eldon Dyer and R.K. Lashof [4] and Browder [3] have used various approaches to compute many special cases. One approach, for which this thesis should be a useful first step, is to discover an appropriate, well-described, CW-structure for the space $\Omega^n S^n(X)$.

If $n = 1$ and X is connected, James [1], in 1955, discovered that $J(X)$, the free topological monoid on the space X, is of the same homotopy type as $\Omega S(X)$. The CW-structure of $J(X)$ is easily describable in terms of the CW-structure of X.

In 1966, R. James Milgram published a paper [2], which gives a general approach to the problem for $n \geq 2$ and X connected. This thesis arose out of an attempt to understand that paper. Making use of insights and methods drawn from [2], we have constructed a CW-complex, $L(X)$, which has the same homotopy type as $\Omega^2 S^2 X$ if X is a connected CW-complex. Furthermore, $L(X)$ has the structure of a topological monoid and the homotopy equivalence from $L(X)$ to $\Omega^2 S^2(X)$ is a monoid homomorphism. The cell-structure of $L(X)$ can be completely described.

Throughout this paper, a familiarity with the elementary terminology and techniques of category theory is assumed. Mac Lane [5] or Mitchell [6] are useful reference works in this regard.

Our approach is to construct $L(X)$ as the colimit over a suitable small, monoidal category, $\mathfrak{G}$, of a certain nicely behaved functor L_X.

We start, in Chapter 1, by making some general remarks on the subject of monoids and monoidal categories. An interesting, and perhaps new, result here (quite elementary) is Proposition 1.4, that the colimit of a monoidal functor is a monoid.

In Chapter 2, we define $\mathfrak{S}$, the category of generalized shuffles, and a general scheme for defining functors with domain $\mathfrak{S}$.

Chapter 3 is concerned with the category of normed topological spaces (denoted Top^{ν}). Most of the work of the paper is done in this category rather than in the more usual category of pointed spaces. The reasons for this are somewhat technical, but are analogous to the reasons for using Moore paths rather than paths of unit length in defining path spaces and loop spaces.

In Chapter 4, the functor $L_X: \mathfrak{S} \to Top^{\nu}$ is defined. This functor is rather complicated in definition, but has many nice properties which are developed in Chapters 4 and 11.

Chapter 5 contains the precise statement of the chief result of this paper. This theorem (5.3) states that if X is a connected CW-complex with a suitable norm then there is a monoid homomorphism from $\operatorname{colim}_{\mathfrak{S}} L_X$ to $\Omega^2 S^2 X$ which is a homotopy equivalence.

The proof of Theorem 5.3 takes up the remainder of the paper. Part II (Chapters 6 through 13) gives various preliminary definitions and results needed for the proof proper, which comprises Part III (Chapters 14 and 15).

Two of the chapters in Part II are perhaps worth a small note. Chapter 7 introduces the concept of a conaturalizer, a categorical construction which seems to arise naturally in the consideration of monoidal categories and functors.

In Chapter 8, a base-space functor B , for normed monoids, is discussed. This is essentially a construction due to Milgram, in the paper [2] mentioned above. Most of the material here is taken from Milgram. However, Proposition 8.3, which observes that B is coadjoint to the Moore loop space functor, is the author's. Although B is well-behaved enough for the purposes of this paper, it is not, generally speaking, a very well-behaved functor. There is clearly more work to be done in this area.

In Part III (Chapters 14 and 15) we give the proof of Theorem 5.3, drawing on all the material in the previous chapters.

PART I

PRELIMINARIES AND STATEMENT OF THEOREM

Chapter 1. Categories of Monoids and Monoidal Categories

Definition 1.1: Let $\mathcal{A}$ be a category, $\otimes: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a functor. Then the pair $\langle \mathcal{A}, \otimes \rangle$ is a monoidal category if

1. $\otimes$ is associative. (We will insist upon coherent associativity, at least. In some applications, we will demand exact associativity.)

2. $\exists\, 0_{\mathcal{A}} \in \mathcal{A}$ such that $0_{\mathcal{A}}$ is an initial object of $\mathcal{A}$ and $0_{\mathcal{A}}$ is a two-sided identity element for $\otimes$.

If $\langle \mathcal{A}, \otimes \rangle$ and $\langle \hat{\mathcal{A}}, \hat{\otimes} \rangle$ are two monoidal categories, then a functor $F: \mathcal{A} \to \hat{\mathcal{A}}$ is a morphism of monoidal categories, or more briefly, a monoidal functor provided:

1. $F(0_{\mathcal{A}}) = 0_{\hat{\mathcal{A}}}$.

2. The diagram below commutes:

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{F \times F} & \hat{\mathcal{A}} \times \hat{\mathcal{A}} \\ \downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \hat{\otimes}} \\ \mathcal{A} & \xrightarrow{F} & \hat{\mathcal{A}} \end{array} \quad .$$

Definition 1.2: Let $\langle \mathcal{A}, \otimes \rangle$ be a monoidal category. Then the category of $\langle \mathcal{A}, \otimes \rangle$ -monoids is the category whose objects are pairs (A, μ) where A is an object of $\mathcal{A}$ and $\mu \in \mathcal{A}(A \otimes A, A)$ such that the following 2 diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes 1_A} & A \otimes A \\ \downarrow{\scriptstyle 1_A \otimes \mu} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

$$\begin{array}{ccc}
A \otimes 0_{\mathcal{G}} & \xrightarrow{1_A \otimes 0_A} & A \otimes A \\
\updownarrow \cong & & \downarrow \mu \\
A & \xrightarrow{1_A} & A \\
\updownarrow \cong & & \uparrow \mu \\
0_{\mathcal{G}} \otimes A & \xrightarrow{0_A \otimes 1_A} & A \otimes A
\end{array}$$

Let (A,μ) and (A',μ') be $\langle \mathcal{G},\otimes \rangle$ -monoids. Then a <u>monoid homomorphism</u> $\varphi\colon (A,\mu) \to (A',\mu')$ is an $\mathcal{G}$-morphism, $\varphi\colon A \to A'$ such that the following diagram commutes:

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\varphi \otimes \varphi} & A' \otimes A' \\
\downarrow \mu & & \downarrow \mu' \\
A & \xrightarrow{\varphi} & A'
\end{array} \quad .$$

<u>Definition 1.3</u>: Let $\mathcal{G}$ be a monoidal category. $\mathcal{G}$ is <u>monoidally complete</u> if the following condition is satisfied:

Let $\mathcal{C}$, $\mathcal{C}'$ be small monoidal categories. Let $F\colon \mathcal{C} \to \mathcal{G}$, $F'\colon \mathcal{C}' \to \mathcal{G}$ be monoidal functors such that $\operatorname*{colim}_{\mathcal{C}} F$ and $\operatorname*{colim}_{\mathcal{C}} F'$ exist in $\mathcal{G}$. Let $F \otimes F'$ be the monoidal functor given by the composition:

$$\mathcal{C} \times \mathcal{C}' \xrightarrow{F \times F'} \mathcal{G} \times \mathcal{G} \xrightarrow{\otimes} \mathcal{G} \quad .$$

Then $\operatorname*{colim}_{\mathcal{C}\times\mathcal{C}'} F \otimes F'$ exists and the naturally induced map

$$\operatorname*{colim}_{\mathcal{C}\times\mathcal{C}}(F \otimes F') \longrightarrow (\operatorname*{colim}_{\mathcal{C}} F) \otimes (\operatorname*{colim}_{\mathcal{C}'} F')$$

is an isomorphism.

Note: For $\mathcal{A}$ to be monoidally complete, it is certainly sufficient that the functors, $X\otimes\_$ and $\_\otimes X$, from $\mathcal{A}$ to $\mathcal{A}$, be cocontinuous, $\forall\, X \in \mathcal{A}$.

<u>Proposition 1.4</u>: Let $\mathcal{A}$ be a monoidally complete monoidal category, let $\mathcal{C}$ be a small monoidal category, and let $F\colon \mathcal{C} \to \mathcal{A}$ be a monoidal functor such that $\underset{\mathcal{C}}{\operatorname{colim}} F$ exists. Then $\underset{\mathcal{C}}{\operatorname{colim}} F$ is, in a natural way, an $\mathcal{A}$-monoid.

<u>Proof</u>: Since $\mathcal{A}$ is monoidally complete, there is an isomorphism:

$$\underset{\mathcal{C}\times\mathcal{C}}{\operatorname{colim}}(F \otimes F) \xrightarrow[\cong]{M} (\underset{\mathcal{C}}{\operatorname{colim}}\, F) \otimes (\underset{\mathcal{C}}{\operatorname{colim}}\, F)\ .$$

Let $i_F\colon \underset{\mathcal{C}\times\mathcal{C}}{\operatorname{colim}} F\circ\otimes_{\mathcal{C}} \to \underset{\mathcal{C}}{\operatorname{colim}} F$ be the morphism induced by $\otimes_{\mathcal{C}}\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$.

Since F is monoidal, we have a commuting diagram:

$$\begin{array}{ccc}
\mathcal{C}\times\mathcal{C} & \xrightarrow{F \times F} & \mathcal{A}\times\mathcal{A} \\
{\scriptstyle \otimes_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \otimes_{\mathcal{A}}} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{A}
\end{array}$$

and hence $F\circ\otimes_{\mathcal{C}} = \otimes_{\mathcal{A}}(F \times F) = F \otimes F$. Let μ_F be the composition:

$$(\underset{\mathcal{C}}{\operatorname{colim}}\, F) \otimes (\underset{\mathcal{C}}{\operatorname{colim}}\, F) \xrightarrow{M^{-1}} \underset{\mathcal{C}\times\mathcal{C}}{\operatorname{colim}}(F\otimes F) \xrightarrow{i_F} \underset{\mathcal{C}}{\operatorname{colim}}\, F\ .$$

It is then straightforward to check that μ_F is associative and if one notes that the zero of $\underset{\mathcal{C}}{\operatorname{colim}} F$ is the map: $0_{\mathcal{A}} = F(0_{\mathcal{C}}) \to \underset{\mathcal{C}}{\operatorname{colim}} F$ it is straightforward to check that diagram 2 of (1.2) commutes.

The $\mathfrak{a}$-monoid structure on $\operatorname{colim}_{\mathbb{C}} F$ is natural in the sense that, if F and F' are two such functors and $\varphi: F \to F'$ is a natural transformation such that $\forall\ \alpha,\beta \in \mathbb{C}$, $\varphi_{\alpha \otimes_{\mathbb{C}} \beta} = \varphi_\alpha \otimes_{\mathfrak{a}} \varphi_\beta$, then $\operatorname{colim} \varphi$: $\operatorname{colim} F \to \operatorname{colim} F'$ is an $\mathfrak{a}$-monoid homomorphism. This is easily checked. □

<u>Definition 1.5</u>: Let $\mathbb{C}$ be a small category with initial object $0_{\mathbb{C}}$. Then <u>$\mathbf{J}(\mathbb{C})$</u> will denote the <u>free monoidal category on $\mathbb{C}$</u>; that is, there is a monoidal category $\mathbf{J}(\mathbb{C})$, with a functor $i: \mathbb{C} \to \mathbf{J}(\mathbb{C})$ such that

1) $i(0_{\mathbb{C}}) = 0_{\mathbf{J}(\mathbb{C})}$.

2) If $F: \mathbb{C} \to \mathfrak{M}$ is any functor from $\mathbb{C}$ to a monoidal category $\mathfrak{M}$, such that $F(0_{\mathbb{C}}) = 0_{\mathfrak{M}}$ then there is a unique monoidal functor $\bar{F}: \mathbf{J}(\mathbb{C}) \to \mathfrak{M}$, such that $\bar{F}i = F$.

3) $i: \mathbb{C} \to \mathbf{J}(\mathbb{C})$ is universal with respect to properties 1) and 2).

<u>Remark 1.6</u>: If $\mathbb{C}$ is a small category with initial object, the existence of $\mathbf{J}(\mathbb{C})$ is a straightforward exercise. That is

$$\mathbf{J}(\mathbb{C}) = \bigcup_{i=1}^{\infty} \mathbb{C}^i$$

modulo the equivalence relation necessary to have $0_{\mathbb{C}}$ act as an identity element.

<u>Definition 1.7</u>: Let $\langle \mathbb{C}, \otimes \rangle$ be a small monoidal category, then $\langle \mathbb{C}, \otimes \rangle$ is an <u>enriched free monoidal category</u> if

1) The set of objects of $\mathbb{C}$ forms a free monoid under the operation $\otimes$;

2) If $\varphi \in \mathbb{C}(\alpha, \beta_1 \otimes \beta_2)$, then $\exists !\ \alpha_1, \alpha_2 \in \mathbb{C}$ and $\exists !\ \varphi_i \in \mathbb{C}(\alpha_i, \beta_i)$, $i = 1,2$, such that $\alpha = \alpha_1 \otimes \alpha_2$ and $\varphi = \varphi_1 \otimes \varphi_2$.

Remark 1.8: Let G be the full subcategory of $(\mathbb{C},\otimes)$ whose objects are the generating set for the set of objects of $\mathbb{C}$ as a free monoid. It is easy to see that $\mathbf{J}(G)$ is naturally a subcategory of $\mathbb{C}$. Thus, $\mathbb{C}$ is the free monoidal category $\mathbf{J}(G)$ "enriched" by additional morphisms. We can say that $\mathbb{C}$ is an enrichment of $\mathbf{J}(G)$.

Chapter 2. The Category of Generalized Shuffles

Definition 2.1: We will let $\mathfrak{G}$ stand for the category of generalized shuffles, defined as follows:

For n a non-negative integer, let $\underline{n}$ denote the finite ordered set $<1,\ldots,n>$ [or $\emptyset$, if $n = 0$]. Then an object of $\mathfrak{G}$ is a set function $f: \underline{n} \to \underline{m}$ where $n \geq 0$ and $m \geq 0$ such that f is surjective and order-preserving (i.e., $i \leq j \Rightarrow f(i) \leq f(j)$). Let $f_i: \underline{n_i} \to \underline{m_i}$, $i = 1,2$, be two such objects; then $\mathfrak{G}(f_1,f_2)$ is the set of all pairs $(\varphi,\bar{\varphi})$ where φ is a set function $\underline{n_1} \to \underline{n_2}$ and $\bar{\varphi}$ is a set function $\underline{m_1} \to \underline{m_2}$ such that:

i) $f_2\varphi = \bar{\varphi}f_1$,

ii) φ is a monomorphism,

iii) $0 < i < j \leq m_1 \Rightarrow \bar{\varphi}(i) \leq \bar{\varphi}(j)$,

iv) if $0 < i < j \leq n_1$ and $f_1(i) = f_1(j)$, then $\varphi(i) < \varphi(j)$.

Composition of $\mathfrak{G}$-morphisms is given by composition as set functions. There is no difficulty in seeing that $\mathfrak{G}$ is in fact a category.

Proposition 2.2: $\mathfrak{G}$ is an enriched free monoidal category.

Proof: Let f_1, f_2 be objects of $\mathfrak{G}$, $f_i: \underline{n_i} \to \underline{m_i}$. Define $f_1 \otimes f_2$: $\underline{n_1+n_2} \to \underline{m_1+m_2}$ by the rule:

$$f_1 \otimes f_2(i) = \begin{cases} f_1(i), & 1 \leq i \leq n_1 \\ f_2(i-n_1)+m_1, & n_1+1 \leq i \leq n_1+n_2. \end{cases}$$

If $\varphi_i \in \mathfrak{G}(f_i,g_i)$, $i = 1,2$, there is an obvious map, $\varphi_1 \otimes \varphi_2$: $f_1 \otimes f_2 \to g_1 \otimes g_2$. $\varphi_1 \otimes \varphi_2$ can easily be seen to be an $\mathfrak{G}$-morphism. Then "$\otimes$" is easily seen to be an associative bifunctor on $\mathfrak{G}$.

Let e_0 be the unique map $\underline{0} \to \underline{0}$. Clearly e_0 is an initial object of $\mathfrak{G}$ and a 2-sided identity element for $\otimes$.

Therefore $(\mathfrak{G},\otimes)$ is a monoidal category.

Let e_n denote the unique set surjection $\underline{n} \to \underline{1}$, $\forall\, n \geq 1$. Then if $f\colon \underline{n} \to \underline{m}$ is an object of $\mathfrak{G}$, clearly $f = e_{n_1} \otimes \ldots \otimes e_{n_m}$, where $n_i = |f^{-1}(i)|$, $\forall\, i \in \underline{m}$. Hence, we can see that the objects of $\mathfrak{G}$ form a free monoid with identity element e_0 and generating set $G = \{e_n | n \geq 1\}$.

To complete the proof we must demonstrate that $\mathfrak{G}$ satisfies property 2 of Definition 1.7.

Let $\varphi \in \mathfrak{G}(g, f_1 \otimes f_2)$ where $f_i\colon \underline{n_i} \to \underline{m_i}$ and $g\colon \underline{k} \to \underline{p}$. Then let

$$k_1 = |\varphi^{-1}(1,\ldots,n_1)|\ ,\ k_2 = |\varphi^{-1}(n_1+1,\ldots,n_1+n_2)|$$
$$p_1 = |\bar{\varphi}^{-1}(1,\ldots,m_1)|,\ p_2 = |\bar{\varphi}^{-1}(m_1+1,\ldots,m_1+m_2)|\ .$$

Define $g_1\colon \underline{k_1} \to \underline{p_1}$ by $g_1(i) = g(i)$, $\forall\, i \in \underline{k_1}$.

Define $g_2\colon \underline{k_2} \to \underline{p_2}$ by $g_2(i) = g(i+k_1) - p_1$.

Then clearly $g = g_1 \otimes g_2$. Then define $\varphi_1\colon \underline{k_1} \to \underline{n_1}$ by $\varphi_1(i) = \varphi(i)$, $\forall\, i \in \underline{k_1}$ and define $\varphi_2\colon \underline{k_2} \to \underline{n_2}$ by $\varphi_2(i) = \varphi(i+k_1) - n_1$, $\forall\, i \in \underline{k_2}$.

It is straightforward to check that φ_1 and φ_2 are $\mathfrak{G}$-morphisms (note that the definition of φ_i forces the definition of $\bar{\varphi}_i$, since f_i is always a surjection). Clearly $\varphi = \varphi_1 \otimes \varphi_2$, and φ_1 and φ_2 are unique. Therefore $\mathfrak{G}$ is an enriched free monoidal category. $\square$

<u>Remark 2.3</u>: An $\mathfrak{G}$-morphism: $e_{n_1} \otimes \ldots \otimes e_{n_p} \to e_m$ where $\Sigma_{i=1}^{p} n_i = m$, is called a <u>p-shuffle</u>.

Then a 2-shuffle is, in fact, a shuffle in the usual sense of the term; that is, a partition of $\underline{m}$ into two sets and a bijection $\underline{m} \to \underline{m}$ which preserves the order within each set.

Remark 2.4: Let $\mathbf{N}$ denote the category whose objects are the ordered sets $\underline{n}$, for $n \geq 0$ and whose morphisms are set monomorphisms which preserve the order relation. If we embed $\mathbf{N}$ in $\mathfrak{S}$ by the obvious functor $\underline{n} \to e_n$ then in the sense of Remark 1.8, $\mathfrak{S}$ is an enrichment of $\mathbf{J}(\mathbf{N})$. The next two paragraphs demonstrate that, loosely speaking, the "enriched structure" of $\mathfrak{S}$ is generated by the 2-shuffles of $\mathfrak{S}$ (see (2.3)).

Definition 2.5: Let $\mathfrak{A}$ be a monoidal category. An $\mathfrak{S}$-prefunctor in $\mathfrak{A}$, is a collection of data F as follows:

1. A functor $F : \mathbf{N} \to \mathfrak{A}$ such that $F(\underline{0}) = 0_{\mathfrak{A}}$;

2. $\forall$ 2-shuffle φ: $e_i \otimes e_j \to e_{i+j}$, an $\mathfrak{A}$-morphism F_φ: $F(e_i) \otimes_{\mathfrak{A}} F(e_j) \to F(e_{i+j})$ such that:

A. If $\lambda_1 \in \mathfrak{S}(e_n, e_{n'})$, $\lambda_2 \in \mathfrak{S}(e_m, e_{m'})$, $\lambda \in \mathfrak{S}(e_{n+m}, e_{n'+m'})$, $\alpha \in \mathfrak{S}(e_n \otimes e_m, e_{n+m})$, $\beta \in \mathfrak{S}(e_{n'} \otimes e_{m'}, e_{n'+m'})$ and $\beta(\lambda_1 \otimes \lambda_2) = \lambda\alpha$, then $F_\beta(F(\lambda_1) \otimes_{\mathfrak{A}} F(\lambda_2)) = F(\lambda)F_\alpha$.

B. If $\alpha \in \mathfrak{S}(e_i \otimes e_j, e_{i+j})$, $\beta \in \mathfrak{S}(e_j \otimes e_k, e_{j+k})$, $\gamma \in \mathfrak{S}(e_{i+j} \otimes e_k, e_{i+j+k})$, and $\delta \in \mathfrak{S}(e_i \otimes e_{j+k}, e_{i+j+k})$ such that $\delta(1_{e_i} \otimes \beta) = \gamma(\alpha \otimes 1_{e_k})$, then $F_\delta(1_{F(e_i)} \otimes_{\mathfrak{A}} F_\beta) = F_\gamma(F_\alpha \otimes 1_{F(e_k)})$.

Proposition 2.6: Let F be an $\mathfrak{S}$-prefunctor in $\mathfrak{A}$, then F has a unique extension to a monoidal functor from $\mathfrak{S}$ to $\mathfrak{A}$.

The proof of this proposition is rather tedious and has been deferred to the appendix. (See Appendix I).

Chapter 3. The Category Top^{ν}

Definition 3.1: Let Top^{ν} denote the following category (called the category of normed topological spaces):

An object of Top^{ν} is a pair (X,ν_X) where X is a compactly generated topological space with basepoint $*$ and ν_X is a continuous map of X into $\mathbb{R}^+$, the space of non-negative real numbers, such that $\nu_X^{-1}(0) = *$. ν_X is called the norm of X . A morphism in Top^{ν} is a continuous map of the spaces which commutes with the norms.

Remark 3.2: Let (X,ν_X) , $(Y,\nu_Y) \in Top^{\nu}$. Let $X \times Y$ be the usual topological product, then we can define $\nu_{X\times Y}: X \times Y \to \mathbb{R}^+$, by $\nu_{X\times Y}(x,y) = \nu_X(x) + \nu_Y(y)$. Thus $\times$ is a bifunctor on the category Top^{ν} . [One should note, however, that "$\times$" is not the product in Top^{ν} in a categorical sense].

Proposition 3.3: $\langle Top^{\nu},\times\rangle$ is a monoidally complete category. (See Definition 1.3.)

Proof: Clearly "$\times$" is an associative bifunctor on Top^{ν} . The one point space, with the obvious trivial norm, is an initial object for Top^{ν} and an identity for $\times$. Hence $\langle Top^{\nu},\times\rangle$ is clearly a monoidal category. We must show that $\langle Top^{\nu},\times\rangle$ is a monoidally complete monoidal category. Let $F: C \to Top^{\nu}$ and $F': C' \to Top^{\nu}$ be functors such that $\underset{C}{\text{colim}}\, F$ and $\underset{C'}{\text{colim}}\, F'$ both exist. We need the following lemma.

Lemma 3.4: Let $X \in Top^{\nu}$; let $F_X: C \to Top^{\nu}$ be the functor $F_X(\alpha) = F_{\alpha} \times X$, then $\underset{C}{\text{colim}}\, F_X \cong (\underset{C}{\text{colim}}\, F) \times X$.

<u>Proof</u>: Ignoring the norm for the moment and just working in the category of compactly generated spaces, we show that $(\operatorname{colim}_{\mathcal{C}} F) \times X \cong \operatorname{colim} F_X$ in that category.

Let $\{f_\alpha: F(\alpha) \times X \to Y \mid \alpha \in \mathcal{C}\}$ be a coherent family of maps. Then we have a coherent family $\{\bar{f}_\alpha: F(\alpha) \to (X,Y) \mid \alpha \in \mathcal{C}\}$ where (X,Y) has the compact-open topology. Hence, $\exists!\ \bar{f}: \operatorname{colim}_{\mathcal{C}} F \to (X,Y)$ commuting with the $\bar{f}_\alpha$'s . Hence $\exists!\ f: \operatorname{colim}_{\mathcal{C}} F \times X \to Y$, commuting with the f_α's . □

In order to prove Proposition 3.3 we now note that $\forall\, \alpha' \in \mathcal{C}'$, $(\operatorname{colim}_{\mathcal{C}} F) \times F'(\alpha') \cong \operatorname{colim}_{\mathcal{C}} F_{F'(\alpha')}$, and that $\operatorname{colim}_{\mathcal{C}} F \times \operatorname{colim}_{\mathcal{C}} F' = \operatorname{colim}_{\mathcal{C}'}(F'_{\operatorname{colim}_{\mathcal{C}} F})$ by the lemma . That $\operatorname{colim}_{\mathcal{C}} F \times \operatorname{colim}_{\mathcal{C}'} F' \cong \operatorname{colim}_{\mathcal{C}\times\mathcal{C}'} F \times F'$ then follows immediately. □

Chapter 4. The Functor L_X

Definition 4.1: Let $X \in Top^\nu$. We define an $\mathfrak{S}$-prefunctor, L_X, in Top^ν (see 2.5) as follows:

1. $L_X(e_0) = * =$ the basepoint of X. For $n \geq 1$,

$$L_X(e_n) = \{(t_1,x_1,\ldots,t_n,x_n) \in (\mathbb{R}^+ \times X)^n \,|\, \Big(\sum_{i\in I} \nu(x_i)\Big)^2 \leq \sum_{i\in I} t_i\ ,\ \forall I \subseteq \underline{n};$$

and $\Big(\sum_{i\in \underline{n}} \nu(x_i)\Big)^2 = \sum_{i\in\underline{n}} t_i\}$.

2. Let $\varphi\colon \underline{n} \to \underline{m}$ be an order-preserving monomorphism (that is: $\varphi \in \mathbb{M}(e_n,e_m) = \mathfrak{S}(e_n,e_m))$, then $L_X(\varphi)\colon L_X(e_n) \to L_X(e_m)$ is the map $L_X(\varphi)(t_1x_1,\ldots,t_nx_n) = (s_1y_1,\ldots,s_my_m)$ where

$$(s_iy_i) = \begin{cases} (t_jx_j)\ , & \text{if } i = \varphi(j) \\ (0,*)\ , & \text{if } i \notin \operatorname{Im}\varphi\ . \end{cases}$$

This is clearly well-defined and continuous and norm preserving, where the norm on $L_X(e_n)$ is defined by:

$$\nu(s_1x_1,\ldots,s_nx_n) = \sum_{i=1}^{n} \nu_X(x_i)\ .$$

3. Let $\alpha\colon (e_n \otimes e_m) \to e_{n+m}$ be a 2-shuffle. Then $L_X(\alpha)\colon L_X(e_n) \times L_X(e_m) \to L_X(e_{n+m})$ is defined by:

$$L_X(\alpha)((t_1,x_1,\ldots,t_r,x_n),(s_1,y_1,\ldots,s_m,y_m)) = (r_1,z_1,\ldots,r_{n+m},z_{n+m})$$

where

$$(r_k,z_k) = \begin{cases} (t_j,x_j) & k = \alpha(j),\ j \in \underline{n} \\ (s_i+2\nu(y_i)\cdot \sum_{j=1}^{n} \nu(x_j),y_i), & k = \alpha(i),\ i \in \underline{m}\ . \end{cases}$$

The next three propositions will establish that $L_X(\alpha)$ is a morphism with those properties required for L_X to be an $\mathfrak{G}$-prefunctor in Top^{ν} .

<u>Proposition 4.2</u>: The point $(r_1, z_1, \ldots, r_{n+m}, z_{n+m})$ as defined above, is an element of $L_X(e_{n+m})$.

<u>Proof</u>: Notation: If $x = (x_1, \ldots, x_k)$ is a k-tuple in X and $I \subseteq \underline{k}$, let $\|x\|_I = \sum_{i \in I} \nu(x_i)$. If $t = (t_1, \ldots, t_k)$ is a k-tuple in $\mathbb{R}^+$, let $\|t\|_I = \sum_{i \in I} t_i$.

Then we wish to show that if $I \subseteq \underline{n+m}$ then $\|r\|_I - \|z\|_I^2 \geq 0$ and $\|r\|_{\underline{n+m}} = (\|z\|_{\underline{n+m}})^2$. Let $I_1 = I \cap \alpha(\underline{n})$, $I_2 = I \cap \alpha(\underline{m})$, $J = \alpha^{-1}(I_1)$, $K = \alpha^{-1}(I_2)$, then

$$\begin{aligned}\|r\|_I - \|z\|_I^2 &= \|r\|_{I_1} + \|r\|_{I_2} - (\|z\|_{I_1} + \|z\|_{I_2})^2 \\ &= \|t\|_J + \|s\|_K + 2\|y\|_K\|x\|_{\underline{n}} - \|x\|_J^2 - 2\|x\|_J\|y\|_K - \|y\|_K^2 \\ &= (\|t\|_J - \|x\|_J^2) + (\|s\|_K - \|y\|_K^2) + 2\|y\|_K(\|x\|_{\underline{n}} - \|x\|_J),\end{aligned}$$

but by definition of $L_X(e_n)$ and $L_X(e_m)$, each of these terms is positive or zero, and if $I = \underline{n+m}$ then $J = \underline{n}$, $K = \underline{m}$ and all three must be zero. Hence the proposition. □

<u>Proposition 4.3</u>: $L_X(\alpha)$ as defined in Definition 4.1(3) satisfies condition (2A) of Definition 2.5.

<u>Proof</u>: Let $\lambda_1 \in \mathfrak{G}(e_n, e_{n'})$, $\lambda_2 \in \mathfrak{G}(e_m, e_{m'})$, $\lambda \in \mathfrak{G}(e_{n+m}, e_{n+m'})$, $\alpha \in \mathfrak{G}(e_n \otimes e_m, e_{n+m})$, $\beta \in \mathfrak{G}(e_{n'} \otimes e_{m'}, e_{n'+m'})$ such that the following diagram commutes:

$$\begin{array}{ccc} e_n \otimes e_m & \xrightarrow{\lambda_1 \otimes \lambda_2} & e_{n'} \otimes e_{m'} \\ \downarrow \alpha & & \downarrow \beta \\ e_{n+m} & \xrightarrow{\lambda} & e_{n'+m'} \end{array} .$$

We must show that the following diagram commutes in Top^{ν} :

$$\begin{array}{ccc} L_X(e_n) \times L_X(e_m) & \xrightarrow{L_X(\lambda_1)\times L_X(\lambda_2)} & L_X(e_{n'}) \times L_X(e_{m'}) \\ \downarrow L_X(\alpha) & & \downarrow L_X(\beta) \\ L_X(e_{n+m}) & \xrightarrow{L_X(\lambda)} & L_X(e_{n'+m'}) \end{array} .$$

Let $w = (t_1x_1,\ldots,t_n,x_n,s_1y_1,\ldots,s_my_m) \in L_X(e_n) \times L_X(e_m)$, then $L_X(\lambda_1) \times L_X(\lambda_2)(z) = (t'_1x'_1,\ldots,t'_{n'},x'_{n'},s'_1y'_1,\ldots,s'_{m'},y'_{m'})$ where

$$(t'_i,x'_i) = \begin{cases} (t_j,x_j), & i = \lambda_1(j) \\ (0,*) , & i \notin \operatorname{Im} \lambda \end{cases} , \quad s'_iy'_i = \begin{cases} s_jy_j & , i = \lambda_2(j) \\ 0,* & , i \notin \operatorname{Im} \lambda_2 \end{cases} .$$

Then

$$L_X(\beta) \cdot (L_X(\lambda_1) \times L_X(\lambda_2))(z) = (r_jz_j \mid 1 \le j \le n'+m') .$$

$$(r_j,z_j) = \begin{cases} (0,*) & \text{if} \quad j \notin \operatorname{Im} \beta(\lambda_1 \otimes \lambda_2) \\ (t_i,x_i) & \text{if} \quad j = \beta\, \lambda_1(i) \ , \ i \in \underline{n} \\ (s_i+\nu(y_i)\|x\|_{\underline{n}'},y_i) & , \ j = \beta\, \lambda_2(i) \ , \ i \in \underline{m} \end{cases}$$

and

$$L_X(\lambda) \cdot L_X(\alpha)(z) = (q_j v_j \mid 1 \le j \le n'+m')$$

$$(q_j, v_j) = \begin{cases} (0,x) , & j \notin \mathrm{Im}(\lambda) \\ (t_i, x_i) , & j = \lambda\,\alpha(i),\ i \in \underline{n} \\ (s_i + \nu(y_i)\|x\|_{\underline{n}}, y_i) , & j = \lambda\,\alpha(i),\ i \in \underline{m} \end{cases} .$$

But $\|x\|_{\underline{n}'} = \|x\|_{\underline{n}}$ (abusing notation a little), and clearly $\mathrm{Im}\,\lambda = \mathrm{Im}\,\beta(\lambda_1 \otimes \lambda_2)$ since α is bijective and $\lambda\,\alpha = \beta(\lambda_1 \otimes \lambda_2)$. Clearly $(q_j v_j) = (r_j z_j)$, $\forall\ j \le n+m$. Thus the proposition is true. □

<u>Proposition 4.4</u>: $L_X(\alpha)$ as defined in Definition 4.1(3) satisfies condition (2B) of Definition 2.5.

<u>Proof</u>: Let $\alpha \in \mathfrak{G}(e_n \otimes e_m, e_{n+m})$, $\beta \in \mathfrak{G}(e_m \otimes e_k, e_{m+k})$, $\gamma \in \mathfrak{G}(e_{n+m} \otimes e_k, e_{n+m+k})$, $\delta \in \mathfrak{G}(e_n \otimes e_{m+k}, e_{n+m+k})$ such that the following diagram commutes in $\mathfrak{G}$:

$$\begin{array}{ccc} e_n \otimes e_m \otimes e_k & \xrightarrow{1_{e_n} \otimes \beta} & e_n \otimes e_{m+k} \\ \Big\downarrow{\scriptstyle \alpha \otimes 1_{e_k}} & & \Big\downarrow{\scriptstyle \gamma} \\ e_{n+m} \otimes e_k & \xrightarrow{\delta} & e_{n+m+k} \end{array} .$$

We wish to show that $L_X(\delta)\cdot(L_X(\alpha) \times 1_{L_X(e_k)}) = L_X(\gamma)\cdot(1_{L_X(e_n)} \times L_X(\beta))$.

Let $w = (t_1 x_1, \ldots, t_n x_n, s_1, y_1, \ldots, s_m, y_m, r_1 z_1, \ldots, r_n z_k)$, an element of $L_X(e_n) \times L_X(e_m) \times L_X(e_k)$, then $L_X(\delta)\cdot(L_X(\alpha) \times 1)(w) = (q_1 v_1, \ldots, q_{n+m+k} v_{n+m+k})$ where

$$(q_i,v_i) = \begin{cases} (t_j x_j) \;; & i = \delta(1 \times \beta)(j)\ ,\ j \in e_n \\ (s_j + 2\nu(y_j)\|x\|_{\underline{n}}, y_j) \;; & i = \delta(1 \times \beta)(j),\ j \in e_m \\ (r_j + 2\nu(z_j)\|x\|_{\underline{n}} + 2\nu(z_j)\|y\|_{\underline{m}}, z_j)\ , & \\ & i = \delta(1 \times \beta)(j),\ j \in e_k \end{cases}$$

(where $j \in e_m$, etc. has the obvious interpretation), and

$$L_X(\gamma)(L_X(\alpha) \times 1)(w) = (p_1 u_1, \ldots, p_{n+m+k}, u_{n+m+k})$$

where

$$(p_i u_i) = \begin{cases} (t_j x_j) \;; & i = \gamma(\alpha \times 1)(j),\ j \in e_n \\ (s_j + 2\nu(y_j)\|x\|_{\underline{n}}, y_j)\ , & i = \gamma(\alpha \times 1)(j),\ j \in e_m \\ (r_j + 2\nu(z_j)\|x,y\|_{\underline{n+m}}, z_j), & i = \gamma(\alpha \times 1)(j),\ j \in e_k \end{cases}$$

but since $\gamma(\alpha \times 1) = \delta(1 \times \beta)$ and $\|x\|_{\underline{n}} + \|y\|_{\underline{m}} = \|x,y\|_{\underline{n+m}}$ these are equal, establishing the proposition. □

Hence we have a well-defined $\mathfrak{S}$-prefunctor in Top^{ν} and therefore by Proposition 2.6 we conclude:

Corollary 4.5: There is a unique monoidal functor $L_X: \mathfrak{S} \to \mathrm{Top}^{\nu}$ whose values on $\mathbb{J}(\mathbb{N})$ and on all 2-shuffles in $\mathfrak{S}$ are given by Definition 4.1.

Chapter 5. Statement of the Theorem

Notation 5.1: Let Top° denote the category of pointed, compactly generated topological spaces. Let Mon^{ν} denote the category of Top^{ν}-monoids in the sense of Definition 1.2. If $X \in Top^{\circ}$, then ΩX is the Moore space of variable-length loops in X. Ω is clearly a functor from Top° to Mon^{ν}, where the norm is given by path length. $\Omega^2 X$ denotes Ω of the underlying pointed space of ΩX. S will denote the usual reduced suspension functor $Top^{\circ} \to Top^{\circ}$.

Definition 5.2: Let X be an explicit cell-complex with basepoint; i.e., X is a CW-complex with a specific set of attaching maps which give rise to the cell-structure. We will define the standard norm $\nu_X : X \to \mathbb{R}^+$ as follows.

Define ν_X on the 0-skeleton of X by setting $\nu_X(y) = 0$, if $y = *$ and $\nu_X(y) = 1$ otherwise.

Assume ν_X is defined on $X^{(n-1)}$, this (n-1)-skeleton of X. Let e_n be an n-cell and $f: S^{n-1} \to X^{(n-1)}$ the attaching map. Let z_0 be the center of e_n. Then extend ν_X to $X^{(n-1)} \cup_f e_n$ by defining $\nu_X(z_0) = n+1$ and extending linearly along radial lines. Thus we can extend ν_X to all of X.

Theorem 5.3: Let X be a connected explicit cell-complex. Then there is a Mon^{ν}-morphism:

$$\varphi: \operatorname{colim}_{\mathcal{G}} L_X \longrightarrow \Omega^2 S^2(X)$$

which is a homotopy equivalence.

The proof of this theorem will occupy the remainder of this paper.

PART II

PRELIMINARIES TO THE PROOF OF THEOREM 5.3

Introduction

Parts II and III comprise the proof of Theorem 5.3. The proof is intricate and involves many notions not yet introduced to the reader. For this reason, the actual proof is not addressed until Part III.

Chapters 6 through 13 introduce the various notions and propositions necessary to the proof proper.

This arrangement of the material, though necessary to the logical exposition of the proof, is somewhat bewildering for the reader, as he is often shunted from one topic to another with no sense of connection or relationship between the two topics.

I will attempt here to erect a few small guide posts which may be of some aid against confusion.

Chapters 6 and 7 are concerned with those notions of a purely categorical nature which the proof will require. In Chapter 6, the category $\mathfrak{G}$, defined in Chapter 2, is discussed further. The propositions therein are of a rather technical nature and could be passed over without pain in a first reading.

Chapter 7 deals with the construction of "conaturalizers of transformations". This notion is rather central to the proof and should be understood before venturing into Part III.

Chapter 8 describes a base-space construction for normed monoids, due to R. James Milgram. This construction is crucial to the argument. The chief result being that if M is a "cellular" normed monoid, then $M \simeq \Omega B(M)$, where B is Milgram's base-space functor.

Chapter 9 discusses the functors J (free monoid) and Σ (normed suspension) and relates them to the material in Chapters 7 and 8.

Chapter 10 is a digression into the theory of convex compact subsets of Euclidean space. It is necessary for technical reasons.

Chapter 11 discusses the functor L_X , first defined in Chapter 4, at great length. Here we see how nicely L_X reflects the structure of the category $\mathfrak{G}$ and, using techniques from Chapter 10, we see that if X is a CW-complex then L_X takes its values in the category of CW-complexes.

Chapters 12 and 13 introduce a new category (double edged spaces) and a new functor (D_X), which will, in Chapter 14, enable us to examine in great detail the structure of the space $J(\Sigma X)$.

I would suggest that for a first reading of this paper one should glance first at Chapters 7, 8, 9 and 11, skipping the long proofs and then plunge into Part III referring back for necessary ideas.

Chapter 6. Further Properties of the Category $\mathfrak{G}$

Notation 6.1: We establish the following terminology and notation. If $\alpha: \underline{n} \to \underline{k}$ is an object of $\mathfrak{G}$, then the grade of α is n and the range of α is k. Put another way, if $\alpha = e_{i_1} \otimes \ldots \otimes e_{i_k}$ then the grade of α is $\sum_{j=1}^{k} i_j$. $(\mathfrak{G},\alpha)$ denotes the category of $\mathfrak{G}$-morphisms whose target is α. $\overline{(\mathfrak{G},\alpha)}$ is the full subcategory of $(\mathfrak{G},\alpha)$ with 1_α omitted. $(\mathfrak{G},\alpha)_0$ will denote the full subcategory of $\overline{(\mathfrak{G},\alpha)}$ whose objects are morphisms $\beta \to \alpha$ such that β and α have the same grade. $(\mathfrak{G},\alpha)_M$ denotes the set of maximal objects in $\overline{(\mathfrak{G},\alpha)}$.

Note that $(\mathfrak{G},e_n)_M$ is just the set of 2-shuffles of order n.

Proposition 6.2: $(\mathfrak{G},\alpha)$ is a partial-order category, $\forall\ \alpha \in \mathfrak{G}$.

This is a straightforward consequence of the definition of $\mathfrak{G}$; particularly of the fact that the morphisms of $\mathfrak{G}$ are set monomorphisms.

Definition 6.3: Let $a \in \mathfrak{G}$ and let $\beta_1, \beta_2 \in (\mathfrak{G},a)_0$. Then β_1 and β_2 are coherent (denoted β_1 coh β_2) if there exists $\gamma \in (\mathfrak{G},a)_0$ such that $\gamma \leqq \beta_1$ and $\gamma \leqq \beta_2$ where $\leqq$ is the partial ordering on $(\mathfrak{G},a)$.

Definition 6.4: An admissible filtration of $\underline{n}$ of length k $(k \geq 0)$ is a sequence, $I = \{I_1, I_2, \ldots, I_k\}$, of subsets of $\underline{n}$ such that $\emptyset \neq I_1 \subset I_2 \subset \ldots \subset I_k \subseteq \underline{n}$.

Proposition 6.5: If $n \geq 1$, there is a one to one correspondence between the objects in $(\mathfrak{G},e_n)$ and the set of admissible filtrations of $\underline{n}$.

Proof: Let $\alpha: \underline{m} \to \underline{k}$ be an object of $\mathfrak{G}$, and let $\varphi \in \mathfrak{G}(\alpha,e_n)$. Then φ is a monomorphism from $\underline{m}$ to $\underline{n}$. Define I^φ, a filtration of $\underline{n}$ of length k, as follows:

$$I_i^{\varphi} = \{j \in \underline{n} \mid \alpha(\varphi^{-1}(j)) \leqq i\} \quad \text{for} \quad i = 1,\ldots,k \; .$$

(Note that $I_k^{\varphi} = I_m(\varphi)$.) It is then easy to see that $\varphi \to I^{\varphi}$ is the advertised one to one correspondence.

Lemma 6.6: Let $\varphi, \psi \in (\mathfrak{G}, e_n)_0$. Then φ coh ψ if and only if the set $I^{\varphi} \cup I^{\psi}$ can be simply ordered by inclusion (i.e., if and only if $I_i^{\varphi} \subseteq I_j^{\psi}$ or $I_j^{\psi} \subseteq I_i^{\varphi}$, for each choice of i and j).

Proof: Let I and J be two admissible filtrations of $\underline{n}$. I is said to be a refinement of J $(I < J)$ if

1. $\forall\, j \geq 1$, $\exists\, i_j \geq 1$ such that $I_j - I_{j-1} \subseteq I_{i_j} - I_{i_j-1}$, where $I_0 = \emptyset = J_0$, and

2. If $j > j'$, then $i_j \geq i_{j'}$.

Then if $\gamma, \lambda \in (\mathfrak{G}, e_n)$, we have that γ is a predecessor of λ if and only if I^{γ} is a refinement of I^{λ} . Further, if $\gamma \in \overline{(\mathfrak{G}, e_n)}$, then $\gamma \in (\mathfrak{G}, e_n)_0$ if and only if $I_k^{\varphi} = \underline{n}$ where k is the length of I^{φ} .

Suppose that $I^{\varphi} \cup I^{\psi}$ can be simply ordered by inclusion. Then this simple ordering yields an admissible filtration J of $\underline{n}$. It is obvious that J is a refinement of I^{φ} and of I^{ψ} , and that if k = length of J , then $k \leqq$ (length of I^{φ}) + (length of I^{ψ}) . It is also clear that $J_k = \underline{n}$. Therefore J corresponds to a $\gamma \in (\mathfrak{G}, e_n)_0$ such that γ is a common predecessor of φ and ψ . Hence φ and ψ are coherent.

Suppose φ and ψ are coherent; then there exists $\gamma \in (\mathfrak{G}, e_n)_0$ such that γ is a predecessor of φ and ψ . Hence I^{γ} is a refinement of I^{φ} and I^{ψ} . Now, suppose that $I^{\varphi} \cup I^{\psi}$ cannot be simply ordered by inclusion. Then there exist i and j such that $I_i^\varphi \not\subseteq I_j^\psi$ and $I_j^\psi \not\subseteq I_i^\varphi$. Let $p \in I_i^\varphi - I_j^\psi$ and let $q \in I_j^\psi - I_i^\varphi$. Let k_p be such that $p \in I_{k_p}^\gamma - I_{k_p-1}^\gamma$ and k_q such that $q \in I_{k_q}^\gamma - I_{k_q-1}^\gamma$.

Without loss of generality we can assume that $k_p \leq k_q$. Since $(I_{k_q}^\gamma - I_{k_q-1}^\gamma) \cap I_j^\psi$ contains q and is therefore not empty, it follows from the definition of refinement that $I_{k_q}^\gamma - I_{k_q-1}^\gamma \subseteq I_j^\psi$. It then follows that $I_{k_q}^\gamma \subseteq I_j^\psi$. But $k_p \leqq k_q$ implies that $I_{k_p}^\gamma \subseteq I_{k_q}^\gamma$ and hence that $p \in I_j^\psi$. This is a contradiction since $p \in I_i^\varphi - I_j^\psi$. Therefore $I^\varphi \cup I^\psi$ can be simply ordered by inclusion.

<u>Proposition 6.7</u>: Let $a \in \mathfrak{G}$, let $\beta_1, \beta_2 \in (\mathfrak{G},a)_0$. Then β_1 coh β_2 if and only if β_1 and β_2 have a unique maximal common predecessor in $\overline{(\mathfrak{G},a)}$.

<u>Proof</u>: This is an immediate corollary of Lemma 6.6, if $a = e_n$ for some n. If $a = e_{n_1} \otimes \ldots \otimes e_{n_k}$, then $(\mathfrak{G},a) = (\mathfrak{G},e_{n_1}) \times \ldots \times (\mathfrak{G},e_{n_k})$ as a partially ordered set. It is easy to see that if $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_k$ and $\psi = \psi_1 \otimes \ldots \otimes \psi_k$ in $(\mathfrak{G},a)$, then φ coh $\psi \Leftrightarrow \varphi_i$ coh ψ_i for $i = 1,\ldots,k$. The proposition then follows immediately.

<u>Notation 6.8</u>: Let $\beta_1, \beta_2 \in (\mathfrak{G},a)_0$ such that β_1 coh β_2. Then the unique maximal common predecessor in $\overline{(\mathfrak{G},a)}$ (which clearly must lie in $(\mathfrak{G},a)_0$) will be denoted $\beta_1 \wedge \beta_2$.

<u>Proposition 6.9</u>: Let α, β, γ be pairwise coherent in $(\mathfrak{G},a)_0$. Then $\alpha \wedge \beta$, $\beta \wedge \gamma$ and $\alpha \wedge \gamma$ are pairwise coherent.

<u>Proof</u>: If $a = e_n$ for some n, this is an immediate consequence of

Lemma 6.6. The general case follows from this special case in the same manner as in Proposition 6.7.

Chapter 7. Conaturalizers

Definition 7.1: Let $F: \mathcal{C} \to \mathcal{A}$, $G: \mathcal{C} \to \mathcal{A}$ be covariant functors, where $\mathcal{C}$ is a small category. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ such that obj $\mathcal{C}'$ = obj $\mathcal{C}$. Let $\varphi: F \to G$ be a transformation that is natural with respect to $\mathcal{C}'$-morphisms. Then a conaturalizer of φ is a functor $H: \mathcal{C} \to \mathcal{A}$ and a transformation $\rho: G \to H$ such that:

1) ρ is $\mathcal{C}$-natural (that is, natural with respect to all $\mathcal{C}$-morphisms),

2) $\rho\varphi$ is $\mathcal{C}$-natural,

3) ρ is universal with respect to properties 1) and 2).

Proposition 7.2: Let φ be as in Definition 7.1. If $\mathcal{A}$ is cocomplete, then the conaturalizer of φ exists and is unique. Furthermore, if $\rho: G \to H$ is the conaturalizer of φ , and if $\mathcal{D}$ is any initial subcategory of $\mathcal{C}$, then $\rho|_{\mathcal{D}}: G|_{\mathcal{D}} \to H|_{\mathcal{D}}$ is the conaturalizer of the transformation: $\varphi|_{\mathcal{D}}: F|_{\varphi} \to G|_{\mathcal{D}}$.

Proof: If $\beta \in \mathcal{C}$, let $(\mathbb{C},\beta)$ denote the category of $\mathcal{C}$-morphisms over β . If $f \in (\mathbb{C},\beta)$, let $d(f)$ denote the domain of f .

Let $\beta \in \mathcal{C}$, define $\rho_\beta: G(\beta) \to H(\beta)$ as the following coequalizer:

$$\begin{array}{ccccc}
\coprod_{f\in(\mathbb{C},\beta)} F(d(f)) & \xrightarrow{\coprod_{f\in(\mathbb{C},\beta)} F(f)} & F(\beta) & & \\
\Big\downarrow {\scriptstyle \coprod_{f\in(\mathbb{C},\beta)} \varphi_{d(f)}} & & \Big\downarrow {\scriptstyle \varphi_\beta} & & \\
\coprod_{f\in(\mathbb{C},\beta)} G(d(f)) & \xrightarrow[\coprod_{f\in(\mathbb{C},\beta)} G(f)]{} & G(\beta) & \xrightarrow{\rho_\beta} & H(\beta)
\end{array}$$

Let $g \in \mathcal{C}(\alpha,\beta)$, then one can verify that the composition $\rho_\beta \cdot G(g)$ coequalizes the diagram that defines ρ_α . Hence the universal property of a coequalizer yields a map $H(g)\colon H(\alpha) \to H(\beta)$ such that $H(g)\cdot\rho_\alpha = \rho_\beta \cdot G(g)$. It is easy to verify then that H is a functor. Clearly ρ is natural with respect to all $\mathcal{C}$-morphisms.

The verification that ρ is the conaturalizer of φ is straightforward and omitted. The final assertion of the proposition is an immediate consequence of the above construction. The definition of H and ρ at β is made with reference only to the behavior of φ , G and F on predecessors of β . Hence the construction for any initial subcategory will be exactly the same.

<u>Definition 7.3</u>: Let $\mathcal{A}$ be a monoidal category with coproducts, let $\mathcal{B}$ be any category with coproducts. Let $T\colon \mathcal{A} \to \mathcal{B}$ be a colimit-preserving functor. T is said to have property (PJH) if there is a natural transformation of bifunctors: $\Lambda_T\colon T(\,.\otimes_{\mathcal{A}} ..) \to T(.) \coprod_{\mathcal{B}} T(..)$ such that the following diagrams commute, $\forall\, A, B, C \in \mathcal{A}$.

i)

$$\begin{array}{ccc}
T(A \otimes B \otimes C) & \xrightarrow{\Lambda_T} & T(A) \coprod T(B \otimes C) \\
\Big\downarrow{\scriptstyle \Lambda_T} & & \Big\downarrow{\scriptstyle 1 \coprod \Lambda_T} \\
T(A \otimes B) \coprod T(C) & \xrightarrow[\Lambda_T \coprod 1]{} & T(A) \coprod T(B) \coprod T(C)
\end{array}$$

ii)

$$\begin{array}{ccc}
T(A \coprod B) & \xrightarrow{T(i)} & T(A \otimes B) \\
& {\scriptstyle I}\searrow{\scriptstyle \cong} & \Big\downarrow \\
& & T(A) \coprod T(B)
\end{array}$$

where $i: A \coprod B \to A \otimes B$ is the map induced on the coproduct by the maps $A = A \otimes 0_{\mathcal{C}} \xrightarrow{1\otimes 0} A \otimes B$, $B = 0_{\mathcal{C}} \otimes B \xrightarrow{0\otimes 1} A \otimes B$ and $I: T(A \coprod B) \longrightarrow T(A) \coprod T(B)$ is the natural isomorphism given by the colimit-preserving nature of T .

Example 7.3A: If $\mathcal{C}$ is a monoidal category and $\mathcal{B}$ is the category of $\mathcal{C}$-monoids, let $T: \mathcal{C} \to \mathcal{B}$ be a "free monoid functor" or, in other words, T is coadjoint to the forgetful functor. Then T is a (PJH)-functor.

First, if $\mathcal{C}$ has a coproduct, then $T(X \coprod Y) = T(X) \coprod T(Y)$ whether $\mathcal{B}$ has coproducts in general or not.

Second, we can define $\Lambda_T: T(X \otimes Y) \to T(X) \coprod T(Y)$ as the coadjoint of the composition:

$$X \otimes Y \xrightarrow{a_x \otimes a_y} T(X \coprod Y) \otimes T(X \coprod Y) \xrightarrow{\mu} T(X \coprod Y) = T(X) \coprod T(Y),$$

where a_x is the composition: $X \to X \coprod Y \to T(X \coprod Y)$. It is a straightforward categorical exercise to see that Λ_T satisfies conditions i) and ii).

Definition 7.4: Let $\mathfrak{M}$ be an enriched free monoidal category, let $\mathcal{C}$ be a cocomplete monoidal category, and let $\mathcal{B}$ be a cocomplete category. Let $F: \mathfrak{M} \to \mathcal{C}$ be a monoidal functor and let $T: \mathcal{C} \to \mathcal{B}$ be a functor with property (PJH). We will define the T-reduction of F to be the following functor $F_T: \mathfrak{M} \to \mathcal{B}$.

First, define $F^{\amalg}: \mathfrak{M} \to \mathcal{C}$ as follows: $F^{\amalg}(e) = F(e)$ if e is a generator of obj$(\mathfrak{M})$. If $\alpha = e_1 \otimes \ldots \otimes e_n$, where e_i is a generator, $\forall\, i$, then $F^{\amalg}(\alpha) = \coprod_{i=1}^{n} F(e_i)$. If $\varphi \in \mathfrak{M}(e_1 \otimes \ldots \otimes e_n, e)$, then $F^{\amalg}(\varphi)$ is the morphism given by the composition:

$$F(e_1) \coprod \cdots \coprod F(e_n) \xrightarrow{i} F(e_1) \otimes_{\mathfrak{a}} \cdots \otimes_{\mathfrak{a}} F(e_n) \xrightarrow{F(\varphi)} F(e)$$

where i is as described in 7.3.ii). Since $\mathfrak{m}$ is an enriched free monoid, every morphism is the product, in a unique way, of such morphisms, hence we can extend $F^{\amalg}$ to all $\mathfrak{m}$-morphisms by defining $F^{\amalg}(\varphi_1 \otimes \ldots \otimes \varphi_m) = F^{\amalg}(\varphi_1) \coprod \ldots \coprod F^{\amalg}(\varphi_m)$. It is somewhat tedious, but completely straightforward to check that $F^{\amalg}$ is indeed a functor. It should be clear that $F^{\amalg}$ is actually a monoidal functor from $\mathfrak{m}$ to the monoidal category $\langle \mathfrak{a}, \coprod \rangle$.

Let Λ_T be the natural transformation associated with the (PJH)-functor T . Λ_T induces a transformation $\bar{\Lambda}_T$: $TF \to T \cdot F^{\amalg}$ which is natural on $\mathbf{J}(G)$, where G is the full subcategory of $\mathfrak{m}$ whose objects are the generating set for obj $\mathfrak{m}$.

However, there is no reason, in general, to expect $\bar{\Lambda}_T$ to be natural with respect to all the morphisms in $\mathfrak{m}$. Let $TF^{\amalg} \xrightarrow{\rho_T} F_T$ be the conaturalizer of $\bar{\Lambda}_T$. Then F_T is the T-reduction of F .

<u>Proposition 7.5</u>: Let $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{B}$, F and T be as in the previous paragraph. Assume further that $\mathfrak{B}$ is the category of $\mathfrak{a}$-monoids and that T is coadjoint to the forgetful functor: $|\ |: \mathfrak{B} \to \mathfrak{a}$. Then there is an $\mathfrak{a}$-monoid isomorphism:

$$\operatorname*{colim}_{\mathfrak{m}} F \cong \operatorname*{colim}_{\mathfrak{m}} F_T ,$$

and this isomorphism is natural in F .

Recall that by Proposition 1.4 , $\operatorname*{colim}_{\mathfrak{m}} F$ is an $\mathfrak{a}$-monoid.

<u>Proof</u>: The proof is straightforward, long, and quite tedious. It is deferred to the Appendix. See Appendix II.

Remarks: In what follows, we will restrict ourselves to the category $\mathcal{O}$, rather than consider enriched free monoidal categories in general. Proposition 7.7 is equally valid in the more general case; however, in the present work our only application will be to the category $\mathcal{O}$. The more general statement of the proposition does not seem, therefore, worth the considerable escalation of notation necessary for the statement and proof.

Notation 7.6: Let $\mathcal{O}_n$ denote the full subcategory of $\mathcal{O}$ consisting of those objects of $\mathcal{O}$ with grade $\leqq n$. Note that $\{\mathcal{O}_n | n \geq 0\}$ is a filtration of $\mathcal{O}$ and that $\mathcal{O}_n = \{\alpha \in \mathcal{O} | \mathcal{O}(\alpha, e_n) \neq \emptyset\}$.

Let $F: \mathcal{O} \to \mathfrak{A}$ be a monoidal functor. Let $T: \mathfrak{A} \to \mathfrak{B}$ be a (PJH)-functor as described in Definition 7.3. Let $Q(F,T)$ denote $\operatorname{colim}_{\mathcal{O}} F_T$ and let $Q_n(F,T)$ denote $\operatorname{colim}_{\mathcal{O}_n} F_T$, where F_T is the T-reduction of F. Note that the inclusion of categories $\mathcal{O}_n \to \mathcal{O}_{n+1}$ induces a map $q_n: Q_n(F,T) \to Q_{n+1}(F,T)$ and that $Q(F,T) = \operatorname{colim}_{n\to\infty} Q_n(F,T)$.

Proposition 7.7: For each $n \geq 1$, there is a map $\xi_n: \operatorname{colim}_{\overline{(\mathcal{O},e_n)}} T\cdot F\pi \to Q_{n-1}(F,T)$ such that the following diagram is a pushout in $\mathfrak{B}$:

(7.7.1)
$$\begin{array}{ccc} \operatorname{colim}_{\overline{(\mathcal{O},e_n)}} TF\pi & \xrightarrow{\ h\ } & TF(e_n) \\ \Big\downarrow{\scriptstyle \xi_n} & & \Big\downarrow{\scriptstyle p} \\ Q_{n-1}(F,T) & \longrightarrow & Q_n(F,T) \end{array} \quad ;$$

where π is the domain functor: $(\mathcal{O},e_n) \to \mathcal{O}$; h is the naturally induced map and p is the composition

$$TF(e_n) = TF^{\amalg}(e_n) \xrightarrow{\rho_T(e_n)} F_T(e_n) \longrightarrow \operatorname{colim}_{\mathcal{O}_n} F_T = Q_n(F,T) .$$

<u>Proof</u>: The proof is in three steps with some straightforward but tedious detail omitted.

<u>Step 1</u>: The construction of ξ_n .

Let $\lambda \in \overline{(\mathfrak{G},e_n)}$, define ξ_n^λ: $TF\pi(\lambda) \to Q_{n-1}(F,T)$ as the composition:

$$\begin{array}{l}
TF\pi(\lambda) = TF(e_{i_1} \otimes \dots \otimes e_{i_k}) = T(F(e_{i_1}) \otimes \dots \otimes F(e_{i_k})) \\
\qquad \downarrow \Lambda_T \\
TF(e_{i_1}) \coprod \dots \coprod TF(e_{i_k}) \\
\qquad \downarrow \bar{\rho}_{e_{i_1}} \coprod \dots \coprod \bar{\rho}_{e_{i_k}} \\
F_T(e_{i_1}) \coprod \dots \coprod F_T(e_{i_k}) \\
\qquad \downarrow \\
Q_{i_1}(F,T) \coprod \dots \coprod Q_{i_k}(F,T) \\
\qquad \downarrow \\
Q_{n-1}(F,T)
\end{array}$$

where $\bar{\rho}$: $TF \to F_T$ is the natural transformation $\rho \circ \bar{\Lambda}_T$ and where the unlabeled morphisms are the obvious ones. The ξ_n^λ form a coherent family of maps. This is a straightforward naturality argument, making use of the naturality of $\bar{\rho}$: $TF^{\amalg} \to F_T$ and the limited naturality of Λ_T: $TF \to TF^{\amalg}$. I omit the details.

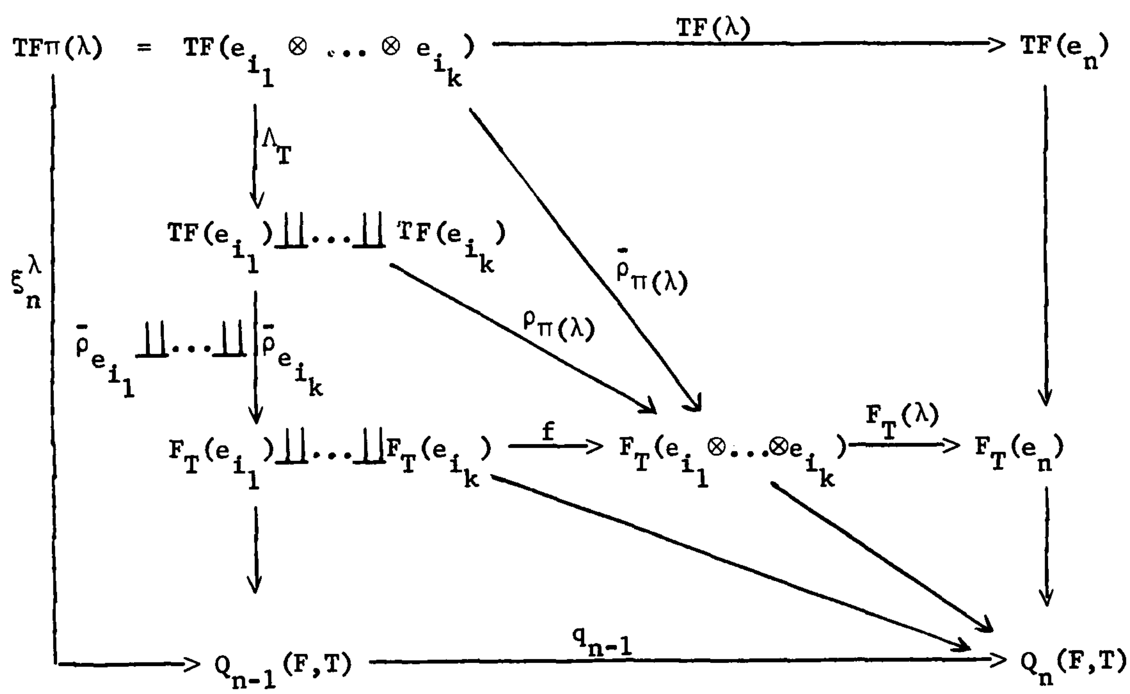
Thus we have $\xi_n = \underset{\overline{\lambda \in (\mathfrak{G},e_n)}}{\text{colim}}\ \xi_n^\lambda : \underset{\overline{(\mathfrak{G},e_n)}}{\text{colim}}\ TF\pi \to Q_{n-1}(F,T)$.

<u>Step 2</u>: To show that diagram (7.7.1) commutes.

In view of the definition of ξ_n , it is sufficient to show that the following diagram commutes $\forall\ \lambda \in \overline{(\mathbb{O},e_n)}$:

(7.7.2)
$$\begin{array}{ccc} TF\pi(\lambda) & \xrightarrow{TF(\lambda)} & TF(e_n) \\ \Big\downarrow{\scriptstyle \xi_n^{\lambda}} & & \Big\downarrow{\scriptstyle p_0} \\ Q_{n-1}(F,T) & \xrightarrow{q_{n-1}} & Q(F,T) \end{array}$$

The commutativity of diagram (7.7.2) can be deduced from the following diagram.



where $f = f_1 \amalg \ldots \amalg f_k$, $f_j = F_T(0 \otimes \ldots \otimes 1_{e_{i_j}} \otimes \ldots \otimes 0)$. The outer rectangle is diagram (7.7.2) and each inner triangle or quadrilateral commutes by naturality or simply by definition. Hence diagram (7.7.2) commutes, and therefore so does diagram (7.7.1).

<u>Step 3</u>: To show that diagram (7.7.1) is a pushout in $\mathfrak{B}$.

Let $Y \in \mathfrak{B}$, and let $g_1 : Q_{n-1}(F,T) \to Y$ and $g_2 : TF(e_n) \to Y$ be $\mathfrak{B}$-morphisms such that $g_1 \xi_n = g_2 h$. We must show the existence of a unique $G: Q_n(F,T) \to Y$ such that $Gq_{n-1} = g_1$ and $Gp = g_2$.

Consider the constant functor $\mathcal{O}_n \to \mathfrak{B}$ with value Y . (We will also denote this functor by Y). Define a transformation $\bar{G}: TF^{\amalg}|_{\mathcal{O}_n} \to Y$ as follows:

$$\bar{G}(e_n): TF^{\amalg}(e_n) \longrightarrow Y$$

is the map

$$TF^{\amalg}(e_n) = TF(e_n) \xrightarrow{\ g_2\ } Y \quad .$$

If $\alpha \in \mathcal{O}_n$, $\alpha \neq e_n$, then $\alpha = e_{i_1} \otimes \dots \otimes e_{i_k}$ for some $k \geq 1$ such that $i_j < n$, for every j . $\bar{G}(\alpha)$ is the composition:

$$\begin{array}{ccc}
TF^{\amalg}(\alpha) = TF(e_{i_1}) \amalg \dots \amalg TF(e_{i_k}) & \xrightarrow{\bar{\rho} \amalg \dots \amalg \bar{\rho}} & F_T(e_{i_1}) \amalg \dots \amalg F_T(e_{i_k}) \\
 & & \downarrow \\
 & \searrow^{\bar{G}(\alpha)} & Q_{n-1}(F,T) \\
 & & \downarrow{\scriptstyle g_1} \\
 & & Y
\end{array}$$

It is then straightforward (and tedious) to check that $\bar{G}: TF^{\amalg}|_{\mathcal{O}_n} \to Y$ is natural and also that $\bar{G}\cdot\bar{\Lambda}_T : TF|_{\mathcal{O}_n} \to Y$ is natural. Then the second assertion of Proposition (7.2) says that $F_T|_{\mathcal{O}_n}$ is the conaturalizer of $\bar{\Lambda}_T : TF|_{\mathcal{O}_n} \to TF^{\amalg}|_{\mathcal{O}_n}$. Hence $\exists!\ \hat{G}: F_T|_{\mathcal{O}_n} \to Y$ such that $\hat{G}\cdot\rho = \bar{G}$. Define $G = \operatorname{colim}_{\mathcal{O}_n} \hat{G} : \operatorname{colim}_{\mathcal{O}_n} F_T = Q_n(T,F) \to Y$. It is immediate that $Gq_{n-1} = g_1$ and $Gp = g_2$. The uniqueness of G follows from the uniqueness of $\hat{G}$. The proposition is thus established.

Chapter 8. Milgram's "Classifying Space" for Normed Monoids

Notation 8.1: The following notation will be used throughout the paper.

Let $X \in Top^{\nu}$, then ΦX will denote the presuspension of X. That is, $\Phi X = \{(t,x) \in \mathbb{R} \times X \mid 0 \le t \le \|x\|\}$. ΣX will denote the suspension of the normed space X. That is, $\Sigma X = \Phi X/_{\{(t,x) \mid t=0 \text{ or } t=\|x\|\}}$. Note that Σ can be thought of as a functor from Top^{ν} to Top°.

If X has a cofibered basepoint, then SX and ΣX are of the same homotopy type. (SX denotes the usual reduced suspension of X as a pointed topological space). This is a straightforward exercise in cofibration theory.

Let $X \in Top^{\circ}$, then $J(X)$ will denote the free topological monoid on the space X. If X is a normed space, then $J(X)$ will be a normed monoid. It should cause no confusion if we let the symbol J represent both the functor from Top° to Mon° and the functor from Top^{ν} to Mon^{ν}.

It is easy to see that $J(X)$ can be thought of as the colimit of the system:

$$X \underset{f_1^2}{\overset{f_1^1}{\rightrightarrows}} X \times X \underset{f_2^3}{\overset{f_2^1}{\Rrightarrow}} X \times X \times X \underset{f_3^4}{\overset{f_3^1}{\rightrightarrows\!\!\rightrightarrows}} \cdots\ X^{n-1} \underset{f_{n-1}^{n}}{\overset{f_{n-1}^{1}}{\vdots}} X^n \ \cdots$$

where $f_i^j\colon X^i \to X^{i+1}$ is the map:

$$f_i^j(x_1,\ldots,x_n) = (x_1,\ldots,x_{j-1},*,x_j,\ldots,x_n) .$$

$J(X)_n$ will denote the colimit of the above system terminated at X^n. Let $j_n\colon J(X)_n \to J(X)_{n+1}$ be the map induced by inclusion of diagrams. Then j_n is clearly an injection and $J(X) = \operatorname{colim}_{n\to\infty} J_n(X)$.

<u>Definition 8.2</u>: Let $M \in \mathrm{Mon}^{\nu}$ and define $E(M) = \Phi M/R$ where R is the equivalence relation generated by the relation: $(t,x) \sim (t,xy)$, $\forall (t,x) \in \Phi M$, $\forall\, y \in M$. Define an action $\mu\colon M \times E(M) \to E(M)$ by the rule: $\mu(x,\overline{(t,y)}) = \overline{(t+\nu(x),xy)}$. This is easily seen to be well-defined and continuous. Define $B(M)$ as the space of maximal orbits of the action μ, given the quotient topology. $\rho_M\colon E(M) \to B(M)$ is the projection.

Let $\varphi\colon M \to M'$ be a Mon^{ν}-morphism, let $B(\varphi)$ be the obvious induced morphism $B(M) \to B(M)'$. Then B is clearly a functor: $\mathrm{Mon}^{\nu} \to \mathrm{Top}^{\circ}$.

<u>Proposition 8.3</u>: B is coadjoint to Ω; i.e., if $X \in \mathrm{Top}^{\circ}$ and $M \in \mathrm{Mon}^{\nu}$, then there is a set isomorphism: $\mathrm{Top}^{\circ}(B(M),X) \cong \mathrm{Mon}^{\nu}(M,\Omega X)$.

<u>Proof</u>: Define $\xi'\colon \Phi\Omega(X) \to X$ by the rule: $\xi'(t,\sigma) = \sigma(t)$. If $\sigma,\tau \in \Omega X$ and if $t \le \nu(\sigma)$, then from the definition of the addition of loops in ΩX, $\sigma\cdot\tau(t) = \sigma(t)$ and $\tau\cdot\sigma(t+\nu(\sigma)) = \sigma(t)$. Hence ξ' induces a map $\xi\colon B\Omega X \to X$.

Let $\varphi \in \mathrm{Top}^{\circ}(B(M),X)$, define $\bar{\varphi} \in \mathrm{Mon}^{\nu}(M,\Omega X)$ by

$$\bar{\varphi}(m)(t) = \begin{cases} \varphi\overline{(t,m)}, & 0 \le t \le \nu(m) \\ \varphi\overline{(\nu(m),m)}, & t \ge \nu(m) \end{cases} .$$

$\bar{\varphi}_{(m)}$ has length equal to $\nu(m)$. One quickly verifies that $\bar{\varphi}$ is continuous, norm-preserving and a homomorphism.

Let $\psi \in \mathrm{Mon}^{\nu}(M,\Omega X)$, then define $\hat{\psi}\colon B(M) \to X$ to be the composition:

$$B(M) \xrightarrow{B(\psi)} B\Omega X \xrightarrow{\xi} X .$$

$\hat{\psi}$ is clearly in $\mathrm{Top}^{\circ}(B(M),X)$. It is then immediate that $\bar{\hat{\psi}} = \psi$ and that $\hat{\bar{\varphi}} = \varphi$. Hence the proposition.

Corol'ary 8.4: B preserves colimits.

Proposition 8.5: There is a natural equivalence of functors $BJ \cong \Sigma$: $Top^{\nu} \to Top^{\circ}$.

Proof: Let $f: X \to \mathbf{J}(X)$ be the inclusion.

Define $\psi: \Sigma X \to B(JX)$, by $\psi\overline{(t,x)} = \overline{(t,f(x))}$. This is clearly well-defined and continuous. If $z \in J(X)$ then $z = x_1, x_2, \ldots, x_n$, where $x_i \in X$. Let $0 \leq t \leq \nu(z)$, then $\exists\ i \leq n$ such that $t \geq \Sigma_{j=1}^{i} \nu(x_i)$ and $t < \Sigma_{j=1}^{i+1} \nu(x_i)$. It follows that

$$\overline{(t,z)} = \overline{(t - \sum_{j=1}^{i} \nu(x_i), x_{i+1})}$$

and hence that ψ is onto. ψ is clearly a monomorphism, and hence the proposition will follow if we show that ψ^{-1} is continuous.

To see that ψ^{-1} is continuous, it is sufficient to show that the composition:

$$\Phi J(X) \longrightarrow E(J(X)) \xrightarrow{\rho_{JX}} B(J(X)) \xrightarrow{\psi^{-1}} \Sigma X$$

is continuous. This map is given by the rule:

$$(t, x_1, \ldots, x_n) \longrightarrow \overline{(t - \sum_{j=1}^{i-1} \nu(x_j), x_i)} \ ,$$

where

$$\sum_{j=1}^{i-1} \nu(x_j) \leq t \leq \sum_{j=1}^{i} \nu(x_j) \ .$$

This can easily be seen to be the colimit of a coherent system of maps on the system given in 8.1. Hence ψ^{-1} is continuous and the proposition is proved.

Definition 8.6: Let $M \in Mon^{\nu}$. M is cellular means that M is a "free H-space with adapted CW-structure" in the terminology of Milgram ([2], p.387). That is, M is free as a monoid (though not necessarily as a topological monoid) and M has a CW-structure that is compatible with the monoid structure. I.e., if $\mu: M \times M \to M$ is the multiplication in M, and e and f are cells of dim n and m in M, then μ is a relative homeomorphism of the interior of $e \times f$ onto the interior of some $(n+m)$-cell in M.

Proposition 8.7: (Milgram): Let $M \in \mathrm{Mon}^{\vee}$ such that M is cellular and connected. Then $\rho_M: E(M) \to B(M)$ is a quasi-fibration.

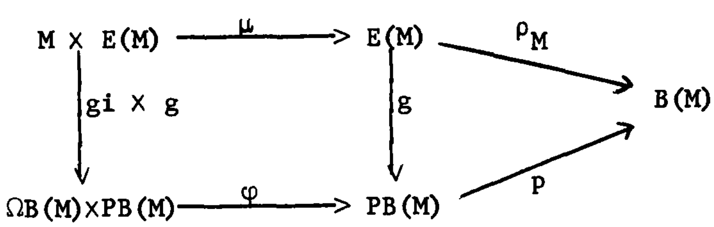
<u>Proof</u>: Since M is cellular, M, considered as a set theoretic monoid, is free. This is enough to ensure that the fibre of ρ_M over any point b in $B(M)$ is M. For the remainder of the proof see [2] Theorem 2.4.

<u>Proposition 8.8</u>: Let $i: M \to E(M)$ be the inclusion given by $i(m) = \overline{(\nu(m),m)}$. Let $PB(M)$ denote the Moore space of variable length paths in $B(M)$. Then there is a map $g: E(M) \to PB(M)$ such that

1) gi maps M into $\Omega B(M)$,

2) gi is adjoint to the identity map on $B(M)$,

3) the following diagram commutes:



where φ is composition of paths and p is projection onto the endpoint.

4) gi is a homotopy equivalence.

5) g and gi are natural in M.

Proof: Define $g: E(M) \to PB(M)$ by

$$g(\overline{(t,m)})(s) = \begin{cases} \rho_M\overline{(t,m)}, & t \leqq s \\ \rho_M\overline{(s,m)}, & s \leqq t \end{cases} .$$

It is easily checked that g is well-defined, continuous, and satisfies properties 1, 2 and 3 above. To see that gi is a homotopy equivalence, we note that, first, since ρ is a quasifibration, we have a long exact sequence of homotopy groups:

$$\longrightarrow \pi_{i+1}(B(M)) \xrightarrow{\xi} \pi_i(M) \xrightarrow{i_*} \pi_i(E(M)) \xrightarrow{\rho_*} \pi_i(B(M)) \xrightarrow{\xi} \pi_{i-1}(M)$$

where $\xi: \pi_{i+1}(B(M)) \to \pi_i(M)$ is the composition

$$\pi_{i+1}(B(M)) \xrightarrow[\simeq]{\rho_*^{-1}} \pi_{i+1}(E(M),M) \xrightarrow{\partial} \pi_i(M) \quad .$$

The commutativity of the diagram in 3) gives us a morphism of long exact sequences:

$$\begin{array}{ccccccccc}
\longrightarrow \pi_{i+1}(B(M)) & \xrightarrow{\xi} & \pi_i(M) & \xrightarrow{i_*} & \pi_i(E(M)) & \xrightarrow{\rho_*} & \pi_i(B(M)) & \longrightarrow \\
\Vert & & \downarrow (gi)_* & & \downarrow g_* & & \Vert & \\
\longrightarrow \pi_{i+1}(B(M)) & \xrightarrow{\partial} & \pi_i(\Omega(M)) & \longrightarrow & \pi_i(PB(M)) & \longrightarrow & \pi_i(B(M)) & \longrightarrow
\end{array}$$

And finally, $E(M)$ is contractible by the map $I \times E(M) \to E(M)$ by $(t,\overline{(s,m)}) \to \overline{((1-s)t,m)}$ and hence by the 5-lemma, $(gi)_*$ is an isomorphism.

Hence gi is a weak homotopy equivalence and since M is a CW-complex, gi is a homotopy equivalence.

<u>Theorem 8.9</u> (James): If $X \in Top^{\vee}$ and is a connected CW-complex, then there is a homotopy equivalence $J(X) \simeq \Omega\Sigma(X)$,

<u>Proof</u>: This is an immediate consequence of 8.5 and 8.8(4).

Chapter 9. The Functors J and Σ

Proposition 9.1: The functor $J\colon Top^{\vee} \to Mon^{\vee}$ (as defined in 8.1) is a (PJH)-functor (as defined in 7.3).

Proof: Since J is coadjoint to the forgetful functor: $Mon^{\vee} \to Top^{\vee}$, it follows that J is cocontinuous. Let $X,Y \in Top^{\vee}$. Then define $\Lambda_J(X,Y)\colon J(X \times Y) \to J(X) \coprod J(Y)$ as follows: first, for $(x,y) \in X \times Y$ set $\Lambda_J(X,Y)(x,y) = \bar{x} \circ \bar{y}$, where $\bar{x}$ is the image of x under the map $X \to J(X)$ and "$\circ$" is multiplication in the monoid $J(X) \coprod J(Y)$. (Recall that the coproduct in $Mon^{\vee}$ is the free product.) Extend $\Lambda_J(X,Y)$ to all of $J(X \times Y)$ by the universality of J.

It is easily verified that Λ_J is a natural transformation of bifunctors satisfying i) and ii) of 7.3. Hence J is a (PJH)-functor.

Proposition 9.2: The functor $\Sigma\colon Top^{\vee} \to Top^{\circ}$ (as described in 8.1) is a (PJH)-functor.

Proof: Let $\bar{\Omega}\colon Top^{\circ} \to Top^{\vee}$ be the composition of Ω with the forgetful functor. Then $\bar{\Omega}$ is adjoint to Σ. Hence Σ is cocontinuous.

Let $X,Y \in Top^{\vee}$. Define $\Lambda_{\Sigma}(X,Y)\colon \Sigma(X \times Y) \to \Sigma X \vee \Sigma Y$ by the following rule:

$$\Lambda_{\Sigma}(X,Y)\overline{(t,(x,y))} = \begin{cases} \overline{(t,x)}, & \text{if } 0 \le t \le \|x\| \\ \overline{(t-\|x\|,y)}, & \text{if } \|x\| \le t \le \|x\| + \|y\| . \end{cases}$$

This is clearly a natural transformation of bifunctors satisfying i) and ii) of 7.3. Hence Σ is a (PJH)-functor.

Proposition 9.3: The following diagram commutes, $\forall\, X,Y \in Top^{\vee}$:

$$\begin{array}{ccc}
BJ(X\times Y) & \xrightarrow{B(\Lambda_J)} & B(J(X) \coprod J(Y)) \\
 & & \Big\downarrow g \ \cong \\
\Psi_{X\times Y} \Big\uparrow \cong & & BJ(X) \vee BJ(Y) \\
 & & \Psi_X \vee \Psi_Y \Big\uparrow \cong \\
\Sigma(X\times Y) & \xrightarrow{\Lambda_\Sigma} & \Sigma X \vee \Sigma Y
\end{array} \quad ;$$

where Ψ is the isomorphism of Proposition 8.5, and g is the isomorphism arising from the cocontinuity of B .

Proof: This is an immediate consequence of the definition of the various maps involved.

Proposition 9.4: Let $\mathfrak{m}$ be an enriched free monoidal category. Let $F: \mathfrak{m} \to (\mathrm{Top}^{\vee}, \times)$ be a monoidal functor. Then there is a Top°-isomorphism (natural in F): $B(\operatorname{colim}_{\mathfrak{m}} F_J) \cong \operatorname{colim}_{\mathfrak{m}} F_\Sigma$. (See 7.4).

Proof: By Proposition 9.3 the following diagram commutes, $\forall\ \alpha \in \mathfrak{m}$:

$$\begin{array}{ccc}
B(JF(\alpha)) & \xrightarrow{B(\bar{\Lambda}_J(\alpha))} & B(JF^{\sqcup}(\alpha)) \\
\Big\uparrow \cong & & \Big\uparrow \cong \\
\Sigma F(\alpha) & \xrightarrow{\bar{\Lambda}_\Sigma(\alpha)} & \Sigma F^{\sqcup}(\alpha)
\end{array} \quad .$$

(See 7.4 for the notation.)

Then it follows, since B is cocontinuous, that $BF_J \cong F_\Sigma$. Therefore, applying cocontinuity once again, we have that $B(\operatorname{colim}_{\mathfrak{m}} F_J) \cong$

$\operatorname{colim}_{\mathfrak{m}} BF_J \cong \operatorname{colim}_{\mathfrak{m}} F_\Sigma$.

<u>Remark 9.5</u>: In following chapters, it will be convenient to iterate the functor Σ . In order to make this possible, we somewhat arbitrarily define a "standard" norm on ΣX as follows: If $\overline{(t,x)} \in \Sigma X$, then $\|\overline{(t,x)}\| = 4t(\|x\| - t)$.

Note that this norm is natural in X and permits the definition of a "canonical inclusion" $X \to \Sigma X$ by $x \mapsto \overline{(\|x\|/2,x)}$. This is a $Top^\vee$-morphism.

It is clear that we can now iterate the functor Σ and that if X has a good basepoint, S^nX and Σ^nX are of the same homotopy type, $\forall\, n \geq 1$.

Chapter 10. Continuous Families of Convex Compact Subsets of $\mathbb{R}^n$

Definition 10.1: Let $\mathbb{K}^n$ denote the set of convex, compact subsets of $\mathbb{R}^n$. Define the Hausdorff metric D on $\mathbb{K}^n$ as follows:

If $X \in \mathbb{K}^n$ and $\epsilon > 0$, let $X_\epsilon = \{r \in \mathbb{R}^n | d(r,X) \leq \epsilon\}$ where d is the usual distance metric on $\mathbb{R}^n$. Then if $X,Y \in \mathbb{K}^n$, we set $D(X,Y) = \inf\{\epsilon > 0 | X \subseteq Y_\epsilon$ and $Y \subseteq X_\epsilon\}$. This can easily be shown to define a metric on $\mathbb{K}^n$. We consider $\mathbb{K}^n$ to be topologized by this metric.

We will let $\overline{\mathbb{K}}^n$ denote the space of compact convex bodies in $\mathbb{R}^n$, i.e., $\overline{\mathbb{K}}^n = \{A \in \mathbb{K}^n | \text{int } A \neq \emptyset\}$.

If $A \in \mathbb{K}^n$, we will let $c(A)$ denote the centroid of A. Then $c: \mathbb{K}^n \to \mathbb{R}^n$ is a continuous map. If $\mathbb{K}_0^n = \{A \in \mathbb{K}^n | c(A) = \text{origin}\}$ it is easy to see that $\mathbb{K}_0^n$ is a strong deformation retract of $\mathbb{K}^n$. The retraction, $g: \mathbb{K}^n \to \mathbb{K}_0^n$ is given by translation to the origin. Indeed, one has $\mathbb{K}^n \cong \mathbb{K}_0^n \times \mathbb{R}^n$.

Definition 10.2: Let X be a topological space and $f: X \to \mathbb{K}^n$ a continuous map. Then X_f denotes the graph of f; that is,

$$X_f = \{(x,v) \in X \times \mathbb{R}^n | v \in f(x)\}.$$

Proposition 10.3: Let $f: X \to \mathbb{K}^n$ be continuous and let $\bar{f}$ be the composition

$$X \longrightarrow \mathbb{K}^n \xrightarrow{g} \mathbb{K}_0^n \subseteq \mathbb{K}^n.$$

Then there is a homeomorphism $X_f \cong X_{\bar{f}}$.

Proof: There is an obvious isomorphism $A \to g(A)$, $\forall A \in \mathbb{K}^n$. These isomorphisms clearly induce a homeomorphism $X_f \cong X_{\bar{f}}$.

Proposition 10.4: Let A be a convex m-cell. Let $a_0 \in \text{int } A$. Let

$f\colon A \to \mathbb{K}_0^n$ be a continuous map such that

1) $f(a) \in \overline{\mathbb{K}^n}$, $\forall\ a \in \text{int}(A)$,

2) if $0 \le s \le 1$ and $a \in A$, then $f(a) \subseteq f(sa+(1-s)a_0)$;

then A_f is an $(n+m)$-cell.

<u>Proof</u>: The proof depends on the following lemma.

<u>Lemma 10.5</u>: Let X be a compact subset of $\mathbb{R}^k$. Let $x_0 \in \text{int}(X)$ such that $\forall\ x \in X$ and $\forall\ t \in [0,1)$, $tx + (1-t)x_0 \in \text{int}(X)$. Then X is a k-cell.

<u>Proof</u>: We will assume, without loss of generality, that x_0 is the origin of $\mathbb{R}^k$. Define $\nu\colon X - \{0\} \to \mathbb{R}^k$ by $\nu(x)$ is that y in $\text{Bd}(X)$ such that $y = \alpha x$, for some $\alpha \ge 1$. The hypothesis insures that $\nu(x)$ is well-defined. To see that ν is continuous one notes that if $x_i \to x$ in $X - \{0\}$, then the sequence $\{\nu(x_i)\}_i$ is bounded and any convergent subsequence of $\{\nu(x_i)\}_i$ must converge, on the one hand, to a point on the ray from 0 through x and, on the other hand, to a point in $\text{BD}(X)$. Hence $\{\nu(x_i)\}_i \to \nu(x)$.

Let $D^k = \{v \in \mathbb{R}^k \mid \|v\| \le 1\}$. Define $\eta\colon X \to D^k$ by

$$\eta(x) = \begin{cases} \dfrac{x}{\|\nu(x)\|} & , x \ne 0 \\ 0 & , x = 0 \end{cases} ,$$

then η is continuous since $\|\nu(x)\|$ is bounded away from 0 . Clearly η is bijective. Therefore X is homeomorphic to D^k and thus a k-cell.

We now return to the proof of 10.4. We can assume that A is a convex subset of $\mathbb{R}^m$ and that a_0 is the origin. Then A_f is a compact subset of $\mathbb{R}^{n+m}$.

Let $(a,v) \in A_f$, let $s \in [0,1)$; then $s(a,v) = (sa,sv)$ and $sv \in \text{int } f(a) \subseteq \text{int } f(sa)$. Therefore A_f satisfies the hypothesis of Lemma 10.5 and is therefore an $(n+m)$-cell.

<u>Lemma 10.6</u>: There is a continuous map $\rho: \mathbb{K}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\rho(A,v) \in A$, $\forall (A,v) \in \mathbb{K}^n \times \mathbb{R}^n$ and $\rho(A,v) = v$ if $v \in A$.

<u>Proof</u>: First define $\delta: \mathbb{K}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\delta(A,v)$ = distance from v to A . I.e., $\delta(A,v) = \inf_{x \in A}\{d(x,v)\}$. δ is continuous since, if $D(A,A') < \epsilon$ and $d(v,v') < \epsilon$, then $\delta(A,v) < \delta(A',v') + 2\epsilon$ and $\delta(A',v') < \delta(A,v) + 2\epsilon$, by a simple application of the triangle inequality.

Since A is compact, $\exists\ y \in A$ such that $\delta(A,v) = d(y,v)$ and since A is convex, this y is unique for each choice of A and v . Define $\rho(A,v) = y$. This map is certainly the one we need if we can show it to be continuous.

Assume $D(A,A') < \epsilon$ and $d(v,v') < \epsilon$. Then let $y = \rho(A,v)$, $y' = \rho(A',v)$ and further let $d = \delta(A,v) = d(y,v)$ and $d' = \delta(A',v') = d(y',v')$. From our discussion of the function δ we know that $|d - d'| < 2\epsilon$. Since $D(A,A') < \epsilon$, there exists $y'' \in A'$ such that $d(y'',y) < \epsilon$. Let $d'' = d(x',y'')$.

Since y' is the point of A' closest to x' , it follows that the triangle whose vertices are x' , y' and y'' must have an obtuse angle at y' . Then by the law of cosines,

$$d(y',y'') = \sqrt{(d'')^2 - (d')^2 + 2d''d' \cos(y')} \leq \sqrt{(d'')^2 - (d')^2} .$$

But $|d - d''| < 2\epsilon$, so $|d' - d''| < 4\epsilon$. Therefore

$$d(y',y'') < \sqrt{(d' + 4\epsilon^2 - (d')^2} = \sqrt{8\epsilon d' + 16\epsilon^2} .$$

Therefore $d(y,y') \le d(y,y'') + d(y',y'') < \epsilon + \sqrt{8\epsilon d' + 16\epsilon^2}$, and ρ must be continuous. (Note that in the above argument, we assumed that x' , y' and y'' are distinct points. If they are not, the argument is trivial.)

Proposition 10.7: Let $A \xrightarrow{i} Y$ be a cofibration. Let $p: Y \to X$ be continuous. Let $f: X \to \mathbb{R}^n$ be continuous and denote the projection $X_f \to X$ by π_1 . Let $\varphi: A \to X_f$ be a map such that $\pi_1\varphi = pi$. Then $\exists\ \bar{\varphi}: Y \to X_f$ extending φ such that $\pi_1\bar{\varphi} = p$.

Proof: Let $\pi_2: X_f \to \mathbb{R}^n$ be the projection. Then $\pi_2\varphi: A \to \mathbb{R}^n$ can be extended to a map $\hat{\varphi}: Y \to \mathbb{R}^n$ since i is a cofibration and $\mathbb{R}^n$ is contractible. Then define $\bar{\varphi}: Y \to X_f$ by the rule:

$$\bar{\varphi}(y) = (p(y), \rho(f(p(y)), \hat{\varphi}(y))) \quad .$$

This map is the desired extension of φ .

Chapter 11. Properties of the Functor L_X

General Remarks 11.1: In defining the functor L_X in Chapter 4 we assumed a fixed normed space X . It would be just as sensible to define a bifunctor $L\colon \mathrm{Top}^{\vee} \times \mathfrak{G} \to \mathrm{Top}^{\vee}$ by $L(X,\alpha) = L_X(\alpha)$. It is an easy consequence of the definition of L_X that L is functorial in X. In what follows we will assume a fixed X ; however, it should be kept in mind that our constructions are, in fact, functorial in X .

We now proceed to analyze the structure of L_X more carefully.

Notation 11.2: Let $\alpha \in \mathfrak{G}$, i.e., α is a surjection $\underline{m} \to \underline{k}$. Let $\varphi \in \mathfrak{G}(\alpha, e_n)$, i.e., φ is a monomorphism $\underline{m} \to \underline{n}$ with certain properties. Let $I_j^{\varphi} = \{i \in \underline{n} \mid \alpha(\varphi^{-1}(i)) \leqq j\}$ for $1 \le j \le k$. Let $I_0^{\varphi} = \{i \in \underline{n} \mid i \notin \mathrm{Im}\ \varphi\}$. Finally, let $L_X(e_n)_{\varphi} = \{(t,x) \in L_X(e_n) \mid \|t\|_{I_j^{\varphi}} = (\|x\|_{I_j^{\varphi}})^2$, $i \le j \le k$; and $x_i = *$, $\forall\ i \in I_0^{\varphi}\}$ where the notation is as in Proposition 4.2.

Proposition 11.3: $L_X(\varphi)\colon L_X(\alpha) \to L_X(e_n)$ is a homeomorphism onto $L_X(e_n)_{\varphi}$.

Proof: A) First we must show that $\mathrm{Im}(L_X(\varphi)) \subseteqq L_X(e_n)_{\varphi}$.

For this purpose it is clearly sufficient to assume that $\alpha = e_p \otimes e_q$, for some non-negative p and q such that $p + q \leqq n$, since every $\mathfrak{G}$-morphism with target e_n factors through such an object.

Therefore let $J_1 = \{1,\dots,p\}$, $J_2 = \{p+1,\dots,p+q\}$ and we may set $L_X(\alpha) = \{(t,x) \in (\mathbb{R}^+ \times X)^{p+q} \mid \|t\|_I \geqq (\|x\|_I)^2$ if $I \subseteq J_i$, $i = 1$ or 2 ; and $\|t\|_{J_i} = (\|x\|_{J_i})^2$ for $i = 1$ or $2\}$. If $(t,x) \in L_X(\alpha)$ then

$L_X(\varphi)(t,x) = (s,y) \in L_X(e_n)$ where:

$$(s_j,y_j) = \begin{cases} (0,*) \ ; & j \notin \operatorname{Im}\varphi \\ (t_i,x_i) \ ; & i \in J_1 \text{ and } j = \varphi(i) \\ (t_i+2\|x\|_{J_1}\|x_i\|,x_i) \ ; & i \in J_2 \text{ and } j = \varphi(i) \ . \end{cases}$$

but then $I_1^\varphi = \varphi(J_1)$ and $I_2^\varphi = \varphi(J_1 \cup J_2)$ and it follows immediately from the definition of L_X that $\|t\|_{I_1^\varphi} = (\|x\|_{I_1^\varphi})^2$ and immediately from Proposition 4.2 that $\|t\|_{I_2^\varphi} = (\|x\|_{I_2^\varphi})^2$. Thus $\operatorname{Im}(L_X(\varphi)) \subseteq L_X(e_n)_\varphi$.

B) Now we wish to construct an inverse for $L_X(\varphi)$ from $L_X(e_n)_\varphi$ to $L_X(\alpha)$. Again we can assume that $\alpha = e_p \otimes e_q$, $p+q \leqq m$, for the factorization of any $\mathfrak{G}$-morphism into products and compositions of 2-shuffles and identity maps gives an obvious induction process.

Having made this assumption we can now define $\psi\colon L_X(e_n)_\varphi \to (\mathbb{R}^+ \times X)^{p+q}$ by $\psi(s,y) = (t,x)$, where

$$(t_i,x_i) = \begin{cases} (s_{\varphi(i)},y_{\varphi(i)}), & i \le i \le p \ ; \\ (s_{\varphi(i)} - 2\|y\|_{I_1^\varphi}\cdot\|y_{\varphi(i)}\|,y_{\varphi(i)}), & p+1 \le i \le p+q \ . \end{cases}$$

To see that this is a well-defined map into $L_X(\alpha)$, one must do some straightforward but tedious checking of the appropriate inequalities. This is omitted.

It is clear that ψ and $L_X(\varphi)$ are inverse to each other. Hence the lemma is true.

<u>Proposition 11.4</u>: $L_X\colon \mathfrak{G} \to \mathrm{Top}^\nu$ has the following "faithfulness" property:

If $\varphi \in (\mathfrak{S},a)$, we set $P(\varphi)$ = set of all predecessors of φ in $(\mathfrak{S},a)$. Then if $\varphi_1,\varphi_2 \in (\mathfrak{S},a)$, we have

$$\mathrm{Im}(L_X\varphi_1) \cap \mathrm{Im}(L_X\varphi_2) = \bigcup_{\lambda\in P(\varphi_1)\cap P(\varphi_2)} \mathrm{Im}(L_X\lambda) \quad .$$

Proof: First, it is sufficient to prove the theorem for the case $a = e_n$. This simplification is an immediate consequence of the following easy facts: If $a = e_{n_1} \otimes \ldots \otimes e_{n_k}$, and if $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_k$ and $\lambda = \lambda_1 \otimes \ldots \otimes \lambda_k$ are in $(\mathfrak{S},a)$, then $\lambda \in P(\varphi)$ if and only if $\lambda_i \in P(\varphi_i)$, $\forall_i$. $L_X(a) = L_X(e_{n_1}) \times \ldots \times L_X(e_{n_k})$ and $\mathrm{Im}(L_X\varphi) = \mathrm{Im}(L_X\varphi_1) \times \ldots \times \mathrm{Im}(L_X\varphi_k)$.

Hence we will assume that $a = e_n$. The inclusion:

$$\bigcup_{\lambda\in P(\varphi_1)\cap P(\varphi_2)} \mathrm{Im}(L_X\lambda) \subseteq \mathrm{Im}(L_X\varphi_1) \cap \mathrm{Im}(L_X\varphi_2)$$

follows from functoriality of L_X . We wish therefore to show the inclusion in the other direction.

We will need the following lemma.

Lemma 11.4.1: Let $(t,x) \in L_X(e_n)$. Let K,K' be subsets of $\underline{n}$ such that $\|t\|_K = \|x\|_K^2$, $\|t\|_{K'} = \|x\|_{K'}^2$ and $\forall\ i \in K \cup K'$, $x_i \neq *$. Then either $K \subseteq K'$ or $K' \subseteq K$.

Proof of Lemma: Let $I = K \cap K'$, $J = K - (K \cap K')$, $J' = K' - (K \cap K')$. Then we wish to show that either J or J' is empty. Then

$$\|t\|_{K\cup K'} = \|t\|_K + \|t\|_{K'} - \|t\|_I$$
$$= \|x\|_K^2 + \|x\|_{K'}^2 - \|t\|_I \leq \|x\|_K^2 + \|x\|_{K'}^2 - \|x\|_I^2 ,$$

so $\|t\|_{K\cup K'} \leqq (\|x\|_I + \|x\|_J)^2 + (\|x\|_I + \|x\|_{J'})^2 - \|x\|_I^2$.

But also $\|t\|_{K\cup K'} \geqq (\|x\|_{K\cup K'})^2 = (\|x\|_I + \|x\|_J + \|x\|_{J'})^2$, so $(\|x\|_I + \|x\|_J + \|x\|_{J'}) \leqq (\|x\|_I + \|x\|_J)^2 + \|x\|_I + \|x\|_{J'})^2 - \|x\|_I^2$ which gives $2\|x\|_J\|x\|_{J'} \leq 0$.

Therefore either $J = \emptyset$ or $J' = \emptyset$ and the lemma is proved. □

Now returning to the proof of the proposition, let $\varphi_1, \varphi_2 \in (\mathcal{O}, e_n)$ and let $(t,x) \in \mathrm{Im}(L_X\varphi_1) \cap \mathrm{Im}(L_X\varphi_2)$. Let I^1 and I^2 be the admissible filtrations of $\underline{n}$ (of lengths k_1 and k_2 respectively) associated with φ_1 and φ_2 , as in 6.4.

By Proposition 11.3, $\mathrm{Im}(L_X\varphi_i) = L_X(e_n)\varphi_i$ for $i = 1,2$ and $(t,x) \in \mathrm{Im}(L_X\varphi_i) \Leftrightarrow \|t\|_{I^i_j} = \|x\|_{I^i_j}$, $\forall\ I^i_j \in I^i$. Let $J = \{j \in \underline{n} \mid x_j = *\}$. For every $I^i_j \in I^i$, $i = 1,2$, let $\hat{I}^i_j = I^i_j - J$, then $\|t\|_{\hat{I}^i_j} = (\|x\|_{\hat{I}^i_j})^2$. Let $K = \{\hat{I}^i_j \mid j = 1,\ldots,k_i;\ i = 1,2\}$. Then the lemma says that K can be simply ordered by inclusion,yielding an admissible filtration of $\underline{n}$. This filtration is clearly a refinement of I^1 and of I^2 .

Let λ be the element of $(\mathcal{O}, e_n)$ associated with K . Then λ is a predecessor of φ_1 and of φ_2 and it is clear that $(t,x) \in L_X(e_n)_\lambda = \mathrm{Im}(L_X\lambda)$. Therefore,

$$\mathrm{Im}(L_X\varphi_1) \cap \mathrm{Im}(L_X\varphi_2) \subseteq \bigcup_{\lambda \in P(\varphi_1) \cap P(\varphi_2)} \mathrm{Im}(L_X\lambda) . \qquad \square$$

<u>Corollary 11.5</u>: Let $\partial L_X(e_n)$ denote the set $\{(t,x) \in L_X(e_n) \mid \|t\|_I = (\|x\|_I)^2$ for some $I \subsetneqq \underline{n}\}$. Then the induced map:

$$\underset{\overline{(\mathcal{O}, e_n)}}{\mathrm{colim}}\ L_X \longrightarrow L_X(e_n)$$

is a homeomorphism onto $\partial L_X(e_n)$.

Proof: This is an immediate consequence of propositions 11.3 and 11.4.

General Remarks 11.6: For the remainder of this chapter we shall assume that X is a CW-complex with standard norm as described in Definition 5.2. It will simplify several descriptions if we consider the base-point of X to be a cell of dimension -1 .

The following paragraphs will establish and describe the structure of $L_X(\alpha)$ as a CW-complex, $\forall\ \alpha \in \mathcal{O}$.

Definition 11.7: Let $\lambda \in \mathcal{O}(\alpha,\beta)$ where β has grade n .

Let E_i be a cell of X of dim d_i for $i = 1,\dots,n$. Assume that if $i \notin \mathrm{Im}(\lambda) \leq \underline{n}$, then $E_i = *$. Let $f_i\colon E_i \to X$ be the attaching map for E_i . Let $\pi\colon L_X(\beta) \to X^n$ be the projection. Then define $L(\lambda;E_1,\dots,E_n)$ as the pullback:

$$\begin{array}{ccc} L(\lambda;E_1,\dots,E_n) & \xrightarrow{\ f^*\ } & L_X(\alpha) \\ & & \Big\downarrow L_X(\lambda) \\ \Big\downarrow \pi^* & & L_X(\beta) \\ & & \Big\downarrow \pi \\ E_1\times\dots\times E_n & \xrightarrow{\ f_1\times\dots\times f_n\ } & X^n \end{array}$$

If $\lambda \in \mathcal{O}(\alpha,\beta)$, then we define the dimension of λ (dim λ) to be (grade of β) - (range of α) .

Lemma 11.8: $L(\lambda;E_1,\dots,E_n)$ is a cell of dimension $d_1 + \dots + d_n + \dim\lambda$.

Proof: Let m = grade of α . Then there is an obvious homeomorphism $L(\lambda;E_1,\dots,E_n) \to L(1_\alpha;E_{\lambda(1)},\dots,E_{\lambda(m)})$ where we think of λ as a monomorphism from $\underline{m}$ to $\underline{n}$. Further, if $\alpha = \alpha_1 \otimes \alpha_2$, with

m_1 = grade (α_1) , then $L(1_\alpha;E_1,\ldots,E_m) \cong L(1_{\alpha_1};E_1,\ldots,E_{m_1}) \times L(1_{\alpha_2};E_{m_1+1},\ldots,E_m)$. Therefore it is sufficient to prove the lemma for the case $\lambda = 1_{e_n}$. We can also assume that $E_i \neq *$, $\forall\ i$. For if not, we can apply an easy induction from a lower dimensional case.

We will consider $\mathbb{R}^{n-1}$ as the hyperplane of $\mathbb{R}^n$ defined by the equation $r_1 + \ldots + r_n = 0$. If $x = (x_1,\ldots,x_n) \in X^n$, let $L(x) = \pi^{-1}(x)$, where π is the projection $L_X(e_n) \to X^n$. Let $T_x\colon \mathbb{R}^n \to \mathbb{R}^n$ be the translation:

$$r_i \longmapsto r_i - (\|x\| \cdot \|x_i\|), \quad i = 1,\ldots,n .$$

Define $F\colon E_1 \times \ldots \times E_n \to \mathbb{K}_0^{n-1}$ by $F(y) = T_{f(y)}(L(f(y)))$ where $f = f_1 \times \ldots \times f_n\colon E_1 \times \ldots \times E_n \to X^n$. We must show that this makes sense.

From the definition of $L_X(e_n)$, the following facts can easily be deduced:

1) $L(x)$ is a convex subset of the hyperplane of $\mathbb{R}^n$ defined by $r_1 + \ldots + r_n = \|x\|^2$.

2) The centroid of $L(x) = (\|x_1\|\ \|x\|,\ldots,\|x_n\|\ \|x\|)$. (This follows from consideration of the symmetry of L_X ; i.e., the above point lies midway between each pair of limiting hyperplanes).

3) If x , $x' \in X^n$ such that $\|x_i\| \geq \|x_i'\|$ for each $i \in \underline{n}$, then $T_x(L(x)) \supseteq T_{x'}(L(x'))$.

Then it follows from 1) and 2) that $F(y) \in \mathbb{K}_0^{n-1}$. It is obvious that $L\colon X^n \to \mathbb{K}^n$ and $T\colon X^n \to (\mathbb{R}^n, \mathbb{R}^n)$ are continuous, so F must be continuous.

Then $L(1_{e_n};E_1,\ldots,E_n)$ is the graph of F (in the sense of 10.2). The norm of $E_1 \times \ldots \times E_n$ is induced by the norm on X^n .

Let z_i be the center of E_i and let $z = (z_1,\dots,z_n) \in E_1 \times \dots \times E_n$. Since X has the standard norm, as defined in 5.2, it follows that if $a = (a_1,\dots,a_n) \in E_1 \times \dots \times E_n$ and if $0 \le s \le 1$, then $\|a_i\| \le \|sa_i + (1-s)z_i\|$, $\forall\, i \in \underline{n}$. Then, from fact 3) above it follows that $F(a) \subseteq F(sa + (1-s)z)$.

From the definition of $L_X(e_n)$ we can also readily see that L_X is of dimension $(n-1)$ unless $x_i = *$ for some $i \in \underline{n}$. Hence $F(y)$ will be $(n-1)$dimensional for all $y \in \operatorname{int}(E_1 \times \dots \times E_n)$.

Therefore F fits the hypothesis of Proposition 10.4 and it follows that $L(1_{e_n};E_1,\dots,E_n)$ is a cell of dimension $d_1 + \dots + d_n + (n-1)$. This establishes the lemma. □

This lemma surmounts the only real difficulty in the proof of the following theorem.

Theorem 11.9: Let $\alpha \in \mathfrak{G}$ of grade n. Then $L_X(\alpha)$ is a CW-complex. The cells of $L_X(\alpha)$ are exactly those cells $L(\lambda;E_1,\dots,E_n)$ where $\lambda \in (\mathfrak{G},\alpha)$ and $\langle E_1,\dots,E_n\rangle$ is an n-tuple of cells of X such that $E_i = *$ if and only if $i \notin \operatorname{Im}\lambda \subseteq \underline{n}$. Further, if $\varphi \in \mathfrak{G}(\alpha,\beta)$ then $L_X\varphi\colon L_X(\alpha) \to L_X(\beta)$ is the inclusion of a subcomplex.

Proof: The proof is a straightforward verification of the necessary facts. The problems are essentially notational rather than conceptual and it is doubtful that an attempt to write down the details would be of much use to anyone.

Corollary 11.10: If C is any initial subcategory of $(\mathfrak{G},\alpha)$ then the induced map: $\operatorname{colim}_C L_X \to L_X(\alpha)$ is a cofibration. In particular, $L_X(\varphi)$ is a cofibration, for every $\mathfrak{G}$-morphism φ.

Corollary 11.11: $\operatorname{colim}_{\mathfrak{G}} L_X$ is a cellular monoid.

Proof: $\operatorname{Colim}_{\mathfrak{G}} L_X$ is the colimit of a system whose objects are CW-complexes and whose morphisms are inclusions of subcomplexes. Hence $\operatorname{colim}_{\mathfrak{G}} L_X$ is a CW-complex.

To see that the cell structure of $\operatorname{colim}_{\mathfrak{G}} L_X$ is adapted to the monoid structure, one must merely look at the definition of the monoid structure on $\operatorname{colim}_{\mathfrak{G}} L_X$. If e is a cell of $L_X(\alpha)$ and f is a cell of $L_X(\beta)$, then their product in the monoid structure is just the cell $e \times f$ in $L_X(\alpha) \times L_X(\beta) = L_X(\alpha \otimes \beta)$.

In Chapter 14, we will need the following rather technical lemmas.

Lemma 11.12: If $a \in \mathfrak{G}$ and m = grade of a, then the projection $L_X(a) \xrightarrow{\pi_a} X^m$ is a homotopy equivalence.

Proof: Define $\sigma_n : X^n \to L_X(e_n)$ by the rule $\sigma_n(x_1,\ldots,x_n) = (\|x\|\cdot\|x_1\|,x_1,\ldots,\|x\|\cdot\|x_n\|,x_n)$. Then σ_n is a left inverse for π_{e_n} and $\sigma_n(X^n)$ is clearly a strong deformation retract of $L_X(e_n)$. Hence π_{e_n} is a homotopy equivalence.

If $a \in \mathfrak{G}_n$, then $a = e_{i_1} \otimes \ldots \otimes e_{i_k}$ and $\pi_a = \pi_{e_{i_1}} \times \ldots \times \pi_{e_{i_k}}$, hence π_a is the product of homotopy equivalences and therefore a homotopy equivalence.

Corollary 11.13: Using the notation of 11.12, if $Y \subseteq X^m$, then

$$\pi_a\big|_{\pi_a^{-1}(Y)} : \pi_a^{-1}(Y) \to Y$$

is a homotopy equivalence.

<u>Proof</u>: The proof of 11.12 can be applied without modification.

<u>Lemma 11.14</u>: Let $a \in \mathcal{S}$. Let $C_a = \bigcup_{\lambda \in \overline{(\mathcal{S},a)} - (\mathcal{S},a)_0} \operatorname{Im} L_X(\lambda) \subseteq L_X(a)$.

If $\beta \in (\mathcal{S},a)_0$, let $F_\beta = \operatorname{Im} L_X(\beta) \cup C_a$. Let f_β be the composition:

$$F_\beta \subseteq L_X(a) \xrightarrow{\pi_a} X^k \quad .$$

Then f_β is a homotopy equivalence.

<u>Proof</u>: Let $W = \{x \in X^k \mid x_i = * \text{ for some } i\}$. Then $C_a = \pi_a^{-1}(W)$.

Let b be the domain of β . Then we have a commutative diagram:

$$\begin{array}{ccc} L_X(b) & \xrightarrow{\cong} & \operatorname{Im} L_X(\beta) \\ \Big\downarrow \pi_b & & \Big\downarrow \pi_a|_{\operatorname{Im}(L_X(\beta))} \\ X^k & \xrightarrow[\beta^*]{\cong} & X^k \end{array}$$

where β^* is the permutation of coordinates given by β . But then we have $\operatorname{Im} L_X(\beta) \cap C_a \cong \pi_b^{-1}(W)$. From 11.12 and 11.13 it follows that $\pi_a|_{\operatorname{Im} L_X(\beta)}$, $\pi_a|_{C_a}$ and $\pi_a|_{C_a \cap \operatorname{Im} L_X(\beta)}$ are homotopy equivalences.

Since C_a and $\operatorname{Im}(L_X(\beta))$ are subcomplexes of $L_X(\alpha)$ and W is a subcomplex of X^k , $f_\beta = \pi_a|_{C_a \cup \operatorname{Im}(L_X(\beta))}$ must be a homotopy equivalence.

<u>Lemma 11.15</u>: Let $a \in \mathcal{S}$, let $\beta \in (\mathcal{S},a)_0$ and let $\mathcal{B}_\beta = \{\lambda \in (\mathcal{S},a)_0 \mid \lambda$ and β are coherent$\}$. Let $\mathcal{A} \subseteq \mathcal{B}_\beta$ such that $\beta \in \mathcal{A}$. Let $F_{\mathcal{A}} = C_a \cup \bigcup_{\lambda \in \mathcal{A}} \operatorname{Im} L_X(\lambda)$. Then $f_{\mathcal{A}} = \pi_a|_{F_{\mathcal{A}}} : F_{\mathcal{A}} \to X^k$ is a homotopy equivalence.

<u>Proof</u>: The proof is an induction on $|\mathcal{A}|$.

If $|\mathcal{A}| = 1$, then $\mathcal{A} = \{\beta\}$ and $F_{\mathcal{A}} = F_\beta$ and Lemma 11.14 gives the result.

Assume $|\mathfrak{a}| > 1$ and the lemma is true for all a', β' and $\mathfrak{a}'$ such that $|\mathfrak{a}'| < |\mathfrak{a}|$. Let $\alpha \in \mathfrak{a}$ such that $\alpha \neq \beta$, and let $\bar{\mathfrak{a}} = \mathfrak{a} - \{\alpha\}$. Then $f_{\bar{\mathfrak{a}}} : F_{\mathfrak{a}} \xrightarrow{\simeq} X^k$ by induction, and $F_\alpha : F_\alpha \xrightarrow{\simeq} X^k$ by Lemma 11.14. Since $F_{\mathfrak{a}} = F_{\bar{\mathfrak{a}}} \cup F_\alpha$ and $F_{\bar{\mathfrak{a}}}$ and F_α are subcomplexes of $F_{\mathfrak{a}}$, it will be sufficient to show that $f_{\mathfrak{a}}|_{F_{\bar{\mathfrak{a}}} \cap F_\alpha} : F_{\bar{\mathfrak{a}}} \cap F_\alpha \to X^k$ is a homotopy equivalence.

Now, $F_{\bar{\mathfrak{a}}} \cap F_\alpha = C_a \cup \bigcup_{\lambda \in \bar{\mathfrak{a}}} (\operatorname{Im} L_X(\alpha) \cap \operatorname{Im} L_X(\lambda))$, but by the definition of coherence and by Proposition 11.4, $\operatorname{Im} L_X(\alpha) \cap \operatorname{Im} L_X(\lambda) \subseteq C_a$ if α and λ are not coherent and $\operatorname{Im} L_X(\alpha) \cap \operatorname{Im} L_X(\lambda) = \operatorname{Im} L_X(\alpha \wedge \lambda)$ if α and λ are coherent. Let $\bar{\mathfrak{a}}_\alpha = \{\lambda \in \bar{\mathfrak{a}} | \lambda \text{ coh } \alpha\}$ (note that $\beta \in \bar{\mathfrak{a}}_\alpha$). Then we have

$$F_{\bar{\mathfrak{a}}} \cap F_\alpha = C_a \cup \bigcup_{\lambda \in \bar{\mathfrak{a}}_\alpha} \operatorname{Im} L_X(\lambda \wedge \alpha) \quad .$$

Let $\mathfrak{a}' = \{\lambda \wedge \alpha | \lambda \in \bar{\mathfrak{a}}_\alpha\}$. By Proposition 6.9, $\mathfrak{a}' \subseteq \mathfrak{B}_{\beta \wedge \alpha}$. Then $F_{\bar{\mathfrak{a}}} \cap F_\alpha = F_{\mathfrak{a}'}$. Clearly, $\beta \wedge \alpha \in \mathfrak{a}'$ and $|\mathfrak{a}'| \leq |\bar{\mathfrak{a}}| < |\mathfrak{a}|$. Hence our induction hypothesis gives that $f_{\mathfrak{a}'}$ is a homotopy equivalence, and therefore $f_{\mathfrak{a}}|_{F_{\bar{\mathfrak{a}}} \cap F_\alpha}$ is also a homotopy equivalence.

This proves the lemma.

Chapter 12. The Category $\mathcal{E}(\mathrm{Top}^\vee)$

Definition 12.1: We define the category $\mathcal{E}(\mathrm{Top}^\vee)$ (which, for the want of a better name, will be called the category of double-edged normed spaces) as follows:

The objects of $\mathcal{E}(\mathrm{Top}^\vee)$ are triples (X, X_0, X_1) where $X \in \mathrm{Top}^\vee$, $X_0 \subseteq X$, $X_1 \subseteq X$ and $* \in X_0 \cap X_1$. The morphisms are $\mathrm{Top}^\vee$-morphisms $\varphi\colon X \to X'$ such that $\varphi(X_0) \subseteq X_0'$ and $\varphi(X_1) \subseteq X_1'$.

Define a bifunctor $\#$ on $\mathcal{E}(\mathrm{Top}^\vee)$ by the rule

$$(X,X_0,X_1) \# (Y,Y_0,Y_1) = ((X \times Y_0) \cup (X_1 \times Y), X_0 \times Y_0, X_1 \times Y_1) .$$

Since $\times$ is coherently associative in $\mathrm{Top}^\vee$ it follows that $\#$ is coherently associative on $\mathcal{E}(\mathrm{Top}^\vee)$. The object $(*,*,*)$ is clearly an initial object and an identity element for $\#$. Hence $(\mathcal{E}(\mathrm{Top}^\vee),\#)$ is a monoidal category.

Let $\Phi\colon \mathrm{Top}^\vee \to \mathcal{E}(\mathrm{Top}^\vee)$ be the functor:

$$\Phi(X) = (\Phi X, \Phi_0 X, \Phi_T X) ,$$

where ΦX is the presuspension of X as defined in 8.1. The norm on ΦX is given by projection onto X and the norm on X .

$$\Phi_0 X = \{(t,x) \in \Phi X \mid t = 0\}$$

and

$$\Phi_T X = \{(t,x) \in \Phi X \mid t = \|x\|\} .$$

Proposition 12.2: $\Phi\colon (\mathrm{Top}^\vee,\times) \to (\mathcal{E}(\mathrm{Top}^\vee),\#)$ is a monoidal functor.

Proof: We must show a natural isomorphism:

$$\Phi(X \times Y) \cong \Phi X \# \Phi Y .$$

We define $\varphi\colon \Phi(X \times Y) \to \Phi X \# \Phi Y$ by

$$\varphi(t,x,y) = \begin{cases} ((t,x),(0,y)) , & 0 \le t \le \|x\| ; \\ ((\|x\|,x),(t-\|x\|,y)), & \|x\| \le t \le \|x\|+\|y\| . \end{cases}$$

φ is clearly well-defined and continuous and natural in X and Y. Define $\psi\colon \Phi X \# \Phi Y \to \Phi(X \times Y)$ as the composition:

$$\Phi X \# \Phi Y \subseteq \Phi X \times \Phi Y \xrightarrow{\bar{\psi}} \Phi(X \times Y)$$

where $\bar{\psi}((t,x),(s,y)) = (t+s,x,y)$. This is clearly continuous and norm preserving.

It is immediate that φ and ψ are inverse to each other and the proposition is therefore true.

<u>Proposition 12.3</u>: Φ is cocontinuous.

<u>Proof</u>: Define the functor $P\colon \mathcal{E}(\mathrm{Top}^{\nu}) \to \mathrm{Top}^{\nu}$ by the rule $P(X,X_0,X_1)$ is the space of Moore paths in X which start in X_0 and terminate in X_1 . The norm is given by path length.

It is straightforward to verify that Φ is coadjoint to P . Therefore Φ is cocontinuous.

<u>Proposition 12.4</u>: Let $T\colon \mathcal{E}(\mathrm{Top}^{\nu}) \to \mathrm{Top}^{\circ}$ be the functor $T(X,X_0,X_1) = X/X_0 \cup X_1$. Then T is a (PJH)-functor in the sense of 7.3.

<u>Proof</u>: First, note that the coproduct in $\mathcal{E}(\mathrm{Top}^{\nu})$ is given by the coproduct of the underlying normed spaces. I.e., $(X,X_0,X_1) \coprod (Y,Y_0,Y_1) = (X \vee Y,\ X_0 \vee Y_0,\ X_1 \vee Y_1)$. Clearly T preserves coproducts.

Define $\Lambda_T\colon T((X,X_0,X_1) \# (Y,Y_0,Y_1)) \to T(X,X_0,X_1) \vee T(Y,Y_0,Y_1)$ by

$$\Lambda_T(x,y) = \begin{cases} \bar{x} & , \ y \in Y_0 \\ \bar{y} & , \ x \in X_1 \end{cases}$$

where $\bar{x}$ is the image of x under the projection:

$$X \to T(X) .$$

Then Λ_T clearly satisfies conditions i) and ii) of 7.3. Hence T is a (PJH)-functor.

Remark 12.5: If $\Sigma\colon \mathrm{Top}^{\vee} \to \mathrm{Top}^{\bullet}$ is the normed suspension functor as described in 8.1, it is clear that there is an equality of functors

$$\Sigma = T\Phi : \mathrm{Top}^{\vee} \to \mathrm{Top}^{\bullet} .$$

Remark 12.6: If X is a CW-complex, then ΦX is a CW-complex and $\Phi_0 X$ and $\Phi_T X$ are subcomplexes.

Chapter 13. The Functor D_X

Definition 13.1: Let $X \in Top^{\vee}$. We define an $\mathfrak{S}$-prefunctor (see 2.5) in $(\mathcal{C}(Top^{\vee}),\#)$ as follows:

$$D_X(e_n) = ((\Phi X)^n, (\Phi_0 X)^n, (\Phi_T X)^n) .$$

Let $\varphi \in \mathfrak{S}(e_n, e_m)$, then define $D_X(\varphi)\colon (\Phi X)^n \to (\Phi X)^m$ by the rule: $D_X(\varphi)(t_1x_1,\dots,t_nx_n) = (s_1y_1,\dots,s_my_m)$, where $(s_i,y_i) = (t_{\varphi^{-1}(i)}, x_{\varphi^{-1}(i)})$ if $i \in \operatorname{Im}\varphi$ and $(0,*)$, if $i \notin \operatorname{Im}\varphi$.

Let $\psi \in \mathfrak{S}(e_n \otimes e_m, e_{n+m})$, then $D_X(\psi)\colon D_X(e_n) \# D_X(e_m) \to D_X(e_{n+m})$ is the composition:

$$((\Phi X)^n \times (\Phi_0 X)^m) \cup ((\Phi_T X)^n \times (\Phi X)^m) \subset (\Phi X)^n \times (\Phi X)^m \xrightarrow{\psi^*} (\Phi X)^{n+m},$$

where ψ^* is the permutation of coordinates induced by ψ.

It is an easy verification that this actually does define an $\mathfrak{S}$-prefunctor in the sense of 2.5. Then D_X can be extended uniquely, to a monoidal functor (by Proposition 2.6) which we will also denote by D_X.

Proposition 13.2: Let $\partial(\Phi X)^n = \{(t,x) \in (\Phi X)^n \mid t_j = 0 \text{ or } \|x_j\|, \text{ for some } j \in \underline{n}\}$. Let Γ_n be the naturally induced map $\operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} D_X\pi \to D_X(e_n) = (\Phi X)^n$, where π is the domain functor $(\mathfrak{S},e_n) \to \mathfrak{S}$. Then Γ_n is a homeomorphism onto $\partial(\Phi X)^n$.

Proof: If $\varphi \in \overline{(\mathfrak{S},e_n)}$, it is clear from the definition of D_X that $\operatorname{Im}(D_X(\varphi)) \subseteq \partial(\Phi X)^n$. Hence, $\operatorname{Im}\Gamma_n \subseteq \partial(\Phi X)^n$.

We will construct a map

$$\Psi\colon \partial(\Phi X)^n \longrightarrow \operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} D_X\pi$$

which will be inverse to Γ_n .

For each $i \in \underline{n}$, let $A_i = \{(t,x) \in (\Phi X)^n | t_i = \|x_i\|\}$ and let $B_i = \{(t,x) \in (\Phi X)^n | t_i = 0\}$. Then $\{A_i, B_i\}_{i \le n}$ is a closed cover of $\partial(\Phi X)^n$.

For simplicity, denote $\underset{\overline{(\mathfrak{G},e_n)}}{\text{colim}}\, D_X\pi$ by D_n . Define $\Psi|_{A_i} : A_i \to D_n$ as the composition

$$A_i \xrightarrow{F_i} D_X(e_1) \# D_X(e_{n-1}) \equiv D_X(e_1 \otimes e_{n-1}) \xrightarrow{G_i} D_n \ ;$$

where $F_i(t,x) = ((\|x_i\|, x_i), (t_1x_1, \ldots, \widehat{t_ix_i}, \ldots, t_nx_n))$ and G_i is the map: $D(e_1 \otimes e_{n-1}) \to \underset{\overline{(\mathfrak{G},e_n)}}{\text{colim}}\, D_X\pi$ induced by the $\mathfrak{G}$-map $\eta_i: e_1 \otimes e_{n-1} \to e_n$ which maps the single element of e_1 to the number i in $\underline{n}$.

Define $\Psi|_{B_i} : B_i \to D_n$ as the composition

$$B_i \xrightarrow{\bar{F}_i} D_X(e_{n-1}) \# D_X(e_1) \xrightarrow{\equiv} D_X(e_{n-1} \otimes e_1) \xrightarrow{\bar{G}_i} D_n \ ;$$

where $\bar{F}_i(t,x) = ((t_1x_1, \ldots, \widehat{t_ix_i}, \ldots, t_nx_n), (0, x_i))$ and $\bar{G}_i$ is the map: $D(e_{n-1} \otimes e_1) \to \underset{\overline{(\mathfrak{G},e_n)}}{\text{colim}}\, D_X\pi$ induced by the $\mathfrak{G}$-map $\bar{\eta}_i: e_{n-1} \otimes e_1 \to e_n$ which maps the single element of e_1 to the number i in $\underline{n}$.

We must show that this construction is well-defined.

Let $(t,x) \in A_i \cap A_k$, $i < k$. We must show that $\Psi|_{A_i}(t,x) = \Psi|_{A_k}(t,x)$.

Let $\lambda: e_2 \otimes e_{n-2} \to e_n$ be the shuffle which takes the two elements of e_2 onto i and k in $\underline{n}$. We have the following commutative diagram in $\mathfrak{G}$:

$$\begin{array}{ccccc}
e_1 \otimes e_{n-1} & \xrightarrow{\eta_i} & e_n & \xleftarrow{\eta_k} & e_1 \otimes e_{n-1} \\
\uparrow {\scriptstyle 1\otimes\gamma_{k-1}} & & \uparrow {\scriptstyle \lambda} & & \uparrow {\scriptstyle 1\otimes\gamma_i} \\
e_1 \otimes e_1 \otimes e_{n-2} & \xrightarrow{I\otimes 1} & e_2 \otimes e_{n-2} & \xleftarrow{T\otimes 1} & e_1 \otimes e_1 \otimes e_{n-2}
\end{array}$$

where $\gamma_j \colon e_1 \otimes e_{n-2} \to e_{n-1}$ takes e_1 to $j \in \underline{n}$,

$I\colon e_1 \otimes e_1 \to e_2$ is the trivial shuffle, and

$T\colon e_1 \otimes e_1 \to e_2$ is the non-trivial shuffle.

Let $z = ((\|x_i\|,x_i,\|x_k\|,x_k),(t_1x_1,\ldots,\widehat{t_ix_i},\ldots,\widehat{t_kx_k},\ldots,t_nx_n))$ in $D_X(e_2) \# D_X(e_{n-2})$, let $z' = ((\|x_i\|,x_i),(\|x_k\|,x_k),(t_1x_1,\ldots,\widehat{t_ix_i},\ldots,\widehat{t_kx_k},\ldots,t_nx_n))$ in $D_X(e_1) \# D_X(e_1) \# D_X(e_{n-2})$ and let $z'' = ((\|x_k\|,x_k),(\|x_i\|,x_i),(t_1x_1,\ldots,\widehat{t_ix_i},\ldots,\widehat{t_kx_k},\ldots,t_nx_n))$ in $D_X(e_1) \# D_X(e_1) \# D_X(e_{n-2})$. Then $D_X(1 \times \gamma_{k-1})(z') = F_i(t,x)$, $D_X(I \otimes 1)(z') = z$, $D_X(T \otimes 1)(z'') = z$ and $D_X(1 \otimes \gamma_i)(z'') = F_k(t,x)$. Hence the commutativity of the above diagram and the definition of colimit gives us that $G_iF_i(t,x) = G_kF_k(t,x)$. Hence $\Psi|_{A_i}(t,x) = \Psi|_{A_k}(t,x)$.

By an entirely analogous argument $\Psi|_{B_i}$ and $\Psi|_{B_k}$ agree on $B_i \cap B_k$.

Let $(t,x) \in A_i \cap B_k$, we must show that $\Psi|_{A_i}(t,x) = \Psi|_{B_k}(t,x)$.

<u>Case 1</u>. Suppose $i = k$, then $\|x_i\| = t_i = t_k = 0$; hence $x_i = *$. Let $z = (t_1x_1 \ldots \widehat{t_ix_i} \ldots t_nx_n) \in B_X(e_{n-1})$. Then we have a commutative diagram in $\mathcal{G}$:

$$\begin{array}{ccc}
e_{n-1} & \xrightarrow{1\otimes 0_{e_1}} & e_{n-1}\otimes e_1 \\
{\scriptstyle 0_{e_1}\otimes 1}\downarrow & & \downarrow{\scriptstyle \bar{\eta}_i} \\
e_1\otimes e_{n-1} & \xrightarrow[\eta_i]{} & e_n
\end{array}$$

and $D_X(0_{e_1}\otimes 1)(z) = F_i(t,x)$ and $D_X(1\otimes 0_{e_1})(z) = \bar{F}_i(t,x)$. Hence in the colimit, $G_iF_i(t,x) = \bar{G}_i\bar{F}_i(t,x)$, and hence, $\Psi|_{A_i}$ agrees with $\Psi|_{B_i}$ on $A_i \cap B_i$.

<u>Case 2</u>. Suppose $i < k$. Then we have the following diagram in $\mathfrak{S}$:

$$\begin{array}{ccc}
e_1\otimes e_{n-2}\otimes e_1 & \xrightarrow{\gamma_{k-1}\otimes 1} & e_{n-1}\otimes e_1 \\
{\scriptstyle 1\otimes\gamma_i}\downarrow & & \downarrow{\scriptstyle \bar{\eta}_k} \\
e_1\otimes e_{n-1} & \xrightarrow[\eta_i]{} & e_n
\end{array}$$

and if $z = ((\|x_i\|,x_i)(t_1x_1,\ldots,\widehat{t_ix_i},\ldots,\widehat{t_kx_k},\ldots,t_nx_n)(0,x_k))$ then $D_X(\gamma_{k-1}\otimes 1)(z) = \bar{F}_k(t,x)$ and $D_X(1\otimes\gamma_i)(z) = F_i(t,x)$. And hence as before $\Psi|_{A_i}$ agrees with $\Psi|_{B_k}$ on $A_i\cap B_k$. The proof is exactly analogous if $k > i$.

Therefore $\Psi\colon \partial(\Phi X)^n \to \operatorname{colim}_{(\overline{\mathfrak{S},e_n})} D_X\pi$ is well-defined and continuous. It is a straightforward verification that Ψ and Γ_n are inverse to each other. Therefore the proposition is true.

<u>Proposition 13.3</u>: The projection $D_X(e_n) \xrightarrow{\pi} X^n$ is a homotopy equivalence.

<u>Proof</u>: Define $p: X^n \to D_X(e_n)$ by $p(x_1,\ldots,x_n) = (0,x_1,0,x_1,\ldots,0,x_n)$.

Define $F: \mathrm{I\!I} \times D_X(e_n) \to D_X(e_n)$ by

$$F(t,(s_1,x_1,\ldots,s_n,x_n)) = ((1-t)s_1,x_1,\ldots,(1-t)s_n,x_n) \ .$$

Then F is clearly a strong deformation retraction of $D_X(e_n)$ onto $p(X^n) \cong X^n$.

PART III

THE PROOF OF THEOREM 5.3

Chapter 14. The Start of the Proof

We will now bring together the disparate notions and propositions of the preceeding eight chapters in order to prove Theorem 5.3.

As a first step, we will use the material of Chapters 7, 8 and 9 to make a significant reduction in the problem. We first establish some notation.

Notation 14.1: X will always denote a CW-complex with standard norm as defined in 5.1. Since $L_X\colon \mathfrak{G} \to \mathrm{Top}^{\vee}$ is a monoidal functor (Corollary 4.5) and $\Sigma\colon \mathrm{Top}^{\vee} \to \mathrm{Top}^{\circ}$ is a (PJH)-functor (Proposition 9.2) we can consider the Σ-reduction of L_X, $(L_X)_\Sigma\colon \mathfrak{G} \to \mathrm{Top}^{\circ}$. Let $Q_n(X)$ denote $Q_n(L_X,\Sigma)$ as defined in 7.6 and let $q_n\colon Q_n(X) \to Q_{n+1}(X)$ be the canonical map. Finally let $Q(X) = \operatorname{colim}_{n\to\infty} Q_n(X)$.

Proposition 14.2: Suppose there is a Top°-morphism $\psi_X\colon Q(X) \to J(\Sigma X)$, which is natural in X and which is a homotopy equivalence. Then there is a $\mathrm{Mon}^{\vee}$-morphism $\varphi_X\colon \operatorname{colim}_{\mathfrak{G}} L_X \to \Omega^2 S^2 X$, natural in X, which is a homotopy equivalence.

Proof: The functor $J\colon \mathrm{Top}^{\vee} \to \mathrm{Mon}^{\vee}$ is coadjoint to the forgetful functor and is a (PJH)-functor (Proposition 9.2). Therefore, by Proposition 7.5, there is a $\mathrm{Mon}^{\vee}$-isomorphism, natural in X, $\operatorname{colim}_{\mathfrak{G}} L_X \to \operatorname{colim}_{\mathfrak{G}}(L_X)_J$.

Therefore, there is a Top°-isomorphism, $B(\operatorname{colim}_{\mathfrak{G}} L_X) \to B(\operatorname{colim}_{\mathfrak{G}}(L_X)_J)$. Then by Proposition 9.4, it follows that $B(\operatorname{colim}_{\mathfrak{G}}(L_X)_J) \cong \operatorname{colim}_{\mathfrak{G}}(L_X)_\Sigma$ and hence there is a Top°-isomorphism (natural in X), $F_X\colon B(\operatorname{colim}_{\mathfrak{G}} L_X) \to$

$\operatorname{colim}_{\mathfrak{G}} (L_X)_\Sigma = Q(X)$.

Therefore, if $\Psi_X: Q(X) \to J(\Sigma X)$ is a homotopy equivalence, then $\Psi_X \cdot F_X: B(\operatorname{colim}_{\mathfrak{G}} L_X) \to J(\Sigma X)$ is a homotopy equivalence and hence $\Omega(\Psi_X \cdot F_X): \Omega B(\operatorname{colim}_{\mathfrak{G}} L_X) \to \Omega J(\Sigma X)$ is a $\mathrm{Mon}^{\vee}$-morphism and a homotopy equivalence.

By Proposition 11.11, $\operatorname{colim}_{\mathfrak{G}} L_X$ is a cellular monoid and hence by Proposition 8.8, we have a homotopy equivalence, $\operatorname{colim}_{\mathfrak{G}} L_X \simeq \Omega B(\operatorname{colim}_{\mathfrak{G}} L_X)$ which is natural in X . By Theorem 8.9 we have a homotopy equivalence $J(\Sigma X) \xrightarrow{\simeq} \Omega\Sigma(\Sigma X)$, and further (see 9.5), we have $\Sigma^2 X \simeq S^2 X$.

Putting all this together we have a string of homotopy equivalences, each one natural in X and a $\mathrm{Mon}^{\vee}$-morphism:

$$\operatorname{colim}_{\mathfrak{G}} L_X \to \Omega B(\operatorname{colim}_{\mathfrak{G}} L_X) \to \Omega(\operatorname{colim}_{\mathfrak{G}}(L_X)_\Sigma) \to \Omega(J(\Sigma X)) \to \Omega(\Omega\Sigma(\Sigma X)) \to \Omega^2 S^2 X \ .$$

This establishes the proposition.

<u>Corollary 14.3</u>: If the maps $q_i: Q_i(X) \to Q_{i+1}(X)$ and $j_i: J(\Sigma X)_i \to J(\Sigma X)_{i+1}$ are cofibrations and if $\exists\ \Psi_i: Q_i(X) \to J(\Sigma X)_i$ such that $\Psi_{i+1} q_i = j_i \Psi_i$ and Ψ_i is a homotopy equivalence and natural in X , then the desired homotopy equivalence: $\operatorname{colim}_{\mathfrak{G}} L_X \to \Omega^2 S^2 X$ exists.

<u>Proof</u>: If $\Psi = \operatorname{colim}_{n \to \infty} \Psi_n: Q(X) \to J(\Sigma X)$ then a standard argument of cofibration-homotopy theory gives that Ψ is a homotopy equivalence. Thus we can apply the previous proposition to get the conclusion.

In light of this corollary, we will now concentrate on showing the existence of such maps $\Psi_i: Q_i(X) \to J(\Sigma X)_i$. To this end, we investigate in a very detailed way, the structure of $Q_i(X)$ and of $J_i(X)$. This detailed structure is the subject of the next two propositions.

<u>Proposition 14.4</u>: Let $F_n(X) = \Phi_0 L_X(e_n) \cup \Phi(\partial L_X(e_n)) \cup \Phi_\tau L_X(e_n)$. (Notation as in 11.5 and 12.1). Let $i_n\colon F_n(X) \to \Phi L_X(e_n)$ be the inclusion. Then Ξ maps $\zeta_n\colon F_n(X) \to Q_{n-1}(X)$ and $\hat{\zeta}_n\colon \Phi L_X(e_n) \to Q_n(X)$ such that the following diagram is a pushout in Top° :

$$\begin{array}{ccc} F_n(X) & \xrightarrow{\ i_n\ } & \Phi L_X(e_n) \\ {\scriptstyle \zeta_n}\big\downarrow & & \big\downarrow{\scriptstyle \zeta'_n} \\ Q_{n-1}(X) & \xrightarrow{\ q_n\ } & Q_n(X) \end{array}$$

<u>Proof</u>: If we define a functor $\bar{F}\colon (\mathfrak{S}, e_n) \to \mathrm{Top}^\circ$ by $\bar{F}(\lambda) = \mathrm{Im}[\Phi L_X(\lambda)] \cup \Phi_0 L_X(e_n) \cup \Phi_\tau L_X(e_n)$, then it is an immediate consequence of 11.5 that $F_n(X) = \operatorname*{colim}_{\overline{(\mathfrak{S}, e_n)}} \bar{F}$. There is an obvious natural transformation $\bar{f}\colon \bar{F} \to \Sigma L_X \pi$ given by collapsing $\Phi_0 L_X(e_n) \cup \Phi_\tau L_X(e_n)$ to a point. Thus we have a map $f_n = \operatorname*{colim}_{\overline{(\mathfrak{S}, e_n)}} \bar{f} : F_n(X) \to \operatorname*{colim}_{\overline{(\mathfrak{S}, e_n)}} \Sigma L_X \pi$, and the following diagram is a pushout.

$$\begin{array}{ccc} F_n(X) & \xrightarrow{\ i_n\ } & \Phi L_X(e_n) \\ {\scriptstyle f_n}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}(e_n)} \\ \operatorname*{colim}_{\overline{(\mathfrak{S}, e_n)}} \Sigma L_X \pi & \xrightarrow{\ h\ } & \Sigma L_X(e_n) \end{array}$$

From Proposition 7.7 we have a pushout

$$\begin{array}{ccc}
\operatorname*{colim}_{\overline{(\Theta,e_n)}} \Sigma L_X \pi & \xrightarrow{\quad h \quad} & \Sigma L_X(e_n) \\
\Big\downarrow \xi_n & & \Big\downarrow p_n \\
Q_{n-1}(X) & \xrightarrow{\quad q_{n-1} \quad} & Q_n(X)
\end{array}$$

Thus we set $\zeta_n = \xi_n f_n$ and $\zeta_n' = p_n \cdot \bar{f}(e_n)$ and the proposition follows.

Proposition 14.5: Let $\chi_n' : (\Phi X)^n \to J(\Sigma X)_n$ be the composition $(\Phi X)^n \to (\Sigma X)^n \to J(\Sigma X)_n$. Let $\partial(\Phi X)^n$ be as in Proposition 13.2. Then there is a map $\chi_n : \partial(\Phi X)^n \to J(\Sigma X)_{n-1}$ such that the following diagram is a pushout:

$$\begin{array}{ccc}
\partial(\Phi X)^n & \xrightarrow{\quad \subseteq \quad} & (\Phi X)^n \\
\Big\downarrow \chi_n & & \Big\downarrow \chi_n' \\
J(\Sigma X)_{n-1} & \xrightarrow{\quad j_{n-1} \quad} & J(\Sigma X)_n
\end{array} .$$

Proof: Let $\partial(\Sigma X)^n = \{(z_1,\dots,z_n) \in (\Sigma X)^n \mid z_i = * \text{ for some } i\}$.

Then from the definition of J we have a pushout:

$$\begin{array}{ccc}
\partial(\Sigma X)^n & \longrightarrow & (\Sigma X)^n \\
\Big\downarrow & & \Big\downarrow \\
J(\Sigma X)_{n-1} & \xrightarrow{\quad j_n \quad} & J(\Sigma X)_n
\end{array} ,$$

and we also have a pushout:

$$\begin{array}{ccc}
\partial(\Phi X)^n & \longrightarrow & (\Phi X)^n \\
\Big\downarrow & & \Big\downarrow \\
\partial(\Sigma X)^n & \longrightarrow & (\Sigma X)^n
\end{array} .$$

The proposition follows.

Corollary 14.6: $\forall\ n \geq 1$, $j_n: J(\Sigma X)_{n-1} \to J(\Sigma X)_n$, and $q_n: Q_{n-1}(X) \to Q_n(X)$ are cofibrations.

Proof: If X is a cell complex, it is clear that ΦX is also a cell complex. $\Phi_0 X$ and $\Phi_\tau X$ are subcomplexes and if A is a subcomplex of X then ΦA will be a subcomplex of ΦX .

It follows that $\partial(\Phi X)^n \subseteq (\Phi X)^n$ is a subcomplex and $F_n(X) \subseteq \Phi L_X(e_n)$ is a subcomplex (since by 11.10 $\partial L_X(e_n) \subseteq L_X(e_n)$ is a subcomplex). The inclusion of a subcomplex is always a cofibration, hence j_n and q_n are cofibrations since they are pushouts of cofibrations.

Remarks 14.7: The consequence of Propositions 14.4 and 14.5 is that, loosely speaking, we have described the structure of $Q_n(X)$ in terms of the monoidal functor $\Phi L_X: \mathfrak{G} \to \mathcal{C}(\mathrm{Top}^\vee)$ and we have described the structure of $J(\Sigma X)_n$ in terms of the functor $D_X: \mathfrak{G} \to (\mathrm{Top}^\vee)$.

We must now construct a natural transformation between these two functors (ΦL_X and D_X) which will enable us to define (in an inductive manner) the maps $\Psi_i: Q_i(X) \to J(\Sigma X)_i$, which, by Corollary 14.3, we must construct.

Notation 14.8: Let $\alpha \in \mathfrak{G}$ and let $k = \mathrm{grade}\ (\alpha)$. Let $\pi_\alpha: \Phi L_X(\alpha) \to X^k$ be the projection given by

$$\Phi L_X(\alpha) \subseteq \mathbb{R}^+ \times (\mathbb{R}^+ \times X)^k \xrightarrow{(\mathrm{proj})^k} X^k$$

and let $\hat{\pi}_\alpha: \Phi L_X(\alpha) \to \Phi(X^k)$ be the map: $\hat{\pi}_\alpha(t,z) = (t,\pi_\alpha(t,z))$.

Let $p_\alpha: D_X(\alpha) \to X^k$ be the composition :

$$D_X(\alpha) \subseteq (\Phi X)^k \xrightarrow{(\mathrm{proj})^k} X^k ,$$

and $\hat{p}_\alpha: D_X(\alpha) \to \Phi(X^k)$ the map:

$$\hat{p}_\alpha(t_1x_1,\dots,t_kx_k) = (\sum_{i=1}^{k} t_i, x_1,\dots,x_k) .$$

Proposition 14.9: There is a natural transformation $\varphi_X: \Phi L_X \to D_X$ with the properties:

1) φ_X is a homomorphism of monoidal functors, i.e., $\varphi_X(\alpha \otimes \beta) = \varphi_X(\alpha) \# \varphi_X(\beta)$, $\forall\ \alpha,\beta \in \mathfrak{G}$.

2) $\forall\ \alpha \in \mathfrak{G}$, $\hat{p}_\alpha\ \varphi_X(\alpha) = \hat{\pi}_\alpha$.

3) $\varphi_X(e_1)$ is the homeomorphism $\Phi L_X(e_1) \cong \Phi X = D_X(e_1)$.

Proof: φ_X will be constructed inductively, using the grading of $\mathfrak{G}$ as the basis of induction.

Step 1. To define $\varphi_X(e_1)$.

From the definition of L_X and D_X we have $\Phi L_X(e_1) = \Phi X = D_X(e_1)$. Hence we define $\varphi_X(e_1) = 1_{\Phi X}$. Then $\varphi_X(e_1)$ is a $\mathcal{C}(\mathrm{Top}^\nu)$-morphism and satisfies condition 2). Since $\Phi L_X(e_0) = * = D_X(e_0)$, we define $\varphi_X(e_0) = 1_*$ and φ_X has been defined on $\mathfrak{G}_1$ satisfying conditions 1) and 2).

We will now assume that φ_X has been constructed as a natural transformation from $\Phi L_X|_{\mathfrak{G}_{n-1}}$ to $D_X|_{\mathfrak{G}_{n-1}}$ satisfying 1) and 2).

Step 2. To extend φ_X to $\mathfrak{G}_n - \{e_n\}$.

Let $\beta \in (\mathfrak{G}_n-\{e_n\}) - \mathfrak{G}_{n-1}$. Then $\beta = \alpha_1 \otimes \alpha_2$ where $\alpha_i \in \mathfrak{G}_{n-1}$ for $i = 1,2$. We are forced to define $\varphi_X(\beta) = \varphi_X(\alpha_1) \# \varphi_X(\alpha_2)$. Note that this definition of $\varphi_X(\beta)$ is independent of the choice of α_1 and α_2 , since if $\beta = \alpha_1' \otimes \alpha_2'$, then either $\exists\gamma$ such that $\alpha_1 = \alpha_1' \otimes \gamma$ and $\alpha_2' = \gamma \otimes \alpha_2$ or $\exists\gamma'$ such that $\alpha_1' = \alpha_1 \otimes \gamma'$ and $\alpha_2 = \gamma' \otimes \alpha_2'$. In either case, the asociativity of $\#$ and condition 2) on $\varphi_X|_{\mathfrak{S}_{n-1}}$ insure that $\varphi_X(\alpha_1) \# \varphi_X(\alpha_2) = \varphi_X(\alpha_1') \# \varphi_X(\alpha_2')$.

If $\lambda \in (\mathfrak{S}_n, \beta)$, then $\lambda = \lambda_1 \otimes \lambda_2$ where $\lambda_i \in (\mathfrak{S}_{n-1}, \alpha_i)$. Hence the naturality of φ_X with respect to λ follows from the naturality of φ_X on $\mathfrak{S}_{n-1}$. φ_X is therefore extended to $\mathfrak{S}_n - \{e_n\}$. Property 1) is clearly satisfied and property 2) can be verified easily.

Step 3. To extend φ_X to $\mathfrak{S}_n$.

First, since $\varphi_X|_{\mathfrak{S}_n - \{e_n\}}$ is natural, we can define

$$\partial\overline{(\varphi_X(e_n))} = \operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} \varphi_X : \operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} \Phi L_X \pi \to \operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} D_X \pi \ .$$

But

$$\operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} \Phi L_X \pi \cong \Phi \operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} L_X \pi \cong \Phi(\partial L_X(e_n)) \ ,$$

by 12.3 and 11.5 and $\operatorname*{colim}_{\overline{(\mathfrak{S},e_n)}} D_X \pi \cong \partial(\Phi X)^n$. Hence $\partial\overline{(\varphi_X(e_n))}$ will be thought of as a map: $\Phi(\partial L_X(e_n)) \to \partial(\Phi X)^n$. Let $\partial_0 \colon \Phi_0 L_X(e_n) \to (\Phi X)^n$, and $\partial_\tau \colon \Phi_\tau L_X(e_n) \to (\Phi X)^n$ be the unique maps such that

$$\hat{\pi}_{e_n}|_{\Phi_0 L_X(e_n)} = \hat{p}_{e_n} \circ \partial_0$$

and

$$\hat{\pi}_{e_n}|_{\Phi_\tau L_X(e_n)} = \hat{p}_{e_n} \circ \partial_\tau \ .$$

It is easy to see that ∂_0, ∂_τ and $\partial\overline{(\varphi_X(e_n))}$ "fit together" to define a map: $\partial(\varphi_X(e_n)) \colon F_n(X) \to (\Phi X)^n$. (Recall from 14.4 that $F_n(X) = \Phi(\partial L_X(e_n)) \cup \Phi_0 L_X(e_n) \cup \Phi_\tau L_X(e_n)$.)

It is clear that $\hat{\pi}_{e_n}|_{F_n} = p_{e_n} \circ \partial(\varphi_X(e_n))$.

We will now extend $\partial(\varphi_X(e_n))$ to all of $\Phi L_X(e_n)$ in an appropriate fashion.

Let $\mathbb{K}^n$ be the space of convex, compact subsets of $\mathbb{R}^n$ as defined in Chapter 10. We will consider $\mathbb{K}^{n-1}$ to be the set of all compact convex subsets of the hyperplane H of $\mathbb{R}^n$, where

$$H = \{(r_1,\ldots,r_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} r_i = 0\} .$$

We will construct a map $f\colon \Phi(X^n) \to \mathbb{K}^{n-1}$ such that $(\Phi X)^n$ is the graph of f in the sense of 10.2. Let $A\colon (\Phi X)^n \to \mathbb{R}^n$ be the composition: $(\Phi X)^n \subseteq (\mathbb{R} \times X)^n \xrightarrow{\text{proj}} \mathbb{R}^n$. Define $\bar{f}\colon \Phi(X^n) \to \mathbb{K}^n$ by $\bar{f}(z) = A(\hat{\pi}_{e_n}^{-1}(z))$. (I.e.,

$$\bar{f}(t,x_1,\ldots,x_n) = \{(t_1,\ldots,t_n) \in \mathbb{R}^n \mid 0 \le t_i \le \|x_i\|, \forall i \text{ and } \sum_{i=1}^{n} t_i = t\}) .$$

Define $g\colon \Phi(X^n) \to \mathbb{R}^n$ by the rule:

$$g(t,x_1,\ldots,x_n) = \begin{cases} \left(\dfrac{t\|x_1\|}{\|x\|},\ldots,\dfrac{t\|x_n\|}{\|x\|}\right), & \|x\| > 0 \\ (0,0,\ldots,0) , & \|x\| = 0 . \end{cases}$$

This is clearly continuous. If $\alpha \in \mathbb{R}^n$, let T_α denote the translation of $\mathbb{R}^n$ which takes α to the origin. Then define $f\colon \Phi(X^n) \to \mathbb{K}^{n-1}$ by $f(z) = T_{g(z)}(\bar{f}(z))$.

To see that f is well-defined, notice that $\|g(t,x_1,\ldots,x_n)\| = t$, and hence, since $\bar{f}(t,x_1,\ldots,x_n)$ lies in the hyperplane determined by $\Sigma_{i=1}^n t_i = t$, we will have $T_{g(z)}\bar{f}(z)$ lying in the hyperplane determined by the equation $\Sigma_{i=1}^n t_i = 0$.

The continuity of f is just a consequence of the continuity of the norm on X. It should be clear that $(\Phi X)^n$ is the graph of f.

We are now in a position to apply Proposition 10.7. Note that since $\partial L_X(e_n) \to L_X(e_n)$ is an inclusion of a subcomplex we have that

$F_n(X) \to L_X(e_n)$ is the inclusion of a subcomplex and, a fortiori, a cofibration. Hence $\partial(\varphi_X(e_n))$ can be extended to all of $\Phi L_X(e_n)$. This extension we shall call $\varphi_X(e_n)$. (Thus justifying our notation.) It is immediate that $\varphi_X(e_n)$ is an $\mathcal{E}(\mathrm{Top}^\vee)$-morphism, natural with respect to all λ in $(\mathfrak{S}, e_n)$, and that $\hat{\pi}_{e_n} = \hat{p}_{e_n} \circ \varphi_X(e_n)$. Hence φ_X has been extended to $\mathfrak{S}_n$. The proposition is proved. □

In the next chapter, the transformation φ_X will be shown to induce homotopy equivalences $\psi_i\colon Q_i(X) \to J(\Sigma X)_i$, for every $i \geq 1$. Thus, by Corollary 14.3, Theorem 5.3 will be established.

Chapter 15. The Conclusion of the Proof

Remarks 15.1: In what follows, we shall show that the natural transformation φ_X , as constructed in 14.9 has certain "nice" homotopy properties. Further, we shall show that φ_X therefore induces a coherent system of homotopy equivalences $\psi_i\colon Q_i(X) = J(\Sigma X)_i$.

This, by Corollary 14.3 will complete the proof of Theorem 5.3.

Proposition 15.2: Let $\varphi\colon \Phi L_X \to D_X$ be a natural transformation such that:

1) $\varphi(\alpha_1 \otimes \alpha_2) = \varphi(\alpha_1) \# \varphi(\alpha_2)$, for every α_1 and α_2 in $\mathfrak{G}$,

2) $\hat{\pi}_\alpha = \hat{p}_\alpha \circ \varphi(\alpha)$, for every α in $\mathfrak{G}$.

Then the map $\bar{\varphi}_n = \varphi(e_n)|_{F_n(X)}\colon F_n(X) \to D_X(e_n)$ has its image lying in $\partial(\Phi X)^n$ and is a homotopy equivalence: $F_n(X) \to \partial(\Phi X)^n$.

Proof: The first assertion - that the image of $\bar{\varphi}_n$ lies in $\partial(\Phi X)^n$ - can be readily demonstrated.

Since $\varphi(e_n)$ is an $\mathcal{E}(\mathrm{Top}^\nu)$-morphism, $\varphi(e_n)(\Phi_0 L_X(e_n)) \subseteq (\Phi_0 X)^n$ and $\varphi(e_n)(\Phi_T L_X(e_n)) \subseteq (\Phi_T X)^n$. The naturality of φ insures that $\varphi(e_n)\ [\Phi(\partial L_X(e_n))] \subseteq \partial(\Phi X)^n$, since $\Phi(\partial L_X(e_n)) = \operatorname{colim}_{\overline{(\mathfrak{G},e_n)}} \Phi L_X \pi$ and $\partial(\Phi X)^n = \operatorname{colim}_{\overline{(\mathfrak{G},e_n)}} D_X \pi$. Therefore, $\varphi(e_n)(F_n(X)) \subseteq \partial(\Phi X)^n$.

The second assertion - that $\bar{\varphi}_n$ is a homotopy equivalence - is rather difficult to prove and will require a number of lemmas. One essential tool will be the following well-known theorem of homotopy theory which we quote without proof.

Lemma 15.2.1: Let $f\colon X \to Y$ be a continuous map of CW-complexes.

Let $\{X_1,\ldots,X_n\}$ and $\{Y_1,\ldots,Y_n\}$ be collections of subcomplexes of X and Y respectively such that:

1. $X = \bigcup_{i=1}^{n} X_i$, $Y = \bigcup_{i=1}^{n} Y_i$;

2. $f(X_i) \subseteq Y_i$, $i = 1,\ldots,n$;

3. $f|_{\bigcap_{i\in I} X_i} : \bigcap_{i\in I} X_i \to \bigcap_{i\in I} Y_i$ is a homotopy equivalence,

$\forall\ I \subseteq \underline{n}$.

Then f is a homotopy equivalence: $X \simeq Y$. □

In order to apply this lemma to the proof of 15.2, we must first find suitable finite coverings of $F_n(X)$ and $\partial(\Phi X)^n$ by subcomplexes. The following paragraph will define such a covering for $F_n(X)$.

<u>Definition 15.2.2</u>: Recall from 6.1 that $(\mathbb{O},e_n)_M$ denotes the set of maximal elements of $\overline{(\mathbb{O},e_n)}$. If $\alpha \in (\mathbb{O},e_n)_M$ then α is a "shuffle": $e_i \otimes e_{n-i} \to e_n$. We will denote by I_1^α the "image" of e_i in $\underline{n}$ under the bijection α . (This is compatible with the notation of 6.4).

We will abuse notation somewhat and for $\alpha \in (\mathbb{O},e_n)$ denote the image of $L_X(\alpha)$ in $L_X(e_n)$ by $L_X(\alpha)$. Define, for every $\alpha \in (\mathbb{O},e_n)$, $\Phi_\ell L_X(\alpha) = \{(t,z) \in \Phi L_X(\alpha) \,|\, 0 \le t \le \|\pi(z)\|_{I_1^\alpha}\}$ where π is the projection: $L_X(e_n) \to X^n$. We will be interested in the following subset of $\Phi L_X(e_n)$: define

$$A = \Phi_0 L_X(e_n) \cup \Phi_T L_X(e_n) \cup \bigcup_{\alpha\in(\mathbb{O},e_n)_M} \Phi_\ell L_X(\alpha) \quad .$$

For each i in $\{1,\ldots,n\}$, let $\sigma_i\colon e_1 \otimes e_{n-1} \to e_n$ be the unique shuffle which takes e_1 to $i \in \underline{n}$. (I.e., if σ_i is thought of as a bijection: $\underline{n} \to \underline{n}$, then $\sigma_i(1) = i$.)

Let $\mathcal{B}_i = \{\beta \in (\mathfrak{G},e_n)_M | \beta$ is coherent with $\sigma_i\}$. (Put another way, $\mathcal{B}_i = \{\beta \in (\mathfrak{G},e_n)_M | i \in I_1^\beta\}$.) If $I \subseteq \underline{n}$, let $\mathcal{B}_K = \bigcap_{i\in K} \mathcal{B}_i$. (Note that $\mathcal{B}_{\underline{n}} = \emptyset$.)

Define $B_K = \bigcup_{\beta\in\mathcal{B}_K} \Phi L_X(\beta)$, and let $B_i = B_{\{i\}}$, $\forall\, i \in \underline{n}$.

We are now in a position to define a "suitable" covering of $F_n(X)$. $\forall\, K \subseteq \underline{n}$, let $F_n(X)_K = A \cup B_K$ and $\forall\, i \in \underline{n}$, let $F_n(X)_i$ denote $F_n(X)_{\{i\}}$. Since every maximal element of $\overline{(\mathfrak{G},e_n)}$ lies in $\mathcal{B}_i$ for some i, it is immediate that $\{F_n(X)_i | 1 \le i \le n\}$ is a cover of $F_n(X)$.

<u>Lemma 15.2.3</u>: If $K \subseteq \underline{n}$, then $\bigcap_{i\in K} F_n(X)_i = F_n(X)_K$.

<u>**Proof**</u>: $\bigcap_{i\in K} F_n(X)_i = \bigcap_{i\in K} (A \cup B_i) = A \cup \bigcap_{i\in K} B_i$.

It is clear that $B_K \subseteq \bigcap_{i\in K} B_i$, since $B_K = \bigcup_{\beta\in\mathcal{B}_K} \Phi L_X(\beta)$ and $\beta \in \mathcal{B}_K \Rightarrow \beta \in \mathcal{B}_i$, $\forall\, i \in K$. Hence $F_n(X)_K = A \cup B_K \subseteq \bigcap_K F_n(X)_i$. We must show that $\bigcap_K B_i \subseteq A \cup B_K$. This will be done by induction on $|K|$.

If $|K| = 1$, there is nothing to prove. So assume $|K| > 1$, let $j \in K$, and set $\bar{K} = K - \{j\}$. Then we can assume, by induction, that $\bigcap_{\bar{K}} B_i \subseteq A \cup B_{\bar{K}}$. It will be sufficient to show that $B_{\bar{K}} \cap B_j \subseteq B_K$.

Let $\beta \in \mathcal{B}_{\bar{K}}$, $\beta' \in \mathcal{B}_j$ and $z \in \Phi L_X(\beta) \cap \Phi L_X(\beta')$. We must show that $z \in \Phi L_X(\beta'')$, for some $\beta'' \in \mathcal{B}_K$, or $z \in A$. By 11.4, $\exists\, \gamma \in \overline{(\mathfrak{G},e_n)}$ such that γ is a predecessor of β and of β' and $z \in \Phi L_X(\gamma)$. We consider two cases:

<u>Case 1</u>. $\gamma \in (\mathfrak{G},e_n)_0$. Then by definition 6.3, β and β' are coherent and by 6.6 either $I_\beta \subseteq I_{\beta'}$ or $I_{\beta'} \subseteq I_\beta$. If $I_\beta \subseteq I_{\beta'}$, then

$j \in I_{\beta'}$ and $\beta' \in \mathcal{B}_j$ and hence $\beta' \in \mathcal{B}_K$. If $I_{\beta'} \subseteq I_\beta$, then $i \in I_\beta$, $\forall\ i \in \bar{K}$, and hence $\beta \in \mathcal{B}_K$. Hence $z \in B_K$.

<u>Case 2.</u> $\gamma \in \overline{(\mathcal{O},e_n)} - (\mathcal{O},e_n)_0$. Let m be the grade of the domain of γ. Then it is a straightforward matter to see that γ factors through a shuffle $\hat{\gamma}: e_m \otimes e_{n-m} \to e_n$ in such a way that $\Phi L_X(\gamma) \subseteq \Phi_\ell L_X(\hat{\gamma})$. Hence $\Phi L_X(\gamma) \subseteq A$, and, a fortiori, $z \in A$. This establishes the lemma. □

We now wish to define a corresponding "suitable" cover for $\partial(\Phi X)^n$.

<u>Definition 15.2.4</u>: We again will abuse notation slightly: if $\alpha \in \overline{(\mathcal{O},e_n)}$, we will denote the image of $D_X(\alpha)$ in $D_X(e_n) = (\Phi X)^n$ by $D_X(\alpha)$.

If $\alpha \in (\mathcal{O},e_n)_M$, we define $D_X^\ell(\alpha) = \{(t,x) \in D_X(\alpha) \mid t_i=0,\ \forall\ i \in \underline{n} - I_1^\alpha\}$.

Let

$$G = (\Phi_0 X)^n \cup (\Phi_\tau X)^n \cup \bigcup_{\alpha \in (\mathcal{O},e_n)_M} D_X^\ell(\alpha) \subseteq \partial(\Phi X)^n .$$

For each $K \subseteq \underline{n}$, define $H_K = \bigcup_{\beta \in \mathcal{B}_K} D_X(\beta)$, and $H_i = H_{\{i\}}$. We now proceed, as in 15.2.2, to define our suitable cover of $\partial(\Phi X)^n$.

For each $K \subseteq \underline{n}$, define $E_K = G \cup H_K$ and define $E_i = E_{\{i\}}$, for each $i \in n$. The set $\{E_i \mid i = 1,\ldots,n\}$ is certainly a cover for $\partial(\Phi X)^n$, since, in fact, the H_i's form a cover for $\partial(\Phi X)^n$.

<u>Lemma 15.2.5</u>: If $K \subseteq \underline{n}$, then $\bigcap_{i \in K} E_i = E_K$.

<u>Proof</u>: As in the proof of 15.2.3 it is clear that $H_K \subseteq \bigcap_K H_i$ and hence $E_K = G \cup H_K \subseteq G \cup (\bigcap_K H_i) = \bigcap_K (G \cup H_i) = \bigcap_K E_i$. Hence, we must show that $\bigcap_K E_i \subseteq E$. It is clearly sufficient to show that $\bigcap_K H_i \subseteq G \cup H_K$.

We induct on $|K|$. If $|K| = 1$, there is nothing to prove.

Assume $|K| > 1$, let $j \in K$ and let $\bar{K} = K - \{j\}$. Then we may assume, inductively, that $\bigcap_{\bar{K}} H_i \subseteq H_{\bar{K}}$ and we must show that $H_{\bar{K}} \cap H_j \subseteq G \cup H_K$.

Let $(s,y) \in H_{\bar{K}} \cap H_j$, then $\alpha \in \mathcal{B}_{\bar{K}}$ and $\alpha' \in \mathcal{B}_j$ such that $(s,y) \in D_X(\alpha) \cap D_X(\alpha')$.

From the definition of D_X, it follows that if $\alpha \in \mathcal{B}_i$ and $(t,x) \in D_X(\alpha)$ such that $t_j > 0$, $\forall j \in \underline{n}$, then $t_i = \|x_i\|$.

We therefore have two cases to consider:

<u>case 1</u>. $s_k = 0$, for some $k \in \underline{n}$. Then $(s,y) \in D_X^{\ell}(\gamma)$ for some $\gamma \in (\mathfrak{S}, e_n)_M$, hence $(s,y) \in G$.

<u>case 2</u>. $s_i = \|y_i\|$, $\forall i \in K$. Then let α be the unique shuffle such that $I_1^\alpha = K$. Then $(s,y) \in D_X(\alpha)$, from the definition of D_X and clearly $\alpha \in \mathcal{B}_K$. Therefore $(s,y) \in H_K$. The lemma is established. □

<u>Lemma 15.2.6:</u> $F_n(X)_i$ and E_i are subcomplexes of $F_n(X)$ and $\partial(\Phi X)^n$ respectively, for $i = 1,\ldots,n$.

<u>Proof</u>: It is easy to see that $\Phi_0 L_X(e_n)$, $\Phi_T L_X(e_n)$ are subcomplexes of $F_n(X)$. It is also clear that $\Phi L_X(\beta)$ is a subcomplex of $F_n(X)$, $\forall \beta \in \overline{(\mathfrak{S}, e_n)}$. Hence $F_n(X)_i$ will be a subcomplex of $F_n(X)$ if we can show that $\Phi_\ell L_X(\alpha)$ is a subcomplex of $F_n(X)$, $\forall \alpha \in (\mathfrak{S}, e_n)_M$. If α is a shuffle $e_m \otimes e_{n-m} \to e_n$, then there is an isomorphism

$$\Phi L_X(\alpha) \cong \Phi L_X(e_m) \# \Phi L_X(e_n) = (\Phi L_X(e_m) \times \Phi_0 L_X(e_{m-n})$$
$$\cup\ (\Phi_T L_X(e_m) \times \Phi L_X(e_{m-n}))$$

and under this isomorphism $\Phi_\ell L_X(\alpha)$ is carried onto $\Phi L_X(e_m) \times \Phi_0 L_X(e_{n-m})$. But $\Phi L_X(e_m) \times \Phi_0 L_X(e_{m-n})$ is clearly a subcomplex of $\Phi L_X(e_m) \# \Phi L_X(e_{n-m})$. Hence $\Phi_\ell L_X(\alpha)$ is a subcomplex of $\Phi L_X(\alpha)$ and, therefore, of $F_n(X)$. Hence $F_n(X)_i$ is the union of subcomplexes and is a subcomplex.

To see that E_i is a subcomplex of $\partial(\Phi X)^n$, one carries out a similar argument, working with the functor D_X instead of ΦL_X . We omit this.

We now have coverings of $F_n(X)$ and of $\partial(\Phi X)^n$ by subcomplexes with rather nicely described intersections. The naturality of $\varphi\colon \Phi L_X \to D_X$ and properties 1) and 2) of the hypothesis of 15.2 give us that $\bar{\varphi}_n(\Phi_0 L_X(e_n)) \subseteq (\Phi_0 X)^n$, $\bar{\varphi}_n(\Phi_\tau L_X(e_n)) \subseteq (\Phi_\tau X)^n$, $\bar{\varphi}_n(\Phi L_X(\beta)) \subseteq D_X(\beta)$ and $\bar{\varphi}_n(\Phi_\ell L_X(\alpha)) \subseteq D_X^\ell(\alpha)$. Hence $\bar{\varphi}_n(F_n(X)_i) \subseteq E_i$ for $i = 1,\ldots,n$.

Let $\bar{\varphi}_n^K = \bar{\varphi}_n|_{F_n(X)_K}\colon F_n(X)_K \to E_K$, for each $K \subseteq \underline{n}$. We will be in a position to apply Lemma 15.2.1 to $\bar{\varphi}_n$ if we can show that $\bar{\varphi}_n^K$ is a homotopy equivalence, $\forall$ K . By condition 2) of the hypothesis of 15.2 we have a commuting diagram, $\forall\ K \subseteq \underline{n}$:

$$\begin{array}{ccccc}
F_n(X)_K & \subseteq & \Phi L_X(e_n) & \xrightarrow{\ \pi\ } & \\
\downarrow{\scriptstyle \bar{\varphi}_n^K} & & \downarrow{\scriptstyle \varphi(e_n)} & & X^n \\
E_K & \subseteq & (\Phi X)^n & \xrightarrow{\ p\ } &
\end{array}$$

Therefore, to show that $\bar{\varphi}_n^K$ is a homotopy equivalence it is sufficient to show that the maps $E_K \xrightarrow[p_K]{} X^n$ and $F_n(X)_K \xrightarrow[\pi_K]{} X^n$ are homotopy equivalences. The next several lemmas will show this for $K \neq \underline{n}$.

<u>Lemma 15.2.7</u>: If $K \subset \underline{n}$, then $p_K\colon E_K \to X^n$ is a homotopy equivalence.

<u>Proof</u>: $E_K = \{(t,x) \in (\Phi X)^n | \exists\ i \in \underline{n}$ such that $t_i = 0$, or $t_i = \|x_i\|$, $\forall\ i \in K\}$. Let $\hat{E}_K = \{(t,x) \in (\Phi X)^n | t_i = 0\ ,\ \forall\ i \in \underline{n} - K\}$. Then, since $K \neq \underline{n}$, we have $\hat{E}_K \subseteq E_K$.

Define $f: \mathbb{I} \times E_K \to E_K$ by $f(s,(t,x)) = (r,x)$ where $r_i = t_i$ if $i \in K$ and $r_i = (1-s)t_i$, $i \in \underline{n} - K$. Then f is a strong deformation retract of E_K onto E_K.

Define $g: \mathbb{I} \times \hat{E}_K \to \hat{E}_K$ by $g(s,(t,x)) = (r,x)$ where $r_i = (1-s)t_i$. Then g is a strong deformation retract of $\hat{E}_K$ onto $(\Phi_0 X)^n$.

But $p_K: E_K \to X^n$ has a left inverse $q: X^n \to E_K$ where $q(x) = (0,x)$ and hence q is a homeomorphism of X^n onto $(\Phi_0 X)^n$. Hence p_K is a homotopy equivalence.

<u>Lemma 15.2.8</u>: If $K \subseteq \underline{n}$, then $\pi_K: F_n(X)_K \to X^n$ is a homotopy equivalence.

<u>Proof</u>: Let $P_K = \Phi_0 L_X(e_n) \cup \bigcup_{\alpha \in (\mathfrak{G},e_n)_M} \Phi_\ell L_X(\alpha) \cup \bigcup_{\beta \in \mathcal{B}_K} \Phi L_X(\beta)$. Then we set

$F_n(X)_K = P_K \cup \Phi_T L_X(e_n)$.

There is a deformation d of P_K onto $\Phi_0 L_X(e_n)$ given by $d_t(s,z) = ((1-t)s,z)$. Clearly $\pi_K|_{\Phi_0 L_X(e_n)}$ and $\pi_K|_{\Phi_T L_X(e_n)}$ are homotopy equivalences (see Lemma 11.12). Hence $\pi_K|_{P_K}$ is a homotopy equivalence.

P_K and $\Phi_T L_X(e_n)$ are subcomplexes of $F_n(X)_K$ and therefore it is sufficient to show that $\pi_K|_{P_K \cap \Phi_T L_X(e_n)}$ is a homotopy equivalence.

It is straightforward to see that

$$\Phi_T L_X(e_n) \cap \bigcup_{\alpha \in (\mathfrak{G},e_n)_M} \Phi_\ell L_X(\alpha) = \bigcup_{\gamma \in \overline{(\mathfrak{G},e_n)} - (\mathfrak{G},e_n)_0} \Phi_T L_X(\gamma) .$$

Therefore

$$\Phi_T L_X(e_n) \cap P_K = \Big[\bigcup_{\gamma \in \overline{(\mathfrak{G},e_n)} - (\mathfrak{G},e_n)_0} \Phi_T L_X(\gamma) \Big] \cup \Big[\bigcup_{\beta \in \mathcal{B}_K} \Phi_T L_X(\beta) \Big]$$

which is homeomorphic to

$$\Big[\bigcup_{\gamma \in \overline{(\mathfrak{G},e_n)} - (\mathfrak{G},e_n)_0} L_X(\gamma) \Big] \cup \Big[\bigcup_{\beta \in \mathcal{B}_K} L_X(\beta) \Big]$$

Lemma 11.18 then gives that $\pi_K|_{P_K \cap \Phi_T L_X(e_n)}$ is a homotopy equivalence. Lemma 15.2.8 follows. □

Remarks 15.2.9: From the discussion just prior to 15.2.7 we now can conclude that $\bar{\varphi}_n^K \colon F_n(X)_K \to E_K$ is a homotopy equivalence if $K \subset \underline{n}$. We must treat separately the case $\bar{\varphi}_n^{\underline{n}}$.

First notice that $\mathcal{B}_{\underline{n}} = \emptyset$. Hence $F_n(X) = A$ and $E_{\underline{n}} = G$ (in the notation of 15.2.2 and 15.2.4.)

Let $P = \Phi_0 L_X(e_n) \cup \bigcup_{\alpha \in (\mathcal{B}, e_n)_M} \Phi_\ell L_X(\alpha)$ and $R = (\Phi_0 X)^n \cup \bigcup_{\alpha \in (\mathcal{B}, e_n)_M} D_X^\ell(\alpha)$.

Then, by straightforward arguments which we have used before, we have $\bar{\varphi}_n|_P \colon P \xrightarrow{\simeq} R$, $\bar{\varphi}_n|_{\Phi_T L_X(e_n)} \colon \Phi_T L_X(e_n) \xrightarrow{\simeq} (\Phi_T X)^n$, and $\bar{\varphi}_n|_{P \cap \Phi_T L_X(e_n)} \colon P \cap \Phi_T L_X(e_n) \xrightarrow{\simeq} R \cap (\Phi_T X)^n$. So, using the same type of pushout argument, we have $\bar{\varphi}_n|_{P \cup \Phi_T L_X(e_n)} \colon P \cup \Phi_T L_X(e_n) \xrightarrow{\simeq} R \cup (\Phi_T X)^n$ and thus $\bar{\varphi}_n^{\underline{n}}$ is a homotopy equivalence.

We finally have everything we need in order to apply Lemma 15.2.1. We apply it, and conclude that $\bar{\varphi}_n \colon F_n(X) \to \partial(\Phi X)^n$ is a homotopy equivalence, establishing Proposition 15.2. □

Let us recall where we are. We are working toward the construction of maps $\Psi_i \colon Q_i(X) \to J(\Sigma X)_i$ which are homotopy equivalences and such that $\Psi_i \circ q_{i-1} = j_{i-1}\Psi_{i-1}$. We are nearly ready to do this; the next proposition provides the essential induction step.

Proposition 15.3: Let $\varphi \colon \Phi L_X \to D_X$ be a natural transformation satisfying conditions 1) and 2) and 3) in the statement of Proposition 14.9.

Suppose $f \colon Q_{n-1}(X) \to J(\Sigma X)_{n-1}$ is a continuous map such that the following diagram commutes:

$$\begin{array}{ccc}
F_n(X) & \xrightarrow{\bar{\varphi}_n} & \partial(\Phi X)^n \\
\big\downarrow \zeta_n & & \big\downarrow \chi_n \\
Q_{n-1}(X) & \xrightarrow{f} & J(\Sigma X)_{n-1}
\end{array} \quad ,$$

where $\bar{\varphi}_n = \varphi(e_n)|_{F_n(X)}$.

Let $g\colon Q_n(X) \to J(\Sigma X)_n$ be the map induced in the diagram:

$$\begin{array}{ccccccccc}
F_n(X) & \xrightarrow{\subseteq} & \Phi L_X(e_n) & \xrightarrow{\zeta_n'} & Q_n(X) & \xleftarrow{q_{n-1}} & Q_{n-1}(X) & \xleftarrow{\zeta_n} & F_n(X) \\
\big\downarrow \bar{\varphi}_n & & \big\downarrow \varphi(e_n) & & \vdots\, g & & \big\downarrow f & & \big\downarrow \bar{\varphi}_n \\
\partial(\Phi X)^n & \xrightarrow{\subseteq} & \Phi X(e_n) & \xrightarrow{\chi_n'} & J(\Sigma X)_n & \xleftarrow{j_n} & J(\Sigma X)_{n-1} & \xleftarrow{\chi_n} & \partial(\Phi X)^n
\end{array} \quad ,$$

by the fact that the top row is a pushout.

Then the following diagram also commutes:

$$\begin{array}{ccc}
F_{n+1}(X) & \xrightarrow{\bar{\varphi}_{n+1}} & \partial(\Phi X)^{n+1} \\
\big\downarrow \zeta_{n+1} & & \big\downarrow \chi_{n+1} \\
Q_n(X) & \xrightarrow{g} & J(\Sigma X)_n
\end{array} \quad .$$

<u>Proof</u>: $F_{n+1}(X) = \Phi_0 L_X(e_{n+1}) \cup \Phi_T L_X(e_{n+1}) \cup \bigcup_{\alpha \in (\Theta, e_{n+1})_M} \Phi L_X(\alpha)$.

$$\zeta_{n+1}(\Phi_0 L_X(e_{n+1}) \cup \Phi_T L_X(e_{n+1})) = *$$

and

$$\chi_{n+1} \cdot \bar{\varphi}_{n+1}(\Phi_0 L_X(e_{n+1}) \cup \Phi_T L_X(e_{n+1})) = * \quad .$$

Therefore we must check that for $\alpha \in (\Theta, e_{n+1})_M$ the following diagram commutes:

$$
(15.3.1)\qquad
\begin{array}{ccc}
\Phi L_X(\alpha) & \xrightarrow{\varphi(\alpha)} & D_X(\alpha) \\
\downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} \\
F_{n+1}(X) & & \partial(\Phi X)^{n+1} \\
\downarrow{\scriptstyle\zeta_{n+1}} & & \downarrow{\scriptstyle\chi_{n+1}} \\
Q_n(X) & \xrightarrow{f} & J(\Sigma X)_n
\end{array}
$$

Since $\alpha \in (\mathfrak{S}, e_{n+1})_M$, the domain of $\alpha = e_m \otimes e_{n-m+1}$ for some $m \le n$.

<u>Claim 1</u>: The following diagram commutes:

$$
(15.3.2)\qquad
\begin{array}{ccc}
\Phi L_X(\alpha) & \xrightarrow{\subseteq} & F_{n+1}(X) \\
\Phi L_X(\alpha)\ \downarrow{\scriptstyle\cong} & & \\
\Phi L_X(e_m) \# \Phi L_X(e_{n-m+1}) & & \downarrow{\scriptstyle\zeta_{n+1}} \\
\downarrow & & \\
\Sigma L_X(e_m) \vee \Sigma L_X(e_{n-m+1}) & & \\
\downarrow{\scriptstyle p_m \vee p_{n-m+1}} & & \\
Q_m(X) \vee Q_{n-m+1}(X) & \longrightarrow & Q_n(X)
\end{array}
\quad .
$$

To see that (15.3.2) commutes note first that, by 14.4, ζ_{n+1} is the composition: $F_n(X) \to \underset{\overline{(\mathfrak{S}, e_{n+1})}}{\operatorname{colim}} \Sigma L_X \xrightarrow{\zeta_{n+1}} Q_n(X)$; now examine the following

diagram:

$$\begin{array}{ccccc}
\Phi L_X(e_m) \# \Phi L_X(e_{n-m+1}) = & \Phi L_X(e_m \otimes e_{n-m+1}) & \xrightarrow{\Phi L_X(\alpha)} & F_n(X) \\
\searrow & \downarrow & & \downarrow \\
 & \Sigma L_X(e_m \otimes e_{n-m+1}) & \longrightarrow & \operatorname{colim}_{\overline{(\Theta, e_{n+1})}} \Sigma L_X \\
 & \downarrow \Lambda_\Sigma & & \\
 & \Sigma L_X(e_m) \vee \Sigma L_X(e_{n-m+1}) & & \downarrow \xi_{n+1} \\
 & \downarrow p_m \vee p_{n-m+1} & & \\
 & Q_m(X) \vee Q_{n-m+1}(X) & \longrightarrow & Q_n(X)
\end{array}$$

The triangle and upper rectangle certainly commute and the lower rectangle commutes by the definition of ξ_{n+1} in 7.7. Since the outer diagram is exactly 15.3.2, it commutes.

<u>Claim 2</u>: The following diagram also commutes:

(15.3.3)

$$\begin{array}{ccc}
D_X(\alpha) & \longrightarrow & \partial(\Phi X)^n \\
\downarrow \cong & & \\
(\Phi X)^m \# (\Phi X)^{n-m+1} & & \\
\downarrow & & \downarrow \\
(\Sigma X)^m \vee (\Sigma X)^{n-m+1} & & \\
\downarrow & & \\
J(\Sigma X)_m \vee J(\Sigma X)_{n-m+1} & \longrightarrow & J(\Sigma X)_n
\end{array}$$

This is obvious.

To see that (15.3.1) commutes, (15.3.2) and (15.3.3) imply that it is sufficient to see that the following diagram commutes:

(15.3.4)

$$\begin{array}{ccc}
\Phi L_X(e_m) \# \Phi L_X(e_{n-m+1}) & \xrightarrow{\varphi_{e_m} \# \varphi_{e_{n-m+1}}} & (\Phi X)^m \# (\Phi X)^{n-m+1} \\
\downarrow & & \downarrow \\
\Sigma L_X(e_m) \vee \Sigma L_X(e_{n-m+1}) & & (\Sigma X)^m \vee (\Sigma X)^{n-m+1} \\
\Big\downarrow {\scriptstyle p_m \vee p_{n-m+1}} & & \Big\downarrow \\
Q_m(X) \vee Q_{n-m+1}(X) & \xrightarrow{g|_{Q_m} \vee g|_{Q_{n-m+1}}} & J(\Sigma X)_m \vee J(\Sigma X)_{n-m+1}
\end{array}$$

But $g|_{Q_m} = g$ if $m = n$ and $g|_{Q_m} = f|_{Q_m}$ if $m < n$ and the commutativity of (15.3.4) is thus an immediate consequence of the hypotheses. The proposition is proved.

We can now apply the finishing touches to our proof of Theorem 5.3.

<u>Proposition 15.4</u>: $\forall\, n \geq 1$, $\exists\, \Psi_n : Q_n(X) \to J(\Sigma X)_n$ such that

1) $\Psi_{n+1} q_n = j_n \cdot \Psi$, $\forall\, n \geq 1$ and

2) Ψ_n is a homotopy equivalence, $\forall\, n \geq 1$.

<u>Proof</u>: Let $\varphi_X : \Phi L_X \to D_X$ be the natural transformation whose existence was demonstrated in 14.9. Note that $Q_1(X) \cong \Sigma X \cong J(\Sigma X)_1$ and that the following diagram commutes:

$$\begin{array}{ccc}
F_2(X) & \xrightarrow{\bar{\varphi}_2} & \partial(\Phi X)^2 \\
\Big\downarrow {\scriptstyle \zeta_2} & & \Big\downarrow {\scriptstyle \chi_2} \\
Q_1(X) & \cong & J(\Sigma X)_1
\end{array}$$

(This is just a matter of checking the definitions and using properties 1), 2) and 3) of 14.9).

Hence we define $\Psi_1 : Q_1(X) \to J(\Sigma X)_1$ to be the homeomorphism $Q_1(X) \cong \Sigma X \cong J(\Sigma X)_1$.

Suppose $\Psi_{n-1} : Q_{n-1}(X) \to J(\Sigma X)_{n-1}$ has been defined such that the following diagram commutes:

$$\begin{array}{ccc} F_n(X) & \xrightarrow{\zeta_n} & Q_{n-1}(X) \\ \downarrow{\scriptstyle \bar{\varphi}_X(e_n)} & & \downarrow{\scriptstyle \Psi_{n-1}} \\ \partial(\Phi X)^n & \xrightarrow{\chi_n} & J(\Sigma X)_{n-1} \end{array} .$$

Then we can define $\Psi_n : Q_n(X) \to J(\Sigma X)_{n-1}$ by "pushing out" - i.e., consider the diagram:

(15.4.1)

$$\begin{array}{ccccccccc} F_n(X) & \xrightarrow{\subseteq} & \Phi L_X(e_n) & \xrightarrow{\zeta'_n} & Q_n(X) & \xleftarrow{q_{n-1}} & Q_{n-1}(X) & \xleftarrow{\zeta_n} & F_n(X) \\ \downarrow{\scriptstyle \bar{\varphi}_X(e_n)} & & \downarrow{\scriptstyle \varphi_X(e_n)} & & \downarrow{\scriptstyle \Psi_n} & & \downarrow{\scriptstyle \Psi_{n-1}} & & \downarrow{\scriptstyle \bar{\varphi}_X(e_n)} \\ \partial(\Phi X)^n & \xrightarrow{\subseteq} & (\Phi X)^n & \xrightarrow{\chi'_n} & J(\Sigma X)_n & \xleftarrow{j_{n-1}} & J(\Sigma X)_{n-1} & \xleftarrow{\chi_n} & \partial(\Phi X)^n \end{array}$$

The top row is the pushout diagram of Proposition 14.4 and the bottom row is the pushout diagram of Proposition 14.5. Ψ_n is the unique map induced by the commutativity of the rest of the diagram.

Proposition 15.3 now gives that the following diagram commutes:

$$\begin{array}{ccc} F_{n+1}(X) & \xrightarrow{\zeta_{n+1}} & Q_n(X) \\ \downarrow{\scriptstyle \bar{\varphi}_X(e_{n+1})} & & \downarrow{\scriptstyle \Psi_n} \\ \partial(\Phi X)^{n+1} & \xrightarrow{\chi_{n+1}} & J(\Sigma X)_n \end{array} .$$

We have therefore defined $\Psi_n: Q_n(X) \to J(\Sigma X)_n$, $\forall\, n \geq 1$, such that $\Psi_{n+1} q_n = j_n \Psi_n$, $\forall\, n \geq 1$.

To see that Ψ_n is a homotopy equivalence $\forall\, n$, first note that Ψ_1 is a homeomorphism and, a fortiori, a homotopy equivalence. If Ψ_{n-1} is a homotopy equivalence, then diagram (15.4.1) implies that Ψ_n is a homotopy equivalence. (Since $\bar{\varphi}_X(e_n)$, $\varphi_X(e_n)$ and Ψ_{n-1} are homotopy equivalences and $F_n(X)$ and $\partial(\Phi X)^n$ are subcomplexes of $\Phi L_X(e_n)$ and $(\Phi X)^n$ respectively).

Therefore Ψ_n is a homotopy equivalence, $\forall\, n \geq 1$.

<u>Corollary 15.5</u>: Theorem 5.3 is true.

This is now an immediate consequence of Corollaries 14.3 and 14.6 and Proposition 15.4.

APPENDIX I. The Proof of Proposition 2.6

We are given the following data:

1) A functor $F: \mathbb{N} \to \mathcal{G}$ such that $F(e_0) = 0_{\mathcal{G}}$,

2) $\forall$ 2-shuffle $\varphi: e_i \otimes e_j \to e_{i+j}$, an $\mathcal{G}$-morphism

$$F_\varphi: F(e_i) \otimes_{\mathcal{G}} F(e_j) \to F(e_{i+j})$$

such that

A) If $\lambda_i \in \mathfrak{S}(e_{n_i}, e_{m_i})$, $i = 1,2$; $\lambda \in \mathfrak{S}(e_{n_1+n_2}, e_{m_1+m_2})$;

$\alpha \in \mathfrak{S}(e_{n_1} \otimes e_{n_2}, e_{n_1+n_2})$; $\beta \in \mathfrak{S}(e_{m_1} \otimes e_{m_2}, e_{m_1+m_2})$ are

such that $\beta(\lambda_1 \otimes \lambda_2) = \lambda\alpha$, then $F_\beta(F(\lambda_1) \otimes_{\mathcal{G}} F(\lambda_2)) = F(\lambda)F_\alpha$.

B) If $\alpha \in \mathfrak{S}(e_i \otimes e_j, e_{i+j})$, $\beta \in \mathfrak{S}(e_j \otimes e_k, e_{j+k})$
$\gamma \in \mathfrak{S}(e_{i+j} \otimes e_k, e_{i+j+k})$, $\delta \in \mathfrak{S}(e_i \otimes e_{j+k}, e_{i+j+k})$

such that $\delta(1_{e_i} \otimes \beta) = \gamma(\alpha \otimes 1_{e_k})$, then $F_\delta \cdot (1_{F(e_i)} \otimes_{\mathcal{G}} F_\beta) = F_\gamma \cdot (F_\alpha \otimes_{\mathcal{G}} 1_{F(e_k)})$.

We wish to show that this data can be extended, in a unique fashion, to a monoidal functor $\bar{F}: \mathfrak{S} \to \mathcal{G}$.

The proof:

We define $\bar{F}$ on objects of $\mathfrak{S}$ in the only way possible:

$\bar{F}(e_{i_1} \otimes \dots \otimes e_{i_k}) = F(e_{i_1}) \otimes \dots \otimes F(e_{i_k})$.

To define $\bar{F}$ on morphisms, we will need to establish some notation. Let $\lambda: \alpha \to \beta$ be an $\mathfrak{S}$-morphism, where $\alpha: \underline{n} \to \underline{m}$ and $\beta: \underline{k} \to \underline{p}$. λ is <u>homogeneous</u> if $n = k$. λ is r-<u>simple</u> if $p = 1$ and $m = r$.

First, we will define $\bar{F}$ on homogeneous morphisms. Since every morphism is the product of simple morphisms in a unique manner and since $\bar{F}$ has to be monoidal it is sufficient to define $\bar{F}$ on simple morphisms.

If λ is a homogeneous 1 or 2-simple morphism we define $\bar{F}(\lambda) = F(\lambda)$. Suppose $n > 2$ and that F has been defined for all $(n-1)$-simple morphisms. Let λ be an n-simple morphism $\lambda\colon e_{i_1} \otimes \ldots \otimes e_{i_n} \to e_m$. Let $m_1 = \sum_{j=1}^{n-1} i_j$ and let $m_2 = \sum_{j=2}^{n} i_j$. There exist

$$\lambda_L\colon e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \to e_{m_1} \text{ , } \lambda_R\colon e_{i_2} \otimes \ldots \otimes e_{i_n} \to e_{m_2}$$

$$\gamma_L\colon e_{m_1} \otimes e_{i_n} \to e_m \quad \text{and} \quad \gamma_R\colon e_{i_1} \otimes e_{m_2} \to e_m$$

such that $\lambda = \gamma_L(\lambda_L \otimes 1_{e_{i_n}}) = \gamma_R(1_{e_{i_1}} \otimes \lambda_R)$. Define $\bar{F}_L(\lambda) = \bar{F}(\gamma_L) \circ (\bar{F}(\lambda_L) \otimes 1_{F(e_{i_n})})$; and $\bar{F}_R(\lambda) = \bar{F}(\gamma_R) \circ (1_{F(e_{i_1})} \otimes \bar{F}(\lambda_R))$.

Suppose that $\forall\, r \le n-1$, every homogeneous r-simple $\mathfrak{G}$-morphism η has the property that $\bar{F}_L(\eta) = \bar{F}_R(\eta)$.

If λ is homogeneous and n-simple we have a commuting diagram in $\mathfrak{G}$ as follows:

$$\begin{array}{ccccc}
 & \xleftarrow{\lambda_L \otimes 1} & e_{i_1} \otimes \ldots \otimes e_{i_n} & \xrightarrow{1 \otimes \lambda_R} & \\
\downarrow & & \downarrow 1 \otimes \delta \otimes 1 & & \downarrow \\
e_{m_1} \otimes e_{i_n} & \xleftarrow{\nu_R \otimes 1} & e_{i_1} \otimes e_{m_3} \otimes e_{i_n} & \xrightarrow{1 \otimes \nu_L} & e_{i_1} \otimes e_{m_2} \\
\gamma_L \downarrow & & & & \downarrow \gamma_R \\
 & \longrightarrow & e_m & \longleftarrow &
\end{array}$$

where $m_3 = \sum_{j=2}^{n-1} i_j$, $\delta = (\lambda_R)_L = (\lambda_L)_R$. By our assumption above, $\bar{F}(\lambda_L) = \bar{F}(\nu_R)(1 \otimes \bar{F}(\delta)), \bar{F}(\nu_L)(\bar{F}(\delta) \otimes 1) = \bar{F}(\lambda_R)$ and by condition B) of the hypothesis , $\bar{F}(\gamma_L)(\bar{F}(\nu_R) \otimes 1) = \bar{F}(\gamma_R)(1 \otimes \bar{F}(\nu_L))$.

Putting these together we have that $\bar{F}_L(\lambda) = \bar{F}_R(\lambda)$ and we therefore define $\bar{F}(\lambda)$ to be $\bar{F}_L(\lambda)$.

Thus we have defined, by induction, $\bar{F}(\lambda)$ for every simple, homogeneous $\mathfrak{G}$-morphism and hence for every homogeneous $\mathfrak{G}$-morphism.

Suppose λ, λ_1 and λ_2 are homogeneous such that $\lambda = \lambda_1 \circ \lambda_2$ ($\circ$ denotes composition). We must show that $\bar{F}(\lambda) = \bar{F}(\lambda_1) \circ \bar{F}(\lambda_2)$.

It is sufficient to consider only the case when λ and λ_1 are simple.

If λ is r-simple for $r \leq 3$, then condition B) will immediately give $\bar{F}(\lambda) = \bar{F}(\lambda_1) \circ \bar{F}(\lambda_2)$. Suppose that $\bar{F}(\lambda') = \bar{F}(\lambda_1') \circ \bar{F}(\lambda_2')$ for any r-simple λ , for $r \leq n-1$. Now let λ be n-simple;then λ_1 is m-simple for some $m \leq n$. If $m = n$, then $\lambda_2 = 1 \otimes \underset{m}{\dots} \otimes 1$ and there's nothing to prove.

We can do a further induction, this time on the number n-m . If $n-m > 0$, then $\lambda_2 = \nu_1 \otimes \dots \otimes \nu_m$ and $\exists\, k \in \{1,\dots,m\}$ such that ν_k is at least 2-simple. Then

$$\lambda_2 = (\nu_1 \otimes \dots \otimes \underset{(k)}{1 \otimes} \dots \otimes \nu_m)(1 \otimes \dots \otimes \nu_k \otimes \dots \otimes 1) .$$

By our induction on n , $\bar{F}(\lambda_1 \circ (\nu_1 \otimes \dots \otimes 1 \otimes \dots \otimes \nu_m)) =$

$\bar{F}(\lambda) \circ \bar{F}(\nu_1 \otimes \dots \otimes 1 \otimes \dots \otimes \nu_m)$ and by our induction on n-m , we have $\bar{F}(\lambda) = \bar{F}(\lambda_1 \circ (\nu_1 \otimes \dots \otimes 1 \otimes \dots \otimes \nu_m)) \circ F(1 \otimes \dots \otimes \nu_k \otimes \dots \otimes 1)$. Thus it follows that $\bar{F}(\lambda) = \bar{F}(\lambda_1) \circ \bar{F}(\lambda_2)$. Therefore everything is wonderful on the homogeneous morphisms of $\mathfrak{G}$.

To extend $\bar{F}$ to non-homogeneous morphisms, one should first notice that any simple, non-homogeneous $\mathfrak{G}$-morphism can be written uniquely as the composition of a homogeneous simple $\mathfrak{G}$-morphism and an $\mathbf{N}$-morphism.

Thus, if λ is homogeneous and simple, and ν is an $\mathbf{N}$-morphism and $\eta = \nu \cdot \lambda$, we define $\bar{F}(\eta) = F(\nu) \circ \bar{F}(\lambda)$. To show that $\bar{F}$ preserves composition of morphisms, one proceeds much as in the homogeneous case, but making use of condition A) rather than condition B). We omit the details.

APPENDIX II. <u>The Proof of Proposition 7.5</u>

We wish to construct an isomorphism of $\mathfrak{G}$-monoids:

$$\operatorname*{colim}_{\mathfrak{M}} F \cong \operatorname*{colim}_{\mathfrak{M}} F_T .$$

<u>Step 1</u>. To construct an ($\mathfrak{G}$-monoid)-morphism

$$\Psi: \operatorname*{colim}_{\mathfrak{M}} F_T \cong \operatorname*{colim}_{\mathfrak{M}} F .$$

We will make use of the conaturalizer properties of F_T. First, define a transformation $\zeta: TF^{\amalg} \to \operatorname*{colim}_{\mathfrak{M}} F$ by the rule: if $\alpha \in \mathfrak{M}$, then $\zeta_\alpha: TF^{\amalg}(\alpha) \to \operatorname*{colim}_{\mathfrak{M}} F$ is the composition:

$$\zeta_\alpha: TF^{\amalg}(\alpha) \xrightarrow{T(i_\alpha)} TF(\alpha) \xrightarrow{\chi_\alpha} \operatorname*{colim}_{\mathfrak{M}} F ,$$

where $i: F^{\amalg} \to F$ is the natural transformation of 7.3, and χ_α is the adjoint of the natural map $F(\alpha) \to \operatorname*{colim}_{\mathfrak{M}} F$.

Since $T(i_\alpha)$ and χ_α are $\mathfrak{G}$-monoid maps, so is ζ_α. Since Ti and χ are natural over $\mathfrak{M}$, so is ζ. I claim further that the composition $\zeta \circ \bar{\Lambda}_T$ is equal to χ. To see this, by adjointness it is enough to show that if $\alpha \in \mathfrak{M}$, then

$$\zeta_\alpha \circ \bar{\Lambda}_T(\alpha)\big|_{F(\alpha)}: F(\alpha) \to \operatorname*{colim}_{\mathfrak{M}} F$$

is just the usual map into the colimit. This, in turn, is a straightforward application of the definition of Λ_T as defined in 7.3A. I omit the details.

But now, the definition of conaturalizer gives us a unique natural transformation $\bar{\Psi}: F_T \to \operatorname*{colim}_{\mathfrak{M}} F$ such that $\bar{\Psi} \cdot \rho_T = \zeta$. Define $\Psi = \operatorname*{colim}_{\mathfrak{M}} \bar{\Psi}: \operatorname*{colim}_{\mathfrak{M}} F_T \to \operatorname*{colim}_{\mathfrak{M}} F$.

Step 2. To define an ($\mathcal{G}$-monoid)-morphism

$$\varphi\colon \operatorname*{colim}_{\mathfrak{M}} F \to \operatorname*{colim}_{\mathfrak{M}} F_T \quad .$$

Let $\varphi_\alpha\colon F(\alpha) \to \operatorname*{colim}_{\mathfrak{M}} F_T$ be the composition:

$$F(\alpha) \xrightarrow{\ i_\alpha\ } TF(\alpha) \xrightarrow{\ \bar{\rho}_\alpha\ } F_T(\alpha) \longrightarrow \operatorname*{colim}_{\mathfrak{M}} F_T \ ,$$

where i_α is the map adjoint to the identity map on $TF(\alpha)$. This is clearly a natural transformation: $F \to \operatorname*{colim}_{\mathfrak{M}} F_T$ and so we define

$$\varphi = \operatorname*{colim}_{\alpha\in\mathfrak{M}} \varphi_\alpha\colon \operatorname*{colim}_{\mathfrak{M}} F \to \operatorname*{colim}_{\mathfrak{M}} F_T \quad .$$

We must show that φ is an $\mathcal{G}$-monoid morphism. In light of the definition of the monoid structure on $\operatorname*{colim}_{\mathfrak{M}} F$ (see 1.4), it is sufficient to show that the following diagram commutes $\forall\ \alpha,\beta \in \mathfrak{M}$.

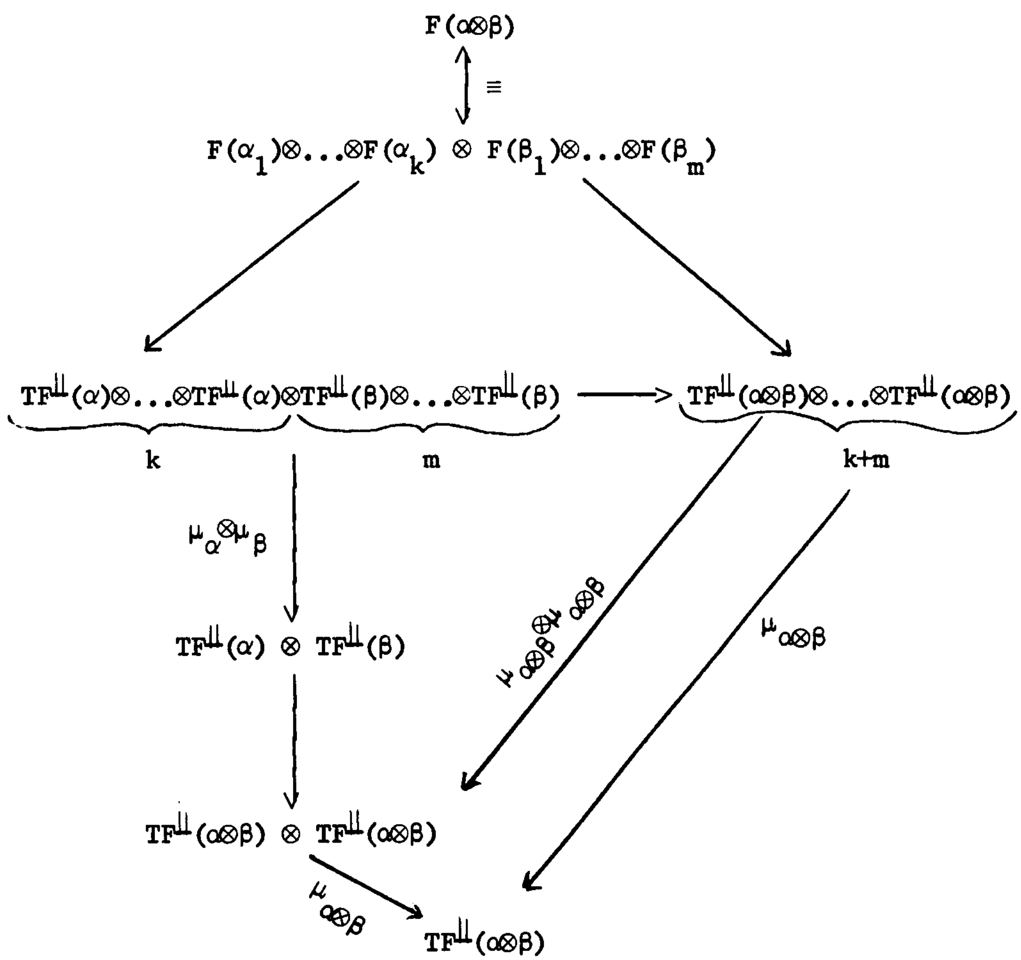
$$\begin{array}{ccc}
F(\alpha)\otimes F(\beta) & \xrightarrow{\ \varphi_\alpha \otimes \varphi_\beta\ } & (\operatorname*{colim}_{\mathfrak{M}} F_T) \otimes (\operatorname*{colim}_{\mathfrak{M}} F_T) \\
\big\updownarrow \equiv & & \big\downarrow \mu \\
F(\alpha\otimes\beta) & \xrightarrow{\ \varphi_{\alpha\otimes\beta}\ } & \operatorname*{colim}_{\mathfrak{M}} F_T
\end{array}$$

where μ is the monoid structure on $\operatorname*{colim}_{\mathfrak{M}} F_T$. But this diagram can be factored into the following diagram:

$$
\begin{array}{ccccccccc}
F(\alpha)\otimes F(\beta) & \xrightarrow{\subseteq} & TF(\alpha)\otimes TF(\beta) & \xrightarrow{\Lambda_\alpha\otimes\Lambda_\beta} & TF^{\amalg}(\alpha)\otimes TF^{\amalg}(\beta) & \longrightarrow & (\operatorname*{colim}_{\mathfrak{m}} TF^{\amalg})\otimes(\operatorname*{colim}_{\mathfrak{m}} TF^{\amalg}) & \xrightarrow{\rho_T\otimes\rho_T} & (\operatorname*{colim}_{\mathfrak{m}} F_T)\otimes(\operatorname*{colim}_{\mathfrak{m}} F_T) \\
\Big\downarrow \cong & & & & \Big\downarrow & & & & \\
 & & & & TF^{\amalg}(\alpha\otimes\beta)\otimes TF^{\amalg}(\alpha\otimes\beta) & & \Big\downarrow \mu^{\amalg} & & \Big\downarrow \mu \\
 & & & & \Big\downarrow & & & & \\
F(\alpha\otimes\beta) & \xrightarrow{\subseteq} & TF(\alpha\otimes\beta) & \xrightarrow{\Lambda_{\alpha\otimes\beta}} & TF^{\amalg}(\alpha\otimes\beta) & \longrightarrow & \operatorname*{colim}_{\mathfrak{m}} TF^{\amalg} & \xrightarrow{\rho_T} & \operatorname{colim} F_T
\end{array}
$$

(AII.1)

The last two rectangles clearly commute. To see that the first one commutes, let $\alpha = \alpha_1 \otimes \ldots \otimes \alpha_k$, $\beta = \beta_1 \otimes \ldots \otimes \beta_m$, where α_i, β_j are generators of $\mathfrak{m}$, $\forall$ i,j . Then $F(\alpha) = F(\alpha_1) \otimes \ldots \otimes F(\alpha_k)$ and $F(\beta) = F(\beta_1) \otimes \ldots \otimes F(\beta_k)$ and we can rewrite the first rectangle of (AII.1) as follows (using the definition of Λ_T from 7.3A):



It is (believe it or not) rather easy to see that each triangle of this diagram commutes. μ_γ is the monoid operation on $TF^{\sqcup\sqcup}(\gamma)$, $\forall\ \gamma \in \mathfrak{m}$ and the unmarked morphisms are all maps defined in the manner of a_x and a_y in 7.3A. The commutativity is just a consequence of the naturality of the transformations and functors involved, and the associativity of μ.

It therefore follows that φ is an $\mathfrak{S}$-monoid morphism.

<u>Step 3</u>. To show that $\Psi\varphi = 1_{\underset{\mathfrak{m}}{\mathrm{colim}}\, F}$.

From the definitions of Ψ and φ, we have that for each $\alpha \in \mathfrak{m}$, the following diagram commutes:

$$\begin{array}{ccccc} F(\alpha) & \xrightarrow{\varphi_\alpha} & F_T(\alpha) & \xrightarrow{\Psi_\alpha} & \underset{\mathfrak{m}}{\mathrm{colim}}\, F \\ & {\scriptstyle i_\alpha}\searrow & \uparrow {\scriptstyle \bar{\rho}_T(\alpha)} & \nearrow {\scriptstyle \chi_\alpha} & \\ & & TF(\alpha) & & \end{array} .$$

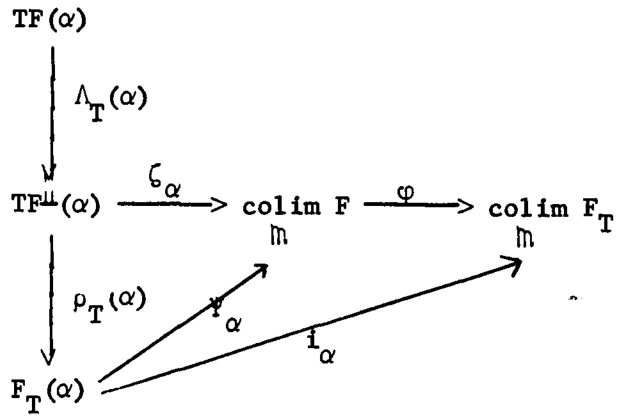
But $\chi_\alpha i_\alpha$ is the usual map: $F(\alpha) \to \underset{\mathfrak{m}}{\mathrm{colim}}\, F$. Hence $\underset{\alpha\in\mathfrak{m}}{\mathrm{colim}}\, \Psi_\alpha\varphi_\alpha = 1_{\underset{\mathfrak{m}}{\mathrm{colim}}\, F}$ and $\Psi\varphi = \underset{\alpha\in\mathfrak{m}}{\mathrm{colim}}\, \Psi_\alpha\varphi_\alpha$.

<u>Step 4</u>. $\varphi\Psi = 1_{\underset{\mathfrak{m}}{\mathrm{colim}}\, F}$.

Let $j_\alpha \colon F_T(\alpha) \to \underset{\mathfrak{m}}{\mathrm{colim}}\, F_T$ be the usual map. We must show that the following diagram commutes for each $\alpha \in \mathfrak{m}$:

$$\begin{array}{ccc} F_T(\alpha) & \xrightarrow{\Psi_\alpha} & \underset{\mathfrak{m}}{\mathrm{colim}}\, F \\ & {\scriptstyle j_\alpha}\searrow & \downarrow {\scriptstyle \varphi} \\ & & \underset{\mathfrak{m}}{\mathrm{colim}}\, F_T \end{array} .$$

Consider the following diagram:



From the definition of Ψ_α , we know that $\Psi_\alpha \rho_T(\alpha) = \zeta_\alpha$. If $i_\alpha \rho_T(\alpha) = \varphi \cdot \zeta_\alpha$, then the conaturalizer nature of F_T implies the existence of a <u>unique</u> map, $f\colon F_T(\alpha) \to \operatorname*{colim}_{\mathfrak{m}} F_T$ such that $f\rho_\alpha = \varphi \cdot \zeta_\alpha$ (since $\varphi\zeta$ conaturalizes Λ_T) , and hence $\varphi \cdot \Psi_\alpha$ will equal i_α .

Since ζ_α , φ , i_α and ρ_α are all $\mathfrak{G}$-monoid homomorphisms, it is enough to check the equality of $\varphi\zeta_\alpha$ and $i_\alpha \rho_\alpha$ on $F^{\amalg}(\alpha)$. But checking the definitions of the various maps, $\varphi\zeta_\alpha|_{F^{\amalg}(\alpha)}$ is the composition:

$$F^{\amalg}(\alpha) \xrightarrow{i_\alpha} F(\alpha) \xrightarrow{\subseteq} TF(\alpha) \xrightarrow{\bar{\Lambda}_T} TF^{\amalg}(\alpha) \xrightarrow{\rho_\alpha} F_T(\alpha) \xrightarrow{i_\alpha} \operatorname*{colim}_{\mathfrak{m}} F_T$$

and $\bar{\Lambda}_T \circ F(i_\alpha) = 1_{TF}$. Therefore $\varphi\zeta_\alpha\Big|_{F^{\amalg}(\alpha)} = \rho_\alpha i_\alpha\Big|_{F^{\amalg}(\alpha)}$. Hence

$\varphi\zeta_\alpha = \rho_\alpha i_\alpha$, $\forall\, \alpha \in \mathfrak{m}$ and hence $\varphi\Psi_\alpha = i_\alpha$. Hence

$$\varphi\Psi = \operatorname*{colim}_{\mathfrak{m}} \varphi\Psi_\alpha = 1_{\operatorname*{colim}_{\mathfrak{m}} F_T} .$$

This concludes the proof of the proposition.

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AUTOBIOGRAPHICAL STATEMENT

Peter Van Zandt Cobb was born in 1943 in Northfield, Minnesota, but grew up in Vermillion, South Dakota, where his father was on the faculty of the University of South Dakota. He attended Pomona College in Claremont, California; receiving his B.A. cum laude and Phi Beta Kappa in 1965. After one year of graduate study at Wesleyan University, he entered the graduate mathematics program at The City University of New York/Graduate Center. Since February 1970, he has been an Instructor in the mathematics department at Brooklyn College.

In December, 1971, he married Nina Kressner, a doctoral candidate in history at The City University of New York.