Conformally Natural Extensions of Continuous Circle Maps

by

Oleg Muzician

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Jun Hu

Date

Chair of Examining Committee

Jozef Dodziuk

Date

Executive Officer

Jozef Dodziuk

Jun Hu

Dragomir Saric

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Advisor: Jun Hu

Conformally natural and continuous extensions were originally introduced by Douady and Earle for circle homeomorphisms, and later by Abikoff, Earle and Mitra for continuous degree ±1 monotone circle maps. The first main result of this thesis shows that conformally natural and continuous extensions exist for all continuous circle maps. The second main result shows that if f is a continuous circle map and is M-quasisymmetric on some arc on the unit circle $\mathbb{S}^1$, then such an extension of f is locally K-quasiconformal on a neighborhood of the arc in the open unit disk $\mathbb{D}$, where the neighborhood and K depend only on M.

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Chapter 1

Introduction

In 1986, by using conformal barycenters, Douady and Earle extended homeomorphisms f of the unit circle to homeomorphisms $\Phi(f)$ of the closed unit disk satisfying:

1. the extension of the identity map on the circle is the identity map on the unit disk;

2. for any homeomorphism f of the unit circle and any conformal isometries A and B of the unit disk,

$$\Phi(A \circ f \circ B) = A \circ \Phi(f) \circ B. \tag{1.1}$$

Such an extension procedure is called a *conformally natural extension operator*, and such extensions are called *conformally natural extensions.*

Definition. Given any probability measure μ on $\mathbb{S}^1$, a point $w \in \mathbb{D}$ is defined to be *the conformal barycenter* of μ if

$$\int_{\mathbb{S}^1} \frac{\zeta - w}{1 - \zeta \bar{w}} d\mu(\zeta) = 0.$$

We denote such a barycenter by $B(\mu)$.

Remark 1. There exists a unique conformal barycenter for any probability measure as long as there are no atoms of weight more than or equal to 1/2. A measure with such a property is called an *admissible measure.*

Douady and Earle defined an extension $\Phi(f)$ of a circle homeomorphism f as follows. Given any point $a \in \mathbb{D}$, $\Phi(f)(a)$ is defined to be the conformal barycenter of the push-forward measure $f_*(\eta_a)$ of η_a by f (that is, $B(f_*(\eta_a))$), where η_a is the harmonic measure on $\mathbb{S}^1$ with respect to a, normalized to have total measure equal to 1.

A natural problem is to investigate if conformally natural extensions exist for orientation preserving endomorphisms of the circle. Such an extension may have applications in the study of Teichmüller spaces of such endomorphisms. Furthermore, we may ask whether conformally natural extensions exist for all continuous maps. Applications of conformally natural extensions of circle homeomorphisms have been found in physics, biochemistry and bioengineering. One can expect applications of conformally natural extensions of arbitrary continuous circle maps in these fields if they exist. Another related question is to study local and global properties $\Phi(f)$ inherits from f in these larger families of circle maps.

It is easy to see that the two properties of the Douady-Earle extensions of circle homeomorphisms imply that if f is the restriction to the circle of a conformal isometry A of the unit disk, then $\Phi(f) = A$. To consider extensions of more general circle maps, we generalize the concept of *a conformally natural extension operator* as follows:

Definition (Conformal naturality)**.** An extension operator Φ from the space of continuous circle maps to the space of maps of the closed disk is said to be *conformally natural* provided that

1. if f is the restriction to the circle of a complex analytic map Ψ defined on a neighborhood of $\overline{\mathbb{D}}$, then $\Phi(f) = \Psi$;

2. for any continuous map f and any two conformal isometries A and B of the unit disk,

$$\Phi(A \circ f \circ B) = A \circ \Phi(f) \circ B.$$

In [1], Abikoff gave an alternative interpretation of conformal barycenters in terms of the unique attractors of the so-called MAY iterators for admissible measures. Then, in [3], Abikoff, Earle and Mitra showed that the MAY iterator also has a unique attractor on $\mathbb{S}^1$ for any non-admissible measure with only one atom of measure $\geq \frac{1}{2}$. They called this attractor *a conformal barycenter for such a non-admissible measure.* Furthermore, they used these conformal barycenters to construct conformally natural extensions for continuous degree ±1 monotone circle maps ([3]). Recently in [10], Prof. Hu and I continued to use conformal barycenters of admissible measures to construct conformally natural extensions for those continuous circle maps f under which the push-forwards of Lebesque measure on the circle have no atoms (such maps are called *maps without atoms*).

Let f be a non-constant continuous circle map. For any point $z \in \mathbb{D}$, let $f_*(\eta_z)$ be the push-forward under f of the normalized harmonic measure η_z on $\mathbb{S}^1$ with respect to z. Then $f_*(\eta_z)$ is a measure with at most one atom of measure $\geq \frac{1}{2}$ and the associated MAY iterator has a unique attractor w on the closed unit disk $\overline{\mathbb{D}}$. One may define an extension map by assigning w to z. Indeed, this is the method used by Abikoff, Earle and Mitra in [3] to introduce and study conformally natural extensions for continuous degree ±1 monotone circle maps. However, the study of such extensions for arbitrary continuous circle maps has not been developed. The main goal of this thesis is to study conformally natural extensions for such a large class of circle maps.

At first, we introduce an abstract but much simpler way to define extensions for all continuous circle maps. Then, we show that such extensions are conformally natural, continuous on the closed disk and also real-analytic on some regions.

Given a non-constant continuous circle map f, we divide the points of the unit disk $\mathbb{D}$ into two sets. Let β_f be the set of all points $z \in \mathbb{D}$ for which $f_*(\eta_z)$ is an admissible measure, and let $\delta_f = \mathbb{D} \backslash \beta_f$. For each $z \in \delta_f$, one can show that $f_*(\eta_z)$ has a unique atom $\gamma_z \in \mathbb{S}^1$ of measure $\geq 1/2$. Therefore, we define

$$\Phi(f)(z) = \begin{cases} f(z), & z \in \mathbb{S}^1 \\ \gamma_z, & z \in \delta_f \\ B(f_*(\eta_z)), & z \in \beta_f \end{cases} .$$

Remark 2. (1) If f is a constant circle map, then we simply define the extension to be equal to the constant map. In this case, β_f is empty and $\delta_f = \mathbb{D}$.

(2) If f is not a constant continuous circle map, then given any point $z \in \mathbb{D}$, $\Phi(f)(z)$ is indeed equal to the attractor of the MAY iterator associated with the measure $f_*(\eta_z)$.

(3) It will be shown that β_f is an open subset of $\mathbb{D}$.

The following two theorems are obtained:

Theorem 1. *The extensions* $\Phi(f)$ *of continuous circle maps* f *are conformally natural.*

Theorem 2. *Let f be a continuous circle map. Then $\Phi(f)$ is a continuous map on $\overline{\mathbb{D}}$ and is real-analytic on β_f.*

In the second half of the thesis, we generalize the local regularities of conformally natural extensions near the boundary circle from the maps studied in [9] and [10] to non-constant continuous circle maps.

Let f be a circle map. Assume that I is a connected circular arc on $\mathbb{S}^1$ and the restriction $f|_I$ of f to I is continuous, injective and orientation-preserving. The *cross-ratio distortion norm* of $f|_I$ is defined to be

$$||f|_I||_{cr} = \sup_{cr(Q)=1} |\ln cr(f(Q))|,$$

where $Q = \{a, b, c, d\} \subset I$, $f(Q) = \{f(a), f(b), f(c), f(d)\}$ and $cr(Q)$ is a cross-ratio of Q defined as

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

We say f is a *quasisymmetric* map on I if there is a constant $M > 0$ such that $||f|_I||_{cr} \leq M$.

In [9], it was shown that the quasiconformality of the extension $\Phi(f)$ of a homeomorphism f is a local property in the following sense: If f is a circle homeomorphism that is quasisymmetric on an arc I, then the extension $\Phi(f)$ of f is quasiconformal on D, where D is a hyperbolic half-plane with I as its outer boundary. Moreover, the maximal dilatation of $\Phi(f)$ on D depends linearly on the cross-ratio distortion norm of f on I.

In [10], it was shown that if f is a continuous circle map without atoms and is M-quasisymmetric on an arc I, then there exists a neighborhood U of I in $\overline{\mathbb{D}}$ and a constant K only depending on M such that $\Phi(f)$ is locally quasiconformal on U with maximal dilatations uniformly bounded by K. Furthermore, it is also shown in [10] that if f is a locally quasisymmetric orientation preserving circle endomorphism of degree d with local cross-ratio distortion norms bounded by a positive constant M, then there exist two annulus neighborhoods U and V of $\mathbb{S}^1$ in $\overline{\mathbb{D}}$ such that $\Phi(f)|_U : U \to V$ is a quasiregular covering map of degree d with maximal dilatation bounded by a constant K depending only on M.

In this thesis, we recapitulate the techniques of [10] to show that the same local regularity holds for the extensions of all non-constant continuous circle maps. That is,

Theorem 3. *Let f be a continuous circle map and $I \subset \mathbb{S}^1$ be an open arc. If f is quasisymmetric on I with the cross-ratio distortion norm of $f|_I$ bounded by a constant $M > 0$, then $\Phi(f)$ is locally K-quasiconformal on D, where $D = \{z \in \mathbb{D} : \eta_z(I) > 1 - \rho/2\pi\}$, and ρ and K depend only on M.*

Remark 3. In [10], after obtaining the local quasiconformality of the extension map $\Phi(f)$ on a neighborhood D of I, we are able to show that if the boundary map f is an orientation-preserving endomorphism of the circle, then the extension is an injective map on a smaller neighborhood D' of I in $\overline{\mathbb{D}}$. But the scheme leading to the injectivity of $\Phi(f)$ on D' for an endomorphism f can not be applied to obtain the same result for extensions of either the continuous circle maps considered in [10] or the arbitrary continuous circle maps considered in this thesis. It remains an interesting question to study whether the extension $\Phi(f)$ is quasiconformal on a neighborhood D' of I in $\overline{\mathbb{D}}$ provided that a continuous circle map f is quasisymmetric on an arc I on the circle.

Remark 4. It is known that the dynamics of an orientation-preserving circle homeomorphism f is rigid if the cross-ratio distortions under f are controlled in some magnitudes (see [13] and [12]). The cross-ratio distortion norm $||f||_{cr}$ of f has been employed to study quantitative relationships between the quasisymmetry of f and the quasi-isometry of the Thurston earthquake representation E_f ([14]) of f (see [5], [6], [7], [11], and etc.). Some ideas and techniques given in [13], [12], [5] and [6] are motivations to develop quantitative relationships between the quasisymmetry (resp. local quasisymmetry) of f and the quasiconformality (resp. local) of the Douady-Earle extension

$\Phi(f)$ in [8] and [9], and then to develop quantitative relationships between the local quasisymmetry of a continuous circle map f and the local quasiconformality of the conformal natural extension $\Phi(f)$ in [10] and this thesis.

Chapter 2

The Douady-Earle Extension for Homeomorphisms

2.1 Preliminaries

In this chapter, we give a brief account of the Douady-Earle extensions for circle homeomorphisms. The key to construct such extensions is the existence and uniqueness of the conformal barycenters for certain probability measures on the unit circle $\mathbb{S}^1$. In the case of homeomorphisms, one only need to consider probability measures without atoms. In [4], Douady and Earle showed the existence and uniqueness of the conformal barycenter for such measures. They also pointed out that existence and uniqueness of the conformal barycenter is also true for probability measures with atoms of measure less than 1/2 (i.e. admissible measures) but the proof needs more work. In this chapter we recall Douady-Earle's proof for the case when the measure has no atoms in this chapter. Then in Chapter 5, we give a proof to the case when the measure is admissible .

Let μ be a probability measure with no atoms on the unit circle $\mathbb{S}^1$. Let

$$\xi_\mu(w) = (1-|w|^2)\int_{\mathbb{S}^1} g_w(\zeta)d\mu(\zeta),$$

where $g_w(z) = \frac{z-w}{1-\bar{w}z}$.

Then we define the conformal barycenter of μ to be the unique point $w = B(\mu) \in \mathbb{D}$ such that $\xi_\mu(w) = 0$. It needs to be proven that such a barycenter exists and is, in fact, unique. To show this we need to describe a few properties of the vector field ξ_μ.

The assignment of a vector field ξ_μ to a measure μ is conformally natural in the following sense: Let $L : Prob(\mathbb{S}^1) \to VF(\mathbb{D})$ be a map that maps each probability measure $\mu \in Prob(\mathbb{S}^1)$ on $\mathbb{S}^1$ to a vector field $\xi_\mu \in VF(\mathbb{D})$ on $\mathbb{D}$, and let G be a group of conformal maps from the unit disk to itself preserving $\mathbb{S}^1$. Then G acts on $Prob(\mathbb{S}^1)$ and on $VF(\mathbb{D})$ as follows:

$$\begin{aligned} g\cdot\mu &= g_*(\mu) \\ g\cdot v(g(z)) &= g'(z)v(z), \end{aligned}$$

where $v \in VF(\mathbb{D})$, $\mu \in Prob(\mathbb{S}^1)$ and $g \in G$. Then we say L is *conformally natural* if $L(g\cdot(\mu)) = g\cdot(L(\mu))$. That is,

Lemma 1 (Conformal naturality of ξ_μ)**.** *For all* $g \in G$*,*

$$\xi_{g_*(\mu)}(g(z)) = g'(z)\xi_\mu(z).$$

Proof. Let $g \in G$. By using the definition of the vector field ξ_μ,

$$\begin{aligned}
\xi_{g_*(\mu)}(g(z)) &= (1-|g(z)|^2)\int_{\mathbb{S}^1} g_{g(z)}(\zeta)dg_*(\mu)(\zeta)\\
&= (1-|g(z)|^2)\int_{\mathbb{S}^1} g_{g(z)}(g(\zeta))d\mu(\zeta)\\
&= (1-|g(z)|^2)\int_{\mathbb{S}^1} \frac{g(\zeta)-g(z)}{1-\overline{g(z)}g(\zeta)}d\mu(\zeta)\\
&= (1-|g(z)|^2)e^{i\theta}\int_{\mathbb{S}^1} \frac{\zeta-z}{1-\overline{z}\zeta}d\mu(\zeta)
\end{aligned}$$

for some θ.

Note that $\frac{g(\zeta)-g(z)}{1-\overline{g(z)}g(\zeta)} = e^{i\theta}\frac{\zeta-z}{1-\overline{z}\zeta}$. Taking derivatives of both sides with respect to ζ and evaluating them at z, we obtain

$$\begin{aligned}
\frac{d}{d\zeta}\left(\frac{g(\zeta)-g(z)}{1-\overline{g(z)}g(\zeta)}\right)\Big|_{\zeta=z} &= \frac{g'(\zeta)(1-\overline{g(z)}g(\zeta))+\overline{g(z)}g'(\zeta)(g(\zeta)-g(z))}{(1-\overline{g(z)}g(\zeta))^2}\Bigg|_{\zeta=z}\\
&= \frac{g'(z)}{1-|g(z)|^2}
\end{aligned}$$

and

$$\frac{d}{d\zeta}\left(e^{i\theta}\frac{\zeta-z}{1-\overline{z}\zeta}\right)\Big|_{\zeta=z} = e^{i\theta}\frac{(1-\overline{z}\zeta)+\overline{z}(\zeta-z)}{(1-\overline{z}\zeta)^2}\Bigg|_{\zeta=z} = \frac{e^{i\theta}}{1-|z|^2}.$$

Then

$$\frac{g'(z)}{1-|g(z)|^2} = \frac{e^{i\theta}}{1-|z|^2}.$$

Therefore,

$$(1-|g(z)|^2)e^{i\theta}=(1-|z|^2)g'(z).$$

Thus,

$$\begin{aligned}\xi_{g_*(\mu)}(g(z)) &= (1-|g(z)|^2)e^{i\theta}\int_{\mathbb{S}^1}\frac{\zeta-z}{1-\overline{z}\zeta}d\mu(\zeta)\\ &= g'(z)(1-|z|^2)\int_{\mathbb{S}^1}g_z(\zeta)d\mu(\zeta)\\ &= g'(z)\xi_\mu(z).\end{aligned}$$

□

Lemma 2. *$Re(\xi_\mu(0))>0$ if $\mu(\widehat{[e^{-i\pi/4},e^{i\pi/4}]})\geq 2/3$.*

Proof. $Re(\xi_\mu(0))=\int_{\mathbb{S}^1}Re(\zeta)d\mu(\zeta)\geq(-1)\frac{1}{3}+\left(\frac{\sqrt{2}}{2}\right)\frac{2}{3}>0$. □

Theorem 4 (Poincare-Hopf Index Theorem)**.** *Let M be a differentiable, compact, orientable manifold with a boundary. Let V be a differentiable vector field over M with isolated singular points p_i and assume that V points inside M on the boundary of M. Let $I(p_i)$ denote the index of V at p_i. Then*

$$\sum_i I(p_i)=\chi(M).$$

Corollary 1. *Let V be a differentiable vector field on $\mathbb{D}$ with $Jac(V(z))>0$ for all $z\in\mathbb{D}$. If D is a closed topological disk contained in $\mathbb{D}$ with piecewise smooth boundary and V points inside at any point on the boundary of D, then V has exactly one singularity in D.*

Proof. Since $Jac(V(z)) > 0$ for all $z \in \mathbb{D}$, V is an orientation preserving and locally injective vector field, all singularities p_i of V are isolated and have index $I(p_i) = 1$. By the Poincare-Hopf Index Theorem,

$$\sum_{p_i \in D} I(p_i) = \sum_{p_i \in D} 1 = \chi(D) = 1.$$

Therefore, there is exactly one singularity. □

Proposition 1 (Douady-Earle). *Suppose μ is a probability measure with no atoms. Then ξ_μ has a unique zero in $\mathbb{D}$.*

Proof. It is easy to see

$$\begin{aligned}
\xi_\mu(w) &= (1-|w|)^2 \int_{\mathbb{S}^1} \frac{\zeta - w}{1 - \bar{w}\zeta} d\mu(\zeta) \\
&= (1-|w|)^2 \int_{\mathbb{S}^1} (\zeta - w)(1 + \bar{w}\zeta) d\mu(\zeta) + o(w) \\
&= \xi_\mu(0) - w + \bar{w} \int_{\mathbb{S}^1} \zeta^2 d\mu(\zeta) + o(w).
\end{aligned}$$

The Jacobian of ξ_μ at $w = 0$ is

$$\begin{aligned}
Jac(\xi_\mu)(0) &= |(\xi_\mu)_w(0)|^2 + |(\xi_\mu)_{\bar{w}}(0)|^2 \\
&= 1 - \int \int_{\mathbb{S}^1 \times \mathbb{S}^1} \zeta^2 \bar{z}^2 d\mu(\zeta) d\mu(z) \\
&= \frac{1}{2} \int \int_{\mathbb{S}^1 \times \mathbb{S}^1} |z^2 - \zeta^2|^2 d\mu(\zeta) d\mu(z) > 0.
\end{aligned}$$

By the conformal naturality of ξ_μ, the Jacobian $Jac(\xi_\mu)$ is positive at every other point of $\mathbb{D}$.

If we show that there exists an $r_0 < 1$, but close to 1 such that for all $r \in (r_0, 1)$ the vector field ξ_μ has no zeros on $C_r = \{w : |w| = r\}$ and points inward, then using Corollary 1, we will conclude that ξ_μ has a unique zero in $\mathbb{D}$.

Let $\alpha > 0$ be such that $\mu(J) \leq \frac{1}{3}$ for any arc $J \subset \mathbb{S}^1$ of length $\leq \alpha$. Take $r_0 < 1$ such that the arc J_α of length α centered at 1 so that the visual measure at r_0 with respect to the Poincare metric is $3\pi/2$. If $|w| = r \geq r_0$, let g be the a Mobius map taking w to 0 and $\frac{-w}{|w|}$ to 1 (Figure 2.1). Let $\nu = g_*(\mu)$. Then,

$$\nu(g(J_\alpha^c)) = \mu(J_\alpha^c) \geq 2/3.$$

Since $\nu(\widehat{[e^{-i\pi/4}, e^{i\pi/4}]}) \geq \nu(g(J_\alpha^c)) \geq 2/3$, by Lemma 2, $Re(\xi_\nu(0)) > 0$, and so $\xi_\nu(0)$ points into $g(C_r)$. By conformal naturality, $\xi_\mu(w)$ points into C_r.

$\square$

2.2 Extension and Conformal Naturality

Let η_a denote a harmonic measure on $\mathbb{S}^1$ with respect to $a \in \mathbb{D}$ normalized to have a total measure equal to 1. More explicitly, for any Borel set $A \subset \mathbb{S}^1$ we have

$$\eta_a(A) = \frac{1}{2\pi} \int_A \frac{1 - |a|^2}{|a - \zeta|^2} |d\zeta|.$$

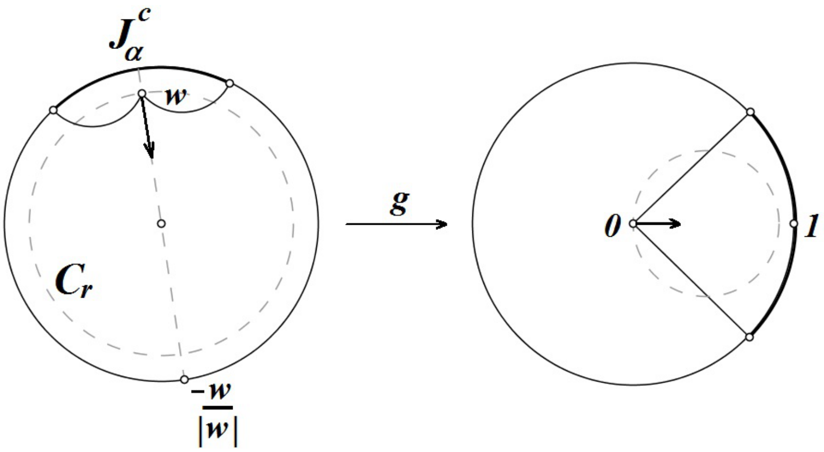

Figure 2.1: An illustration for Proposition 1.

A harmonic measure η_a can be also viewed as a pushforward of the Lebeque measure by g_a, where $g_a(z) = \frac{z-a}{1-\bar{a}z}$.

Now, we can define extensions $\Phi(f)$ for a circle homeomorphisms f to the closed unit disk $\overline{\mathbb{D}}$ as follows:

$$\Phi(f)(z) = \begin{cases} f(z), & z \in \mathbb{S}^1 \\ B(f_*(\eta_z)), & z \in \mathbb{D} \end{cases} .$$

Proposition 2 ([4]). *Given a circle homeomorphism f, the extension $\Phi(f)$ is conformally natural.*

Proof. Let f be a circle homeomorphism and let A and B be two Mobius maps preserving the unit disk. Let $z \in \mathbb{D}$ and $b = B(z)$. Then, $\Phi(f)(b) = w$

means

$$\int_{\mathbb{S}^1} g_w \circ f \circ g_b^{-1}(\zeta)|d\zeta| = 0.$$

On the other hand, $\Phi(A \circ f \circ B)(z) = w^*$ means

$$0 = \int_{\mathbb{S}^1} g_{w^*} \circ A \circ f \circ B \circ g_z^{-1}(\zeta)|d\zeta| = \int_{\mathbb{S}^1} g_{w^*} \circ A \circ f \circ B \circ g_{B^{-1}(b)}^{-1}(\zeta)|d\zeta|.$$

If $w^* = A(w)$, then

$$\begin{aligned}
& \int_{\mathbb{S}^1} g_{w^*} \circ A \circ f \circ B \circ g_{B^{-1}(b)}^{-1}(\zeta)|d\zeta| \\
= & \int_{\mathbb{S}^1} (g_{A(w)} \circ A) \circ f \circ (B \circ g_{B^{-1}(b)}^{-1})(\zeta)|d\zeta| \\
= & \int_{\mathbb{S}^1} (e^{i\theta_1} g_w) \circ f \circ (g_{B^{-1}(b)} \circ B^{-1})^{-1}(\zeta)|d\zeta| \\
= & \int_{\mathbb{S}^1} (e^{i\theta_1} g_w) \circ f \circ (g_b)^{-1}(e^{i\theta_2}\zeta)|d\zeta| \\
= & \ e^{i\theta_1} \int_{\mathbb{S}^1} (g_w) \circ f \circ (g_b)^{-1}(\zeta)|d\zeta| = 0,
\end{aligned}$$

for some real numbers θ_1 and θ_2.

By uniqueness of the conformal barycenter,

$$\Phi(A \circ f \circ B)(z) = A \circ \Phi(f) \circ B(z) = A(w).$$

□

2.3 Continuity

Next, we show continuity of $\Phi(f)$. Inside $\mathbb{D}$, $\Phi(f)$ is continuous because η_z changes continuously with z, and so we only need to show continuity on $\mathbb{S}^1$.

Proposition 3. *If f is a circle homeomorphism, then $\Phi(f)$ is a continuous map at every point of $\mathbb{S}^1$.*

Proof. Let J be an arc in $\mathbb{S}^1$. Let

$$V(f(J)) = \{\zeta \in \mathbb{D} : \eta_\zeta(f(J)) \geq 1/4\} \cup f(J)$$

and

$$U(J) = \{\zeta \in \mathbb{D} : \eta_\zeta(J) \geq 2/3\}.$$

Let $z \in U(J)$ and $\mu = f_*(\eta_z)$, then $\mu(f(J)) \geq 2/3$.

Let $w \in \Gamma = \partial V(f(J)) \cap \mathbb{D}$. There exists $g \in Mob(\mathbb{D})$ such that $g(f(J)) = \widehat{(e^{-i\pi/4}, e^{i\pi/4})}$ and $g(w) = 0$. Let $\nu = g_*(\mu)$. Then $\nu(g(f(J))) \geq 2/3$ and by Lemma 2, $Re(\xi_\nu(0)) > 0$. By conformal naturality, $\xi_\mu(w)$ points inside of $V(f(J))$. Since ξ_μ points into $\mathbb{D}$ near the boundary of $\mathbb{D}$ and it is a continuous vector field in $\mathbb{D}$, it follows that $B(\mu) \in int(V(f(J)))$. Hence, $\Phi(U(J)) \subset V(f(J))$.

Now, let $z \in \mathbb{S}^1$ and J be any neighborhood of z. Since f is a homeomorphism, $f(z) \in f(J)$ and $f(J)$ shrinks to $f(z)$ as J shrinks to z. Therefore, $J \cup U(J)$ is a neighborhood of $z \in \overline{\mathbb{D}}$ and the sets $f(J) \cup V(f(J))$ span a fundamental system of neighborhoods of $f(z) \in \overline{\mathbb{D}}$. Thus $\Phi(f)$ is continuous at z. $\square$

Chapter 3

Background on Generalizations of Conformally Natural Extensions

3.1 Continuous Monotone Degree ±1 Maps

It is possible to generalize conformally natural extensions to a larger family of circle maps. Growing out of ideas in [2] and unpublished ideas of Milnor, an effective algorithm, called the MAY iterator, was formally introduced in [1] to find the images of points under $\Phi(f)$. By using the MAY iterator Abikoff, Earle and Mitra in [3] generalized conformally natural extensions to the case of continuous monotone degree one circle maps. The MAY iterator is an anti-holomorphic map h_μ from $\mathbb{D}$ to itself that depends on a probability measure μ whose support contains more than two points. Such a measure can have at most one atom with measure more than or equal to $1/2$. Iterations of

h_μ will produce a dynamical system with one attractor that will be defined to be the conformal barycenter of continuous monotone degree ±1 map.

Let P be the space of probability measures on $\mathbb{S}^1$. Let $P' \subset P$ be the space of admissible measures. Let P'' be the space of measures whose support contain more than 2 points. Following the terminology of Abikoff, an atom is called a strong atom if its measure is $1/2$ or more. Note that measures in $P'' \backslash P'$ have exactly one strong atom.

Theorem 5 ([3]). *Let $\mu \in P''$. Then there exists a unique map $h_\mu : \mathbb{D} \to \mathbb{D}$ with the following properties:*

$$h_{\gamma_*(\mu)} = \gamma \circ h_\mu \circ \gamma^{-1},$$

for all Mobius maps $\gamma : \mathbb{D} \to \mathbb{D}$; and

$$h_\mu(0) = \int_{\mathbb{S}^1} \zeta d\mu(\zeta).$$

Each h_μ is an anti-holomorphic map and is contractible with respect to the hyperbolic metric.

The explicit definition of $h_\mu(w)$ for $w \in \mathbb{S}^1$ is:

$$h_\mu(w) = \frac{\int_{\mathbb{S}^1} \zeta(1-\zeta\bar{w})^{-1} d\mu(\zeta)}{\int_{\mathbb{S}^1} (1-\zeta\bar{w})^{-1} d\mu(\zeta)} = -g_w\left(-\int_{\mathbb{S}^1} g_w(\zeta) d\mu(\zeta)\right),$$

where $g_w(z) = \frac{z-w}{1-z\bar{w}}$.

For any $\mu \in P''$, the iterator h_μ is a contraction with respect to the hyperbolic metric on $\mathbb{D}$. Then by the Denjoy-Wolff Theorem, the iterates h_μ^n converge locally uniformly in $\mathbb{D}$ to a unique point w_0 in $\overline{\mathbb{D}}$. Moreover, $w_0 \in \mathbb{D}$ if and only if h_μ has a fixed point in $\mathbb{D}$. In this case, the fixed point is unique and equal to w_0. Furthermore, it is easy to see from the above equation, that such a fixed point is the conformal barycenter defined by Doaudy and Earle. From this perspective, one can define the attractor of the MAY iterator as the conformal barycenter of the corresponding measure.

Definition. For a probability measure $\mu \in P''$, a conformal barycenter of μ is the point

$$B(\mu) = \lim_{n \to \infty} h_\mu^n(z), z \in \mathbb{D}.$$

This definition is independent of the choice of z and so we may pick any point in $\mathbb{D}$.

Theorem 6 ([3])**.** *Let $\mu \in P''$. If μ is an admissible measure (i.e. $\mu \in P'$), then $B(\mu) \in \mathbb{D}$. Otherwise, if μ has a strong atom at ζ_0, then $B(\mu) = \zeta_0$.*

Proof. **Part 1.**

Suppose $\mu \in P'$. Then we can choose positive numbers $a < 1/2$ and $\epsilon < 1$ such that for every $w \in \mathbb{S}^1$, an open arc $\alpha_w = \{z \in \mathbb{S}^1 : |z - w| < \epsilon\}$ satisfies

$\mu(\alpha_w) < a$. Consider a function

$$l_\mu(w) = \frac{1}{2}\int_{\mathbb{S}^1} \log\left(\frac{1-|w|^2}{|w-\zeta|^2}\right) d\mu(\zeta), w \in \mathbb{D}.$$

For any $w \in \mathbb{D}$,

$$\begin{aligned} l_\mu(w) &= \frac{1}{2}\log(1-|w|^2) - \int_{\alpha_w} \log|w-\zeta| d\mu(\zeta) - \int_{\mathbb{S}^1\backslash\alpha_w} \log|w-\zeta| d\mu(\zeta) \\ &\leq \frac{1}{2}\log(1-|w|^2) - \log(1-|w|)\mu(\alpha_w) - \log(\epsilon)\mu(\mathbb{S}^1\backslash\alpha_w) \\ &\leq \frac{1}{2}\log(1-|w|^2) - a\log(1-|w|) - \log(\epsilon) \\ &= \frac{1}{2}\log(1+|w|) + \left(\frac{1}{2}-a\right)\log(1-|w|) - \log(\epsilon). \end{aligned}$$

So, $l_\mu(w) \to -\infty$ as $|w| \to 1$, and l_μ achieves its maximum at some point $w_0 \in \mathbb{D}$. Therefore,

$$0 = \frac{\partial l_\mu}{\partial \bar{w}}(w_0) = \frac{1}{2}\frac{1}{(1-|w_0|^2)}\int_{\mathbb{S}^1} \frac{\zeta - w_0}{1-\bar{w}_0\zeta} d\mu(\zeta),$$

and w_0 is the conformal barycenter for μ, hence $B(\mu) = w_0 \in \mathbb{D}$.

Part 2.

Suppose μ has a strong atom at ζ_0. By conformal naturality, we may assume $\zeta_0 = 1$. Let w be any point in $\mathbb{D}$ and let g_w be the Mobius map defined as before. Note that the inverse of $-g_w$ is $-g_w$. Let ξ be a Euclidian barycenter of $(-g_w)_*\mu$. Then $h_\mu(w) = -g_w(\xi)$. Let $\zeta_1 = -g_w(1)$. Then,

$$\xi = \zeta_1\mu(1) + \int_{\mathbb{S}^1\backslash\{1\}} \frac{w-\zeta}{1-\bar{w}\zeta} d\mu.$$

Since $\mu(1) \geq 1/2$,

$$\left|\xi - \frac{\zeta_1}{2}\right| \leq |\xi - \zeta_1\mu(1)| + \left|\zeta_1\left(\mu(1) - \frac{1}{2}\right)\right| < (1 - \mu(1)) + \left(\mu(1) - \frac{1}{2}\right) = \frac{1}{2}.$$

So, ξ lies in the interior of the horocycle Γ of radius $1/2$ at ζ_1 that has 0 on its boundary. Then, $h_\mu(w) = -g_w(\xi)$ lies in the interior of the horocycle $-g_w(\Gamma)$ at 1 which has w on its boundary. Since w was arbitrary point, h_μ maps any horocycle at 1 into its interior. Therefore, by the Denjoy-Wolff Theorem there is a limiting value inside these invariant horocycles. Hence, $\lim_{n\to\infty} h_\mu^{(n)}(w) = 1$ for all $w \in \mathbb{D}$.

$\square$

In [3], extensions $\Phi(f)$ are defined for monotone continuous degree ±1 circle map as follows:

$$\Phi(f)(z) = \begin{cases} f(z), & z \in \mathbb{S}^1 \\ B(f_*(\eta_z)), & z \in \mathbb{D} \end{cases}.$$

This definition coincides with the Douady-Earle definition in the case of homeomorphisms. In fact, $\Phi(f)$ is still continuous in $\mathbb{D}$. It is important to note that for a degree ±1 continuous monotone map we understand the collection of points $z \in \mathbb{D}$ for which $f_*(\eta_z)$ have strong atoms very well. If $I = f^{-1}(a)$ is an arc in $\mathbb{S}^1$ for some point $a \in \mathbb{S}^1$ such that $Leb(I) > 0$, then the whole hyperbolic half-plane bounded by the geodesic connecting the endpoints of I is mapped to a by $\Phi(f)$. This is a direct consequence of Theorem

6. This observation was an inspiration for the definition of the sets δ_f and β_f for each continuous circle map before introducing a conformally natural extensions for f in Chapter 5.

3.2 Continuous Maps with No Atoms

Recently, Prof. Hu and I developed new techniques to generalize conformally natural extensions to the family of continuous circle maps whose push-forward of Lebesque measure has no atoms([10]). Since the push-forward of a harmonic measure by such f doesn't have atoms as well, it will have the conformal barycenter. Moreover, the original definition of the conformal barycenter can be applied to define the extension of f to $\overline{\mathbb{D}}$. One of the useful techniques that was used in [10] was the definition of a modified vector field $\tilde{\xi_\mu} = (1 - |w|^2)^{-1}\xi_\mu$, where ξ_μ is defined as in Chapter 2. This vector field $\tilde{\xi_\mu} = (1 - |w|^2)^{-1}\xi_\mu$ is no longer conformally natural, however it will have a unique zero in $\mathbb{D}$, which is also a zero of ξ_μ. Moreover, as w approaches the boundary of $\mathbb{D}$, $\tilde{\xi_\mu}(w)$ will no longer approach 0. This vector field will be very useful in proving continuity of the conformally natural extension of a continuous circle map in Chapter 5.

The following theorem proves that the extension of such f is continuous in $\overline{\mathbb{D}}$ and real-analytic in $\mathbb{D}$. However, since the proof is very similar to

Proposition 3 (Proof of continuity in the homeomorphism case), I will show only a little extra step.

Theorem 7 ([10]). *Let f be a continuous circle map with no atoms. Then $\Phi(f)$ is a continuous map from $\overline{\mathbb{D}}$ onto itself and is real-analytic in the interior $\mathbb{D}$.*

Proof. Let $z \in \mathbb{S}^1$ and let I be a closed arc on $\mathbb{S}^1$ containing $f(z)$ in its interior. By the continuity of f on $\mathbb{S}^1$, there exists a closed arc J on $\mathbb{S}^1$ containing z in its interior such that $f(J) \subset I$. Now define

$$U(J) = \left\{ \zeta \in \mathbb{D} : \eta_\zeta(J) \geq \frac{2}{3} \right\} \text{ and } V(I) = \left\{ \zeta \in \mathbb{D} : \eta_\zeta(I) \geq \frac{1}{4} \right\},$$

where η_ζ is a normalized harmonic measure on $\mathbb{S}^1$ with respect to ζ.

Since $V(I)$ shrinks to the point $f(z)$ as I shrinks to $f(z)$, it suffices to prove $\Phi(U(J)) \subset V(I)$. Now using a similar technique to that in the proof of Proposition 3, it is clear that the conformal barycenter of $f_*\eta_\zeta$ belongs to $V(I)$. Therefore, $\Phi(f)$ is continuous at every point of $\mathbb{S}^1$.

Let $w = \Phi(f)(z)$, $z \in \mathbb{D}$. Then w and z satisfy the following equation:

$$F(z, w) = 0,$$

where

$$F(z, w) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{f(\xi) - w}{1 - \bar{w} f(\xi)} \cdot \frac{1 - |z|^2}{|z - \xi|^2} |d\xi|. \tag{3.1}$$

By conformal naturality we may assume that $\Phi(f)(0) = 0$. Then,

$$F'_w(0,0) = -1 \text{ and } F'_{\bar{w}}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f^2(\zeta)|d\zeta|.$$

Hence,

$$|F'_w(0,0)|^2 - |F'_{\bar{w}}(0,0)|^2 = \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{S}^1} |f^2(z) - f^2(\zeta)|^2|dz||d\zeta| > 0.$$

By the Implicit Function Theorem and by conformal naturality $\Phi(f)$ is continuous and real-analytic in $\mathbb{D}$.

□

Chapter 4

Background on Local Regularity

In [9], it was shown that quasiconformality of the conformally natural extension of a homeomorphism is a local property. In other words:

Theorem 8 ([9]). *Let f be an orientation-preserving homeomorphism of the unit circle $\mathbb{S}^1$ and $\Phi(f)$ be the Douady-Earle extension of f to the closed unit disk $\overline{\mathbb{D}}$. Let $p \in \mathbb{S}^1$ and I_p be an open arc on $\mathbb{S}^1$ with p in the middle. If $||f|_{I_p}||_{cr} < \infty$, then there exists an open hyperbolic half plane U_p with p in the middle of its boundary on $\mathbb{S}^1$ such that*

$$\ln K(\Phi(f)|_{U_p}) \leq C_1 ||f|_{I_p}||_{cr} + C_2$$

for two universal positive constants C_1 and C_2, where $K(\Phi(f)|_{U_p})$ is the maximal dilatation of $\Phi(f)$ on U_p.

In [10], it was shown that if f is a continuous map with no atoms, then the quasiconformality of $\Phi(f)$ is a local property. Moreover, if f is an orientation-preserving endomorphism then $\Phi(f)$ is quasiconformal on annulus neighborhood of $\mathbb{S}^1$ in $\overline{\mathbb{D}}$. More precisely,

Theorem 9 ([10])**.** *Let f be a continuous circle map with no atoms, and let I be an arc on $\mathbb{S}^1$. If f is injective, orientation-preserving on I and $||f|_I||_{cr}$ is finite, then there exists a neighborhood U of I in $\mathbb{D}$ and a constant $K > 0$ depending only on $||f|_I||_{cr}$ such that f is locally K-quasiconformal on U.*

We say an orientation-preserving endomorphism f of $\mathbb{S}^1$ is locally quasisymmetric if for any point $z \in \mathbb{S}^1$, there is a circular arc I containing z in its interior such that f is quasisymmetric on I.

Theorem 10 ([10])**.** *Let M be a positive real number and d be a natural number. If f is a locally quasisymmetric orientation-preserving endomorphism of $\mathbb{S}^1$ of degree d with a local cross-ratio distortion norm uniformly bounded by M, then there exist two annulus neighborhoods U and V of $\mathbb{S}^1$ in $\overline{\mathbb{D}}$ such that $\Phi(f)|_U : U \to V$ is a quasiregular covering map of degree d with maximal dilation bounded by a constant K only depending on M.*

Before we try to provide a flavor of the arguments in the proofs of the above theorems, we need to provide some background.

Let $z \in \mathbb{D}$ and $w = \Phi(f)(z)$. Then w and z satisfy $F(z, w) = 0$, where $F(z, w)$ is given by (3.1). By using the Implicit Function Theorem, one may express the Jacobian of $w = \Phi(f)(z)$ in terms of the partial derivatives of F to z, $\bar{z}$, w and $\bar{w}$. Furthermore, one can express the maximal dilatation of $\Phi(f)$ at z through these partial derivatives if the Jacobian of $\Phi(f)$ at z is not zero. Again by using the conformal naturality, we assume that $\Phi(f)(0) = 0$ and then work on the Jacobian of $\Phi(f)$ at 0 and the maximal dilatation of $\Phi(f)$ at 0. At first, the partial derivatives of $F(z, w)$ at $(0, 0)$ are:

$$c_1 = \frac{\partial F}{\partial z}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \bar{\xi} f(\xi)|d\xi|, \ \ c_{-1} = \frac{\partial F}{\partial \bar{z}}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \xi f(\xi)|d\xi|$$

and

$$d_1 = \frac{\partial F}{\partial w}(0,0) = -1, \ \ d_{-1} = \frac{\partial F}{\partial \bar{w}}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(\xi)^2|d\xi|,$$

then

$$\frac{\partial w}{\partial \bar{z}}(0) = -\frac{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial \bar{z}}(0,0) - \frac{\partial F}{\partial \bar{w}}(0,0)\overline{\frac{\partial F}{\partial z}(0,0)}}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \bar{w}}(0,0)|^2}$$

and

$$\frac{\partial w}{\partial z}(0) = -\frac{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial z}(0,0) - \frac{\partial F}{\partial \bar{w}}(0,0)\overline{\frac{\partial F}{\partial \bar{z}}(0,0)}}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \bar{w}}(0,0)|^2}.$$

So the Jacobian of $\Phi(f)(z)$ at $z = 0$ is equal to

$$\left|\frac{\partial w}{\partial z}(0)\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}(0)\right|^2 = \frac{|\frac{\partial F}{\partial z}(0,0)|^2 - |\frac{\partial F}{\partial \bar{z}}(0,0)|^2}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \bar{w}}(0,0)|^2} = \frac{|c_1|^2 - |c_{-1}|^2}{|d_1|^2 - |d_{-1}|^2}.$$

Let $z = e^{is}$, $\xi = e^{it}$, and $f(e^{iu}) = e^{ih(u)}$, where $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism with $h(u + 2\pi) = h(u) + 2\pi d$, where d is a degree of f. Then

$$\begin{aligned} |d_{-1}|^2 &= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} f(z)^2\overline{f(\xi)}^2 |dz||d\xi| \\ &= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} e^{2ih(s)}e^{-2ih(t)} dsdt \\ &= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \cos 2(h(s) - h(t)) dsdt, \end{aligned}$$

and hence

$$|d_1|^2 - |d_{-1}|^2 = 2\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \sin^2(h(s) - h(t)) dsdt. \tag{4.1}$$

Similarly,

$$\begin{aligned} |c_1|^2 &= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \bar{\xi} z f(\xi)\overline{f(z)} |d\xi||dz| \\ &= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} e^{i(s-t)}e^{i(h(t)-h(s))} dsdt \\ &= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} [\cos(s-t)\cos(h(s)-h(t)) \\ &\qquad + \sin(s-t)\sin(h(s)-h(t))] dsdt \end{aligned}$$

and

$$\begin{aligned} |c_{-1}|^2 = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} &[\cos(s-t)\cos(h(s)-h(t)) \\ &- \sin(s-t)\sin(h(s)-h(t))] dsdt. \end{aligned}$$

Thus,

$$\begin{aligned}
|c_1|^2 - |c_{-1}|^2 &= 2\left(\frac{1}{2\pi}\right)^2 \int_{s=0}^{2\pi}\int_{t=0}^{2\pi} \sin(s-t)\sin(h(s)-h(t))dsdt \\
&= 2\left(\frac{1}{2\pi}\right)^2 \int_{u=0}^{2\pi} \sin u \int_{t=0}^{2\pi} \sin(h(t+u)-h(t))dtdu \\
&= 2\left(\frac{1}{2\pi}\right)^2 \int_{u=0}^{\pi} \sin u \int_{t=0}^{2\pi} [\sin(h(t+u)-h(t)) \\
&\qquad\qquad + \sin(h(t+2\pi)-h(t+u+\pi))]dtdu.
\end{aligned}$$

Finally, $|c_1|^2 - |c_{-1}|^2$ can be expressed as:

$$|c_1|^2 - |c_{-1}|^2 = \left(\frac{1}{2\pi}\right)^2 \int_{u=0}^{\pi} \sin u \int_{t=0}^{2\pi} H(t,u)dtdu \tag{4.2}$$

with

$$\begin{aligned}
H(t,u) &= \sin(h(t+u)-h(t)) + \sin(h(t+2\pi)-h(t+u+\pi)) \\
&\quad + \sin(h(t+\pi+u)-h(t+\pi)) + \sin(h(t+\pi)-h(t+u)).
\end{aligned}$$

For an extension $\Phi(f)$ satisfying the normalization condition $\Phi(f)(0) = 0$, one result in [8] shows that if there exist two positive constants δ_1 and δ_2 such that $|d_1|^2 - |d_{-1}|^2 > \delta_1 > 0$ and $|c_1|^2 - |c_{-1}|^2 > \delta_2 > 0$, then the maximal dilatation of $\Phi(f)$ at 0 is bounded from above by a positive constant K only depending δ_1 and δ_2.

In [8], it was also shown that if f is a quasisymmetric homeomorphism with the cross-ratio distortion norm $||f||_{cr}$ bounded by a constant $M > 0$ and

$\Phi(f)$ satisfies the normalization condition, then there exist $\delta_1 > 0$ and $\delta_2 > 0$ only depending on $||f||_{cr}$ such that $|d_1|^2 - |d_{-1}|^2 > \delta_1 > 0$ and $|c_1|^2 - |c_{-1}|^2 > \delta_2 > 0$. Hence the maximal dilatation $\Phi(f)$ at 0 is bounded by a constant K only depending on $||f||_{cr}$. By the conformal naturality of $\Phi(f)$ and the invariance of the cross-ratio distortion norm of f under precomposition and postcomposition by conformal isometries of $\mathbb{D}$, it is concluded in [8] that $\Phi(f)$ is quasiconformal if $||f||_{cr}$ is finite.

In [9] and [10], Hu and I considered certain continuous circle maps that are quasisymetric on an arc $I \subset \mathbb{S}^1$. By taking a point $z \in \mathbb{D}$ close enough to I and mapping it to zero by a conformal map g, they obtain a new map $g \circ f$ that is quasisymmetric on the very large arc $g(I)$ (almost filling the whole circle). It is shown that in this case, $\delta_1 > 0$ and $\delta_2 > 0$ exist for these kinds of maps, which implies that $\Phi(f)$ is locally quasiconformal on some neighborhood of I in $\overline{\mathbb{D}}$.

After we construct and study the extensions $\Phi(f)$ for arbitrary continuous circle maps f in Chapter 5, we will recapitulate the scheme and techniques of [10] in Chapter 6 to show that if a continuous circle map f is also quasisymmetric on an arc I on $\mathbb{S}^1$, then $\Phi(f)$ is locally quasiconformal on some neighborhood of D of I in $\mathbb{D}$.

Chapter 5

Conformally Natural Extensions of Continuous Circle Maps

5.1 Preliminaries

Before introducing conformally natural extensions for continuous circle maps, we need to generalize the proof given by Earle and Douady in [4] for the existence and uniqueness of the conformal barycenter to the case of admissible measures. As before, we define a vector field

$$\xi_\mu(w) = (1 - |w|^2) \int_{\mathbb{S}^1} \frac{\zeta - w}{1 - \bar{w}\zeta} d\mu(\zeta).$$

First we need two small lemmas about the properties of the vector field ξ_μ:

Lemma 3. *$Re(\xi_\mu(0)) > 0$ if $\mu(\widehat{[e^{-i\epsilon}, e^{i\epsilon}]}) \geq \frac{1}{2} + \epsilon$ for some $0 < \epsilon \leq 1/4$.*

Proof. Given $0 < \epsilon \leq 1/4$,

$$\begin{aligned} Re(\xi_\mu(0)) &= \int_{\mathbb{S}^1} Re(\zeta) d\mu(\zeta) \\ &\geq (-1)\left(\frac{1}{2} - \epsilon\right) + \cos(\epsilon)\left(\frac{1}{2} + \epsilon\right) \\ &\geq (-1)\left(\frac{1}{2} - \epsilon\right) + (1-\epsilon)\left(\frac{1}{2} + \epsilon\right) \\ &= \frac{3}{2}\epsilon - \epsilon^2 > 0. \end{aligned}$$

□

Lemma 4. *Suppose μ is an admissible probability measure. Then there exists $0 < R < 1$ such that for any $R < r < 1$ and any $w \in C_r = \{z : |z| = r\}$,*

1. *$\xi_\mu(w)$ points inside of C_r, and*
2. *$\inf_{z \in C_r} |\xi_\mu(z)| \neq 0$.*

Proof. **Step 1.**

Let $\epsilon = \frac{1}{4} - \frac{a}{2}$ where a is the maximal atom weight for μ. Note that $0 \leq a < \frac{1}{2}$ and so $0 < \epsilon \leq \frac{1}{4}$. Let $\alpha > 0$ be such that $\mu(J) \leq \frac{1}{2} - \epsilon$ for any arc $J \subset \mathbb{S}^1$ of length $\leq \alpha$. Note that our choice of ϵ guaranties that α is not zero.

Take $R < 1$ such that the arc J of arclength α centered at 1 so that the visual measure at R with respect to Poincare metric is $2\pi - 2\epsilon$.

Suppose $|w| = r \geq R$, then let J_w be an arc with arclength equal to α and a midpoint at $\frac{w}{|w|}$. Let g be a Mobius map taking w to 0 and $\frac{-w}{|w|}$ to 1.

Let $\nu = g_*(\mu)$. Then,

$$Leb(g(J_w^c)) = \eta_w(J_w^c) \leq 2\epsilon.$$

Since, $\nu(\widehat{[e^{-i\epsilon}, e^{i\epsilon}]}) \geq \nu(g(J_w^c)) \geq 1/2+\epsilon$, then by Lemma 3, $Re(\xi_{g_*(\mu)}(0)) > 0$, and so $\xi_{g_*(\mu)}(0)$ points into $g(C_r)$. By conformal naturality, $\xi_\mu(w)$ points into C_r.

Step 2.

Suppose $\inf_{z \in C_r} |\xi_\mu(z)| = 0$, then there exists a sequence $\{z_n \in C_r\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} |\xi_\mu(z_n)| = 0$. Let $z_0 = \lim_{n\to\infty} z_n$. Since ξ_μ is a continuous vector field in $\mathbb{D}$, then $\xi_\mu(z_0) = 0$. This is a contradiction to part 1 of this lemma. □

Now we can prove the existence and uniqueness of a conformal barycenter for any admissible measure. The following proposition is similar to Proposition 1 in [4]. It should be noted, however, that this proof uses the previous lemma which is more general than the corresponding lemma in [4].

Proposition 4. *Suppose μ is an admissible probability measure. Then ξ_μ has a unique zero in $\mathbb{D}$. We will denote it by $B(\mu)$ and call it the* conformal barycenter *of μ.*

Proof. With the same reasoning as in the proof of Proposition 1, we can see that

$$Jac(\xi_\mu)(0) = \frac{1}{2}\int\int_{\mathbb{S}^1\times\mathbb{S}^1} |z^2-\zeta^2|^2 d\mu(\zeta)d\mu(z) > 0.$$

If $\xi_\mu(0) = 0$, then 0 is an isolated singular point of index 1. By conformal naturality, we know that every zero of the vector field ξ_μ is an isolated zero of index one. By Lemma 4, the vector field ξ_μ has no zeros on the circle $C_r = \{w : |w| = r\}$ of radius r sufficiently close to 1 and points inward. By the corollary to the Poincare-Hopf Index Theorem, we conclude that ξ_μ has a unique zero in $\mathbb{D}$. □

5.2 The Sets β_f and δ_f

Let f be a continuous circle map. Considering proposition 4 we divide $\mathbb{D}$ into two sets:

Definition. Let β_f be a set of all points $z \in \mathbb{D}$ for which $f_*(\eta_z)$ is an admissible measure, and let $\delta_f = \mathbb{D} \setminus \beta_f$.

Remark 5. If f is a circle homeomorphism or a continuous circle map without atoms, then δ_f is empty. If f is a continuous degree ±1 monotone circle map with some arcs collapsed to points, then δ_f is not empty. If f is a constant map, then $\delta_f = \mathbb{D}$. In general, if δ_f is not empty for a non-constant continuous circle map f, then for any point $z \in \delta_f$, $f_*(\eta_z)$ has exactly one atom γ_z with weight $\geq 1/2$ (called a *strong atom* for $f_*(\eta_z)$).

Note that both β_f and δ_f are conformally natural sets in the following sense: if A and B are two Mobius maps preserving the unit circle and $z \in \beta_f$, then $A^{-1}(z) \in \beta_{B\circ f\circ A} = A^{-1}(\beta_f)$, and similarly for the set δ_f.

Let $B(z, r)$ denote a ball in $\mathbb{D}$ centered at $z \in \mathbb{D}$ and of radius r. The following property of a harmonic measure is useful.

Lemma 5. *Let f be a continuous circle map. Given $z_0 \in \mathbb{D}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for any $z \in B(z_0, \delta)$,*

$$|f_*(\eta_z)(E) - f_*(\eta_{z_0})(E)| < \epsilon Leb(f^{-1}(E))$$

for any measurable set $E \subset \mathbb{S}^1$.

Proof. Since $z_0 \in \mathbb{D}$, g'_z converges to g'_{z_0} uniformly on $\mathbb{S}^1$ as z approaches z_0. That is, given $\epsilon > 0$, there exists $\delta > 0$ such that, for any $z \in B(z_0, \delta)$, $|g'_z(\zeta) - g'_{z_0}(\zeta)| < \epsilon$ for any point $\zeta \in \mathbb{S}^1$. Therefore,

$$\begin{aligned}
|f_*(\eta_z)(E) - f_*(\eta_{z_0})(E)| &= |\eta_z(f^{-1}(E)) - \eta_{z_0}(f^{-1}(E))| \\
&= \left|\int_{f^{-1}(E)} d\eta_z - \int_{f^{-1}(E)} d\eta_{z_0}\right| \\
&= \left|\int_{f^{-1}(E)} |g'_z(\zeta)||d\zeta| - \int_{f^{-1}(E)} |g'_{z_0}(\zeta)||d\zeta|\right| \\
&\leq \int_{f^{-1}(E)} |g'_z(\zeta) - g'_{z_0}(\zeta)||d\zeta| \leq \epsilon Leb(f^{-1}(E))
\end{aligned}$$

□

Let f be a continuous map from $\mathbb{S}^1$ into itself, and let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $\mathbb{D}$ converging to a point $z_0 \in \mathbb{D}$. Let $\mu_n = f_*(\eta_{z_n})$ and $\mu_0 = f_*(\eta_{z_0})$.

Corollary 2. *Let μ_n and μ_0 be the probability measures defined as above, then for any $\epsilon > 0$ there exists a large enough N such that for any $n > N$ and any measurable set $E \subset \mathbb{S}^1$,*

$$|\mu_n(E) - \mu_0(E)| < \epsilon Leb(f^{-1}(E)).$$

Proposition 5. *The set β_f is an open subset of $\mathbb{D}$, and δ_f is a closed subset of $\mathbb{D}$.*

Proof. Let $z_0 \in \beta_f$. Then for all $w \in \mathbb{S}^1$, $f_*(\eta_{z_0})(w) < 1/2$. Let

$$m = \sup_{w \in \mathbb{S}^1} f_*(\eta_{z_0})(w).$$

Then $m < 1/2$; otherwise there exists a sequence of atoms whose weights under $f_*(\eta_{z_0})$ approach $1/2$. Thus, there exist three atoms with weights more than $1/3$, and then the total measure of $\mathbb{S}^1$ is more than 1. This contradicts the fact that η_{z_0} is a probability measure.

Let $0 < \epsilon < 1/2 - m$. By Lemma 5, there exists a neighborhood $B(z_0)$ of z_0 such that for all sets $E \subset \mathbb{S}^1$ and for all $z' \in B(z_0)$,

$$|f_*(\eta_{z_0})(E) - f_*(\eta_{z'})(E)| \leq Leb(f^{-1}(E))\epsilon \leq \epsilon.$$

By the triangle inequality, $f_*(\eta_z)(E) < m + \epsilon < 1/2$ for all $z \in B(z_0)$ and all $E \subset \mathbb{S}^1$. Hence $B(z_0) \subset \beta_f$. Therefore, the set β_f is open. Since δ_f is the complement of β_f, it is closed. $\square$

The set δ_f can be divided further into components $\delta_f(\gamma_z)$ consisting of all points $z \in \mathbb{D}$ such that the measures $f_*(\eta_z)$ have γ_z as a strong atom. These components are not necessarily connected. In the next lemma, we show that each such component is also a closed set in $\mathbb{D}$.

Lemma 6. *Each component $\delta_f(\gamma_z)$ is a closed subset of $\mathbb{D}$.*

Proof. Let $a = \gamma_z$ and suppose that $\delta_f(a)$ is not empty. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $\delta_f(a)$ converging to a point $z_0 \in \mathbb{D}$. Consider a function $h(z) = f_*(\eta_z)(a)$. By Corollary 2, $h(z_n)$ converges to $h(z_0)$. Since $h(z_n) \geq 1/2$ for all $n \geq 1$, $h(z_0) \geq 1/2$. This implies that a is a strong atom for $f_*(\eta_{z_0})$. Thus $z_0 \in \delta_f(a)$. Hence $\delta_f(a)$ is a closed set. $\square$

Remark 6. Two different components $\delta_f(a)$ and $\delta_f(b)$ are disjoint closed subsets of $\mathbb{D}$; otherwise, if z_0 belongs to their intersection, then η_{z_0} has two strong atoms a and b, which is impossible.

The following lemma will be used to prove the continuity of the extension in this chapter.

Lemma 7. *Let* $\{z_n\}_{n=1}^{\infty}$ *be a sequence of points in* δ_f *converging to a point* $z_0 \in \delta_f(a)$ *for some point* $a \in \mathbb{S}^1$. *Then there is a large enough* N *such that for all* $n > N$, $z_n \in \delta_f(a)$.

Proof. If there is a large enough N such that for all $n > N$, $z_n \in \delta_f(b)$ for some atom $b \in \mathbb{S}^1$, then by Lemma 6, $z_0 \in \delta_f(b)$. Hence, $a = b$.

Otherwise, there is a subsequence $\{z_{n_k}\}_{n_k=1}^{\infty}$ such that $z_{n_k} \in \delta_f(b_{n_k})$, where $b_{n_k} \neq a$ for any k. If all b_{n_k} are eventually equal to a constant b, then the above argument shows that $b = a$. Hence we may assume that all b_{n_k} are distinct.

Note that $\mu_0(a) = f_*(\eta_{z_0})(a) \geq 1/2$. Let $\epsilon > 0$. By Corollary 2, there exists $K > 0$ such that for all $n_k > K$, $|f_*(\eta_{z_{n_k}})(E) - f_*(\eta_{z_0})(E)| < \epsilon$ for any measurable set $E \subset \mathbb{S}^1$. By letting $E = \{b_{n_k}\}$, we obtain $\mu_0(b_{n_k}) > 1/2 - \epsilon$ for all $n_k > K$, which contradicts the fact that μ_0 is a probability measure.

□

Corollary 3. *Let* z_0 *be an interior point of* δ_f *and* $z_0 \in \delta_f(a)$. *Then there exists a neighborhood* $U(z_0)$ *of* z_0 *in* δ_f *that is contained in* $\delta_f(a)$.

5.3 Extension and Conformal Naturality

Now we can define an extension $\Phi(f)$ of f as follows.

Definition. Let f be a continuous circle map. We define

$$\Phi(f)(z) = \begin{cases} f(z), & z \in \mathbb{S}^1 \\ \gamma_z, & z \in \delta_f \\ B(f_*(\eta_z)), & z \in \beta_f \end{cases},$$

where γ_z is the point on $\mathbb{S}^1$ with $f_*(\eta_z)(\gamma_z) \geq 1/2$.

Theorem 1. *The extensions $\Phi(f)$ of continuous circle maps f are conformally natural.*

Proof. **Part 1.** We first prove for any continuous circle map f and any two conformal isometries A and B of the unit disk,

$$\Phi(A \circ f \circ B) = A \circ \Phi(f) \circ B.$$

From the previous section, we know $\beta_{A\circ f\circ B} = B^{-1}(\beta_f)$ and $\delta_{A\circ f\circ B} = B^{-1}(\delta_f)$. Now we break the proof into two steps. In the first step, we show that the conformal naturality holds for any point in $B^{-1}(\beta_f)$, and in the second step, we show that the conformal naturality holds for any point in $B^{-1}(\delta_f)$.

Step 1.

Let $z \in B^{-1}(\beta_f)$, and $b = B(z)$, then $b \in \beta_f$. Therefore, $\Phi(f)(b) = w$ such that

$$\int_{\mathbb{S}^1} g_w \circ f \circ g_b^{-1}(\zeta)|d\zeta| = 0.$$

Note that $g_z = R \circ g_b \circ B$ for some $R(z) = e^{i\theta}z$. Hence if $b \in \beta_f$, then $z \in \beta_{f\circ B} = \beta_{A\circ f\circ B}$. So, $\Phi(A \circ f \circ B)(z) = w^*$ such that

$$0 = \int_{\mathbb{S}^1} g_{w^*} \circ A \circ f \circ B \circ g_z^{-1}(\zeta)|d\zeta| = \int_{\mathbb{S}^1} g_{w^*} \circ A \circ f \circ B \circ g_{B^{-1}(b)}^{-1}(\zeta)|d\zeta|.$$

If $w^* = A(w)$, then

$$\begin{aligned} & \int_{\mathbb{S}^1} g_{w^*} \circ A \circ f \circ B \circ g_{B^{-1}(b)}^{-1}(\zeta)|d\zeta| \\ = \; & \int_{\mathbb{S}^1} (g_{A(w)} \circ A) \circ f \circ (B \circ g_{B^{-1}(b)}^{-1})(\zeta)|d\zeta| \\ = \; & \int_{\mathbb{S}^1} (e^{i\theta_1} g_w) \circ f \circ (g_{B^{-1}(b)} \circ B^{-1})^{-1}(\zeta)|d\zeta| \\ = \; & \int_{\mathbb{S}^1} (e^{i\theta_1} g_w) \circ f \circ (g_b)^{-1}(e^{i\theta_2}\zeta)|d\zeta| \\ = \; & e^{i\theta_1} \int_{\mathbb{S}^1} (g_w) \circ f \circ (g_b)^{-1}(\zeta)|d\zeta| = 0. \end{aligned}$$

By the uniqueness of the conformal barycenter, $\Phi(A \circ f \circ B)(z) = A \circ \Phi(f) \circ B(z) = A(w)$.

Step 2.

Let $z \in B^{-1}(\delta_f)$, and $b = B(z)$. Then $b \in \delta_f$, and so $\Phi(f)(b) = \gamma_b$. Let $I = \{\zeta \in \mathbb{S}^1 : \Phi(f) \circ B(\zeta) = \gamma_b\}$. Note that $\eta_z(I) \geq 1/2$.

Since $f(\zeta) = \Phi(f)(\zeta)$ for $\zeta \in \mathbb{S}^1$ and $I \subset \mathbb{S}^1$, we have

$$\Phi(A \circ f \circ B)(I) = A \circ f \circ B(I) = A \circ \Phi(f) \circ B(I) = A(\gamma_b).$$

Since $\eta_z(I) \geq 1/2$, $z \in \delta_{A\circ f\circ B}$ and $\Phi(A \circ f \circ B)(z) = A(\gamma_b)$. Thus

$$\Phi(A \circ f \circ B)(z) = A \circ \Phi(f) \circ B(z).$$

Part 2. Next, we need to show that if f is the restriction to the circle of a complex analytic map Ψ defined on a neighborhood of $\overline{\mathbb{D}}$, then $\Phi(f) = \Psi$. In this case, such a map Ψ is a Blashke product and for $f = \Psi|_{\mathbb{S}^1}$, δ_f is empty. This part was proved in [10]. For completeness, we provide a proof here as well.

Given a point $a \in \mathbb{D}$, let $b = \Psi(a)$. Let $\tilde{f} = g_b \circ f \circ g_a^{-1}$ and $\tilde{F} = g_b \circ \Psi \circ g_a^{-1}$. By the first part of this theorem, it suffices to show that $\Phi(\tilde{f})(0) = 0$; that is, $\int_{\mathbb{S}^1} \tilde{f}(z)|dz| = 0$. By Cauchy Integral Formula, we obtain

$$\int_{\mathbb{S}^1} \tilde{f}(z)|dz| = \int_{\mathbb{S}^1} \tilde{F}(z)|dz| = \frac{1}{i}\int_{\mathbb{S}^1} \frac{\tilde{F}(z)}{z} dz = 2\pi \tilde{F}(0) = 0.$$

□

5.4 Continuity

Next, we show that $\Phi(f)$ is a continuous function from $\overline{\mathbb{D}}$ to itself for any continuous circle map f. Noting again that the set β_f is open and the set δ_f is closed, we divide the proof of the continuity of $\Phi(f)$ at a point $z \in \overline{\mathbb{D}}$ into four cases according to where z is:

1. $z \in \beta_f$.
2. $z \in int(\delta_f)$.
3. $z \in \mathbb{S}^1$.

4. $z \in \partial\delta_f$, where $\partial\delta_f$ denotes the boundary of δ_f in $\mathbb{D}$.

The proofs for the first two cases are straightforward:

1. For $z \in \beta_f$, we note that if $w = B(f_*(\eta_z))$ for a continuous non-constant function f then from Chapter 4 we know that

$$0 = F(z, w) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{f(\xi) - w}{1 - \bar{w} f(\xi)} \cdot \frac{1 - |z|^2}{|z - \xi|^2} |d\xi|,$$

By the conformal naturality, we may assume $\Phi(f)(0) = 0$. Then by using the formula 4.1 in Chapter 4,

$$\left| \frac{\partial F}{\partial w}(0,0) \right|^2 - \left| \frac{\partial F}{\partial \bar{w}}(0,0) \right|^2 > 0.$$

By applying the Implicit Function Theorem, $w = \Phi(f)(z)$ is continuous in a neighborhood of z. Furthermore, we can see that $\Phi(f)(z)$ is real-analytic in β_f.

2. For $z \in int(\delta_f)$, by Corollary 3 and continuity method, we conclude that $\Phi(f)$ is constant on each connected component of $int(\delta_f)$.

Now we consider the third case when $z \in \mathbb{S}^1$.

Let $J \in \mathbb{S}^1$ be an arc and $A = f(J)$. Let

$$U(J) = \left\{ \zeta \in \mathbb{D} : \eta_\zeta(J) \geq \frac{2}{3} \right\}$$

and

$$V(f(J)) = V(A) = \left\{ \zeta \in \mathbb{D} : \eta_\zeta(A) \geq \frac{1}{4} \right\} \cup A.$$

Lemma 8. *Let f be a continuous circle map, $J \subset \mathbb{S}^1$ be any arc and $A = f(J)$, then $\Phi(f)(U(J)) \subset V(A)$.*

Proof. Let $z_0 \in U(J)$ and $\mu = f_*(\eta_{z_0})$. Then $\mu(A) \geq 2/3$. The proof will be broken into two parts: $z_0 \in \beta_f$ and $z_0 \in \delta_f$. Notice that if A is a point, then $z_0 \in \delta_f$ since $f_*(\eta_{z_0})(A) \geq 2/3 > 1/2$ and in this case $V(f(J)) = f(J) = A$.

Part 1.

Let $z_0 \in \beta_f$, then A is an arc and $V(f(J))$ has an interior in $\mathbb{D}$. Let $w \in \Gamma = \partial V(f(J)) \cap \mathbb{D}$. There exist $g(z) \in Mob(\mathbb{D})$ such that $g(f(J)) = (\widehat{e^{-i\pi/4}, e^{i\pi/4}})$ and $g(w) = 0$. Let $\nu = g_*(\mu)$. Then $\nu(g(A)) \geq 2/3$ and by Lemma 2, $Re(\xi_\nu(0)) > 0$. By conformal naturality, $\xi_\mu(w)$ points inside of $V(f(J))$. Hence, since ξ_μ points into $\mathbb{D}$ near the boundary of $\mathbb{D}$ and it is a continuous vector field in $\mathbb{D}$, $B(\mu) \in V(f(J))$ and consequently

$$\Phi(f)(U(J) \cap \beta_f) \subset V(f(J)).$$

Part 2.

Let $z_0 \in \delta_f$. Then there is an atom $\alpha \in \mathbb{S}^1$ such that $\mu(\alpha) \geq 1/2$ and $\Phi(f)(z_0) = \alpha$. Since $\mu(A^c) < 1/3$, $\alpha \in A \subset V(f(J))$. Note that if A is a point, then $\alpha = A$. Therefore, $\Phi(f)(U(J) \cap \delta_f) \subset V(f(J))$.

Combining both parts, we get $\Phi(f)(U(J)) \subset V(f(J))$.

□

Proposition 6. *Let f be a continuous circle map. Then $\Phi(f)$ is a continuous map on $\mathbb{S}^1$.*

Proof. Let $z_0 \in \mathbb{S}^1$ and $w = f(z_0)$. Let A be any neighborhood of w on $\mathbb{S}^1$. Since f is continuous, there is a neighborhood $J \subset \mathbb{S}^1$ of z_0 such that $f(J) \subset A$. Let $A' = f(J)$. By Lemma 8 and the fact that $A' \subset A$,

$$\Phi(f)(U(J)) \subset V(A') \subset V(A)$$

Therefore $\Phi(f)$ is continuous at z_0. $\square$

Finally, by showing that $\Phi(f)$ is continuous on the boundary of δ_f, we can conclude the global continuity of $\Phi(f)$. If $z \in \partial\delta_f$, then the measure $\mu = f_*(\eta_z)$ contains an atom of weight equal to $1/2$. Suppose $\mu(1) = 1/2$. Since f is a continuous circle map, then for any open arc $A \subset \mathbb{S}^1$, $\mu(A) > 0$. Moreover, if $1 \in A$ then $\mu(A) > 1/2$. Before we prove the continuity of $\Phi(f)$ at $z \in \partial\delta_f$, we first develop properties of the modified vector field $\tilde{\xi}_\mu(w) = (1 - |w|^2)^{-1}\xi_\mu(w)$ for such a measure μ. In this case, both $\xi_\mu(w)$ and $\tilde{\xi}_\mu(w)$ have no singularities in $\mathbb{D}$ and point in the same direction at each $w \in \mathbb{D}$. However, $\tilde{\xi}_\mu$ can not be continuously extended to the closed disk $\overline{\mathbb{D}}$. Therefore, we first study the vector field $\tilde{\xi}_\mu(w)$ when w stays on a horocycle at 1 and the limiting behavior as w approaches 1 along the horocycle.

Lemma 9. *Let* Γ *be any horocycle in* $\mathbb{D}$ *at 1 and* μ *be a probability measure defined as above with* $\mu(1) = 1/2$. *Then*

1. $\tilde{\xi}_\mu(z)$ *points inside* Γ *at any* $z \in \Gamma \backslash \{1\}$.
2. $\lim_{z \in \Gamma, z \to 1} \tilde{\xi}_\mu(z) = -1$.
3. $\inf_{z \in \Gamma \backslash \{1\}} |\tilde{\xi}_\mu(z)| \neq 0$.

Proof. **Part 1.**

Let $z_0 \in \Gamma \backslash \{1\}$. Let A be a geodesic going through 1 and z_0, and let $a \in \mathbb{S}^1$ be the other end of A. Note that A is perpendicular to Γ at z_0. Let $\phi \in Mob(\mathbb{D})$ be such that $\phi(1) = 1$, $\phi(a) = -1$, and $\phi(z_0) = 0$. Note that $\phi(\Gamma)$ is a horocycle at 1 going through 0. Let $\nu = \phi_*(\mu)$. Then $\nu(1) = 1/2$ and so $\nu(R) = 1/2 + \epsilon$ for some $\epsilon \geq 0$, where R is a right half circle. Therefore, $Re(\xi_\nu)(0) > 0$ and so $\xi_\nu(0)$ points inside $\phi(\Gamma)$. Hence, by conformal naturality $\xi_\mu(z_0)$ points inside Γ and so $\tilde{\xi}_\mu(z_0)$ points inside Γ.

Part 2.

Given $\epsilon > 0$, let $I(z, \epsilon)$ be an open arc on $\mathbb{S}^1$ centered at $w = \frac{z}{|z|}$ and with $\eta_z(I(z, \epsilon)) = 1 - 2\epsilon$. At first we show that there exists an arc Γ^*_ϵ on the upper half of Γ with one endpoint at 1 such that for any $z \in \Gamma^*_\epsilon$, $1 \notin I(z, \epsilon)$.

Let $z \in \Gamma$ be in the upper half plane. Let $I_z \subset I(z, \epsilon)$ be the half of $I(z, \epsilon)$ starting from w and going clock-wise. Then $\eta_z(I_z) = 1/2 - \epsilon$. Let J_z be an open arc from w to 1 in the clock-wise direction (see Figure 5.1). Note that

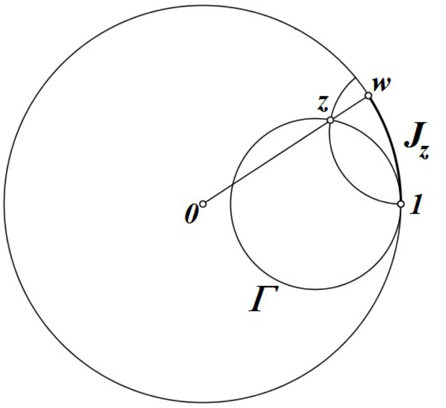

Figure 5.1: An illustration for the proof of Part 2 of Lemma 9.

as z approaches 1 in the clock-wise direction, $\eta_z(J_z)$ is approaching 1/2. So, for all z that are close enough to 1, $\eta_z(I_z) < \eta_z(J_z)$ and so $I_z \subset J_z$. Hence, I_z doesn't contain 1. Γ_ϵ^* is an open arc with 1 on the boundary.

Then, let $\Gamma_\epsilon \subset \Gamma_\epsilon^*$ be an arc such that for all $z \in \Gamma_\epsilon$, $\mu(I(z,\epsilon)) < \epsilon$. Similarly, Γ_ϵ is an open arc with 1 on the boundary. In order to show this, suppose the opposite. Then, there exist a sequence of $\{z_n\}_{n=1}^\infty \in \Gamma_\epsilon^*$ such that z_n approach 1 and for all n, $\mu(I(z_n,\epsilon)) > \delta$ for some $\delta > 0$. Then since each $I(z_n,\epsilon)$ is an open arc, there exists a subsequence $\{z_{n_j}\}$ such that all $I(z_{n_j},\epsilon)$ are disjoint. Then $\mu\left(\bigcup_{n_j} I(z_{n_j},\epsilon)\right)$ is infinite which contradicts the fact that μ is a probability measure.

Note that $g_z(-w) = -w$, then for all $z \in \Gamma_\epsilon$,

$$\begin{aligned}
& |\tilde{\xi}_\mu(z) - (-w)| \\
= & \left| \int_{\mathbb{S}^1} \frac{\zeta - z}{1 - \bar{z}\zeta} d\mu(\zeta) - \int_{\mathbb{S}^1} -w d\mu(\zeta) \right| \\
= & \left| \int_{\mathbb{S}^1} \left[\frac{\zeta - z}{1 - \bar{z}\zeta} - (-w) \right] d\mu(\zeta) \right| \\
= & \left| \int_{I(z,\epsilon)} \left[\frac{\zeta - z}{1 - \bar{z}\zeta} - (-w) \right] d\mu(\zeta) + \int_{I^c(z,\epsilon)} \left[\frac{\zeta - z}{1 - \bar{z}\zeta} - (-w) \right] d\mu(\zeta) \right| \\
\leq & |2\mu(I(z,\epsilon))| + \sup_{\zeta \in I^c(z,\epsilon)} \left[\frac{\zeta - z}{1 - \bar{z}\zeta} - (-w) \right] \mu(I^c(z,\epsilon)) \\
\leq & |2\mu(I(z,\epsilon)) + \epsilon\mu(I^c(z,\epsilon))| \\
\leq & |2\epsilon + \epsilon| = 3\epsilon
\end{aligned}$$

As $z \to 1$ along Γ, $w \to 1$, and hence $\tilde{\xi}_\mu(z) \to -1$.

Part 3.

Suppose $\inf_{z \in \Gamma \setminus \{1\}} |\tilde{\xi}_\mu(z)| = 0$, then there is a sequence of points $\{z_n\}_{n=1}^\infty$ such that $\tilde{\xi}_\mu(z_n)$ approaches a zero vector as $n \to \infty$. Let $z_0 = \lim_{n \to \infty} z_n$. Thus, either $z_0 \in \mathbb{D}$ or $z_0 = 1$. Since $\tilde{\xi}_\mu$ is a continuous vector field in $\mathbb{D}$, then in the first case $\tilde{\xi}_\mu(z_0) = 0$. This contradicts part 1 of this lemma. In the second case, we have a contradiction to part 2 of this lemma.

□

Let f be a continuous circle map. Let $z_0 \in \mathbb{D}$, and let $\{z_n\}_{n=1}^\infty$ be a sequence of points in $\mathbb{D}$ converging to z_0. Let $\mu_0 = f_*(\eta_{z_0})$ and $\mu_n = f_*(\eta_{z_n})$.

The final thing we need to show before proving continuity is that $\tilde{\xi}_{\mu_n}$ converges to $\tilde{\xi}_{\mu_0}$ uniformly in $\mathbb{S}^1$.

Lemma 10. *Let f be a continuous map on $\mathbb{S}^1$ and let μ_n and μ_0 be the probability measures defined above, then for any $\epsilon > 0$ there exists a large enough $N > 0$ such that for all $n > N$, $|\tilde{\xi}_{\mu_n}(z) - \tilde{\xi}_{\mu_0}(z)| < \epsilon$ for all $z \in \mathbb{D}$.*

Proof. By Corollary 2, μ_n converge to the measure μ_0 in the following sense: for any $\epsilon > 0$, there exists a large enough $N > 0$ such that for all $n > N$,

$$|\mu_n(E) - \mu_0(E)| < \epsilon Leb(f^{-1}(E))$$

for any measurable set $E \subset \mathbb{S}^1$. Hence,

$$\begin{aligned} |\tilde{\xi}_{\mu_n}(z) - \tilde{\xi}_{\mu_0}(z)| &= \left| \int_{\mathbb{S}^1} \frac{\zeta - z}{1 - \bar{z}\zeta} d\mu_n(\zeta) - \int_{\mathbb{S}^1} \frac{\zeta - z}{1 - \bar{z}\zeta} d\mu_0(\zeta) \right| \\ &\leq \int_{\mathbb{S}^1} \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right| |d\mu_n(\zeta) - d\mu_0(\zeta)| \\ &= \int_{\mathbb{S}^1} |d\mu_n(\zeta) - d\mu_0(\zeta)|. \end{aligned}$$

The convergence of μ_n to μ_0 in the above sense implies

$$|\tilde{\xi}_{\mu_n}(z) - \tilde{\xi}_{\mu_0}(z)| < \int_{\mathbb{S}^1} |d\mu_n(\zeta) - d\mu_0(\zeta)| < \epsilon Leb(f^{-1}(\mathbb{S}^1)) = \epsilon.$$

□

Now we show the continuity $\Phi(f)$ at every point on the boundary of δ_f.

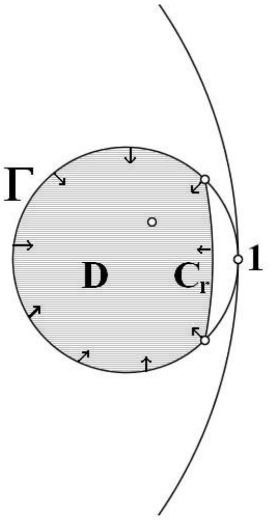

Figure 5.2: An illustration for the proof of Proposition 7.

Proposition 7 (Hu-Muzician). *Let f be a continuous circle map, then $\Phi(f)$ is continuous at every point on $\partial\delta_f$.*

Proof. Let $z_0 \in \partial\delta_f$. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $\mathbb{D}$ converging to z_0. We have two cases to consider:

1. For any subsequence $\{z_{n_k}\}_{n_k=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ contained in δ_f, we show that $\Phi(f)(z_{n_k})$ is eventually equal to $\Phi(f)(z_0)$. By Lemma 7, there is a large enough N such that for all $n_k > N$, z_{n_k} belongs to the same component of δ_f as z_0 and therefore is mapped by $\Phi(f)$ to $\Phi(f)(z_0)$.

2. For any subsequence $\{z_{n_k}\}_{n_k=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ contained in β_f, we show that $\Phi(z_{n_k})$ converges to $\Phi(z_0)$ as z_{n_k} converges to z_0. For brevity of notation we will continue to use $\{z_n\}_{n=1}^{\infty}$ to denote such a subsequence.

Let $\mu_n = f_*(\eta_{z_n})$ for $n \geq 0$. Since $z_0 \in \partial\delta_f$, there exists $w \in \mathbb{S}^1$ such that $\mu_0(w) = 1/2$ and so $\Phi(f)(z_0) = w$.

Let Γ be a horocycle at w. By Lemmas 9 and 10, there exists a large enough N such that for all $n > N$, $\tilde{\xi}_{\mu_n}(z)$ points inside Γ at each $z \in \Gamma\backslash\{1\}$. Now we show that for each $n > N$, the conformal barycenter of μ_n belongs to the open disk bounded by Γ. By Lemma 4, for each $n > N$, there exists $0 < r < 1$, depending on n, such that $\tilde{\xi}_{\mu_n}(z)$ points inside the circle $C_r = \{z \in \mathbb{D} : |z| = r\}$ at each point on C_r and C_r intersects Γ at two points. Let D be the domain bounded by C_r and Γ (see Figure 5.2). Then $\tilde{\xi}_{\mu_n}$ points inside D at each point on the boundary of D. Note that $\tilde{\xi}_{\mu_n}$ is a continuous vector field in $\mathbb{D}$ with a unique zero in $\mathbb{D}$. On the other hand, by the corollary to the Poincare-Hopf Theorem, $\tilde{\xi}_{\mu_n}$ has a unique zero in D and then $\Phi(f)(z_n) \in D$. Thus $\Phi(f)(z_n)$ belongs to the disk bounded by Γ. As the horocycle Γ shrinks to the point w, $\Phi(f)(z_n)$ approaches w as $n \to \infty$.

By combining the two cases, we have shown that $\Phi(f)$ is continuous at any point $z_0 \in \partial\delta_f$. $\square$

5.5 Examples

To better understand this extension as well as the sets β_f and δ_f, we consider a few examples:

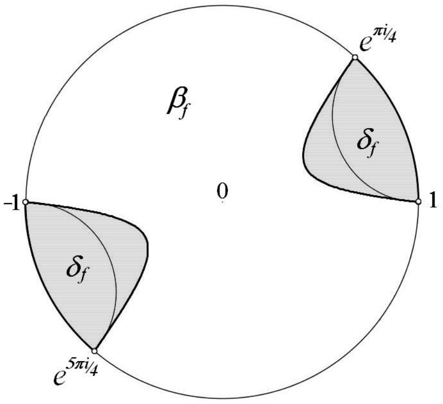

Figure 5.3: $\Phi(f_{1/4})(z)$.

Example 1: Consider a one parameter family of continuous degree two circle maps $f_\lambda(z)$ mapping two arcs $\widehat{[1, e^{i\pi\lambda}]}$ and $\widehat{[-1, -e^{i\pi\lambda}]}$ to 1, and mapping the other two arcs onto $\mathbb{S}^1\backslash\{1\}$ evenly.

Figure 5.3 shows the domain of $\Phi(f_{1/4})(z)$. Two $\delta_{f_{1/4}}$ sets are disjoint and they are clearly not bounded by geodesics connecting end points of collapsing arcs. Zero is a critical point of degree two. If λ starts approaching 1/2, $\Phi(f_\lambda)(0)$ will approach 1.

Figure 5.4 shows examples of $\Phi(f_\lambda)(z)$ with $\lambda = 1/2$ and $\lambda = 3/4$. It is easy to see that in both cases the critical point $0 \in \delta_{f_\lambda}$, and β_{f_λ} has two disjoint components each mapping onto $\mathbb{D}$.

Example 2:

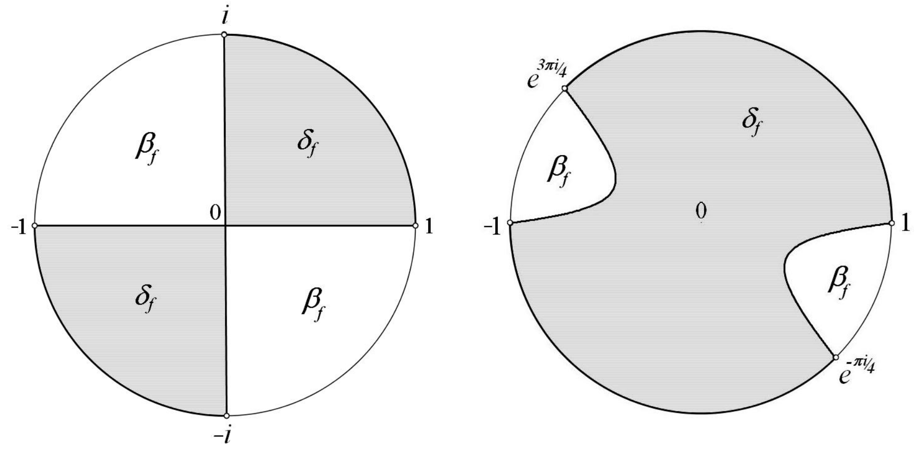

Figure 5.4: $\Phi(f_{1/2})(z)$ and $\Phi(f_{3/4})(z)$.

Suppose

$$f(z) = \begin{cases} z, & Im(z) \geq 0 \\ \bar{z}, & Im(z) < 0. \end{cases}$$

The map f isn't surjective in this example, and so $\Phi(f)$ isn't surjective as well. Figure 5.5 shows the image of $f(\overline{\mathbb{D}})$. Both the upper half-disk and lower half-disk map onto the shaded region, and $\Phi(f)(0) \approx 0.42i$.

5.6 Relation to the MAY Iterator

Note that if f is a continuous circle map, then $f_*(\eta_z)$ has at most 1 strong atom for all $z \in \mathbb{D}$. In other words, $f_*(\eta_z) \in P''$, where P'' is as defined in Chapter 3. Therefore, since the MAY iterator finds a conformal barycenter for any measure in P'', it can be used for finding conformally natural ex-

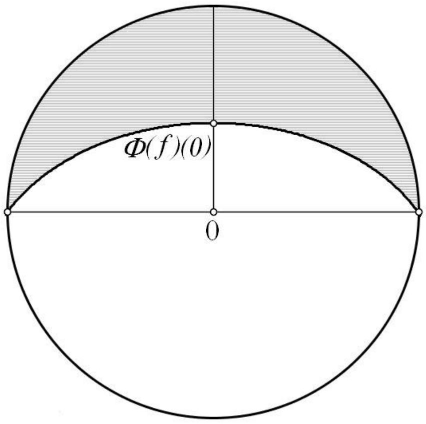

Figure 5.5: Degree zero map .

tensions for any continuous map. Moreover, it completely agrees with the definition of conformally natural extensions for any continuous map given in Chapter 5:

1. $z \in \beta_f$ if and only if $f_*(\eta_z) \in P'$. In this case, the MAY iterator produces the same conformal barycenter as Douady-Earle's definition as was shown in Chapter 3.

2. $z \in \delta_f$ if and only if $f_*(\eta_z) \in P'' \backslash P'$. As shown in Theorem 6, $\lim_{n\to\infty} h^{(n)}_{f_*(\eta_z)}$ is equal to a strong atom which is the same as γ_z.

This enables us to use computer software to generate conformally natural extensions for any continuous circle map.

Chapter 6

Local Regularity

6.1 Preliminaries

Let f be a continuous circle map and $\Phi(f)$ be its conformally natural extension to the unit disk. In this chapter we prove the following theorem. Although the scheme and techniques to derive this theorem are almost identical to those used in [10] to develop the same result for the extensions of the maps considered in that paper, we wish to recapitulate them here to provide a self-contained exposition on the proof.

Theorem 3. *Let* f *be a continuous circle map and* $I \subset \mathbb{S}^1$ *be an open arc. If* f *is quasisymmetric on* I *with the cross-ratio norm bounded by a constant* $M > 0$*, then* $\Phi(f)$ *is locally* K*-quasiconformal on* D*, where* $D = \{z \in \mathbb{D} : \eta_z(I) > 1 - \rho/2\pi\}$*, and* ρ *and* K *depend only on* M*.*

Given two points X and Y on the unit circle $\mathbb{S}^1$, we let $\widehat{XY}$ stand for the closed circular arc on $\mathbb{S}^1$ from X to Y in the counterclockwise direction and $int(\widehat{XY})$ for the open circular arc defined in the same way. Let $|\widehat{XY}|$ stand for the length of $\widehat{XY}$ and $|XY|$ for the Euclidean distance between X and Y on the complex plane. Given several points X_1, X_2, $\cdots$, X_n on $\mathbb{S}^1$ arranged in the counterclockwise direction, denote the circular arc on $\mathbb{S}^1$ from X_1 to X_n through other points in the counterclockwise direction by $\widehat{X_1X_2\cdots X_n}$.

Let A, B, C and D be the endpoints of two perpendicular diameters of $\mathbb{S}^1$, arranged in the counterclockwise direction. The following concept is intended to be defined for a semicircle on $\mathbb{S}^1$ in an arbitrary position. In order to introduce numerical values relatively easily, we assume that A is located at the point -1 (see Figure 6.1). Then any rotation of the semicircle $\widehat{BCD}$ has the property required by the following concept with respect to the configuration after the rotation.

Let E be the point on $\mathbb{S}^1$ such that the hyperbolic geodesic on $\mathbb{D}$ connecting E to B is perpendicular to the one connecting A to D (see Figure 6.1). By using the cross-ratio condition for perpendicularity, we work out that $E = \frac{-4+3i}{5}$. Then the length of the shorter circular arc between E and A is strictly between $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Let τ be the length of the shorter circular arc

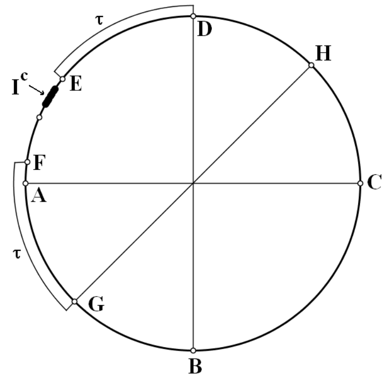

Figure 6.1: An illustration for the scale of I^c.

between D and E, then $\frac{\pi}{4} < \tau < \frac{\pi}{3}$. Now let $H = e^{i\frac{\pi}{4}}$ and $G = e^{i\frac{3\pi}{4}}$, and let F be the point on the unit circle such that the arc $\widehat{FG}$ in the counterclockwise direction from F to G is equal to τ. Then the length of the arc $\widehat{EF}$ from E to F in the counterclockwise direction is equal to $4\alpha = \frac{3\pi}{4} - 2\tau$, which is between $\frac{\pi}{12}$ and $\frac{\pi}{4}$. Note that $\frac{\pi}{48} < \alpha < \frac{\pi}{16}$.

Let J (denoted by I^c) be a closed circular arc contained in $\widehat{EF}$ and with its length less than $\frac{3\pi}{4} - 2\tau$, and let J^c (that is I) be the complement of J in $\mathbb{S}^1$. Then the semicircle $\widehat{BCD}$ has the following two properties:

1. $\widehat{BCD}$ is contained in I; and
2. the length of one component of $I \setminus \widehat{BCD}$ is greater than τ and the length of the other is greater than $\frac{\pi}{4} + \tau$.

Two concepts are introduced in [9]. First, referring to Figure 6.1, $\widehat{BCD}$ is said to be *τ-contained in I* if the above two properties are satisfied. Secondly, given three points A, B and C on $\mathbb{S}^1$, we call the other endpoint of the geodesic passing through C and perpendicular to the geodesic connecting A to B the *hyperbolic middle point* between A and B with respect to C.

Note that since $\widehat{BCD}$ is a semicircle τ-contained in I, the other endpoint A of the diameter that passes through C and is perpendicular to the diameter connecting B to D belongs to I (see Figure 6.1 again).

Remark 7. Note that the condition $\Phi(f)(0) = 0$ is preserved under pre-composition and post-composition by rotation around the origin. Moreover, the cross-ratio distortion norm is preserved as well. Because of this, in the following lemmas we may pre-compose and post-compose f by rotations around the origin to move considered points in desired positions. For brevity, we will say *by rotational invariance* whenever we apply this action.

In the following, the scheme and techniques used to prove Theorem 9 of [10] are applied to provide an exposition on a self-contained proof of Theorem 3. In order to do so, we first restate several lemmas of [10] for arbitrary continuous circle maps, and then provide their proofs although they are identical to the proofs of the same results for continuous maps considered in [10].

Lemma 11. *Suppose that f is a continuous map of $\mathbb{S}^1$ and the conformally* *natural extension* $\Phi(f)$ *of* f *fixes the origin. Assume that* f *is injective on* I *and* $||f|_I||_{cr}$ *is finite, where* I *is an open circular arc on* $\mathbb{S}^1$ *with arc length* $\geq 2\pi - \frac{1}{16}$*. Then for any semicircle* $\widehat{BCD}$ *(with* C *at the middle)* τ*-contained in* I*,* $f(\widehat{BCD})$ *has a length greater than or equal to a positive constant* ϵ_3*, where* $\epsilon_3 = 2\arcsin(\delta_3/2)$*,* $\delta_3 = \frac{e^{-M}}{2/\sqrt{3}+e^{-M}}$ *and* $M = ||f|_I||_{cr}$*.*

Furthermore, let A *be the other endpoint of the diameter passing through* C *and perpendicular to the diameter* BD*. Then the semicircle* $\widehat{ABC}$ *is contained in* I *(but not* τ*-contained) and* $f(\widehat{ABC})$ *also has a length greater than or equal to* ϵ_3*. Finally both* $\mathbb{S}^1\backslash f(\widehat{BCD})$ *and* $\mathbb{S}^1\backslash f(\widehat{ABC})$ *have a length greater than or equal to* ϵ_3 *too.*

Proof. Note that $\alpha > \frac{1}{16}$. Therefore, with $|I^c| \leq \frac{1}{16}$, the circular arc I is big enough to τ-contain a semicircle inside.

Let I and B, C, D, A be the notations defined in the statement of the lemma. Denote their images under f by A', B', C' and D' respectively. By rotational invariance, we may assume that AC is horizontal and BD is vertical (see the left part in Figure 6.2).

Step 1. In this step, we show that one of the two arcs $\widehat{A'B'C'}$ and $\widehat{B'C'D'}$ have an arc length of at least $\frac{\pi}{3}$. The proof only requires the normalization condition that $\Phi(f)(0) = 0$. By rotational invariance, we may further assume in this step that the two images A' and D' are complex conjugates.

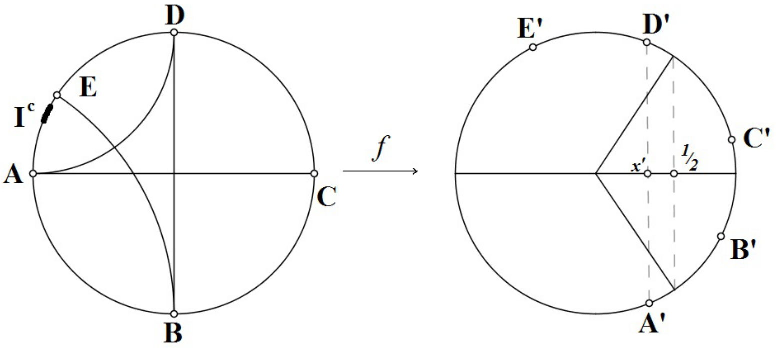

Figure 6.2: An illustration for Step 1 in the proof of Lemma 11.

Note that for any circular arcs I_1 and J_1 on $\mathbb{S}^1$, if $J_1 = f(I_1)$, then $\mu(J_1) \geq Leb(I_1)$, where μ is the pushforward of the Lebeque measure by f.

Let x' be the x-coordinate of A'. Since the conformal barycenter is at the origin,

$$0 = \int_{\mathbb{S}^1} z d\mu(z) = \int_{\mathbb{S}^1} x d\mu(z) + i \int_{\mathbb{S}^1} y d\mu(z).$$

Then

$$\begin{aligned} 0 &= \int_{\mathbb{S}^1} x d\mu(z) \geq (-1)\mu(\widehat{D'A'}) + x'\mu(\widehat{A'B'C'D'}) \\ &= (-1)\left(\frac{1}{4} - \rho\right) + x'\left(\frac{3}{4} + \rho\right) \\ &= (-1)\left(\frac{1}{4}\right) + x'\left(\frac{3}{4}\right) + \rho(1 + x') \geq (-1)\left(\frac{1}{4}\right) + x'\left(\frac{3}{4}\right), \end{aligned}$$

where ρ is the total measure of the set of all points in I^c that are mapped into $\widehat{A'B'C'D'}$. By solving $x' \cdot 3/4 + (-1) \cdot 1/4 \leq 0$, we obtain $x' \leq 1/3 < 1/2$.

Thus $|\widehat{A'B'C'D'}| \geq 2\pi/3$. Then $|\widehat{A'B'C'}| \geq \pi/3$ or $|\widehat{B'C'D'}| \geq \pi/3$. In order to obtain a lower bound for $|\widehat{B'C'D'}|$, we only need to handle the situation that $|\widehat{A'B'C'}| \geq \frac{\pi}{3}$ and $|\widehat{B'C'D'}| < \frac{\pi}{3}$. Now we go to Step 2.

Step 2. We show that there exists a constant $\delta_3 > 0$ such that $|\widehat{B'C'D'}| \geq \delta_3$. The proof makes use of the assumptions that $\Phi(f)(0) = 0$, $||f|_I||_{cr}$ is finite, and the semicircle $\widehat{BCD}$ is τ-contained in I. We divide the proof into two cases according to either $|\widehat{C'D'A'}| \leq \pi$ or $|\widehat{C'D'A'}| > \pi$, but the proofs are similar.

Case 1: Suppose $|\widehat{C'D'A'}| \leq \pi$. Let E be the hyperbolic middle point between A and D with respect to B and E' be the image of E under f. Since the semicircle $\widehat{BCD}$ is τ-contained in I, the point E belongs to I. For convenience in this case, by rotational invariance, we assume that B' and E' are complex conjugates.

Note that $|\widehat{DE}| > \pi/4$ and so $\mu(\widehat{B'C'D'E'}) \geq |\widehat{BCDE}| > 5/8$ (Refer to the left part of Figure 6.3).

Let $\tilde{x}$ be the x-coordinate of B'. The normalization condition $\Phi(f)(0) = 0$ implies that

$$0 = \int_{\mathbb{S}^1} x d\mu(z) \geq (-1)\mu(\widehat{E'D'}) + \tilde{x}\mu(\widehat{B'C'D'E'}).$$

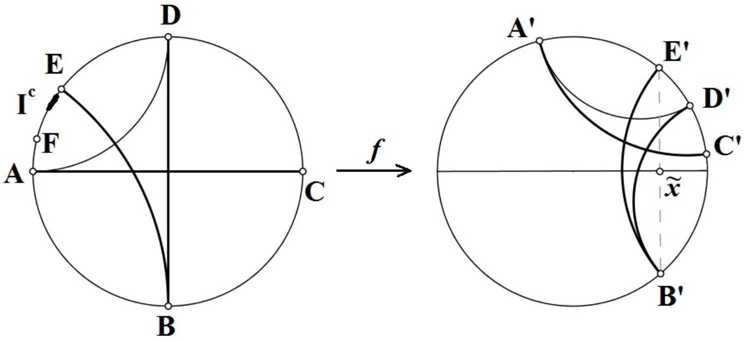

Figure 6.3: An illustration for Case 1 in Step 2 of the proof for Lemma 11.

Then

$$\tilde{x} \leq \frac{\mu(\widehat{E'B'})}{\mu(\widehat{B'C'D'E'})} < \frac{3/8}{5/8} = \frac{3}{5} < \frac{\sqrt{3}}{2}.$$

Thus $|\widehat{B'C'D'E'}| > \pi/3$. On the other hand, using the assumptions that $|\widehat{B'C'D'}| < \frac{\pi}{3}$ and $|\widehat{C'D'A'}| \leq \pi$ we obtain

$$|\widehat{B'C'D'E'}| < |\widehat{B'C'D'A'}| < |\widehat{B'C'D'}| + |\widehat{C'D'A'}| \leq \frac{\pi}{3} + \pi = \frac{4\pi}{3}. \tag{6.1}$$

Thus the Euclidean distance $|B'E'|$ between B' and E' is greater than or equal to 1; that is, $|B'E'| \geq 1$.

Using the same method from Step 1 and the fact that the arc length of I^c is less than 1/16, we can also show that $|\widehat{B'C'D'A'}| > 2\pi/3$ as follows:

First by rotational invariance we may assume that A' and B' are complex conjugates. Let x' be the x-coordinate of A'. Using the normalization condition $\Phi(f)(0) = 0$,

$$\begin{aligned} 0 &= \int_{\mathbb{S}^1} x d\mu(z) \geq (-1)\mu(\widehat{A'B'}) + x'\mu(\widehat{B'C'D'A'}) \\ &= (-1)\left(\frac{1}{4} + \rho'\right) + x'\left(\frac{3}{4} - \rho'\right) = (-1)\left(\frac{1}{4}\right) + x'\left(\frac{3}{4}\right) - \rho'(1 + x') \\ &\geq (-1)\left(\frac{1}{4}\right) + x'\left(\frac{3}{4}\right) - 2\rho' \geq (-1)\left(\frac{1}{4}\right) + x'\left(\frac{3}{4}\right) - \frac{1}{8} \\ &= -\frac{3}{8} + x'\left(\frac{3}{4}\right), \end{aligned}$$

where ρ' is the total measure of the set of all points in I^c that are mapped into $\widehat{B'C'D'A'}$, which is less than or equal to $\frac{1}{16}$.

Thus $x' \leq 1/2$ and then $|\widehat{B'C'D'A'}| \geq 2\pi/3$.

By (6.1), $|\widehat{B'C'D'A'}| \leq \frac{4\pi}{3}$. Thus $|B'A'| \geq \sqrt{3}$.

Let $M = ||f|_I||_{cr}$. By considering the cross-ratio distortion of f on the quadruple $\{E, A, B, D\}$, we obtain

$$e^{-M} \leq \frac{|E'A'||B'D'|}{|A'B'||D'E'|} \leq \frac{2|B'D'|}{\sqrt{3}|D'E'|}.$$

Then

$$\frac{e^{-M}}{2/\sqrt{3} + e^{-M}} \leq \frac{|B'D'|}{|B'D'| + |D'E'|} \leq \frac{|B'D'|}{|B'E'|} \leq |B'D'|;$$

that is,

$$|B'D'| \geq \delta_3$$

with $\delta_3 = \frac{e^{-M}}{2/\sqrt{3}+e^{-M}}$.

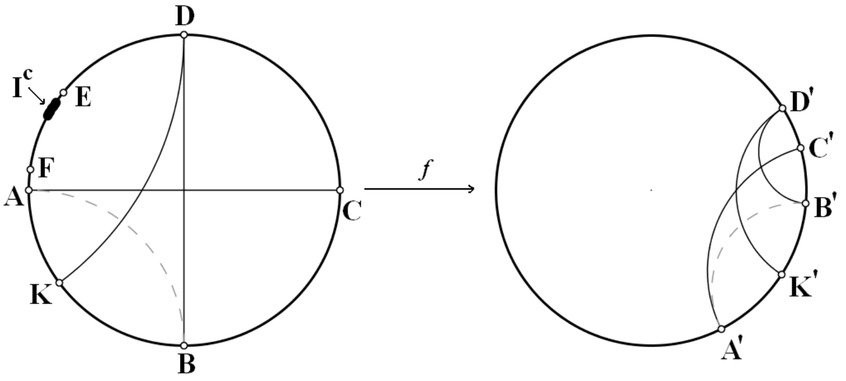

Figure 6.4: An illustration for Case 2 in Step 2 of the proof for Lemma 11.

Now let $\epsilon_3 = 2\arcsin(\delta_3/2)$. Then, the length of $f(\widehat{BCD})$ is greater than or equal to ϵ_3.

Case 2: Suppose $|\widehat{C'D'A'}| > \pi$. Then $|\widehat{A'B'C'}| < \pi$. Let K be the other endpoint of the geodesic passing through D and perpendicular to the geodesic connecting A to B and $K' = f(K)$. By rotational invariance, we assume here that D' and K' are complex conjugates. With similar work as in Case 1, we also obtain that the length of $f(\widehat{BCD})$ is greater than or equal to ϵ_3 (referring to Figure 6.4).

Now we show that the length of $f(\widehat{ABC})$ is also greater than or equal to ϵ_3. Since the hyperbolic middle point between C and D with respect to A and the one between D and A with respect to C belong to I, one can show that $|f(\widehat{ABC})| \geq \epsilon_3$ in a similar way as for $f(\widehat{BCD})$.

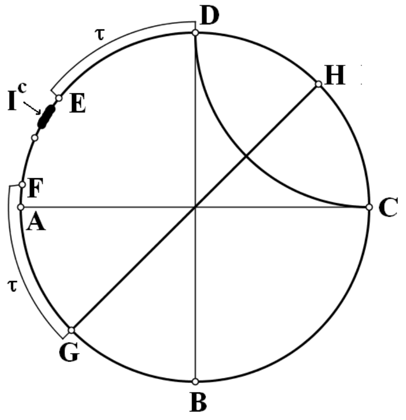

Figure 6.5: An illustration for the proof of Corollary 4.

Finally, the proof of $|\mathbb{S}^1 \setminus f(\widehat{BCD})| \geq \epsilon_3$ follows the same method by knowing that the hyperbolic middle point between C and D with respect to B or the one between B and C with respect to D does not fall into the region I^c. For the same reason, $|\mathbb{S}^1 \setminus f(\widehat{ABC})| \geq \epsilon_3$.

We have completed the proof. $\square$

Corollary 4. *With the same assumptions as in Lemma 11, set* $M = ||f||_{cr}$, $\delta_4 = \frac{e^{-5M}}{64(2/\sqrt{3}+e^{-M})^3}$ *and* $\epsilon_4 = 2\arcsin(\delta_4/2)$. *Suppose that* β *is a circular arc in* $\mathbb{S}^1$ *with arc-length between* $\frac{\pi}{4}$ *and* $\frac{\pi}{2}$ *and that can be extended in one direction to a semicircle* BCD *that is* τ*-contained in* I. *Then the arc-length of* $f(\beta)$ *is greater than or equal to* ϵ_4.

Proof. It suffices to show that the conclusion holds when β has length equal to $\frac{\pi}{4}$. We divide the proof into two steps. In Step 1, we show that it is true for β with length equal to $\frac{\pi}{2}$; and in Step 2 we show the case for β with length equal to $\frac{\pi}{4}$.

Step 1. Let A, B, C, D, A', B', C' and D' be the same points given in Lemma 11 or its proof, and assume that β is the quarter between C and D. Let ϵ_3 and δ_3 be the constants obtained in Lemma 11. Note that $\epsilon_3 = 2\arcsin(\delta_3/2)$. By applying the conclusions in the previous lemma, $|f(\widehat{BCD})| \geq \epsilon_3$ and $|\mathbb{S}^1 \setminus f(\widehat{BCD})| \geq \epsilon_3$ imply that $|B'D'| \geq \delta_3$. Similarly, $|f(\widehat{ABC})| \geq \epsilon_3$ and $|\mathbb{S}^1 \setminus f(\widehat{ABC})| \geq \epsilon_3$ imply that $|A'C'| \geq \delta_3$.

We show that if $|C'D'| \leq \delta_3/2$, then it is greater than or equal to $e^{-M}{\delta_3}^2/8$. By the triangle inequality,

$$|B'C'| \geq |B'D'| - |C'D'| \geq \delta_3 - \frac{\delta_3}{2} = \frac{\delta_3}{2}.$$

Similarly,

$$|D'A'| \geq |C'A'| - |C'D'| \geq \frac{\delta_3}{2}.$$

Considering the cross-ratio distortion under f on the quadruple $\{A, B, C, D\}$ and using the definition of $||f|_I||_{cr}$, we obtain

$$e^{-M} \leq \frac{|A'B'||C'D'|}{|B'C'||D'A'|} \leq \frac{2|C'D'|}{{\delta_3}^2/4} \leq \frac{8|C'D'|}{{\delta_3}^2}.$$

Thus $|C'D'| \geq e^{-M}{\delta_3}^2/8$. Hence, $|C'D'| \geq \delta = \min\{e^{-M}{\delta_3}^2/8, \delta_3/2\} =$

$e^{-M}{\delta_3}^2/8$ since $\delta_3 = \frac{e^{-M}}{2/\sqrt{3}+e^{-M}} < 1$. Clearly, δ is less than δ_3, and hence also less than 1.

Step 2. Continue to use the symbols in Step 1 and let H and G be the Euclidean middle points on $\mathbb{S}^1$ between C and D and assume that H is on the short arc between C and D (see Figure 6.5). Without loss of generality, it suffices to show in this step that $|H'D'|$ can not be too small.

Suppose that $|H'D'| \leq \delta/2$. From the previous step, $|C'D'| \geq \delta$. By the triangle inequality we obtain

$$|C'H'| \geq |C'D'| - |H'D'| \geq \delta - \delta/2 = \delta/2.$$

Since the semicircle $\widehat{GBH}$ is also τ-contained in I, Lemma 11 implies that $|G'H'| \geq \delta_3$. Since $\delta < \delta_3$, we obtain

$$|D'G'| \geq |G'H'| - |H'D'| \geq \delta_3 - \delta/2 \geq \delta_3/2.$$

From the definition of $M = ||f||_{cr}$, we obtain

$$e^{-M} \leq \frac{|G'C'||H'D'|}{|C'H'||D'G'|} \leq \frac{2|H'D'|}{\delta\delta_3/4} \leq \frac{8|H'D'|}{\delta\delta_3}.$$

Thus $|H'D'| \geq e^{-M}\delta\delta_3/8$.

Hence, $|H'D'| \geq \delta_4$, where $\delta_4 = \min\{e^{-M}\delta\delta_3/8, \delta/2\} = e^{-M}\delta\delta_3/8$ since $\delta < 1$. Then

$$\delta_4 = e^{-2M}\delta_3^3/64 = \frac{e^{-5M}}{64(2/\sqrt{3} + e^{-M})^3}$$

Now let $\epsilon_4 = 2\arcsin(\frac{\delta_4}{2})$. Then the arc-length of the arc $\widehat{H'D'}$ is greater than or equal to ϵ_4. □

6.2 Proof of Local Regularity

In the next two lemmas, we prove that $|d_1|^2 - |d_{-1}|^2 > \delta_1 > 0$ and $|c_1|^2 - |c_{-1}|^2 > \delta_2 > 0$, where d_1, d_2, c_1 and c_2 are constants defined in Chapter 4.

Note that the definition of a homeomorphism $h(u)$ from Chapter 4 is still applicable for a continuous map $f(z)$. In fact, we will call $f(z)$ a degree d continuous circle map where d is equal to the winding number of $f(z)$ around the origin.

Lemma 12. *Let f be a continuous circle map. Suppose that $\Phi(f)(0) = 0$, f is injective on I, and $||f|_I||_{cr} \leq M$, where I is an arc on $\mathbb{S}^1$ with the arc-length of I^c less than $\frac{1}{16}$. Then there exists a constant δ_1 depending only on M such that $|d_1|^2 - |d_{-1}|^2 > \delta_1$, where d_1 and d_{-1} are as the same as introduced in Chapter 4.*

Proof. Let Γ be the union of two circular arcs of arc-length τ on $\mathbb{S}^1$: one starting at i and going counter-clockwise and the other starting at $e^{i\frac{5\pi}{4}}$ and going clockwise; that is, $\Gamma = \widehat{DE} \cup \widehat{FG}$ in Figure 6.6. Since τ is a particular value chosen at the beginning of this chapter, the shorter circular arc between

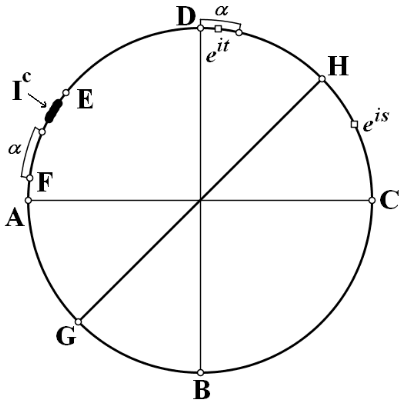

Figure 6.6: An illustration for the proof of Lemma 12.

the two components of Γ has the arc-length equal to $\frac{3\pi}{4} - 2\tau$, which is greater than $\frac{\pi}{12}$. Let $\alpha = (\frac{3\pi}{4} - 2\tau)/4$. Then $\alpha \geq \frac{\pi}{48}$. In counterclockwise order, we add to each component of Γ small circular arcs $\widehat{MD}$ and $\widehat{NF}$, of length equal to α and 2α respectively (see Figure 6.6). Since the length of I^c is less then $\frac{1}{16}$, which is less than $\frac{\pi}{24}$, by rotational invariance we may assume that I^c is contained in the shorter arc between $\widehat{DE}$ and $\widehat{NF}$. By (4.1), we know

$$\begin{aligned}
|d_1|^2 - |d_{-1}|^2 &= 2\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} \sin^2(h(s) - h(t))dsdt \\
&= 2\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_{t-2\pi}^{t} \sin^2(h(s) - h(t))dsdt \\
&\geq 2\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_{t-\frac{\pi}{4}-\alpha}^{t-\frac{\pi}{4}} \sin^2(h(s) - h(t))dsdt.
\end{aligned}$$

Furthermore, the double integral in the previous line is greater than or equal to

$$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}}\int_{t-\frac{\pi}{4}-\alpha}^{t-\frac{\pi}{4}}\sin^2(h(s)-h(t))dsdt+\int_{\frac{3\pi}{2}-\alpha}^{\frac{3\pi}{2}}\int_{t-\frac{\pi}{4}-\alpha}^{t-\frac{\pi}{4}}\sin^2(h(s)-h(t))dsdt,$$

which can be combined as

$$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}}\int_{t-\frac{\pi}{4}-\alpha}^{t-\frac{\pi}{4}}[\sin^2(h(s)-h(t))+\sin^2(h(s+\pi)-h(t+\pi))]dsdt.$$

Note that if $\frac{\pi}{2}-\alpha \le t \le \frac{\pi}{2}$ and $t-\frac{\pi}{4}-\alpha \le s \le t-\frac{\pi}{4}$, then in counterclockwise order, the arc between e^{is} and e^{it}, or $e^{i(s+\pi)}$ and $e^{i(t+\pi)}$, or $B=e^{i3\pi/2}$ and $C=e^{i2\pi}$ (refer to Figure 6.6), can be extended in one direction to a semicircle τ-contained in I and has arc-length between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. By Corollary 4, we conclude that all three values of $|h(s)-h(t)|$, $|h(s+\pi)-h(t+\pi)|$ and $|h(\frac{3\pi}{2})-h(2\pi)|$ are greater than or equal to ϵ_4. Thus

$$|h(s)-h(t)|+|h(s+\pi)-h(t+\pi)| \le 2\pi-\epsilon_4.$$

Therefore, one of these two summands is less than or equal to $\pi-\frac{\epsilon_4}{2}$, and hence is between $\frac{\epsilon_4}{2}$ and $\pi-\frac{\epsilon_4}{2}$. It follows that one of the values of $\sin^2(h(s)-h(t))$ and $\sin^2(h(s+\pi)-h(t+\pi))$ is greater than or equal to $\sin^2(\epsilon_4/2)$ for any $\frac{\pi}{2}-\alpha \le t \le \frac{\pi}{2}$ and $t-\frac{\pi}{4}-\alpha \le s \le t-\frac{\pi}{4}$. Thus

$$|d_1|^2-|d_{-1}|^2 \ge 2\left(\frac{1}{2\pi}\right)^2\alpha^2\sin^2\left(\frac{\epsilon_4}{2}\right)=\frac{\alpha^2\delta_4^2}{8\pi^2}.$$

By letting $\delta_1=\frac{\alpha^2\delta_4^2}{8\pi^2}$ with $\delta_4=\frac{e^{-5M}}{64(2/\sqrt{3}+e^{-M})^3}$, we complete the proof. $\square$

Lemma 13. *Suppose that $M > 0$, f is a continuous degree d circle map and $\Phi(f)(0) = 0$. There exist two positive constants ρ and δ_2, only depending on M, such that if there exists a circular arc I on $\mathbb{S}^1$ with arc-length of I^c less than ρ on which f is injective and $||f|_I||_{cr} \leq M$, then $|c_1|^2 - |c_{-1}|^2 > \delta_2$, where c_1 and c_{-1} are the same as defined in Chapter 4.*

Proof. Similar to the proof of the previous lemma, the estimate in Corollary 4 is the main resource to derive the estimate in this lemma. We therefore require first that $\rho \leq \frac{1}{16}$.

Let τ and α be the same constants introduced in the proof of the previous lemma and before. For convenience, by rotational invariance, we may assume that the arc I^c is contained in the open arc on $\mathbb{S}^1$ from 1 to $e^{i\rho}$ in the counterclockwise order.

Given $t \in [0, 2\pi)$ and $u \in [0, \pi)$, define β_1 to be the arc on $\mathbb{S}^1$ from e^{it} to $e^{i(t+u)}$ in counterclockwise order, β_2 from $e^{i(t+u+\pi)}$ to e^{it}, β_3 from $e^{i(t+\pi)}$ to $e^{i(t+u+\pi)}$, and finally β_4 from $e^{i(t+u)}$ to $e^{i(t+\pi)}$.

Let us recall the expression denoted by $|c_1|^2 - |c_{-1}|^2$ given in Chapter 4; that is,

$$|c_1|^2 - |c_{-1}|^2 = \left(\frac{1}{2\pi}\right)^2 \int_0^\pi \int_0^{2\pi} \sin(u) H(t, u) dt du,$$

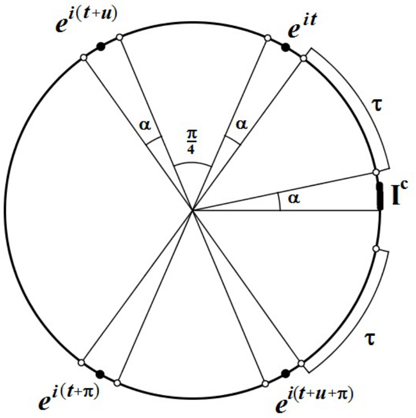

Figure 6.7: An illustration for Lemma 13.

where

$$\begin{aligned} H(t,u) &= \sin(h(t+u)-h(t)) + \sin(h(t+\pi)-h(t+u)) \\ &\quad + \sin(h(t+\pi+u)-h(t+\pi)) + \sin(h(t+2\pi)-h(t+u+\pi)) \\ &= \sin(\alpha_1)+\sin(\alpha_2)+\sin(\alpha_3)+\sin(\alpha_4), \end{aligned}$$

where $h:\mathbb{R}\to\mathbb{R}$ is a homeomorphism with $h(u+2\pi)=h(u)+2\pi d$, and α_j denotes the arc-length of $f(\beta_j)$ which is equal to the difference in the sine in order for $j=1,2,3,4$.

Suppose e^{it}, $e^{i(t+u)}$, $e^{i(t+\pi)}$ and $e^{i(t+u+\pi)}$ do not belong to I^c. Then I^c is contained in one of β_j. For example, assume that I^c is contained β_4. Since f is injective on I and is a degree-d continuous map of $\mathbb{S}^1$,

$$\sum_{j=1}^{4}\alpha_j = 2\pi d$$

and $\alpha_4 > 2\pi(d-1)$. We let $\alpha_4' = \alpha_4 - 2\pi(d-1)$, then $\alpha_1+\alpha_2+\alpha_3+\alpha_4' = 2\pi$.

Using trigonometric identities it is possible to show that if $\sum_{j=1}^4 \alpha_j = 2\pi$, then

$$\sum_{j=1}^{4} \sin\alpha_j = 4\sin\frac{\alpha_1+\alpha_2}{2}\sin\frac{\alpha_1+\alpha_3}{2}\sin\frac{\alpha_2+\alpha_3}{2}.$$

Thus, for such values of t and u, $H(t,u) \geq 0$. For the same reason, we can see that such inequality holds as soon as I^c is contained in any of the other β_j. Therefore, we separate such points in the product space $[0,2\pi) \times [0,\pi)$ from others; that is, we let

1. B be the set of all pairs (t,u) in the product space such that at least one of the four points e^{it}, $e^{i(t+u)}$, $e^{i(t+\pi)}$ and $e^{i(t+u+\pi)}$ belongs to I^c, and
2. G be the complement of B in the product space. (Refer to Figure 6.8.)

We have just shown that $H(t,u) \geq 0$ for each point $(t,u) \in G$. Furthermore, it is easy to see that the box $[\tau+\alpha, \tau+2\alpha] \times [\pi/4, \pi/4+\alpha]$ is contained in G by checking $(\tau+2\alpha) + (\pi/4+\alpha) < \pi$. Then

$$\left(\frac{1}{2\pi}\right)^2 \int\int_G \sin(u)H(t,u)dtdu \geq \left(\frac{1}{2\pi}\right)^2 \int_{\pi/4}^{\pi/4+\alpha}\int_{\tau+\alpha}^{\tau+2\alpha} \sin(u)H(t,u)dtdu.$$

For any point $(t,u) \in [\tau+\alpha, \tau+2\alpha]\times[\pi/4, \pi/4+\alpha]$, $I^c \subset \beta_4$, $\frac{\pi}{4} \leq |\beta_1|, |\beta_3| \leq \frac{\pi}{2}$ and $\frac{\pi}{4} \leq |\beta_2| \leq \frac{3\pi}{4}$.

Since the semicircle $\beta_1 \cup \beta_2$ is τ-contained in I, by using Lemma 11 we

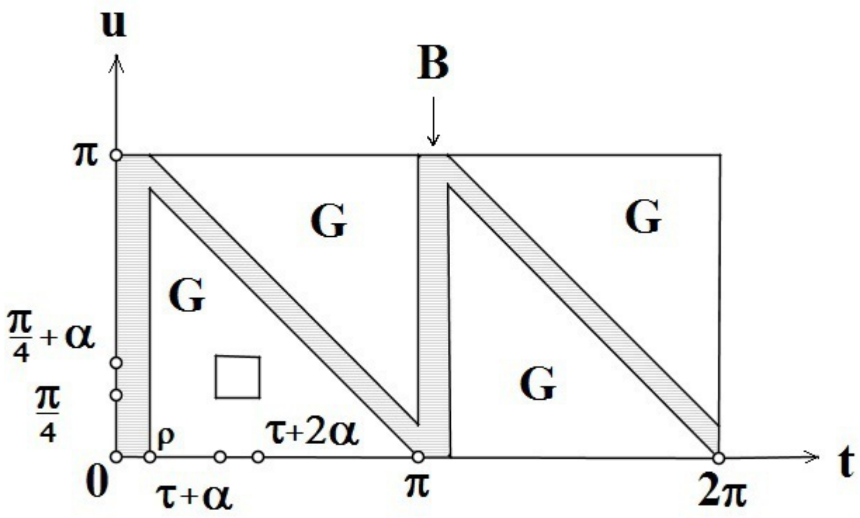

Figure 6.8: An illustration for the scale of I^c.

obtain

$$\epsilon_3 \leq \alpha_1 + \alpha_2 \leq 2\pi - \epsilon_3.$$

Similarly by using the semicircle $\beta_2 \cup \beta_3$ and the same lemma, we obtain

$$\epsilon_3 \leq \alpha_2 + \alpha_3 \leq 2\pi - \epsilon_3.$$

Both β_1 and β_3 can be extended in one direction to a semicircle that is τ-contained in I. Therefore, by Corollary 4, $\alpha_k \geq \epsilon_4$ for $k = 1, 3$. On the other hand, since $\frac{\pi}{2} < |\beta_2| < \frac{3\pi}{4}$, β_2 contains a quarter of the unit circle that extends to the semicircle $\beta_2 \cup \beta_3$, which is τ-contained in I as well. Applying Corollary 4 again to that quarter of the unit circle, one can conclude that

$\alpha_2 \geq \epsilon_4$. Putting the estimates together, we obtain

$$\epsilon_4 < 2\epsilon_4 \leq \alpha_1 + \alpha_3 \leq 2\pi - \alpha_2 \leq 2\pi - \epsilon_4.$$

Thus for such a point (t, u), the product expression of $H(t, u)$ satisfies

$$H(t, u) \geq 4\sin^2(\epsilon_3/2)\sin(\epsilon_4/2) = 4(\delta_3/2)^2(\delta_4/2).$$

Hence

$$\begin{aligned}
\left(\frac{1}{2\pi}\right)^2 \iint_G \sin(u)H(t,u)dtdu &\geq \left(\frac{1}{2\pi}\right)^2 \int_{\pi/4}^{\pi/4+\alpha} \int_{\tau+\alpha}^{\tau+2\alpha} \sin(u)H(t,u)dtdu \\
&\geq \frac{1}{\pi^2} \int_{\pi/4}^{\pi/4+\alpha} \int_{\tau+\alpha}^{\tau+2\alpha} \sin(u)(\delta_3/2)^2(\delta_4/2)dtdu \\
&= \frac{1}{\pi^2} \int_{\pi/4}^{\pi/4+\alpha} \sin(u)(\delta_3/2)^2(\delta_4/2)\alpha du \\
&\geq \frac{1}{\pi^2}(\delta_3/2)^2(\delta_4/2)\alpha\frac{\sqrt{2}}{2}\alpha = \frac{\sqrt{2}\delta_3^2\delta_4\alpha^2}{16\pi^2}
\end{aligned}$$

Let $\delta' = \frac{\sqrt{2}\delta_3^2\delta_4\alpha^2}{16\pi^2}$.

Next we estimate the integration on the set B. Since the area $area(B)$ of B is equal to $4\pi\rho - 2\rho^2$,

$$\begin{aligned}
\left|\left(\frac{1}{2\pi}\right)^2 \iint_B \sin(u)H(t,u)dtdu\right| &\leq \left(\frac{1}{2\pi}\right)^2 max(H(t,u))area(B) \\
&= \left(\frac{1}{2\pi}\right)^2 4(4\pi\rho - 2\rho^2).
\end{aligned}$$

Let ρ be small enough such that $(\frac{1}{2\pi})^2 4(4\pi\rho - 2\rho^2) = \delta'/2$. Then

$$\begin{aligned}
& |c_1|^2 - |c_{-1}|^2 \\
= & \left(\frac{1}{2\pi}\right)^2 \int_0^{\pi} \int_0^{2\pi} \sin(u) H(t,u) dt du \\
= & \left(\frac{1}{2\pi}\right)^2 \iint_G \sin(u) H(t,u) dt du + \left(\frac{1}{2\pi}\right)^2 \iint_B \sin(u) H(t,u) dt du \\
\geq & \ \delta' - \left| \left(\frac{1}{2\pi}\right)^2 \iint_B \sin(u) H(t,u) dt du \right| \\
\geq & \ \delta' - \delta'/2 = \delta'/2.
\end{aligned}$$

By setting $\delta_2 = \delta'/2$, we complete the proof. □

To have a sense of the magnitude of ρ, we work out an upper bound for ρ from the proof of the previous lemma. From Lemma 11 and Corollary 4, $\delta_3 = \frac{e^{-M}}{2/\sqrt{3}+e^{-M}}$ and $\delta_4 = \frac{e^{-5M}}{64(2/\sqrt{3}+e^{-M})^3}$. Since $\alpha < \pi/16$, by Lemma 13,

$$\delta' = \frac{\sqrt{2}\delta_3^2 \delta_4 \alpha^2}{16\pi^2} < \frac{\sqrt{2}e^{-7M}}{64(2/\sqrt{3}+e^{-M})^5 16^3}.$$

Then $\rho = \pi\left(1 - \sqrt{1-\delta'/4}\right) \approx \frac{\pi\delta'}{8}$.

Now we prove Theorem 3.

Proof. Let $||f|_I||_{cr} < M$ and let ρ be a positive constant obtained in the previous lemma. Let $D = \{z \in \mathbb{D} : \eta_z(I) > 1 - \rho/2\pi\}$. Let $z \in D$ and $w = \Phi(f)(z)$. Let $\tilde{f} = g_w \circ f \circ g_z^{-1}$, then $\Phi(\tilde{f})(0) = 0$, $\tilde{f}$ is quasisymmetric on arc $\tilde{I} = g_z(I)$ with $||f(z)|_{\tilde{I}}||_{cr} < M$ and the length of $\tilde{I}^c$ is less than

$\rho < 1/16 < (3\pi/4 - 2\tau)/4$. Therefore, $\tilde{f}$ and $\tilde{I}$ satisfy conditions of Lemma 12 and Lemma 13.

The Jacobian of $w = \Phi(\tilde{f})(z)$ at 0 is

$$\left|\frac{\partial w}{\partial z}(0)\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}(0)\right|^2 = \frac{|c_1|^2 - |c_{-1}|^2}{|d_1|^2 - |d_{-1}|^2}.$$

Since $0 < |d_1|^2 - |d_{-1}|^2 \leq |d_1|^2 = 1$ and $|c_1|^2 - |c_{-1}|^2 \geq \delta_2 > 0$ by Lemma 13,

$$\left|\frac{\partial w}{\partial z}(0)\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}(0)\right|^2 \geq \delta_2 > 0.$$

Thus, by conformal naturality, $\Phi(f)$ is a local homeomorphism in D. Then,

$$\left|\frac{\frac{\partial w}{\partial \bar{z}}(0)}{\frac{\partial w}{\partial z}(0)}\right|^2 \leq 1 - \frac{\delta_2}{|\frac{\partial w}{\partial z}(0)|^2}.$$

By the definition of $\frac{\partial w}{\partial z}(0)$, it is easy to see that the numerator has its absolute value less than or equal to 2. Then by Lemma 12, we obtain

$$\left|\frac{\partial w}{\partial z}(0)\right|^2 \leq \frac{2}{|d_1|^2 - |d_{-1}|^2} \leq \frac{2}{\delta_1}.$$

Therefore,

$$\left|\frac{\frac{\partial w}{\partial \bar{z}}(0)}{\frac{\partial w}{\partial z}(0)}\right|^2 \leq 1 - \frac{\delta_1 \cdot \delta_2}{2}.$$

Then,

$$K(\Phi)(0) \leq \frac{1+k}{1-k},$$

where $k = \sqrt{1 - \frac{\delta_1 \cdot \delta_2}{2}}$.

Since $1+k \leq 2$ and $\alpha \geq \frac{\pi}{48}$, by the expressions of δ_1 and δ_2 in Lemma 12 and Lemma 13 we obtain

$$K(\Phi(\tilde{f})(0)) \leq \frac{(1+k)^2}{1-k^2} \leq \frac{8}{\delta_1\delta_2} = \frac{16(8)(8)\pi^4}{\sqrt{2}\alpha^4\delta_3^2\delta_4^3} \leq \frac{\sqrt{2}(8^3)(48^4)}{\delta_3^2\delta_4^3},$$

where $\delta_3 = \frac{e^{-M}}{2/\sqrt{3}+e^{-M}}$, $\delta_4 = \frac{e^{-5M}}{64(2/\sqrt{3}+e^{-M})^3}$ and $M = ||\tilde{f}|_{\tilde{I}}||_{cr} = ||f|_I||_{cr}$.

Thus

$$\ln K(\Phi(\tilde{f})(0)) \leq 17M + C$$

for a positive constant C. Therefore

$$\ln K(\Phi(f)(z)) = \ln K(\Phi(\tilde{f})(0)) \leq 17M + C$$

for any $z \in D$.

$\square$

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