Normal Families and Monodromies of Holomorphic Motions

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2012

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

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We explore some generalizations of results in holomorphic motions that result from Earle's infinite-dimensional generalization of Montel's Theorem. We then investigate topological obstructions to extending holomorphic motions. We finish with some miscellaneous facts.

Acknowledgements

I am grateful to my advisors, Yunping Jiang and Sudeb Mitra, for their suggestions and advice. I would like to thank my other teachers in the Math program, such as Ara Basmajian, Jozef Dodziuk, Frederick Gardiner, Zheng Huang, Linda Keen, Christina Sormani, Nils Tongring and Zhe Wang. I also thank my fellow students, Tao Chen, Dustin Mulcahey, Samir Shah and Lou Thrall, who were always there to listen to me. Robert Landsman, the Math Department secretary, was invaluable during the years I was there. Special thanks to Professor Hiroshige Shiga of the Tokyo Institute of Technology, Japan.

Most of all I would like to thank my parents, Richard and Sheila, without whose financial and emotional support I could never have completed this program.

Contents

Chapter 1

Introduction and Prerequisites

1.1 Quasiconformal Mappings

Throughout this dissertation, $\mathbb{C}$ will denote the complex plane, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ will denote the Riemann sphere, and $\Delta := \{z : |z| < 1\}$ will denote the open unit disc.

$$cr(a,b,c,d) := \frac{a-c}{a-d}\frac{b-d}{b-c}$$

will denote the **cross-ratio**, and if viewed as a function of a alone (with b, c, d all fixed) it is the unique Möbius map such that $b \mapsto 1, c \mapsto 0, d \mapsto \infty$.

We begin with a review of quasiconformal mappings. Proofs of all results given, and much more, can be found in [1], [15], [19] and [20].

Definition 1. *Let D be a region in $\mathbb{C}$, that is, a connected open set in the complex plane. A function $w = f(z) : D \to \mathbb{C}$ is called **quasiconformal**, or a **quasiconformal mapping**, if the following hold:*

1. *f is a sense-preserving homeomorphism onto its image*

2. *The complex distributional derivatives of f,*

$$f_z = \frac{1}{2}(f_x - if_y) \qquad \text{and} \qquad f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$$

are locally square integrable functions on D.

3. *There is a $0 \leq k < 1$ such that $|f_{\overline{z}}| \leq k|f_z|$ almost everywhere on D.*

If f is a quasiconformal mapping, we denote by μ_f its **Beltrami coefficient**

$$\mu_f(z) := \frac{f_{\overline{z}}(z)}{f_z(z)}.$$

It is a property of quasiconformal maps (see any of the above references) that f_z vanishes on a set of areal measure 0, so μ_f is a well-defined function, which is furthermore measurable. If $||\mu_f||_\infty \leq k < 1$, set $K := \frac{1+k}{1-k}$ and then we call f **K-quasiconformal**. If $K_1 < K_2$, and f is K_1-quasiconformal, then f is K_2-quasiconformal. Clearly given a quasiconformal map there is a minimal $K \geq 1$ such that f is K-quasiconformal, we will denote this minimal number by K_f. Also, since the analytic properties of the definition must only hold almost everywhere we may speak of quasiconformal mappings between open subsets of $\widehat{\mathbb{C}}$, rather than merely of $\mathbb{C}$, in the obvious manner.

Quasiconformal mappings have many interesting properties, some of which are given below.

Proposition 1.1.1 (Properties of Quasiconformal Maps).

1. *If f is K-quasiconformal, f^{-1} is also K-quasiconformal.*
2. *If f_i is K_i-quasiconformal for $i = 1, 2$, then $f_2 \circ f_1$ is K_1K_2-quasiconformal*
3. *A map f is quasiconformal with $\mu_f \equiv 0 \Leftrightarrow f$ is conformal, i.e., injective and holomorphic.*

4. *If f and g are quasiconformal mappings on D such that $\mu_f = \mu_g$, there is a conformal mapping $h : f(D) \to g(D)$ such that $g = h \circ f$.*

5. *For any given $M \geq 1$, a family of quasiconformal maps f on D with $K_f \leq M$, and which misses at least two points in the plane, is compact (see [1]).*

Thus a quasiconformal map f defines a measurable function μ_f with $||\mu_f||_\infty < 1$, and this function (the complex dilatation) describes the quasiconformal map f up to precomposition by a conformal map. It is logical to try and reverse this question – given such a function, is there a quasiconformal map having it as its complex dilatation? The answer is, surprisingly, yes. This result is referred to as the **Measurable Riemann Mapping Theorem**.

Let S be any measurable subset of $\widehat{\mathbb{C}}$, and let μ be a complex-valued measurable function on S. Set $L^\infty(S) := \{\mu : ||\mu||_\infty < \infty\}$, and $M(S) := \{\mu \in L^\infty(S) : ||\mu||_\infty < 1\}$. If S has measure zero, both these spaces will be trivial.

Theorem 1.1.1 (Measurable Riemann Mapping Theorem). *Let $\mu \in M(\mathbb{C})$ be given, and consider the* ***Beltrami equation***

$$f_{\bar{z}} = \mu f_z$$

Then there exists a unique quasiconformal mapping $w^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that w^μ has μ as its Beltrami coefficient and w^μ fixes 0,1 and ∞.

In addition we have

Theorem 1.1.2 (Ahlfors-Bers Theorem). *Let all notation be as in Theorem 1.1.1. Then for every $z \in \mathbb{C}$ the evaluation map $\mu \mapsto w^\mu(z)$ defines a holomorphic map.*

For proofs, see [1], [2], [15] or [19].

A quasiconformal self-homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes $0, 1$ and ∞ is called **normalized**. The difficult parts of this result are the existence and the holomorphic dependence, for uniqueness follows easily from the observation that if f and g have the same complex dilatation on $\widehat{\mathbb{C}}$, there is as noted above a conformal $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $h \circ f = g$. Since the only conformal maps on the entire Riemann sphere are the Möbius maps, and since the only Möbius map that fixes $0, 1$ and ∞ (which h must do if f and g are normalized) is the identity, uniqueness follows.

Although the argument was carried out for coefficients in $M(\mathbb{C})$, if instead the domain of definition D of μ is any other region the result goes through by taking μ to be identically 0 in D^c.

1.2 Teichmüller Spaces of Plane Regions

Recall that if X and Y are normed vector spaces, then a map $f : X \to Y$ is called a **split surjection** if it is continuous, linear, surjective and has a continuous linear right inverse $g : Y \to X$. If instead X and Y are complex Banach manifolds, then a function $f : X \to Y$ is called a **holomorphic split submersion** if f is holomorphic. This is equivalent to saying f has local holomorphic right inverses, also called **local holomorphic sections**. See [19] for more details.

Let D be any region in the complex plane whose complement $\mathbb{C} \setminus D$ contains at least two points. By definition, two quasiconformal mappings f and g on D belong to the same **Teichmüller class** if there is a conformal map h from $f(D)$ onto $g(D)$ such that the selfmapping $g^{-1} \circ h \circ f : D \to D$ is isotopic to the identity modulo ∂D. This means $g^{-1} \circ h \circ f$ extends to a homeomorphism on $\overline{D}$ which is isotopic to $Id(\overline{D})$ by an isotopy which fixes the boundary pointwise. This is an equivalence relation on the set of quasiconformal mappings on D, and the set of equivalence classes is called the **Teichmüller space** of D, denoted $Teich(D)$. Much more on these concepts can be found in [15], [19], [20], [30] and [18].

The **standard projection** $\Phi : M(D) \to Teich(D)$ maps $\mu \in M(D)$ to the Teichmüller class of any quasiconformal map whose domain is D and whose Beltrami coefficient is μ (this makes sense by our comments in the previous section). The **basepoints** of $M(D)$ and $Teich(D)$ are defined to be 0 and $\Phi(0)$, respectively. $Teich(D)$ has a complex Banach manifold structure such that Φ is a holomorphic split submersion. It emerges from the definition this property determines the structure uniquely. See [19], [20] or [30].

Another construction associated with the Teichmüller projection is the **Teichmüller distance**. We begin by defining the **Teichmüller distance** on $Teich(D)$ by

$$d(p, q) = \frac{1}{2} \inf\{\log(K_f \circ K_{g^{-1}}) : \Phi(\mu_f) = p, \Phi(\nu_g) = q\}$$

This is indeed a distance, and its induced topology is the same as that of the canonical complex structure on $Teich(D)$. Moreover d is equal to the Kobayashi pseudodistance on $Teich(D)$. These were shown for finite-dimensional Teichmüller spaces in [31], and for infinite-dimensional ones in [16]. See also [11].

1.3 Teichmüller Spaces of Closed Subsets of the Riemann Sphere

We now address the Teichmüller space of a closed set in the Riemann sphere. Throughout this section E will denote a closed subset of $\widehat{\mathbb{C}}$ containing the points $0, 1$ and ∞. The results in this section are proven in [13] and [24].

Definition 2. *Two normalized quasiconformal self-homeomorphisms f and g of $\widehat{\mathbb{C}}$ are called* ***E-equivalent****, denoted $f \sim_E g$, if f and g are isotopic relative to E.*

That this defines an equivalence relation is self-evident. A necessary (but by no means sufficient) condition for $f \sim_E g$ is $f|E = g|E$. By Theorem 1.1.2, the set of all normalized quasiconformal self-homeomorphisms of $\widehat{\mathbb{C}}$ is identified with $M(\mathbb{C})$, so it makes sense to say that if $\mu, \nu \in M(\mathbb{C})$, then $\mu \sim_E \nu$ when $w^\mu \sim_E w^\nu$. We will denote the quotient map associated with this equivalence relation by $P_E(\mu) := [w^\mu]_E$, where $[w^\mu]_E$ denotes the E-equivalence class of the normalized quasiconformal map w^μ.

Since E is closed, $E^c = \widehat{\mathbb{C}} \setminus E$ consists of countably many connected components, which we number $D_1, D_2, ...$ (it will emerge the specific numbering is irrelevant). We now define $Teich(E^c)$. If the number of components is finite, we define $Teich(E^c)$ to be the Cartesian product $Teich(D_1) \times \ldots \times Teich(D_n)$. If the number of components is infinite, we proceed as follows.

First, let 0_n be the basepoint of $Teich(D_n)$ and let d_n be the Teichmüller distance there. By definition, the **product Teichmüller space** $Teich(E^c)$ is the set of all sequences $t = (t_n)_{n \geq 1}$ such that

$$\sup\{d_n(0_n, t_n)\} < \infty$$

The basepoint of $Teich(E^c)$ is the sequence 0 whose every term is 0_n. Similarly,

$$L^\infty(E^c) := \{\mu = \{\mu_n \in M(D_n)\}_{n\geq 1} : ||\mu||_\infty = \sup ||\mu_n||_\infty < \infty\}.$$

Put another way, $L^\infty(E^c)$ is the complex Banach space of all sequences $\{\mu_n\}_{n\geq 1}$, with $\mu_n \in M(D_n)$ for all n, such that the essential suprema of the μ_n have a uniform upper bound. $M(E^c) := \{\mu \in L^\infty(E^c) : ||\mu||_\infty < 1\}$. Observe that if $\mu \in M(E^c)$, then $\mu_n \in M(D_n)$ for all $n \geq 1$, but not conversely.

For each $n \geq 1$, let Φ_n be the standard projection $M(D_n) \to Teich(D_n)$ described in Sec. 1.2. For $\mu \in M(E^c)$ set $\Phi(\mu)$ to be the sequence $\{\Phi_n(\mu_n)\}_{n\geq 1}$. It is plain $\Phi(\mu) \in Teich(E^c)$, and the mapping Φ is surjective. We call Φ the **standard projection** from $M(E^c)$ to $Teich(E^c)$. Analogously to the planar domain case, there is a complex structure on $Teich(E^c)$ such that Φ is a holomorphic split submersion, see [13] and [24].

Now, let $M(E)$ be the open unit ball in $L^\infty(E)$, then $Teich(E^c) \times M(E)$ is a complex Banach manifold. If $m(E) = 0$, then $M(E)$ is the trivial Banach space and $Teich(E^c)\times M(E)$ is isomorphic to $Teich(E^c)$.

Given $\mu \in M(\mathbb{C})$, we define a mapping $\widetilde{P_E} : M(\mathbb{C}) \to Teich(E^c) \times M(E) : \mu \mapsto (\Phi(\mu|E^c), \mu|E)$. One can show, see [13], that $\widetilde{P_E}$ is a holomorphic split submersion.

Consider now the mapping $\Theta : T(E) \to Teich(E^c) \times M(E) : [\mu]_E \mapsto (\Phi(\mu|E^c), \mu|E)$.

Theorem 1.3.1 (Lieb's Isomorphism Theorem)**.** *Given $\mu, \nu \in M(\mathbb{C})$, $P_E(\mu) = P_E(\nu) \Leftrightarrow \widetilde{P_E}(\mu) = \widetilde{P_E}(\nu)$*

Since $Teich(E^c) \times M(E)$ has a complex Banach manifold structure which makes $\mu \mapsto (\Phi(\mu|E^c), \mu|E)$ a holomorphic split submersion, by pulling this structure back via Θ we give

$T(E)$ a complex Banach manifold structure such that P_E is a holomorphic split submersion. That is, if (U, f) is a holomorphic chart on $T(E)$ if and only if $(\Theta^{-1}(U), \Theta^{-1} \circ f)$ is a complex chart on $Teich(E^c) \times M(E)$.

When $E = \widehat{\mathbb{C}}$, $T(\widehat{\mathbb{C}})$ is canonically identified with $M(\mathbb{C})$ because $E^c = \emptyset$. On the other hand, whenever E is finite, E^c is a planar region so that $Teich(E^c)$ and $T(E)$ are canonically biholomorphic. If E has order n, $Teich(E^c)$ is usually denoted $T(0, n)$ (the "0" refers to the genus of the Riemann sphere, which is 0).

If $E \subset \widehat{E}$, with $\widehat{E}$ also closed, then $\sim_{\widehat{E}}$ is a stronger equivalence relation than $\sim_E$. Thus we can define a forgetful map $p_{\widehat{E},E} : T(\widehat{E}) \to T(E)$ such that $p_{\widehat{E},E} \circ P_{\widehat{E}} = P_E$. To be more explicit, since $\sim_{\widehat{E}}$ is a stronger equivalence relation each $\sim_{\widehat{E}}$-class is contained in a $\sim_E$-class, and $p_{\widehat{E},E}$ takes each $\sim_{\widehat{E}}$-class to the unique $\sim_E$-class which contains it. It follows from the definitions $p_{\widehat{E},E}$ is a basepoint-preserving holomorphic split submersion.

1.4 The Basics of Motion Theory

Definition 3. *Let E be any set in the Riemann sphere. If V is a connected Hausdorff space with basepoint t_0, a function $\varphi : V \times E \to \widehat{\mathbb{C}}$ is called a* ***motion*** *if the following properties hold.*

1. $\varphi(t_0, z) = z$ *for all* $z \in E$,
2. $z \mapsto \varphi(t, z)$ *defines an injection* $E \to \widehat{\mathbb{C}}$ *for all* $t \in V$.

V is called the **parameter space** of φ. φ_t denotes the function $E \to \widehat{\mathbb{C}} : z \mapsto \varphi(t, z)$ (φ_t is always injective), and φ^z denotes the function $V \to \widehat{\mathbb{C}} : t \mapsto \varphi(t, z)$. We may say "$\varphi$ is a motion of E" in cases where V is understood.

If V and W be connected Hausdorff spaces with basepoints t_0 and u_0 respectively, let $f : W \to V$ be any function taking u_0 to t_0 and let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be any motion. Then the **pullback of φ by f**, denoted $f^*\varphi$, is the motion $f^*\varphi : W \times E \to \widehat{\mathbb{C}}$ given by $(u, z) \mapsto \varphi(f(u), z)$. This is clearly a well-defined motion.

If $E \subset E'$ are subsets of $\widehat{\mathbb{C}}$, and $\varphi : V \times E \to \widehat{\mathbb{C}}$ is a motion, another motion $\varphi' : V \times E' \to \widehat{\mathbb{C}}$ is called an **extension** of φ if $\varphi'(t, z) = \varphi(t, z)$ whenever $z \in E$.

We have now defined the most general concept of motion, but in order to make use of it additional regularity conditions must be imposed.

Definition 4. *Let V be a connected complex Banach manifold with basepoint t_0, let E be a subset of $\widehat{\mathbb{C}}$ containing at least 3 points. A **holomorphic motion** is a motion of $V \times E$ which is holomorphic in the first variable. Explicitly, for each $z \in E$ the function $\varphi^z := \varphi(\cdot, z) : V \to \widehat{\mathbb{C}}$ is holomorphic.*

Several examples show the definition is not vacuous.

1. Let V be any complex Banach manifold with a basepoint, E any subset of $\widehat{\mathbb{C}}$, and define $\varphi : V \times E \to \widehat{\mathbb{C}} : (t, z) \mapsto z$. This is called the **trivial** holomorphic motion. For some parameter spaces, such as the complex plane or a compact Riemann surface, this is the only holomorphic motion there is (in a sense to be made clear shortly).

2. The next example is more interesting. Take $V := \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n : i \neq j \Rightarrow z_i \neq z_j\}$. Let $E = \{1, 2, \dots, n\}$, and for any $Z = (z_1, \dots, z_n) \in V$ define $\varphi(Z, j) := z_j$. It is clear this is a holomorphic motion with basepoint $(1, 2, \dots, n)$.

3. Finally we consider the case $V = M(\mathbb{C})$ with basepoint the origin, and $E = \widehat{\mathbb{C}}$. If $\mu \in V$,

let w^μ be the normalized solution to the Beltrami equation with coefficient μ. Then $\Psi_{\widehat{\mathbb{C}}}(t,z) := w^\mu(z)$ is a holomorphic motion. That $\Psi_{\widehat{\mathbb{C}}}(0,z) = z$ follows from the fact $w^0 = Id$, injectivity in the second coordinate comes from the fact quasiconformal maps are by definition injective, and holomorphicity in the first coordinate is the regularity statement in Theorem 1.1.2.

For convenience we assume our holomorphic motions are **normalized**, meaning E is assumed to contain $\{0,1,\infty\}$ and $\varphi(t,\zeta) = \zeta$ for all $t \in V$ and all $\zeta \in \{0,1,\infty\}$. This condition is not as strict as it may appear, as the following argument shows.

Now let E be an arbitrary subset of $\widehat{\mathbb{C}}$ containing at least 3 points, call them z_1, z_2, z_3. Define $\gamma(z) := cr(z, z_1, z_2, z_3)$ for notational convenience, then replacing φ with

$$\varphi' : V \times \gamma(E) \to \widehat{\mathbb{C}} : (t,z) \mapsto cr(\varphi(t,\gamma^{-1}(z)), \varphi(t,z_1), \varphi(t,z_2), \varphi(t,z_3))$$

defines another holomorphic motion which *is* normalized.

We now show that if φ is any normalized holomorphic motion $\varphi : \mathbb{C} \times E \to \widehat{\mathbb{C}}$, φ is the trivial motion. Given $z \in E$, if $z = 0, 1, \infty$ then $\varphi(t,z) = z$ for all t by normalization. If $z \neq 0, 1, \infty$, then φ^z defines an entire function whose range omits two points, specifically 0 and 1, so it is constant by Picard's Little Theorem. Hence for any $t \in \mathbb{C}$, $\varphi(t,z) = \varphi(t_0,z) = z$, as desired.

If $\varphi : R \times E \to \widehat{\mathbb{C}}$ is a normalized holomorphic motion with R a compact Riemann surface, then if $z \in E, z \neq 0, 1, \infty$, then φ^z is a holomorphic map of a compact Riemann surface into the thrice-punctured sphere, and so must be constant. So for any $t \in \mathbb{C}$, $\varphi(t,z) = \varphi(t_0,z) = z$, meaning the motion is trivial.

A vital question in the theory of holomorphic motions, is when the function φ can be extended from $V \times E$ to a function $\widetilde{\varphi}$ on a larger set $V \times \widetilde{E}$, with $E \subsetneq \widetilde{E} \subset \widehat{\mathbb{C}}$ and $\widetilde{\varphi}$ also a holomorphic motion. In particular, we are interested in the case when $\widetilde{E}$ can be taken to be all of $\widehat{\mathbb{C}}$. The study of conditions under which φ can be so extended, and the properties such an extension maintains from the original motion, is a major theme in this dissertation.

Theorem 1.4.1 (Infinite-Dimensional Montel's Theorem)**.** *Let V be any connected complex manifold, in particular we allow for V to be infinite-dimensional. Let $\mathcal{F}$ be any collection of holomorphic maps $f : V \to \widehat{\mathbb{C}}$ such that the range of any such f never contains $0, 1$ or ∞. Then $\mathcal{F}$ is a normal family, meaning that if (f_α) is any net in $\mathcal{F}$ there is a subnet (f_β) which converges in the compact-open topology.*

A proof can be found in [10].

A fundamental result on holomorphic motions is the λ-Lemma, so named because in [25], where the concept of holomorphic motion first appeared using Δ as the parameter space, the variable λ was used for the time parameter. The following proof is slightly different, and takes advantage of Theorem 1.4.1. It is modeled on the one found in [3].

Proposition 1.4.1 (λ-Lemma)**.** *Let V be any connected complex Banach manifold with basepoint t_0, let $E \subset \widehat{\mathbb{C}}$, and let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. Let ρ_V denote the Kobayashi pseudometric on V. Then:*

1. *$\varphi(t, z)$ is jointly continuous, with respect to the ρ_V topology on V and the subspace topology on E.*
2. *$\varphi(t, z)$ extends to a holomorphic motion of $\overline{E}$ over V.*

3. $\varphi_t : E \to \widehat{\mathbb{C}}$ *is the restriction of a quasiconformal mapping of the Riemann sphere.*

Proof. We assume without loss of generality E contains the points $0, 1$ and ∞ and the holomorphic motion φ is normalized. Let ρ, without subscript, be the Poincaré distance on the Riemann sphere punctured at $0, 1$ and ∞. Note that if $z, w \in \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ are a bounded hyperbolic distance apart, and $|z| \to 0$, then $|w| \to 0$. Define $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\eta(M, \epsilon) := \sup\{|w| : \rho(z, w) \leq M, |z| \leq \epsilon\}$. Evidently this function is continuous, increasing and unbounded in ϵ for each fixed M, and moreover $\eta(M, \epsilon) \to 0$ as $\epsilon \to 0$ and $|w| \leq \eta(M, |z|)$ whenever $\rho(z, w) \leq M$.

For any four distinct points $a, b, c, d \in E$ define:

$$g(t) := cr(\varphi_t(a), \varphi_t(b), \varphi_t(c), \varphi_t(d)).$$

the cross-ration of the points $\varphi_t(a), \varphi_t(b), \varphi_t(c)$ and $\varphi_t(d)$. Since φ is injective in the second coordinate, this gives a mapping $g : V \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Since φ is holomorphic in the first coordinate, g is holomorphic and thus $\rho(g(t), g(u)) \leq \rho_V(t, u)$ for all $t, u \in V$. Since $g(t_0)$ is equal to $cr(a, b, c, d)$, we have:

$$|cr(\varphi_t(a), \varphi_t(b), \varphi_t(c), \varphi_t(d)| \leq \eta(\rho_V(t, t_0), |cr(a, b, c, d)|) \tag{1.1}$$

Keep b and d fixed, and let $a \to c$. Then $cr(a, b, c, d) \to 0$, whence it follows $\varphi_t(a) \to \varphi_t(c)$ uniformly with modulus of continuity dependent only on $\rho_V(t, t_0)$. Since φ is continuous in the first coordinate this gives the first statement, that of joint continuity.

For the second statement, using the above arguments we note that for any fixed t φ_t is uniformly continuous on E. Hence it can be extended to a continuous function φ_t on $\overline{E}$ (since this extension is unique, no ambiguity occurs when we reuse notation). For any fixed $z \in \overline{E}$, $z \neq 0, 1, \infty$, let $z_n \to z$, where (z_n) is a sequence in E. Since $\varphi^{z_n}(t)$ is holomorphic for each z_n, and $z_n \neq 0, 1, \infty$ for large enough n, (φ^{z_n}) is a normal family by Theorem 1.4.1. It follows there is a convergent subsequence $(\varphi^{z_{n_i}}(t))$, the limit will be $\varphi^z(t)$ and φ^z is holomorphic since the set of holomorphic maps is closed in the compact-open topology. (See Chapter 17 in [6]). Clearly $\varphi_{t_0}(z) = z$ for all $z \in \overline{E}$. For injectivity, since for any $z \neq w \in \overline{E}$, the cross-ratio $cr(0, \varphi(t, z), \varphi(t, w), \infty)$ is bounded we have $\varphi(t, z) \neq \varphi(t, w)$.

For the final statement, consider any point $z \in \overline{E}$, and any other two points $w_1, w_2 \in \overline{E}$ such that $cr(z, w_1, w_2, \infty) = 1$. Then $cr(\varphi_t(z), \varphi_t(w_1), \varphi_t(w_2), \infty) \leq \rho_V(t_0, t)$, this implies φ_t is the restriction of a quasiconformal mapping of all of $\widehat{\mathbb{C}}$.

$\square$

Remark: The ρ_V topology on V is coarser than the original topology on V, so each holomorphic motion is continuous on V in the usual sense as well. To see this, let τ_1 be the original topology, and let τ_2 be the Kobayashi topology. It suffices to show that if $C \subset V$ is τ_2 closed, it is also τ_1 closed. Since both topologies are first-countable, closure can be detected by convergent sequences. With this in mind, let $t_n \to t$ in the τ_1 topology, with each t_n in C. Let X be the model Banach space, let B be the unit ball in X, and let (U, ψ) be a holomorphic chart with $\psi(U) = B$ and $\psi(t) = 0$. Then $\psi(t_n) \to 0$, for if ρ_B is the Kobayashi metric on B then $\rho_B(\psi(t_n), 0) \to 0$ since $\rho_B(x, 0) = \rho_\Delta(||x||, 0)$ ((see the next paragraph). Now, $\psi^{-1} : B \to V$ is a holomorphic map and so $\rho_V(t_n, t) \leq \rho_B(\psi(t_n), 0)$, since the second sequence converges to 0 the first must do so as well. Hence $t \in C$ by the fact C is τ_2 closed, hence C is τ_1 closed as well.

To see $\rho_B(x,0) = \rho_\Delta(||x||,0)$, (assuming $x \neq 0$, otherwise there is nothing to prove) consider first the complex-linear function $h : \Delta \to B : \zeta \mapsto \frac{\zeta x}{||x||}$. Then $h(0) = 0$ and $h(||x||) = x$, demonstrating $\rho_B(x,0) \leq \rho_\Delta(||x||,0)$. On the other hand, given $x \in B$ by a standard application of the Hahn-Banach Theorem there is an A in the dual space of X such that $A(x) = ||x||$ and $||A|| = 1$. Then A restricts to a holomorphic map $B \to \Delta$ which carries 0 to 0 and x to $||x||$, implying $\rho_B(x,0) \geq \rho_\Delta(||x||,0)$ and proving the claim.

The ρ_V topology may be *strictly* coarser, as demonstrated by the complex plane where it is the trivial topology.

Proposition 1.4.2 (Slodkowski's Theorem)**.** *Let E be any subset of $\widehat{\mathbb{C}}$, and let $\varphi : \Delta \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. Then φ has an extension $\widetilde{\varphi}$ to all of $\Delta \times \widehat{\mathbb{C}}$ which is also a holomorphic motion.*

A proof of Slodkowski's Theorem, as well as much more on the history and applications of holomorphic motions, can be found in [17] and in [19]. Slodkowski's original proof, using very different methods, is in [33].

Two concepts related to holomorphic motions are the following.

Definition 5. *Let V be a connected Hausdorff space with basepoint t_0, let E be a subset of $\widehat{\mathbb{C}}$. A motion $\varphi : V \times E \to \widehat{\mathbb{C}}$ is called a* ***continuous motion*** *if the following properties hold:*

1. *φ is a (jointly) continuous map.*

2. *$\varphi_t : E \to \widehat{\mathbb{C}}$ is a topological embedding for all $t \in V$.*

Once more, all of our continuous motions are taken to be normalized. While the first condition automatically implies φ_t is continuous, we want the much stronger condition φ_t be a homeomorphism onto its image. If E is a *closed* subset of $\widehat{\mathbb{C}}$, in particular if $E = \widehat{\mathbb{C}}$, the second condition is automatically satisfied when the first is (note φ_t is injective by the definition of a motion, then apply some basic point-set topology).

We now need some notation. If $\varphi : V \times E \to \widehat{\mathbb{C}}$ is a motion, and $a, b, c, d \in E$, all distinct, then define $\varphi_t(a, b, c, d) := cr(\varphi_t(a), \varphi_t(b), \varphi_t(c), \varphi_t(d))$. Remember ρ, without subscript, denotes the Poincarè distance on the thrice-punctured sphere. It is equal to the Kobayashi pseudometric $\rho_{\widehat{\mathbb{C}}\setminus\{0,1,\infty\}}$.

Definition 6. *Let V be a connected Hausdorff space with basepoint t_0, and let E be a subset of $\widehat{\mathbb{C}}$. A motion $\varphi : V \times E \to \widehat{\mathbb{C}}$ is called a* ***quasiconformal motion*** *if the following property holds.*

- *Let $\varepsilon > 0$ be given, let $t \in V$ be given. Then there is an open neighborhood $U_t \subset V$ of t such that the following property holds: if $x, y \in U_t$ and $a, b, c, d \in E$ are arbitrary, with a, b, c, d all distinct, then we have $\rho(\varphi_x(a, b, c, d), \varphi_y(a, b, c, d)) < \varepsilon$*

All holomorphic motions are quasiconformal motions under Kobayashi topology, and therefore the original topology, on V. To see this, let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. That the first property of being a quasiconformal motion is satisfied is definitional. Next, recall that if $a, b, c, d \in E$ are fixed then $s \mapsto \varphi_s(a, b, c, d)$ defines a holomorphic map on V, and in particular it is distance-decreasing with respect to the Kobayashi pseudometrics. So if $t \in V$ and $\varepsilon > 0$ are chosen, and $U_t := \{x \in V : \rho_V(x, t) < \varepsilon/2\}$, then it is self-evident $\rho(\varphi_x(a,b,c,d), \varphi_y(a,b,c,d)) \leq \rho_V(x,y) < \varepsilon$ for all $x, y \in U_t$ and all distinct points $a, b, c, d \in E$. Hence every holomorphic motion is a quasiconformal motion.

We have:

Proposition 1.4.3. *Let I denote the unit interval with 0 taken as basepoint, and let E be a closed subset of $\widehat{\mathbb{C}}$ which contains 0,1 and ∞. Let $\varphi : I \times E \to \widehat{\mathbb{C}}$ be a quasiconformal motion. Then φ extends to a quasiconformal motion of $\widehat{\mathbb{C}}$.*

A proof can be found in [34]. We also observe:

- If W and V are complex Banach manifolds, and f is a basepoint-preserving holomorphic map, and φ a holomorphic motion, then $f^*\varphi$ is a holomorphic motion.
- If W and V are connected Hausdorff spaces, and f is a basepoint-preserving continuous map, and φ is a quasiconformal motion, then $f^*\varphi$ is a quasiconformal motion.
- If W and V are connected Hausdorff spaces, and f is a basepoint-preserving continuous map, and φ is a continuous motion, then $f^*\varphi$ is a continuous motion.

All three of these facts are obvious.

We later require the following definition.

Definition 7. *A compact subset $K \subset \Delta$ is called* ***AB-removable*** *if every bounded holomorphic map on $\Delta \setminus K$ can be extended to a holomorphic map on Δ.*

For brevity we use the notation $\Delta_K := \Delta \setminus K$ henceforth. Observe the extended function need not be bounded, only the original function must be. The extension will be unique by the Identity Theorem. Such K's exist – for example, the Riemann Removable Singularities

Theorem shows every singleton set in Δ, and more generally every finite set in Δ, is AB-removable. Less elementary is the fact every compact $K \subset \Delta$ of Lebesgue measure zero is AB-removable, this is shown in [8] in Section 1.5. Note the authors do not use our terminology.

Any such K has empty interior, for if K contains a disc $\Delta(a, r)$, the function $f(z) = 1/(z - a)$ is bounded on Δ_K yet f cannot be extended to holomorphic map on all of Δ, as this function would be f and $f(a) = \infty$. As K is necessarily closed, we can rephrase this by saying K must be nowhere dense. We also note an AB-removable K is necessarily full in Δ, meaning Δ_K is connected. For suppose contrariwise we can write $\Delta_K = U_1 \cup U_2$, where U_1 and U_2 are disjoint, nonempty and open. Define a function f by $f(z) = 1, \forall z \in U_1$ and $f(z) = 0, \forall z \in U_2$. Then f is clearly holomorphic and bounded on Δ_K, but if f could be extended to all of Δ holomorphically we would surely violate the Identity Theorem. It is virtually definitional that every subset of an AB-removable set is AB-removable.

The relevance of this concept to holomorphic motions is the following result, proven in [29].

Proposition 1.4.4. *Let $K \subset \Delta$ be compact and AB-removable, let $\varphi : \Delta_K \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. Then the following are equivalent.*

1. *φ can be extended to a continuous motion $\widetilde{\varphi} : \Delta_K \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

2. *φ can be extended to a holomorphic motion $\widehat{\varphi} : \Delta_K \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

3. *φ can be extended to a holomorphic motion $\varphi_0 : \Delta \times E \to \widehat{\mathbb{C}}$, i.e., $\varphi_0(t, z) = \varphi(t, z)$ if $t \in \Delta_K$.*

1.5 Universal Holomorphic Motions

The most important application of $T(E)$ is the following definition. Other applications of the ideas in this section were given in [21].

Definition 8. *Let E be a closed subset of $\widehat{\mathbb{C}}$, as usual we assume $0, 1, \infty \in E$. Then the* ***universal holomorphic motion*** *is the function $\Psi_E : T(E) \times E \to \widehat{\mathbb{C}}$ given by the formula $([\mu]_E, z) \mapsto w^\mu(z)$.*

This is well-defined, because if $\mu \sim_E \nu$ then $w^\mu|E = w^\nu|E$. We now show it to be a normalized holomorphic motion, justifying the terminology. Since all quasiconformal maps are injective, Ψ_E is clearly injective in the second variable. That Ψ_E is holomorphic in the first variable is a natural consequence of the definition and the standard fact w^μ depends holomorphically on μ, combined with the fact P_E has local holomorphic subsections. Using $0 \in T(E)$ as basepoint, it is clear $\Psi_E(0, z) = z$ for all $z \in E$. So Ψ_E is a holomorphic motion, and that it is normalized emerges directly from the construction.

The term "universal holomorphic motion" originates from the following fact, first proved by Lieb in [24] when the parameter space is Δ. The form given here was proven by Mitra in [27].

Proposition 1.5.1. *Let V be a simply connected complex Banach manifold with basepoint t_0, and let $E \subset \widehat{\mathbb{C}}$ be closed and contain the points $0, 1, \infty$. Then if $\varphi : V \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion, there is a unique basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*\Psi_E = \varphi$.*

Mitra also proved the following, in [27].

Proposition 1.5.2. *Let* V *be a connected complex Banach manifold with basepoint* t_0*, and let* $E \subset \widehat{\mathbb{C}}$ *be closed and contain the points* $0, 1, \infty$*. If* $\varphi : V \times E \to \widehat{\mathbb{C}}$ *be a holomorphic motion, the following are equivalent:*

1. *There exists a continuous motion* $\widetilde{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *extending* φ*.*
2. *There exists a quasiconformal motion* $\widehat{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *extending* φ*.*
3. *There exists a basepoint-preserving holomorphic map* $F : V \to T(E)$ *such that* $F^*(\Psi_E) = \varphi$*.*

Corollary 1.5.1. *Let* V *be a simply connected complex Banach manifold with basepoint* t_0*, let* $E \subset \widehat{\mathbb{C}}$ *be closed and contain the points* $0, 1, \infty$*. Then if* $\varphi : V \times E \to \widehat{\mathbb{C}}$ *is a holomorphic motion, there is a quasiconformal motion* $\widehat{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *extending* φ*.*

Proof. There is a basepoint preserving holomorphic map $f : V \to T(E)$ such that $f^*\Psi_E = \varphi$ by Proposition 1.5.1, and by Proposition 1.5.2 this means the desired quasiconformal motion exists. □

Proposition 1.5.3. *Let* V *be a connected complex Banach manifold with a basepoint, and let* $\varphi : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *be a holomorphic motion. Then there exists a unique basepoint preserving continuous map* $f : V \to M(\mathbb{C})$ *such that* $f^*\Psi_{\widehat{\mathbb{C}}} = \varphi$*.*

The above, and the following corollary, were proven in [28].

Corollary 1.5.2. *Let* V *be a connected Hausdorff space with a basepoint. A motion* $\varphi : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is quasiconformal if and only if it satisfies:*

1. *The map* $\varphi_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is quasiconformal for each* $t \in V$*, and*

2. *the map from* V *to* $M(\mathbb{C})$ *which sends* t *to the Beltrami coefficient of* φ_t *for each* $t \in V$ *is continuous.*

Corollary 1.5.3. *Every quasiconformal motion of* $\widehat{\mathbb{C}}$ *over a connected Hausdorff space* V *is a continuous motion.*

Proof. If φ is such a quasiconformal motion of $\widehat{\mathbb{C}}$, there is a basepoint-preserving continuous map $f : V \to M(\mathbb{C})$ such that $f^*\Psi_{\widehat{\mathbb{C}}} = \varphi$. Since $\Psi_{\widehat{\mathbb{C}}}$ is a holomorphic motion, it is a continuous motion, and the pullback of a continuous motion under a continuous map is also a continuous motion. □

The following lemma is proven in [27], (Section 13).

Lemma 1.5.1. *Let* V *be a connected complex Banach manifold with basepoint and let* $\{0, 1, \infty\} \subset E \subsetneq \widehat{E} \subset \widehat{\mathbb{C}}$, *where both* E *and* $\widehat{E}$ *are closed. Suppose* $f : V \to T(E)$ *and* $\widehat{f} : V \to T(\widehat{E})$ *are basepoint-preserving holomorphic maps, and set* $\varphi := f^*\Psi_E$ *and* $\widehat{\varphi} := \widehat{f}^*\Psi_{\widehat{E}}$. *Then* $\widehat{\varphi}$ *is an extension of* φ *if and only if* $p_{\widehat{E},E} \circ \widehat{f} = f$.

Chapter 2

Extensions of Holomorphic Motions and G-equivariance

2.1 Preliminaries

The results in this section are based on [3].

Definition 9. *Let G be a group of Möbius transformations, let $E \subset \widehat{\mathbb{C}}$ be G-invariant (i.e, $g(E) = E, \forall g \in G$), and let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. We say φ is* ***G-equivariant*** *if for any $t \in V, g \in G$ there is a Möbius transformation, denoted $\theta_t(g)$, such that the following equation holds for all $z \in E$:*

$$\varphi(t, g(z)) = \theta_t(g)(\varphi(t, z)) \tag{2.1}$$

Lemma 2.1.1 (Properties of Equivariant Holomorphic Motions)**.** *Let V be a connected complex Banach manifold with basepoint t_0, E a subset of $\widehat{\mathbb{C}}$ containing at least three points, and let G be a group of Möbius transformations. Suppose $\varphi : V \times E \to \widehat{\mathbb{C}}$ is a G-equivariant holomorphic motion. Then the following hold:*

1. *$\theta_t(g)$ is uniquely determined.*

2. *$g \mapsto \theta_t(g)$ defines a group homomorphism from G into Möb.*

3. This homomorphism is always injective.

4. θ_{t_0} *is the inclusion homomorphism.*

5. For each fixed g*, the map* $t \mapsto \theta_t(g)$ *is a holomorphic mapping* $V \to M\ddot{o}b$*, where* $M\ddot{o}b$ *is identified with* $PSL(2, \mathbb{C})$ *in the usual way.*

Proof. If t and g are given, and $z_1, z_2, z_3 \in E$ are distinct, we have from Definition 9 that $\theta_t(g)(\varphi(t, z_i)) = \varphi(t, g(z_i))$. Thus $\theta_t(g)$ is uniquely determined at three distinct points, and since $\theta_t(g)$ is by definition a Möbius map this suffices to describe it completely.

If $g, h \in G$, then for any $z \in E$:

$$\theta_t(g)(\theta_t(h)(\varphi(t, z))) = \theta_t(g)(\varphi(t, h(z))) = \varphi(t, g(h(z))) = \theta_t(gh)(\varphi(t, z))$$

by the definition of G-equivariance. It then follows immediately θ_t defines a homomorphism on the group G.

To show the homomorphisms are injective, hence isomorphisms, we suppose $\theta_t(g_1) = \theta_t(g_2)$ for some t, g_1 and g_2. We will show $g_1 = g_2$. Given $z \in E$, we have

$$\varphi(t, g_1(z)) = \theta_t(g_1)(\varphi(t, z)) = \theta_t(g_1)(\varphi(t, z)) = \varphi(t, g_1(z)) \Rightarrow g_1(z) = g_2(z) \Rightarrow g_1 = g_2.$$

With the last equality exploiting that E contains at least three points.

Given $z \in E$, $\theta_{t_0}(g)(z) = \theta_{t_0}(g)(\varphi(t_0, z)) = \varphi(t_0, g(z)) = g(z)$, proving θ_{t_0} is indeed the inclusion homomorphism.

We will now show $t \mapsto \theta_t(g)$ is a holomorphic function $V \to PSL(2, \mathbb{C})$ for every $g \in G$. Choose distinct points $z_1, z_2, z_3 \in E$. For each $t \in V$, let h_t be the unique Möbius transformation such that

$$h_t(z_i) = \varphi(t, z_i) \qquad i = 1, 2, 3 \tag{2.2}$$

and

$$\varphi(t, g(z_1)) = \theta_t(g)\varphi(t, z_i) \Rightarrow \theta_t(g)((h_t(z_i)) = \theta_t(g)\varphi(t, z_i) \qquad i = 1, 2, 3 \tag{2.3}$$

The right-hand sides of (2.2) and (2.3) are both holomorphic functions of t for each i. It follows that $t \mapsto h_t$ is holomorphic and $t \to \theta_t(g) \circ h_t$ is holomorphic. Thus $t \mapsto \theta_t(g)$ is holomorphic.

□

Definition 10. *Let G be a subgroup of Möb, and suppose $E \subset \widehat{\mathbb{C}}$ is invariant under G. An injective homomorphism $\eta : G \to$ Möb is said to be* ***induced by an injection*** *$f : E \to \widehat{\mathbb{C}}$ if*

$$f(g(z)) = \eta(g)(f(z)), \forall z \in E, g \in G \tag{2.4}$$

If we can furthermore take f to be the restriction of a quasiconformal self-map of $\widehat{\mathbb{C}}$ then η is said to be a ***quasiconformal deformation*** *of G.*

A quasiconformal map inducing a given quasiconformal deformation is not necessarily uniquely determined. Now, suppose there is a f_t a quasiconformal map inducing θ_t. Since $f_t \circ g \circ f_t^{-1} = \theta_t(g)$, each homomorphism θ_t will be type-preserving.

Theorem 2.1.1. *Let G be a group of Möbius maps and let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion, where as usual V is a connected complex Banach manifold with basepoint t_0 and E is a G-invariant closed subset of the Riemann sphere containing $0, 1$ and ∞. If the motion is G-equivariant, then each θ_t is a quasiconformal deformation of G.*

For a proof, see [22] or [29].

2.2 Statement of Results

Our first two results are technical lemmas necessary for what follows. We will closely follow [3].

Lemma 2.2.1. *Let V be a connected complex Banach manifold with basepoint t_0, let E be a finite subset of $\widehat{\mathbb{C}}$ containing $0, 1$ and ∞. Let $\Phi := \{$all normalized holomorphic motions $\varphi : V \times E \to \widehat{\mathbb{C}}$ with basepoint $t_0\}$. Then Φ is a compact set in the compact-open topology.*

Remark: The requirement E be finite is crucial for this lemma. Since $\mathcal{O}(V)$ (the space of holomorphic maps on V) is not first-countable when the dimension of V is infinite (see Chapter 16 in [6]), we must detect compactness using nets rather than sequences. Since diagonalization arguments are not possible using general nets, we cannot pass from the finite case to the infinite case. However, this additional hypothesis on E will not be of importance.

Lemma 2.2.2. *Let $\{E_n\}$ be an ascending sequence of finite subsets of $\widehat{\mathbb{C}}$ such that $E_1 \supset \{0, 1, \infty\}$, and let $E = \overline{\bigcup_n E_n}$. For each n, let $\varphi_n : V \times E_n \to \widehat{\mathbb{C}}$, be a holomorphic motion, where as usual V is a complex connected Banach manifold with basepoint t_0. Then there is a subsequence φ_{n_j}, and a holomorphic motion $\varphi : V \times E \to \widehat{\mathbb{C}}$, such that φ_{n_j} converges compactly to φ on each $V \times E_n$.*

Next we require the following terminology.

Definition 11. *Let V be a complex manifold with a basepoint t_0. We say that V* ***satisfies the holomorphic axiom of choice*** *if for any finite subset E of $\widehat{\mathbb{C}}$ containing $\{0, 1, \infty\}$,* *any* $y \in \widehat{\mathbb{C}} \setminus E$*, and any holomorphic motion of* E *over* V*,* φ *extends to a holomorphic motion of* $E \cup \{y\}$ *over* V*.*

Theorem 2.2.1. *Let* V *be any connected complex Banach manifold with basepoint* t_0 *which satisfies the holomorphic axiom of choice. Then if* E *is any subset of* $\widehat{\mathbb{C}}$*, and* $\varphi : V \times E \to \widehat{\mathbb{C}}$ *a holomorphic motion, there is a holomorphic motion* $\widehat{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *extending* φ*.*

Proposition 2.2.1. *Let* V *be a connected complex Banach manifold with basepoint* t_0*,* $\{0, 1, \infty\} \subset E \subset \widehat{\mathbb{C}}$*, and let* $\varphi : V \times E \to \widehat{\mathbb{C}}$ *be a holomorphic motion with the following property:*

If E_0 *is a finite subset of* E*, and* $Y \subset \widehat{\mathbb{C}} \setminus E_0$ *is finite, there is a holomorphic motion* $\widetilde{\varphi} : V \times (E_0 \cup Y) \to \widehat{\mathbb{C}}$ *whose restriction to* $V \times E_0$ *agrees with* φ*.*

Then there is a holomorphic motion $\widehat{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *which extends* φ*.*

While Theorem 2.2.1 and Proposition 2.2.1 are similar in statement and (we will see) in proof, they are not equivalent, for the first involves a hypothesis concerning *all* holomorphic motions over the parameter space V, and the second involves a single holomorphic motion over V.

For reasons of brevity, if G is a group of Möbius transformations and $z \in \widehat{\mathbb{C}}$ then the stabilizer subgroup $Stab_G(z)$ shall be denoted G_z for the remainder of the dissertation.

Proposition 2.2.2. *Let* V *be a connected complex Banach manifold with basepoint* t_0*, let* G *be a group of Möbius transformations, let* E *be a* G*-invariant subset of* $\widehat{\mathbb{C}}$ *containing*

$\{0,1,\infty\}$, and let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a G-equivariant holomorphic motion. Let $F = \{z \in \widehat{\mathbb{C}} : G_z \neq 0\}$. Then φ has an extension $\widetilde{\varphi} : V \times (E \cup F) \to \widehat{\mathbb{C}}$ which is also a G-equivariant holomorphic motion.

Theorem 2.2.2. *Let V be a connected complex Banach manifold with basepoint t_0 which satisfies the holomorphic axiom of choice. Let G be a group of Möbius transformations, let $E \subset \widehat{\mathbb{C}}$ be G-invariant, closed, and contain $0, 1$ and ∞. Then if $\varphi : V \times E \to \widehat{\mathbb{C}}$ is a G-equivariant holomorphic motion, φ can be extended to a G-equivariant holomorphic motion of all of $\widehat{\mathbb{C}}$ over V.*

Note we did not specify any extensions are unique. The description "holomorphic axiom of choice" is due to Sullivan and Thurston in [34], where they proved Theorem 2.2.1 (see also [5] , and hypothesized Theorem 2.2.2, for Δ. This was proven in [11].

Note that the papers cited above did not prove the holomorphic axiom of choice holds for the unit disc, this was shown by Slodkowski in [33], and was the crucial step in proving Slodkowski's Theorem.

2.3 Proofs of the Lemmas

Proof of Lemma 2.2.1. By the λ-Lemma, each element of Φ is a jointly continuous function so speaking of Φ as a subset of the compact-open topology makes sense. To show Φ is compact, we must show every net in Φ has a subnet converging to a limit in Φ. Let $z_1, \ldots, z_n$ be the elements of $E \setminus \{0, 1, \infty\}$.

Let (φ_α) be a net in Φ, and consider the net of holomorphic maps $(\varphi^1_\alpha)(t) := \varphi_\alpha(t, z_1)$. This defines a family of holomorphic maps on V which miss $0, 1$ and ∞, so by Theorem 1.4.1

there is a convergent subnet (φ^1_β). Consider next the net (φ_β) in Φ, and the corresponding net (φ^2_β), with like notation. By the same result there is a subnet (φ^2_γ) which converges compactly, (φ^1_γ) converges compactly as well. Repeating this argument we obtain a net (φ_δ) such that each (φ^k_δ) converges compactly. Setting $\varphi(t, z_k) := \lim_\delta \varphi^k_\delta(t)$, and setting $\varphi(t, \zeta) := \zeta$ if $\zeta = 0, 1, \infty$, defines a function $\varphi : V \times E \to \widehat{\mathbb{C}}$.

It is evident from the finiteness of E that $(\varphi_\delta$ converges to φ in the compact-open topology. Once we show $\varphi \in \Phi$, we will be done. That $\varphi(t_0, z) = z$ for all $z \in E$ is obvious. That φ is holomorphic in the first coordinate follows from the fact each φ^k_δ is holomorphic, and the collection of holomorphic maps is closed in the compact-open topology. Also, the limit function φ is evidently normalized. Showing φ is injective in the second coordinate is done as follows.

Fix $t \in V$, since each $\varphi_\delta(t, z) \in \Phi$ there by the λ-Lemma an η, independent of δ, such that:

$$|cr(\varphi_{\delta,t}(z), 1, 0, \varphi_{\delta,t}(z'))| \leq \eta(|cr(z, 1, 0, z')|)$$

with all notations as in the proof of the λ-Lemma, and with z and z' distinct elements of E not equal to 0 or 1. Passing to the limit gives:

$$|cr(\varphi_t(z), 1, 0, \varphi_t(z'))| \leq \eta(|cr(z, 1, 0, z')|)$$

The cross-ratio on the RHS will be $< \infty$, so the cross-ratio on the LHS will be $< \infty$, implying $\varphi_t(z) \neq \varphi_t(z')$, thus proving injectivity in the second coordinate in this case. The possibility z or z' is equal to 0 or 1 is dealt with by permuting the elements in the cross-ratios above. ☐

Proof of Lemma 2.2.2. Denote $\bigcup_n E_n$ by E' for convenience. Since $\varphi_n|(V \times E_1)$ is a collection of holomorphic motions of E_1, and E_1 is finite, by Lemma 2.2.1 there is a subsequence $\varphi_{n(1)}$ which converges compactly on $V \to E_1$. Since $\varphi_{n(1)}|(V \times E_2)$ is a sequence of holomorphic motions of E_2 over V there is by the same lemma a further subsequence $\varphi_{n(2)}$ which converges compactly on $V \times E_2$, and therefore on $V \times E_1$ as well. Continuing in like manner, and then applying a diagonalization argument, shows there is a sequence φ_{n_j} which converges compactly on each $V \to E_n$. This motion extends to one of $\overline{E'}$ over V by the λ-Lemma.

☐

2.4 Proofs of the First Two Results

Proof of Theorem 2.2.1. **Step 1:** Use the λ-Lemma to assume E is closed. Let $\{E_n\}$ be an ascending sequence of finite subsets of E whose union E' dense in E, and let $y \in \widehat{\mathbb{C}} \setminus E$. We claim φ has an extension $\varphi' : V \times (E \cup \{y\}) \to \widehat{\mathbb{C}}$ which is also a holomorphic motion.

Let $\varphi_n : V \times E_n \to \widehat{\mathbb{C}}$ be the holomorphic motion obtained by restricting φ, and let $\varphi'_n : \times(E_n \cup \{y\}) \to \widehat{\mathbb{C}}$ be a holomorphic motion which extends φ_n (such an extension exists by hypothesis). By Lemma 2.2.2, there is a subsequence φ'_{n_j} which converges at each point of $E' \cup \{y\}$ to a holomorphic motion of $E' \cup \{y\})$ over V. This holomorphic motion can, by the λ-Lemma, be extended to a holomorphic motion $\varphi' : E \cup \{y\} \to \widehat{\mathbb{C}}$, and since it agrees with φ on the dense subset $V \times E'$ of $V \times E$, and since both are continuous, φ' is the extension desired.

Step 2: Let $E \subset \widehat{\mathbb{C}}$ be any closed set, and let $Y = \{y_1, y_2, \ldots\}$ be a countable dense subset of $\widehat{\mathbb{C}} \setminus E$. Let $E_0 = E$, let $F_1 = E \cup \{y_1\}$, let $F_2 = E_1 \cup \{y_2\}$, and so on. Let $\varphi_0 = \varphi$. By Step 1 there is an extension φ_1 to $V \times F_1$ of φ_0 which is also a holomorphic motion. By Step 1 again, there is an extension φ_2 to $V \times F_2$ of φ_1 which is also a holomorphic motion. Continuing inductively, we obtain a sequence $\varphi_n : V \times F_n \to \widehat{\mathbb{C}}$ of holomorphic motions, all of which extend φ. Since each holomorphic motion is an extension of the one before, a holomorphic motion $\varphi' : V \times (E \cup Y) \to \widehat{\mathbb{C}}$ clearly exists. Apply the λ-Lemma, and we are done by choice of Y. $\square$

Proof of Proposition 2.2.1. Assume E is closed, without loss of generality, by the λ-Lemma, and let $\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \ldots ...$ be an ascending sequence of finite subsets of E whose union E' is dense in E. Let $Y = \{y_1, y_2, ...\}$ be a countable dense subset of $\widehat{\mathbb{C}}$, and let $F_n := E_n \cup \{y_1, y_2, \ldots, y_n\}$ for every n. By hypothesis there is for each n a holomorphic motion $\varphi_n : V \times F_n \to \widehat{\mathbb{C}}$ whose restriction to $V \times E_n$ coincides with φ. By Lemma 2.2.2, if $F' := \cup F_n$ there is a holomorphic motion $\varphi' : V \times F' \to \widehat{\mathbb{C}}$ such that φ' agrees with φ on $V \times E'$. Let $\widehat{\varphi}$ be the extension of this motion to the closure of F', it will extend φ and since $\overline{F'} = \widehat{\mathbb{C}}$ this is the extension desired. $\square$

2.5 Proofs of the Next Two Theorems

Proof of Proposition 2.2.2. Since E is closed and G-invariant and contains at least three points, it contains all fixed points of parabolic or loxodromic (including hyperbolic) elements of G. This follows from the fact any such fixed point is an attractor of the transformation itself (in the parabolic and attracting loxodromic case) or its inverse (in the case the fixed point is a repeller of a loxodromic element). Thus, if $z \in F \setminus E$, then the stabilizer subgroup

G_z contains only the identity and elliptic transformations. This also holds for all $\theta_t(G_z)$ because each θ_t is type preserving as stated previously.

If $g, h \in M\ddot{o}b$ are nonidentity elements and do not have the same fixed point set, $ghg^{-1}h^{-1}$ is parabolic (see 9G in Chapter 2 of [23]). It follows every element of G_z has the same two fixed points. The same is true for each $\theta_t(G_z)$. Since $\theta_t(g)$ depends holomorphically on t for each $g \in G$, for each $z \in F \setminus E$ there is a unique holomorphic map ψ_z on V such that $\psi_z(t_0) = z$ and $\psi_z(t)$ is fixed by $\theta_t(g)$ for all $g \in G_z$ and all $t \in V$.

We extend φ to $E \cup F$ by setting $\widetilde{\varphi}(t, z) := \psi_z(t)$ if $t \in V$ and $z \in F \setminus E$. We claim this extended map is a G-equivariant holomorphic motion. For any $z \in F \cup E$, $\widetilde{\varphi}(t_0, z) = z$ by construction. That $\widetilde{\varphi}$ is holomorphic in the first coordinate also follows directly from construction.

Showing $\widetilde{\varphi}$ is G-equivariant is only slightly more involved. Note $E \cup F$ is G-invariant; for E is G-invariant by hypothesis, and F is G-invariant by elementary abstract algebra. If $z \in E$, $\theta_t(g)(\widetilde{\varphi}(t, z)) = \widetilde{\varphi}(t, g(z))$ for all $g \in G$ by hypothesis. If $z \in F \setminus E$, then the result follows from the definition of ψ_z and elementary algebraic facts about group actions.

The injectivity property is more delicate. We first prove the preliminary result:

Lemma 2.5.1.

$$\textit{If } \widetilde{\varphi}(s, z) = \widetilde{\varphi}(s, g(z)) \textit{ for some } g \in G, s \in V \textit{ and some } z \in E \cup F, \textit{ then } g \in G_z \qquad (2.5)$$

Proof. Assume there is some combination of g, z and s for which (2.5) is false. If $z \in E$ this cannot happen, so assume $z \in F \setminus E$ henceforth. For simplicity's sake let $w := \widetilde{\varphi}(s, z)$, and by our hypothesis and G-equivariance of $\widetilde{\varphi}$ we have $\theta_s(g)(w) = w$. Choose a quasiconformal homeomorphism f_s of the Riemann sphere inducing θ_s, and observe g fixes the point $z' := f_s^{-1}(w)$ because

$$g(z') = g \circ f_s^{-1}(w) = f_s^{-1} \circ f_s \circ g \circ f_s^{-1}(w) = f_s^{-1} \circ \theta_s(g)(w) = f_s^{-1}(w) = z'$$

That is, $g \in G_{z'}$. If $z = z'$ there is nothing to prove, so we assume this is not the case henceforth. If $h \in G_z$, then by the G-equivariance we have

$$h(z') = h \circ f_s^{-1}(w) = f_s^{-1} \circ f_s \circ h \circ f_s^{-1}(w) = f_s^{-1}\theta_s(h)(w) = f_s^{-1}(w) = z'$$

implying $G_z \subset G_{z'}$. Recall we assumed g was not in G_z, and choose a nontrivial $h \in G_z$. The commutator $h^* = hgh^{-1}g^{-1}$ is parabolic, so it can have only one fixed point, which will of course be z' since both g and h fix it. The transformation $\theta_s(h^*)$ is also parabolic, and it fixes $\widetilde{\varphi}(s, z')$ by the G-equivariance and it fixes w because f_s induces θ_s. Therefore $\widetilde{\varphi}(s, z') = w = \widetilde{\varphi}(s, z)$. Since $h \in G_z$, and $G_z \subset G_{z'}$, G-equivariance implies $\theta_s(h)$ fixes both $\widetilde{\varphi}(s, z')$ and $\widetilde{\varphi}(s, z)$ for every $t \in V$. But $\theta_s(h)$ is always elliptic, and its fixed points are given by two holomorphic maps of s on V with disjoint graphs (as subsets of $V \times \widehat{\mathbb{C}}$).

It then follows from the definition of $\widetilde{\varphi}$ that $\widetilde{\varphi}(t, z)$ and $\widetilde{\varphi}(t, z')$, as functions of t, either agree everywhere or agree nowhere. But we have already seen that when $t = s$, $\widetilde{\varphi}(t, z) = \widetilde{\varphi}(t, z')$. But this contradicts the fact $\widetilde{\varphi}(t_0, z) \neq \widetilde{\varphi}(t_0, z')$, since $z \neq z'$ by assumption. Hence we have a contradiction. Hence (2.5) is indeed true.

□

Now we can prove the injectivity property of $\widetilde{\varphi}$. Suppose $\widetilde{\varphi}(t, z) = \widetilde{\varphi}(t, z')$, where $t \in V$ is fixed. We need to show $z = z'$. If both are in E, this is true by hypothesis. Assume, then,

$z \in F \setminus E$. Then for all $g \in G$ we have

$$\widetilde{\varphi}(t, g(z)) = \theta_t(g)(\widetilde{\varphi}(t, z)) = \theta_t(g)(\widetilde{\varphi}(t, z')) = \widetilde{\varphi}(t, g(z'))$$

So if $g \in G_{z'}$, $\widetilde{\varphi}(t, z) = \widetilde{\varphi}(t, g(z))$. This, by (2.5), implies $g \in G_z$. Thus $G_{z'} \subset G_z$, and $G_z = G_{z'}$ follows because the argument is symmetric. Since $z \in F \setminus E$, G_z is a nontrivial group consisting only of elliptic elements all of which share the same fixed points. If $z \neq z'$, they must be these fixed points. So $\widetilde{\varphi}(s, z')$ and $\widetilde{\varphi}(s, z)$ are the two fixed points of $\theta_s(g)$ for any $s \in V$ and nontrivial $g \in G$ (this follows from the argument about disjoint graphs above), contradicting our assumption $\widetilde{\varphi}(t, z) = \widetilde{\varphi}(t, z')$. So $z = z'$, and the proof is complete. □

Proof of Theorem 2.2.2. We showed in the proof of Theorem 2.2.1 the hypothesis has the implication that if A is *any* subset of $\widehat{\mathbb{C}}$, and $y \in \widehat{\mathbb{C}} \setminus A$, and $\psi : V \times A \to \widehat{\mathbb{C}}$ is a holomorphic motion, then there is an extension of ψ to a holomorphic motion of $(A \cup \{y\})$ over V. With this in mind, let φ and E be as in the hypothesis, and let F be as in Proposition 2.2.2. Then φ has a G-equivariant extension to $V \times (E \cup F)$, denote this extension by φ as well for simplicity. Note the definition of G-equivariance of a motion of a set clearly extends to the closure of that set. If $E \cup F$ is dense in $\widehat{\mathbb{C}}$, we are done, as φ extends to $V \times \widehat{\mathbb{C}}$ by the Lambda-Lemma. Otherwise let A be a G-invariant subset of $\widehat{\mathbb{C}}$ on which there is a G-equivariant holomorphic extension of φ, denoted φ once more for simplicity, and further assume $(E \cup F) \subset A$. Again, if A is dense in $\widehat{\mathbb{C}}$ we are done.

If not, take $y \in \widehat{\mathbb{C}} \setminus A$, and extend φ to $\varphi' : V \times (A \cup \{y\}$, this can be done by the comment above. Now extend φ' to all of $V \times (A \cup G(y))$ by the formula

$$\varphi'(t, g(y)) := \theta_t(g)(\varphi'(t, y))$$

where $g \in G, t \in V$. Here $G(y)$ denotes the G-orbit of y, and this is well-defined because G_y is trivial (after all, y is not in F). We claim this extended φ' is a G-equivariant holomorphic motion. $A \cup G(y)$ is plainly G-invariant. Since $\theta_{t_0}(g) = g$, $\varphi'(t_0, g(y)) := \theta_{t_0}(g)(\varphi'(t_0, y)) = g(y)$, we have property (1) of Definition 3. Since for fixed g $\theta_\cdot(g)$ is holomorphic on V, and $\varphi'(\cdot, y)$ is holomorphic on V by construction, for $g(y) \in G(Y)$ we have $\varphi'(\cdot, g(y))$ is the product of two holomorphic map, and so holomorphic itself. That φ' is G-equivariant is self-evident.

Before verifying injectivity, we make some general comments about fixed points of transformations in $\theta_t(G)$, where $t \in V$ is given. For any subset $D \subset A$ we define

$$\varphi(t, D) := \{\zeta \in \widehat{\mathbb{C}} : \zeta = \varphi(t, z) \text{ for some } z \in D\}$$

For any nontrivial $g \in M\ddot{o}b$, let $Fix(g)$ be the set of fixed points of g. We claim that if $g \in G$, $\varphi(t, Fix(g)) = Fix(\theta_t(g))$. Since θ_t is type-preserving, both $Fix(g)$ and $Fix(\theta_t(g))$ contain the same finite number of points. Now say $a \in Fix(g)$. Then $\varphi(t, a) = \varphi(t, g(a)) = \theta_t(g)(\varphi(t, a))$, implying $\varphi(t, Fix(g)) \subset Fix(\theta_t(g))$, and equality follows.

Finally fix $t \in V$, we need to show $\varphi'(t, z) = \varphi'(t, z') \Rightarrow z = z'$. If both z and z' are in $A \cup \{y\}$, this is true by construction. So assume $z \in A$, and $z' \in G(y)$. There is a $g \in G$ such that $g(y) = z'$, and by G-invariance of A there is a $\zeta \in A$ such that $g(\zeta) = z$. Then we have, by G-equivariance:

$$\theta_t(g)(\varphi'(t, \zeta)) = \varphi'(t, z) = \varphi'(t, z') = \theta_t(g)(\varphi'(t, y)) \Rightarrow \varphi'(t, \zeta) = \varphi'(t, y)$$

Since the last statement is untrue, we have a contradiction. Finally, assume both points are in $G(y)$, then there are distinct $g, h \in G$ such that $g(y) = z, h(y) = z'$, and $g \neq h$. Then $\theta_t(g)(\varphi'(t, y)) = \theta_t(h)(\varphi'(t, y))$. $\theta_t(gh^{-1})$ thus fixes $\varphi'(t, y)$. It follows from the above comments $y \in Fix(gh^{-1})$, implying $y \in F \subset A$, a contradiction.

Step 2: Take Y a countable subset of $\widehat{\mathbb{C}}$ such that

1. Any two distinct elements of Y are in distinct G-orbits.
2. Y does not intersect A
3. $A \cup G(Y)$ is dense in $\widehat{\mathbb{C}}$, where $G(Y)$ is the G-orbit of the entire set Y.

By applying the logic in Step 1 repeatedly, we obtain a G-equivariant extension of φ to all of $V \times (A \cup G(Y))$, and then apply the density argument we could not use before.

□

Chapter 3

Monodromies for Holomorphic Motions

3.1 Preliminaries

The results in this section are based on [4].

Let V be a connected complex Banach manifold with a basepoint t_0, and let E be a closed subset of the Riemann sphere containing $\{0, 1, \infty\}$. We fix a holomorphic motion $\varphi : V \times E \to \widehat{\mathbb{C}}$ until further notice, and describe two algebraic-topological constructions associated with it.

For the first note that if $z \in E \setminus \{0, 1, \infty\}$ then by definition the function $\varphi^z : V \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is holomorphic, and in particular continuous. Therefore it has a pushforward homomorphism $\varphi^z_* : \pi_1(V, t_0) \to \pi_1(\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}, z)$, defined by $[\gamma] \mapsto [\varphi^z \circ \gamma]$ in the usual way . The collection of all such homomorphisms is called the **trace monodromy** of φ, and if all of these are all the trivial homomorphism we say φ has **trivial trace monodromy**. We usually suppress the basepoints when discussing trace monodromy.

The next construction is more involved. Let $p : \widetilde{V} \to V$ be the holomorphic universal cover, choose a point $\widetilde{t_0} \in p^{-1}(t_0)$ as basepoint, and let $\widetilde{\varphi} := p^*\varphi$. This is a holomorphic motion of E over $\widetilde{V}$, and since $\widetilde{V}$ is simply connected Corollary 1.5.1 says there is a quasiconformal motion $\widehat{\varphi} : \widetilde{V} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ extending $\widetilde{\varphi}$, and so there is by Proposition 1.5.3 a unique basepoint-preserving continuous map $f : \widetilde{V} \to M(\mathbb{C})$ such that $f^*\Psi_{\widehat{\mathbb{C}}} = \widehat{\varphi}$.

We set G_V to be the deck group of p and identify it with $\pi_1(V, t_0)$ via the basepoint t_0 in the canonical manner. If c is a loop in V based at t_0, let g_c be the corresponding deck group element.

If $z \in E, g \in G_V$, then we have:

$$w^{f\circ g(\widetilde{t_0})}(z) = \widetilde{\varphi}(g(\widetilde{t_0}), z) = \varphi(p(g(\widetilde{t_0})), z) = \varphi(t_0, z) = z$$

Therefore $w^{f\circ g(\widetilde{t_0})}$ keeps every point of E fixed. We claim the homotopy class for $w^{f\circ g(\widetilde{t_0})}$ is well defined.

Lemma 3.1.1. *The homotopy class of $w^{f\circ g(\widetilde{t_0})}$ relative to E does not depend on the choice of continuous mapping f.*

Proof. Let $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ be two such extensions, $f_1, f_2 : \widetilde{V} \to M(\mathbb{C})$ the corresponding maps, and $\rho_1, \rho_2 : \pi_1(V) \to QC(E)$ the corresponding group maps. Let γ be a path in $\widetilde{V}$ from $\widetilde{t_0}$ to $\widetilde{g(t_0)}$, and define:

$$H(z, s) := [w^{f_1(g(\widetilde{t_0}))} \circ (w^{f_1(\gamma(s))})^{-1} \circ w^{f_2(\gamma(s))}](z)$$

Evidently the function H is jointly continuous on $\widehat{\mathbb{C}} \times [0, 1]$, $H(\cdot, 0) = w^{f_1\circ g(\widetilde{t_0})}(\cdot)$ and $H(\cdot, 1) = w^{f_2\circ g(\widetilde{t_0})}(\cdot)$. So H is a homotopy from $w^{f_1\circ g(\widetilde{t_0})}$ to $w^{f_1\circ g(\widetilde{t_0})}(\cdot)$. Moreover, given $z \in E$ and $0 \le s \le 1$, $w^{f_k(\gamma(s))}(z) = \widehat{\varphi}_k(\gamma(s), z)$ for $k = 1, 2$, but these are equal because both extend $\widetilde{\varphi}$. Hence $H(z,s) = w^{f_1(g(\widetilde{t_0}))} = z$. So the homotopy H is relative to E. Thus $\rho_1(g)$ is homotopic rel E to $\rho_2(g)$ for all $g \in \pi_1(V)$, and the independence from $\widehat{\varphi}$ follows. $\square$

We now assume that E is a finite set and that it contains $n(>3)$ points including $0, 1$ and ∞. Let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. The map $w^{f \circ g(\widetilde{t_0})}$ is a quasiconformal selfmap of the Riemann surface X_E, and therefore represents a mapping class of X_E. So we have, by Lemma 3.1.1, a homomorphism $\rho_\varphi : \pi_1(V, t_0) \to Mod(0,n)$ given by

$$\rho_\varphi(c) = [w^{f \circ g_c(\widetilde{t_0})}]$$

where $Mod(0,n)$ is the mapping class group of the $n+3$-times punctured sphere (see [30], $g_c \in G_V$ is an element corresponding to $c \in \pi_1(V, t_0)$, and $[w]$ denotes the mapping class of X_E for w. We call the function ρ_φ the **monodromy** of the holomorphic motion φ of the finite set E. The monodromy is **trivial** if it maps every element of $pi_1(V, t_0)$ to the identity of $Mod(0,n)$.

Lemma 3.1.2. *Let $\varphi : V \times E \to \widehat{\mathbb{C}}$ which extends to a continuous motion of $\widehat{\mathbb{C}}$ over V. Then ρ_φ is trivial.*

Proof. By Proposition 1.5.2, there is a quasiconformal motion $\varphi' : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ extending φ. By Corollary 1.5.2, each φ'_t is quasiconformal, and if $\mu(t)$ denotes the Beltrami coefficient of φ'_t then the function $h : V \to M(\mathbb{C}) : t \mapsto \mu(t)$ is continuous. Let $\widetilde{h} := p^*(h)$, i.e. let $\widetilde{h}(u)) = h(p(u))$, and let $\widetilde{\varphi}' := p^*\varphi'$. It is self-evident $\widetilde{\varphi}'$ is a quasiconformal motion of $\widetilde{V} \times \widehat{\mathbb{C}}$ that extends $\widetilde{\varphi}$, it is our claim the unique $f : \widetilde{V} \to M(\mathbb{C})$ associated with it is given by $\widetilde{h}$. To see this, observe that $(u,z) \in \widetilde{V} \times \widehat{\mathbb{C}}$, we have

$$\widetilde{\varphi}'(u,z) = \varphi'(p(u),z) = w^{h(p(u))}(z) = w^{\widetilde{h}(u)}(z)$$

as required. Now let γ be a loop in V based at t_0, let g be the element of the deck group corresponding to γ. Unwinding the definitions gives $\rho_\varphi(g) = [w^{\widetilde{h}(g(\widetilde{t_0)})}] = [w^{h(p(g(\widetilde{t_0})))}] = [w^{h(t_0)}] = [Id]$, which implies the desired result. $\square$

3.2 Statement of results

In this section we let V be a connected complex Banach manifold with basepoint t_0, and we let E be a closed subset of $\widehat{\mathbb{C}}$ containing $\{0,1,\infty\}$ as a subset. Let $\varphi : V \times E \to \widehat{\mathbb{C}}$ denote a holomorphic motion. Let E^n be the n-fold Cartesian product of E, and let X be any topological space. For $n = 1$ define $Y_1 := \mathbb{C} \setminus \{0,1\}$, and for $n \geq 2$ define $Y_n := \mathbb{C}^n \setminus \{\text{the diagonal of } \mathbb{C}^n\}$, in other words Y_n is the collection of all n-tuples of elements of $\mathbb{C}$ with no two entries equal. Let $F_n : Y_n \to X$ be a continuous map. Define the function $\varphi_n : V \times (E^n \cap Y_n) \to \widehat{\mathbb{C}}^n$ by the formula

$$\varphi_n(t,(z_1,\ldots,z_n)) = (\varphi(t,z_1),\ldots,\varphi(t,z_n))$$

That the range of this function is contained within Y_n is self-evident. Now consider the function $G_n : V \times (E^n \cap Y_n) \to X$ given by $G_n := F_n \circ \varphi_n$. Given a closed curve γ in V defined on $[0,1]$, and a $z \in E^n \cap Y_n$, we abuse notation slightly and define $G_n(\gamma, Z) := G_n(\gamma(s), Z)$ for all $0 \leq s \leq 1$. Here we use the convention the capital letter Z stands for an n-tuple $(z_1,\ldots,z_n)$

We can now state our first theorem in this section.

Theorem 3.2.1. *If $\varphi : V \times E \to \widehat{\mathbb{C}}$ can be extended to a continuous motion of $\widehat{\mathbb{C}}$ over V, then the path $G_n(\gamma, Z)$ is homotopic to $G_n(t_0, Z)$ (so is null-homotopic).*

Corollary 3.2.1. *Let V and E be as above, let $Z \in E^4 \cap Y_4$, let $F_4(z_1, z_2, z_3, z_4) = cr(z_1, z_2, z_3, z_4)$. Set $H_Z(t) := G_4(t, Z) = F_4 \circ \varphi_4$. If φ extends to a continuous motion $\widetilde{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, then the map $H_Z(t)$ is null-homotopic. In particular, if γ is any closed curve in V based at t_0, then $s \mapsto \varphi_{\gamma(s)}(z_1, z_2, z_3, z_4)$ is null-homotopic.*

Corollary 3.2.2. *Let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion which extends to a continuous motion of $\widehat{\mathbb{C}}$ over V. Then φ has trivial trace monodromy, that is, the homomorphism $\varphi_*^z : \pi_1(V) \to \pi_1(\mathbb{C} \setminus \{0, 1\})$ is trivial for all $z \in E, z \neq 0, 1, \infty$.*

Before the next Corollary we create some notation. Let $n = 2$, and for X we use $\mathbb{Z}$. For any two closed curves $\delta_1, \delta_2 : [0, 1] \to \mathbb{C}$ of bounded variation, such that $\delta_1(s) \neq \delta_2(s), \forall 0 \leq s \leq 1$, we still continue with the notation F_2, which is defined as follows:

$$F_2(\delta_1, \delta_2) := \frac{1}{2\pi} \int_0^1 d(arg(\delta_1(s) - \delta_2(s)))$$

This is another way of saying that F_2 is the winding number of the curve $\delta_1 - \delta_2$, that this winding number is an element of $\mathbb{Z}$ is a fact found in any elementary book on complex analysis. In fact F_2 is not technically defined on Y_2, but rather on a subset of the collection of pairs of closed curves in $\mathbb{C}$. However, we shall see in the proof of the following corollary this abuse of notation is of no consequence.

Corollary 3.2.3. *Let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion. For any distinct $z_1, z_2 \in E$,* *and any closed curve* γ *in* V*, we define:*

$$H(\gamma) := F_2(\varphi(\gamma, z_1), \varphi(\gamma, z_2)) \tag{3.1}$$

If φ *extends to a continuous motion* $\widetilde{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$*, then* $H(\gamma) = 0$*. Unwinding the definitions, this means the winding number of the closed curve* $s \mapsto \varphi(\gamma(s), z_1) - \varphi(\gamma(s), z_2)$ *is always 0.*

Remark: This proves one direction of Chirka's claim, see [7].

Remark: Corollary 3.2.2 showed that if a holomorphic motion $\varphi : V \times E \to \widehat{\mathbb{C}}$ has an extension to all of $V \times \widehat{\mathbb{C}}$ as a continuous motion, it has trivial trace monodromy and monodromy. It natural to ask whether the converse is true. In this direction we have the following result.

Theorem 3.2.2. *Let* $E = \{0, 1, \infty, a\} \subset \widehat{\mathbb{C}}$ *be a four-point set, and let* $\varphi : V \times E \to \widehat{\mathbb{C}}$ *be a holomorphic motion. TFAE:*

1. φ *can be extended to a holomorphic motion* $\widetilde{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
2. φ *has trivial trace monodromy.*
3. *The monodromy* $\rho_\varphi : \pi_1(V) \to Mod(0, 4)$ *is trivial.*

Theorem 3.2.3. *Let* $K \subset \Delta$ *be AB-removable, and let* $E \subset \widehat{\mathbb{C}}$ *be finite and contain* $0, 1$ *and* ∞*. Let* $\varphi : \Delta_K \times E \to \widehat{\mathbb{C}}$ *be a holomorphic motion. Then* φ *can be extended to a holomorphic motion* $\widetilde{\varphi}$ *of* $\widehat{\mathbb{C}}$ *over* Δ_K *if and only if the monodromy* ρ_φ *is trivial.*

Corollary 3.2.4. *Let* $K \subset \Delta$ *be AB-removable, and let* $E \subset \widehat{\mathbb{C}}$ *be closed and contain* $0, 1$ *and* ∞*. Let* $\varphi : \Delta_K \times E \to \widehat{\mathbb{C}}$ *be a holomorphic motion. If* E' *is a finite subset of* E*, still* *containing* 0.1 *and* ∞*, let* φ' *denote the restriction of* φ *to* $\Delta_K \times E'$*. Then* φ *can be extended to a holomorphic motion* $\widehat{\varphi}$ *of* $\widehat{\mathbb{C}}$ *over* Δ_K *if and only if each monodromy* $\rho_{\varphi'}$ *is trivial.*

Theorem 3.2.4. *Let* E *be a closed subset of* $\widehat{\mathbb{C}}$ *containing* $0, 1, \infty$ *and which contains at least five points. Let* $K \subset \Delta$ *be AB-removable, and let* Δ^* *denote the punctured unit disc, i.e.,* $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$*. Then the following hold:*

1. *If there exists a connected component of* $\widehat{\mathbb{C}} \backslash E$ *which is conformally equivalent to neither* Δ *nor* Δ^**, then there exists a holomorphic motion* φ_0 *of* E *over* Δ_K *which has trivial trace monodromy but which cannot be extended to a holomorphic motion of all of* $\widehat{\mathbb{C}}$ *over* Δ_K*.*

2. *If every connected component of* $\widehat{\mathbb{C}} \setminus E$ *is conformally equivalent to either* Δ *or* Δ^**, then every holomorphic motion* $\varphi : \Delta_K \times E \to \widehat{\mathbb{C}}$ *can be extended to a holomorphic motion of* $\widehat{\mathbb{C}}$ *over* Δ_K*.*

3.3 Proofs

Proof of Theorem 3.2.1. Let V, γ and φ be as in the statement of the theorem. By Proposition 1.5.2, there is a quasiconformal motion $\widehat{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ extending φ. By Corollary 1.5.2, each $\widehat{\varphi}_t$ is a normalized quasiconformal self-homeomorphism of $\widehat{\mathbb{C}}$. If $\mu(t)$ denotes it's Beltrami coefficient, then Corollary 1.5.2 also states $t \mapsto \mu(t)$ is a continuous mapping $V \to M(\mathbb{C})$. To be more explicit, for any $z \in E$ we have $\varphi(t, z) = \widehat{\varphi}(t, z) = w^{\mu(t)}(z)$ and this is a continuous function in t.

For any $Z \in E^n \cap Y_n$, define a function $\Gamma : I \times I \to X : (s, r) \mapsto F_n(w^{r\gamma(s)}(z_1), \ldots, w^{r\gamma(s)}(z_n))$. This is evidently a continuous map, and moreover $\Gamma(s, 1) = F_n(w^{\gamma(s)}(z_1), \ldots, w^{\gamma(s)}(z_n)) =$

$G_n(\gamma, Z)$, and $\Gamma(s,0) = F_n(Z) = G_n(t_0, Z)$. So Γ is the desired homotopy. $\square$

Proof of Corollary 3.2.1. Let γ be any loop in V based at t_0, we wish to show the loop $H_Z(\gamma)$, based at Z, is homotopic to the constant path at $H_Z(t_0) = cr(z_1, z_2, z_3, z_4)$.. Since $H_Z(\gamma) = G_4(\gamma, Z)$ is by Theorem 3.2.1 homotopic to $G_4(t_0, Z) = H_Z(t_0)$, there is really nothing to do here. $\square$

Proof of Corollary 3.2.2. Again, let γ be a loop in V based at t_0. We need to show the curve $\varphi(\gamma(s), z)$ is homotopic to $\varphi(t_0, z)$, for any $z \in E, z \neq 0, 1, \infty$. But if $n = 1$, and $F_1 = Id$, this is just another way of saying $G_1(\gamma, z)$ is homotopic to $G_1(t_0, z)$, which is exactly the content of Theorem 3.2.1. We are done. $\square$

Proof of Corollary 3.2.3. Arguing as in the proof of Theorem 3.2.1, and using the same notation, we have:

$$\Gamma(s,r) := F_2(\varphi_{r\gamma(s)}(z_1), \varphi_{r\gamma(s)}(z_2)) = F_2(w^{r\gamma(s)}(z_1), w^{r\gamma(s)}(z_2))$$

This is continuous, thus a homotopy, and $\Gamma(s,1) = H(\gamma(s))$, while $\Gamma(s,0) = H(t_0)$, using t_0 to denote the constant path at t_0. Since two homotopic paths have the same winding number, and since $H(t_0)$ is the winding number of a constant path, $H(\gamma) = 0$ is now clear. $\square$

Proof of Theorem 3.2.2. $1 \Rightarrow 2$: This is just Corollary 3.2.2.

$2 \Rightarrow 1$: Now suppose 2 holds, this means φ_*^a is the trivial homomorphism. Let $p : \Delta \to \mathbb{C} \setminus \{0, 1\}$ be a holomorphic universal covering such that $p(0) = a$. Since φ_*^a is trivial, we have $\varphi_*^a(\pi_1(V) \subset p_*\pi_1(\Delta)$ (in fact they are equal since Δ is simply connected, but all we need here is the containment). By elementary topology, there exists a holomorphic lift $\widetilde{\varphi^a} : V \to \Delta$ such that $\widetilde{\varphi^a}(t_0) = a$ (by lift we mean $p \circ \widetilde{\varphi^a} = \varphi^a$).

Now define a holomorphic motion $\psi : \mathbb{C} \setminus \{0,1\} \times E \to \widehat{\mathbb{C}}$ by the formula

- $\psi(t,\zeta) = \zeta$ for $\zeta = 0, 1, \infty$ and $\forall t \in V$.
- $\psi(t,a) = t, \forall t \in V$

That ψ is indeed a holomorphic motion is self-evident. Now, let $\psi' = p^*(\psi)$, this is a holomorphic motion of E over Δ, and by Slodkowski's Theorem it extends to a holomorphic motion $\widetilde{\psi} : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Finally define $\widetilde{\varphi} = \widetilde{\varphi^a}^*(\widetilde{\psi})$. Then $\widetilde{\varphi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic motion, and by unwinding the construction we see that for $t \in V$ we have:

$$\widetilde{\varphi}(t,a) = \widetilde{\psi}(\widetilde{\varphi^a}(t), a) = \psi'(\widetilde{\varphi^a}(t), a) = \psi(p(\widetilde{\varphi^a}(t)), a) = \psi(\varphi(t,a), a) = \varphi(t,a).$$

It follows immediately that $\widetilde{\varphi}$ is the extension desired.

$1 \Rightarrow 3$: This is implied by Lemma 3.1.2.

$3 \Rightarrow 1$: This follows from the same logic used for $2 \Rightarrow 1$.

□

Example: If we allow E to contain more than four points, there is a family of counterexamples to (2) $\Rightarrow$ (1) in this theorem. For let $K \subset \Delta$ be an AB-removable subset which contains 0, for example the singleton set $\{0\}$ itself. Since K is by definition compact, we can take a point $a \in \Delta$ such that neither a nor $-a$ is an element of K. Let $E \subset \widehat{\mathbb{C}}$ be a proper closed subset of $\widehat{\mathbb{C}}$ containing $\{0, 1, \infty\}$. Since E is closed, we can find a simply connected open subset U of $\widehat{\mathbb{C}} \setminus E$, let $f : \Delta \to U$ be a Riemann map. We now enlarge E to contain $f(a)$ and $f(-a)$ and call the new set E', clearly E' contains at least five points. Define a function $\varphi : \Delta_K \times E' \to \widehat{\mathbb{C}}$ by:

- $\varphi(t, \zeta) = \zeta$ if $\zeta \neq f(a), f(-a)$.
- $\varphi(t, f(a)) = f(t)$
- $\varphi(t, f(-a)) = f(-t)$

With a as basepoint, it is evident $\varphi : \Delta_K \times E' \to \widehat{\mathbb{C}}$ is indeed a holomorphic motion. However, it cannot be extended to all of $\Delta \times E'$ as a holomorphic motion φ_0, for then continuity would imply $\varphi_0(0, f(a)) = f(0) = \varphi_0(0, f(-a))$, violating injectivity. By Proposition 1.4.4, this implies no extension of φ to $\Delta_K \times \widehat{\mathbb{C}}$ as a holomorphic motion, or even a continuous motion, exists.

Finally, φ has trivial trace monodromy. To see this, let $z \in E'$ be $\neq 0, 1, \infty$, and let γ be a loop in Δ_K based at a. If $z \neq f(a), f(-a)$, the loop $\varphi^z \circ \gamma$ is constant and certainly null-homotopic. If $z = f(a)$ or $f(-a)$, then the loop has image contained in the simply connected open set U and is again be null-homotopic. This completes the example.

Theorem 3.2.4, proven below, gives a different family of counterexamples which will not require us to enlarge the original set E.

Proof of Theorem 3.2.3. Suppose first the desired extension $\widetilde{\varphi}$ exists. Then by Lemma 3.1.2, ρ_φ is indeed trivial.

Conversely, assume the monodromy $\rho : \pi_1(\Delta_K) \to Mod(0, n)$ is trivial and write $E = \{z_1, \dots, z_n\}$. Let $p : \Delta \to \Delta_K$ be a holomorphic universal cover taking $0 \mapsto t_0$, and let

Γ_K be the corresponding Fuchsian group. Define $\psi := p^*\varphi$, so that $\psi(u,z) = \varphi(p(u),z)$ for all $(u,z) \in \Delta \times E$. By Slodkowski's Theorem, ψ has an extension to a holomorphic motion $\widetilde{\psi} : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. So by Proposition 1.5.1, there exists a unique holomorphic map $f : \Delta \to M(\mathbb{C})$ such that $f^*\Psi_{\widehat{\mathbb{C}}} = \widetilde{\psi}$. Then $\widetilde{\psi}(u,z) = \psi(u,z) = w^{f(u)}(z)$ for all $u \in \Delta, z \in E$ and so we get a holomorphic map $\widetilde{f} : \Delta \to T(E) = T(0,n) : u \mapsto [w^{f(u)}]_E$, the Teichmüller space of $\widehat{\mathbb{C}}$ with n punctures.

Since ρ is by assumption trivial, $\widetilde{f} \circ g = \widetilde{f}$ for any $g \in \Gamma_K$. This means $\widetilde{f}$ projects to a function $\widehat{f} : \Delta_K \to T(E)$, and this new function is by construction holomorphic and basepoint preserving. We now claim $\widehat{f}$ can be extended to a holomorphic map $\Delta \to T(0,n)$, where $T(0,n)$ is the Teichmüller space of the sphere with n punctures, recall $T(E)$ and $T(0,n)$ are canonically identified.

Let $\zeta \in K$, and introduce the notation c_V to mean the Carathêodory pseudodistance on a complex Banach manifold V. Since K is AB-removable, it is self-evident $c_{\Delta_K} = c_\Delta|\Delta_K$. Since K is nowhere dense, there is a sequence $\{x_k\}$ in Δ_K such that $x_k \to \zeta$ as $k \to \infty$. Clearly $\{x_k\}$ is a Cauchy sequence with respect to $c_\Delta = \rho_\Delta$, and by the previous comment $\{x_k\}$ is a Cauchy sequence with respect to c_{Δ_K}. By the distance-decreasing property of the Carathéodory pseudodistance, $c_{T(0,n)}(\widehat{f}(x_k), \widehat{f}(x_l))$ also forms a Cauchy sequence, and since as pointed out in [9] and [32] the Carathéodory metric in Teichmüller spaces is complete we can define $\widehat{f}(\zeta) := \lim x_k$. If $\{y_k\}$ is some other sequence tending to ζ, whence $c_{\Delta_K}(x_k, y_k) \to 0$ as $k \to \infty$, $c_{T(0,n)}(\widehat{f}(x_k), \widehat{f}(y_k)) \to 0$ as $k \to \infty$, showing this extension is well-defined.

Finally, let $\varphi_0 := \widehat{f}^*(\Psi_E)$. This is a holomorphic motion of E over Δ. If we can show φ_0 extends φ, then the desired result will follow from Proposition 1.4.4. Given $t \in \Delta_K$ and

$z \in E$, we have, using the notation s for the global continuous subsection of P_E and taking $u \in p^{-1}(t)$ arbitrarily,

$$\varphi_0(t,z) = \Psi_E(\widehat{f}(t), z) = w^{s(\widehat{f}(t))}(z) = w^{s(\widetilde{f}(u))}(z) = w^{f(u)}(z) = \psi(u,z) = \varphi(t,z)$$

$\square$

Proof of Corollary 3.2.4. Suppose φ has the desired extension, then each φ' can be extended to all of $\Delta_K \times \widehat{\mathbb{C}}$ trivially, so each $\rho_{\varphi'}$ by Theorem 3.2.3. Conversely, suppose the monodromy condition holds. By the other direction of Theorem 3.2.3, the condition in Proposition 2.2.1 is trivially satisfied, so combining these two facts Proposition 2.2.1 gives the result. $\square$

Proof of Theorem 3.2.4. **Proof of (1)** We construct a concrete counterexample. Let K be an AB-removable compact subset of Δ, and assume $0 \in K$ without loss of generality. Let Ω be a connected component of $\widehat{\mathbb{C}} \setminus E$ which is conformally equivalent to neither Δ nor to Δ^*. Since E contains at least five points, there exists a simply connected domain D such that $\partial D \subset \Omega$, D contains at least two points of E, say z_1, z_2, and $D^c \cap E$ contains at least three points. We may assume that $0, 1, \infty$ are not in D. We take z_0 in $D \setminus E$.

Take z_1 and z_2 two points of E inside D, $0, 1, \infty \in E \cap D^c$, and let z_0 be an element of D not in E. Let $h : \Delta \to D$ be a Riemann map such that $h(0) = z_0$. Now, there is a positive number $r < 1$ such that $h(\{r < |t| < 1\}) \cap E = \emptyset$ (this is because $E \cap D$ is clearly compact), and by increasing r if need be we may assume r is not in K. We construct a holomorphic motion $\varphi_0 : E \times \Delta_K \to \widehat{\mathbb{C}}$ as follows:

- $\varphi_0(t,z) = z$ if $z \in E \setminus D$

- $\varphi_0(t,z) = h((th^{-1}(z))/r)$ if $z \in E \cap D$

If $r \in \Delta_K$ is taken as basepoint, it is not hard to see this indeed defines a holomorphic motion. Moreover, if γ is a loop in Δ_K based at r and $z \in E \setminus \{0,1,\infty\}$, the loop $\varphi_0^z(\gamma)$ is either constant or contained the simply connected subset D of $\widehat{\mathbb{C}} \setminus \{0,1,\infty\}$, hence φ_0 has trivial trace monodromy. On the other hand, φ_0 cannot be extended to $\Delta_K \times \widehat{\mathbb{C}}$ as a holomorphic motion because it does not satisfy the winding number condition of Corollary 3.2.3.

Indeed, let γ be a loop in Δ_K based at r which is not null-homotopic in Δ_K, after a simple normalization we may assume γ has winding number 1 about the origin and $h(\gamma) \ni z_1, z_2$. Then for a holomorphic map $f(t) = \varphi_0(t,z_1) - \varphi_0(t,z_2) = h((th^{-1}(z_1))/r) - h((th^{-1}(z_2))/r)$, $H(\gamma)$ as defined in (3.1) is the winding number of $f(\gamma)$ about the origin, which by the Argument Principle is the number of zeros of f inside γ, which is exactly 1. To see this, note:

$$\begin{aligned} f(t) = 0 \Leftrightarrow h((th^{-1}(z_1))/r) = h((th^{-1}(z_2))/r) \Leftrightarrow \\ th^{-1}(z_1))/r = th^{-1}(z_2))/r \Leftrightarrow th^{-1}(z_1)) = th^{-1}(z_2)) \Leftrightarrow t = 0. \quad (3.2) \end{aligned}$$

With the last equivalence following from the fact $h^{-1}(z_1) \neq h^{-1}(z_2)$.

Proof of (2): Let E be as closed subset of $\widehat{\mathbb{C}}$, containing $\{0,1,\infty\}$ such that every connected component of $\widehat{\mathbb{C}} \setminus E$ is conformally equivalent to either Δ or Δ^*. Let φ be a holomorphic motion of $\Delta_K \times E$ with basepoint t_0, and let $c \in \pi_1(\Delta_K, t_0)$. Associate with c a quasiconformal homeomorphism $w_c : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as in Chapter 1.4. We showed w_c fixes every point of E in that same subsection. It is a purely topological result that any orientation-preserving homeomorphism of $\widehat{\mathbb{C}}$ fixes every component of the complement of its fixed point set, a proof can be found in [13]. In particular, w_c fixes every connected component of $\widehat{\mathbb{C}} \setminus E$. Using Lemma 3.3.1 given below, it follows from our hypothesis on E that w_c is isotopic to the identity relative to E (since such an isotopy class was shown in Lemma 3.1.1 not to depend on the w_c chosen.) Therefore if we restrict the holomorphic motion to any finite subset $E' \subset E$, the monodromy of this restricted holomorphic motion must be trivial. By Corollary 3.2.4, this is sufficient to show φ can be extended to a holomorphic motion of $\widehat{\mathbb{C}}$ over Δ_K. $\square$

Lemma 3.3.1. *Let g be a quasiconformal self-homeomorphism of $\widehat{\mathbb{C}}$ and let D be an open subset of $\widehat{\mathbb{C}}$ either simply connected or conformally equivalent Δ^*, and further suppose g fixes every point of ∂D. Then $g|_D$ is isotopic to the identity relative to ∂D.*

Proof. We must address two separate cases.

Case 1: If D is either $\widehat{\mathbb{C}}$ or $\mathbb{C}$, there is nothing to prove. So assume D is conformally equivalent to Δ, and take a $z_0 \in D$ as basepoint. We claim that we can assume $g(z_0) = z_0$ without loss of generality, the proof is as follows.

Assume first there is a disk $\widetilde{D}$ (a true disk, not a conformal disk) contained in D with $g(z_0)$ at the center and z_0 in the disc. If there is a quasiconformal homeomorphism h of the plane which is the identity outside this disk, and takes $g(z_0)$ to z_0, postcomposing g to get $h \circ g$ will clearly not change any aspect of the statement of the Lemma. To construct such an h, first conformally map the disc to the infinite horizontal strip $S := \{(x, y) \in \mathbb{C} : 0 < y < 2\}$, so that $g(z_0)$ is mapped to the point (0,1) and z_0 is mapped to the point $(t, 1)$. Then all I need to do is apply the map $h'(x, y) := (x + y(y-2)t, y)$. This map is quasiconformal on S, and it is clearly the identity on the boundary of S. Pulling it back to the disc $\widetilde{D}$, and then extending outside $\widetilde{D}$ by the identity, evidently gives the desired h.

For the case where no such $\widetilde{D}$ exists, we can by a standard lemma connected $g(z_0)$ and z_0 by a finite sequence of such disks, each overlapping the one before. This will give us a finite sequence of such hs, and the result will follow by composition. This proves the claim.

Now, let $f : \Delta \to D$ be a Riemann map taking 0 to z_0, and set $h := f^{-1} \circ g|D \circ f : \Delta \to \Delta$. Let $\zeta \in \partial D$ be a point accessible from D, meaning there is a Jordan curve $c : [0, 1) \to D$ such that $\lim_{s\to 1} = \zeta$. We assume henceforth $c(0) = z_0$ without loss of generality. By postcomposing with a Möbius map if need be, we also assume ∂D is a bounded subset of $\widehat{\mathbb{C}}$ without loss of generality. (Note we are not assuming D is bounded, for example if $D = \widehat{\mathbb{C}} \setminus [0, 1]$ then ∂D is bounded but D is not). Then f is bounded near $\partial\Delta$. We claim $f^{-1} \circ (c)$ lands at a single point on $\partial\Delta$.

We denote $f^{-1} \circ c$ by c' henceforth for notational convenience. Now, suppose contrariwise the set of accumulation points of c' on $\partial\Delta$ contains more than one element. Then by considering the angle of $c'(s)$, we see by an elementary geometric argument this set contains an open arc I. Let $x \in I$ be arbitrary, then c' must cross the radial line from the origin to x arbitrarily close to x by the same connectivity argument. Let $p_1, p_2, ...$ be a sequence on this radial line, all of them in c', such that $p_n \to x$. Then by construction, $f(p_n) \to \zeta$ (as $f \circ c' = c$ by definition). So if f has a radial limit at any point in I, then this radial limit is necessarily ζ. By a theorem of Fatou (see [26], Subsection 17), f has radial limits at almost every point on $\partial\Delta$. Hence it follows the set of points on $\partial\Delta$ where f's radial limit is ζ has positive measure. Yet it is a theorem of F. and M. Riesz (see [26] once more) that if f is any conformal function on Δ, the set of all points in $\partial\Delta$ where the radial limit of f assumes any given value is either empty or has measure zero. Thus we have a contradiction, and so c' lands at a single point on $\partial\Delta$.

It follows from how we constructed h that $h(c')$ also lands at a single point on $\partial\Delta$. We now claim the landing points of c' and $h(c')$ are in fact the same. For suppose they are different, and call them x_1 and x_2 respectively. Then there clearly exists a Jordan domain Ω_0 in Δ bounded by subarcs of $c', h(c')$ and a subarc I of $\partial\Delta$ between x_1 and x_2. Write $L := f(\partial\Omega_0 \cap \Delta$. Then $\widehat{L} := L \cup \{\zeta\}$ is a simple closed curve in $\widehat{\mathbb{C}}$. Let D_0 be the simply connected domain bounded by $\widehat{L}$ with $D_0 \supset f(\Omega_0)$.

Since h is a quasiconformal self-homeomorphism of Δ, it extends homeomorphically to $\overline{\Delta}$ and is orientation-preserving. Hence there exists a small arc $\delta \subset I$ near x_1 such that $h(\delta) \cap I = \emptyset$. Since f has radial limits a.e. on $\partial\Delta$ by the theorem of Fatou mentioned before, we may find a point $x_0 \in \delta$ such that f has a radial limit there. In other words, if l_0 is the line segment from the origin to x_0 then $f(l_0)$ lands at a point $\zeta_0 \in \partial D$. Since $f \circ h = g \circ f$ and g keeps every point of ∂D fixed, $f(h(l_0))$ also lands at ζ_0. This implies $f(h(l_0))$ eventually belongs to $f(\Omega_0) \subset D_0$. But this is absurd because $h(l_0)$ terminates at $h(\zeta_0) \in h(\delta)$. Hence $h \circ c'$ and c' must terminate at the same point on $\partial\Delta$, as claimed.

The above argument shows $h(x) = x$ if for the radius l_x from the origin to $x \in \partial\Delta$, $f(l_x)$ lands at a single point in ∂D. Since this is true for almost every such x, h fixes almost every point of $\partial\Delta$. Since it is a homeomorphism of $\overline{\Delta}$, h fixes every point of $\partial\Delta$ by a standard argument. Hence it follows from Corollary 2.4 of [12] that $g|D$ is isotopic to the identity relative to ∂D.

Case 2: Now assume D is conformally equivalent to Δ^*, and let a be the puncture of D. Since $g(z) = z$ for all $z \in \partial D$ and $D' := D \cup \{a\}$ is simply connected, it follows from Case 1 that $g|D'$ is isotopic to the identity rel $\partial D'$. So there is a continuous map $H : \overline{D'} \times [0, 1] \to \overline{D'}$ such that:

1. $H(z, 0) = z \ \forall z \in \overline{D'}$, and $H(z, 1) = g(z) \ \forall z \in \overline{D'}$
2. $H(\cdot, t)$ defines a homeomorphism $\overline{D'} \to \overline{D'}$
3. Each $H(\cdot, t)$ leaves each point $\partial\overline{D'}$ fixed.

Which is exactly what we wanted.

□

Remark: In the above Lemma, it is crucial that the map g is quasiconformal on the whole Riemann sphere. In fact there exists a simply connected domain D and an orientation-preserving homeomorphism $g : D \to D$ such that the map $h : \Delta \to \Delta$ given in the above proof is homotopic to the identity rel $\partial\Delta$ while g does not have a continuous extension to $\overline{D}$. See [12].

Example: Let R be a Riemann surface, and suppose there exists a holomorphic map $f : R \to \Delta^*$ such that $f_*(\pi_1(R))$ is nontrivial. Then, for any closed set E satisfying the condition (1) of Theorem 3.2.4 there exists a holomorphic motion φ of E over R such that the trace monodromy φ_*^z is trivial but φ cannot be extended to a holomorphic motion of $\widehat{\mathbb{C}}$. Indeed, we define a holomorphic motion $\varphi_0 : \Delta^* \times E \to \widehat{\mathbb{C}}$ by (3.3) and set $\varphi := f^*\varphi_0$. Since

$f_*(\pi_1(R))$ is nontrivial, there exists a closed curve c in R such that $f(c)$ is not null-homotopic in Δ^*. Therefore, by the same argument as in part (1) of the proof of Theorem 3.2.4, we can verify φ cannot be extended to a holomorphic motion of $\widehat{\mathbb{C}}$.

Chapter 4

Miscellaneous results

4.1 Results on quasiconformal motions

Given a topological space V, and two points p, q in V, a **path** in V from p to q is a continuous function $\gamma : [0,1] \to V$ such that $\gamma(0) = p$ and $\gamma(1) = q$. A **arc** from p to q is a path which is a topological embedding. A space is **path connected** if any two points have a path connecting them, and it is **arc connected** if any two distinct points have an *arc* connecting them.

Clearly arc-connected spaces are path connected, but the converse does not hold. The classical counterexample is the so-called "line with two origins". Take $\mathbb{R}$, and attach another point which we call $0'$, and say $x \in \mathbb{R} \setminus \{0\}$ is $<$ or $>$ $0'$ according to whether or not it is $<$ or $>$ than 0. Giving $\mathbb{R} \cup \{0'\}$ the induced order topology, it is clear this space is path-connected yet there is no arc from 0 to $0'$ – even though this space is locally Euclidean.

The crucial property of the line with two origins is it is not Hausdorff. If V is Hausdorff then path-connectedness of V implies arc-connectedness. See [14].

Theorem 4.1.1. *Let V be a path-connected Hausdorff space with basepoint t_0, and let E be a subset of the Riemann sphere containing 0,1 and ∞. Let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a normalized* *quasiconformal motion. Then, given $t \in V$, there is a normalized quasiconformal mapping of the sphere whose restriction to E is φ_t.*

Proof. As noted above, each path-connected Hausdorff space is arc-connected. If $t = t_0$, the desired extension is just the identity map. So assume $t \neq t_0$, and let $\gamma : I \to V$ be an arc from t_0 to t. Let δ be the image of γ, then consider the restriction $\varphi' : \delta \times E \to \widehat{\mathbb{C}}$ of φ. That φ' is a quasiconformal motion emerges directly from the definition. Then $\psi := \gamma^* \varphi' : I \times E \to \widehat{\mathbb{C}}$ is also a quasiconformal motion if we take the basepoint of I to be 0.

By Proposition 1.4.3 there is an extension Ψ of ψ to $I \times \widehat{\mathbb{C}}$ which is also a quasiconformal motion, then $\widetilde{\varphi'} = (\gamma^{-1})^* \Psi : \delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is also a quasiconformal motion which by construction restricts to $\varphi : V \times E \to \widehat{\mathbb{C}}$, as $(\gamma^{-1})^* (\gamma)^* \varphi = \varphi$. In particular, $\widetilde{\varphi'}_t | E = \varphi_t$, and so via Corollary 1.5.2 we see $\widetilde{\varphi'}_t$ is a quasiconformal mapping of the whole sphere, so it is the desired extension. ☐

Theorem 4.1.2. *Let V be a connected Hausdorff space with basepoint t_0, let $E \subset \widehat{\mathbb{C}}$ be finite and contain 0, 1 and ∞, and let $\varphi : V \times E \to \widehat{\mathbb{C}}$ be a function such that $\varphi(t_0, z) = z$ for all $z \in E$. Assume there is a continuous basepoint-preserving map $f : V \to M(\mathbb{C})$ such that $\varphi_t = w^{f(t)} | E$ for all $t \in V$. Then φ is a quasiconformal motion.*

Proof. Because f is basepoint preserving for all $z \in E$ we have $\varphi(t_0, z) = w^{f(t_0)}(z) = w^0(z) = z$. Thus the first requirement for a quasiconformal motion is satisfied. Fix $t \in V$ for the rest of the proof. Let a_1, a_2, a_3, a_4 be distinct points in E, and let $u \to t$ in V in the sense of nets (we need to use nets because V was not assumed to be first-countable). Then by continuity $f(u) \to f(t)$, so $w^{f(u)}(a_i) \to w^{f(t)}(a_i)$ for each i by the dependence on Beltrami differentials. It follows from the definition of cross-ratio $\varphi_u(a_1, a_2, a_3, a_4) \to \varphi_t(a_1, a_2, a_3, a_4)$

in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$, so given $\varepsilon > 0$ there is a neighborhood U_{a_1,a_2,a_3,a_4} of t such that if $u \in U_{a_1,a_2,a_3,a_4}$,

$$\rho(\varphi_t(a_1, a_2, a_3, a_4), \varphi_u(a_1, a_2, a_3, a_4)) < \varepsilon \backslash 2$$

By the Triangle Inequality, $u, u' \in U_{a_1,a_2,a_3,a_4}$ then

$$\rho(\varphi_u(a_1, a_2, a_3, a_4), \varphi_{u'}(a_1, a_2, a_3, a_4)) < \varepsilon$$

Now, take the intersection of all the U_{a_1,a_2,a_3,a_4} as the a_is vary through E. By the finiteness of E, this intersection, denote it U, is an open set and from construction it is the U specified in the definition of quasiconformal motion. Since t was arbitrary, we are done. □

4.2 Mappings into $Teich(\mathbb{H}) \times Teich(\mathbb{H}^-)$

We generalize a result of Lieb (see [24]) from the unit disc into any connected complex Banach manifold. Recall from Section 1.3 that $\Theta : T(E) \to Teich(E^c) \times M(E) : [\mu]_E \mapsto (\Phi(\mu|E^c), \mu|E)$, and that it is by definition biholomorphic.

Theorem 4.2.1. *Let V be a connected complex Banach manifold with basepoint t_0, and let $G : V \to Teich(\mathbb{H}) \times Teich(\mathbb{H}^-)$ be any function. Then define a function $G^* : V \times \widehat{\mathbb{R}} \to \widehat{\mathbb{C}}$ by the formula*

$$G^*(t, z) = [(\Theta \circ G)(t)](z)$$

G^* is a normalized motion, and if G is holomorphic then G^* is a holomorphic motion.

Proof. We begin by showing G^* is always a motion. Let $z \in \widehat{\mathbb{R}}$ be arbitrary, then using the hypothesis G is basepoint preserving we obtain

$$G^*(t_0, z) = [(\Theta \circ G)(t_0)](z) = [\boldsymbol{B}(0)](z) = w^0(z) = z$$

So the first requirement of being a motion is satisfied. Next, fix $t \in V$ and let $\mu \in M(\mathbb{C})$ be such that $P_{\widehat{\mathbb{R}}}(\mu) = \Theta \circ G)(t)$. By unwinding the definitions we see $G^*(t, \cdot) = w^{\mu}(\cdot)$, yielding the second requirement of being a motion. The construction automatically yields the normalization requirement.

Now we assume G is holomorphic. Fix $z \in \widehat{\mathbb{R}}$, we want to show $G^*(\cdot, z)$ is a function holomorphic in the first variable. Fix $t \in V$, it suffices to show there is a neighborhood U of t such that the restriction to U is holomorphic. There is by Lieb's work a neighborhood W of $G(t)$ such that on this neighborhood W there is a holomorphic map $\boldsymbol{S}$ from $Teich(\mathbb{H}) \times Teich(\mathbb{H}^-)$ to $M(\mathbb{H}) \times M(\mathbb{H}^-) \cong M(\mathbb{C})$ which is the inverse of the product of the relevant Teichmüller projection maps. Let U be the inverse of W under G. Then on U the function

$$(\boldsymbol{S} \circ G) : V \to M(\mathbb{C})$$

is holomorphic, and recall G^*_t is the restriction of $w^{\boldsymbol{S} \circ G}$ to the real line. Since point evaluation of the solution to the Beltrami equation is a holomorphic function from $M(\mathbb{C})$ to $\mathbb{C}$ we see $G^*(t, z)$ is a holomorphic map of t for any fixed z. $\square$

We now address the converse.

Theorem 4.2.2. *Let V be a simply connected complex Banach manifold with basepoint t_0, let $G : V \to Teich(\mathbb{H}) \times Teich(\mathbb{H})$ be any basepoint-preserving function and let G^* be the* *induced motion as above. Assume the induced motion G^* is a holomorphic motion. Then G is a holomorphic map.*

Proof. Since V is simply connected, by Proposition 1.5.2 there exists a basepoint-preserving holomorphic map $F : V \to T_{\widehat{\mathbb{R}}}$ such that $F^* : \Psi_{\widehat{\mathbb{R}}} = G^*$. We claim $\boldsymbol{B} \circ G = F$, this will prove the result since $\boldsymbol{B}$ is by definition a biholomorphism. Let $t \in V$ and $z \in \widehat{\mathbb{R}}$ be given, and let $\mu \in M(\mathbb{C})$ be such that $P_{\widehat{\mathbb{R}}}(\mu) = F(t)$. Then it follows $w^\mu(z) = G^*(t, z)$, by the construction of F. □

Bibliography

[1] L. Ahlfors. *Lectures on Quasiconformal Mappings. Second Edition.* University Lecture Series 38. American Mathematical Society, Providence, R.I., 2006.

[2] L. Ahlfors and L. Bers. Riemann's Mapping Theorem for Variable Metrics. *Annals of Math.*, 72:pp. 345–404, 1960.

[3] M. Beck, Y. Jiang, and S. Mitra. Normal Families and Holomorphic Motions over Infinite-Dimensional Parameter Spaces. Accepted by Contemporary Mathematics, Proceedings of the Conference on Conformal Dynamics and Hyperbolic Geometry Honoring Linda Keen.

[4] M. Beck, Y. Jiang, S. Mitra, and H. Shiga. Holomorphic Motions and Monodromy. *Annales Academiae Scientiarum Fennicae Mathematica*, Volume 37:pp. 53–67, 2012.

[5] L. Bers and H. L. Royden. Holomorphic Families of Injections. *Acta Math.*, Volume 157:pp. 243–257, 1986.

[6] S. B. Chae. *Holomorphy and Calculus in Normed Spaces.* Pure and Applied Mathematics 92. Marcel Dekker Inc., 1985.

[7] E. M. Chirka. On the Extension of Holomorphic Motions. *Doklady Mathematics*, Volume 70 No. 1:pp. 516–519, 2004.

[8] E. F. Collingwood and A. J. Lohwater. *The Theory of Cluster Sets.* Cambridge University Press, 1966.

[9] C. J. Earle. On the Carathéodory Metric in Teichmüller Spaces. In *Discontinuous Groups and Riemann Surfaces, 1973 Maryland Conference*, Ann. Math. Studies 79, pages pp. 99–103. Princeton University Press, 1974.

[10] C. J. Earle. A Montel Theorem for Holomorphic Functions on Infinite Dimensional Spaces that Omit the Values 0 and 1. *Computational Methods and Function Theory*, Volume 8:pp. 195–198, 2008.

[11] C. J. Earle, I. Kra, and S. L. Krushkal. Holomorphic Motions and Teichmüller Spaces. *Trans. Amer. Math. Soc.*, Volume 343(no. 2):pp. 927–948, 1994.

[12] C. J. Earle and C. McMullen. Quasiconformal Isotopies. In *Quasiconformal Functions and Moduli I.* Springer-Verlag, 1988.

[13] C. J. Earle and S. Mitra. Variation of Moduli Under Holomorphic Motions. In *In the Tradition of Ahlfors and Bers, Proceedings of the First Ahlfors-Bers Colloquium*, volume Volume 256, pages pp. 39–67, 2000.

[14] R. Engelking. *General Topology.* Helderman Verlag, Berlin, 1989.

[15] A. Fletcher and V. Markovic. *Quasiconformal Maps and Teichmüller Theory.* Number 11 in Oxford Gradute Texts in Mathematics. Oxford University Press, 2007.

[16] F. Gardiner. Approximation of Infinite-Dimensional Teichmüller Spaces. *Trans. Amer. Math. Soc.*, Volume 282(no. 1):pp. 367–383, 1984.

[17] F. Gardiner, Y. Jiang, and Z. Wang. Holomorphic Motions and Related Topics. In *Geometry of Riemann Surfaces*, number No. 368 in London Mathematical Society Lecture Note Series, pages pp. 156–193. London Mathematical Society, 2010.

[18] F. Gardiner and N. Lakic. *Quasiconformal Teichmüller Theory.* Number Vol. 76 in Mathematical Surveys and Monographs. American Mathematical Society, 2000.

[19] J. H. Hubbard. *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics, Volume I: Teichmüller Theory.* Matrix Editions, 2006.

[20] Y. Imayoshi and M. Taniguchi. *An Introduction to Teichmüller Spaces.* Springer, 1992.

[21] Y. Jiang and S. Mitra. Some Applications of Universal Holomorphic Motions. *Kodai Mathematics Journal*, Volume 30:pp. 85–96, 2007.

[22] Y. Jiang, S. Mitra, and H. Shiga. Quasiconformal Motions and Isomorphisms of Möbius Groups. *Israel Journal of Mathematics*, Volume 188:pp. 177–194, 2012.

[23] J. Lehner. *Discontinuous Groups and Automorphic Functions.* Math Surveys, VIII. American Mathematical Society, Providence, R.I., 1964.

[24] G. Lieb. *Holomorphic Motions and Teichmuller Space.* PhD thesis, Cornell University, 1990.

[25] R. Mañé, P. Sad, and D. Sullivan. On the Dynamics of Rational Maps. *Ann. Sci. cole. Norm. Sup. (4)*, Volume 16:pp. 193–217, 1983.

[26] J. Milnor. *Dynamics in One Complex Variable, Third Edition.* Annals of Mathematics Studies 160. Princeton University Press, 2006.

[27] S. Mitra. Teichmüller Spaces and Holomorphic Motions. *J. d'Analyse Math.*, Volume 81:pp. 1–33, 2000.

[28] S. Mitra. Extensions of Holomorphic Motions to Quasiconformal Motions. *Contemporary Mathematics*, Volume 432:pp. 199–208, 2007.

[29] S. Mitra and H. Shiga. Extensions of Holomorphic Motions and Holomorphic Families of Mobius Groups. *Osaka Journal of Mathematics*, Volume 432:pp. 1167 – 1187, 2010.

[30] S. Nag. *Complex Analytic Theory of Teichmüller spaces.* Canadian Mathematical Society Series of Monographs and Advanced Texts. Wiley Interscience, 1988.

[31] H. Royden. Automorphisms and Isometries of Teichmüller space. *Advances in the Theory of Riemann Surfaces*, pages pp. 369–384, 1971.

[32] H. Shiga. On Analytic and Geometric Properties of Teichmüller Spaces. *J. Math Kyoto University*, Volume 24:pp. 441–452, 1984.

[33] Z. Slodkowski. Holomorphic Motions and Polynomial Hulls. *Proceedings of the American Mathematical Society*, Volume 111(No. 2):pp. 347–355, 1991.

[34] D. Sullivan and W. Thurston. Extending Holomorphic Motions. *Acta Mathematica*, Volume 157:pp. 243 –257, 1986.