Order Number 9009795

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City University of New York, 1989

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THE LIGHT-CONE GAUGE IN POLYAKOV'S THEORY OF STRINGS

AND ITS RELATION TO THE CONFORMAL GAUGE

by

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A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1989

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

7/10/89
date

Chairman of Examining Committee

9/11/89
date

Executive Officer

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The City University of New York

Abstract

THE LIGHT-CONE GAUGE IN POLYAKOV'S THEORY OF STRINGS AND ITS RELATION TO THE CONFORMAL GAUGE

by
Rodanthy Tzani

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We study the string theory as a gauge theory. The analysis includes the formulation of the interacting bosonic string by fixing the Gervais-Sakita light-cone gauge in Polyakov's path-integral formulation of the theory and the study of the problem of changing gauge in string theory in the context of the functional formulation of the theory. The main results are the following: Mandelstam's picture is obtained from the light-cone gauge fixed Polyakov's theory. Due to the off-diagonal nature of our gauge the calculation of the determinants differs from the usual (conformal gauge) case. The regularization of the functional integrals associated with these determinants is done by using the conformal-invariance principle. We then show that the conformal anomaly associated with this new gauge fixing is canceled at dimensions of space-time d = 26. Studying the problem of changing gauge in string theory, we show the equivalence between the light-cone and conformal gauge in the path-integral formulation of the theory. In particular, by performing a proper change of variables in the commuting and ghost fields in the Polyakov path-integral, the string theory in the conformal gauge is obtained from the light-cone gauge fixed expression. Finally, the problem of changing gauge is generalized to the higher genus surfaces. It is shown that the string theory in the conformal gauge is equivalent to the light-cone gauge fixed theory for surfaces with arbitrary number of handles.

Acknowledgements

Many people have contributed with their presence or memory to my carrying on in life and science to this point. I am grateful and obliged to all of them. Since it is almost impossible to correctly credit them, due to the long list and diversity of their contribution; I will not name them individually. I wish only to explicitly express my deep gratitude to my advisor in City College, Prof. Bunji Sakita whose intellectual, moral and material support and assistance was invaluable in the pursuit and completion of the present work. I am indebted to him for generously sharing with me his many insights and ideas.

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CHAPTER 1

Introduction

Feynman's [1] path-integral for a free particle is defined to be the sum over all possible paths connecting the initial and final positions of the particle, weighted by the exponential of the action. The action is proportional to the length of the trajectory.

For the case of an one-dimensional object, namely, for the case of a string, its successive positions for different times generate a two-dimensional surface Σ in space-time. The obvious generalization of the point particle case is to take the action to be proportional to the area of this surface. It is given by

$$S = \int_{\Sigma} d^2\sigma \, L \, (X \, (\sigma) \,), \qquad 1.1$$

where

$$L = \sqrt{\det h} \quad , \qquad h_{ab} = \partial_a X^{\mu} \partial_b X_{\mu} \qquad 1.2$$

is the Nambu-Goto Lagrangian [2]. The fields $X^{\mu}(\sigma)$, $\mu = 1, \cdots, d$ are the coordinates of the string in the d-dimensional space-time and σ is a collective notation for the coordinates (τ , σ) of the two-dimensional surface (not to be confused with the component σ).

The crucial point is that the area of the surface does not depend on its parametrization. One can check that the action S is invariant under the two-dimensional general coordinate transformation

$$\sigma \rightarrow \tilde{\sigma}^{\epsilon} \, (\sigma) \qquad 1.3$$

where $\epsilon^a(\sigma)$, $a=1,2$ is an infinitesimal vector parameter, which specifies its infinitesimal transformation matrix. The significance of this invariance is that one can use the two reparametrizations to gauge away two field-degrees of freedom. Indeed Goddard, Goldstone, Rebbi and Thorn (G.G.R.T.) [3] used this invariance to gauge away the two longitudinal coordinates X^+ and X^- of the string variables, leaving thus only the d - 2 transverse components X^i as independent dynamical variables describing the theory. In their work, which was a canonical formulation of the free relativistic string, the condition for the critical dimensions (d=26) was also obtained. In their gauge X^+ was essentially chosen to be the time component τ of the two-dimensional coordinates, defining this way the light-cone gauge.

Gervais and Sakita [4] were the first to formulate the interacting string as a functional integral. They fixed the light-cone gauge in the Nambu-Goto Lagrangian. Their path-integral is an integration over the transverse components of the string. In order to include interactions, they chose the X^+ component of the string field to be a function of (τ, σ). Namely,

$$X^+ = f(\sigma) \qquad 1.4$$

where σ is two-dimensional. $f(\sigma)$ is an arbitrary function in the stage of gauge fixing, but later is chosen to satisfy the inhomogeneous Laplace equation in two dimensions in order to derive the N-string amplitude and therefore it depends on the external momenta.

In Mandelstam's picture [5] the X^+ component is chosen to be just τ, introducing the G.G.R.T. light-cone gauge in the path-integral formalism. Mandelstam's formulation is a light-cone gauge fixed theory. In this approach the propagation of the string is given by a functional integral over the transverse coordinates of the field X^μ, which are the physical degrees of freedom of the string. The difference between Gervais-Sakita (G.-S.) and Mandelstam's approach is that in this later case, through a conformal transformation the integration variables are defined on Mandelstam's plane (fig.1).

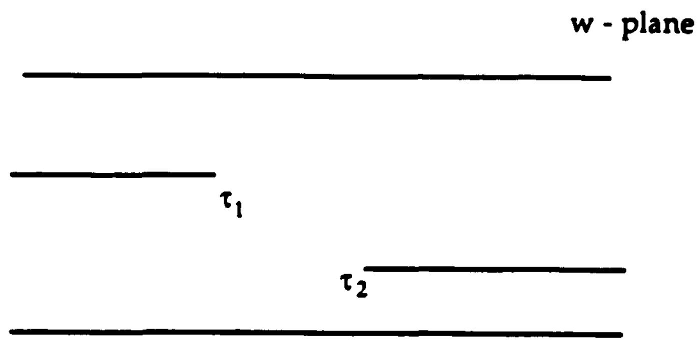


Fig. 1

Fig. 1: Mandelstam's plane

Mandelstam's conformal mapping is defined to be essentially the function $f(\sigma)$, chosen by G.-S., in such a way that the light-cone gauge condition $X^+ = \tau$ coincides with (1.4). The topology of Mandelstam's plane specifies the interaction process of the strings. To obtain the transition amplitude the initial and final wave functions of the string have to be integrated over. In this formalism the interactions are specified by the interaction times, which are specific values of the time variable τ. Since τ corresponds to the real time and the actual motion of the string is described by the evolution of the transverse modes, which are the physical degrees of freedom, Mandelstam's approach appears as a physical picture.

The disadvantage of this approach is that the quantum theory is formulated in a gauge-fixed manner. Specifically, the Nambu-Goto Lagrangian involves a square-root and therefore the path-integral quantization of the theory is highly non-trivial. Choosing, however, the light-cone gauge, as above, one obtains a simple Lagrangian and the quantization is possible. On the other hand, since this is a non-covariant gauge the theory is not manifestly Lorentz invariant.

An alternative formulation which is based on the general covariant Brink-De Vecchia-Howe Lagrangian [6] is proposed by Polyakov [7]. The new point of this approach is the introduction of a 2x2 metric g_{ab} , which characterizes the two-dimensional surface. The conceptual picture of this approach, is essentially the same as before but the appearance is slightly different from Mandelstam's. In this case, the two-dimensional surface is viewed as parameter space (the world-sheet) with coordinates σ. Then $X^{\mu}(\sigma)$ is a mapping function from the parameter space to a two-dimensional surface in the d-dimensional space-time (fig.2).

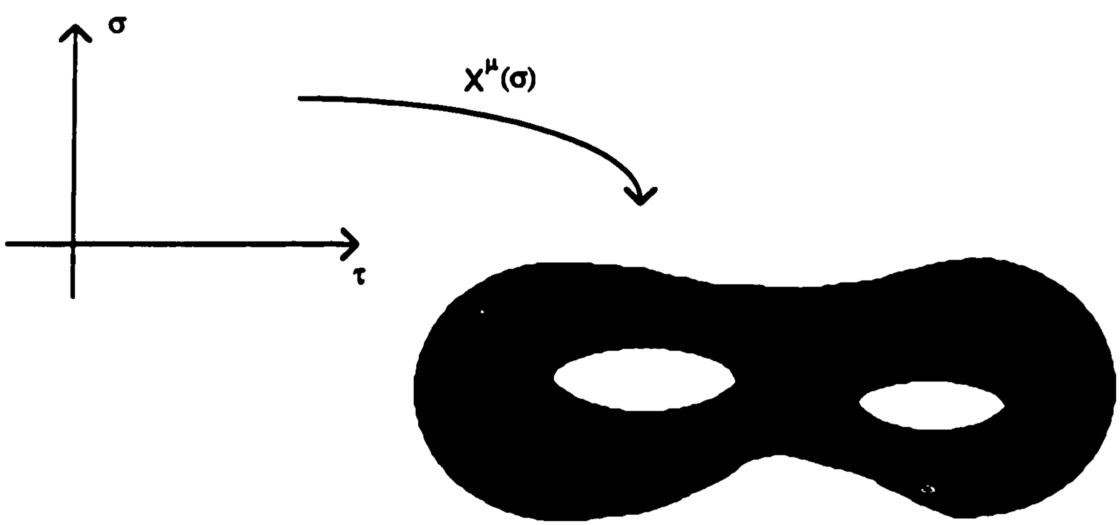


Fig. 2

Fig. 2: Two dimensional manifold embedded in the d-dimensional space-time.

The topology of these surfaces determines the interaction processes of the strings. Polyakov's path-integral is an integration over all possible mappings $X^{\mu}(\sigma)$ and all possible metrics $g_{ab}(\sigma)$. In order to include all possible interactions the summation over all topologies of the surfaces is necessary and the theory is described by the following sum over random surfaces swept by the strings

$$Z = \sum_{topologies} \int DX^{\mu} \int Dg_{ab} \exp[-S]. \qquad 1.5$$

where

$$S = \frac{1}{2} \int d^2\sigma \sqrt{-g}\, g^{ab}\, \partial_a X^{\mu} \partial_b X_{\mu}. \qquad 1.6$$

g_{ab} which is treated as dynamical variable, is an intrinsic metric characterizing the surface, independent of the induced metric

$$h_{ab} = \partial_a X^{\mu} \partial_b X_{\mu}. \qquad 1.7$$

Only in the classical level g_{ab} equals h_{ab} through the equation of motion. The action (1.6) has extra degrees of freedom - the three independent components of the metric g_{ab} - and an extra symmetry, other than the reparametrization invariance defined, as in the previous case, by (1.3). It is invariant under Weyl transformations

$$g_{ab} \rightarrow g'_{ab} = g_{ab}(\sigma)\, e^{\phi(\sigma)} \qquad 1.8$$

These are scaling transformations of the metric, which leave the action invariant in the case that the theory is defined in a two-dimensional manifold . Polyakov's theory is a non-gauge-fixed theory, since it is based on the non-gauge-fixed covariant Lagrangian. It has both symmetries - reparametrization and Weyl - built in. Its reparametrization transformations (diffeomorphisms of the surface) act on the fields X^μ as follows

$$X^\mu(\sigma) \to \tilde{X}^\mu(\sigma) : \tilde{X}^\mu(\tilde{\sigma}(\sigma)) = X^\mu(\sigma). \qquad 1.9$$

while their action on the metric g_{ab} is given by

$$g_{ab}(\sigma) \to g'_{ab}(\sigma') = \frac{\partial \sigma^\alpha}{\partial \sigma^a} \frac{\partial \sigma^\beta}{\partial \sigma^b} g_{\alpha\beta}(\sigma) \qquad 1.10$$

That the covariant Brink-DeVecchia-Howe Lagrangian describes classically the same theory as the Nambu-Goto Lagrangian is proven through the equations of motion. Assuming that the metric field satisfies the equations of motion, the action (1.6) reduces to the Nambu-Goto form. However, to establish the equivalence of the theories at the quantum level is more involved. It has been stated [8] that Mandelstam's picture of interacting strings "is probably equivalent to that of the G.-S. and Polyakov pictures with a specific choice of parametrization". To answer precisely the above question one has to study the related gauges. The method which has been mainly used in studying Polyakov's theory is fixing the conformal (diagonal) gauge [9]. In this method, one represents the complex structure by a metric of constant curvature. Namely, one chooses

$$g_{ab} = e^\phi \hat{g}_{ab} \qquad 1.11$$

where

$$\hat{g}_{ab} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

is the flat two-dimensional Minkowski metric. This locally can always be done in the two dimensions. (1.11) defines the conformal gauge, in which the two diagonal components of the metric $g_{\alpha\beta}$ are fixed using the two reparametrizations invariances of the action. ϕ represents the third component of the metric and can be associated with the Weyl symmetry. Then assuming there are no anomalies involved, the action (1.6) is independent of the metric. The integration over $X^{\mu}(\sigma)$ is Gaussian and the integral (1.5) easily reduces to an integral over the moduli space (for the definition of the moduli space look at ref. [9], [12], [31]) times the determinant of the Laplacian operator, which depends on the dimensions of space-time. Since this is a covariant formulation with respect to the space-time index μ, Lorentz invariance is manifest in this approach.

There are two subtle points in this discussion. Firstly, as Polyakov pointed out, there is an anomaly associated with Weyl symmetry. Even if the action is invariant under the reparametrization and Weyl symmetry, one can not write down a measure which is invariant under both symmetries [10]. One can not do better than writing a reparametrization invariant measure which, however, will be non-invariant under Weyl transformations. Therefore, for the correct quantization of Polyakov's theory, one has to regularize the functional integrals associated with the determinants and to compute the anomaly; namely, a ϕ-dependent factor.

Secondly, the drawback of this method is that the moduli space has a complex structure by itself. To exploit the complex structure of the moduli space Giddings and D'Hoker [11] represented the complex structure by period matrices. Their formulation is based on the theory of Abelian differentials on Riemann surfaces and is rather abstract.

A different approach, which has not been investigated, is fixing the light-cone gauge in Polyakov's theory. It is the establishment of this last approach which constitutes this thesis. In particular, in this thesis, we are formulating the interacting strings by fixing the same light-cone gauge conditions, as Gervais-Sakita did for the Nambu-Goto Lagrangian. The significance of choosing X^+ to be a function of σ and not the usual τ will be apparent later in the course of the work. This choice is crucial for the inclusion of interactions.

The power and beauty of this last idea stem from its ability to relate and connect the physical picture of Mandelstam's approach and the covariant looking formalism of Polyakov's theory. These two formulations appear dissimilar but, as is indicated in ref. [12], are probably related by a coordinate transformation. In this last ref. [12], it is suggested how Mandelstam's picture is viewed in Polyakov's theory of strings. Motivated by this work we are investigating the precise relationship between these two formulations. Since Mandelstam's picture is a light-cone gauge fixed theory, the study of the light-cone gauge in the non-gauge-fixed Polyakov's theory will precisely relate the two formulations. The intuitive expectation is that the light-cone gauge fixed Polyakov's theory coincides with Mandelstam's picture.

The method is appealing for several other reasons. In the standard literature of strings [13], it is stated that X^+ can be chosen to be the time τ coordinate of the two-dimensional parameter space, by using the residual symmetry which remains after fixing the conformal gauge. In this view the light-cone gauge appears as a part of the conformal gauge. On the other hand, in the original G.G.R.T. light-cone-gauge paper [3] the light-cone gauge was imposed in the context of Dual models independently of the conformal gauge. In this thesis, we introduce the light-cone gauge in the theory of strings independently of the conformal gauge. In particular, we establish the light-cone gauge in Polyakov's path-integral formulation of strings in the same level as the standard conformal gauge-fixed covariant Lagrangian path-integral formalism has been investigated.

Another motivation for this work, is the work of Kato and Ogawa [14]. They reformulated the quantized theory of the bosonic string, in a covariant operator formalism based on Becchi-Rouet-Stora- Tyutin (BRST) symmetry [15]. They fixed the conformal (diagonal) gauge for Polyakov's Lagrangian. It seems interesting to us to establish the standard canonical quantization in the light-cone gauge, using BRST symmetry as in the case of Kato and Ogawa. This could shed light in the understanding of the BRST-symmetry and the role of the BRST ghost (antighost) fields, which seem so crucial in the second quantized formalism. A first attempt, however, to investigate this question in the canonical formalism met considerable difficulties. One does not have a guiding principle in this formalism and it seems that the correct number of fields, which describe the gauge-fixed theory is rather arbitrary. For this reason, it is not clear to us yet how one can obtain a nilpotent BRST charge in the light-cone gauge. Since the path-integral approach does not contain arbitrariness of that kind, in this thesis, we are investigating the analogous problem in the functional formalism, as Fujikawa did for the conformal gauge [16].

Moreover, the precise relationship between the two gauges (light-cone and conformal) in the theory of strings remains obscure. One way to precisely investigate the relationship between these two gauges is by changing gauge from one to the other. In this treatise, we study the changing of gauge problem in the functional formalism of the theory of strings. In particular, we change gauge from the light-cone to the conformal gauge in Polyakov's path-integral of the bosonic string. Using this same formalism we investigate the precise relationship between these two gauges.

In connection with the second quantized formulation, the establishment of this approach is important for the following reason: Witten's [17] field theory of strings inherits the covariant canonical formulation of Kato and Ogawa, while the Kaku- Kikkawa [18] light-cone string field theory is based on Mandelstam's formulation. The understanding of the interconnection between the two gauges could help the investigation of the relationship between the two field theories (those of Witten and of Kaku-Kikkawa) of strings.

In the functional formulation of gauge theories the gauge symmetry is a symmetry of the functional integration. The gauge parameters are the symmetry variables just as the angle variables are the symmetry parameters in the central potential problem in ordinary analytic mechanics. Gauge symmetry in this language means a freedom in choosing an appropriate body fixed coordinate system in a specific reparametrization of the coordinates. A different body fixed coordinate system corresponds to a different gauge fixing. The change of gauge in this view is the change of variables from one body fixed coordinate system to the other.

In string theory after fixing the original reparametrization invariance by picking the conformal gauge, the Lagrangian is invariant under the more restrictive residual transformation [13]

$$\sigma^+ \rightarrow \tilde{\sigma}^+(\sigma^+), \qquad \sigma^- \rightarrow \tilde{\sigma}^-(\sigma^-). \qquad 1.12$$

These are some of the diffeomorphisms, those which preserve the angle and are thus conformal transformations. Conformal symmetry is immediately related to the interesting problem of the elimination of the unphysical degrees of freedom [19]. However, the mechanism which is responsible for the decoupling of the unphysical degrees of freedom and the role of the conformal symmetry in it is not deeply understood. On the other hand, fixing the light-cone gauge, as we discussed above, the gauge conditions alone make complete use of the invariance of the Lagrangian. The elimination of the unphysical degrees of freedom, in this case, can be done by integrating over the corresponding variables, in a similar way as in the original Gervais-Sakita paper. Changing gauge from the conformal to light-cone is one way to understand the role of the conformal symmetry in the elimination of the unphysical states in the theory of strings, since choosing the light-cone gauge leads to the Hilbert space with only the physical degrees of freedom (transverse components) of the string.

The change of gauge in string theory can be viewed, in a way analogous to the gauge theories, as a change of variables in the functional integral. We are examining these ideas in Polyakov's theory. The material to be covered is arranged as follows:

In chapter 2, we perform the light-cone gauge fixing in Polyakov's functional integral of the bosonic string. We show by explicit calculations how Polyakov's theory reduces to Mandelstam's picture, apart from a ϕ-dependent factor, after imposing Gervais-Sakita light-cone conditions.

In chapter 3, we discuss the regularization of the determinants and the calculation of the conformal anomaly in the light-cone gauge. Due to off-diagonal nature of our gauge, the calculation of the determinants differs from the usual (conformal gauge) case, and special care is needed in regularizing the functional integrals associated with them. Finally, we show that the conformal anomaly associated with this new gauge fixing cancels at dimensions $d = 26$.

In chapter 4, we study the change of gauge problem in string theory. In particular, by performing a coordinate transformation in the functional variables of Polyakov's expression, we change gauge from the light-cone to the conformal gauge. This study relates precisely the two gauges (light-cone and conformal) in string theory.

Subtle points related with the Faddeev- Popov ghosts are discussed in both the third and fourth chapters. Light-cone gauge ghosts are introduced in chapter 3 in order to express our off-diagonal Faddeev-Popov determinant, while changing the gauge, in chapter 4, we define the coordinate transformation needed for the ghost fields. In the chapters 2, 3 and 4 of this thesis we restrict ourselves to the genus zero case for simplicity. The generalization to the higher genus surfaces is discussed in chapter 5.

In particular, in the last chapter 5, it is shown that for the higher genus surfaces, the problem of changing gauge between the light-cone and conformal gauge in string theory is reduced to solving the Green's function equation for surfaces with more complicated topology. The case of torus is discussed in detail and it is shown how Mandelstam's picture is obtained from a gauge-fixed Polyakov's theory. The method used in this chapter, shows that the light-cone gauge fixing in Polyakov's functional formulation of strings can be done for the higher genus case, in a similar way as for the case of the sphere discussed in chapter 2.

In this work, we are confining the discussion to the bosonic string and the first quantized language. The degrees of freedom are the displacements of the string, not the operators that create or destroy strings. Since we are working in the light-cone gauge it is necessary to prove that our light-cone gauge fixing describes a Lorentz invariant theory. This question is not discussed in this thesis.

CHAPTER 2

Light-cone gauge in Polyakov's theory.

I. Introduction

The problem we would like to study in this chapter of the thesis is in the same spirit of that studied by Gervais and Sakita [4]. Namely, they formulated the interacting strings in the path-integral approach by fixing the light-cone gauge in the Nambu-Goto Lagrangian. They chose the gauge

$$X^{+} = f(\sigma), \qquad 2.1$$

where σ is a collective coordinate in the two-dimensional surface generated by the successive positions of the string, i.e., $\sigma = (\tau, \sigma)$ and

$$(\dot{X} - X')^2 = 0. \qquad 2.2$$

Since the Nambu-Goto Action is given by

$$I = i \int_{\Sigma} d\sigma d\tau \sqrt{h} \qquad 2.3$$

where

$$h_{ab} = \partial_a X^{\mu} \partial_b X_{\mu}, \qquad 2.4$$

the condition (2.2) corresponds to

$$h_{--} = 0 \qquad 2.5$$

in the light-cone basis.

In what follows, we formulate the interacting string by fixing the light-cone gauge in Polyakov's theory. Our choice of gauge is

$$X^{+} = f\ (\sigma) \qquad 2.6$$

and

$$g_{--} = 0. \qquad 2.7$$

The second condition (2.7) corresponds to (2.5), because for Polyakov's action

$$S = \int d^{2}\sigma \sqrt{g}\ g^{ab}\ \partial_{a} X^{\mu} \partial_{b} X_{\mu}$$

$g^{ab} = h^{ab}$ is a solution of the classical equation of motion.

Choosing X^{+} to be a function of (σ,τ) is crucial for the inclusion of the interacting string. As we shall see later the function $f\ (\sigma)$ satisfies the homogeneous Laplace equation except from the points of interaction. These are the a finite number of singular points in the world sheet, where the light-cone time X^{+} becomes infinite. However, the function X^{-} will be τ after performing Mandelstam's conformal transformation.

The sections of this chapter are arranged as follows: In section II, we perform the gauge fixing using the Faddeev-Popov procedure and derive the Virasoro-Shapiro amplitude from the light-cone gauge fixed path-integral expression.

In section III, we show by explicit calculations how Polyakov's theory is reduced to Mandelstam's picture, apart from a ϕ - dependent factor, after imposing the above gauge conditions. Finally, we conclude with some remarks.

II. Light-cone formalism and derivation of Virasoro-Shapiro amplitude.

We start from Polyakov's path-integral formulation of Virasoro amplitude [7], [20], given as follows:

$$\prod_j \int \cdots \int d^2\sigma_j \sqrt{-g(\sigma_j)}\, DX^{\mu} Dg_{ab} \exp\left\{ \frac{i}{2}\int d^2\sigma \sqrt{-g}\, g^{ab}\, \partial_a X^{\mu} \partial_b X_{\mu} - i\int d^2\sigma\, j^{\mu}(\sigma) X_{\mu}(\sigma) \right\}. \qquad 2.8$$

where $\sigma = (\sigma^0, \sigma^1)$ are coordinates in the 2-dimensional Minkowski parameter space, which take values on the whole complex plane. Note here that we use the 0, 1 indices in the σ coordinates for convenience, but (σ^0, σ^1) is the same as the (τ, σ) of our previous notation; from now on we are going to interchange freely between these two notations, without distinction. $X^{\mu}(\sigma)$ defines a map from the 2-dimensional parameter space to a surface in the d-dimensional physical space-time. The metric g_{ab} characterizes the parameter space. The reparametrization invariant functional integration measure is defined by:

$$DX^{\mu} = \prod_{\sigma} dX^{+}(\sigma)\, dX^{-}(\sigma)\, DX^{i}(\sigma)$$

$$Dg_{ab} = \prod_{\sigma} dg_{++}\, dg_{+-}\, dg_{--}\, (-g)^{-\frac{3}{2}}. \qquad 2.9$$

$\prod_{\sigma}$ is the reparametrization invariant infinite product such that for a given scalar density $\rho(\sigma)$, i.e., $\rho'(\sigma') = \rho(\sigma)$ for a reparametrization $\sigma \to \sigma' = \Omega(\sigma)$,

$\prod_\sigma \rho(\sigma)$ is invariant: namely,

$$\prod_\sigma \rho'(\sigma) = \prod_\sigma \rho(\Omega^{-1}(\sigma)) = \prod_{\Omega^{-1}(\sigma)} \rho(\Omega^{-1}(\sigma)) = \prod_\sigma \rho(\sigma). \qquad 2.10$$

We define the δ -functional by

$$\prod_\sigma \delta(A(\sigma)) = \int \prod_\sigma d\Psi(\sigma) \exp\left\{ i \int d^2\sigma \sqrt{-g}\, A\, \Psi \right\} \qquad 2.11$$

so that

$$\int \prod_\sigma dA(\sigma) \prod_\sigma \delta(A(\sigma)) = 1. \qquad 2.12$$

$j^\mu(\sigma)$ is an external source term defined by

$$j^\mu(\sigma) = \sum_i \delta^2(\sigma - \sigma_i) k_i^\mu. \qquad 2.13$$

The substitution of (2.13) into (2.8) gives the following expression for the source term:

$$\exp\left\{ i \int d^2\sigma\, j^\mu(\sigma) X_\mu(\sigma) \right\} =$$

$$\exp\left\{ i \sum_i k_i^\mu X_\mu(\sigma_i) \right\} = \prod_i \exp\left\{ i k_i^\mu X_\mu(\sigma_i) \right\}. \qquad 2.14$$

We use Minkowski metric for the parameter space, for reasons which will be clear later. Keeping the imaginary unit (i) in the exponent assures the appearance of δ -functions in our later calculations.

In writing the integral representation of Virasoro amplitude (2.8), we should factor out the Mobius volume and the reparametrization volume (diffeomorphism). The integrand is invariant under 2-dimensional general coordinate transformations, after the factor $\sqrt{-g(\sigma_i)}$ assures the invariance of the measure. The invariance of the source term is assured because of the integration over the Koba-Nielsen variables (for an explicit discussion of this point see below the relation (2.24)). The Mobius transformation is defined by

$$z \rightarrow z' = z^g = \frac{az + b}{cz + d} \qquad 2.15$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

Here a, b, c, d are complex parameters and z, defined by $z = \sigma^0 + \sigma^1$, is the variable on the complex plane. These transformations form the SL(2,C) group. For a proof of the invariance of Virasoro-Shapiro amplitude under Mobius transformations see ref. [12]. Although we do not specify it, the division by these volumes should be understood. The momentum conservation factor $(2\pi)^d \delta^{(d)}(\sum_i k_i)$, which multiplies the integral, is hidden in the integration over the zero modes of the functional integration variables $X^\mu(\sigma)$. In the path-integral expression (2.8), we consider only the covariant Brink, Di Vecchia and Howe Lagrangian [6], [7]. In Polyakov's theory, however, the inclusion of the cosmological term is necessary for the renormalization, which will be discussed at the end of the third chapter.

We choose the light-cone gauge conditions as in ref. 4 :

$$X^+ = f(\sigma)$$

and

$$g_{--} = 0. \qquad 2.16$$

Here $f(\sigma)$ is a function of σ^0 and σ^1. As we shall see later in (2.16), we can always add a constant factor C to $f(\sigma)$. This means that the gauge fixing does not include the zero mode of the variable X^+.

We fix the gauge using the invariance of the action under general coordinate transformations in 2-dimensional parameter space:

$$\sigma^a \rightarrow \sigma^{(\epsilon)a}(\sigma) = \sigma^a + \epsilon^a(\sigma). \qquad 2.17$$

Under infinitesimal reparametrization transformations the variables X^μ and g_{ab} transform as

$$X^\mu \rightarrow X^{(\epsilon)\mu}(\sigma) = X^\mu(\sigma) - \epsilon^a \partial_a X^\mu(\sigma)$$

$$g_{ab} \rightarrow g_{ab}^{(\epsilon)} = g_{ab} - \epsilon^c \partial_c g_{ab} - \partial_a \epsilon^c g_{cb} - \partial_b \epsilon^c g_{ac}. \qquad 2.18$$

We change the integration variables dX^+ and dg_{--} in (2.8), to the two parameters of the transformations $d\epsilon^+$ and $d\epsilon^-$. We use the following light-cone basis:

$$\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^1)$$

$$g_{\pm\pm} = \frac{1}{2}(g_{11} + g_{00} \pm 2g_{01})$$

$$g_{+-} = \frac{1}{2}(g_{11} - g_{00}). \qquad 2.19$$

Following Faddeev-Popov procedure, we insert in the integral the identity

$$\int D\epsilon\, \Delta_f(X,g) \prod_{\sigma} \delta(X^{(\epsilon)+} - f(\sigma)) \prod_{\sigma} \delta(g^{(\epsilon)}_{--}) = 1. \qquad 2.20$$

where $D\epsilon$ is the invariant measure of the transformations (2.17) and $\Delta_f(X,g)$ is the Faddeev-Popov determinant given by

$$\Delta_f = \begin{vmatrix} -\partial_+ X^+ & -\partial_- X^+ \\ -(\partial_+ g_{--} + 2g_{+-}\partial_-) & -(2g_{--}\partial_- + \partial_- g_{--}) \end{vmatrix}. \qquad 2.21$$

Then we obtain

$$\prod \int \cdots \int d^2\sigma_i \sqrt{-g(\sigma_i)}\, DX^{\mu} Dg_{ab}\, D\epsilon\, \Delta_f\, \delta(X^+ - f)\, \delta(g_{--})$$

$$\exp\left\{ \frac{i}{2} \int d^2\sigma \sqrt{-g}\, g^{ab}\, \partial_a X^{\mu}\, \partial_b X^{\mu} - ik^{\mu} X_{\mu}(\sigma_i) \right\}. \qquad 2.22$$

Next, we interchange the order of integration in (2.22) and change the integration variables as follows

$$X^{\mu}(\sigma) \to X^{\mu(\epsilon^{-1})}(\sigma), \qquad g_{ab}(\sigma) \to g_{ab}^{(\epsilon^{-1})}(\sigma). \qquad 2.23$$

The action is invariant under (2.23). The invariance of the source term can be discussed as follows [21]: We make an inverse reparametrization transformation in X^{μ} variable. Then the source term becomes

$$\int d^2\sigma\, j.X^{(\epsilon)^{-1}}(\sigma) = \int d^2\sigma\, j.X(\sigma^{\epsilon}) = \int d^2\sigma\, j.X(\sigma). \qquad 2.24$$

The last equality holds because, due to the specific form of $j^\mu(\sigma)$ (contains δ -functions), the argument of the field X is integrated out, since the functional integral involves reparametrization invariant integration over the interaction points σ_i. Since the Faddeev-Popov determinant is invariant under the gauge transformations, the measure is also invariant under (2.23) and we obtain the expression

$$\prod_i \int \cdots \int D\epsilon\, d^2\sigma_i \sqrt{-g(\sigma_i)} DX^\mu Dg_{ab}\, \Delta_F\, \delta(X^+ - f)\, \delta(g_{--})$$
$$\exp\left\{ \frac{i}{2}\int d^2\sigma \sqrt{-g}\, g^{ab}\, \partial_a X^\mu \partial_b X_\mu - ik_i^\mu X_\mu(\sigma_i) \right\}. \qquad 2.25$$

The last integral is independent of the parameters of the transformation and the integration over ϵ^+, ϵ^- is factored out as the volume of the reparametrization group.

Next, we separate the transverse and longitudinal components and we have

$$\prod_i \int \cdots \int d^2\sigma_i \sqrt{-g(\sigma_i)} \prod_\sigma dX^+ dX^- DX^i \prod_\sigma dg_{++} dg_{+-} dg_{--} (-g)^{-3/2} \Delta_F(X, g)$$
$$\prod_\sigma \delta(X^+ - f(\sigma)) \prod_\sigma \delta(g_{--}) \exp\left\{ -\frac{i}{2}\int d^2\sigma \sqrt{-g}\, g^{ab}\, \partial_a X^i \partial_b X^i + i\int d^2\sigma\, j_i X^i \right.$$
$$\left. + i\int d^2\sigma \sqrt{-g}\, g^{ab}\, \partial_a X^+ \partial_b X^- - i\int d^2\sigma\, j^+ X^- - i\int d^2\sigma\, j^- X^+ \right\} \qquad 2.26$$

The integrations over X^+ and g_{--} are trivially carried out. Next, we choose $f(\sigma)$ such that it satisfy the following equation

$$2\partial_+\partial_- f(\sigma) = -j^+(\sigma) \qquad 2.27$$

where $j^+(\sigma)$ is given as in (2.13) by

$$j^+(\sigma) = \sum_j \delta(\sigma' - \sigma'_j)\, \delta(\sigma^0 - \sigma^0_j)\, k_j^+ . \qquad 2.28$$

The purpose of the choice (2.27) is to cancel the linear terms in the action of (2.26). Then (2.26) becomes

$$\prod_j \int \cdots \int d^2\sigma_j \sqrt{-g(\sigma_j)} \prod_\sigma dX^- DX^i \prod_\sigma dg_{++} dg_{+-} (-g)^{-1/2} \Delta_F$$

$$\exp\Big\{ -\frac{1}{2}\int d^2\sigma [\sqrt{-g}\, g^{--}(\partial_- X^i)^2 + 2\partial_+ X^i \partial_- X^i] + ik_j^i X^i(\sigma_j) +$$

$$i\int d^2\sigma \sqrt{-g}\, g^{--} \partial_- f\, \partial_- X^- - i\sum_j k_j^- f(\sigma_j) \Big\} \qquad 2.29$$

where we used $\sqrt{-g}\, g^{+-} = 1$, since in our choice of gauge $\sqrt{-g} = g_{+-}$. The last expression (2.29) defines the light-cone gauge fixed Polyakov's theory of strings. The gauge fixing determinant Δ_f, after substituting the gauge conditions in, is given by

$$\Delta_f = \begin{vmatrix} -\partial_+ f & -\partial_- f \\ -2g_{+-}\partial_- & 0 \end{vmatrix} .$$

The function $f(\sigma)$ is the solution of (2.27) and because of (2.28) it depends on the external momentum k^+. It is given by

$$f(\sigma) = \sum_i k_i^+ \Delta(\sigma, \sigma_i) , \qquad 2.30$$

where $\Delta(\sigma, \sigma')$ is the Green's function of the Laplacian. If the points σ and σ_i do not coincide it is given by

$$f(\sigma) = \sum_i k_i^{+} \ln | \sigma - \sigma_i | . \qquad 2.31$$

However, when σ equals σ_i, the expression (2.31) becomes singular. So the last factor of (2.29) contains an infinity. We extract, from the summation of this factor, the singular term which is given by

$$-ik_i^{-}k_i^{+} \Delta(\sigma_i, \sigma_i) . \qquad 2.32$$

Next, the integration over the variable X^- is done as follows: We multiply the measure dX^- by the factor $(-\partial_- f \partial_-)$ and the part of (2.29) which involves the integration over X^- becomes

$$\frac{1}{\det(-\partial_- f \partial_-)} \int d(-\partial_- f \partial_- X^-)(-g)^{-1/2} \ldots$$
$$\exp\left\{ i \int d^2\sigma \sqrt{-g}\, g^{--} \partial_- f \partial_- X^- + \cdots \right\}. \qquad 2.33$$

Then performing the integration we obtain the expression

$$[\det(-\partial_- f \partial_-)]^{-1} \delta(g^{--}) , \qquad 2.34$$

where we used the definition (2.11). Because of this δ function the integration over g_{++} picks only the $g_{++} = 0$ sector and cancels the g^{--} - dependence of the exponent of (2.29). Notice that the relation between the upper and the lower indices of the metric is

$$\sqrt{g}\;\; g^{ab} = (-1)^{ab} \frac{g_{\bar{a}\bar{b}}}{\sqrt{g}}$$

where $\bar{a}$, $\bar{b}$ are defined to be + or - whenever a, b are - or + respectively. Thus $g^{--} = 0$ corresponds to $g_{++} = 0$. It is important to notice here that in a natural way in our light-cone gauge we have recovered the second of the conformal gauge conditions, which we did not impose in this gauge. This can be understood as the existence of some sector of coincidence between the light-cone and the conformal gauge. The net result of the integrations over X^- and g_{++} is the following determinant factor

$$[\det(-\partial_-^{\cdot} f \; \partial_-)]^{-1} . \qquad 2.35$$

which appears in the denominator since it results from the integration over two bosonic fields. Then the original expression takes the form

$$\prod_j \int \cdots \int d^2\sigma_j \sqrt{-g(\sigma_j)} \prod_\sigma dg_{+-} DX^i \frac{\Delta_F}{\det(-\partial_- f \; \partial_-)}$$

$$\exp\left\{ -i \int d^2\sigma \partial_+ X^i \partial_- X^i + i \sum_j k_j^i X^i(\sigma_j) - i {\sum_j}' k_j^- f(\sigma_j) \right\} . \qquad 2.36$$

Here the prime in the summation of the last factor, means that we have excluded the singular term defined by (2.32). The last expression (2.36) defines the Virasoro-Shapiro amplitude in the light-cone gauge. In order to recover the Virasoro-Shapiro amplitude from (2.36), we have to perform the integration over X^i variables. For this purpose we complete the square between the first and second terms of the exponent. The relevant part can be expressed as

$$\int DX^i \exp[\, i \int d^2\sigma\, (\, X^i D X^i \; + \; j^i X^i \,)\,] \,. \qquad 2.37$$

where $D = \partial_+\partial_-$. In the last expression we have changed the notation of the momentum in the source term back to the original j^μ. Then the exponent of (2.37) can be written as follows

$$i \int d^2\sigma (X^i D X^i \; + \; \sqrt{D} X^i \frac{1}{\sqrt{D}} j^i \; + \; j^i \frac{1}{D} j^i \,) - j^i \frac{1}{D} j^i \;. \qquad 2.38$$

The term in the parenthesis of (2.38) is a complete square, while the extra term is formally written as

$$-i \int d^2\sigma\, j^i \frac{1}{D} j^i \;. \qquad 2.39$$

$\frac{1}{D}$ is the propagator given by the Green's function $\Delta(\sigma-\sigma')$ and (2.39) takes the form

$$-i \int d^2\sigma d^2\sigma' \sum_{j,k} k_j^i \; \delta^2(\sigma-\sigma_j\,)\, \Delta(\sigma-\sigma')\, k_k^i \; \delta^2(\sigma'-\sigma'_k\,) =$$

$$= -i \; \sum_{j,k} k_j^i \; \Delta(\sigma_j - \sigma_k\,)\, k_k^i \;. \qquad 2.40$$

The last expression gives the extra term, which we need to add and subtract in the action in order to complete the square. This becomes singular when $\sigma_j = \sigma_k$. Therefore, there is also a singular term in the summation over the transverse components in the second factor in the exponent of (2.36), which is given by (2.40) for j = k. We combine the j = k term of (2.40) with that of (2.32) and obtain the following expression for the final singular factor contained in the exponent of the path-integral (2.29)

$$-ik_i^{\mu}k_{i\mu}\Delta(\sigma_i,\sigma_i) \qquad 2.41$$

By a conformally-invariant regularization (see regularization procedure in the following chapter) the finite part of (2.41) is proportional to $\phi(\sigma_i)$. Then expressing the factor $\sqrt{-g(\sigma_i)} = e^{\phi(\sigma_i)}$, the $\phi(\sigma_i)$-dependence cancels provided that the external lines are on shell, i.e. $k_i^{\mu}k_{i\mu} = -2$.

Then, after the integration over X', the exponential of (2.36), apart from a determinant factor, becomes

$$\exp\left\{-\sum_{i\neq j} k_i k_j \ln|z_i - z_j|\right\} = \prod_{i\neq j} |z_i - z_j|^{-k_i k_j} . \qquad 2.42$$

where we have used (2.31). The expression (2.42) is essentially the Virasoro-Shapiro amplitude, apart from the integration over Koba-Nielsen variables. In the next section, we explicitly obtain Mandelstam's picture from the light-cone gauge fixed Polyakov's path-integral.

III. Derivation of Mandelstam's picture from the light-cone gauge fixed Polyakov's theory.

We start again from the expression (2.36). In order to obtain the path-integral in Euclidean metric and get rid of the i in the exponent, we perform the following Wick rotation:

$$\sigma^0 = i\,\xi^2, \qquad \sigma^1 = \xi^1 \tag{2.43}$$

and we have

$$\prod \int \cdots \int d^2 z_j \prod_\sigma dg_{+-} \prod_\sigma DX^i \frac{\Delta_F}{\det(-\partial_{\bar z}\, f\, \partial_{\bar z})}$$

$$\exp\left\{ -\frac{1}{2}\int d^2\xi\, (\nabla_\xi X^i)^2 + i\sum_{j} k_j^i X^i(\xi_j) - \sum_{j} k_j^- \, \mathrm{Re}F(z_j) \right\} \tag{2.44}$$

where

$$F(z) = \frac{1}{\pi}\sum_i k_i^+ \ln(z - z_i),$$

is the analytic function whose the Real part is given by (2.31).

$$DX = \prod_z dX^i(z)$$

where z stands for z and $\bar z$ defined as $z = \xi^1 + i\,\xi^2$ and $\bar z = \xi^1 - i\,\xi^2$. Note that any g-dependence in the transverse part of our measure is included in our definition of the reparametrization invariant product. In the last expression the variables ξ and z are defined on the entire complex plane. In the light-cone picture of strings the integration variables are defined on the Mandelstam's plane. In order for the variables to be defined on Mandelstam's plane, we make the following conformal transformation

$$z \rightarrow w = F(z) \equiv \tau + i\,\sigma. \tag{2.45}$$

This defines a map from the entire plane to the Mandelstam's "tube" (see fig. 3). Under the transformation (2.45) the points z_i transform into the strings (i) at initial or final times $\tau_i = \pm\infty$, depending on the values of k_i^+: $k_i^+ < 0$ for outgoing and $k_i^+ > 0$ for incoming strings.

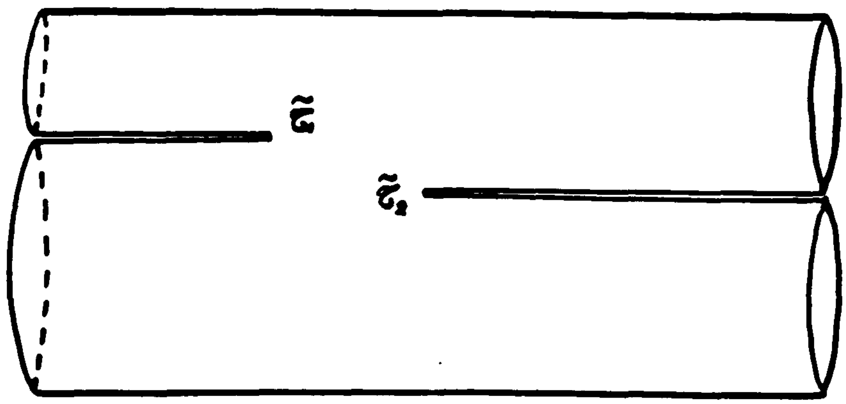


Fig. 3

Fig. 3: Mandelstam's "tube".

After the transformation (2.45) the last factor of (2.44) becomes

$$\exp\left\{ - \sum_i k_i^- \tau_i \right\}. \qquad 2.46$$

Changing z_i reflects a change of the interaction times $\tilde{\tau}_i$, which are defined by

$$\tilde{\tau}_i = f(\tilde{z}_i), \quad \text{where} \quad \frac{\partial f(z)}{\partial z}\Big|_{z = \tilde{z}_i} = 0 . \qquad 2.47$$

We must perform the same transformation (2.45) on the functional integration variables X', such that they be defined on Mandelstam's tube. Then we change the Koba-Nielsen variables z_i to the variables $\tilde{\tau}_i$. The Jacobian of the last transformation times the Jacobian due to the conformal transformation on the functional integration variables X' has been calculated by Mandelstam [8]. It has been shown that it gives a constant at dimensions d=26. Then the integral (2.44) becomes

$$\Pi \int \cdots \int d\tilde{\tau}_i \prod_{\sigma} DX' \, dg_{+-} \frac{\Delta_f}{\det(-\partial_{\bar{w}} f \, \partial_{\bar{w}})}$$

$$\exp \left\{ -\frac{1}{2} \int_D d^2 w \, (\nabla_w X')^2 + ik' . X'(\tau_i) - k_i^- \tau_i \right\} \qquad 2.48$$

where $DX' = \prod_w dX'(w)$.

Now all functional integration variables and determinant operators are defined on Mandelstam's "tube". In the last expression, we denote this by explicitly writing the dependence of the operators on the new variable w, which through (2.45) defines the light-cone plane. The integral (2.48), except from the determinant factor, which depends on g_{+-}, coincides with Mandelstam's picture of strings. The g_{+-}-dependence of the integral defines the conformal anomaly. Conformal anomaly is a result of the non-invariance of the measure under scaling of the metric (Weyl transformation) [10]. Since the change in the ϕ-dependent factor under the conformal transformation (2.45) is a constant (ϕ-independent) [21] [see (3.4) below for the definition of ϕ], the physical result does not change if we calculate the conformal anomaly after performing Mandelstam's transformation. In the next chapter we compute the anomaly factor defined on Mandelstam's tube and we show that it cancels at dimensions d=26.

IV. Conclusion

In this chapter, we have formulated the interacting strings by fixing the light-cone gauge in Polyakov's theory. It has been shown that fixing the light-cone gauge in Polyakov's theory can be done in an equivalent way as fixing the conformal gauge. One can start from the non-gauge-fixed Polyakov's formulation of strings and choose one or the other gauge, depending on the problem one wishes to study. Mandelstam's picture is obtained from the light-cone gauge fixed Polyakov's theory.

The main point we would like to emphasize is that the light-cone and conformal are two independent, equivalent gauges. One is not included in the other and light-cone conditions can be fixed independently of the conformal ones. Obtaining one gauge from the other can be done by a proper coordinate transformation.

Notice that to obtain Mandelstam's picture from Polyakov's theory, the specific choice of gauge seems crucial. The function $f(\sigma)$ is simply τ for the non-interacting strings case, while here, it being the solution of the equation

$$\nabla^2 f(\sigma) = -i j^+(\sigma), \qquad 2.49$$

it depends on the external momenta. After Mandelstam's conformal transformation, however, $f(\sigma)$ becomes τ.

CHAPTER 3

Regularization of the determinants and Conformal Anomaly in the light-cone gauge.

I. Introduction

In the path-integral approach, anomalies are now well understood as due to the non-invariance of the measure under a specific symmetry of the action. In Polyakov's theory of the bosonic string the action is invariant under general coordinate transformations and rescalings of the metric (Weyl symmetry) in two dimensions. As we already discussed earlier, in this case one can not write down a measure which possesses both invariances of the theory. The best one can do is write a reparametrization invariant measure in which case there will be an anomaly due to the non-invariance of it under the Weyl symmetry. Fujikawa recently proved that, equivalently, one can choose to work with a Weyl invariant measure. In this last case the anomaly appears as a result of the non-invariance of the measure under the general coordinate transformation. In our case the measure (2.9) is invariant under reparametrization transformations. Therefore, there is an anomaly hidden in the g_{+-} -dependence of the measure and the determinant factor, due to the non-invariance of (2.9) under the scaling transformation of the metric

$$g_{ab} \rightarrow g'_{ab} e^{\phi(z)}. \qquad 3.1$$

The conformal anomaly is defined as the ϕ -dependence which in our case is hidden in the determinant factor

$$\frac{\Delta_f}{\det(-\partial_{\bar{z}}\, f\, \partial_{\bar{z}})}. \qquad 3.2$$

where Δ_f after the gauge fixing is given by

$$\Delta_f = \begin{vmatrix} -\partial_+ f & -\partial_- f \\ -g_{+-}\partial_- & 0 \end{vmatrix}. \qquad 3.3$$

and also in the scalar Laplacian operator ∇_ϕ of the integral expression (2.48). The determinants involved are now defined on Mandelstam's "w-tube". We define

$$g_{+-}(w) = e^{\phi(w)} \qquad 3.4$$

Under the conformal transformation (2.45), the function $f(\sigma)$ transforms into τ, in the "w-tube". Therefore, $\partial_- f$ and $\partial_+ f$ become simply constants. We use the notation χ_- and χ_+ for the constant fields in which $\partial_- f$ and $\partial_+ f$ respectively transform under (2.45). Then (3.2) is expressed in terms of χ_+ and χ_- as

$$\frac{\det \begin{vmatrix} -\chi_+ & -\chi_- \\ -g_{+-}\partial_- & 0 \end{vmatrix}}{\det(-\chi_-\partial_-)}. \qquad 3.5$$

In order to compute the determinants of the infinite-by-infinite non-diagonal matrices of the last expression, one needs to regularize the functional intergations associated with them. The regularization procedure extracts the ϕ-dependence.

There are a few points we would like to emphasize about the nature of those determinants. Firstly, these are infinite-by-infinite matrices, since the operators depend on σ which is a continuous variable labeling all the points of the string. The Faddeev-Popov determinant, which appears in the numerator of (3.5), is not diagonal and since the operators of the upper right corner and the lower left corner operate in different spaces (acting on vector space they give scalar and two-tensor respectively), the multiplication of those two operators is not allowed. Each operator is a big matrix with different in general dimensionality. One would maybe think that since the fields $\chi_{\pm}$ are constants they do not contribute in the calculations of the determinants and therefore we can ignore them, obtaining all contribution only from the operator $(-g_{+-}\partial_{-})$. However, as it will be apparent later, we must keep the fields $\chi_{\pm}$ until the end of the calculations and in fact they play an essential role in this computation.

Secondly, there is also a determinant in the denominator of the expression (3.5). This results from the integration over two bosonic fields. Since we are working in the light-cone gauge the expectation is that this last determinant corresponds to the two reduced dimensions, going from 26 (space-time dimensions of the bosonic string) to the 24 (light-cone dimensions of space-time). However, the situation is not exactly such. Due to the mixing of the X-variables with the g-variables in our non-diagonal gauge, our Faddeev-Popov determinant does not exactly correspond to the Faddeev-Popov determinant of the conformal gauge. Therefore, if we independently try to regularize the numerator and denominator we shall not obtain the anomaly factor in the d-power from the regularization of the numerator and to the (- 2) power from the regularization of the denominator. For consistent regularization we need to regularize the whole expression in such a way that the infinities between the numerator and denominator will cancel. Another reason that this is crucial is because of the nature of the infinities involved. Even if both numerator and denominator are infinite-by-infinite matrices, the numerator is formally a 2x2 matrix with complicated operators, while the denominator consists of one operator. Naively we can say that there are "two infinities" in the numerator and "one infinity" in the denominator. An independent regularization of the numerator and denominator naturally can not lead to the cancellation of those infinities unambiguously. Complications of such kind do not appear in the conformal gauge. In that case, the determinant factor consists only of the Faddeev-Popov Jacobian, which is diagonal and the operators are of the same nature, namely, they both operate on vectors and give two-tensors.

Another point we like to emphasize is that these operators (and this is so even for the conformal gauge) are of the nature of the chiral Dirac operator, i.e. they map a space V_1 to a different space V_2 [22].

$$A : V_1 \longrightarrow V_2 \qquad 3.6$$

There is no meaning of the determinant of such an operator. One way to analyze these operators is to deduct the operator which sends the space back to itself. In other words, we must compute the adjoint A^+ :

$$A^+ : V_2 \longrightarrow V_1 \qquad 3.7$$

Thus we have to make sense out of objects like

$$\det (A^+ A) \qquad 3.8$$

and finally extract square roots in order to obtain $\det A$. This last discussion makes obvious the need of a metric.

In general, in order for someone to be able to calculate the adjoint operator one needs to define an inner product. However, there is a problem in this gauge. In our case, because of the off-diagonal nature of the gauge, we can not introduce the usual g_{zz} metric as in the conformal (diagonal) gauge case. Thus, the inner product defined in the case of the conformal gauge is not suitable for our calculations. That this is an effect due entirely to the non-diagonal gauge choice can be seen as follows: Since our gauge conditions mix the space of the X-variables with the space of the g-metric, the reparametrization invariance is not maintained in the course of our calculations in chapter 2. One can check this by looking for instance at the expression (2.29). The functional integrations performed are such that they break reparametrization invariance, while in the ordinary conformal gauge the reparametrization invariance is respected throughout the calculations and the regularization procedure is such as to preserve the reparametrization invariance. Therefore we need to find a principle in our gauge in order to define inner product and perform the regularization. It is necessary to uncover some hidden symmetry which is going to be preserved in our regularization and play a role analogous to the reparametrization symmetry of the conformal gauge case.

The organization of this chapter is as follows: In section II, we show how we use the BRST-invariance to recover a principle for the regularization procedure.

In section III, we define consistent inner products and compute the adjoints of the determinant operators.

In section IV, the calculation of the Faddeev-Popov determinant and the determinant factor, appearing after performing the integration over X^- and g_{+-} variables, on Mandelstam's plane is given. The method used for the regularization is the Pauli-Villars regularization procedure [23]. Finally, it is shown that the conformal anomaly cancels at the critical dimensions (d = 26).

II. Use of BRST-invariance to recover a regularization principle.

The BRST-invariance of Polyakov's action can be proven through its invariance under the general coordinate transformation. The change of the Lagrangian (1.6) under the reparametrization transformation is a total divergence, which leads to an invariant action. The BRST-transformation [14], [15] is defined by replacing the parameter $\epsilon^a(\sigma)$ of the general coordinate transformation by $\lambda c^a(\sigma)$, where $c^a(\sigma)$ is the Faddeev-Popov anticommuting ghost field and λ is an anticommuting c-number parameter. It is given as follows

$$\delta X^{\mu} = -\lambda c^{a}\,\partial_a X^{\mu}$$

$$\delta g^{ab} = -\lambda\partial_c\,(c^c g^{ab}) + \lambda\partial_c c^a g^{cb} + \lambda\partial_c c^b g^{ac} \,. \qquad 3.9$$

The BRST-transformation for the ghost field is determined by the requirement that the BRST-transformation be nilpotent. It is given by

$$\delta c^{a} = -\lambda c^{b}\,\partial_b c^{a} \,. \qquad 3.10$$

One can then show, by directly applying the transformations (3.9) and (3.10), that the total action (the conformal gauge fixed Polyakov's action incorporating the Faddeev-Popov ghost action) exhibits BRST-symmetry. In Kato-Ogawa approach the total Lagrangian (the conformal gauge fixed plus the ghost Lagrangian) was obtained using the BRST-invariance as the guiding principle. The Faddeev-Popov ghosts (antighosts) were introduced in their method by the Kugo-Uehara gauge fixing procedure [24].

According to this procedure one adds to the original Lagrangian a BRST-invariant term which can be used as the gauge fixing and the FP-ghost Lagrangian. It is defined by

$$L_{GF+FP} = -i\,\delta(\bar{c}_a F^a)\,. \qquad 3.11$$

δ stands for the BRST-transformation, which by definition is nilpotent assuring thus the BRST-invariance of L_{GF+FP}. F^a is an arbitrary function of the fields with zero ghost number. It can be taken to be the gauge conditions. $\bar{c}_a$ are antighost variables with transformation property under BRST given by

$$\delta\bar{c}_a = i\,\lambda B_a\,, \qquad 3.12$$

where B_a are auxiliary fields. They transform as follows

$$\delta B_a = 0\,. \qquad 3.13$$

We first notice that our gauge fixing of chapter 2 can be written in a form of the Kugo-Uehara gauge fixing procedure. The Faddeev-Popov determinant Δ_f appeared as the Jacobian of the transformation of dX^+ and dg_{--} to $d\epsilon^+$ and $d\epsilon^-$. Thinking of it as a 2x2 matrix, the two components of the determinant operator do not have the same "signature". They both operate on vectors and give a scalar or two-tensor field respectively. In terms of ghost and antighost fields, it can be written

$$\Delta_f =$$

$$\int d\tilde{h}\,d\tilde{h}^{--}dc^+dc^-\exp\left\{-i\int d^2\sigma\sqrt{-g}\,(\tilde{h},\tilde{h}^{--})\begin{pmatrix}-\chi_+ & -\lambda_- \\ -2g_{+-}\partial_- & 0\end{pmatrix}\begin{pmatrix}c^+ \\ c^-\end{pmatrix}\right\} =$$

$$\int d\tilde{h}\,d\tilde{h}^{--}dc^+dc^-\exp\left\{-i\int d^2\sigma\sqrt{-g}\left(\tilde{h}\,\chi_+c^+ + \tilde{h}\,\lambda_-c^- + 2\tilde{h}^{--}g_{+-}\partial_{+-}\partial_-c^+\right)\right\}.$$

$$3.14$$

where we have introduced two antighost fields $\tilde{b}$, $\tilde{b}^{--}$ scalar and two-tensor respectively. The peculiar appearance of the antighost fields is due to our non-diagonal choice of gauge. There is a scalar antighost $\tilde{b}$ corresponding to the gauge condition which fixes the field X^+ and a two-tensor $\tilde{b}^{--}$ which results from the fixing of the field g_{--}. c^a are the two vector ghost fields corresponding to the two parameters of the general coordinate transformations ϵ^a, as in the conformal gauge case and B, B^{--} are the auxiliary fields, defined by (3.12). The BRST-transformation of the last anticommuting fields is given by

$$\delta c^a = -c^b \partial_b c^a , \qquad \delta \tilde{b} = iB ,$$
$$\delta \tilde{b}^{--} = iB^{--}, \qquad \delta B = 0. \qquad 3.15$$

The indices a and b are + or - in the light-cone notation. We write the BRST transformation for the variables X^+ and g_{--}

$$\delta X^+ = -c^a \partial_a X^+$$
$$\delta g_{--} = -c^c \partial_c g_{--} - 2g_{+-}\partial_- c^+ - 2g_{--}\partial_- c^- . \qquad 3.16$$

Next, using the Kugo-Uehara prescription described by the relations (3.11) through (3.13), we write the sum of the Faddeev-Popov and the gauge-fixing Lagrangian in our gauge. It is given by

$$L_{GF+FP} = -i\,\delta[\tilde{b}(X^+ - f(\sigma)) + \tilde{b}^{--} g_{--}] =$$
$$B(X^+ - f(\sigma)) + B^{--} g_{--} - i\tilde{b}c^a \partial_a X^+ - i\tilde{b}^{--}[(2\partial_- c^- - c^c \partial_c) g_{--} + 2g_{+-}\partial_- c^+] ,$$
3.17

where F^a are replaced by the gauge conditions $X^+ - f(\sigma)$ and g_{--}. This Lagrangian is BRST-invariant due to the nilpotency of the BRST transformations; δ here stands for the BRST variation. We can obtain the same BRST-invariant Lagrangian in our path-integral approach if in the very early stage of our calculations - expression (2.22) - we express the δ functions as functional intergations over the auxiliary fields B and the determinant Δ_f in terms of anticommuting variables:

$$\delta(X^+ - f(\sigma)) = \int dB \ \exp\left\{ i \int d^2\sigma \sqrt{-g}\, B\,(X^+ - f(\sigma)) \right\}$$

$$\delta(g_{--}) = \int dB^{--} \ \exp\left\{ i \int d^2\sigma \sqrt{-g}\, B^{--} g_{--} \right\}$$

$$\Delta_f = \int d\tilde{b}\, d\tilde{b}^{--} dc^+ dc^- \exp\left\{ i \int d^2\sigma \sqrt{-g}\, (\tilde{b}\ \tilde{b}^{--}) \right.$$

$$\left. \begin{pmatrix} -\partial_+ X^+ & -\partial_- X^+ \\ -(\partial_+ g_{--} + 2g_{+-}\partial_-) & -(2g_{--}\partial_- + \partial_- g_{--}) \end{pmatrix} \begin{pmatrix} c^+ \\ c^- \end{pmatrix} \right\}. \qquad 3.18$$

It is now easy to see that the Lagrangian appearing in the exponent of (3.18) coincides with L_{GF+FP} of (3.17). Therefore it is BRST invariant. Moreover, the last of the relations (3.18) takes the form given by (3.14), after we fix the gauge (performing the functional integrations) and substitute χ_+ and χ_- for $\partial_+ f$ and $\partial_- f$ respectively.

Now we observe that BRST-transformations and the BRST-invariant Lagrangian are c o v a r i a n t under the following transformations:

$$\sigma^+ \to \sigma'^{\epsilon +}(\sigma^+), \quad \sigma^- \to \sigma'^{\epsilon -}(\sigma^-). \qquad 3.19$$

These are the conformal transformations, since after the Wick rotation

$\sigma^+(\sigma^-)$ become essentially $z(\bar{z})$. So conformal transformations on integration variables induce the transformation property of χ_+ and χ_- as vector fields with the signature + or -, respectively. Then BRST-transformations and the BRST-invariant Lagrangian manifest conformal covariance. We use the conformal invariance as a guiding principle for the regularization of the determinants. We like to emphasize at this point the connection between BRST and conformal symmetry. The fact that the invariance which we preserve during the regularization of the determinants is the conformal symmetry makes sense in connection with the conformal anomaly. the Liouville action, which is the conformal anomaly factor, is invariant under conformal transformation. It is therefore natural in the regularization procedure, which extracts the conformal anomaly factor, to preserve the symmetry which is a symmetry of this same factor. In the next section we define inner products using the conformal symmetry as a guiding principle.

III. Definition of inner products and computation of the adjoints of the operators.

Following the procedure discussed in the introduction, we first compute the adjoints of the operators of the determinants. We define the conformal invariant scalar product of 2-tensor fields by:

$$(\Psi_{++}, X_{++}) = \int d\sigma^+ d\sigma^- g_{+-} \Psi^*_{++} X_{++} g^{+-2} . \qquad 3.20$$

where $\Psi^*_{++} = \Psi_{--}$, while the inner product of scalar and vector fields are defined by

$$(\Psi, X) = \int d\sigma^+ d\sigma^- g_{+-} \Psi^* X \qquad 3.21$$

and

$$(\Psi^{*},X^{*}) = \int d\sigma^{+} d\sigma^{-} g_{+-} \Psi^{*'} X^{*} g_{+-} \qquad 3.22$$

respectively. We use the metric g_{+-} to lower or raise indices, while the * operation means interchange between + and - indices. Then if $T = -\chi_{-}\partial_{-}$, the adjoint T^{+} is defined by the relation

$$(\delta g_{--}, T\delta X) = (T^{+}\delta g_{--}, \delta X). \qquad 3.23$$

Using the definition (3.20) we write the left hand side of (3.23) as follows

$$\begin{aligned}
(\delta g_{--}, T\delta X) &= \int d\sigma^{+} d\sigma^{-} g_{+-} \delta g_{--}^{*} (-\chi_{-}\partial_{-}) \delta X \; g^{+-'} \\
&= \int d\sigma^{+} d\sigma^{-} g^{+-} \delta g_{--}^{*} (-\chi_{-}\partial_{-}) \delta X = \\
&\int d\sigma^{+} d\sigma^{-} \partial_{-}(\chi_{-} g^{+-}) \delta g_{--}^{*} \; \delta X \\
&= \int d\sigma^{+} d\sigma^{-} g_{+-} g^{+-} [\partial_{+}(\chi_{+} g^{+-})]^{*} \delta g_{--} \; \delta X .
\end{aligned} \qquad 3.24$$

Comparing the last relation with (3.23) we obtain the expression for T^{+}

$$T^{+} = g^{+-}\partial_{+}\chi_{+} g^{+-} . \qquad 3.25$$

Next we compute the adjoint of the operator

$$Q = \begin{vmatrix} 0 & -\chi_{-} \\ -g_{+-}\partial_{-} & 0 \end{vmatrix} . \qquad 3.26$$

This is the relevant part of the Faddeev-Popov determinant given by (3.5).

The inner product consistent with (3.20) is defined by

$$(\Psi,\Psi) = \int d\sigma^+ d\sigma^- \Psi^T \; G \; \Psi, \qquad 3.27$$

where

$$\Psi = \begin{vmatrix} \delta X \\ \delta g_{--} \end{vmatrix}$$

and

$$G = \begin{vmatrix} g_{+-} & 0 \\ 0 & g^{+-} \end{vmatrix}.$$

The adjoint Q^+ satisfies the relation

$$(\Psi, Q\epsilon) = (Q^+\Psi, \epsilon), \qquad 3.28$$

where Ψ is given as above and ϵ stands for the two- component vector field

$$\epsilon = \begin{vmatrix} \epsilon^+ \\ \epsilon^- \end{vmatrix}.$$

We use (3.28) to compute the adjoint Q^+. The inner product for the vector field ϵ is defined by the relation

$$(\epsilon, \epsilon) = \int d\sigma^+ d\sigma^- \epsilon^T \; G \; \epsilon, \qquad 3.29$$

where now G is given by

$$G = \begin{vmatrix} g^2_{+-} & 0 \\ 0 & g^2_{+-} \end{vmatrix}.$$

Then

$$(\Psi, Q\epsilon) = \int d\sigma^+ d\sigma^- \Psi^T\, G\, (Q\epsilon) =$$

$$= \int d\sigma^+ d\sigma^- (\delta X, \delta g_{++}) \begin{vmatrix} g_{+-} & 0 \\ 0 & g^{+-} \end{vmatrix} \begin{vmatrix} 0 & -\chi_- \\ -g_{+-}\partial_- & 0 \end{vmatrix} \begin{vmatrix} \delta\epsilon^+ \\ \delta\epsilon^- \end{vmatrix}$$

$$= \int d\sigma^+ d\sigma^- (-2\partial_-\delta g_{++},\ -\chi_- g_{+-}\delta X) \begin{vmatrix} \delta\epsilon^+ \\ \delta\epsilon^- \end{vmatrix}.$$

which can be written as

$$(\Psi, Q\epsilon) =$$

$$\int d\sigma^+ d\sigma^- (-2g^{2}\ \partial_-\delta g_{++},\ -g^{+-}\chi_-\delta X) \begin{vmatrix} g^2_{+-} & 0 \\ 0 & g^2_{+-} \end{vmatrix} \begin{vmatrix} \delta\epsilon^+ \\ \delta\epsilon^- \end{vmatrix}. \qquad 3.30$$

On the other hand the right hand side of (3.28) is expressed as

$$(Q^+\Psi, \epsilon) =$$

$$\int d\sigma^+ d\sigma^- \left| (Q^+\Psi)_1, (Q^+\Psi)_2 \right| \begin{vmatrix} g^2_{+-} & 0 \\ 0 & g^2_{+-} \end{vmatrix} \begin{vmatrix} \delta\epsilon^+ \\ \delta\epsilon^- \end{vmatrix}. \qquad 3.31$$

Comparing the relations (3.30) and (3.31) we obtain the following result

$$(Q^+\Psi)_1 = -2g^{+-2}\partial_+\,\delta g_{--}$$

and

$$(Q^{+}\Psi)_2 = -g^{+-}\chi_+ \delta X$$

or written in a matrix form

$$Q^{+} = \begin{vmatrix} 0 & -g^{+-2}\partial_+ \\ -g^{+-}\chi_+ & 0 \end{vmatrix} . \tag{3.32}$$

Then the operators to be regularized are given by

$$Q^{+}Q = \begin{vmatrix} g^{+-2}\partial_+ g_{+-}\partial_- & 0 \\ 0 & g^{+-}h_{+-} \end{vmatrix} \tag{3.33}$$

and

$$T^{+}T = -g^{+-}\partial_+ h_{+-} g^{+-}\partial_- . \tag{3.34}$$

Here $h_{+-} (\equiv \chi_+\chi_-)$ is independent of $\sigma^{\pm}$, since χ_+ and χ_- are constant fields by definition. The above operators $Q^{+}Q$ and $T^{+}T$ acting on vector and scalar space respectively, are well defined. However, the regularization of these operators is necessary for a meaningful definition of the determinants. Since, in order to obtain the adjoints, we have introduced explicitly a metric, we can interpret the anomalies as the dependence of the operators on the metric. Regularization in this sense means to extract the g_{+-}-dependence of the determinants. In the next section we compute the determinants of the operators $Q^{+}Q$ and $T^{+}T$, as well as the determinant of the Laplacian operator. The determinants of the operators Q and T are then obtained by extracting square roots from the result of the $\det(Q^{+}Q)$ and $\det(T^{+}T)$.

IV. Regularization of the determinants and computation of the conformal anomaly.

There are several methods for the regularization procedure. The most commonly used is the Heat-Kernel method [7]. In this method, one has to cut-off the small time contribution, which corresponds to the zero modes of the determinants, since the determinant of the zero modes is zero. Then the dependence of the determinants on the metric is obtained by computing the effects for very small time. Despite the popularity of this method it lacks physical simplicity. An alternative method, in which the meaning of the regularization is more transparent, is the Pauli-Villars method.

We use the conformally invariant Pauli-Villars regularization method for the regularization of our determinants [21]. In this procedure the regularization of the operator ∇_ξ^2 is done as follows: A number of auxiliary Bose and Fermi fields, with masses M_i , are introduced in order to cut-off the large momentum contribution. The determinant is regularized to be the determinant due to all these fields:

$$\det \nabla_\xi^{2\,reg} \equiv \prod_i \left| \det (\nabla_\xi^2 - M_i^2 e^{\phi(\xi)}) \right|^{C_i} \qquad 3.35$$

where $C_i = \pm 1$ depending on the statistics. We define $C_0 = 1$ and $M_0 = 0$. We multiplied M_i^2 by $e^{\phi(\xi)}$ to keep conformal covariance. At the end of the calculation we set $M_i^2 \rightarrow \infty$ such that

$$\sum_{i=1} C_i = 0, \qquad \sum C_i M_i^2 = 0,$$

$$\sum_i C_i M_i^2 \ln M_i = \text{finite}, \qquad 3.36$$

to assure that (3.35) is formally equal to $\det \nabla_\xi^2$. The ϕ-dependence of the determinant is then computed by

$$\frac{\delta}{\delta\phi(\xi)} \ln \det \nabla_\xi^{2\,reg} = -\sum_i C_i\, M_i^{\,2}\, e^{\phi(\xi)} < \hat{\xi}\ |\ \frac{1}{\nabla_\xi^2 - M_i^{\,2} e^{\phi(\xi)}}\ |\ \hat{\xi} > . \qquad 3.37$$

Following the above prescription we first compute the variation of $\ln \det Q^+Q$ and $\ln \det T^+T$, where Q^+Q and T^+T are expressed as in (3.33) and (3.34). We write the operators as

$$Q^+Q = e^{-2\phi} h_{+-}\, e^{-\phi}\, \partial_+\, e^{\phi}\, \partial_-$$

$$= e^{-2\phi}\, h_{+-} [\, \partial_+\partial_- + (\, \partial_+\phi\,)\, \partial_-\,] \equiv A_1$$

and

$$T^+T = -e^{-\phi}\, h_{+-}\, \partial_+\, e^{-\phi}\, \partial_-$$

$$= -e^{-2\phi}\, h_{+-} [\, \partial_+\partial_- - (\, \partial_+\phi\,)\, \partial_-\,] \equiv A_2\, . \qquad 3.38$$

In writing (3.38) we multiplied the lower-right corner with the upper-left corner in the matrix (3.33), since the multiplied operator is diagonal (i.e., no derivatives) and of conformal weight zero. The last multiplication can be justified by the following argument: in matrix form the operator Q^+Q given by the relation (3.33), can be written as follows

$$Q^+Q = \begin{vmatrix} (g^{+-}h_{+-})^{-1} & 0 \\ 0 & g^{+-}h_{+-} \end{vmatrix} \begin{vmatrix} g^{+-}h_{+-}(\, g^{+-\prime}\partial_+ g_{+-}\partial_-) & 0 \\ 0 & 1 \end{vmatrix} .$$

Then

$$\det Q^+Q = \det 1 \det (\, g^{+-}h_{+-} g^{+-\prime} \partial_+ g_{+-} \partial_-\,), \qquad 3.39$$

since the lower right corner of the last matrix is just the identity operator. It is important to keep the conformal invariance in order for the multiplication to be consistent. The conformal invariance of the operators of (3.33) and (3.34) justifies the presence of the field h_{+-}. h_{+-} is of the same conformal weight as g_{+-}. From now we use the definition

$$h_{+-} = e^{\sigma} \qquad 3.40$$

for the field h_{+-}. h_{+-} was defined as a constant field in the paragraph just above (3.5). Thus, it was taken out of the derivative operators in (3.38). However, it plays an essential role in the regularization procedure below, through its conformal weight. We keep the factor e^{σ} throughout the calculations, since we need to have conformal weight zero operators. Then, following the prescription given by (3.36) and (3.37), we write the regularized quantities to be computed as

$$I_1 = \sum_i C_i \; tr \; \ln (\tilde{\Delta}_1 + M_i^{\,2} e^{2\phi - \sigma})$$

and

$$I_2 = \sum_i C_i \; tr \; \ln (\tilde{\Delta}_2 + M_i^{\,2} e^{2\phi - \sigma}) , \qquad 3.41$$

where $\tilde{\Delta}_1 = \partial_+\partial_- + (\partial_+\phi) \partial_-$ and $\tilde{\Delta}_2 = \partial_+\partial_- - (\partial_+\phi) \partial_-$, since

$$\ln \Delta_1 = const + \ln \tilde{\Delta}_1$$

and

$$\ln \Delta_2 = const + \ln \tilde{\Delta}_2 . \qquad 3.42$$

The factor $e^{2\phi-\sigma}$ multiplying the mass term assures the quantity in the parenthesis to be conformally covariant. In order to compute the ϕ-dependence of the determinant factor, we first vary the log of the determinants with respect to ϕ. Since we are interested to regularize in such a way as to cancel the infinities between the numerator and the denominator we regularize the ratio of the two operators as follows

$$\delta \ln \frac{\det Q^+Q}{\det T^+T} = \delta \ln \det Q^+ Q - \delta \ln \det T^+ T \ . \tag{3.43}$$

Using (3.43) the quantity to be computed becomes

$$\begin{aligned}\frac{\delta I_1}{\delta\phi(\xi)} - \frac{\delta I_2}{\delta\phi(\xi)} = \sum_i C_i \Big[& M_i^2 e^{2\phi}\, 2 < \xi \mid \frac{1}{\Delta_1 + M_i^2 e^{2\phi}} \mid \xi > - \\ & \partial_+ < \xi \mid \partial_- \frac{1}{\Delta_1 + M_i^2 e^{2\phi}} \mid \xi > \Big] \\ - \sum_i C_i \Big[& M_i^2 e^{2\phi}\, 2 < \xi \mid \frac{1}{\Delta_2 + M_i^2 e^{2\phi}} \mid \xi > + \\ & \partial_+ < \xi \mid \partial_- \frac{1}{\Delta_2 + M_i^2 e^{2\phi}} \mid \xi > \Big] .\end{aligned} \tag{3.44}$$

The computation of (3.44) for small ϕ involves the expansion of the function of $\phi(\hat{\xi})$ around a point ξ as

$$\begin{aligned}< \hat{\xi} \mid & \frac{1}{\Delta_1(\hat{\xi}) + M_i^2 e^{2\phi(\hat{\xi})}} \mid \hat{\xi} > = \\ & \sum_{n=0}^{\infty} < \hat{\xi} \mid \frac{1}{\Delta_1 + M_i^2 e^{2\phi}} \Big[(-\Delta\hat{\Delta}_1 - \Delta M) \frac{1}{\Delta_1 + M_i^2 e^{2\phi}} \mid \hat{\xi} > \Big]^n .\end{aligned} \tag{3.45}$$

where

$$\tilde{A}_1(\hat{\xi}) =$$

$$\tilde{A}_1(\xi) + (\hat{\xi}-\xi)^{\alpha}[\, \partial_{\alpha}(\partial_+\phi)\,]\,\partial_- + \frac{1}{2}(\hat{\xi}-\xi)^{\alpha}(\hat{\xi}-\xi)^{\beta}[\, \partial_{\alpha}\partial_{\beta}(2\partial_+\phi)\,]\,\partial_- + \cdots$$

$$\equiv \tilde{A}_1(\xi) + \Delta\tilde{A}_1(\hat{\xi}-\xi;\xi)$$

and

$$M_t^{\,2}e^{2\phi(\hat{\xi})} =$$

$$M_t^{\,2}e^{2\phi(\xi)} + M_t^{\,2}(\hat{\xi}-\xi)^{\alpha}\partial_{\alpha}(2\phi) + \frac{M_t^{\,2}}{2}(\hat{\xi}-\xi)^{\alpha}(\hat{\xi}-\xi)^{\beta}$$

$$[\, \partial_{\alpha}\partial_{\beta}(2\phi) + \partial_{\alpha}(2\phi)\,\partial_{\beta}(2\phi)\,] + \cdots \equiv M_t^{\,2}e^{2\phi(\xi)} + \Delta M\;(\hat{\xi}-\xi\,;\,\xi\,).$$

Next, we insert the complete set of states

$$\int \frac{d^2p}{(2\pi)^2}\;|\,p><p\,|\; = 1 \qquad 3.46$$

and reduce the calculations to simple integrations over the momentum. For instance the $n = 0$ term in the expansion (3.44) becomes

$$\frac{1}{(2\pi)^2}\int d^2p\frac{1}{-p_+p_-+i\,(\partial_+\phi)p_-+M^2e^{2\phi}} =$$

$$\frac{1}{4\pi}\int d^2p\frac{1}{-p_+p_-+M^2e^{2\phi}}. \qquad 3.47$$

where we shifted the momentum p_+ as

$$p_+ \rightarrow p_+ - i\;(\,\partial_+\phi\,).$$

In a similar way, we expand all the terms in (3.44), while the second summation involves the expansion of $\tilde{A}_2(\hat{\xi})$ given by

$$\tilde{A}_2(\hat{\xi}) =$$

$$\tilde{A}_2(\xi) - (\hat{\xi}-\xi)^{\alpha}[\partial_{\alpha}(\partial_+\phi)]\partial_- + \frac{1}{2}(\hat{\xi}-\xi)^{\alpha}(\hat{\xi}-\xi)^{\beta}[\partial_{\alpha}\partial_{\beta}(\partial_+\phi)]\partial_- + \cdots$$

$$\equiv \tilde{A}_2(\xi) + \Delta\tilde{A}_2(\hat{\xi}-\xi;\xi).$$

We do not show all the details of the calculations at this point, because they are tedious and lengthy. Nevertheless, we give an explicit similar derivation of the anomaly factor later for the case of the Laplacian operator, since this is straightforward and shows exactly the method used. For the present case we just state the results: It turns out that the n = 0 terms in the expansion (3.44) cancel each other. Then, from the $n \neq 0$ term we obtain the contribution

$$-\sum C_i \left| M_i^2 e^{2\phi} 2\frac{1}{(2\pi)^2}\int d^2p \frac{1}{-p_+p_-+M_i^2e^{2\phi}}(\Delta\tilde{A}_1+\Delta M-\Delta\tilde{A}_2-\Delta M)\frac{1}{-p_+p_-+M_i^2e^{2\phi}} \right.$$

$$\left. -\partial_+\frac{1}{(2\pi)^2}\int d^2p \frac{ip_-}{-p_+p_-+M_i^2e^{2\phi}}(\Delta\tilde{A}_1+\Delta M+\Delta\tilde{A}_2+\Delta M)\frac{1}{-p_+p_-+M_i^2e^{2\phi}} \right|.$$

3.48

In the last expansion (3.48), the non-zero term is proportional to the factor $\partial_+\partial_-\phi(\xi)$. All other terms vanish either due to the integration over odd powers of the momentum or because they contain the factor $\frac{1}{(M_i^2)^l}$, $l > 0$ and from the regularization $M_i^2 \to \infty$. The term proportional to $\partial_+\phi\,\partial_-\phi$ cancels with the analogous term coming from n = 2 expansion of (3.44). For similar reasons as above all terms in higher-n expansion cancel. The net result is

$$\frac{\delta I_1}{\delta\phi(\xi)} - \frac{\delta I_2}{\delta\phi(\xi)} = -\frac{24}{12\pi}\,\partial_+\partial_-\phi\,. \qquad 3.49$$

Then, from (3.44) and (3.49) we obtain

$$\left\{\frac{\det Q^+Q}{\det P^+P}\right\}^{1/2} = \exp\left\{\frac{24}{24\pi}\int d^2\xi\,\frac{1}{2}\,(\partial_+\phi)(\partial_-\phi)\right\}. \qquad 3.50$$

Next, we compute the ϕ-dependence of the scalar Laplacian operator. The integration over the transverse variables X^i in (2.36) gives the determinant of the Laplacian as

$$[\det(\partial_+\partial_-)]^{-(d-2)/2}, \qquad 3.51$$

where our + (-) notation corresponds to $z\,(\bar{z})$. We use the same regularization procedure as in the previous case. Notice that there is no explicit dependence on the metric in the Laplacian operator. Nevertheless, in the expression of the regularized Laplacian operator, as in (3.35) the ϕ-dependence appears with the mass term. This is an effect of the regularization procedure which we used. The factor e^{ϕ} must multiply the M_i^2 term in order that the terms in the parenthesis be of the same conformal weight. The quantity to be computed is now given by

$$I' = \sum_i C_i\, tr\, \ln(\partial_+\partial_- + M_i^2 e^{\phi(\xi)}) \qquad 3.52$$

and the derivative with respect to ϕ gives

$$\frac{\delta I'}{\delta\phi(\xi)} = \sum C_i\, M_i^2 e^{\phi} < \xi \mid \frac{1}{\partial_+\partial_- + M_i^2 e^{\phi(\xi)}} \mid \xi >. \qquad 3.53$$

We expand as before and we have

$$<\hat{\xi}\,|\,\frac{1}{\partial_+\partial_- + M^2 e^{\phi(\hat{\xi})}}\,|\,\hat{\xi}>=$$

$$<\xi\,|\,\frac{1}{\partial_+\partial_- + M_i^2 e^{\phi(\xi)}}\left\{-\Delta M\,\frac{1}{\partial_+\partial_- + M_i^2 e^{\phi(\xi)}}\,|\,\xi>\right\}^n \qquad 3.54$$

where

$$\Delta M = M_i^2 e^{\phi}[\,(\hat{\xi}-\xi)^{\alpha}\partial_{\alpha}\phi + \frac{1}{2}(\hat{\xi}-\xi)^{\alpha}(\hat{\xi}-\xi)^{\beta}(\partial_{\alpha}\partial_{\beta}\phi + \partial_{\alpha}\phi\partial_{\beta}\phi) + \ldots].$$

Now the zero order term n = 0 in the expansion (3.54) gives logarithmic divergence. We regularize by introducing a cut-off Λ. In momentum-space representation we have:

$$<\hat{\xi}\,|\,\frac{1}{\partial_+\partial_- + M_i^2 e^{\phi}}\,|\,\hat{\xi}>=$$

$$\frac{1}{(2\pi)^2}\int_0^{\Lambda} d^2p\,\frac{1}{-p_+p_- + M_i^2 e^{\phi}} = \frac{1}{2\pi}\ln\left|\frac{p^2}{2} + M_i^2 e^{\phi}\right|\Big|_0^{\Lambda} =$$

$$=\frac{1}{2\pi}\left|\ln(\frac{\Lambda^2}{2}+M_i^2 e^{\phi})-\ln(M_i^2 e^{\phi})\right|. \qquad 3.55$$

In the limit $\Lambda\rightarrow\infty$ the first term does not contribute and from (3.53) and (3.55), using the conditions of our regularization (3.36), we obtain

$$-\sum_i \frac{1}{\pi} C_i\, M_i^2 e^{\phi} \ln M_i = -\frac{\mu^2}{\pi} e^{\phi} \qquad 3.56$$

where

$$\mu^2 = \sum_i C_i \; M_i^{\,2} \ln M_i \; = \; finite \; .$$

The only nonvanishing term of the higher-order expansion of (3.54) is proportional to the factor $\partial_+\partial_-\phi$. All other terms vanish for reasons similar to the previous case. Proceeding as before we compute the first-order non-vanishing contribution. It is given by

$$-\langle\hat{\xi}\,|\,\frac{1}{\partial_+\partial_- + M_i^{\,2}e^{\phi(\xi)}}(\Delta M')\frac{1}{\partial_+\partial_- + M_i^{\,2}e^{\phi(\xi)}}\,|\,\hat{\xi}\rangle . \qquad 3.57$$

where

$$\Delta M' = \frac{1}{2}(\hat{\xi}-\xi)^{\alpha}(\hat{\xi}-\xi)^{\beta}(\partial_{\alpha}\partial_{\beta}\phi)$$

is the part of ΔM which gives non-vanishing contribution. Then we insert the complete set of states (3.46) and (3.57) becomes

$$-\int \frac{dp^2}{(2\pi)^2}\,\frac{M_i^{\,2}e^{\phi}\partial_+\partial_-\phi}{-p_+p_- + M_i^{\,2}e^{\phi}}\left(-\frac{\partial}{\partial p_+}\frac{\partial}{\partial p_-}\right)\frac{1}{-p_+p_- + M_i^{\,2}e^{\phi}} . \qquad 3.58$$

which gives the following result, for the kinetic energy term of the scalar Laplacian operator

$$\frac{\delta I}{\delta\phi(\xi)} = -\frac{1}{12\pi}\,\partial_+\partial_-\phi . \qquad 3.59$$

Then from (3.51),(3.56) and (3.59) we have

$$[\det(\partial_+\partial_-)]^{-(d-2)/2} =$$

$$\exp\left\{-\left|\frac{d-2}{2}\right|\left|\frac{1}{12\pi}\int d^2\xi\, \partial_+\phi\partial_-\phi + \frac{1}{\pi}\int d^2\xi\, \mu^2 e^{\phi}\right|\right\} \quad 3.60$$

Comparing the last expression with (3.50) we obtain the cancellation of the kinetic energy term of the Liouville action, given by the Lagrangian (3.60), with the result d = 26. The term proportional to μ^2 is the divergent cosmological term. The constant μ^2 can be absorbed in the renormalization of the cosmological constant in such a way that the divergent term will cancel at d=26. The usual way of treating this problem [9][21] is to add in the original Lagrangian a cosmological term given by

$$\mu_0^2 \int d^2\sigma \sqrt{-g} \,. \quad 3.61$$

where μ_0^2 is a regulator-dependent constant. Note that only at d=26 after the cancellation of the ϕ-dependence, the expression (2.48) coincides with Mandelstam's picture of strings.

V. Conclusion

The regularization of the Faddeev-Popov gauge-fixing determinant and the computation of the conformal anomaly in the Polyakov functional formulation of strings have been investigated for the case of the conformal (diagonal) gauge. In this chapter we have performed analogous calculations in order to regularize the non-diagonal determinants and compute the conformal anomaly in the light-cone gauge. The interesting point to notice is that in this case we can not regularize by preserving the reparametrization invariance, as is the case for the usual conformal gauge. Due to the non-diagonal nature of our gauge, the reparametrization invariance was not respected in the course of the calculations. The invariance we choose to use as the regularization principle, in this case, is the conformal symmetry. This is justified by the fact that the Liouville action, which is essentially the conformal anomaly factor, possesses conformal symmetry.

We like to stress at this point the use of the BRST-invariance in our work. BRST-symmetry does not play a crucial role in this case. Nevertheless, the covariant form of the BRST-invariant expressions under conformal transformations, induces the transformation properties of the operators $\partial_+ f$ and $\partial_- f$, which are crucial for the calculation of the determinants and guides towards the choice of the regularization principle.

Note also that in order to avoid ambiguities we regularized the whole determinant factor in such a way that the infinities between the numerator and denominator cancel. This was crucial in obtaining the correct result, as we emphasized in the introduction. The Pauli-Villars method is used for the regularization procedure since it shows directly how one can extract the anomaly through the regularization.

CHAPTER 4

The problem of changing gauge in Polyakov's theory and the relation between light-cone and conformal gauge.

I. Introduction

In the second chapter of this thesis, studying the Gervais-Sakita light-cone gauge in Polyakov's functional formulation of the bosonic string, we emphasized the independence between the two gauges (light-cone and conformal). We argued that the light-cone and conformal are two equivalent gauges and one can be obtained from the other by a proper coordinate transformation. The connection between the two gauges is made more transparent through the relation (2.34). Even if, in our light-cone gauge, we fix the X^+ component of the string coordinates and only one component of the g-metric ($g_{--}=0$), we recover the other condition of the conformal gauge through a δ-function; namely the integration over X^- gives $\delta(g^{--})$, which corresponds to $g_{++}=0$. The meaning of this is that there must exist a sector where the two gauges (light-cone and conformal) coincide.

Our definition of the light-cone gauge fixed Polyakov's theory is given (see chapter 2) by

$$\prod_i \int \cdots \int d^2\sigma_i \sqrt{-g(\sigma_i)} \prod_\sigma dg_{+-}(\sigma)(-g)^{-1} \prod_\sigma DX^i(\sigma) \prod_\sigma dX^-(\sigma) \prod_\sigma dg_{++}(\sigma)$$

$$\Delta_{l-c} \exp\Big\{ -\frac{i}{2}\int d^2\sigma [\sqrt{-g}\, g^{--}(\partial_- X^i)^2 + 2\partial_+ X^i \partial_- X^i] + ik^i X^i(\sigma_i)$$

$$-i\sum_j k_j^+ X^-(\sigma_j) + i\int d^2\sigma \sqrt{-g}\, g^{--}\partial_- f\, \partial_- X^- -$$

$$-i\sum_j k_j^- f^-(\sigma_j) + 2i\int d^2\sigma\, \partial_+ X^- \partial_- f^-(\sigma)\Bigg\}. \qquad 4.1$$

where $\prod_\sigma$ is defined in the first chapter to be the reparametrization invariant infinite product. Δ_{f-l} is the Faddeev-Popov determinant of the light-cone gauge, given by the following non-diagonal matrix

$$\Delta_{F-L} = \det\begin{vmatrix} -\partial_- f^- & -\partial_+ f^- \\ 0 & -2g_{+-}\partial_- \end{vmatrix}. \qquad 4.2$$

Note that in the expression (4.1) we have already integrated over the variables X^+ and g_{--} and substituted the source term j^μ by its expression in terms of the external momenta (see relation (2.13)).

On the other hand, one can choose the conformal gauge conditions, namely

$$g_{++}=0$$

and

$$g_{--}=0 \qquad 4.3$$

in Polyakov's theory. Then the conformal gauge-fixed theory in the light-cone notation is given by

$$\prod_j\int\cdots\int d^2\sigma_j\sqrt{-g(\sigma_j)}\prod_\sigma dg_{+-}(\sigma)(-g)^{-1}\prod_\sigma dX^+(\sigma)\prod_\sigma dX^-(\sigma)\prod_\sigma DX^i(\sigma)$$

$$\Delta_{f-c}\exp\left\{i\int d^2\sigma\partial_+X^\mu\partial_-X^\mu - i\int j^\mu X_\mu\right\}. \qquad 4.4$$

where the Faddeev-Popov determinant in this gauge is given by the following diagonal matrix

$$\Delta_{l-c} = \det \begin{bmatrix} -2g_{+-}\partial_+ & 0 \\ 0 & -2g_{+-}\partial_- \end{bmatrix}. \qquad 4.5$$

The invariance of Polyakov's action under the two-dimensional general coordinate transformation

$$\sigma^a \rightarrow \tilde{\sigma}^a = \sigma^a + \epsilon^a(\sigma) \qquad 4.6$$

is used for the gauge fixing, in both cases. The meaning of the last invariance is that one can define Polyakov's theory on any of the parameter spaces of the family σ^a. The members of the family are related to each other by (4.6). The theories so defined are then equivalent. We can, therefore, think of choosing to study the two different gauges (the light-cone and conformal gauge) in Polyakov's theory starting from two different parametrizations (members of the family σ^a) of the two-dimensional parameter space. Namely, we choose to fix the light-cone gauge conditions for the theory which is defined on the parameter space with σ coordinates and the conformal gauge for the case that the parameter space has coordinates denoted by $\tilde{\sigma}$. After the gauge fixing we obtain the two theories defined by (4.1) and (4.4). Due to the reparametrization symmetry if the two gauges are equivalent we must obtain the same theory. In this chapter, we examine the precise relationship between the two gauges (light-cone and conformal) in Polyakov's theory, by performing the gauge fixing in two different parametrizations of the parameter space. The method we are using is the changing of gauge method. Namely, we change gauge from the light-cone to the conformal by making a coordinate transformation in the functional integration variables. The organization of this chapter is as follows:

In section II, we introduce the problem by defining the variables in both gauges and we give the relations between them. A relation which specifies the coordinate of the two-dimensional parameter space of the conformal gauge in terms of the light-cone gauge variables is obtained.

In section III, we obtain the conformal gauge fixed Polyakov's theory, apart from the determinant-factor, by making a simple coordinate transformation in the integration variables of the light-cone gauge fixed Polyakov's theory.

In section IV, we show that by enlarging the coordinate transformation in such a way that it include anticommuting variables the two gauge-fixed theories coincide totally. The light-cone gauge fixing Faddeev-Popov determinant transforms to the conformal-gauge-fixing determinant while the superdeterminant of the transformation results to a constant.

In section V, we show that Mandelstam's picture emerges rather trivially, after integrating out the longitudinal coordinates of the string, from the conformal-gauge-fixed Polyakov's theory.

II. The light-cone and conformal gauge variables and their relations.

Let us denote by $\tilde{\sigma}$ the reparametrization of the two-dimensional parameter space for the conformal-gauge-fixed theory. Then

$$\tilde{g}_{++}(\tilde{\sigma}) = 0, \quad \tilde{g}_{--}(\tilde{\sigma}) = 0 \qquad 4.7$$

are the conformal gauge conditions. The conditions for the light-cone gauge are given by

$$X^+(\sigma) = f(\sigma), \quad g_{--}(\sigma) = 0 \qquad 4.8$$

where σ denotes the specific reparametrization of the two-dimensional parameter space, for the case that we choose to fix the light-cone gauge. The coordinates σ^a and $\tilde{\sigma}^a$ are related by a general coordinate transformation

$$\sigma^a \rightarrow \tilde{\sigma}^a = \sigma^a + \epsilon^a(\sigma). \qquad 4.9$$

Polyakov's theory is invariant under the transformation (4.9). Under the finite reparametrization transformation the variables $X^\mu(\sigma)$ and $g_{ab}(\sigma)$ transform as follows

$$X^\mu(\sigma) \rightarrow \tilde{X}^\mu(\sigma): \quad \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma)$$

$$g_{ab}(\sigma) \rightarrow \tilde{g}_{ab}(\sigma): \quad \tilde{g}_{ab}(\tilde{\sigma})\, d\tilde{\sigma}^a\, d\tilde{\sigma}^b = g_{ab}(\sigma)\, d\sigma^a\, d\sigma^b. \qquad 4.10$$

After the gauge fixing, the integration variables for the light-cone gauge are

$$X^i(\sigma), \quad X^-(\sigma), \quad g_{++}(\sigma), \quad g_{+-}(\sigma); \qquad 4.11$$

while the variables which remain after fixing the conformal gauge are

$$\tilde{X}^\mu(\tilde{\sigma}), \quad \tilde{g}_{+-}(\tilde{\sigma}). \qquad 4.12$$

From now on we shall refer to the variables (4.11) and (4.12) as the light-cone and conformal gauge variables respectively. Throughout this part we are using "tilde" notation for the conformal gauge variables and plain notation for the light-cone gauge variables. The two sets of variables (4.11) and (4.12) are related through (4.10). It is the object of the present chapter of the thesis to show that by changing coordinates between the light-cone and conformal gauge integration variables one can obtain the string theory in either of the two gauges.

We define $y(\sigma)$ to be that specific value of the coordinate $\tilde{\sigma}$ for which the $\tilde{X}^+(\tilde{\sigma})$ component of the $\tilde{X}^\mu(\tilde{\sigma})$ variables of the conformal-gauge-fixed theory takes the light-cone gauge fixed value of the $X^+(\sigma)$ variable, defined by (4.8). Namely,

$$\tilde{X}^+(y(\sigma)) \equiv f(\sigma). \tag{4.13}$$

The relation (4.13) specifies $y(\sigma)$ as a functional of the conformal gauge variable $\tilde{X}^+(\tilde{\sigma})$. Then using the relations (4.10) and (4.13) we express the integration variables of the light-cone gauge fixed theory in terms of the conformal gauge variables, by substituting $\tilde{\sigma} = y(\sigma)$. They are given by the following transformations

$$\begin{aligned}
X^\mu(\sigma) &= \tilde{X}^\mu(y(\sigma)), \quad \mu \neq + \\
f(\sigma) &= \tilde{X}^+(y(\sigma)) \\
g_{++}(\sigma) &= 2\tilde{g}_{+-}(y(\sigma)) \frac{\partial y^+}{\partial \sigma^+} \frac{\partial y^-}{\partial \sigma^+} \\
g_{+-}(\sigma) &= \tilde{g}_{+-}(y(\sigma)) \left[\frac{\partial y^+}{\partial \sigma^+} \frac{\partial y^-}{\partial \sigma^-} + \frac{\partial y^+}{\partial \sigma^-} \frac{\partial y^-}{\partial \sigma^+} \right] \\
g_{--}(\sigma) &= 2\tilde{g}_{+-}(y(\sigma)) \frac{\partial y^+}{\partial \sigma^-} \frac{\partial y^-}{\partial \sigma^-}.
\end{aligned} \tag{4.14}$$

where we have used the conditions $\tilde{g}_{++} = \tilde{g}_{--} = 0$.

The light-cone gauge condition $g_{--}(\sigma) = 0$ results to either $\dfrac{\partial y^+}{\partial \sigma^-} = 0$ or

$\frac{\partial y^-}{\partial \sigma^-} = 0$. We must choose the first case, since the identity transformation should be included in the transformations. Then $y(\sigma)$ can be expressed as follows

$$y^+(\sigma) = \rho_1(\sigma^+)$$

$$y^-(\sigma) = \rho_2(\sigma^-) + h(\sigma^+,\sigma^-), . \qquad 4.15$$

where ρ_1 , ρ_2 and h are arbitrary functions of the argument. In order to further specify the function $y(\sigma)$ we use the relations (4.14). From the third and the last of the relations (4.14) we obtain

$$\frac{2\frac{\partial y^-}{\partial \sigma^+}}{\frac{\partial y^-}{\partial \sigma^-}} = \frac{g_{++}(\sigma)}{g_{+-}(\sigma)} \qquad 4.16$$

Because of (4.15), (4.16) can be written as follows

$$\frac{2\frac{\partial h}{\partial \sigma^+}}{\frac{\partial \rho_2}{\partial \sigma^-} + \frac{\partial h}{\partial \sigma^-}} = \frac{g_{++}(\sigma)}{g_{+-}(\sigma)} . \qquad 4.17$$

The last relation (4.17) specifies the coordinate $y(\sigma)$ in terms of the light-cone gauge variables $g_{++}(\sigma)$, $g_{+-}(\sigma)$. In what follows we shall choose

$$\rho_1(\sigma^+) = \sigma^+ \text{ and } \rho_2(\sigma^-) = \sigma^- , \qquad 4.18$$

since due to the conformal invariance of the conformal gauge-fixed-theory this choice does not change the result. Then $y(\sigma)$ is given by

$$y^+(\sigma) = \sigma^+$$
$$y^-(\sigma) = \sigma^- + h(\sigma^+,\sigma^-), \qquad 4.19$$

where $h(\sigma^+,\sigma^-)$ is related to $g_{++}(\sigma)$ and $g_{+-}(\sigma)$ by

$$\frac{2\dfrac{\partial h}{\partial \sigma^+}}{1+\dfrac{\partial h}{\partial \sigma^-}} = \frac{g_{++}(\sigma)}{g_{+-}(\sigma)} . \qquad 4.20$$

The meaning of the last relation (4.20) is that the $g_{++}(\sigma)$ integration of the light-cone gauge can be transformed to $\dfrac{\partial h}{\partial \sigma^+}$ and hence to the $\tilde{X}^+$ - since X^+ is related to y through (4.13) -, integration of the conformal gauge. This will be clear in the next section.

III. From the light-cone gauge fixed to the conformal-gauge-fixed theory by coordinate transformation.

The relations (4.14) define the transformations of the conformal gauge coordinates to the light-cone gauge coordinates. In what follows, we show that the conformal-gauge-fixed theory defined by (4.4) can be obtained from the light-cone gauge fixed theory by changing the integration variables in Polyakov's functional formulation of strings.

In order to relate the functional integrals (4.1) and (4.4), we express the integral (4.1) in terms of the conformal gauge ("tilde") variables (4.12).

For this purpose we make a change of variables from g_{++}, g_{+-}, X^- and X' to the new variables $\tilde{X}^+$, $\tilde{g}_{+-}$, $\tilde{X}^-$ and $\tilde{X}'$. In order to compute the Jacobian of the transformation, we first compute the variation of g_{++} with respect to $\tilde{X}^+$. $\tilde{X}^+$ is related to $y(\sigma)$ through (4.13). Since $f(\sigma)$ is fixed, a change in $\tilde{X}^+$ results to a change in the argument $y(\sigma)$ to compensate for it. Therefore we must compute the variation of g_{++} with respect to $y(\sigma)$ and then the variation of $y(\sigma)$ with respect to $\tilde{X}^+$. We vary the third of the relations (4.14) and we have

$$\delta g_{++}(\sigma) =$$

$$= 2(\tilde{g}_{+-})_{y(\sigma)}(\partial_+\delta h) + (\partial_+\tilde{g}_{+-})_{y(\sigma)}\delta y^+(\partial_+ h) + (\partial_-\tilde{g}_{+-})_{y(\sigma)}\delta y^-(\partial_+ h) =$$

$$= 2(\tilde{g}_{+-})_{y(\sigma)}(\partial_+\delta h) + (\partial_-\tilde{g}_{+-})_{y(\sigma)}\delta h(\partial_+ h).$$

The last equality holds since because of (4.19) $\delta y^+ = 0$ and $\delta y^- = \delta h$. The above notation $(\partial_+\tilde{g}_{+-})_{y(\sigma)}$ means that we first take derivative with respect to $y^+(\sigma)$ and then evaluate it at the point $y(\sigma)$; while the meaning of $\partial_+ h$ is $\dfrac{\partial h}{\partial \sigma^+}$. We are using this notation throughout this chapter. Then

$$\delta g_{++}(\sigma) = 2(\tilde{g}_{+-})_{y(\sigma)}(\partial_+\delta h) + (\textit{part proportional } \partial_+ h) \qquad 4.21$$

In order to simplify the expressions we denote the dependence on $y(\sigma)$ by a subindex. Terms proportional to $\partial_+ h$ are irrelevant since, as it is shown in chapter 2, after the integration over $X^-(\sigma)$ all the contribution comes from the $g_{++}(\sigma) = 0$ sector, while the condition $g_{++}(\sigma) = 0$ corresponds to $\partial_+ h = 0$ due to the relations (4.14). The vanishing of the $g_{++}(\sigma)$ reflects the invariance of the theory under conformal transformations, since, through (4.15), $y(\sigma)$ defines then a conformal transformation. Alternatively, we can say that after obtaining the condition $g_{++}=0$ we are led to the conformal gauge which possesses conformal symmetry. Note that even if the contribution of the proportional to $\partial_+ h$ terms is vanishing, we are not allowed to set $\partial_+ h=0$ at the beginning of the calculations (relation (4.14)). We first take variation and then substitute $\partial_+ h=0$ in the equation. Indeed (4.21) shows that there exists a non-zero contribution due to the variation of $\partial_+ h$.

We now compute δh in terms of $\delta\tilde{X}^+$ using (4.13). We first substitute (4.19) into (4.13) and obtain

$$f(\sigma)=\tilde{X}^+(y(\sigma))=\tilde{X}^+(\sigma^+,\sigma^-+h(\sigma^+,\sigma^-)). \qquad 4.22$$

Differentiation of (4.22) with respect to σ^+ gives

$$\partial_+ f(\sigma)=(\partial_+\tilde{X}^+)_{y(\sigma)}+(\partial_-\tilde{X}^+)_{y(\sigma)}(\partial_+ h). \qquad 4.23$$

We vary (4.23) and we have

$$0=(\partial_+\delta\tilde{X}^+)_{y(\sigma)}+(\partial_-\tilde{X}^+)_{y(\sigma)}(\partial_+\delta h)+(\partial_-\partial_+\tilde{X}^+)_{y(\sigma)}\delta h+$$
$$+(\textit{part prop.}(\partial_+ h)), \qquad 4.24$$

where again we used $\delta y^+=0$. Then δh is given in terms of $\delta\tilde{X}^+$ by

$$\delta h=-\left[(\partial_-\tilde{X}^+)_{y(\sigma)}\partial_+ +(\partial_-\partial_+\tilde{X}^+)_{y(\sigma)}\right]^{-1}\partial_+\delta\tilde{X}^+$$
$$+(\textit{part prop.}(\partial_+ h)). \qquad 4.25$$

By substituting (4.25) into (4.21) we obtain δg_{++} in terms of $\delta\tilde{X}^+$ as follows

$$\delta g_{++}(\sigma) =$$
$$-2(\tilde{g}_{+-})_{x(\sigma)}\partial_+\left[(\partial_-\tilde{X}^+)_{x(\sigma)}\partial_+ + (\partial_-\partial_+\tilde{X}^+)_{x(\sigma)}\right]^{-1}\partial_+\delta\tilde{X}^+$$
$$+ (part\ \ prop.(\partial_+h)) \qquad 4.26$$

or the matrix element of the transformation from g_{++} to $\tilde{X}^+$ is given by

$$\frac{\delta g_{++}(\sigma)}{\delta\tilde{X}^+(\tilde{\sigma})} =$$
$$= -2(\tilde{g}_{+-})_{x(\sigma)}\partial_+\left[(\partial_-\tilde{X}^+)_{x(\sigma)}\partial_+ + (\partial_-\partial_+\tilde{X}^+)_{x(\sigma)}\right]^{-1}\partial_+\delta(\tilde{\sigma}-x(\sigma)), \qquad 4.27$$

apart from terms proportional to ∂_+h .

It is now straightforward to show that the rest of the transformations do not contribute in the Jacobian. The variations of g_{+-}, X^-, X^i with respect to $\tilde{g}_{+-}$, $\tilde{X}^-$, $\tilde{X}^i$ respectively, give diagonal elements of the transformation matrix equal to the identity. To illustrate this point we compute the variation of X^- with respect to $\tilde{X}^-$. We use the first of the relations (4.14) and obtain

$$\delta X^-(\sigma) = (\delta\tilde{X}^-)_{x(\sigma)} + (\partial_-\tilde{X}^-)_{x(\sigma)}\delta x^-(\sigma)$$
$$= (\delta\tilde{X}^-)_{x(\sigma)} + (\partial_-\tilde{X}^-)_{x(\sigma)}\delta h\ , \qquad 4.28$$

since $\delta x^+ = 0$. δh in the last relation is given in terms of $\delta\tilde{X}^+$ by (4.25). Then the matrix element of this transformation is given by

$$\frac{\delta X^{-}(\sigma)}{\delta X^{-}(\tilde{\sigma})} = 1\,\delta(\tilde{\sigma}-y(\sigma)), \qquad 4.29$$

which is the identity operator. By similar arguments we can show that the transformations of g_{+-} and X^{+} to $\tilde{g}_{+-}$ and $\tilde{X}^{+}$ respectively, give the same identity operator for the diagonal elements.

The resulting Jacobian of the transformations then is a matrix with the off-diagonal upper corner zero and the only diagonal matrix element different than the identity given by (4.27). Then the above change of variables requires the following Jacobian

$$J = \det \left| -2(\tilde{g}_{+-})_{y(\sigma)}\,\partial_{+} \left[(\partial_{-}\tilde{X}^{+})_{y(\sigma)}\,\partial_{+} + (\partial_{-}\partial_{+}\tilde{X}^{+})_{y(\sigma)} \right]^{-1} \partial_{+} \right| \qquad 4.30$$

since, as we already discussed, only the change from g_{++} to $\tilde{X}^{+}$ gives a non-trivial contribution. Using the definition (4.13) we express $(\partial_{-}\tilde{X}^{+})_{y(\sigma)}$ and $(\partial_{-}\partial_{+}\tilde{X}^{+})_{y(\sigma)}$ in the last Jacobian as follows

$$(\partial_{-}\tilde{X}^{+})_{y(\sigma)} = \partial_{-} f(\sigma)$$

$$(\partial_{-}\partial_{+}\tilde{X}^{+})_{y(\sigma)} = \partial_{-}\partial_{+} f(\sigma), \qquad 4.31$$

where we have set $\partial_{+}h = 0$. Consistency with our definition of $\partial_{+} f(\sigma)$ as a constant field x_{+}, given in chapter 3, requires $\partial_{-}\partial_{+} f(\sigma)$ to be set zero in the Jacobian. Then (4.30) takes the form

$$J = \det \left| -2(\tilde{g}_{+-})_{y(\sigma)} (\partial_{-} f)^{-1}\,\partial_{+} \right|. \qquad 4.32$$

After performing the above change of variables in the functional integration of (4.1) we obtain

$$\Pi \int \cdots \int d^2\sigma \ \sqrt{-g(\sigma_\cdot)} \prod_\sigma (d\tilde{g}_{--})_{(\sigma)} (-\tilde{g})^{-1} \prod_\sigma (D\tilde{X}^i)_{(\sigma)} \prod_\sigma (d\tilde{X}^+)_{(\sigma)}$$

$$\prod_\sigma (d\tilde{X}^-)_{(\sigma)} J \Delta_{f-L} \exp \left[-\frac{i}{2}\int d^2\sigma \left(\sqrt{-g}\, g^{--}(\partial_- X^i)^2 + 2\, \partial_+ X^i \partial_- X^i \right) + \right.$$

$$+ i \int d^2\sigma \sqrt{-g}\, g^{--}\, \partial_- f \ \partial_- X^- + ik_j^i X^i(\sigma_j) -$$

$$\left. - i\sum_j k_j^- f(\sigma_j) - i\sum_j k_j^+ \tilde{X}^-(\sigma_j) - 2i \int d^2\sigma\, X^- \partial_- \partial_+ f \ \right]. \qquad 4.33$$

In the last expression, only the functional integration variables are in terms of the conformal gauge ("tilde") variables. Next, we express the rest of the light-cone variables in (4.33) in terms of the "tilde" conformal gauge variables using (4.14). For this purpose, we first show that the light-cone gauge action given by the exponent of (4.33) coincides with the conformal gauge action after substituting the relations (4.14) in it. We write the light-cone gauge action without the source term as follows

$$S =$$

$$-\frac{1}{2}\int d^2\sigma \sqrt{-g}\, g^{--}[(\partial_- X^i)^2 - 2\partial_- f \ \partial_- X^-] - \int d^2\sigma \partial_+ X^i \partial_- X^i - 2\int d^2\sigma X^- \partial_- \partial_+ f$$

$$4.34$$

Next, using the relations (4.14), we substitute in (4.34) $f(\sigma)$ by $\tilde{X}^+(y(\sigma))$ and $X^\mu(\sigma)$ by $\tilde{X}^\mu(y(\sigma))$ for $\mu \neq +$ and we obtain

$$S =$$

$$-\frac{1}{2}\int d^2\sigma\sqrt{-g}\,g^{--}\left[(\partial_-\tilde{X}^i(y(\sigma)))^2-2\partial_-\tilde{X}^+(y(\sigma))\partial_-\tilde{X}^-(y(\sigma))\right]$$

$$-\int d^2\sigma\partial_+\tilde{X}^i(y(\sigma))\partial_-\tilde{X}^i(y(\sigma))-\int d^2\sigma\tilde{X}^-(y(\sigma))(2\partial_-\partial_+\tilde{X}^+(y(\sigma))),$$

where the partial derivatives in the last expression are with respect to σ. Next, we express the integration measure in terms of the new parameter $(y(\sigma))$ of the two-dimensional parameter space, chosen for the conformal-gauge-fixed theory in the last section. It is as follows

$$d\sigma^+d\sigma^-=dy^+dy^-\left[\frac{\partial\sigma^+}{\partial y^+}\frac{\partial\sigma^-}{\partial y^-}+\frac{\partial\sigma^+}{\partial y^-}\frac{\partial\sigma^-}{\partial y^+}\right]$$

$$=dy^+dy^-\frac{\partial\sigma^+}{\partial y^+}\frac{\partial\sigma^-}{\partial y^-}.$$

Then the action becomes

$$S=$$

$$\frac{1}{2}\int d^2y\frac{\partial\sigma^-}{\partial y^-}\left[\sqrt{-g}\,g^{--}(\partial_-\tilde{X}^\mu(y(\sigma)))^2+2\partial_+\tilde{X}^\mu(y(\sigma))\partial_-\tilde{X}_\mu(y(\sigma))\right].$$

4.35

The derivatives of the variable $\tilde{X}^\mu$ with respect to σ are related to those with respect to $y(\sigma)$ as follows

$$\partial_-\tilde{X}^\mu(y(\sigma))\equiv\frac{\partial\tilde{X}^\mu(y(\sigma))}{\partial\sigma^-}=\frac{\partial y^-}{\partial\sigma^-}(\partial_-\tilde{X}^\mu)_{y(\sigma)},$$

since $\dfrac{\partial y^+}{\partial\sigma^-}=0$

and

$$\partial_+ \tilde{X}^{\mu}(y(\sigma)) \equiv \frac{\partial \tilde{X}^{\mu}(y(\sigma))}{\partial \sigma^+} = \frac{\partial y^+}{\partial \sigma^+}(\partial_+ \tilde{X}^{\mu})_{y(\sigma)} + \frac{\partial y^-}{\partial \sigma^+}(\partial_- \tilde{X}^{\mu})_{y(\sigma)} =$$

$$= (\partial_+ \tilde{X}^{\mu})_{y(\sigma)} + (\partial_- \tilde{X}^{\mu})_{y(\sigma)} \frac{\partial y^-}{\partial \sigma^+}. \qquad 4.36$$

Then using the relations (4.36) we write the expression (4.35) as follows

$$S =$$

$$\int d^2y \left|\frac{\partial y^-}{\partial \sigma^-}\right|^{-1} \left\{ - \frac{\frac{\partial y^-}{\partial \sigma^+}}{\frac{\partial y^-}{\partial \sigma^-}} \left|\frac{\partial y^-}{\partial \sigma^-}\right|^2 (\partial_- \tilde{X}^{\mu})^2_{y(\sigma)} + \right.$$

$$\left. + \left[(\partial_+ \tilde{X}^{\mu})_{y(\sigma)} + \frac{\partial y^-}{\partial \sigma^+}(\partial_- \tilde{X}^{\mu})_{y(\sigma)}\right] \left[\frac{\partial y^-}{\partial \sigma^-}(\partial_- \tilde{X}^{\mu})_{y(\sigma)}\right] \right\}, \qquad 4.37$$

where we have also used (4.16) and the relation $g^{--} = -(g^{+-})^2 g_{++}$. Then (4.37) becomes

$$S =$$

$$\int d^2y\, (\partial_+ \tilde{X}^{\mu})_{y(\sigma)} (\partial_- \tilde{X}^{\mu})_{y(\sigma)} + \frac{\partial y^-}{\partial \sigma^+}(\partial_- \tilde{X}^{\mu})_{y(\sigma)} (\partial_- \tilde{X}^{\mu})_{y(\sigma)} - \frac{\partial y^-}{\partial \sigma^+}(\partial_- \tilde{X}^{\mu})^2_{y(\sigma)}$$

$$= \int d^2y\, (\partial_+ \tilde{X}^{\mu})_{y(\sigma)} (\partial_- \tilde{X}^{\mu})_{y(\sigma)}, \qquad 4.38$$

which is the conformal gauge action. Is is now easy to see that we can perform the same change of variables from the light-cone to the conformal gauge ones in the source term of the action and the light-cone gauge action with the source term, given by the exponent of (4.33), takes the form of the conformal gauge action given by

$$S =$$

$$i \int d^2y \, (\partial_+ \tilde{X}^\mu)_y \, (\partial_- \tilde{X}^\mu)_y + ik_j^i \tilde{X}^i(y_j) - ik_j^- \tilde{X}^+(y_j) - ik_j^+ \tilde{X}^-(y_j),$$

4.39

where we defined $y(\sigma_j) \equiv y_j$. The action (4.39) coincides with the action of the conformal-gauge-fixed Polyakov's theory for $\tilde{\sigma} = y(\sigma)$. This was to be expected since the string action is reparametrization invariant and the "tilde" variables are related to the light-cone variables by a reparametrization transformation.

Then by substituting the expression (4.39) for the action of (4.33) the integral (4.33) is expressed in terms of the conformal gauge variables as follows

$$\prod_j \int \cdots \int d^2\tilde{\sigma}_j \sqrt{-\tilde{g}(\tilde{\sigma}_j)} \prod_{\sigma} d\tilde{g}_{+-}(\tilde{\sigma})(-\tilde{g})^{-1} \prod_{\tilde{\sigma}} D\tilde{X}^i(\tilde{\sigma}) \prod_{\sigma} d\tilde{X}^+(\tilde{\sigma}) \prod_{\tilde{\sigma}} d\tilde{X}^-(\tilde{\sigma}) \, J$$

$$\Delta_{F-P} \exp\left\{ i \int d^2\tilde{\sigma} \, \partial_+ \tilde{X}^\mu \partial_- \tilde{X}^\mu + ik_j^i \tilde{X}^i(\tilde{\sigma}_j) - i k_j^- \tilde{X}^+(\tilde{\sigma}_j) - ik_j^+ \tilde{X}^-(\tilde{\sigma}_j) \right\}.$$

4.40

where we have renamed $y(\sigma)$ to be $\tilde{\sigma}$ and we used the fact that the integration over the Koba-Nielsen variables, i.e. the factor $d^2\tilde{\sigma}\sqrt{-\tilde{g}(\tilde{\sigma}_j)}$, is invariant under reparametrization transformation. $\prod_{\sigma}$ is by definition (see chapter 2) the reparametrization invariant infinite product. It is

$$\prod_{\sigma} d\tilde{X}(y(\sigma)) = \prod_{y} d\tilde{X}(y) \qquad 4.41$$

for $\tilde{X}$ being a scalar under reparametrization. The last integral (4.40) coincides with the conformal-gauge-fixed Polyakov's path-integral of strings, apart from the determinant-factor ($J\ \Delta_{l-l}$). In the next section we show that by a proper coordinate transformation in the anticommuting (ghost-antighost) fields, the light-cone determinant Δ_{l-l} transforms into the conformal gauge determinant Δ_{l-c} . The Jacobian (J) coming from the change of commuting field variables combines with a Jacobian coming from the change of variables in the anticommuting fields and results merely to a constant.

IV. Change of variables in the ghost fields.

The light-cone gauge fixing Faddeev-Popov determinant Δ_{l-l} , in terms of anticommuting fields, is defined in chapter 3 (relation (3.14)). Keeping only the relevant part in the ghost Lagrangian, we express the determinant as follows

$$\Delta_{l-l} = \int d\tilde{h} d\tilde{h}^{--} dc^{-} dc^{+}$$
$$\exp\left[-i\int d^2\sigma \sqrt{-g}\,(\tilde{h}\,\partial_- l\, c^- + 2\,\tilde{h}^{--} g_{+-}\partial_- c^+)\right]. \qquad 4.42$$

where all the variables are functions of σ. c^-, c^+ and $\tilde{h}, \tilde{h}^{--}$ are the light-cone ghost and antighost fields respectively, defined in the first part. We express the conformal-gauge-fixing Faddeev-Popov determinant Δ_{l-c} in terms of ghost (antighost) variables, for $\tilde{\sigma} = y(\sigma)$ as follows

$$\Delta_{FP} = \int d\tilde{\eta}^{++} d\tilde{\eta}^{--} d\theta^{-} d\theta^{+}$$

$$\exp\left[i \int d^2 y \sqrt{-g}\, (2\tilde{\eta}^{++} \tilde{g}_{+-} \partial_{+} \theta^{-} + 2\tilde{\eta}^{--} \tilde{g}_{+-} \partial_{-} \theta^{+}) \right]. \qquad 4.43$$

where now the variables are functions of $y(\sigma)$. The anticommuting variables $\theta^{-}, \theta^{+}, \tilde{\eta}^{++}, \tilde{\eta}^{--}$ are the Faddeev-Popov ghost and antighost fields of the conformal-gauge-fixing determinant - they correspond to the usual $c^{-}, c^{+}, \tilde{\eta}^{++}, \tilde{\eta}^{--}$ [9][13] -. Note that the "tilde" notation in this case refers to the antighost variables.

We define the following change of variables from the conformal gauge to the light-cone gauge anticommuting fields

$$\tilde{b}(\sigma) = 2\, \tilde{\eta}^{++}(y(\sigma))\, \tilde{g}_{+-}(y(\sigma)) \frac{\partial}{\partial y^{+}} \left[\frac{\partial y^{-}}{\partial \sigma^{-}} \frac{\partial}{\partial y^{-}} \tilde{\lambda}^{-}(y(\sigma)) \right]^{-1}$$

$$\tilde{b}^{--}(\sigma) = \tilde{\eta}^{--}(y(\sigma)) \left[\frac{\partial y^{-}}{\partial \sigma^{-}} \right]^{-2}$$

$$c^{-}(\sigma) = \theta^{-}(y(\sigma))$$

$$c^{+}(\sigma) = \theta^{+}(y(\sigma)). \qquad 4.44$$

Next, we make a change of variables in the integral (4.42), from $\tilde{b}, \tilde{b}^{--}, c^{-}$ and c^{+} to the new variables $\tilde{\eta}^{++}, \tilde{\eta}^{--}, \theta^{-}$ and θ^{+}. We need to compute only the variation of the field $\tilde{b}$ with respect to $\tilde{\eta}^{++}$. The rest of the transformations give diagonal matrix elements equal to the identity, while the resulting Jacobian is a matrix with the off-diagonal upper corner zero. Therefore, the transformation from $\tilde{b}$ to $\tilde{\eta}^{++}$ is the only one which gives a non-trivial contribution in the Jacobian. This becomes almost obvious by looking at the transformation equations (4.44) and taking in account the "signature" of the fields involved: The first of the transformation equations is a non-trivial one, since it is a change of a scalar under reparametrization field $\tilde{h}$ to a two tensor $\tilde{\eta}^{++}$; while the rest of the transformations transform a field to another one of the same nature. Using the first of the relations (4.44) we compute the variation of $\tilde{h}(\sigma)$ as follows

$$\delta\tilde{h}(\sigma) =$$

$$= 2\delta\tilde{\eta}^{++}\tilde{g}_{+-}\partial_+(1+\partial_- h)^{-1}(\partial_-\tilde{V}^{+})^{-1} + 2(\partial_-\tilde{\eta}^{++})(\delta h)\tilde{g}_{+-}\partial_+(1+\partial_- h)^{-1}(\partial_-\tilde{V}^{+})^{-1} +$$
$$+ 2\tilde{\eta}^{++}\delta\tilde{g}_{+-}\partial_+(1+\partial_- h)^{-1}(\partial_-\tilde{V}^{+})^{-1} + 2\tilde{\eta}^{++}(\partial_-\tilde{g}_{+-})(\delta h)\partial_+(1+\partial_- h)^{-1}(\partial_-\tilde{V}^{+})^{-1} +$$
$$+ 2\tilde{\eta}^{++}\tilde{g}_{+-}\partial_-\partial_+(1+\partial_- h)^{-1}(\partial_-\tilde{V}^{+})^{-1}\delta h - 2\tilde{\eta}^{++}\tilde{g}_{+-}\partial_+(\partial_-\tilde{V}^{+})^{-1}\partial_-\delta h =$$

$$= 2\tilde{g}_{+-}\partial_+(\partial_-\tilde{V}^{+})^{-1}\delta\tilde{\eta}^{++} + 2(\partial_-\tilde{\eta}^{++})\tilde{g}_{+-}\partial_+(\partial_-\tilde{V}^{+})^{-1}\delta h +$$
$$+ 2\tilde{\eta}^{++}\partial_+(\partial_-\tilde{V}^{+})^{-1}\delta\tilde{g}_{+-} + 2\tilde{\eta}^{++}(\partial_-\tilde{g}_{+-})\partial_+(\partial_-\tilde{V}^{+})^{-1}\delta h +$$
$$+ 2\tilde{\eta}^{++}\tilde{g}_{+-}\partial_-\partial_+(\partial_-\tilde{V}^{+})^{-1}\delta h - 2\tilde{\eta}^{++}\tilde{g}_{+-}\partial_+(\partial_-\tilde{V}^{+})^{-1}\partial_-\delta h.$$

Here all the derivatives are with respect to the argument. Then setting $\partial_- h$ to be zero, as in the previous section, and rearranging the terms we obtain

$$\delta\tilde{h}(\sigma) =$$

$$= 2\tilde{g}_{+-}\partial_+(\partial_-\tilde{V}^{+})^{-1}\delta\tilde{\eta}^{++} + 2\tilde{\eta}^{++}\partial_+(\partial_-\tilde{V}^{+})^{-1}\delta\tilde{g}_{+-} +$$
$$+ \Big[2(\partial_-\tilde{\eta}^{++})\tilde{g}_{+-}\partial_+(\partial_-\tilde{V}^{+})^{-1} + 2\tilde{\eta}^{++}(\partial_-\tilde{g}_{+-})\partial_+(\partial_-\tilde{V}^{+})^{-1} +$$
$$+ 2\tilde{\eta}^{++}\tilde{g}_{+-}\partial_-\partial_+(\partial_-\tilde{V}^{+})^{-1} - 2\tilde{\eta}^{++}\tilde{g}_{+-}\partial_+(\partial_-\tilde{V}^{+})^{-1}\partial_-\Big]\delta h \, .$$

Assuming that the field $\tilde{g}_{+-}$ does not change we obtain the following expression for the variation of $\tilde{h}(\sigma)$

$$\delta\tilde{h}(\sigma) =$$

$$2(\tilde{g}_{+-})_{y(\sigma)}\partial_+(\partial_- f)^{-1}\delta\tilde{\eta}^{++} + 2(\partial_-\tilde{\eta}^{++})_{y(\sigma)}(\tilde{g}_{+-})_{y(\sigma)}\partial_+(\partial_- f)^{-1}\delta h\,. \qquad 4.45$$

where we have used (4.31) and the assumption that $\partial_- f$ is a constant field as in previous section. Therefore it has been taken out of the derivative operators in the computation of the variation of $\tilde{h}(\sigma)$.

Then we observe that the variation of the anticommuting field $\tilde{h}(\sigma)$ depends on the variation of $\tilde{X}^+$ through δh. δh is given in terms of $\delta\tilde{X}^+$ by (4.25), which after using the assumptions of the last paragraph takes the following form

$$\delta h = -(\partial_- f)^{-1}\delta\tilde{X}^+\,. \qquad 4.46$$

Then the matrix elements are

$$\frac{\delta\tilde{h}(\sigma)}{\delta\tilde{\eta}^{++}(\tilde{\sigma})} = 2(\tilde{g}_{+-})_{y(\sigma)}\partial_+(\partial_- f)^{-1}\delta(\tilde{\sigma} - y(\sigma))$$

$$\frac{\delta\tilde{h}(\sigma)}{\delta\tilde{X}^+(\tilde{\sigma})} = -2(\partial_-\tilde{\eta}^{++})_{y(\sigma)}(\tilde{g}_{+-})_{y(\sigma)}\partial_+(\partial_- f)^{-1}(\partial_- f)^{-1}\delta(\tilde{\sigma} - y(\sigma)). \qquad 4.47$$

Because of the mixing of $\tilde{X}^+$ field in the variation of the anticommuting field, we consider the last change of variables simultaneously with the change of variables in the commuting fields, described in section III. Then the Jacobian of the transformation is given by the superdeterminant coming from the transformations of the light-cone variables $g_{++}, \tilde{h}$ to the conformal variables $\tilde{X}^+, \tilde{\eta}^{++}$. All other transformations contribute to the

Jacobian by an identity operator. The final expression for the Jacobian is then given as follows

$$J =$$

$$\det \begin{vmatrix} -2(\tilde{g}_{+-})_{x(\sigma)}\partial_+(\partial_- f)^{-1} & 0 \\ -2(\partial_-\tilde{\eta}^{++})_{x(\sigma)}(\tilde{g}_{+-})_{x(\sigma)}\partial_+(\partial_- f)^{-1}(\partial_- f)^{-1} & 2(\tilde{g}_{+-})_{x(\sigma)}\partial_+(\partial_- f)^{-1} \end{vmatrix} \qquad 4.48$$

where we have used (4.32) and the relations (4.47).

In general, the superdeterminant (sdet) of a supermatrix M

$$M = \begin{vmatrix} A & X \\ Y & B \end{vmatrix}, \qquad 4.49$$

where A and B are matrices with even Grassmann entries while X and Y have odd Grassmann entries, is defined [25] as follows

$$sdet M = \det(A - XB^{-1}Y)\det B^{-1}. \qquad 4.50$$

We compute the determinant of the supermatrix (4.48) using the definition (4.50). It is given by

$$sdet\ J = \det A\ \det B^{-1}, \qquad 4.51$$

where A is the upper left corner of (4.48) and defines the Jacobian of the transformations between the commuting field variables, while B is the lower right corner of (4.48) resulting from the transformations between the anticommuting variables.

The inverse matrix B^{-1} in our case is the inverse of an element and is given trivialy by

$$B^{-1} = \left| 2(\tilde{g}_{--})_{x(\sigma)} \partial_+(\partial_- f)^{-1} \right|^{-1}. \qquad 4.52$$

Then (4.51) becomes

$$sdetJ = \det \left| -2\tilde{g}_{+-} \partial_+(\partial_- f)^{-1} \right| \det \left| 2\tilde{g}_{+-} \partial_+(\partial_- f)^{-1} \right|^{-1}, \qquad 4.53$$

where in the last expression we have suppressed the dependence on $y(\sigma)$. Then (4.53) gives

$$sdet\ J = 1. \qquad 4.54$$

Next, we perform the change of variables in the ghost and commuting fields of the integral (4.33) simultaneously. We first show that under the transformations (4.44) the light-cone ghost action of (4.42) transforms into the conformal gauge ghost action given by the exponent of (4.43). The light-cone ghost action is

$$I_{L-G} = -\int d^2\sigma\sqrt{-g}\left[\tilde{b}\,\partial_- f c^- + 2\tilde{b}^{--} g_{+-}\partial_- c^+\right]. \qquad 4.55$$

By substituting the relations (4.44) in the last expression, we express the light-cone ghosts in terms of the conformal ghosts as follows

$$L_{L-C} =$$

$$-\int d^2\sigma\sqrt{-g}\ \Big[2\tilde\eta^{++}(y(\sigma))\tilde g_{+-}(y(\sigma))\frac{\partial}{\partial y^+}\Big(\frac{\partial}{\partial y^-}\tilde X^+(y(\sigma))\Big)^{-1}\Big(\frac{\partial y^-}{\partial \sigma^-}\Big)^{-1}(\partial_- f)\theta^-(y(\sigma)) +$$

$$+ 2\tilde\eta^{--}(y(\sigma))\Big(\frac{\partial y^-}{\partial \sigma^-}\Big)^{-2}\tilde g_{+-}\partial_-\theta^+(y(\sigma))\Big] \qquad 4.56$$

Then we change the integration variable σ to the new variable y. The measure changes as follows

$$d^2\sigma\sqrt{-g} = d^2y\sqrt{-\tilde g}\ .$$

Then, since

$$\left[\frac{\partial y^-}{\partial \sigma^-}\frac{\partial}{\partial y^-}\tilde X^+(y(\sigma))\right]^{-1} = (\partial_- f)^{-1}. \qquad 4.57$$

using the third of the relations (4.14) we obtain

$$L_{L-C} = -\int d^2y\sqrt{-\tilde g}\ \Big[2\tilde\eta^{++}\tilde g_{+-}\partial_+\theta^- + 2\tilde\eta^{--}\tilde g_{+-}\partial_-\theta^+\Big] \qquad 4.58$$

Where now all the variables are functions of y and the partial derivatives are with respect to the argument y. The last expression (4.58) coincides with the conformal gauge ghost action given by (4.43). Therefore, since the measure of (4.42) transforms into the measure of (4.43) times the Jacobian (4.54) whose determinant is one, Δ_{L-L} transforms totally to Δ_{L-C}. Putting these results together with the results of the section III for the change in the commuting variables, we obtain the path-integral (4.33) in terms of the conformal gauge variables as follows

$$\prod \int \cdots \int d^2\tilde{\sigma}_j \sqrt{-\tilde{g}(\tilde{\sigma}_j)} \prod_{\sigma} d\tilde{g}_{+-}(\tilde{\sigma}) \prod_{\tilde{\sigma}} D\tilde{X}^{i}(\tilde{\sigma}) \prod_{\sigma} d\tilde{X}^{+}(\tilde{\sigma}) \prod_{\tilde{\sigma}} d\tilde{X}^{-}(\tilde{\sigma})(-\tilde{g})^{-1}$$

$$\Delta_{f-c} \exp\left[i \int d^2\tilde{\sigma} (\partial_+ \tilde{X}^{\mu})(\partial_- \tilde{X}_{\mu}) + ik_j^{i} \tilde{X}^{i}(\tilde{\sigma}_j) - ik_j^{-} \tilde{X}^{+}(\tilde{\sigma}_j) - ik_j^{+} \tilde{X}^{-}(\tilde{\sigma}_j) \right]$$

4.59

This completes the proof of the equivalence of the two ways of gauge fixing (the light-cone and conformal) in string theory. The last integral (4.59) coincides with the conformal gauge fixed Polyakov's theory.

V. From conformal to light-cone gauge in Polyakov's theory by direct integrations.

In this last section of this chapter, we show how one can obtain the light-cone gauge fixed Mandelstam's theory from the conformal-gauge-fixed Polyakov's path- integral expression by direct integrations. This at first appears to be very surprising. Nevertheless, as we shall see in this analysis below, using as a guide the light-cone gauge fixing procedure of chapter 2 we can obtain the result by simply integrating out the unphysical degrees of freedom. Although, this shows the power of the path-integral method, in general one does not know which are the unphysical degrees of freedom. In this case we use the knowledge we gained from the previous work of chapter 2. We start from the conformal-gauge-fixed Polyakov's theory of strings given by

$$\prod_j \int \cdots \int d^2\tilde{\sigma}_j \sqrt{-\tilde{g}(\tilde{\sigma}_j)} \prod_{\sigma} d\tilde{g}_{+-}(\tilde{\sigma})(-\tilde{g})^{-1} \prod_{\sigma} d\tilde{X}^{+}(\tilde{\sigma}) \prod_{\tilde{\sigma}} d\tilde{X}^{-}(\tilde{\sigma})$$

$$\prod_{\sigma} D\tilde{X}^{i}(\tilde{\sigma}) \Delta_{f-c} \exp\left[2i \int d^2\tilde{\sigma}\, \partial_+ \tilde{X}^{+}\, \partial_- \tilde{X}^{-} - i \int d^2\tilde{\sigma}\, \partial_+ \tilde{X}^{i}\, \partial_- \tilde{X}^{i}\right.$$

$$+ i \int d^2\tilde{\sigma}\, j^i \tilde{X}^i - i \int d^2\tilde{\sigma}\, j^+ \tilde{X}^- - i \int d^2\tilde{\sigma}\, j^- \tilde{X}^+ \Big| \qquad 4.60$$

where in the last expression, after fixing the conformal gauge, we have separated transverse and longitudinal components. First, we perform the integration over $\tilde{X}^-(\tilde{\sigma})$ and obtain the following δ-function

$$\delta\,(\, 2\partial_-\partial_+\tilde{X}^+(\tilde{\sigma}) + j^+(\tilde{\sigma})\,). \qquad 4.61$$

The last expression specifies the function $\tilde{X}^+(\tilde{\sigma})$ to be the solution of the following familiar equation

$$2\partial_-\partial_+\tilde{X}^+(\tilde{\sigma}) = -j^+(\tilde{\sigma}). \qquad 4.62$$

Comparison of (4.62) with the equation (2.27) gives

$$\tilde{X}^+(\tilde{\sigma}) = f\,(\tilde{\sigma}), \qquad 4.63$$

which is the light-cone gauge condition chosen in chapter 2. It is important to notice here that the equation (4.62) was imposed, in the light-cone gauge fixing of chapter 2, in order to cancel the linear terms in the exponent of the path-integral (2.26); here it appears naturally after integrating over $\tilde{X}^-$. $f\,(\tilde{\sigma})$ is the solution of (4.62) and is given by

$$f\,(\tilde{\sigma}) = \sum_i k_i^+ \Delta(\tilde{\sigma},\tilde{\sigma}_i\,), \qquad 4.64$$

where $\Delta(\tilde{\sigma},\tilde{\sigma}_i\,)$ is the Green's function of the Laplacian.

Next, we perform the integration over $\tilde{X}^{+}(\tilde{\sigma})$. Because of the δ-function (4.61), we substitute $\tilde{X}^{+}(\tilde{\sigma})$ in the exponent of (4.60) by $f(\tilde{\sigma})$. Then (4.60) becomes

$$\prod \int \cdots \int d^2\tilde{\sigma}_i \sqrt{-\tilde{g}(\tilde{\sigma})} \prod_{\sigma} d\tilde{g}_{+-}(\tilde{\sigma})(-\tilde{g})^{-1} \prod_{\sigma} D\tilde{X}^{i}(\tilde{\sigma})$$

$$\Delta_{f-t} \left| \det(2\tilde{g}_{+-}\partial_{+}\partial_{-}) \right|^{-1} \exp\left\{ -i \int d^2\tilde{\sigma}\, \partial_{+}\tilde{X}^{i}\, \partial_{-}\tilde{X}^{i} + \right.$$

$$\left. + i \int d^2\tilde{\sigma}\, j^{i}\tilde{X}^{i} - i \int d^2\tilde{\sigma}\, j^{-} f(\tilde{\sigma}) \right\}. \qquad 4.65$$

Notice that the $\tilde{g}_{+-}$ factor, which multiplies the operator of the resulting determinant, is due to our definition of the δ-function (see relation (2.11)). The relation (4.65) coincides with the light-cone expression (2.36) of the second chapter, except from the determinant-factor. We would expect that the determinant-factor coincide apart maybe from an irrelevant constant. Namely,

$$\frac{\det \begin{vmatrix} -2\tilde{g}_{+-}\partial_{+} & 0 \\ 0 & -2\tilde{g}_{+-}\partial_{-} \end{vmatrix}}{\det(2\tilde{g}_{+-}\partial_{-}\partial_{+})} = \frac{\det \begin{vmatrix} \partial_{-} f & 0 \\ 0 & -2g_{+-}\partial_{-} \end{vmatrix}}{\det(\partial_{-} f\, \partial_{-})}. \qquad 4.66$$

In fact, the relation (4.66) can be read off from the previous calculations (sect. III and IV). One way to see that (4.66) is true is the following: We multiply the numerator and denominator of the second term of (4.66) by a determinant-factor as follows

$$\frac{\det \begin{vmatrix} -2g_{+-}\partial_{+}(\partial_{-} f)^{-1} & 0 \\ 0 & 1 \end{vmatrix} \det \begin{vmatrix} \partial_{-} f & 0 \\ 0 & -2g_{+-}\partial_{-} \end{vmatrix}}{\det[-2g_{+-}\partial_{+}(\partial_{-} f)^{-1}] \det[\partial_{-} f\, \partial_{-}]}. \qquad 4.67$$

The above multiplication is justified through the relation (3.39) of chapter 3. Since the operators of the 2x2 matrix which multiplies the numerator, are of conformal weight zero, we can multiply the upper left corner with the lower right corner in it. In this sense the multiplied-determinant-factor is the identity operator. Performing the multiplications in the expression (4.67) we obtain the first part of (4.66); namely the conformal gauge determinant-factor. This gives a constant after extracting the ϕ-dependence hidden in the operators.

The expression (4.65) is an integration over the transverse components of the string. The variables $\tilde{\sigma}$ and $\tilde{\sigma}_{.}$ are defined on the parameter space of the specific Riemann surface. One can now proceed, in the same way as in chapter 2, to obtain Mandelstam's picture. Namely, after performing a Wick rotation, in order to have the variables defined on Mandelstam's "tube" one has to perform the following conformal transformation

$$z \to w = F(z) = \tau + i\,\sigma. \qquad 4.68$$

This maps the parameter space (here the entire z-plane) onto Mandelstam's "tube". The mapping function is the analytic function $F(z)$ whose real part is given by $f(\tilde{\sigma})$.

The discussion of this section shows that, after integrating over $\tilde{X}^-$, the coincidence between the conformal-gauge-fixed functional formulation of strings and Mandelstam's picture takes place at the point where $\tilde{X}^+$ component takes the value of the light-cone gauge condition for X^+ i.e. the value of the fixed function $f(\sigma)$. This is the only value of $\tilde{X}^+$ which contributes in the computation of the string amplitude. The light-cone gauge fixed slice was shown in chapter 2 to coincide with Mandelstam's physical picture, after the integration over g_{++} and X^- variables, at the point where g_{++} takes the value zero. Therefore the sector of coincidence of both gauge fixed slices (the conformal and light-cone) in string theory has two of the coordinates fixed at the values: $\tilde{X}^{+} = f(\sigma)$, $g_{++} = 0$.

IV. Conclusion

In this chapter, we have explicitly performed the change of gauge from the light-cone to the conformal gauge in Polyakov's functional formulation of strings. It has been shown that by changing coordinates in the functional integration variables one can obtain the string theory in the light-cone or the conformal gauge. We wish to stress the fact that the coordinate transformations needed for the change of gauge between the light-cone and conformal gauge is a change of variables in the commuting fields as well as in the Faddeev-Popov ghosts. This is a proof of the equivalence of the two different ways of gauge fixing of the theory of strings.

In the last section, we show that Mandelstam's picture emerges from the conformal-gauge-fixed Polyakov's theory, after direct integrations. This is another elaboration of the equivalence of the two ways of gauge fixing of Polyakov's theory, since we proved in chapter 2 that Mandelstam's picture emerges from the light-cone gauge fixed Polyakov's integral formulation of strings. In this last case we showed that the conformal-gauge-fixed Polyakov's theory becomes the Mandelstam's picture only when the $\tilde{X}^{+}$ component of its field variables takes the fixed value chosen for it in the light-cone gauge fixing performed in chapter 2 i. e. $f(\sigma)$. On the other hand, previously we showed that Mandelstam's picture can be obtained from the Gervais-Sakita light-cone gauge fixed theory only when $g_{++}(\sigma)$ becomes zero i.e. takes the fixed value of the conformal gauge condition. Therefore the sector of coincidence of the two gauges has the coordinates

X^+, g_{++} and g_{--} fixed, while all other coordinates can freely take any value. In other words the Mandelstam's light-cone gauge fixed theory has the above three coordinates fixed. It is now clear why the usual (conventional) light-cone gauge fixing of string theory works. Even if the light-cone gauge is an independent gauge and can be studied independently than the conformal gauge in the theory of strings, one has to gauge fix three of the variables in Polyakov's theory in order to obtain Mandelstam's picture. Therefore the usual choice $X^+ = \tau$ after fixing the conformal gauge is allowed within our last analysis.

After the discussion of this last chapter, it is straightforward to generalize the problem for higher genus surfaces. The generalization then can be done in an equivalent way by using either the light-cone or the conformal gauge. In the next chapter, we study the generalization to the higher genus case.

CHAPTER 5

Generalization to the higher genus surfaces.

I Introduction

In the previous chapters, we explicitly showed the equivalence between the light-cone and conformal gauge. In the second chapter, we derived Mandelstam's picture from Polyakov's light-cone gauge fixed theory, while in the fourth, we obtained one gauge from the other by changing coordinates and also we obtained Mandelstam's expression from the conformal-gauge-fixed Polyakov's theory. Therefore, in order to discuss the generalization to the higher genus case, we can start from either the light-cone or the conformal-gauge-fixed path-integral expression.

We follow the procedure of section V of the last chapter, choosing the conformal-gauge-fixed path-integral expression (4.60) as the starting point. Since our method is shown explicitly in the previous chapters, in what follows we emphasize only the points which are relevant to the higher genus case. There are two parts which need modification. The first is related to the solution of the Green's function equation and the second is the integration over the moduli parameters. We first analyze the question related to the Green's function equation for an arbitrary genus surface. The discussion of the moduli space integration is given later.

The interesting point to notice is that in our method the dependence on the genus of the surface is apparent in a natural way in the Green's function equation. The equation which specifies the function $f(\sigma)$ and hence $X^+(\sigma)$ is the following

$$2\,\partial_-\partial_+ X^+(\sigma) = -j^+(\sigma) \qquad 5.1$$

This was obtained in the last chapter in a δ -function, after performing the integration over the variable $X^-(\sigma)$ (see relation (4.61)). For the light-cone gauge fixed theory this equation was imposed as the requirement for the cancellation of the linear terms in the action. Note that in this section we drop the "tilde" notation since our discussion does not depend on the choice of gauge. $j^+(\sigma)$ in (5.1) is given by

$$j^+(\sigma) = \sum_r \delta^{(2)}(\sigma-\sigma_r)k_r^+ . \qquad 5.2$$

Then the function $X^+(\sigma)$ is given by the solution of (5.1)

$$X^+(\sigma) = i\int G(\sigma-\sigma')\,j^+(\sigma')d\sigma', \qquad 5.3$$

where $G(\sigma-\sigma')$ is the Green's function satisfying the following equation

$$\nabla_\sigma^2\, G(\sigma,\sigma') = i\,\delta^{(2)}(\sigma-\sigma'). \qquad 5.4$$

The Green's function depends on the topology of the specific surface. Therefore in order to discuss the problem of higher genus, one has to solve the Laplace equation (5.4) for each Riemann surface with a given topology. The Riemann surface determines uniquely the Green's function. On the other hand, the Green's function contains all the information needed to map the specific Riemann surface onto Mandelstam's light-cone diagram. The mapping function is the analytic function whose real part is given by $X^+(\sigma)$ [26], defined by (5.3). In what follows, we demonstrate the procedure only for the case of torus (g=1). The method applies straightforwardly for any number of genus.

II. The case of torus.

The torus is defined by the following identification [27]

$$z = z + n\lambda_1 + m\lambda_2 \qquad 5.5$$

in the complex z -plane. Here n,m are arbitrary integers and λ_1, λ_2 are complex vectors. The parameter space consists of all parallelograms identical to the fundamental one, which is defined by the vectors λ_1 and λ_2. Then the Green's function of the Laplacian is defined to satisfy the condition

$$\Psi(z) = \Psi(z + n\lambda_1 + m\lambda_2). \qquad 5.6$$

It is given by the following doubly-periodic function

$$G(z, z') = \sum_{n,m} \ln |z - z' - n - m\tau| + \frac{|z'||z|}{|\tau|}. \qquad 5.7$$

where we defined $\frac{\lambda_2}{\lambda_1} = \tau$ (not to be confused with the light-cone "time" variable τ), with $\mathrm{Im}\,\tau > 0$. One can check that this is the correct function which satisfies (5.6). The last term in the equation (5.7) assures the invariance of the Green's function under the transformation (5.5) [8]. Then the function $f(a)$ is given by

$$f(\sigma) = \sum k_i^+ G(z, z_i) =$$

$$\sum_{i,n,m} k_i^+ \ln|z - z_i - n - m\tau| + \sum k_i^+ \frac{|z||z_i|}{|\tau|}. \qquad 5.8$$

We denote by F(z) the analytic function whose real part is given by f(z). Mandelstam's conformal transformation is defined as follows

$$z \to w = F(z) = \tau + i\sigma. \qquad 5.9$$

For the purpose of the mapping we need to study the singularities of the function F(z). It suffices to study the derivative of F(z) given by

$$F'(z) = \sum_{i,n,m} \frac{k_i^+}{(z - z_i - n - m\tau)} + \sum_i k_i^+ \frac{|z_i|}{z|\tau|}. \qquad 5.10$$

$F'(z)$ has simple poles at the points

$$z = z_i + n + m\tau, \quad n, m = \ldots -2, -1, 0, 1, 2, \ldots \qquad 5.11$$

where only the points $z = z_i$, are in the fundamental region. We consider the case with four external states, thus we let $i = 1, \cdots, 4$.

Next, we study the zeros of $F'(z)$. By inspection we see that $F'(z)$ becomes zero at four different points lying in the fundamental region. To actually calculate these points, it is easier to express the mapping function in terms of the Jacobi θ-function. We first show that the Green's function except from the last correction term is given in terms of the first Jacobi θ-function as follows

$$G(z,z') = \mathrm{Re}\left[-2\pi i \ln\theta_1\left(\frac{\frac{\ln z' - \ln z}{2\pi i}}{\tau}\right)\right]. \qquad 5.12$$

We denote the zeros of $F(z)$ by $\hat{z}_i$, where $\hat{z}_i$ are defined by

$$\frac{\partial \Gamma(z)}{\partial z}\Big|_{z=\hat{z}_i} = 0 \qquad 5.13$$

Since Mandelstam's mapping has been described in a number of references [8], [28], in what follows we give only a brief discussion. Our interest is to show how the knowledge of the Green's function alone gives a complete description of Mandelstam's conformal mapping. For simplicity we locate the external states z_i on the boundary AA as shown in fig.4.

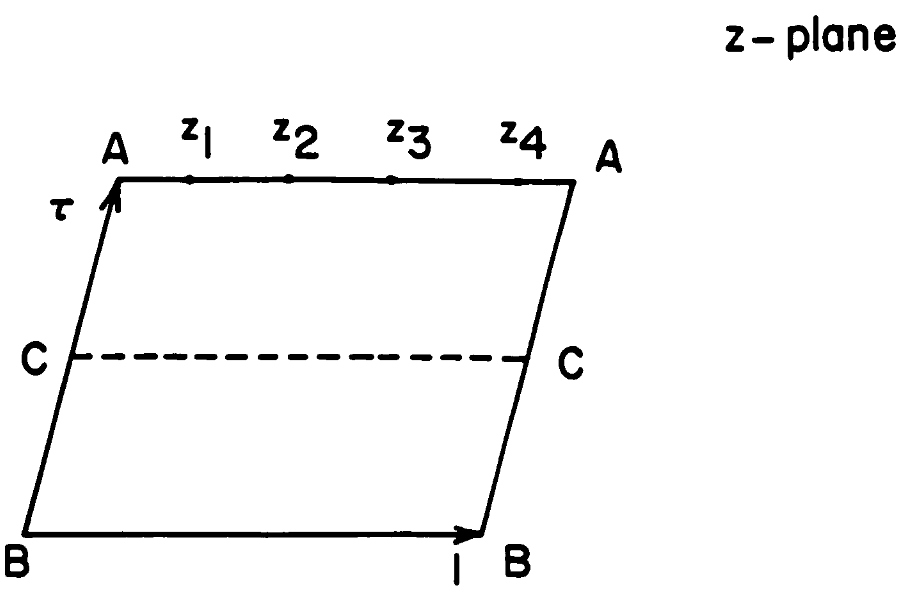


Fig. 4

Fig. 4: Torus diagram in the z-plane.

It is easier to understand the mapping if we consider the closed string diagram as made up by identifying two equivalent open string diagrams. For this purpose we cut the parallelogram by an imaginary line CC, as it is shown in fig.4, and consider each of the two semi-parallelograms to be the parameter space of an open string diagram. The following transformation

$$\zeta = \exp z \qquad 5.14$$

maps the parallelogram of fig.4 to an annulus in the ζ-plane of fig.5.

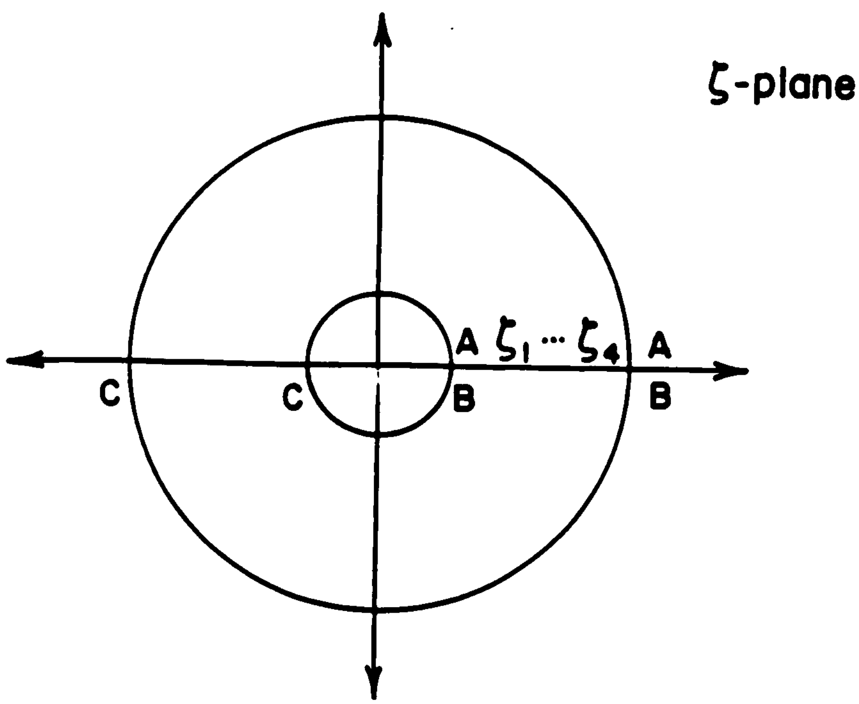


Fig. 5

Fig. 5: The annulus of the torus of fig.4.

The identification (5.5) corresponds to the following in the ζ-plane

$$\zeta = w^{-m} \zeta \qquad 5.15$$

where $w = \rho \exp i \sigma$. The torus is the double [*] of the annulus of fig. 5.

Next, we study the mapping of the boundaries AA and CC of the open string diagram, under F(z). Fig.5 suggests that the boundary points A(C) of the open string diagram can be taken to be the positive (negative) segments of the real axis contained inside the annulus. The external states - points z_i - map on the positive segment of the real axis - points ζ_i -.

The function $F(z)$ maps each semi-annulus of fig.5 onto a finite rectilinear segment parallel to the real axis in the w-plane [26] as it is shown in the following fig.6.

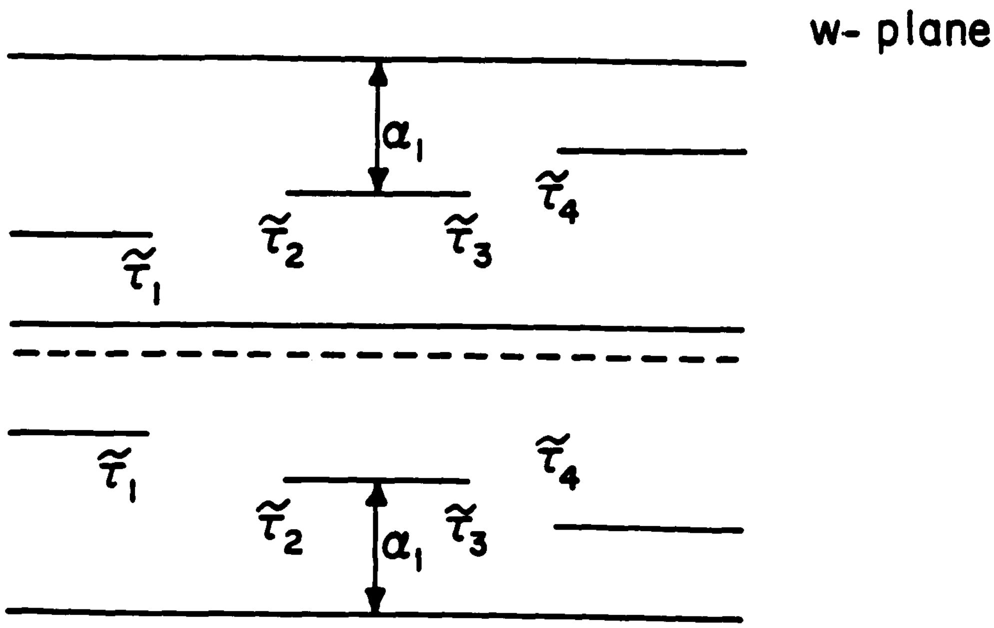


Fig. 6

Fig. 6: Light-cone diagram of the torus.

The positive part of the real axis maps onto the external boundary of Mandelstam's string diagram of fig. 6. The points z_i - poles of $F'(z)$ - map at the infinity in the w-plane. The points $\tilde{z}_i$ - zeros of $F'(z)$ - map at the turning points in the w-plane. There exist four zero-points for the one-loop four point function case. These map at four turning points in the string diagram, denoted by $\tilde{\tau}_i$ in fig. 6. They are the points where the strings split or join. The negative segment of the real axis - points C of fig. 5 - maps onto a finite line parallel to the real axis [8]. This is the slit of Mandelstam's diagram for the one-loop case. The length of the slit is specified by the value of the function at those of the $\tilde{z}_i$'s which map at the relevant turning points. The exact position of the slit in the imaginary axis is given by the value of the imaginary part $\mathrm{Im}F(z)$ at $z = -|z|$. By identifying the points as shown in fig. 6, we obtain Mandelstam's diagram for the one-loop closed string case.

The discussion of the last paragraph leads to the conclusion that, the logarithmic singularities of the Green's function - poles of $F'(z)$- correspond to the external states of the light-cone string diagram, while the points of non-analyticity of the Green's function -zeros of $F'(z)$ - are identified with the interaction times in the physical picture of the string-process. The slit on Mandelstam's picture contains the information of the definition of the torus (5.5). Going from a point z of the fundamental parallelogram to a point in a different parallelogram defined by (5.5) on the parameter space corresponds to going around the slit on Mandelstam's picture. The number of times of cycling the slit is specified by the integers n and m.

Next, we discuss the subtle point of the integration over the moduli space. For this purpose let us refer again back to the expression (4.60). This formally looks the same for any Riemann surface. The dependence on the topology is hidden in the function $F(\sigma)$ and in the integration over the Koba-Nielsen variables. After performing the conformal transformation for a given surface, as we described above, the domain of the integration of the variables $\tilde{\sigma}$ and $\tilde{\sigma}_j$ is defined to be Mandelstam's "tube".

There is a dependence on the topology of the surface hidden in the integration over the Koba-Nielsen variables $\tilde{\sigma}_j$. These are as many as the external states. For the case of the sphere, because of the invariance of the integrand under the Mobius transformation

$$z \rightarrow z' = \frac{az + b}{cz + d}, \qquad 5.16$$

we fix three of the Koba-Nielsen variables. The integration variables are then the remaining n-3, where n is the number of the external states. After performing the Mandelstam's conformal mapping, we must transform the Koba-Nielsen variables to the $\tilde{\tau}_i$, $i = 1, \ldots, n-3$ variables of the light-cone diagram, as we discussed in chapter 2. In this case, since the parameter space is the whole complex plane - the surface has trivial topology -, there is not any moduli parameter involved. Moreover, there is one to one correspondence between the Koba-Nielsen variables and the independent parameters (interaction times τ_i) of the light-cone diagram. Therefore, the integration over the τ_i variables in Mandelstam's plane can be easily shown that it is equivalent to the integration over the Koba-Nielsen variables in the complex plane (the parameter space of the Riemann surface).

For the case of torus the Mobius transformation takes z of the fundamental region, defined by the parallelogram of fig. 4, to a point in a different region in the complex plane. Since the parameter space is defined to be the fundamental region, we can fix only one of the Koba-Nielsen variables, which corresponds to the invariance under the scaling of z [8]. There is one extra integration variable resulting from the fact that tori with different values of the parameter τ cannot be conformally transformed onto one another. This is the moduli parameter of the torus.

There is a symmetry related with this parameter. The integral is invariant under the following transformations

$$\tau \to -\frac{1}{\tau} \quad \text{and} \quad \tau \to \tau + 1. \tag{5.17}$$

This is the modular invariance. Because of this invariance we must integrate over a region in the upper half τ-plane, such that two points in that region are not related by a transformation (5.17), but any other point of the half upper plane can be obtained by a point from this region by a transformation (5.17) [8]. This is the fundamental region defined by [13], [31]

$$-\frac{1}{2} \leqslant \text{Re}\,\tau \leqslant \frac{1}{2}, \quad \text{and} \quad |\tau| > 1. \tag{5.18}$$

The Koba-Nielsen variables, which are n-1 for this case, and the parameter τ are transformed to the n-1 interaction times $\tilde{\tau}$ ($i = 1, \cdots n$ with $\tilde{\tau}_1 = 0$) and the parameter α_1 specifying the position of the slit, in the light-cone diagram. All $\tilde{\tau}_i$'s and α_1 are complex for the closed string case. For any topology different than sphere one has to prove that the integration over the fundamental region of the moduli parameter and over the Koba-Nielsen variables, which take values on the fundamental parallelogram of fig. 4, is equivalent to the integration over the new light-cone parameters in the Mandelstam "tube". In other words, it is not obvious that both integration regions are equal. For the one-loop case it has been explicitly shown [29] that the integration over the moduli parameter τ for specific values of the Koba-Nielsen variables gives the same integration region as the integration over the corresponding parameters on the light-cone diagram. This provides a proof that, for the case of the torus, the integration region of the light-cone diagram coincides with the single cover of the moduli space.

In general for the case of surface with h handles with $h \geqslant 2$ and n external states the number of moduli parameters characterizing the surface is 3h-3 [13], [22], [31] in complex notation. The number of the integration variables are 3h-3+n, which must be transformed to the parameters of the light-cone diagram. These parameters are the interaction times, the diameters of the internal "tubes" and the twist angles of the internal "tubes". They are exactly 3h-3+n and denoted by $\tilde{\tau}$, α_a and θ_a respectively [13], [22]. Therefore the counting of the parameters is correct. In order to explicitly show, however, that the integration region of the light-cone diagram coincides with a single cover of the moduli space, is mush more involved in this case.

The calculation of the conformal anomaly and the determinants involved are the same as in the genus zero case, since the dependence on the topology is only on the Green's function and the integrations and gauge fixing procedure, which we used, do not change with the genus of the surface. The determinant, however, coming from the transformations from the Koba-Nielsen variables to the light-cone parameters, is more complicated to calculate in this case and it can give different result. From Mandelstam's work [8] we would expect that the Jacobian of this transformation times the change in the Laplacian under the conformal transformation of the transverse components X^i of the string gives a constant in the dimensions of space-time equal to 26 for any number of genus surface.

III. Conclusion

In this chapter, we have generalized the problem of the equivalence between the light-cone and conformal gauge to the case of higher genus surfaces. In particular, we have shown how one can obtain Mandelstam's picture from the conformal (or equivalently from the light- cone) gauge-fixed Polyakov's theory for any topology.

The important point to notice is that our method can be immediately generalized to the higher genus surfaces through the Green's function equation, which appears naturally as we discussed in the last chapter and in chapter 2. The prescription is the following: One has to solve the Laplace equation and obtain the Green's function for each surface with any complicated, given topology. Then one uses the analytic function ($F(z)$) whose Real part is the Green's function and maps the Riemann surface to the Mandelstam's plane. As we explicitly showed for the torus, the Green's function alone describes completely the Mandelstam's conformal mapping. Specifically, the poles of the derivative of the function $F(z)$ correspond to the external states in the light-cone diagram, while the zeros of $f'(z)$ map at the interaction times in Mandelstam's plane. Moreover, the slits in Mandelstam's picture give all the information about the topology of the Riemann surface, since they correspond to the handles of the surface.

The change from the Koba-Nielsen variables to the new light-cone parameters can be done in the same way as for the case of the sphere. One, however, now has to include the integ ation over the moduli parameters. A simple way to see this is by counting the parameters: There are 3h-3+n parameters necessary to correctly describe the light-cone diagram. In general for $h \geqslant 2$ there are n Koba-Nielsen variables - equal to the number of the external states - in the Riemann surface. Therefore 3h-3 moduli parameters are needed in order to have the same number of parameters as in the light-cone description. Finally, in general one has to prove that the integration region over the Koba-Nielsen variables and over the moduli parameter on the Riemann surface is equivalent to the integration region over the light-cone parameters on the Mandelstam's "tube".

EPILOGUE

In this last chapter, we present an overview of the whole work included in this treatise and a quick summary of the main results, since the conclusions have been already stated in the relevant sections of the preceding chapters. Moreover, we emphasize a few points which we feel that they are of some unique importance and give a short guideline of possible future directions in which research related to the subject exposed in this thesis might be continued.

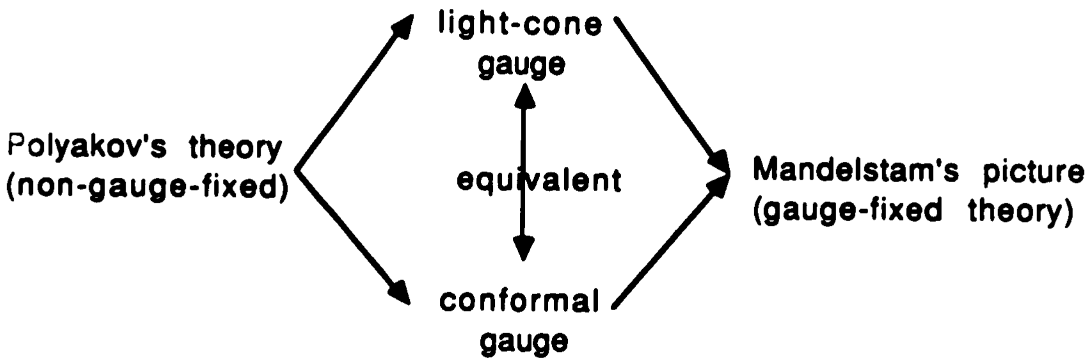
The essence of this treatise is the formulation of the theory of strings as a gauge theory, in the functional integral approach. The main point made in chapters 2 and 3 is that the light-cone gauge can be chosen in string theory independently from the conformal gauge. In other words, one can start from Polyakov's path-integral of strings and choose either the light-cone or the conformal gauge, depending on the specific problem one wishes to study. The method used is fixing the Gervais-Sakita light-cone gauge conditions in Polyakov's functional formulation of the bosonic string. In particular, it is shown by explicit calculations that Mandelstam's picture emerges from Polyakov's theory, after fixing the above light-cone gauge conditions. This indikates that the two gauges are equivalent and we should be able to obtain one from the other by a simple coordinate transformation.

The equivalence between those two gauges has been explicitly shown, in chapter 4, by the changing of gauge method. In particular, the change of gauge from the light-cone to the conformal gauge has been performed, by changing integration variables in Polyakov's functional formulation of strings. At this point, we like to stress the fact that the coordinate transformation needed for changing gauge between the light-cone and conformal gauge is a change of variables in the commuting fields as well as in Faddeev-Popov ghost variables.

It has also been shown, in the last section of this same chapter, that Mandelstam's picture can be obtained from the conformal-gauge-fixed Polyakov's expression, after integrating out the unphysical field variables. This is another proof of the equivalence of the two different ways of gauge fixing the theory of strings, since, as it is shown in chapter 2,, Mandelstam's picture emerges from the light-cone gauge fixed Polyakov's theory.

The main point, we wish to emphasize here, is that, in general, the light-cone and conformal gauge coincide in most of the variables but few. After the integration over the minus-component of the field (which is an unphysical degree of freedom of the string) the total coincidence of the two gauge fixed slices takes place at the point where the plus-component of the string field at the conformal gauge fixed theory takes the fixed value chosen for this same variable at the light-cone gauge fixed theory; and the component of the metric, which is not fixed in the light-cone gauge fixed theory, takes the value of the conformal-gauge-fixed theory, i.e. zero. Therefore the sector of coincidence of the two gauges has, after integrating out the minus component of the string field, three of the coordinates fixed, namely the "plus-component" of the string field and the two diagonal components of the metric, while all other (physical) coordinates can freely take any value. The fact that even after the initial gauge fixing of the theory there are residual unphysical degrees of freedom indicates the existence of a residual gauge freedom. Indeed, in both gauges (our light-cone gauge and the usual conformal gauge) we are free to perform a conformal transformation. This extra freedom is used in the traditional light-cone approach to set the plus-component of the field equal to τ, after fixing the conformal gauge. In our method this corresponds to the fact that Mandelstam's picture is obtained from eigher of the two gauge-fixed theories, only when the gauge fixed slice becomes the "sector of coincidence" which has three of the coordinates fixed. The resulting theory does not contain extra degrees of freedom.

Summarizing we give the essence of this work, by the following diagram



One of the advantages of this approach is that the problem of the generalization to the higher loop case can be essentially discussed in terms of the solution of Green's function equation. Green's function equation appears naturally in our method, as a result of the integration over $\tilde{X}^-$ variable, inside a δ-function. As it is obvious from the discussion of the last section of chapter 4, Green's function is essentially the gauge fixed value of the variable X^+ of the light-cone gauge i.e. $f(\sigma)$. In this spirit, it has been shown for the higher genus case that the Gervais-Sakita light-cone gauge and the conformal gauge in Polyakov's path-integral formulation of strings are equivalent. To obtain Mandelstam's picture from a gauge fixed Polyakov's theory one has to perform the conformal mapping from the parameter space of the Riemann surface onto Mandelstam's "tube". The mapping function is the analytic function whose real part is given by the solution of the Laplace equation for that specific surface.

Before we go on to discuss some of the possible future directions of this work, let us take this opportunity and speculate about the question of the conformal symmetry in connection with the elimination of the unphysical degrees of freedom in string theory. The present study does not include any rigurous argument or specific reasoning concerning this question. Nevertheless, according to the author's opinion, there are several interesting results in this treatise related to the above problem, which we like to bring to the reader's attention. As we discussed in the introduction, conformal invariance, which is responsible for the elimination of the unphysical degrees of freedom, remains after fixing the conformal (diagonal) gauge. Conformal gauge is then formulated with the inclusion of ghosts, while light-cone gauge is known in the literature to be ghost-free gauge.

Firstly, notice the connection of the BRST-symmetry with the conformal invariance in our approach. We pointed out, in chapter 2, that BRST-invariant Lagrangian and BRST-transformations possess conformal symmetry. This same Lagrangian which is shown to be conformal invariant is the BRST invariant Lagrangian of Kugo-Uehara gauge fixing procedure, which introduces the ghosts in the canonical formulation of the theory. On the other hand, conformal invariance in string theory is responsible for the elimination of the ghost fields (unphysical degrees of freedom). It is, therefore, the author's personal opinion based on an intuitive understanding of this question, that the ghost fields in the first quantized approach are artificial objects introduced merely for calculational convenience and have no other significance. Mandelstam's light-cone gauge which depends only on the physical degrees of freedom does not include ghosts and is not conformally invariant. In this treatise, fixing the Gervais-Sakita light-cone gauge we introduced ghosts while keeping two unphysical degrees of freedom (g^{--} and X^-) in the Lagrangian. The total Lagrangian which includes the commuting fields (the transverse physical modes and the two extra fields) plus the ghost Lagrangian is conformally invariant. If we choose to integrate out the unphysical commuting fields, we must combine the determinant appearing because of this integration with the Faddeev- Popov determinant (corresponding to the ghost Lagrangian) in order to extract the ϕ-dependence. What remains then is Mandelstam's expression which depends on the physical modes only and it contains no ghosts. It appears, therefore, to be one's own choice to either keep the extra two unphysical degrees of freedom and introduce ghosts, which means that the theory is formulated in a conformally invariant manner, or formulate the theory with only the physical degrees of freedom. In this last case, however, one has neither ghosts nor conformal symmetry.

An analogous mechanism also holds for the conformal gauge case. This is formulated with two unphysical degrees of freedom and the ghost (antighost) fields, while it possesses conformal symmetry. One can choose to integrate out the two extra degrees of freedom, as it is shown in chapter 4, and obtain the theory with only physical modes. However, in doing so one has to combine the Faddeev-Popov determinant with this extra determinant resulting from the last integration, in order to compute the conformal anomaly, and the final expression does not contain ghosts. The dimensional reduction then is rather trivial, in the first quantized language, as it is evident from the argument of the last few paragraphs.

Few words in connection with the work of Giddings and Wolpert [30] are now in order. They proved the equivalence between Polyakov's theory and Mandelstam's picture, by analyzing the integral representation of multiloop amplitudes in terms of moduli parameters and Koba-Nielsen variables. Specifically, they have shown that there is one to one correspondence between Riemann surfaces and light-cone diagrams. In their work [30],[11], which is based in the existence of a unique Abelian differential, it is argued that the light-cone diagrams cover the moduli space once. In our method we see how, by knowing the Green's function of the specific Riemann surface, we obtain the corresponding light-cone diagram. The Green's function is the real part of the third Abelian integral of their language. In order to answer such possible questions as which is the purpose of our work given the existence of the work of ref. [11], [30], we emphasize that our method has the advantage of being explicit. Moreover, in our work which was already underway before the above papers were published, we are interested in studying the internal connection between the two gauges besides proving the equivalence between the two functional approaches.

Future work on the subject could continue in several directions. It is a very interesting problem to investigate similar questions in the operator formalism. As we stated in the introduction, Kato and Ogawa have formulated the quantized theory of the bosonic string in the covariant canonical formalism. Their work makes essential use of the BRST-symmetry. One could investigate the quantization of Polyakov's Lagrangian in the light-cone gauge in the operator formalism, in a similar way as Kato and Ogawa did for the conformal gauge. This study would shed light in the understanding of the BRST-symmetry in the light-cone gauge and the role of the Faddeev-Popov ghosts. The question of whether the canonical formalism is equivalent to the path-integral formulation is a very important one and relevant to this problem.

Another interesting direction is the application of this method in the second quantized approach. The analogous problem is again stated in the introduction. One can investigate the connection between Witten's covariant field theory of stings with the light-cone string field theory of Kaku-Kikkawa.

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