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CHARACTERIZATIONS OF SOCIAL DECISION FUNCTIONS

by

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TABLE OF CONTENTS

I BASIC NOTIONS AND DEVELOPMENT

A FUNDAMENTALS AND EARLY HISTORY

Social choice is the theory which deals with the processes by which a group of decision makers (henceforth called voters), faced with a set of alternatives, may determine some best "choice." The voters and alternatives, in classical political applications, include such situations as elections to public office, committees, juries, and the like. The theory also extends to economics, where the voters may be consumers and the alternatives, different "bundles of goods." In short, social choice is applicable wherever decisions must be made. The emphasis in this work, however, will be on social choice among a finite set of alternatives.

Since the social choice process involves matching a "choice" to each evaluation by voters of alternatives, it takes the form of a function, whose range and domain depend upon the particular context. Recent investigations have dealt mainly with two major subclasses of social choice processes, distinguished by the type of choice produced (see [3], [11], and [16]). The first of these is the more common social choice function, in which the choice is some non-empty subset of the alternatives. The other, somewhat more general function is the social decision function, wherein the choice consists of permutations (orders of preference) of the alternatives.

Definition 1.1: Let $A = \{a_1, \ldots, a_m\}$ be a finite set of alternatives. Let $D^m = \bigcup_{n=0}^{\infty} \bigotimes_n S_m$, where S_m is the set of permutations $\sigma = (\sigma(a_1), \ldots, \sigma(a_m))$ on m objects. (D^m is the set of all possible tuples of permutations on the m alternatives). Finally, let P(A) be the power set of A. A social choice function (SCF) is a mapping $g_m: D^m \longrightarrow P(A) - \emptyset$.

Each "preference profile" $\bar{d} \in D^m$ represents a set of n orderings, $\sigma_1, \ldots, \sigma_n \in S_m$ of the alternatives, corresponding to the preferences of the n voters, 1, ..., n, respectively. Specifically, $\bar{d}_i = \sigma$ means that voter i has preference order $\sigma = (\sigma(a_1), \ldots, \sigma(a_m))$.

At this time, we note that our approach implicitly restricts the SCFs considered in two ways. First, the range excludes the empty set, $\emptyset$, so that the SCF must actually make some choice. This treatment still allows for "no decision," though, provided that one of the alternatives is called "no decision." Second, the set of alternatives is finite, as mentioned earlier. Let us introduce an example to illustrate these ideas:

Example 1.1: Suppose that there are seven voters, 1, ..., 7, and three alternatives, a_1, a_2, and a_3. Suppose further that the voters have the following preference orders:

1	2	3	4	5	6	7
a_1	a_2	a_3	a_2	a_3	a_3	a_1
a_2	a_1	a_1	a_1	a_1	a_1	a_2
a_3	a_3	a_2	a_3	a_2	a_2	a_3

Let us denote the six elements of S_3 by:

$\eta_1 = e = a_1, a_2, a_3$ is the identity $\qquad \eta_2 = a_1, a_3, a_2$

$\eta_3 = a_2, a_1, a_3 \qquad \eta_4 = a_2, a_3, a_1 \qquad \eta_5 = a_3, a_1, a_2 \qquad \eta_6 = a_3, a_2, a_1$

For this example, then, $\bar{d} = (e, \eta_3, \eta_5, \eta_3, \eta_5, \eta_5, e)$. Consider these SCFs:

1) The dictatorial function which chooses the first choice of voter 2, and chooses A in the absence of voter 2. Here $g(\bar{d}) = \{a_2\}$.

2) The celebrated Borda scoring function, which assigns a value of 0 to

an alternative each time it is ranked last, 1 each time it is ranked next to last, ... , k each time it is ranked k from last. The Borda scoring function then chooses as the winning set those alternatives with the maximum score. Here we would have $g(\bar{d}) = \{a_1\}$.

3) The common simple plurality function, which chooses as the winning set of alternatives those which are placed first a maximum number of times. Here $g(\bar{d}) = \{a_3\}$.

4) The least favored elimination method wherein the alternative which is placed last most often is eliminated from consideration. The process continues on the remaining alternatives until only one alternative is left or a tie ensues, in which case all of the remaining alternatives are declared winners. Here $g(\bar{d}) = \{a_1\}$.

Some assumptions are implicit in our definition of the SCF. First, no assumption is made regarding strengths of preference. Indeed, as is often the case, in practice it is virtually impossible actually to measure strengths of preference. Moreover, a "nonweighted" preference order is in some sense an "average" of possible preference orders.

A second assumption excludes individual intransitivities among alternatives, so that if a voter prefers a_i to a_j and a_j to a_k, we assume that he prefers a_i to a_k. Once again, this assumption may not be universally valid, but it is the kind of "rationality" axiom which naturally finds its way into the theory.

A third assumption is that no voter is indifferent between alternatives. Although it might seem that this assumption is overly restrictive, it has in fact the opposite effect. In most cases it provides for a stronger theory, and admits treatment of indifference as a special case.

The body of this paper deals for the most part with social decision functions. These are now defined:

<u>Definition 1.2</u>: Let A, S_m and D^m be as in Definition 1.1. Let $P(S_m)$ be the power set of S_m. A <u>social decision function</u> (SDF) is a mapping f_m: $D^m \longrightarrow P(S_m) - \emptyset$.

In contrast to SCFs, SDFs produce entire preference orders on the alternatives as the "choice." The two types of functions are related, though, since each SDF defines an associated SCF. Henceforth, we will always drop the subscript m when the context is otherwise clear.

<u>Definition 1.3</u>: The SCF $g^f(\overline{d}) = \{a_i \mid a_i$ appears first in at least one of the preference orders in $f(\overline{d})\}$ is said to be the SCF <u>derived</u> from f.

Historically, the study of social choice began around the time of the French Revolution, when scientists became quite interested in the democratic process and its workings. Jean-Charles de Borda, a pioneer in the field, contributed a major paper to the French Academy of Sciences in 1784. The paper, entitled "Mémoire sur les Élections au Scrutin ," [5] presented the Borda scoring function discussed in Example 1.1, along with a common-sense defense of his ideas. Although it is this work which serves as the principal source of his thoughts, Borda had already in 1770 presented a paper on elections to the Academy.

As a result of these papers, Borda's ideas on elections became widely known throughout the Academy. Indirectly, they bore an influence on the work of the Marquis de Condorcet. He wrote one of the most important papers ever on social choice, entitled "Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix," [7] published in 1785. This work made two major contributions to the theory. First, it outlined the so-called "Condorcet criterion" for elections. This criterion states basically that if there is some alternative a_i which would defeat each of the other alternatives in an election in which the remaining alternatives were excluded, then a_i should be chosen uniquely. In Example 1.1, for instance, a_1 is a Condorcet alternative.

Second, Condorcet considered elections in which there is no such alternative. In doing so, he was the first to explore the famous "voting paradox," in which each of the voters has a transitive ordering of the alternatives, but the consensus ordering is intransitive. Since the probability of such a paradox is, when there are many alternatives, greater than that of a Condorcet alternative, Condorcet opened up a major area of investigation (see [4], for instance).

One of the nineteenth century figures who explored the voting paradox was the Rev. Charles Dodgson (Lewis Carroll), who discussed the deficiencies, as he saw them, of the election systems of the day in [8], [9], and [10]. Dodgson then offered his own system, which he claimed would deal properly with the voting paradox, without the shortcomings of the other systems. After Dodgson's contributions, the field remained fairly dormant until the advent of Kenneth Arrow's work on social welfare functions in the nineteen fifties. (See [1]). In the meantime, there were some advances in related areas; e.g., the Pareto optimality condition [17], and the introduction of game theory by Von Neumann and Morgenstern [19], among others. These works had only an ancillary, albeit important, effect on classical social choice.

It remained for Arrow to exploit the voting paradox. He used the voting paradox (See [1], p. 51) to show that if an election system satisfies certain "fairness" criteria which seemed indispensible to a democracy, that system must be a dictatorship. With the advent of Arrow's work, much of the emphasis in the field has turned to determining consistent and desirable axioms for characterizing SCFs and SDFs.

In particular, this paper examines a recent election system, the Kemeny function [13] , in the light of intuitively appealing axioms. It does so by considering the effect of a "Condorcet-like" principle, while exploiting the symmetries of the problem. The mathematics involved is that of convex analysis.

B DOMAINS FOR SOCIAL FUNCTIONS

We have thus far considered only the most general case of social functions (i.e. SCFs and SDFs), and the most general domain, D^m. For certain classes of social functions we can simplify this domain considerably. In this section, we will identify some of these important classes and discuss their respective domains. It turns out that the distinguishing quality of many such classes is some particular desirable "fairness" characteristic, and we should therefore want to study these characteristics carefully.

First, let us consider "anonymous" social functions. These systems do not recognize the identity of the voters, but depend instead only on the number of voters who prefer the various orderings. Formally, we say:

Definition 1.4: An anonymous SCF (SDF) is invariant under permutations of the voters' identities.

In other words, an anonymous social function has domain $N^{m!}$, where the vectors $(n_1,\ldots,n_{m!})$ can be thought of as representing the number of voters with preference orders $\sigma_1,\ldots,\sigma_{m!} \in S_m$, respectively. The first SCF mentioned in Example 1.1 is not anonymous, but the others are.

Another aspect of social functions is consistency of behavior in similar surroundings. In particular, we should like our social functions to reflect the opinions of smaller component subgroups when determining the decisions of the larger umbrella group. Let us detail these ideas, already explored by Young [20], [21] , [22] :

Suppose that a group of voters splits into two separate subcommittees to discuss and vote upon an issue. Suppose also that among the choice set for the first group are some preference orders chosen by the second group as well. Then, when the two subcommittees regroup to vote together, the election process should guarantee the election of precisely these preference orders. The reasoning is that neither the first nor the second subcommittee prefers any other preference order to these, while every other preference order is rejected by at least one subcommittee. Formally:

Definition 1.5: Let V be a group of voters with associated preference vector $\bar{d} = (\sigma_1, \ldots, \sigma_n) \in D^m$. Suppose that V' and V" are two non-empty and disjoint subsets of V, and suppose that $\bar{d}'$ and $\bar{d}''$ are the preference vectors associated with V' and V", respectively. A social decision function f is said to be consistent if $f(\bar{d}') \cap f(\bar{d}'') \neq \emptyset$ implies $f(\bar{d}) \quad f(\bar{d}') \cap f(\bar{d}'')$.

Anonymous SDFs that are consistent display the homogeneity property:

Definition 1.6: An anonymous SDF f is said to be homogeneous if, for any $\bar{x}$ in its domain, and for all positive integers n, $f(n\bar{x}) \quad f(\bar{x})$.

Some decision systems, notably those of constitutional amendment, tend to treat the status quo more favorably than other alternatives. In a sense, such elections are unfair in that they reflect not the opinions of the electorate on the alternatives alone, but also reflect factors concerning the names of the alternatives (e.g., status quo, proposed amendment). A case in point, the Prohibition Amendment, was treated differently both times it was considered.

A fair election system ought to consider only the alternatives themselves, rather than their names or identities. Systems which do this are called neutral:

<u>Definition 1.7</u>: Let $\bar{d} \in D^m$ represent the preference orders for a given set of voters. Let $\bar{d}^{\sigma}$ denote the vector which results from the same set of voters if we relabel the alternatives by a permutation $\sigma \in S_m$; that is, if $\bar{d}_i = \pi$, then $\bar{d}_i^{\sigma} = \pi\sigma$. Let $f(d)\sigma = \{\pi\sigma | \pi \in f(\bar{d})\}$. Then, f is said to be <u>neutral</u> if $f(\bar{d}^{\sigma}) = f(\bar{d})\sigma$.

A similar definition applies for SCFs. A social function that is both anonymous and neutral is called <u>symmetric</u>.

The extension of domain for symmetric, consistent SDFs proceeds along the following lines: (see [22]). Define $f(\bar{x}/n)$ $f(\bar{x})$ for all positive integers n. This definition extends the domain from $N^{m!}$ to the positive orthant of $Q^{m!}$, where Q is the set of rational numbers. Next, if $\bar{1}$ denotes the vector in $N^{m!}$ whose components are all 1, then we define $f(\bar{x}-q\bar{1}) = f(\bar{x})$ for every rational positive q. This extends the domain to all of $Q^{m!}$, and the extended function is still symmetric and consistent on the whole domain.

We now introduce another useful property satisfied by some anonymous functions, resembling a "domination by large numbers" idea. Suppose that there are two subcommittees, A and B, which join together in a group C. Now, suppose that some permutation σ remains in the choice set of the group no matter how many subcommittees identical to B join C. Then it is reasonable to assume that σ must have been among the choices of subcommittee B alone. This property is called continuity.

Definition 1.8: An anonymous social function h is said to be continuous if, for any $\bar{x},\bar{y}\in N^{m!}$, whenever there is some positive integer n' such that $\sigma\in f(n'\bar{x}+\bar{y})$ for all $n'\geq n$, then $\sigma\in f(\bar{x})$ holds for every $\sigma\in S_m$.

In the presence of symmetry and consistency, this condition is equivalent to the following: whenever some rational vector sequence $\{\bar{x}_i\}$ converges to $\bar{x}\in Q^{m!}$ and $\sigma\in h(\bar{x}_i)$ for all i, then $\sigma\in h(\bar{x})$ (See Chapter III, page 24).

For many applications, the use of all m! preference orders as input to the social decision process is too cumbersome. Many voting procedures therefore operate by considering only the results of pairwise voting proportions between alternatives. In that case, any non-null electorate can be completely represented by an election matrix E whose entry e_{ij} represents the proportion of voters preferring alternative a_i to a_j minus the proportion who prefer a_j to a_i. For the election situation of Example 1.1, the election matrix is:

$$E \quad \begin{matrix} 0 & 3/7 & 1/7 \\ -3/7 & 0 & 1/7 \\ -1/7 & -1/7 & 0 \end{matrix}$$

Clearly, E is always skew-symmetric. Furthermore, every election matrix can be represented as a convex combination of matrices of the following type:

Definition 1.9: To every permutation $\sigma\in S_m$, there corresponds a skew-symmetric permutation point $E^{\sigma}\in Q^{m^2}$ whose (i,j)th coordinate is 1 iff if and only if the preference order corresponding to σ places a_i before a_j.

Permutation points are simply the election matrices that result when the entire electorate has the same preference order. In general, the election matrix arising from the situation in which voters $1,\ldots,n$ have preference orders $\sigma_1,\ldots,\sigma_n$ is just $1/n(E^{\sigma_1} + \ldots + E^{\sigma_n})$. The set $\mathcal{E}$ of election matrices therefore has a nice "convex polytope" type property over the set Q. We make these remarks somewhat more precise; let $\mathcal{S}$ be the set of $m \times m$ skew-symmetric rational matrices, and let $\overline{\mathcal{S}} = cl(\mathcal{S})$.

<u>Lemma 1.1</u>: The set $\mathcal{E}$ has dimension $\binom{m}{2}$ over Q; in fact, if $X \in \mathcal{S}$, then there is some $a \in Q$, $a > 0$, and some $E \in \mathcal{E}$ such that $X \quad aE$.

<u>Proof</u>: For each (i,j), $1 \leq i \neq j \leq m$, consider an electorate with two voters, the first of which has preference order $(a_i, a_j, a_1, \ldots, a_m)$ and the second of which has order $(a_m, \ldots, a_1, a_i, a_j)$. The resulting election matrix $E(i,j)$ has an entry of 1 in component e_{ij}, -1 in e_{ji} and 0 elsewhere. These matrices span the subspace $\mathcal{S}$ over Q. $\Box$

The set $\mathcal{E}$ and especially its closure, $\overline{\mathcal{E}}$, has been the subject of recent investigations in combinatorial mathematics (see [6] for example). Basically, the problem is that, although $\overline{\mathcal{E}}$ has a simple internal representation as the polytope generated by the m! permutation points, there is no known external characterization in terms of inequalities. In [6], Bowman incorrectly stated that $\overline{\mathcal{E}}$ is the solution set in R^{m^2} of the inequality sets:

(1) $-1 \leq e_{ij} \leq 1$ for all i,j

(2) $e_{ij} = -e_{ji}$ for all i,j

(3) $-1 \leq e_{ij} + e_{jk} + e_{ki} \leq 1$ for mutually distinct i,j,k.

Although these conditions are necessary for $\overline{\mathcal{E}}$ (the last condition is the analog of transitivity of individual preference), they are not sufficient. (A counterexample to that effect has been provided by Dr. Alan J. Hoffman of the I.B.M. Watson Research Center.) The reason that so much effort has gone into determining an external representation for $\overline{\mathcal{E}}$ is that such a representation would be useful for solving certain linear or integer programming problems. Bowman cites examples of practical problems that can be reduced to solving an integer program over "permutation polyhedra," as he calls certain sets which correspond to the sets $\overline{\mathcal{E}}$. To his list we will soon add our own political applications.

II THE KEMENY FUNCTION

A DEVELOPMENT OF F

At this point we turn from social functions in general to concentrate specifically on social decision functions. In particular, we will deal with an important SDF first proposed by John Kemeny in [13] and treated more fully by Kemeny and Snell in [14].

While discussing mathematical models for various phenomena, Kemeny considered the classical social choice problem of aggregating a number of individual preference orders into a group consensus. He proposed a new SDF as a solution, but because his interests lay in model building rather than social choice, he did not explore several noteworthy properties of his system. We will define Kemeny's SDF in our own notation first and then recapitulate his axiomatization.

Let us denote by $(\bar{x}, \bar{y})$ the ordinary inner product of two vectors $\bar{x}$ and $\bar{y}$ in R^n. Keeping Definition 1.9 in mind, we have:

<u>Definition 2.1</u>: The <u>Kemeny</u> <u>SDF</u> <u>F_m</u> is the function defined by: $F_m(E)$ $\{\sigma \mid (E^\sigma, E) \geq (E^\pi, E) \ \forall \pi \in S_m\}$ for all skew-symmetric m x m matrices E. Henceforth, we will drop the subscript m from F_m.

We note immediately that F is defined on the subspace $\bar{\mathcal{S}}$ $cl(\mathcal{S})$ of skew-symmetric m x m matrices. A natural interpretation of F is that, for a given election matrix E, F maximizes "agreement" between E and the permutation points corresponding to the winning preference orders. We might also expect that it displays a "dual" sort of property, involving minimizing a measure of "disagreement." We can in fact prove such a result. Let $d(\bar{x}, \bar{y})$ denote the Euclidean distance between $\bar{x}$ and $\bar{y}$ in R^n.

Lemma 2.1: $F(E) = \{\sigma \mid d(E^{\sigma},E) \leq d(E^{\pi},E) \ \forall \pi \in S_m\}$ for all m x m skew-symmetric matrices E.

Proof: $d(E^{\sigma},E) = (E^{\sigma}-E,E^{\sigma}-E)^{\frac{1}{2}} = ((E^{\sigma},E^{\sigma}) - 2(E^{\sigma},E) + (E,E))^{\frac{1}{2}}$, for all permutation points E^{σ}. Since E is fixed and $(E^{\sigma},E^{\sigma}) = m(m-1)$ for all $\sigma \in S_m$, minimizing the left hand side of the equation is equivalent to maximizing (E^{σ},E) over S_m.

Kemeny's development in [13] followed along lines reminiscent of Lemma 2.1. He was interested in develping an SDF that would minimize a distance measure which he defined, and he produced F as the unique SDF which would do this. Let us begin by defining a notion of "distance" between two permutations, σ and π. Kemeny's original development allowed for individual indifference between alternatives, but for convenience we will assume only strong orders. We could always introduce additional hypothetical voters to achieve the same effect as indifference. Kemeny proposed the following axioms for his measure of "distance:"

1) The distance between two preference orders $\sigma \in S_m$ and $\pi \in S_m$, (denoted by $\Delta(\sigma,\pi)$) satisfies the following three geometric postulates:

(a) $\Delta(\sigma,\pi) \geq 0$ and $\Delta(\sigma,\pi) = 0$ iff $\sigma = \pi$.

(b) $\Delta(\sigma,\pi) = \Delta(\pi,\sigma)$.

(c) $\Delta(\sigma,\pi) + \Delta(\pi,\eta) \geq \Delta(\sigma,\eta)$ and equality holds iff, for every pair of alternatives a_i and a_j, π places a_i and a_j in the same order as either σ or η.

2) $\Delta(\eta\sigma,\eta\pi) = \Delta(\sigma,\pi)$ for all $\eta,\sigma,\pi \in S_m$. In other words, the distance between two permutations should not change just because of relabeling the alternatives.

3) Suppose that $\sigma,\pi \in S_m$ and $\sigma',\pi' \in S_{m-r-s}$ are such that:

$\sigma = (\sigma(a_1),\sigma(a_2),\ldots,\sigma(a_r),\sigma(a_{r+1}),\ldots,\sigma(a_{m-s}),\sigma(a_{m-s+1}),\ldots,\sigma(a_m))$

$\pi = (\sigma(a_1),\sigma(a_2),\ldots,\sigma(a_r),\pi(a_{r+1}),\ldots,\pi(a_{m-s}),\sigma(a_{m-s+1}),\ldots,\sigma(a_m))$

$\sigma' = (\sigma(a_{r+1}),\ldots,\sigma(a_{m-s}))$ and $\pi' = (\pi(a_{r+1}),\ldots,\pi(a_{m-s}))$

then the distance $\Delta(\sigma,\pi) = \Delta(\sigma',\pi')$.

As Kemeny put it, "If two rankings are in complete agreement at the beginning of the list and at the end of the list, and differ only as to the ranking of the k objects in the middle, then this distance is the same as if these k objects were the only objects under consideration."

4) The minimum positive distance is 2.

As Kemeny and Snell show in their book, Mathematical Models in the Social Sciences [14] (p. 13):

Lemma 2.2: **Axioms (1)-(4) are consistent, and, furthermore, there is only one distance measure which satisfies all of them.**

The required distance measure is defined by:

$\Delta(\sigma,\pi) = \sum_{i<j} k_{ij}^{\sigma\pi}$, where $k_{ij}^{\sigma\pi} = 2$ if σ and π disagree about the relative positions of i and j, and $k_{ij}^{\sigma\pi} = 0$ otherwise. In Kemeny's setup, the object is to choose as the social preference some order whose average distance from the individual preferences is a minimum.

We note that the entries $k_{ij}^{\sigma\pi}$ are equal to the absolute values of the corresponding entries in $E^{\sigma} - E^{\pi}$. It is easy to verify that for m alternatives, the function which chooses $\left\{\sigma \,\middle|\, \sum_{i=1}^{n} \Delta(\sigma,\pi_i) \leq \sum_{i=1}^{n} \Delta(\eta,\pi_i)\ \forall \eta \in S_m\right\}$ whenever the voters have preferences $\pi_1,\ldots,\pi_n$ agrees with F, the Kemeny SDF of Definition 2.1.

Kemeny suggested that other SDFs involving minimization of his distance measure might be useful. For example, he presented a second function which chooses preference orders $\left\{\sigma \,\middle|\, \sum_{i=1}^{n} \Delta(\sigma,\pi_i)^2 \leq \sum_{i=1}^{n} (\eta,\pi_i)^2 \;\; \forall \eta \in S_m\right\}$. He offered no reason for preferring any one SDF over another; it was rather his distance measure which he viewed as the major contribution.

In retrospect, we can say that, at least from a social choice viewpoint, Kemeny did not exploit the more important implications of his discovery. By concentrating on the distance minimizing aspect of the problem, he did not recognize that it is the function itself, rather than the distance, which possesses the important properties. As a result, his exposition relies on mathematically sound, but intuitively obscure axioms. Had he been interested in exploring the social choice ramifications of his functions, he might have realized that there is ample cause to prefer one over another.

A few concluding remarks on Kemeny's development are in order here. His line of attack, strictly speaking, did not define an SDF as we did, because his Axiom (3) depends upon comparing distances between permutations of different orders. As a result, his domain for SDFs can properly be regarded as the set $\bigcup_{m=1}^{\infty} D^m$, while we require only that a particular SDF be defined for a fixed number m of alternatives. This is not, however, a material difference.

Second, Kemeny did not make much of a connection between the distance he defined and the decision-making context in which we might want to minimize that distance. One might argue, however, that the idea of choosing as representative those choices which "minimize" disagreements is universally appealing. The difficulty with this argument is that there is a multiplicity of possible "measures of disagreement." Kemeny's axioms uniquely determine such a measure, but the axioms themselves seem to be motivated mathematically, rather than by the context of the problem. The contribution of this paper is to point out that the Kemeny function does in fact have an intuitively appealing axiomatization motivated by the social choice context.

As an example of the Kemeny function, consider once again the election of Example 1.1. The related election matrix E appears earlier in Section A of Chapter I. Using the notation of Example 1.1, we find that the six desired inner products are:

$(E_e, E) = 10/7$	$(E_{\eta_2}, E) = 6/7$	$(E_{\eta_3}, E) = -2/7$
$(E_{\eta_4}, E) = -6/7$	$(E_{\eta_5}, E) = 2/7$	$(E_{\eta_6}, E) = -10/7$

Consequently, F will choose permutation e, the natural order, as the unique winner, and the SCF derived from F will choose a_1 as the unique winning alternative. It is no accident that a_1 is a Condorcet alternative; we will soon show that g^F, the SCF derived from the Kemeny function, always chooses the Condorcet alternative, when one exists. This property is one of the appealing properties of F. Let us begin now to discuss some of these properties.

B PROPERTIES OF F

The Kemeny function possesses quite a number of desirable properties which we might expect fair and reasonable SDFs to possess. The most important of these is the Condorcet property. Apparently this observation was not made by Kemeny.

Definition 2.2: An SDF f is said to possess the Condorcet property if g^f, the SCF derived from f, chooses the Condorcet alternative uniquely whenever one exists.

We note that if a Condorcet alternative, say a_i, exists for some election, then the ith row (i.e., the row indexed by alternative a_i) of the associated election matrix E must be all positive (except for entry e_{ii}). We may therefore formulate the above definition in terms of the election matrix E: f satisfies the Condorcet property if $g^f = \{a_i\}$ whenever row i of E (except e_{ii}) is positive.

Lemma 2.3: F possesses the Condorcet property.

Proof: Suppose that a_c is the Condorcet alternative and suppose that $a_r \in g^F(E)$, $a_r \neq a_c$, where E is some election matrix. Then for some $\sigma \in F(E)$, $\sigma(a_1) = a_r$ while $\sigma(a_k) = a_c$ for some $k > 1$. Let σ' be such that $\sigma'(a_1) = a_c, \sigma'(a_2) = a_r, \sigma'(a_3) = \sigma(a_2), \ldots, \sigma'(a_k) = \sigma(a_{k-1})$, $\sigma'(a_{k+1}) = \sigma(a_{k+1}), \ldots, \sigma'(a_m) = \sigma(a_m)$. Finally, let e^{σ}_{ij} be the (i,j)th entry of E^{σ} and e_{ij} be the (i,j)th entry of E.

Then $(E^{\sigma'},E) - (E^{\sigma},E) = \sum_{i=1}^{m} e^{\sigma'}_{ci}e_{ci} + \sum_{i=1}^{m} e^{\sigma'}_{ic}e_{ic} - \sum_{i=1}^{m} e^{\sigma}_{ci}e_{ci} - \sum_{i=1}^{m} e^{\sigma}_{ic}e_{ic} = 2\sum_{i=1}^{m} e^{\sigma'}_{ci}e_{ci} - 2\sum_{i=1}^{m} e^{\sigma}_{ci}e_{ci} > 0$, since, for all $i \neq c$,

$e^{\sigma'}_{ci} \geq e^{\sigma}_{ci}$, while $e_{ci} > 0$ and $e^{\sigma'}_{cr} e_{cr} > e^{\sigma}_{cr} e_{cr}$, because a_c is a Condorcet alternative. Therefore, $\sigma \notin F(E)$, contradicting our hypotheses. $\square$

From Definition 2.1. it is immediate that F is both neutral and anonymous, so that F is symmetric (i.e., symmetric in its treatment of both different voters and different alternatives). F is also consistent; indeed, let S be a set of n voters with associated election matrix E, and let S' and S" be two non-empty sets partitioning S, where $n' = |S'|$ and $n'' = |S''|$. If E' and E" denote the election matrices corresponding to S' and S", respectively, then $E = \frac{n'}{n'+n''} E' + \frac{n''}{n'+n''} E''$. Thus, if $\sigma \in F(E') \cap F(E'')$, then $(E^{\sigma},E') \geq (E^{\pi},E')$ and $(E^{\sigma},E'') \geq (E^{\pi},E'')$ so that $(E^{\sigma},E) \geq (E^{\pi},E)$ for all $\pi \in S_m$ and hence $\sigma \in F(E)$. Moreover, if $\sigma' \in F(E)$ while $\sigma' \notin F(E') \cap F(E'')$, then $(E^{\sigma},E') \geq (E^{\sigma'},E')$ and $(E^{\sigma},E'') \geq (E^{\sigma'},E'')$, and at least one of these is a strict inequality. This implies, however, that $(E^{\sigma},E) > (E^{\sigma'},E)$, contradicting our hypothesis that $\sigma' \in F(E)$.

At this point, we define a traditional property of SDFs, analogous to the Pareto property for SCFs [17] (Also see [20]):

Definition 2.3: An SDF f is said to be faithful if $f(\bar{d}) = \{\sigma\}$ in the case that there is only one voter with preference order σ. An SDF is anti-faithful if $f(\bar{d}) = \{-\sigma\}$ when $\bar{d}$ consists of a single voter with preference order σ. (By $-\sigma$ we mean the preference order wherein $-\sigma(a_k) = \sigma(a_{m+1-k})$ for all $1 \leq k \leq m$.)

It is clear that F is faithful, since $(E^{\sigma},E^{\sigma}) > (E^{\pi},E^{\sigma})$ for all $\pi \neq \sigma \in S_m$. F is also continuous, a property which follows from the continuity of the inner product functional.

Thus, we have seen that F is symmetric, consistent, continuous and faithful. These properties are natural in the social choice context and are in fact shared by most decision processes actually in use. The special and particularly appealing attribute of F is that it satisfies the Condorcet property. We now explore the ramifications of such behavior in the presence of "ties." We know that a Condorcet alternative can be considered "superior" in some sense to the others, since it can defeat each of them in pairwise voting. Let us consider instead an alternative which ties every other in pairwise comparison voting. A natural extension of the Condorcet principle would be to say that such an alternative is in some sense <u>indifferent</u> to every other alternative.

We wish to express the idea that if a_s is an indifferent alternative, and if σ is any winning permutation chosen by f, then σ' is an equally good outcome, where σ' is obtained from σ by switching a_s with its neighbor above or below.

Let A be a set of m alternatives, and let $B \subset A$. If σ is a permutation on the alternatives in B, let $\sigma\langle A-B\rangle$ be the set of all permutations on A which agree with σ on B; that is, for all $a_i, a_j \in B$, a_i precedes a_j in σ iff a_i precedes a_j in π for every $\pi \in \sigma\langle A-B\rangle$.

<u>Definition 2.4</u>: An SDF f is said to be <u>Condorcet</u> <u>indifferent</u> if, whenever a_k is an indifferent alternative for preference profile $\bar{d}$, then $\pi \in f(\bar{d})$ and $\pi \in \sigma\langle a_k\rangle$ imply that $\sigma\langle a_k\rangle \subseteq f(\bar{d})$.

<u>Lemma 2.4</u>: F is Condorcet indifferent.

<u>Proof</u>: Suppose $\pi, \pi' \in \sigma\langle a_k\rangle$ for some indifferent alternative a_k. Then E^{π} and $E^{\pi'}$ differ only in the kth row and column, and (E^{π}, E) and $(E^{\pi'}, E)$ can likewise differ only in terms involving e_{ik} and e_{ki}, $1 \leq i \leq m$. All of these terms are 0 if row k of E is 0. Hence, $\pi \in F(E)$ iff $\pi' \in F(E)$. $\square$

One of the less obvious properties of Condorcet indifferent SDFs is that if they are consistent, then their effective domain is the set $\mathcal{E}$ of election matrices, and by Lemma 1.1 is therefore extensible to $\mathcal{S}$.

<u>Lemma 2.5</u>: If f is consistent and Condorcet indifferent, then whenever $\bar{d}$ and $\bar{d}'$ are profiles having the same election matrix, $f(\bar{d}) = f(\bar{d}')$.

<u>Proof</u>: Given $\bar{d}$, let W_d be the m x m matrix with entry (i,j) equal to the number of times alternative a_i is preferred to a_j. Let K be the m x m matrix with entries of 1 off of the diagonal and 0 on it. Then if $\bar{d}$ has n components, $E_d = (2/n)W_d - K$. We assert that $W_d = W_{d'}$ implies $f(\bar{d}) = f(\bar{d}')$. Let $\bar{d}$ and $\bar{d}'$ be preference vectors on disjoint voter sets such that $W_d = W_{d'}$, and choose a vector $\bar{d}''$ on a voter set disjoint from those of $\bar{d}$, $\bar{d}'$ such that $W_{d+d''} = W_{d'+d''}$ and $E_{d+d''} = E_{d'+d''} = \bar{0}$. By Condorcet indifference, $f(\overline{d+d''}) = f(\overline{d'+d''}) = S_m$, so that by consistency, $f(\bar{d}) = f(\bar{d}) \cap S_m = f(\bar{d}+(\overline{d''+d'})) = f((\overline{d+d''})+\bar{d}') = S_m \cap f(\bar{d}') = f(\bar{d}')$. From this it follows that whenever $W_d = W_{d'}$, $f(\bar{d}) = f(\bar{d}')$, since we can always find a third $\bar{d}''$ on a set of voters disjoint from those of both $\bar{d}$ and $\bar{d}'$ such that $W_d = W_{d'} = W_{d''}$.

Next, suppose that $E_d = E_{d'}$; then $(2/n)W_d - K = (2/n')W_{d'} - K$, and hence $n'W_d = nW_{d'}$, where $\bar{d}$, $\bar{d}'$ have n, n' components, respectively. Consistency and the remarks above imply that $f(\bar{d}) = f(W_d) = f(n'W_d) = f(nW_{d'}) = f(W_{d'}) = f(\bar{d}')$, and the lemma is proved. □

The major result of the next chapter is that there are only three SDFs which are neutral, consistent and Condorcet indifferent, and that the only one which is faithful is the Kemeny function. Hence the seemingly weaker Condorcet indifference property implies (with the others) the Condorcet property.

III A CHARACTERIZATION OF F

A SOME CONVEXITY RESULTS

Before proceeding to the characterization of F, we will find it rewarding to explore some of the implications of the properties mentioned in the previous chapter. Some of these properties lead to useful mathematical results concerning SDFs that satisfy them. We will begin our analysis by discussing the convexity implications of several of the fairness criteria. Since we will deal only with symmetric SDFs, the domain of definition will always be a subset of $R^{m!}$. We may note, however, that the following two definitions can always be interpreted in the context of the domain concerned.

Definition 3.1: Denote by Ra(f) the range of a given SDF f on m alternatives, and let $S \in Ra(f)$. Then the strong inverse of S is $f_S^{-1} = \left\{ \bar{x} \mid f(\bar{x}) = S \right\}$, and the weak inverse of S is the set $f^{-1}(S) = \left\{ \bar{x} \mid S \subseteq f(\bar{x}) \right\}$. Denote $cl(f^{-1}(S))$ by $\bar{X}_S$ and $ri(\bar{X}_S)$ by X_S.

We will also find it useful to consider the notion of Q-convexity, where $X \subseteq Q^n$ is Q-convex if $\bar{x}, \bar{y} \in X$ imply $\left\{ a\bar{x} + (1-a)\bar{y},\ 0<a<1,\ a \in Q \right\} \subseteq X$. Many properties of convexity can be carried over to Q-convexity, by virtue of the following results proved by Young in [22]:

Lemma 3.1: $C \subseteq Q^n$ is Q-convex iff $C = Q^n \cap cvx(C)$.

Lemma 3.2: If $C \subseteq Q^n$ is Q-convex, then cl(C) is convex.

Note that therefore $C \subseteq cvx(C) \subseteq cl(C)$. Hence, $cl(C) = cl(cvx(C))$ and therefore $ri(cl(C)) = ri(cl(cvx(C))) = ri(cvx(C))$.

We also have the following general fact about convexity:

Lemma 3.3: If $C = \bigcup_{i=1}^{k} S_i$, where $C \subseteq R^n$ is convex and k is finite, then for some i, $\dim(C) = \dim(S_i)$.

We may now state a theorem concerning the structure of inverse sets for appropriate SDFs:

Theorem 3.1: If f is **symmetric** and consistent, and if $C \in Ra(f)$, then both f_C^{-1} and $f^{-1}(C)$ are Q-convex cones. If, moreover, f is continuous, then there is a sequence of proper subsets $C_1 \subsetneq \dots \subsetneq C_r = C$ such that

(a) $C_i \in Ra(f)$ for $1 \le i \le r$,

(b) $\overline{X}_{C_i}$ is polyhedral for $1 \le i \le r$, and

(c) $\dim \overline{X}_{C_1} = m!$ while $\dim \overline{X}_{C_i} \le \dim \overline{X}_{C_{i-1}} - 1$ for $1 < i \le r$.

Proof: By consistency, f is homogeneous (see Definition 1.6) so that both f_C^{-1} and $f^{-1}(C)$ are Q-cones. In fact, if $\overline{y} = a\overline{x} + (1-a)\overline{x}'$, $0 < a < 1$, $a \in Q$, $\overline{x}, \overline{x}' \in f^{-1}(C)$ (or f_C^{-1}), then by consistency, $f(\overline{y}) = f(\overline{x}) \cap f(\overline{x}')$ and $\overline{y} \in f^{-1}(C)$ (or f_C^{-1}). Therefore f_C^{-1} and $f^{-1}(C)$ are Q-convex, and by Lemma 3.2, $\overline{X}_C$ and therefore X_C are also convex. We need two lemmas to proceed:

Lemma 3.4: Suppose f is **symmetric** and consistent and that $C \subsetneq C'$ are both in $Ra(f)$. Then $X_C \cap Q^{m!} \subsetneq f_C^{-1} \subseteq f^{-1}(C) \subseteq cl(f_C^{-1}) \cap Q^{m!} = \overline{X}_C \cap Q^{m!}$, and $f^{-1}(C') \subseteq (f^{-1}(C) - f_C^{-1}) \subseteq bd(\overline{X}_C) \cap Q^{m!}$.

Proof: By definition, $f_C^{-1} \subseteq f^{-1}(C)$. Now suppose that $\overline{x} \in f_C^{-1}$ and $\overline{y} \in X_C \cap Q^{m!}$. Then there is some rational number $u > 1$ such that $\overline{z} = (1-u)\overline{x} + u\overline{y} \in X_C \cap Q^{m!} \subseteq f^{-1}(C)$ by Lemma 3.1. Therefore $f(\overline{y}) = f(\overline{x}) \cap f(\overline{z}) = C$ and $\overline{y} \in f_C^{-1}$. Thus $X_C \cap Q^{m!} \subseteq f_C^{-1}$. But $cl(f^{-1}(C)) = \overline{X}_C$ and therefore $X_C \cap f^{-1}(C) \subseteq f_C^{-1}$ is dense in X_C, so $\overline{X}_C = cl(X_C) \subseteq cl(f_C^{-1}) \subseteq \overline{X}_C$, and $\overline{X}_C = cl(f_C^{-1})$. We know from the definition that $f^{-1}(C')$ is contained in $f^{-1}(C)$ but is disjoint from f_C^{-1}. Since $X_C \cap Q^{m!} \subseteq f_C^{-1}$, if $\overline{x} \in f^{-1}(C) - f_C^{-1}$, $\overline{x}$ must be a rational point in $\overline{X}_C - X_C$; i.e., $(f^{-1}(C) - f_C^{-1}) \subseteq bd(\overline{X}_C) \cap Q^{m!}$. □

We note that we may now easily demonstrate the equivalency of the two definitions of continuity (see Definition 1.8). Suppose that $\{\bar{x}_i\}$ is a sequence in $f^{-1}(\sigma)$ for some $\sigma \in S_m$ which converges to $\bar{x} \in Q^{m!}$; then Lemma 3.4 implies that $\bar{x} \in cl(f^{-1}(\sigma))$. Choose some point $\bar{y} \in X_\sigma \cap Q^{m!} \subseteq f^{-1}(\sigma)$ (such a point exists by the density of $f^{-1}(\sigma)$ in $cl(f^{-1}(\sigma)) = \bar{X}_\sigma$); then the sequence $\{\bar{z}_i = \bar{y}/i + \bar{x}\} \subseteq X_\sigma \cap Q^{m!}$, so that by homogeneity, the desired sequence is $\{i\bar{z}_i = \bar{y} + i\bar{x}\} \subseteq X_\sigma \cap Q^{m!}$.

<u>Corollary 3.4.1</u>: If f is continuous as well as symmetric and consistent, and if $C \in Ra(f)$, then $f^{-1}(C) = \bar{X}_C \cap Q^{m!}$ and $f_C^{-1} = X_C \cap Q^{m!}$.

<u>Proof</u>: By Lemma 3.4, $cl(f_C^{-1}) \cap Q^{m!} = \bar{X}_C \cap Q^{m!}$, so that if $\bar{x} \in \bar{X}_C \cap Q^{m!}$, continuity implies that $C \subseteq f(\bar{x})$. Therefore $\bar{X}_C \cap Q^{m!} \subseteq f^{-1}(C)$, and, coupled with the result above that $f^{-1}(C) \subseteq \bar{X}_C \cap Q^{m!}$, we conclude that $f^{-1}(C) = \bar{X}_C \cap Q^{m!}$. We also know from the lemma that $X_C \cap Q^{m!} \subseteq f_C^{-1}$. Now, suppose that $f_C^{-1} \neq X_C \cap Q^{m!}$; then there exists some $\bar{x} \in f_C^{-1}$ that is on the boundary of $\bar{X}_C$. Fix $\bar{y}$ in $X_C \cap Q^{m!}$, and for any $a > 1$ consider the set $Z_a = \{\alpha\bar{x} + (1-\alpha)\bar{y},\ 1 < \alpha \in Q < a\}$. If $Z_a \cap \bar{X}_C \neq \emptyset$, it follows from the convex analysis that $\bar{x} \in ri(\bar{X}_C)$, and we must therefore conclude that $Z_a \cap \bar{X}_C = Z_a \cap f^{-1}(C) = \emptyset$ for all a. But $Ra(f)$ is finite and f consistent; so there is an a such that for some $C' \neq C$ in $Ra(f)$, $f(\bar{z}) = C'$ for all $\bar{z} \in Z_a$. Now $C' \not\subseteq C$ or else consistency would imply that $f(\bar{x}) = f(Z_a) = C'$. Thus there is some $\sigma \in (C' - C)$ and by continuity, $\sigma \in f(\bar{x})$; consequently, $\bar{x} \notin f_C^{-1}$. This contradiction proves that $f_C^{-1} - X_C = \emptyset$. □

<u>Corollary 3.4.2</u>: If f is symmetric and consistent and if $S_m \in Ra(f)$, then X_{S_m} is affine.

<u>Proof</u>: Let H denote the affine hull of $f^{-1}(S_m)$. If $f^{-1}(S_m)$ consists of

just one point, we are done, so let us assume that $\dim(H) \geq 1$. Now, $f^{-1}(S_m) \subset H \cap Q^{m!}$; suppose that $\bar{z} \in (H \cap Q^{m!}) - f^{-1}(S_m)$ and let $\bar{x}$ be some point within X_{S_m}. By the lemma and the definition of X_{S_m}, there is a relative neighborhood N about $\bar{x}$ such that $N \subseteq X_{S_m}$, and as a consequence, $N \cap Q^{m!} \subseteq f^{-1}(S_m)$. But if $Z = \{ a\bar{x} + (1-a)\bar{z},\ 0 < a \in Q < 1 \}$, then $f(Z) = f(\bar{z}) \cap f(\bar{x}) = f(\bar{z})$ by consistency, and Z has a nonempty intersection with all relative neighborhoods of $\bar{x}$. This contradicts the relative openness of X_{S_m}, so there can be no such $\bar{z}$. Therefore $\bar{X}_{S_m} = cl(f^{-1}(S_m)) = X_{S_m}$ is affine. $\square$

To return to the proof of Theorem 3.1, we need one more lemma, this time concerning polyhedra in general:

Lemma 3.5: Suppose that $C_1, \ldots, C_r$ are closed convex sets of dimension n whose union is a polyhedron C and whose relative interiors are pairwise disjoint. Then $C_1, \ldots, C_r$ are polyhedra.

Proof: Since C is a polyhedron, $C = \bigcap_{i=1}^{s} S_i$ for some finite number of half-spaces S_i. Consider arbitrarily selected C_k, and let us operate in the vector space of dimension n containing C. Since $int(C_k) \cap int(C_j) = \emptyset$, there is a hyperplane H_k^j and an associated half-space S_k^j properly separating C_k from C_j and containing C_k, for every $j \neq k$. Hence, $C_k \subset S_1 \cap \ldots \cap S_s \cap S_k^1 \cap \ldots \cap S_k^r = S$. If $C_k \subsetneq S$, then by closure of C_k, $(S - C_k) \cap int(S) \neq \emptyset$, and we may suppose that $\bar{x} \in (S - C_k) \cap int(S)$. Then $\bar{x} \in C_j$ for some $j \neq k$, but $\bar{x} \in int(S_k^j)$, so that H_k^j does not separate C_j and C_k, a contradiction. $\square$

To complete the proof of Theorem 3.1, we first note that if $\dim(\bar{X}_C) = m!$, we are done. This follows because $Q^{m!}$ is the union of a finite number of sets $f^{-1}(C^j)$ and therefore $R^{m!} = \bigcup_{j=1}^{n<\infty} \bar{X}_{C^j}$. Applying Lemma 3.3, we deduce that $R^{m!}$ is the finite union of all of the sets $\bar{X}_{C^j}$ which have dimension m! Fix j and let $\bar{x},\bar{y}\in X_{C^j}\cap Q^{m!} \subseteq f^{-1}(C^j)$ by Lemma 3.1. By the argument of Lemma 3.4, $f(\bar{x}) \subseteq f(\bar{y})$ and $f(\bar{y}) \subseteq f(\bar{x})$ so there is a $C^{j'}$ such that $f(X_{C^j}\cap Q^{m!}) = C^{j'} \supseteq C^j$. But $cl(f^{-1}(C^j)) = \bar{X}_{C^j}$ so that $X_{C^j}\cap f^{-1}(C^j) \subseteq f^{-1}_{C^{j'}}$ is dense in X_{C^j}. Therefore $cl(f^{-1}_{C^{j'}}) = cl(X_{C^j}) = \bar{X}_{C^{j'}}$, so for all j, $\bar{X}_{C^j} = \bar{X}_{C^{j'}}$, $C^{j'}\in Ra(f)$. Consequently, $\bar{X}_{C^i} \neq \bar{X}_{C^j}$ implies $f^{-1}_{C^i} \cap f^{-1}_{C^j} = \emptyset$ which in turn implies, by density of $f^{-1}_{C^i}$, $f^{-1}_{C^j}$ in X_{C^i}, X_{C^j}, respectively, that $X_{C^i}\cap X_{C^j} = \emptyset$. Thus $\bar{X}_C$ is polyhedral, by Lemma 3.5.

Assume then that $\dim(\bar{X}_C) < m!$, and construct a maximal sequence of proper subsets $C_1 \subsetneq \dots \subsetneq C_r = C$ all of which are in Ra(f). Lemma 3.4 implies that for each $1 < i \leq r$, $f^{-1}(C_i)$ lies on the boundary of $\bar{X}_{C_{i-1}}$, and therefore $\bar{X}_{C_i} = cl(f^{-1}(C_i))$ has dimension no greater than $\dim(\bar{X}_{C_{i-1}}) - 1$.

We proceed by induction to show that $\bar{X}_{C_i}$ is polyhedral for all $1 \leq i \leq r$. We have just shown that $R^{m!} = \bigcup_{i=1}^{n<\infty} cl(f^{-1}_{C^j})$, where X_{C^j} has dimension m! If $\dim(\bar{X}_{C_1}) < m!$, then $\bar{x}\in X_{C_1}\cap Q^{m!} \subseteq f^{-1}_{C_1}$ is the limit point of a sequence $\{\bar{x}_i\} \subseteq f^{-1}_{C^j}$ for some $1 \leq j \leq n$. By continuity, $C_j \subsetneq f(\bar{x}) = C_1$, so the sequence $C_1, \dots, C_r$ is not maximal. Thus $\dim(\bar{X}_{C_1}) = m!$ and by our analysis above, $\bar{X}_{C_1}$ is polyhedral.

Assume then that $\bar{X}_{C_{I-1}}$ is polyhedral for some $1 < I \leq r$. We assert that if K_I is the affine hull of $f^{-1}_{C_I}$, then $K = K_I \cap \bar{X}_{C_{I-1}}$ is equal to $\bar{X}_{C_I}$, wherefore the latter is polyhedral, since it is the intersection of two polyhedra. Clearly, $\bar{X}_{C_I} \subseteq K$ by virtue of Lemma 3.4. Suppose now that $\bar{x},\bar{y}\in ri(K)\cap Q^{m!}$; then there is some $1<\alpha_1\in Q$ for which $\bar{z}^1 = \alpha_1\bar{x} + (1-\alpha_1)\bar{y} \in ri(K)\cap Q^{m!}$ and there is some $0>\alpha_2\in Q$ for which $\bar{z}^2 = \alpha_2\bar{x} + (1-\alpha_2)\bar{y} \in ri(K)\cap Q^{m!}$. Then $C_{I-1} \subseteq f(\bar{x})\cap f(\bar{y})\cap f(\bar{z}^1)\cap f(\bar{z}^2)$ by continuity; hence $f(\bar{x}) = f(\bar{y})\cap f(\bar{z}^1) \subseteq f(\bar{y}) = f(\bar{x})\cap f(\bar{z}^2) \subseteq f(\bar{x})$, so that $f(\bar{x}) = f(\bar{y})$. Finally, there exists some $\bar{x}\in ri(\bar{X}_{C_I})\cap Q^{m!} \subseteq ri(K)\cap Q^{m!}$, so that $ri(K) \cap Q^{m!} \subseteq f_{C_I}^{-1}$. Therefore $cl(ri(K) \cap Q^{m!}) = K \subseteq cl(f_{C_I}^{-1}) = \bar{X}_{C_I}$, since $aff(ri(K)) = aff(f_{C_I}^{-1}) = K_I$. $\square$

We have actually proved the following result as well:

Corollary 3.1.1: Let f be symmetric, consistent and continuous, and let $C \in Ra(f)$. If K is a face of $\bar{X}_C$, then f is constant on $ri(K) \cap Q^{m!}$.

The results of Theorem 3.1 extend analogously to Condorcet indifferent SDFs defined on $\mathcal{E}$

B THE CHARACTERIZATION

To recapitulate, we have shown that SDFs displaying appropriate fairness characteristics also display some pleasing mathematical properties. Thus, knowing which fairness properties are possessed by some SDF may contribute measurably to our knowledge of the behavior of that SDF. The major result of this paper is that four of these properties suffice to characterize the Kemeny function, and that only three of them are needed to characterize a family of three SDFs. The only SDFs that are neutral, consistent and Condorcet indifferent are F, -F and T - the Kemeny function, the "anti-Kemeny" function, and the completely trivial function, respectively. Moreover, only F is faithful.

By the <u>anti-Kemeny function</u>, we mean the function -F which chooses $-\sigma$ iff F chooses σ (see Definition 2.3). The <u>completely trivial SDF T</u> always chooses S_m. We may now state our major theorem:

<u>Theorem 3.2</u>: An SDF f is neutral, consistent and Condorcet indifferent iff it is F, -F or T, and if f is also faithful, f = F.

<u>Proof</u>: We have already demonstrated necessity for F in Chapter II; the same proofs hold for -F, while verification is trivial for T. Of course, only F can be faithful. For sufficiency, we will need some additional definitions and lemmas.

First, we note that since Condorcet indifferent SDFs tend to treat indifferent alternatives "indifferently," they discriminate only among the remaining alternatives. Consequently, they correspond in some sense to SDFs of lower order; i.e., on fewer alternatives, whenever there is an indifferent alternative. We may discuss this relationship somewhat more precisely:

Definition 3.2: Let E be an election matrix indexed by the alternative set $B \subsetneq A$, and let $|A| = m$. Let $E\langle A-B\rangle$ be the m x m matrix derived from E by adding a labeled zero row and zero column for each alternative in A - B. Finally, let f be a consistent and Condorcet indifferent SDF on A. Then the B-derived SDF of f is the function f^B whose domain is the set of $|B| \times |B|$ skew-symmetric rational matrices and whose range is $P(S_{|B|}) - \emptyset$ defined by:
$\sigma \in f^B(E)$ iff $\sigma\langle A-B\rangle \subseteq f(E\langle A-B\rangle)$.

The important question to ask about B-derived functions is: What properties of f do they inherit?

Lemma 3.6: If f is neutral, consistent and Condorcet indifferent, then so is each derived SDF f^B.

Proof: To show that Condorcet indifference is inherited, suppose that f is a function on a set A of m alternatives. Fix j, $1 \leq j \leq m$ and let E be an m-1 x m-1 skew-symmetric rational matrix, indexed by the alternatives a_k in $B = A - \{a_j\}$ such that some row indexed by a_i (i.e., the a_ith row) is 0, for some $i \neq j$. Now $E\langle a_j\rangle$ has row a_j and row a_i zero, so that if $\sigma \in f^B(E)$, then $\sigma \in \pi\langle a_i\rangle$ for some π on $A - \{a_i, a_j\}$, and then $\pi\langle a_i, a_j\rangle \subseteq f(E\langle a_j\rangle)$ by Condorcet indifference. Consequently $\pi\langle a_i\rangle \subseteq f^B(E)$, and f^B must be Condorcet indifferent. Applying the analysis recursively implies the result is true for all $B \subsetneq A$.

To show consistency, suppose that the group of voters producing E splits into two subgroups with respective election matrices E' and E", and let $B = A - \{a_j\}$. Now, if $f^B(E') \cap f^B(E'') = S \neq \emptyset$, then $\sigma \in S$ iff $\sigma\langle a_j\rangle \overset{C}{=} f(E'\langle a_j\rangle) \cap f(E''\langle a_j\rangle)$, and the latter set is equal to $f(E\langle a_j\rangle)$, by consistency of f. But $\sigma \in f^B(E)$ iff $\sigma\langle a_j\rangle \overset{C}{=} f(E\langle a_j\rangle)$ iff $\sigma \in S$. Once again, the proof applies recursively if $|B| < m-1$.

The proof of neutrality is straightforward and follows the same general approach. $\square$

Note that by neutrality, f^B and $f^{B'}$ are essentially the same function, modulo relabeling of the alternatives, if $|B| = |B'|$.

Using the mechanics of the B-derived function, we may now prove a lemma for SDFs satisfying the hypotheses of the theorem. I would like to take this opportunity to acknowledge the significant contribution of Dr. H. P. Young to the proof of the theorem, as well as his useful notational advice for Lemma 3.7.

Lemma 3.7: If f is neutral, consistent and Condorcet indifferent, then f^B are either all faithful, all anti-faithful, or all completely trivial.

Proof: Let us denote by e_r the identity permutation on r objects (which r objects depends on the context). By considering the possible values of f^B on E^{e_2}, we conclude that f^B is either faithful, anti-faithful or $f^B(E^{e_2}) = S_2$, where $|B| = 2$. In the latter case, neutrality and consistency imply that $f^B = T$. By neutrality, f^B is faithful iff $f^{B'}$ is faithful whenever $|B| = |B'| = 2$; similarly for anti-faithful and trivial.

We proceed by induction; assume that the lemma holds for all B such that $|B| < k \leq m$. We wish to show that it holds for all B such that $|B| \leq k$. Fix C such that $|C| = k$.

Let us isolate three cases, based upon f^B, $|B| = 2$.

Case 1: $f^B(E^{e_2}) = \{e_2\}$.

By induction, assume that $f^{B_i}(E^{\hat{e}}) = \{\hat{e}\}$, where $B_i = C - \{a_i\}$, so $|B_i| = k-1$, for $i = 1,\ldots,k$, and where $\hat{e}$ is the identity permutation on the k-1 alternatives in B_i. Denote $E^{\hat{e}}\langle a_i\rangle$ by E_i. By Condorcet indifference, $f^C(E_i) = \hat{e}\langle a_i\rangle$ for each of the k permutation points E_i. By consistency, $f^C(E^{e_k}) = f^C((k-1)E^{e_k}) = f^C(\sum_{i=1}^{k} E_i) = \bigcap_{i=1}^{k} f^C(E_i) = \{e_k\}$. By neutrality, the result holds for all $\sigma \in S_k$.

Case 2: $f^B(E^{e_2}) = \{-e_2\}$. The same argument applies.

Case 3: $f^B(E^{e_2}) = S_2$.

Using the same analysis as in the first case, we find that $f^C(E^{e_k}) = \bigcap_{i=1}^{k} f^C(E_i) = S_k$ by induction, Condorcet indifference, and consistency. By neutrality, $f^C(E^\sigma) = S_k$ for all $\sigma \in S_k$, and since $\mathcal{E}$ is the convex hull of the permutation points, $f^C(E) = S_k$ for all election matrices, so that f = T.

Finally, for any C' such that $|C'| = |C| = k$, neutrality implies that $f^{C'}$ is faithful if and only if f^C is faithful, and so forth. $\square$

Recapitulating our results so far, we know that F, -F and T are all neutral, consistent and Condorcet indifferent. We also know that if f is any neutral, consistent and Condorcet indifferent SDF, f is faithful, anti-faithful or completely trivial, as are all its B-derived SDFs. We would like to show that in fact the only faithful SDF satisfying these conditions is F. From this it follows that the only anti-faithful such SDF is -F. Thus, we need consider only cases 1 and 2 above and show that F is the only SDF in case 1 and -F the only SDF in case 2.

We may begin this analysis with the case m=3 alternatives.

In this case, we note that $\mathcal{E} \cap F^{-1}(e_3)$ is contained in the Q-convex hull of 8 points [15]:

$$E_1 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad E_2 = \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad E_3 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{matrix} \quad E_4 = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix}$$

$$E_5 = \begin{matrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{matrix} \quad E_6 = \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{matrix} \quad E_7 = \begin{matrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{matrix} \quad E_8 = \begin{matrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{matrix}$$

It will suffice to show that f agrees with F on these 8 points, in case 1, for then f agrees with F throughout $F^{-1}(e_3)$ by consistency, and f agrees with F everywhere else on $\mathcal{E}$ by neutrality. In case 2, it likewise suffices to show that f agrees with -F on these 8 points.

Case 1: $f^B(E^{e_2}) = \{e_2\}$, for $|B| = 2$.

$f(E_8) = F(E_8) = \{e_3\}$ since f is faithful by Lemma 3.7. $f(E_1) = F(E_1) = S_3$ by neutrality. Since $E_2 = E^{e_2}\langle a_3\rangle$, Condorcet indifference implies that $f(E_2) = F(E_2) = e_2\langle a_3\rangle$. Likewise, $f(E_3) = F(E_3)$ and $f(E_4) = F(E_4)$. Next, $f(E_5) = f(E_2) \cap f(E_4) = F(E_2) \cap F(E_4) = \{e_3, \eta_2\}$ by consistency, since $E_5 = E_2 + E_4$; since $E_6 = E_3 + E_4$, a similar argument disposes of E_6. (See Example 1.1 for notation $\eta_1, \ldots, \eta_6$.)

We are left only with E_7, which is a fixed point of the permutations η_4 and η_5. By neutrality, we conclude that $f(E_7) = \{e_3, \eta_4, \eta_5\} = F(E_7)$ or S_3 or $S_3 - F(E_7)$. In the last two cases, though, η_3 would have to be a member of $f(E_7)$, and consistency would then imply that $f(E_3) = f(E_7) \cap f(E^{\eta_3}) = \{\eta_3\}$, since $E_3 = \frac{1}{2}(E_7 + E^{\eta_3})$. This contradicts our prior conclusion that $f(E_3)$ is multi-valued. Thus f = F on the eight points, as claimed.

Case 2: $f^B(E^{e_2}) = \{-e_2\}$ for $|B| = 2$. The argument is the same as above.

In order to extend the proof of the theorem to the case of $m \geq 4$ alternatives, we will need some additional results concerning neighboring points on $\bar{\mathcal{E}}$. For convenience in the remainder of this chapter, we will assume that the alternatives $a_1, \ldots, a_m$ are identical with the integers $1, \ldots, m$. Thus a permutation σ of A will be abbreviated by $\sigma(a_1), \ldots, \sigma(a_m) = \sigma_1, \ldots, \sigma_m$.

Definition 3.3: The signature of $\sigma \in S_m$, denoted by $s(\sigma)$, is given by $\{(\sigma_r, \sigma_s) \mid \sigma_r > \sigma_s \text{ and } r < s\}$.

Definition 3.4: The graph of σ, G_σ, is the graph with vertex set $s(\sigma)$ and adjacencies defined as follows:

(1) $i < j < k$ and $\sigma_j < \sigma_k < \sigma_i$ imply that (σ_i, σ_j) is adjacent to (σ_i, σ_k),

(2) $i < j < k$ and $\sigma_k < \sigma_i < \sigma_j$ imply that (σ_j, σ_k) is adjacent to (σ_i, σ_k),

and there are no other adjacencies.

The following result was first observed by Dr. Alan Hoffman:

Lemma 3.8: For any $\sigma \neq e \in S_m$, E^e and E^σ are neighbors on $\bar{\mathcal{E}}$ iff G_σ is connected. (e is the identity permutation in S_m.)

Actually, all we need is that G_σ is connected whenever E^e and E^σ are neighbors on $\bar{\mathcal{E}}$.

Let us now continue with the proof of Theorem 3.2, assuming that $m \geq 4$. The second case, where $f^B(E^{e_2}) = \{-e_2\}$, is proved analogously to the first, and we will therefore consider only the case $f^B(E^{e_2}) =$

$\{e_2\}$, $|B| = 2$, in detail. Since f is faithful, $\pi \notin Ra(f)$ for all $\pi \in S_m$.

Suppose $\sigma \neq e \in S_m$. Then Lemma 3.4 implies that $X_e \cap \int^{C} \subseteq f_e^{-1}$, $X_\sigma \cap \int \subseteq f_\sigma^{-1}$; since $f_e^{-1} \cap f_\sigma^{-1} = \emptyset$, $X_e \cap X_\sigma \cap \int = \emptyset$. If $X_e \cap X_\sigma = S \neq \emptyset$, S is an open subset of $\overline{X}_e$ and since f_e^{-1} is dense in $cl(f_e^{-1}) = \overline{X}_e$ by Lemma 3.4, $S \cap f_e^{-1} \neq \emptyset$. Therefore $S \cap \int \neq \emptyset$, contradicting our previous results, and so $X_e \cap X_\sigma = \emptyset$. By the separation theorem of convex analysis, there is some $U^\sigma \in \overline{\int} = cl(\int)$ such that:

$(U^\sigma, E) \geq 0$ if $E \in \overline{X}_e$ and $(U^\sigma, E) \leq 0$ if $E \in \overline{X}_\sigma$.

Suppose now that E^e and E^σ are neighbors on $\overline{\mathcal{E}}$, and consider $U^\sigma = (u_{ij}^\sigma)$. If $i < j$ and $(j,i) \notin s(\sigma)$, then by considering matrix E with entry of 1 in position (i,j), entry -1 in (j,i) and 0 otherwise, we find that $e, \sigma \in f(E)$ by Condorcet indifference and because $f^{\{a_i, a_j\}}(E^{e_2}) = \{e_2\}$ for case 1. This implies that $(U^\sigma, E) = 0$ and therefore that $u_{ij}^\sigma = u_{ji}^\sigma = 0$.

Suppose then that $i < j$ and $(j,i) \in s(\sigma)$. Lemma 3.8 implies that G_σ is connected; as a result, there are two cases to consider:

<u>case 1a)</u>: (j,i) is the only vertex in G_σ. For E as above, Condorcet indifference and faithfulness imply that $e \in f(E)$ but $\sigma \notin f(E)$. Hence, $(U^\sigma, E) > 0$ and $u_{ij}^\sigma = -u_{ji}^\sigma = \lambda > 0$, $u_{hk}^\sigma = 0$ for all other h,k.

<u>case 1b)</u>: (j,i) is adjacent to some other vertex of G_σ. By Definition 3.4, there are only two classes of such vertices. If (j,i) is adjacent to (k,i) (i.e., there are three numbers $r < s < t$ such that $\sigma_t = i < \sigma_r = j < \sigma_s = k$), let us consider the matrix E', where $e'_{ij} = e'_{jk} = -e'_{ik} = 1$ and all other entries not implied by skew-symmetry are 0. By our conclusions for m=3 alternatives and by Condorcet indifference, $e, \sigma \in f(E')$. Therefore, $(U^\sigma, E) = 0$ and $u_{ij}^\sigma + u_{jk}^\sigma - u_{ik}^\sigma = 0$. Since we already know that $u_{jk}^\sigma = 0$ from the above analysis, we conclude that $u_{ik}^\sigma = u_{ij}^\sigma$.

If (j,i) is adjacent to some (j,k) (i.e., there are three numbers $r < s < t$ such that $\sigma_s = i < \sigma_t = k < \sigma_r = j$), a similar argument follows. Thus we conclude that if v and w are two adjacent vertices of G_σ, then $u_v \quad u_w$. But since E^σ is a neighbor of E^e and therefore by Lemma 3.8 G_σ is connected, we conclude that:

$$u_v = \begin{cases} -\lambda & \text{for all } v \in s(\sigma) \\ \lambda & \text{for all } v = (i,j), \text{ where } (j,i) \in s(\sigma) \\ 0 & \text{otherwise}. \end{cases}$$

Since f is faithful and therefore $f(E^e) = \{e\}$, $(U^\sigma, E^e) \geq 0$. Without loss of generality, choose $\lambda = 2$, and then $U^\sigma = E^e - E^\sigma$; i.e., $(E^\sigma, E) \leq (E^e, E)$ for every $E \in f^{-1}(e)$, for all neighbors E^σ of E^e. But a linear function is maximized at a vertex p of a bounded polytope if it is maximized at p with respect only to the neighbors of p. Regarding E as the linear function, it follows that $(E^\sigma, E) \leq (E^e, E)$ for all $\sigma \in S_m$, whenever $E \in f^{-1}(e)$. Thus $f^{-1}(e) \subseteq F^{-1}(e)$ and by neutrality, $f^{-1}(\sigma) \subseteq F^{-1}(\sigma)$ for every $\sigma \in S_m$. We now show that there is no $\sigma \in S_m$ for which $f^{-1}(\sigma) \subsetneq F^{-1}(\sigma)$.

We first observe that if S is Q-convex, then since S is dense in $cl(S)$ and $S \subseteq cvx(S) \subseteq cl(S)$ by Lemma 3.2, S is dense in $cvx(S)$. Now, suppose that for some $\sigma \in S_m$, E^e and E^σ are neighbors on $\overline{\mathcal{E}}$. Then, in particular, there is some linear function $\hat{E} \in \overline{\int}$ and therefore some $E \in \int$ which is maximized on $ext(\overline{\mathcal{E}})$ only at E^e and E^σ, so that $F^{-1}_{e\sigma} \neq \emptyset$. Let $Y_e = F^{-1}(e,\sigma) \cap f^{-1}(e)$, let $Y_\sigma = F^{-1}(e,\sigma) \cap f^{-1}(\sigma)$, and let $Y^*_e = cvx(Y_e)$ and $Y^*_\sigma = cvx(Y_\sigma)$. Next, let $A = aff(F^{-1}(e,\sigma))$; since $\overline{0} \in F^{-1}(e,\sigma)$, A is a subspace of $\int$. If $\overline{Z} = cl(F^{-1}(e,\sigma))$, $Z = ri(\overline{Z})$, $E \in F^{-1}_{e\sigma}$ and $E' \in Z \cap \int$, then there is some $u > 1$, $u \in Q$ such that $E'' = (1-u)E + uE' \in Z \cap \int$ and $F(E') = F(E) \cap F(E'') = \{e, \sigma\}$, so $Z \cap \int \subseteq F^{-1}_{e\sigma}$. But $cl(F^{-1}(e,\sigma)) = \overline{Z}$, and thus $Z \cap$

$F^{-1}(e,\sigma) \subseteq F^{-1}_{e\sigma}$ is dense in Z, so $cl(Z) \subseteq cl(F^{-1}_{e\sigma}) \subseteq \overline{Z}$ and $\overline{Z} = cl(F^{-1}_{e\sigma})$. Therefore $aff(F^{-1}_{e\sigma}) = aff(F^{-1}(e,\sigma)) = A$.

Suppose now that $E' \in Y_e - Y_\sigma$, and let $E \in ri(Y^*_e) \cap \mathcal{Q}$. Then there is some $u>1$, $u \in Q$ for which $E'' = (1-u)E' + uE \in ri(Y^*_e) \cap \mathcal{Q} \subseteq Y^*_e \cap \mathcal{Q} = Y_e$, by Lemma 3.1. Consistency implies that $f(E) = f(E') \cap f(E'') \supseteq \{e\} \neq \emptyset$, and since $\sigma \notin f(E')$, $\sigma \notin f(E)$. Therefore $ri(Y^*_e) \cap ri(Y^*_\sigma) \cap \mathcal{Q} \subseteq ri(Y^*_e) \cap Y_\sigma = \emptyset$. We assert that $ri(Y^*_e) \cap ri(Y^*_\sigma) = \emptyset$. Note first that $f^{-1}(\pi) \subseteq F^{-1}(\pi)$ for all $\pi \in S_m$; hence $F^{-1}_{e\sigma} \subseteq Y_e \cup Y_\sigma$. Therefore $cl(F^{-1}_{e\sigma}) \subseteq cl(Y_e) \cup cl(Y_\sigma)$. Clearly, $aff(Y_e) \subseteq A$ and $aff(Y_\sigma) \subseteq A$, and since $cl(F^{-1}_{e\sigma})$ is convex by Lemma 3.2 with affine hull A by our remarks above, by Lemma 3.3, one of Y_e, Y_σ has affine hull A and the other has affine hull $A' \subseteq A$. If $ri(Y^*_e) \cap ri(Y^*_\sigma) = S \neq \emptyset$, then S is an open set in the subspace A' and there must be some rational matrix $E \in S$ by the density of Y_e in Y^*_e, Y_σ in Y^*_σ. But then $E \in \mathcal{Q} \cap ri(Y^*_e) \cap ri(Y^*_\sigma)$, contradicting our earlier result that this intersection is null. Consequently, $ri(Y^*_e) \cap ri(Y^*_\sigma) = \emptyset$.

The separation theorem of convex analysis therefore implies that there exists some $U \neq \overline{0}$, $U \in A$ for which $(U,E) \geq 0$ if $E \in Y^*_e$ and $(U,E) \leq 0$ if $E \in Y^*_\sigma$. By definition of F, $\sum_{(j,i) \in s(\sigma)} e_{ij} = 0$ for all $E \in F^{-1}(e,\sigma)$ and therefore for all $E \in A$, including U. Consider the graph G_σ, and suppose that $i<j$. If $(j,i) \notin s(\sigma)$, consider the matrix E with 1 in component (i,j), -1 in component (j,i) and 0 elsewhere. By Condorcet indifference and faithfulness, $\{e,\sigma\} \subseteq f(E) \cap F(E)$, so that $E \in Y_e \cap Y_\sigma$ and $(U,E) = 0$. Hence, $u_{ij} = 0$.

If, on the other hand $i<j$ and $(j,i) \in s(\sigma)$, suppose that (j,i) is adjacent to (k,i) for some $(k,i) \in s(\sigma)$. We may then consider as we did earlier a matrix E with entries of 1 in positions e_{ki}, e_{jk}, and e_{ij}, and all other entries not implied by skew-symmetry, 0. By Condorcet

indifference, we know that $\{e,\sigma\} \stackrel{C}{=} f(E) \cap F(E)$, so that $E \in Y_e \cap Y_\tau$ and $(U,E) = 0$. We conclude that $u_{ij} + u_{jk} - u_{ik} = 0$, and since $u_{jk} = 0$ from the preceding paragraph, we find that $u_{ij} = u_{ik}$. A similar analysis shows that if (j,i) is adjacent to some (j,k), then $u_{ji} = u_{jk}$. We conclude therefore that $u_v = u_w$ for all $v,w \in s(\tau)$, since all v,w are connected in G_τ, by Lemma 3.8. But since $\sum_{(j,i)\in s(\sigma)} e_{ij} = 0$, we conclude that $i < j$ and $(j,i) \in s(\sigma)$ imply $u_{ij} = 0$. We also showed that $i < j$ and $(j,i) \notin s(\tau)$ imply $0 = u_{ij}$, however, so that $U = \overline{0}$, contradicting the separation theorem. We must conclude then that $Y_e - Y_\tau = \emptyset$. It follows in general that if E^σ and E^π are neighbors on $\overline{\xi}$, then $\{\sigma,\pi\} \stackrel{C}{=} F(E)$ and σ or $\pi \in f(E)$ implies $\{\sigma,\pi\} \stackrel{C}{=} f(E)$, $E \in \mathcal{S}$.

Suppose now that for some E, $f(E) \not\stackrel{C}{=} F(E)$. Then for some $\sigma \neq \pi$, we have $\{\sigma,\pi\} \stackrel{C}{=} F(E)$ and $\sigma \in f(E)$ but $\pi \notin f(E)$. Now, the linear function E must attain its maximum on $\overline{\xi}$ at the vertices E^σ and E^π. A theorem of Balinski (see [2]) implies that there is a path from $E^\sigma = E^{\eta_1}$ to ... to $E^{\eta_k} = E^\pi$, where E^{η_i} is a neighbor of $E^{\eta_{i+1}}$ on $\overline{\xi}$, and where $(E^{\eta_i},E) = (E^{\eta_{i+1}},E)$ for $1 \leq i < k$. This implies in turn that $\{\eta_1,\ldots,\eta_k\} \stackrel{C}{=} F(E)$ and by our results above, that $\{\sigma,\pi\} \stackrel{C}{=} \{\eta_1,\ldots,\eta_k\} \stackrel{C}{=} f(E)$, contradicting the hypotheses. Thus we conclude that $f(E) = F(E)$ for all $E \in \mathcal{S}$. $\square$

We note that the four conditions of neutrality, consistency, Condorcet indifference and faithfulness are independent.

(a) All but neutrality: The function on three alternatives which agrees with F at all points except aE_7, $a > 0$, $a \in Q$, where it takes on the value $\{e_3\}$. (See earlier in the proof for notation.)

(b) All but consistency: The function on two alternatives which takes on the values $\{e_2\}$ at E^{e_2}, $\{-e_2\}$ at $-E^{e_2}$, and S_2 elsewhere.

(c) All but Condorcet indifference: The Borda scoring function of Example 1.1. The proof of this assertion will be presented in the following chapter.

(d) All but faithfulness: The functions -F and T.

One difficulty with the Kemeny function is that there is no efficient algorithm known for determining which vertices of $\bar{\mathcal{E}}$ maximize the linear function given by a matrix $E\epsilon\int$. Consequently, when the number of alternatives is large, it may be very tedious to resolve an election. The Kemeny function also presents some other difficulties, which we shall soon discuss. Thus, although it uniquely satisfies some appealing axioms, it may not be the most desirable procedure to use in practice.

IV SCORING FUNCTIONS

We may extend our definition of the social decision function somewhat by considering functions whose domain is the set $R^{m!}$, instead of $Q^{m!}$, as is the case for symmetric and consistent SDFs. Thus, an <u>extended</u> SDF on a set A of m alternatives is a function $f: R^{m!} \longrightarrow P(S_m) - \emptyset$. For $\bar{x} \in R^{m!}$, the components $x_1, \ldots, x_{m!}$ of $\bar{x}$ can be thought of as representing the weights assigned by the electorate to each of the preference orders $\sigma_1, \ldots, \sigma_{m!} \in S_m$, respectively. We will henceforth restrict our dicussion to extended SDFs. and will assume that neutrality and consistency, etc. are defined analogously for such SDFs. An example of an extended SDF that we have already discussed in detail is the Kemeny SDF.

As we have seen, the Kemeny SDF satisfies quite a number of criteria for fairness and reasonableness. It is symmetric, consistent, continuous, faithful and Condorcet indifferent; it also satisfies the Condorcet condition as well as Kemeny's distance minimizing axioms. Yet, as we can see from the following example, it suffers from an important shortcoming: F yields ties on preference orders rather than on alternatives, so that it is often unclear which is the most preferred alternative.

<u>Example 4.1</u>: Suppose that there are six voters, 1,..., 6 , and four alternatives, $a_1, \ldots, a_4$. Suppose further that the voters have the following preferences:

1	2	3	4	5	6
a_1	a_4	a_1	a_2	a_3	a_1
a_2	a_2	a_4	a_3	a_1	a_3
a_3	a_3	a_2	a_4	a_4	a_4
a_4	a_1	a_3	a_1	a_2	a_2

The corresponding election matrix is:

$$\begin{matrix} 0 & 1/3 & 0 & 1/3 \\ -1/3 & 0 & 1/3 & -1/3 \\ 0 & -1/3 & 0 & 1/3 \\ -1/3 & 1/3 & -1/3 & 0 \end{matrix}$$

F yields four winning permutations: a_1, a_2, a_3, a_4, a_1, a_3, a_4, a_2, a_1, a_4, a_2, a_3, and a_3, a_1, a_4, a_2. It is unclear whether a_1 or a_3 is the "most preferred" alternative, or whether they should be considered tied. It is even less clear which alternative should be considered "second most preferred," since each of the alternatives has that distinction in at least one winning preference order.

One way to avoid such difficulties is to insist that all ties in the set of permutations chosen by an SDF be resolvable in some intuitively clear sense into ties among alternatives. We can accomplish this goal by restricting our attention to SDFs which choose, in effect, <u>weak orders</u> of alternatives.

<u>Definition 4.1</u>: A <u>weak order</u> on a set A of m alternatives is an ordered partition $\theta = (\theta(1), \ldots, \theta(k))$ for some $k \leq m$, where the $\theta(i)$ partition A.

Now, if we denote by $\mathcal{O}_m$ the set of all weak orders on A, we note that for every $\theta \in \mathcal{O}_m$ there corresponds a certain subset $T(\theta)$ of linear orders, namely:

$$T(\theta) = \{\sigma \in S_m \mid \text{if } a_i \in \theta(r),\ a_j \in \theta(s) \text{ and } \theta(r) \text{ precedes } \theta(s)$$
$$\text{(i.e., } r < s\text{), then } a_i \text{ precedes } a_j \text{ in } \sigma, \text{ for all } a_i, a_j \in A\}$$

We denote by K_m the set $\{T(\theta) \mid \theta \in \mathcal{O}_m\}$. We may now define the notion of an omnidecisive SDF, closely related to the rationality axioms discussed in Arrow (see [1], page 19).

Definition 4.2: An SDF f is said to be omnidecisive if $Ra(f) = K_m$.

An omnidecisive function, then, can be thought of as always choosing a unique weak order, so that ties among orders can always be resolved into ties among the alternatives alone. Moreover, every weak order is chosen at least once, so that such SDFs are as "discriminating" as possible, given the restriction on their range. Since the range of an omnidecisive function is in correspondence with the set of weak orders, we will find it convenient at times to refer to the latter as the range.

The following result on simple scoring functions (See [21]) is the major conclusion of this paper for omnidecisive functions:

Theorem 4.1: A neutral, consistent and continuous SDF is omnidecisive iff it is a nontrivial simple scoring function.

Before discussing scoring functions, let us observe several facts: First, if f is neutral, consistent and continuous, then arguments similar to those of Theorem 3.1 show that if $C \in Ra(f)$, then $f^{-1}(C)$ is closed, conical and convex; in fact $f^{-1}(C)$ is polyhedral. Moreover, we can also show that if K is a face of $f^{-1}(C)$, then f is constant on ri(K). We assert that if $C \subsetneq C'$, $C, C' \in Ra(f)$ and if there is no $C'' \in Ra(f)$ such that $C \subsetneq C'' \subsetneq C'$, then $\dim(f^{-1}(C')) = \dim(f^{-1}(C)) - 1$. Indeed, as in Lemma

3.4, we deduce that $f^{-1}(C') \overset{C}{=} K$ for some facet K of $f^{-1}(C)$. But if $f^{-1}(C') \neq K$, then by closure of $f^{-1}(C')$, K, we have $(K-f^{-1}(C')) \cap ri(K) \neq \emptyset$, and therefore $f(ri(K)) = C'' \neq C'$, and $C \overset{C}{\neq} C''$ by continuity. But then $\bar{x} \in K = cl(ri(K))$ implies by continuity that $C'' \overset{C}{=} f(\bar{x})$, so that $C \overset{C}{\neq} C'' \overset{C}{\neq} C'$, a contradiction. We now need three definitions:

Definition 4.3: Two permutations in S_m are said to be close neighbors if they differ only by the inversion of two adjacent alternatives.

Definition 4.4: Two convex sets of dimension n are said to be neighbors if their intersection has dimension n-1.

Definition 4.5: For any $\theta \in \mathcal{O}_m$, $\theta = (\theta(1),\ldots,\theta(k))$, the sets $\theta(i)$ are called indifference rows (or classes) of θ, with length $|\theta(i)|$.

Lemma 4.1: If f is neutral, consistent, continuous and omnidecisive, then θ has k indifference rows iff $\dim(f^{-1}(\theta)) = m!-m+k$.

Proof: We argue by induction; if k=m, θ is a permutation and the result follows immediately from our remarks above. Suppose sufficiency holds for all $m \geq k > K$, and let θ be a weak order with K indifference rows. At least one row, say row i, contains more than 1 alternative; let $a_j, a_k \in \theta(i)$. Let θ' be defined by $\theta'(1) = \theta(1),\ldots,\theta'(i-1) = \theta(i-1)$, $\theta'(i) = \theta(i)-\{a_k\}$, $\theta'(i+1) = \{a_k\}$, $\theta'(i+2) = \theta(i+1),\ldots,\theta'(K+1) = \theta(K)$. Our remarks imply that $f^{-1}(\theta) \overset{C}{=} f^{-1}(\theta'') \overset{C}{=} f^{-1}(\theta')$ for some facet $f^{-1}(\theta'')$ of $f^{-1}(\theta')$, where $T(\theta') \overset{C}{=} T(\theta'') \overset{C}{=} T(\theta)$. By omnidecisiveness, we must have $\theta'' = \theta$ or $\theta'' = \theta'$, and the latter is outlawed by continuity. Thus $\theta'' = \theta$, and $\dim(f^{-1}(\theta)) = m!-m+K$ by induction. Necessity is now immediate. $\square$

We know then that if σ and π are close neighbors, there is a unique vector $\bar{u}_\sigma^\pi$, $|\bar{u}_\sigma^\pi| = 1$, <u>separating</u> $f^{-1}(\sigma)$ from $f^{-1}(\pi)$ in the sense that if $\bar{x} \in f^{-1}(\sigma)$, then $(\bar{x}, \bar{u}_\sigma^\pi) \geq 0$ and if $\bar{x} \in f^{-1}(\pi)$, $(\bar{x}, \bar{u}_\sigma^\pi) \leq 0$, for every SDF satisfying the hypotheses of Lemma 4.1. Consequently, $f^{-1}(\sigma) \subset X^\sigma \quad \{\bar{x} \mid (\bar{x}, \bar{u}_\sigma^{\pi_i}) \geq 0$ for the m-1 close neighbors π_i, $1 \leq i \leq m-1$, of $\sigma\}$, for every $\sigma \in S_m$. We assert that in fact $f^{-1}(\sigma) \quad X^\sigma$; if not, $f^{-1}(\sigma)$ has some facet K which is not a facet of X^σ. But f(ri(K)) $\{\sigma, \pi\}$ by Lemma 4.1 for some close neighbor π of σ, and there are two distinct hyperplanes separating $f^{-1}(\sigma)$ and $f^{-1}(\pi)$, coutradicting the lemma. Thus we can write $f^{-1}(\sigma)$ as the solution set to m-1 inequalities, which are pairwise distinct, since f is constant on ri(K) for each facet K of $f^{-1}(\sigma)$, as we pointed out earlier.

At this point, we introduce the concept of a scoring function. Scoring functions were introduced in the 18th century by Borda [5], although some scoring functions like simple plurality date back to antiquity. Recertly much attention has been focused on SCF scoring functions, and Theorem 4.1 parallels a result obtained for SCFs by Young [22]. The following definition is taken from that source:

<u>Definition 4.6</u>: For each $\sigma \in S_m$, let P^σ be the m x m permutation matrix with entry of 1 in component (i, j) iff $\sigma(a_j) \quad a_i$. For each $\bar{x} \quad (x_{\pi_1}, \ldots, x_{\pi_{m!}}) \in R^{m!}$, the <u>scoring matrix</u> $D(\bar{x})$ <u>of</u> $\bar{x}$ is given by:

$$D(\bar{x}) = \sum_{i=1}^{m!} x_{\pi_i} P^{\pi_i}.$$

We denote the m rows of $D(\bar{x})$ by $D_1(\bar{x}), \ldots, D_m(\bar{x})$.

<u>Definition 4.7</u>: The <u>simple scoring SDF f^s with score vector $\bar{s}$</u> $(s_1, \ldots, s_m)$ is defined by:

$f^S(\bar{x}) = T(\Theta)$, where, if $a_i \in \Theta(r)$ and $a_k \in \Theta(t)$, then $\Theta(r)$ precedes $\Theta(t)$ if $(D_i(\bar{x}),\bar{s}) > (D_k(\bar{x}),\bar{s})$, and $r=t$ if $(D_i(\bar{x}),\bar{s}) = (D_k(\bar{x}),\bar{s})$, for all $a_i, a_k \in A$. ($(D_i(\bar{x}),\bar{s})$ is the inner product of row $D_i(\bar{x})$ and $\bar{s}$.)

A concept related to that of the simple scoring function is the composite scoring function:

<u>Definition 4.8</u>: Let S be the m x n matrix consisting of columns $s_1,\ldots,s_n$. The <u>composite scoring function f^S with score matrix S</u> is defined by:

$f^S(\bar{x}) = T(\Theta)$, where, if $a_i \in \Theta(r)$ and $a_k \in \Theta(t)$, then $\Theta(r)$ precedes $\Theta(t)$ if $(D_i(\bar{x}),\bar{s}_j) > (D_k(\bar{x}),\bar{s}_j)$ for some $j \geq 1$ and $(D_i(\bar{x}),\bar{s}_u) \geq (D_k(\bar{x}),\bar{s}_u)$ for all $1 \leq u < j$, and $r=t$ if $(D_i(\bar{x}),\bar{s}_u) = (D_k(\bar{x}),\bar{s}_u)$ for all $1 \leq u \leq n$, for all $a_i, a_k \in A$.

This definition is somewhat more compact than other formulations presented in Young [22], which point out more clearly the relationship of composite scoring functions as refinements of simple scoring functions. It suffices at this stage to say only that if f^{s_1} is not trivial, where $\bar{s}_1$ is the first column of S, then f^S agrees with f^{s_1} almost everywhere.

It is clear that scoring functions are symmetric, since they are not defined in terms of the voters and do not depend on the labeling of the alternatives. We show now that they are also consistent:

<u>Lemma 4.2</u>: f^S is consistent for any score matrix S.

<u>Proof</u>: Suppose that f^S is a scoring function with score matrix S, and that $f^S(\bar{x}) = B \subset P(S_m) - \emptyset$, and suppose that $\bar{x}^1 + \bar{x}^2 = \bar{x}$, where $f^S(\bar{x}^1) \cap f^S(\bar{x}^2) = B'$. We must show that $B = B'$ whenever $B' \neq \emptyset$.

Suppose then, contrary to the lemma, that $\sigma \in (B - B')$; there must be some alternatives a_i, a_k such that a_i precedes a_k in σ, but a_i follows a_k for all $\pi \in B'$. Therefore, there are columns $\bar{s}_{j_1}$, $\bar{s}_{j_2}$ of S for which $(D_k(\bar{x}^r), \bar{s}_{j_r}) > (D_i(\bar{x}^r), \bar{s}_{j_r})$ and $(D_k(\bar{x}^r), \bar{s}_j) = (D_k(\bar{x}^r), \bar{s}_j)$ for $1 \leq j < j_r$, $r = 1,2$, respectively. For $j = \min(j_1, j_2)$, we have $(D_k(\bar{x}), \bar{s}_j) > (D_i(\bar{x}), \bar{s}_j)$, while $(D_k(\bar{x}), \bar{s}_u) \geq (D_i(\bar{x}), \bar{s}_u)$ for all $1 \leq u < j$, since $D_k(\bar{x}) = D_k(\bar{x}^1) + D_k(\bar{x}^2)$ and $D_i(\bar{x}) = D_i(\bar{x}^1) + D_i(\bar{x}^2)$. But then a_k is placed before a_i in every $\eta \in B$ and $\sigma \notin (B - B')$.

If, on the other hand, $\sigma \in (B' - B)$, there are alternatives a_i, a_k for which a_i precedes a_k in σ, but a_k precedes a_i for all $\pi \in B$. Let $\bar{s}_j$ be the first column of S for which $(D_k(\bar{x}), \bar{s}_j) > (D_i(\bar{x}), \bar{s}_j)$; then $(D_k(\bar{x}), \bar{s}_u) \geq (D_i(\bar{x}), \bar{s}_u)$ for all $1 \leq u < j$. But since there is some $\sigma \in B'$ for which a_i precedes a_k, $(D_i(\bar{x}), \bar{s}_t) > (D_k(\bar{x}), \bar{s}_t)$ for some $1 \leq t$ and $(D_i(\bar{x}), \bar{s}_v) \geq (D_k(\bar{x}), \bar{s}_v)$ for all $1 \leq v < t$, by the analysis of the preceding paragraph. This contradiction implies $(B' - B) = \emptyset$. $\square$

We now turn our attention to simple scoring functions. Once again as in the case of the Kemeny function, continuity follows from the fact that the inner product is a continuous functional. By definition, $Ra(f^s) \subset K_m$ (See Definition 4.1) for every simple scoring function f^s; we now show that $Ra(f^s) = K_m$ unless f^s is completely trivial.

Lemma 4.3: Every nontrivial simple scoring function is omnidecisive.

Proof: Let f^s be some arbitrary nontrivial simple scoring function with score vector $\bar{s}$ ($\neq (t, t, \ldots, t)$), and let $\theta = (\theta(1), \ldots, \theta(k)$ be a weak order. We must show that there is some $\bar{x} \in R^{m!}$ for which $f(\bar{x}) = T(\theta)$. By consistency, it suffices to show that there are $\bar{x}^1, \ldots, \bar{x}^{k-1}$ for which $f^s(\bar{x}^1) = T(\theta_1)$, $\ldots$, $f^s(\bar{x}^{k-1}) = T(\theta_{k-1})$, where θ_r

$(\bigcup_{i=1}^{r} \Theta(i), \bigcup_{i=r+1}^{k} \Theta(i))$, $1 \leq r \leq k-1$, since then $f^s(\bar{x}) = \bigcap_{i=1}^{k-1} T(\Theta_i) = T(\Theta)$.

Fix Θ_r, and suppose, without loss of generality, that the alternatives in $\bigcup_{i=1}^{r} \Theta(i)$ are $a_1, \ldots, a_n$. Next, suppose that the components $s_1, \ldots, s_m$ of $\bar{s}$ are ordered $s_{\tau(1)} \geq \ldots \geq s_{\tau(m)}$ for some τ and at least one of the inequalities is strict, by the nontriviality assumption. Let $\bar{x}^*$ be the vector with 1 in component x_σ if $\tau\sigma \in T(\Theta_r)$ and 0 elsewhere. Then $(D_1(\bar{x}^*), \bar{s}) = \ldots = (D_n(\bar{x}^*), \bar{s}) = (m-n)!(n-1)!(s_{\tau(1)} + \ldots + s_{\tau(n)}) > (D_{n+1}(\bar{x}^*), \bar{s}) = \ldots = (D_m(\bar{x}^*), \bar{s}) = (m-n-1)!n!(s_{\tau(n+1)} + \ldots + s_{\tau(m)})$, so $f^s(\bar{x}^*) = T(\Theta_r)$. $\square$

To summarize, we have shown that simple scoring functions are symmetric, consistent, continuous and, if nontrivial, omnidecisive. In order to show that these conditions are sufficient to characterize simple nontrivial scoring functions, it will be necessary first to prove a lemma:

Lemma 4.4: Suppose that f is neutral, consistent, continuous and omnidecisive. Let σ_1 and π_1 be close neighbors which interchange a_i and a_k and let σ_2 and π_2 be close neighbors which interchange a_i and a_k, for some $a_i, a_k \in A$. Then $\bar{u}_{\sigma_1}^{\pi_1} = \pm \bar{u}_{\sigma_2}^{\pi_2}$.

Proof: If $m = 2$, the lemma is trivial, so suppose that $m > 2$. Denote by τ the permutation which interchanges a_1 and a_2 and denote by η the permutation which takes $a_1 \to a_2 \to a_3 \to a_1$. Consider the following six permutations of S_m:

$\varphi_1 = a_1, a_2, a_3, a_4, \ldots, a_m$ $\qquad$ $\varphi_2 = a_1, a_3, a_2, a_4, \ldots, a_m$

$\varphi_3 = a_2, a_1, a_3, a_4, \ldots, a_m = \tau$ $\qquad$ $\varphi_4 = a_2, a_3, a_1, a_4, \ldots, a_m = \eta$

$\varphi_5 = a_3, a_1, a_2, a_4, \ldots, a_m$ $\qquad$ $\varphi_6 = a_3, a_2, a_1, a_4, \ldots, a_m$

There are six separating vectors $\bar{u}^{\pi}$ associated with these permutations:

$\bar{u}^1$ separates $f^{-1}(\psi_1)$ from $f^{-1}(\psi_3)$ (i.e., $(\bar{x},\bar{u}^1) > 0$ if $\bar{x} \in f^{-1}(\psi_1)$ and $(\bar{x},\bar{u}^1) \leq 0$ if $\bar{x} \in f^{-1}(\psi_3)$.)

$\bar{u}^2$ separates $f^{-1}(\psi_2)$ from $f^{-1}(\psi_5)$

$\bar{u}^3$ separates $f^{-1}(\psi_4)$ from $f^{-1}(\psi_6)$

$\bar{u}^4$ separates $f^{-1}(\psi_1)$ from $f^{-1}(\psi_2)$

$\bar{u}^5$ separates $f^{-1}(\psi_3)$ from $f^{-1}(\psi_4)$

$\bar{u}^6$ separates $f^{-1}(\psi_5)$ from $f^{-1}(\psi_6)$.

Note that η and τ induce linear transformations L and L', respectively, on $R^{m!}$, where L: $\bar{u}^1 \rightarrow \bar{u}^3 \rightarrow -\bar{u}^2 \rightarrow \bar{u}^1$ and $\bar{u}^4 \rightarrow -\bar{u}^5 \rightarrow \bar{u}^6 \rightarrow \bar{u}^4$; L': $\bar{u}^1 \leftrightarrow -\bar{u}^1$, $\bar{u}^2 \leftrightarrow \bar{u}^3$, $\bar{u}^4 \leftrightarrow \bar{u}^5$, and $\bar{u}^6 \leftrightarrow -\bar{u}^6$. Next, we observe that if θ is the weak order such that $T(\theta) = \bigcup_{i=1}^{6} \psi_i$, then Lemma 4.1 implies that $f^{-1}(\theta)$ has dimension $m!-2$, and since $\bar{u}^i$ is orthogonal to $f^{-1}(\theta)$ for $1 \leq i \leq 6$, $\dim \{\bar{u}^1, \ldots, \bar{u}^6\} \leq 2$.

Suppose now that $\bar{u}^1$ and $\bar{u}^6$ are linearly independent; then for some α, β we have $\bar{u}^2 = \alpha\bar{u}^1 + \beta\bar{u}^6$. Applying L' to this equation, we find that $\bar{u}^3 = -\alpha\bar{u}^1 - \beta\bar{u}^6$, and by adding the two equations, we deduce $\bar{u}^2 = -\bar{u}^3$. Recursive application of L implies that $\bar{u}^1 = \bar{u}^3 = -\bar{u}^2$. But then $f^{-1}(\psi_1)$, $f^{-1}(\psi_4)$ and $f^{-1}(\psi_5)$ all lie in the same half-space generated by $\bar{u}^1$, while $f^{-1}(\psi_2)$, $f^{-1}(\psi_3)$ and $f^{-1}(\psi_6)$ all lie in the opposite half-space. Then $\bar{u}^1$ separates $f^{-1}(\psi_5)$ from $f^{-1}(\psi_6)$, and we must have $\bar{u}^1 = \bar{u}^6$. Thus, in any case, $\bar{u}^1 = k\bar{u}^6$, and by application of L to both sides of this equation, we have $\bar{u}^2 = k\bar{u}^5$ and $\bar{u}^3 = k\bar{u}^4$ for the same $k = \pm 1$.

By neutrality, then, we have proved the lemma for the case when

$\sigma_1^{-1}(a_i) = a_1 = \pi_1^{-1}(a_k), \sigma_2^{-1}(a_i) = a_2 = \pi_2^{-1}(a_k)$, and $\sigma_1^{-1}(a_r)$ precedes $\sigma_1^{-1}(a_s)$ iff $\sigma_2^{-1}(a_r)$ precedes $\sigma_2^{-1}(a_s)$ for all $a_r, a_s \in A - \{a_i, a_k\}$. We can extend this result further by considering the weak order θ' defined by: $\theta'(1) = \{a_3\}$, $\theta'(2) = \{a_1, a_2, a_4\}$, $\theta'(3) = \{a_5\}, \ldots, \theta'(m-2) = \{a_m\}$. A similar argument on the vectors separating the regions $f^{-1}(\psi'_i)$, $\psi'_i \in T(\theta')$ proves the lemma when $\sigma_1^{-1}(a_i) = a_2 = \pi_1^{-1}(a_k)$, $\sigma_2^{-1}(a_i) = a_3 = \pi_2^{-1}(a_k)$, and $\sigma_1^{-1}(a_r)$ precedes $\sigma_1^{-1}(a_s)$ iff $\sigma_2^{-1}(a_r)$ precedes $\sigma_2^{-1}(a_s)$ for all $a_r, a_s \in A - \{a_i, a_k\}$. Recursively, then the argument generalizes to show that the lemma is valid for all close neighbors σ_1 and π_1, σ_2 and π_2 which differ respectively in the same two alternatives a_i, a_k, provided that $\sigma_1^{-1}(a_r)$ precedes $\sigma_1^{-1}(a_s)$ iff $\sigma_2^{-1}(a_r)$ precedes $\sigma_2^{-1}(a_s)$ for all $a_r, a_s \in A - \{a_i, a_k\}$. To prove the lemma, it suffices to eliminate the restriction of this proviso. (If m = 3, we are done.)

Suppose then that σ_3 and π_3 are close neighbors of σ_1 and π_1, respectively, and of each other; that is, if μ is the permutation that interchanges a_i and a_k, and if ζ is the permutation that interchanges a_r and a_s, for some $a_r, a_s \in A - \{a_i, a_k\}$, then $\sigma_1 = \pi_1\mu \quad \sigma_3\zeta = \pi_3\mu\zeta$. Consider the following 4 separating vectors:

$\bar{v}^1$ separates $f^{-1}(\sigma_1)$ from $f^{-1}(\pi_1)$

$\bar{v}^2$ separates $f^{-1}(\sigma_3)$ from $f^{-1}(\pi_3)$

$\bar{v}^3$ separates $f^{-1}(\sigma_1)$ from $f^{-1}(\sigma_3)$

$\bar{v}^4$ separates $f^{-1}(\pi_1)$ from $f^{-1}(\pi_3)$.

Let L'' be the linear transformation induced by μ; hence, L'': $\bar{v}^1 \leftrightarrow -\bar{v}^1$, $\bar{v}^2 \leftrightarrow -\bar{v}^2$, and $\bar{v}^3 \leftrightarrow \bar{v}^4$.

By arguments similar to those set forth earlier in the proof, $\dim\{\bar{v}^1, \ldots, \bar{v}^4\} \leq 2$. Suppose then that $\bar{v}^1$ and $\bar{v}^2$ are linearly independent; there are then α, β such that $\bar{v}^3 = \alpha\bar{v}^1 + \beta\bar{v}^2$. Applying L'' to both sides of the equation, we find $\bar{v}^4 = -\alpha\bar{v}^1 - \beta\bar{v}^2$ and deduce $\bar{v}^3 = -\bar{v}^4$.

Let Ψ be the permutation which interchanges a_i and a_r as well as a_k and a_s, and let L* be the linear transformation on $R^{m'}$ induced by Ψ. Then $L^*(\bar{v}^3) = L^*(-\bar{v}^4) = -L^*(\bar{v}^4)$, and by our previous results, $\bar{v}^1 = \pm L^*(\bar{v}^3)$ while $\bar{v}^2 = \pm L^*(\bar{v}^4)$. Thus in any case, $\bar{v}^1 = \pm\bar{v}^2$. This conclusion removes the remaining restriction concerning a_r and a_s, so that we argue recursively to prove the lemma in general. □

Although we have established the existence of appropriate relationships between the hyperplanes separating the various neighboring regions, we have not yet established that the <u>directions</u> of those hyperplanes are what we would expect. This result is the next corollary:

<u>Corollary</u> <u>4.4.1</u>: Suppose that f satisfies the hypotheses of Lemma 4.4 and that σ_1 and π_1, σ_2 and π_2 are respective pairs of close neighbors as in the lemma. Then, if both σ_1 and σ_2 place a_i before a_k, while both π_1 and π_2 place a_i after a_k, then $\bar{u}_{\sigma_1}^{\pi_1} = \bar{u}_{\sigma_2}^{\pi_2}$.

<u>Proof</u>: Recall that we established subsequent to Lemma 4.1 that $f^{-1}(\sigma)$ could be represented as $X^\sigma = \{\bar{x} \mid (\bar{x}, \bar{u}_\sigma^{\pi_i}) \geq 0$ for the m-1 close neighbors π_i, $i = 1, \ldots, m$ of $\sigma\}$. Consider the following three regions, defined as the respective solution sets for the inequalities displayed:

<u>$f^{-1}(\varphi_1)$</u>	<u>$f^{-1}(\varphi_3)$</u>	<u>$f^{-1}(\varphi_6)$</u>
$(\bar{x}, \bar{u}^1) \geq 0$	$(\bar{x}, \bar{u}^1) \leq 0$	$(\bar{x}, \bar{u}^6) \leq 0$
$(\bar{x}, \bar{u}^4) \geq 0$	$(\bar{x}, \bar{u}^5) \geq 0$	$(\bar{x}, \bar{u}^3) \leq 0$
$\vdots$	$\vdots$	$\vdots$
$(\bar{x}, \bar{u}_{\varphi_1}^{\eta_i}) \geq 0$	$(\bar{x}, \bar{u}_{\varphi_3}^{\tau_i}) \geq 0$	$(\bar{x}, \bar{u}_{\varphi_6}^{\rho_i}) \geq 0$
$\vdots$	$\vdots$	$\vdots$

By $\bar{u}^{\eta_i}_{\varphi_1}$, $\bar{u}^{\tau_i}_{\varphi_3}$, and $\bar{u}^{\rho_i}_{\varphi_6}$ we mean the vectors separating $f^{-1}(\varphi_1)$, $f^{-1}(\varphi_3)$, and $f^{-1}(\varphi_6)$ from their respective ith neighbors, $f^{-1}(\eta_i)$, $f^{-1}(\tau_i)$, and $f^{-1}(\rho_i)$, $3 \le i \le m-1$; all other notation is as in the lemma.

Consider $f^{-1}(\varphi_1,\varphi_3)$. If $\bar{x} \in f^{-1}(\varphi_1,\varphi_3) - f^{-1}(\varphi_1) \cap f^{-1}(\varphi_3)$, then $(\bar{x},\bar{u}^{\eta_3}_{\varphi_1}) \geq 0$ and $(\bar{x},\bar{u}^{\tau_3}_{\varphi_3}) \geq 0$. By the lemma, we know that $\bar{u}^{\eta_3}_{\varphi_1} = \pm\bar{u}^{\tau_3}_{\varphi_3}$. Suppose by way of contradiction that the sign is negative; then $(\bar{x},\bar{u}^{\tau_3}_{\varphi_3}) = 0$. Let $K = f^{-1}(\varphi_3) \cap \{\bar{y} \mid (\bar{y},\bar{u}^{\tau_3}_{\varphi_3}) = 0\}$. By earlier remarks, f is constant on ri(K), and by Lemma 4.1, $\tau_3 \in f(\bar{y})$ for some $\bar{y} \in \mathrm{ri}(K)$. Therefore, $\tau_3 \in f(\mathrm{ri}(K))$ and by continuity $\tau_3 \in f(\bar{x})$, since $\bar{x} \in K$. We then conclude that there is no $\bar{x}$ for which $f(\bar{x}) = \{\varphi_1,\varphi_3\}$, contradicting omnidecisiveness. Thus $\bar{u}^{\eta_3}_{\varphi_1} = \bar{u}^{\tau_3}_{\varphi_3}$.

Next, consider the weak order Θ described in the proof of the lemma. By our previous results, $\bar{u}^{\eta_i}_{\varphi_1} = \pm\bar{u}^{\rho_i}_{\varphi_6}$ for $i = 4,\dots,m-1$. But an argument similar to the one above shows that unless the sign is positive, we arrive at a contradiction to omnidecisiveness, so that $\bar{u}^{\eta_i}_{\varphi_1} = \bar{u}^{\rho_i}_{\varphi_6}$ for $i = 4,\dots,m-1$.

Suppose now that $\bar{u}^1 = -\bar{u}^6$; then $\bar{u}^3 = L(\bar{u}^1) = L(-\bar{u}^6) = -\bar{u}^4$ for the linear transformation L introduced in the proof of the lemma. Since $\mathrm{int}(f^{-1}(\varphi_1)) \cap \mathrm{int}(f^{-1}(\varphi_6)) = \emptyset$, and since the ith inequality in $f^{-1}(\varphi_1)$ is the same as the ith inequality in $f^{-1}(\varphi_6)$, $i \neq 3$, $f^{-1}(\varphi_1)$ and $f^{-1}(\varphi_6)$ are separated by the vector $\bar{u}^{\rho_3}_{\varphi_6}$. But if $\bar{x} \in f^{-1}(\Theta)$, then $\bar{x} \in f^{-1}(\varphi_1) \cap f^{-1}(\varphi_6)$ so that $(\bar{x},\bar{u}^{\rho_3}_{\varphi_6}) = 0$. Thus $\rho_3 \in f(\bar{x})$ and there is no $\bar{x}$ for which $f(\bar{x}) = \Theta$, contradicting omnidecisiveness. Thus $\bar{u}^1 = \bar{u}^6$.

By neutrality, our results extend to the other alternatives as well, so that in repeating the arguments in Lemma 4.4, we are sure that the directions of the hyperplanes concerned are as claimed in the corollary. $\square$

At this point, let us recall our characterization result for simple scoring functions:

Theorem 4.1: A neutral, consistent and continuous SDF is omnidecisive iff it is a nontrivial simple scoring function.

Proof: We have already established that nontrivial simple scoring functions satisfy these properties.

Denote by $\bar{u}$ the vector which separates $f^{-1}(\varphi_1)$ from $f^{-1}(\varphi_3)$. We wish to show that there is a simple scoring function f^s for which $\bar{u}$ separates ${f^s}^{-1}(\varphi_1)$ from ${f^s}^{-1}(\varphi_3)$, for then neutrality and Corollary 4.4.1 would imply that $\bar{u}^{\pi}_{\sigma} = \bar{u}^{s\pi}_{\sigma}$ for all close neighbors $\sigma, \pi \in S_m$, where $\bar{u}^{s\pi}_{\sigma}$ denotes the vector separating ${f^s}^{-1}(\sigma)$ from ${f^s}^{-1}(\pi)$. But then $f^{-1}(\sigma) = X^{\sigma} = {f^s}^{-1}(\sigma)$ for all $\sigma \in S_m$, and therefore $f = f^s$.

Next, we examine what form $\bar{u}$ must have in order for there to exist some f^s such that $\bar{u}$ separates ${f^s}^{-1}(\varphi_1)$ from ${f^s}^{-1}(\varphi_3)$. We assert that if there are numbers $s_1, \dots, s_m$ for which entry u_σ (the σth component of $\bar{u}$) is given by $s_i - s_j$ whenever $\sigma(a_i) = a_1$ and $\sigma(a_j) = a_2$, then $\bar{u}$ separates ${f^{\bar{s}_\alpha}}^{-1}(\varphi_1)$ from ${f^{\bar{s}_\alpha}}^{-1}(\varphi_3)$, where $\bar{s}_\alpha = (s_1+\alpha, s_2+\alpha, \dots, s_m+\alpha)$, for every $\alpha \in R$. Indeed, let $\bar{x}^{\pi}$ be the vector with component x^{π}_{π} equal to 1 and all other components x^{π}_{σ}, $\sigma \neq \pi$, equal to 0; then if $\bar{x} \in R^{m!}$, we can write $\bar{x} = \sum_{\pi \in S_m} a_\pi \bar{x}^{\pi}$ for some appropriate set of coefficients a_π. Denote by π^{ij}_k the kth permutation, $1 \le k \le (m-2)!$, such that $\pi^{ij}_{k\,-1}(a_i) = a_1$ and $\pi^{ij}_k(a_j) = a_2$. Now, let us fix α and suppose $\bar{x} \in {f^{\bar{s}_\alpha}}^{-1}(\varphi_1)$; then $(D_1(\bar{x}), \bar{s}+\bar{\alpha}) \geq (D_2(\bar{x}), \bar{s}+\bar{\alpha})$, so $(D_1(\bar{x}), \bar{s}) \geq$

$(D_2(\bar{x}),\bar{s})$, where $\bar{\alpha} = (\alpha,\alpha,\dots,\alpha)$, and therefore $(D_1(\bar{x})-D_2(\bar{x}),\bar{s}) \geq 0$. But then $\bar{x} = \sum_{i=1}^{m} \sum_{\substack{j\neq i\\ j=1}}^{m} \sum_{k=1}^{(m-2)!} a_{\pi_k^{ij}} \bar{x}^{\pi_k^{ij}}$ for an appropriate set of coefficients $a_{\pi_k^{ij}}$, and therefore $(D_1(\bar{x})-D_2(\bar{x}),\bar{s}) =$

$$\sum_{i=1}^{m} s_i \Big(\sum_{\substack{j\neq i\\ j=1}}^{m} \Big(\sum_{k=1}^{(m-2)!} a_{\pi_k^{ij}} - a_{\pi_k^{ji}} \Big)\Big) = \sum_{k=1}^{(m-2)!} \Big(\sum_{i=1}^{m} \sum_{\substack{j\neq i\\ j=1}}^{m} (s_i - s_j) a_{\pi_k^{ij}} \Big) = (\bar{u}, x)$$

for the vector $\bar{u}$ with coordinates as described before. A similar argument shows that if $\bar{x} \in f^{s_{\bar\alpha}^{-1}}(\Psi_3)$, then $(\bar{x},\bar{u}) \leq 0$. Thus $\bar{u}$ separates $f^{s_{\alpha}^{-1}}(\varphi_1)$ from $f^{s_{\alpha}^{-1}}(\Psi_3)$.

At this point, we need to introduce the concept of a subclass of permutations:

__Definition 4.9__: A __subclass of S_m of order M__ is a set $C_M = \{\sigma_i \in S_m \mid \sigma_i^{-1}(a_k) = \sigma_j^{-1}(a_k)$ for all $m \geq k > M$ and for each $1 \leq i,j \leq M!\}$. When otherwise clear from the context, C_M will be abbreviated C.

Following is an illustration of two subclasses of S_5 of order 3:

a_1, a_2, a_3, a_4, a_5	a_5, a_1, a_2, a_3, a_4
a_1, a_3, a_2, a_4, a_5	a_5, a_1, a_3, a_2, a_4
a_2, a_1, a_3, a_4, a_5	a_5, a_2, a_1, a_3, a_4
a_2, a_3, a_1, a_4, a_5	a_5, a_2, a_3, a_1, a_4
a_3, a_1, a_2, a_4, a_5	a_5, a_3, a_1, a_2, a_4
a_3, a_2, a_1, a_4, a_5	a_5, a_3, a_2, a_1, a_4

Let C be a subclass of order M given on positions $\rho_1,\dots,\rho_M$ (i.e., if $\sigma \in C$ and $1 \leq k \leq M$, then $\sigma^{-1}(a_k) = a_{\rho_i}$ for some $1 \leq i \leq M$). We assert that there are numbers $s_1,\dots,s_M$ (depending on C) for which $u_\sigma = s_i - s_j$ when $\sigma \in C$ places a_1 in position ρ_i and a_2 in position ρ_j.

We proceed by induction on M, $M \geq 2$. Note that if μ is a permutation with a single cycle $(\mu(a_1),\dots,\mu^k(a_1)=a_2,\dots,\mu^n(a_1)=a_1)$, then (**) if $\sigma\in S_m$, $\sum_{i=1}^{n} u_{\sigma\mu^i}=0$. Indeed, let $\bar{x}=\sum_{i=1}^{n}\bar{x}^{\sigma\mu^i}$; by neutrality, if $\eta\in f(\bar{x})$ and η places a_1 before a_2, then since $\bar{x}$ is a fixed point of μ, $\eta\mu^k\in f(\bar{x})$, and $\eta\mu^k$ places a_2 before a_1. Therefore Lemma 4.4 implies that $0=(\bar{x},\bar{u})=\sum_{i=1}^{n} u_{\sigma\mu^i}$.

If $M=2$, then C consists of just two permutations, σ and $\sigma\tau$, where τ interchanges a_1 and a_2. By (**) above, $u_\sigma+u_{\sigma\tau}=0$, so we can choose $s_1=u_\sigma$ and $s_2=0$ and we are done.

If $M=3$, then for some $1\le\rho_1<\rho_2<\rho_3\le m$, C has the form:

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
a_1	a_1	a_2	a_2	a_3	a_3	position ρ_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
a_2	a_3	a_1	a_3	a_1	a_2	position ρ_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
a_3	a_2	a_3	a_1	a_2	a_1	position ρ_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

With τ as above, we conclude that $u_{\sigma_1}+u_{\sigma_3}=u_{\sigma_2}+u_{\sigma_4}=u_{\sigma_5}+u_{\sigma_6}=0$, since $\sigma_1=\sigma_3\tau$ etc., by (**). Moreover, if μ is the permutation with cycle (a_2,a_3,a_1), then $\sigma_5=\sigma_4\tau=\sigma_1\tau^2$, and we conclude that $u_{\sigma_1}+u_{\sigma_4}+u_{\sigma_5}=0$. Let $s_1=u_{\sigma_2}$, $s_2=u_{\sigma_5}$ and $s_3=0$; substitution into the preceding equation implies that $u_{\sigma_1}+s_2=s_1$

0 and consequently $u_{\sigma_1} = s_1 - s_2$, so we are done.

Suppose $M = 4$. Any subclass of order 4 has 24 members. Let C be such a subclass, determined by positions $\rho_1 < \rho_2 < \rho_3 < \rho_4$. We note that $C = \bigcup_{k=1}^{4} C_k$, where, for each k, C_k is the subclass of order 3 determined by fixing alternative a_4 in position ρ_k. By the remarks above for $M = 3$, we have for C_k the score vector $\bar{s}^k = (s_1^k, s_2^k, s_3^k)$ such that $u_\sigma = s_i^k - s_j^k$ whenever $\sigma \in C_k$ places a_1, a_2, a_k in positions ρ_i, ρ_j, ρ_k, respectively, $1 \leq i, j, k \leq 4$. We are done if we show that these conditions imply the existence of an appropriate vector $\bar{s}$ for which the induction proposition holds on $\bar{u}$ for the subclass C of order 4.

We recall from our previous remarks that the vectors $\bar{s}$ and $\bar{s}+\bar{\gamma}$ produce the same separating vector $\bar{u}$, so that in fact, we may choose the scoring vectors $\bar{s}^k$ such that $s_4^k = 0$ for k=1,2,3, and $\bar{s}^4$ such that $s_3^4 = s_3^1$. Consider now the 3-cycle permutation τ with cycle (a_2, a_4, a_1). Let σ be the permutation in C with a_1 in position ρ_1, a_2 in position ρ_3 and a_4 in position ρ_4; (**) implies that $u_{\sigma\tau} + u_{\sigma\tau^2} + u_{\sigma\tau^3} = 0 = (s_4^3 - s_1^3) + (s_3^1 - s_4^1) + (s_1^4 - s_3^4)$. But since we chose $s_4^3 = s_4^1 = 0$ and $s_3^4 = s_3^1$, we deduce that $s_1^4 = s_1^3$. Next, apply τ to the permutation which places a_1 in position ρ_1, a_2 in position ρ_2 and a_4 in position ρ_4. By (**) we deduce that $(s_4^2 - s_1^2) + (s_2^1 - s_4^1) + (s_1^4 - s_2^4) = 0$; since $s_4^1 = s_4^2$, $s_2^4 = s_1^4 - s_1^2 + s_2^1$. Finally, apply τ to the permutation which places a_1 in position ρ_1, a_2 in position ρ_2 and a_4 in position ρ_3; by (**) we have $(s_3^2 - s_1^2) + (s_2^1 - s_3^1) + (s_1^3 - s_2^3) = 0$. Therefore, $s_2^3 = s_1^3 + s_3^2 - s_1^2 + s_2^1 - s_3^1$.

Consider now the 4-cycle $\zeta = (a_2, a_3, a_4, a_1)$. If $\eta \in C$ is the permutation with a_1, a_2, a_3, a_4 in positions $\rho_1, \rho_2, \rho_3, \rho_4$ respectively, then (**) implies that $0 = u_{\eta\zeta} + u_{\eta\zeta^2} + u_{\eta\zeta^3} + u_{\eta\zeta^4} = (s_1^4 - s_2^4) + (s_4^3 - s_1^3) + (s_3^2 - s_4^2) + (s_2^1 - s_3^1)$. But since $s_1^4 = s_1^3$ and $s_4^2 = s_4^3$, we deduce that

$s_3^2 = s_1^3 - s_1^2 + s_3^1$. Substituting this expression into the equation for s_2^3, we find that $s_2^3 = 2(s_1^3 - s_1^2) + s_2^1$. Finally, if ψ is the permutation which takes $a_1 \leftrightarrow a_2$ and $a_3 \leftrightarrow a_4$, and ξ the permutation which takes $a_1 \leftrightarrow a_3$ and $a_2 \leftrightarrow a_4$, then let $\bar{x} = \bar{x}^{\eta} + \bar{x}^{\eta\psi} + \bar{x}^{\eta\xi} + \bar{x}^{\eta\psi\xi}$. Since $\bar{x}$ is fixed under the double inversions ψ and ξ, symmetry implies that a_1 and a_2 are in the same indifference row in $f(\bar{x})$ so that $(\bar{x}, \bar{u}) = 0$. Consequently, $u_{\eta} + u_{\eta\psi} + u_{\eta\xi} + u_{\eta\psi\xi} = 0 = (s_1^4 - s_2^4) + (s_2^3 - s_1^3) + (s_3^2 - s_4^2) + (s_4^1 - s_3^1)$. Substitution into our previous equations yields $s_1^2 = s_1^3$, which in turn implies that $s_2^3 = s_2^1$ and $s_3^2 = s_3^1$.

Thus, we have concluded that $s_i^k = s_i^n$ for all $1 \leq i \neq k, n \leq 4$. But this is all we had to show

The general case follows the same general approach. If C is a subclass of order $M > 4$, determined by positions $\rho_1 \leq \dots \leq \rho_M$, then $C = \bigcup_{i=1}^{M} C_k$, where C_k is the subclass of order M-1 which places alternative a_M in position ρ_k. By induction, we have for each C_k a score vector $\bar{s}^k$ such that $u_\sigma = s_i^k - s_j^k$ whenever $\sigma \in C_k$ places a_1, a_2, a_k in positions ρ_i, ρ_j, ρ_k, respectively, $1 \leq i, j, k \leq M$. We note as before that we may choose the $\bar{s}^k$ such that $s_M^k = 0, k=1,\dots,M-1$ and $s_{M-1}^M = s_{M-1}^1$.

Let τ be the 3-cycle (a_2, a_M, a_1), and let $\sigma = \sigma(u,v)$ be any permutation which places a_1, a_2, a_M in positions ρ_1, ρ_u, ρ_v, respectively, for some $1 < u < v \leq M$. By (**), $u_{\sigma\tau} + u_{\sigma\tau^2} + u_{\sigma\tau^3} = 0 = (s_1^v - s_u^v) + (s_v^u - s_1^u) + (s_u^1 - s_v^1)$. Next, let γ be the 4-cycle (a_2, a_3, a_M, a_1), and let $\pi = \pi(u,v)$ be any permutation which places a_1, a_2, a_3, a_M in positions $\rho_1, \rho_u, \rho_v, \rho_M$, respectively, for some $1 < u < v < M$. As in the case $M=4$, we conclude that $(s_1^M - s_u^M) + (s_M^v - s_1^v) + (s_v^u - s_M^u) + (s_u^1 - s_v^1) = 0$. By considering all of the equations obtained from the various combinations of u,v in the above two equation forms, we conclude that:

$s_u^v = 2(s_1^v - s_1^u) + s_u^1$ and $s_v^u = (s_1^v - s_1^u) + s_v^1$, for all $1 < u < v < M$,
$s_u^M = (s_1^M - s_1^u) + s_u^1$, for all $1 < u < M$, and
$s_1^M = s_1^{M-1}$.

We know by assumption that $s_M^1 = s_M^u$, for all $1 < u < M$, and that $s_{M-1}^M = s_{M-1}^1$. All that remains is to show that $s_1^u = s_1^v$ for all $1 < u < v < M$.

Consider then any permutation $\eta(u,v)$ which places a_1, a_2, a_3, a_M in positions $\rho_2, \rho_u, \rho_v, \rho_M$, respectively, where $2 < u < v < M$. Applying ζ to $\eta(u,v)$, we conclude, after substituting our previous results, that $s_1^u = 2s_1^v - s_1^2$. If $M > 5$, then by merely varying u and v we conclude that $s_1^u = s_1^v$ for all $1 < u < v < M$, and we are done. If $M = 5$, we conclude as we did for $M = 4$, where ψ takes $a_1 \leftrightarrow a_2$ and $a_3 \leftrightarrow a_5$, etc. Thus, the claim is proved by induction, and $M = m$ is the theorem. $\square$

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