INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

Bell & Howell Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

UMI®

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

POWER RADIATION BY A SCATTERED PLANE WAVE

By

IGOR BALSIM

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2001

UMI Number: 9997072

UMI®

UMI Microform 9997072

Bell & Howell Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

12/19/2000
Date

Richard Sacksteder
Chair of Examining Committee

12/19/2000
Date

Executive Officer

Richard Sacksteder

SUPERVISORY COMMITTEE

THE CITY UNIVERSITY OF NEW YORK

Abstract

POWER RADIATION by a SCATTERED PLANE WAVE

by

Igor Balsim

Adviser: Professor Richard Sacksteder

In this thesis we show how to estimate asymptotically, as k the wave number goes to infinity, the power of a plane wave of unit amplitude which is scattered by a cylinder with its axis perpendicular to the direction of propagation. The result $\frac{E(k)}{k^2} \cong \frac{4}{\pi}$ obtained in chapter VII shows that the power in the scattered wave is for high frequencies (or large wave numbers) asymptotically proportional to the square of the frequency. We also want to estimate asymptotically the normalization factor $D(k)$. The estimate $\frac{D(k)}{k^2} \cong \frac{1}{2}$ shows that the dissipation function (normalization factor $D(k)$) is also proportional to the square of the frequency and therefore E(k) is asymptotically proportional to D(k). Since, in general, both the power and the dissipation are proportional to the square of the amplitude, the latter result holds for plane waves of any amplitude.

The key tool used in obtaining our results was the method of steepest descent. The above problem leads rise to solving a Helmholtz equation in the exterior of the unit disc in two dimensions with boundary conditions given by the derivative of a plane wave.

The solution can be expressed as an infinite sum of Hankel functions. The power can be expressed as a sum of ratios of the derivative of Bessel functions and Hankel functions. In order to get the desired asymptotic estimates we needed to get uniform estimates of the derivative of Hankel functions using the method of steepest descent.

TABLE OF CONTENTS

POWER RADIATION BY A SCATTERED PLANE WAVE

Igor Balsim

I. Introduction

In classical linear acoustics, one of the commonly studied partial differential equations is the Helmholtz equation (or the reduced wave equation)

$$\Delta u + k^2 u = 0,\ \ k \neq 0. \tag{1.1}$$

k is a real number, which is physically interpreted as the wave number of the wave.

Here we want to study solutions of the Helmholtz equation that are defined in the exterior of the unit disc in two-dimensional space and whose unit exterior normal on the boundary of the disc agrees with that of the plane wave:

$$v(r,\vartheta,k) = e^{ikr\cos\vartheta}. \tag{1.2}$$

Here (r,ϑ) are the usual polar coordinates and the disc is given by $r \leq 1$.

We also limit the class of solutions by a boundary condition at infinity, which assures that no energy is extracted from outside the disc as a result of the wave motion. This is known as the Sommerfeld radiation, or outgoing wave, condition.

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} + iku\right) = 0. \tag{1.3}$$

We will assume the Uniqueness Theorem for the above class of solutions, which was proved by Rellich and Kupradse [Colton and Kress, p.78], and we will prove the existence of the solution by constructing an infinite sum of Hankel functions.

The purpose of this paper is to study the power radiation of the solutions of the boundary problem for $k \to \infty$. It will be possible to construct a normalization factor so that the ratio of the power and the normalization factor will be dimensionless and independent of k. As a result of these properties the ratio can be used as a way to standardize the power of sound waves at least in this particular boundary problem. The inverse of the normalization factor is known in the literature as the dissipation function, which we will denote by $D(k)$, which for our problem is given by $D(k) = \int_{S^1} |u_r|^2 ds$.

Here S^1 denotes the boundary r=1 and s is arclength.

II. Computation of the Solutions

The Helmholtz equation can be expressed in polar coordinates in two dimensions as

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2}+k^2 u=0. \qquad (2.1)$$

We compute the solutions to the boundary problem using the method of separation of variables. Consider a solution of the form $u(r,\vartheta)=f(r)g(\vartheta)$. Equation (2.1) separates into the following two equations.

$$g''(\vartheta)+n^2 g(\vartheta)=0 \qquad (2.2)$$

$$f''(r)+\frac{1}{r}f'(r)+(k^2-\frac{n^2}{r^2})f(r)=0. \qquad (2.3)$$

The general solution of (2.2) is a sum of terms of the form $g_n(\vartheta)=c_n e^{in\vartheta}$. We consider only those solutions where n is an integer to assure that the solution will be single valued. The general solution of (2.3) is of the form $f_n(r)=a_n H_n(kr)+b_n \overline{H}_n(kr)$, where a_n and b_n are complex constants. Note that throughout this paper $H_n(z)$ will denote the Hankel function of the first kind. To satisfy the radiation conditions (1.3) it is necessary that $a_n=0$.

We will first look at formal summations, and then we will justify these computations by showing that the sum converges uniformly and absolutely. Let

$$u(r,\vartheta,k)=\sum_{n=-\infty}^{\infty} b_n \overline{H}_n(kr)e^{in\vartheta}. \qquad (2.4)$$

We take the derivative of this solution in the radial direction.

$$u_r(r,\vartheta,k) = k\sum_{n=-\infty}^{\infty} b_n \overline{H}_n'(kr)e^{in\vartheta}. \qquad (2.5)$$

On the boundary $r=1$, so

$$u_r(1,\vartheta,k) = k\sum_{n=-\infty}^{\infty} b_n \overline{H}_n'(k)e^{in\vartheta}. \qquad (2.6)$$

The boundary conditions state that $u_r(1,\vartheta,k) = v_r(1,\vartheta,k)$ where $v(r,\vartheta,k)$ is given by (1.2). Applying these boundary conditions to (2.6) we get

$$k\sum_{n=-\infty}^{\infty} b_n \overline{H}_n'(k)e^{in\vartheta} = ik\cos\vartheta e^{ik\cos\vartheta}.$$

Multiply both sides by $\sin(j\vartheta)$ where $j>0$ is an integer, and integrate from 0 to 2π. We get

$$\pi[b_j\overline{H}_j'(k) - b_{-j}\overline{H}_{-j}'(k)] = i\int_0^{2\pi}\sin(j\vartheta)\cos\vartheta e^{ik\cos\vartheta}d\vartheta.$$

Since $\cos(\vartheta)e^{ik\cos\vartheta}$ is an even function and $\sin(j\theta)$ is an odd function, the integral must be 0. The Hankel functions satisfy $\overline{H}_{-j}(z) = (-1)^j\overline{H}_j(z)$; hence $b_j = (-1)^j b_{-j}$. We insert this into (2.6) and we get

$$u_r(1,\vartheta,k) = b_0 k\overline{H}'(k) + 2k\sum_{n=1}^{\infty} b_n\overline{H}_n'(k)\cos(n\vartheta). \qquad (2.7)$$

We expand e^z into a series of Bessel functions using the generating function of the Bessel function. $e^{\frac{z}{2}(t-\frac{1}{t})} = J_0(z) + \sum_{n=1}^{\infty} J_n(z)[t^n + (-1)^n t^{-n}]$, set $z = kr, t = ie^{i\vartheta}$ and (1.2) becomes $v(r,\vartheta,k) = J_0(kr) + 2\sum_{n=1}^{\infty}(i)^n J_n(kr)\cos(n\vartheta)$, hence

$$\frac{\partial v}{\partial r}\Big|_{r=1} = kJ_0'(k) + 2k\sum_{n=1}^{\infty}(i)^n J_n'(k)\cos(n\vartheta). \qquad (2.8)$$

To satisfy the boundary conditions we equate the coefficients of $\cos(n\vartheta)$ in equations (2.7) and (2.8), and we get

$b_n = (i)^n J_n'(k) / \overline{H}_n'(k)$ for $n \geq 0$.

We substitute these values into our solution (2.4) and we get

$$u(r,\vartheta,k) = J_0'(k)\overline{H}_0(kr)/\overline{H}_0'(k) + 2\sum_{n=1}^{\infty}(i)^n \frac{\overline{H}_n(kr)J_n'(k)\cos(n\vartheta)}{\overline{H}_n'(k)} \tag{2.9}$$

$$u_r(r,\vartheta,k) = k\frac{J_0'(k)\overline{H}_0'(kr)}{\overline{H}_0'(k)} + 2k\sum_{n=1}^{\infty}(i)^n \frac{J_n'(k)\overline{H}_n'(kr)\cos(n\vartheta)}{\overline{H}_n'(k)}. \tag{2.10}$$

Using these two equations we will now compute the power output of the solution [Sacksteder,p.300]. If the velocity of the wave propagation and the area are fixed, the power radiation by a solution u is proportional to

$$E(k) = \frac{-ik}{2\pi}\int_0^{2\pi}[u(1,\vartheta,k)\overline{u}_r(1,\vartheta,k) - u_r(1,\vartheta,k)\overline{u}(1,\vartheta,k)]d\vartheta.$$

We compute $E(k)$ for our solution using (2.9) and (2.10); we get

$$\frac{1}{2\pi}\int_0^{2\pi} u\overline{u}_r(1,\vartheta,k)d\vartheta = k\sum_{n=0}^{\infty}\frac{J_n'(k)^2\overline{H}_n(k)H_n'(k)}{|H_n'(k)|^2},$$ and

$$\frac{1}{2\pi}\int_0^{2\pi} u_r\overline{u}(1,\vartheta,k)d\vartheta = k\sum_{n=0}^{\infty}\frac{J_n'(k)^2 H_n(k)\overline{H}_n'(k)}{|H_n'(k)|^2}.$$ Hence ,

$$E(k) = \frac{4k}{\pi}[2\sum_{n=0}^{\infty}\frac{J_n'(k)^2}{|H_n'(k)|^2} - \frac{J_0'(k)^2}{|H_0'(k)|^2}]. \tag{2.11}$$

We used the Wronskian identity for Hankel functions;

$$\overline{H}_n(z)H_n'(z) - H_n(z)\overline{H}_n'(z) = \frac{4i}{\pi z}.$$

In the next section we will show that for z fixed and bounded and n→∞

$$J_n(z) = O((\frac{ez}{n})^n) \tag{2.12}$$

$$Y_n(z) = -((\frac{n}{ez})^n\frac{\sqrt{2}}{\sqrt{n\pi}})[1 + O(n^{-1/5})] + O[\ln(n/z)] \tag{2.13}$$

$$J_n'(z) = O((\frac{ez}{n})^n \frac{n}{z}) \tag{2.14}$$

$$Y_n'(z) = -(\frac{\sqrt{2n}}{z\sqrt{\pi}}(\frac{n}{ez})^n)[1 + O(n^{-1/5})] + O(\sqrt{\frac{n^2}{z^2} - 1}) \tag{2.15}$$

These estimates imply that (2.9) and (2.10) converge uniformly and absolutely for $1 \le r \le |kr| \le B$. In fact the nth term of (2.9) and of (2.10) is $O((\frac{ek}{n})^n n)$. Similarly the nth term of the series of $E(k)$ is $O(n\ (\frac{ke}{n})^{4n})$.

We now compute the dissipation function $D(k)$.

$$D(k) = \frac{1}{2\pi}\int_{s^1}|u_r|^2 ds = \frac{1}{2\pi}\int_0^{2\pi} u_r\bar{u}_r(1, \vartheta, k)d\vartheta = k^2[2\sum_{n=0}^{\infty}[J_n'(k)]^2 - J_0'(k)^2].$$

Again we can show convergence, because the nth term is $O((\frac{ke}{n})^{2n} n^2)$.

III. Estimates of $H_n(z)$ and $H_n'(z)$ as $n \to \infty$.

We will use the contour integral representation of the $H_n(z)$ and estimate the integral for n large and k fixed, using the saddle point method described, for example, by Courant and Hilbert [C/H, p. 527]. Since our purpose is to show the uniform and absolute convergence of a series of Hankel functions on compact sets of k, it suffices to consider the case where $1 \le |k| \le B < \infty$. The Hankel function of the first kind is represented by

$$H_n(k) = \frac{1}{\pi} \int_\Gamma e^{i[-k \sin x + nx]} dx \ , \tag{3.1}$$

where Γ is a curve that starts at $-\pi + i\infty$ and goes down until $-i\infty$. Γ can be represented as the union of oriented curves, $\Gamma = \bigcup_{i=1}^{5} \gamma_i$. We will describe γ_i below. It follows that

$$H_n'(k) = \frac{-i}{\pi} \int_\Gamma \sin x e^{i[-k \sin x + nx]} dx.$$

For n > 0, let $\alpha = k/n$; $\beta = \alpha^{-1}$.

$P(x,\alpha) = i[-\alpha \sin x + x]$, and

$$H_n(\alpha n) = \frac{1}{\pi} \int_\Gamma e^{nP(x,\alpha)} dx.$$

The main contribution of such an integral comes from a small neighborhood of the critical points of $P(x,\alpha) = i[-\alpha \sin x + x]$,, which are the solutions of $P'(x,\alpha) = 0$. If $P'(x,\alpha) = -\alpha \cos x + 1 = 0$, then $\cos x = \alpha^{-1}$.

For large n with k fixed $\alpha^{-1} > 1$; so the only solutions in $\{x : -\pi \le \mathrm{Re}(x) \le 0\}$ are the complex solutions $x = \pm i \cosh^{-1}(\beta)$. Here we use the notation $y_0 = \cosh^{-1}(\beta) > 0$.

Let $0 < \varepsilon < \frac{1}{2}$ be a fixed constant which will be determined later on in Chapter V (Lemma 5.2 and Theorem 5.3). For the purpose of constructing the curves and obtaining our estimates we will need the following lemma. We will state and prove the lemmas in this chapter under more general conditions than are necessary for the conclusion in this chapter so that these results can be used in the next chapter where both $n \to \infty$ and $k \to \infty$ at a constant ratio i.e. $\beta = \frac{n}{k} \geq 1 + \varepsilon$. Assume that $n \geq 2$ for all the statements in the rest of this chapter.

Lemma 3.0

i) *if* $n \geq k\cosh(1)$ *then* $\varepsilon_n = n^{-2/5} \leq \cosh^{-1}(\beta) = y_0$,

ii) *if* $\beta = \frac{n}{k} > 1 + \varepsilon$ *and* $k^{-1} \leq \varepsilon^{5/2}(1+\varepsilon)[\sinh\ (1)]^{-5/2}$ *then*

$\varepsilon_n = n^{-2/5} \leq \cosh^{-1}(\beta) = y_0$.

Proof:

If n is large enough so that $n \geq k\cosh(1)$, then $n^{-1}\cosh(n^{-2/5}) \leq n^{-1}\cosh(1) \leq k^{-1}$. Thus,

$\cosh(n^{-2/5}) \leq \frac{n}{k} = \beta$ and $n^{-2/5} \leq \cosh^{-1}(\beta)$.

If $\beta > 1 + \varepsilon$ and k is large enough such that $k^{-1} \leq \varepsilon^{5/2}(1+\varepsilon)[\sinh\ (1)]^{-5/2}$, then $\rho(k,\varepsilon) = k^{-2/5}(1+\varepsilon)^{-2/5}\sinh\ (1) \leq \varepsilon$.

Use the Mean Value Theorem to show that $\cosh(x) \leq 1 + \sinh(\bar{x})x \leq 1 + \sinh(1)x$ for $0 < \bar{x} < x \leq 1$. Apply this inequality with $x = (1+\varepsilon)^{-2/5}k^{-2/5}$.

We have the estimate $\cosh((1+\varepsilon)^{-2/5}k^{-2/5}) \leq 1 + \rho(k,\varepsilon) \leq 1 + \varepsilon \leq \beta$, or

$(1+\varepsilon)^{-2/5}k^{-2/5} \leq \cosh^{-1}(\beta)$. Since, $1 + \varepsilon \leq \beta = \frac{n}{k}$, $(1+\varepsilon)^{-2/5}k^{-2/5} \geq n^{-2/5}$.

Thus, we conclude that $n^{-2/5} \leq \cosh^{-1}(\beta) = y_0$.

For the rest of this chapter assume that either i) $n \geq k\cosh(1)$ or

ii) $k^{-1} \leq \varepsilon^{5/2}(1+\varepsilon)[\sinh\ (1)]^{-5/2}$ *and* $\beta = \frac{n}{k} > 1 + \varepsilon$. The first condition will be satisfied by the conditions of Chapter II, where we want to show that the series for the power radiation and the dissipation function converge. The second condition will be satisfied by the conditions of Chapter IV. Thus, we can conclude that in either case $\varepsilon_n \leq \cosh^{-1}(\beta) = y_0$.

The curves γ_t are given as follows oriented with increasing t.

$$\begin{array}{ll}
\gamma_1 = -\pi - it & -\infty < t \le -y_0 \\
\gamma_2 = t + iy_0 & -\pi \le t \le 0 \\
\gamma_3 = -it & -y_0 \le t \le y_0 - \varepsilon_n \\
\gamma_4 = -it & y_0 - \varepsilon_n \le t \le y_0 + \varepsilon_n \\
\gamma_5 = -it & y_0 + \varepsilon_n \le t < \infty
\end{array}$$

Let $G_j(\alpha,n) = \frac{1}{\pi}\int_{\gamma_j} e^{nP(x,\alpha)}dx$ and

$$K_j(\alpha,n) = \frac{-i}{\pi}\int_{\gamma_j} e^{nP(x,\alpha)}\sin x dx \text{ for j = 1,2,3,4,5.}$$

Then $\quad H_n(\alpha n) = \sum_{i=1}^{5} G_i(\alpha,n)$ and $H_n'(\alpha n) = \sum_{i=1}^{5} K_i(\alpha,n)$.

Note that $K_j(\alpha,n) = R_j(\alpha,n) + L_j(\alpha,n)$ where $R_j(\alpha,n) = \frac{-1}{2\pi}\int_{\gamma_j} e^{nP(x,\alpha)+ix}dx$ for j = 1....5

and $L_j(\alpha,n) = \frac{1}{2\pi}\int_{\gamma_j} e^{nP(x,\alpha)-ix}dx$ for j = 1....5.

Lemma 3.1:

$$|G_1(\alpha,n)| \le \frac{e^{-ny_0}}{\pi n}.$$

$$|K_1(\alpha,n)| \le \frac{2e^{-ny_0}}{\pi k}. \tag{3.2}$$

$G_1(\alpha,n)$ and $K_1(\alpha,n)$ are purely imaginary.

Proof:

$$G_1(\alpha,n) = \frac{-i}{\pi}\int_{y_0}^{\infty} e^{n[-\alpha\sinh t - t]}dt.$$

$$|G_1(\alpha,n)| \le \frac{1}{\pi}\int_{y_0}^{\infty} e^{-nt}dt.$$

$|G_1(\alpha,n)| \le \frac{e^{-n(y_0)}}{n\pi}$. Similarly, one can show that,

$$|K_1(\alpha,n)| \le \frac{e^{-ny_0}\cosh y_0}{(n-1)\pi} \le \frac{e^{-ny_0}2}{k\pi}.$$

Let

$$d(\beta) = \cosh^{-1}(\beta) - \sqrt{1-\beta^{-2}}. \tag{3.3}$$

Lemma 3.2:

$$|G_2(\alpha,n)| \le e^{-nd(\beta)} \qquad \text{and}$$

$$|K_2(\alpha,n)| \le \beta e^{-nd(\beta)}. \tag{3.4}$$

Proof:

If $x(t) = t + iy_0(t)$ then $\sin[x(t)] = i\sqrt{\alpha^{-2}-1}\cos t + \alpha^{-1}\sin t$, so

$\alpha\sin[x(t)] = i\sqrt{1-\alpha^2}\cos t + \sin t$, and on γ_2 let

$u(t) = \mathrm{Re}(P(x,\alpha)) = -\sqrt{1-\alpha^2}\cos t + y_0$

$v(t) = \mathrm{Im}(P(x,\alpha)) = \sin t + t.$

Thus, $G_2(\alpha,n) = \frac{1}{\pi}\int_{-\pi}^{0} e^{-n[u+iv]}dt$.

$$|G_2(\alpha,n)| \le \frac{1}{\pi}\int_{-\pi}^{0} e^{n[\sqrt{1-\alpha^2}\cos t - y_0]}dt \;\le \frac{e^{-ny_0}}{\pi}\int_{0}^{\pi} e^{n\sqrt{1-\alpha^2}}\,dt = e^{-nd(\beta)}.$$

Similarly, one can show that $\qquad |K_2(\alpha,n)| \le \beta e^{-nd(\beta)}$.

Lemma 3.3:

$$|G_3(\alpha,n)| \le \frac{e^{nd(\beta)-b\,n^{\frac{1}{5}}}}{\pi}[1+\ln(\beta)][1+O(n^{-1/5})] + \pi^{-1}[1+\ln(\beta)] \qquad \textit{and}$$

$$|K_3(\alpha,n)| \le \frac{e^{nd(\beta)-b\,n^{\frac{1}{5}}}\sqrt{\beta^2-1}}{\pi}[1+O(n^{-1/5})] + \pi^{-1}\sqrt{\beta^2-1}. \tag{3.5}$$

where $b = \frac{1}{2}\sqrt{1-\alpha^2}$. $G_3(\alpha,n)$ *and* $K_3(\alpha,n)$ *are purely imaginary. If* $\beta \ge 1+\varepsilon$, *then we can replace* b *by* b_ε *, where* $b_\varepsilon = \dfrac{\sqrt{2\varepsilon+\varepsilon^2}}{2(1+\varepsilon)}$.

Proof:

$$G_3(\alpha,n) = \frac{-i}{\pi}\int_{-y_0+\varepsilon_n}^{y_0} e^{-ng(t)}dt \qquad \text{where } g(t) = -\alpha\sinh t + t.$$

We divide $G_3(\alpha,n)$ into a sum of two integrals $I_3(\alpha,n) + II_3(\alpha,n)$

$$I_3(\alpha,n) = \frac{-i}{\pi}\int_{0}^{y_0} e^{-ng(t)}dt,$$

$$II_3(\alpha,n) = \frac{-i}{\pi}\int_{-y_0+\varepsilon_n}^{0} e^{-ng(t)}dt.$$

For $0 \le t \le y_0$, g(t) is an increasing function of t, with a minimum value at t =0, therefore

$|I_3(\alpha,n)| \le \frac{1}{\pi}\int_0^{y_0} dt = y_0\pi^{-1}$. We can bound $y_0 = \cosh^{-1}(\beta)$ by ln(β) +1 ,because for x>1,

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) = \ln[x(1 + \sqrt{1 - x^{-2}})] = \ln x + \ln(1 + \sqrt{1 - x^{-2}}) \le \ln x + \ln 2.$$

Since we are assuming that at least one of the conditions i) or ii) of Lemma 3.0 are satisfied, we can conclude that $\beta > 1$ and use the above inequality for $\cosh^{-1}(\beta) = y_0$.

Thus. $|I_3(\alpha,n)| \le [\ln(\beta) + 1]\pi^{-1}$.

The second integral we estimate the same way that we will estimate $G_5(n)$ in Lemma 3.4.

$$|II_3(\alpha,n)| = \frac{1}{\pi}\int_0^{y_0-\varepsilon_n} e^{ng(t)}dt.$$

Since $g''(t) < 0$ for t > 0, g(t) is concave down. Thus we can estimate $g(t)$ by the tangent line of $g(t)$ at $t = y_0 - \varepsilon_n$ which we denote by $f_1(t)$.

$g(t) \le f_1(t)$ for t >0.

$f_1(t) = m_1(t - y_0 + \varepsilon_n) + g(y_0 - \varepsilon_n)$, where $m_1 = g'(y_0 - \varepsilon_n)$.

$g(y_0 - \varepsilon_n) = g(y_0) + \frac{\varepsilon_n^2}{2}g''(y_0) - \frac{\varepsilon_n^3}{6}g'''(\xi_1)$ where $y_0 - \varepsilon_n < \xi_1 < y_0$.

$g(y_0) = d(\beta)$,

$g''(y_0) = -\sqrt{1-\alpha^2}$,

$|g'''(\xi_1)| = \alpha \cosh(\xi_1) \le \alpha \cosh(y_0) = \alpha\beta = 1$,

$g(y_0 - \varepsilon_n) = d(\beta) - \frac{n^{-4/5}}{2}\sqrt{1-\alpha^2} + O(n^{-6/5})$. Because of the above inequality, the constant implied in the remainder term is at most 1.

Let $b = \frac{1}{2}\sqrt{1-\alpha^2}$.

$$|II_3(\alpha,n)| \le \frac{e^{nd(\beta)-n^{\frac{1}{5}}b}}{\pi} \int_0^{y_0-\varepsilon_n} e^{nm_1(t-(y_0-\varepsilon_n))} dt[1+O(n^{-1/5})].$$ Let $-s = t-(y_0-\varepsilon_n)$

$$|II_3(\alpha,n)| \le \frac{e^{nd(\beta)-n^{\frac{1}{5}}b}}{\pi} \int_0^{y_0-\varepsilon_n} e^{-nm_1 s} ds[1+O(n^{-1/5})]$$

$m_1 > 0$, because $g''(t) < 0$ for t > 0 and $g'(y_0) = 0$.

$$|II_3(\alpha,n)| \le \frac{e^{nd(\beta)-n^{1/5}b}}{\pi} \int_0^{y_0-\varepsilon_n} 1 ds[1+O(n^{-1/5})].$$

If $\beta \ge 1+\varepsilon$, $\alpha \le (1+\varepsilon)^{-1}$ and $b \ge b_\varepsilon$, where $b_\varepsilon = \frac{\sqrt{2\varepsilon+\varepsilon^2}}{2(1+\varepsilon)}$. Thus,

in the next chapter where $\beta \ge 1+\varepsilon$ we have uniform estimates by replacing b by b_ε.

$$|II_3(\alpha,n)| \le \frac{e^{nd(\beta)-b\,n^{1/5}}[\ln(\beta)+1][1+O(n^{-1/5})]}{\pi}$$

Similarly, we will show that

$$|K_3(\alpha,n)| \le \frac{e^{nd(\beta)-b\,n^{1/5}}\sqrt{\beta^2-1}\,[1+O(n^{-1/5})]+\sqrt{\beta^2-1}}{\pi}.$$

To evaluate $R_3(\alpha,n)$ and $L_3(\alpha,n)$ we also write them as sums.

$R_3(\alpha,n) = I_{3R}(\alpha,n) + II_{3R}(\alpha,n)$
$L_3(\alpha,n) = I_{3L}(\alpha,n) + II_{3L}(\alpha,n)$

where

$$I_{3R}(\alpha,n)=\frac{i}{2\pi}\int_{0}^{y_0}e^{-ng(t)+t}dt,$$

$$II_{3R}(\alpha,n)=\frac{i}{2\pi}\int_{-y_0+\varepsilon_n}^{0}e^{-ng(t)+t}dt,$$

$$I_{3L}(\alpha,n)=\frac{i}{2\pi}\int_{0}^{y_0}e^{-ng(t)-t}dt,$$

$$II_{3L}(\alpha,n)=\frac{i}{2\pi}\int_{-y_0+\varepsilon_n}^{0}e^{-ng(t)-t}dt.$$

$$|I_{3R}(\alpha,n)|\le\frac{1}{2\pi}\int_{0}^{y_0}e^{t}dt \qquad \text{and} \qquad |I_{3L}(\alpha,n)|\le\frac{1}{2\pi}\int_{0}^{y_0}e^{-t}dt$$

$$|I_{3R}(\alpha,n)|+|I_{3L}(\alpha,n)|\le\frac{\sinh y_0}{\pi}=\pi^{-1}\sqrt{\beta^2-1}.$$

The estimates for $|II_{3R}(\alpha.n)|$ and $|II_{3L}(\alpha,n)|$ are similar to the estimates for $|II_3(\alpha,n)|$.

$$|K_3(\alpha,n)|\le|I_{3R}(\alpha,n)|+|I_{3R}(\alpha,n)|+|II_{3R}(\alpha,n)|+|II_{3L}(\alpha,n)|$$

Thus,

$$|K_3(\alpha,n)|\le\frac{e^{nd(\beta)-b\,n^{1/5}}\sqrt{\beta^2-1}\,[1+O(n^{-1/5})]+\sqrt{\beta^2-1}}{\pi}$$

Thus, $K_3(\alpha,n)$ is also purely imaginary that is bounded as in the above estimates.□

Lemma 3.4:

$$|G_5(\alpha,n)|\le\frac{e^{nd(\beta)-b\,n^{1/5}}\,2}{\pi n^{1/5}}[1+O(n^{-1/5})], \qquad \text{and}$$

$$|K_5(\alpha,n)|\le\frac{e^{nd(\beta)-b\,n^{1/5}}\,2}{\pi(n^{1/5}-1)}[1+O(n^{-1/5})]. \tag{3.6}$$

$G_5(\alpha,n)$ and $K_5(\alpha,n)$ are purely imaginary, and $b=\frac{1}{2}\sqrt{1-\alpha^2}$. If $\beta\ge1+\varepsilon$, then we can replace b by b_ε , where $b_\varepsilon=\frac{\sqrt{2\varepsilon+\varepsilon^2}}{2(1+\varepsilon)}$.

Proof:

$$G_5(\alpha,n)=\frac{-i}{\pi}\int_{-\infty}^{-y_0-\varepsilon_n}e^{-ng(t)}dt \quad =\frac{-i}{\pi}\int_{y_0+\varepsilon}^{\infty}e^{ng(t)}dt.$$

Since $g''(t)<0$, g(t) is concave down. Thus, we can estimate $g(t)$ by the tangent line of $g(t)$ at $t=y_0+\varepsilon_n$ which we denote by $f(t)$. The slope of the line is $m_0=g'(y_0+\varepsilon_n)$.

$g(t)\leq f(t)$ for $t>0$.

$f(t)=m_0(t-y_0-\varepsilon_n)+g(y_0+\varepsilon_n)$.

$$g(y_0+\varepsilon_n)=g(y_0)+\frac{\varepsilon_n^{2}}{2}g''(y_0)+\frac{\varepsilon_n^{3}}{6}g'''(\xi) \quad \text{where} \quad y_0<\xi<y_0+\varepsilon_n.$$

$$|g'''(\xi)|=\alpha\cosh\xi\leq\alpha\cosh(y_0+\varepsilon_n)\leq\frac{\alpha}{2}[e^{y_0+\varepsilon_n}+1]\leq\frac{\alpha}{2}[e^{\ln\beta+\varepsilon_n+1}+1]\leq\frac{\alpha}{2}[e^2\beta+1]\leq e^2+1$$

Thus, we have shown that the implied constant in the remainder term is uniformly bounded.

$g(y_0)=d(\beta)$

$g''(y_0)=-\sqrt{1-\alpha^2}$ and $\varepsilon_n^{2}=n^{-4/5}$

$$g(y_0+\varepsilon_n)=d(\beta)-\frac{n^{-4/5}}{2}\sqrt{1-\alpha^2}+O(n^{-6/5}).$$

Thus, $$|G_5(\alpha,n)|\leq\frac{e^{nd(\beta)-n^{1/5}b}}{\pi}\int_{y_0+\varepsilon_n}^{\infty}e^{nm_0(t-(y_0+\varepsilon_n))}dt[1+O(n^{-1/5})]$$

where $b=\frac{1}{2}\sqrt{1-\alpha^2}$.

To complete the proof of our Lemma, we will show that $m_0\leq\frac{-n^{-4/5}}{2}$.

Expand $g'(t)$ into a Taylor's series about the point y_0

$g^{2i}(y_0)=-\sqrt{1-\alpha^2} \qquad g^{2i+1}(y_0)=-1 \qquad$ for $i=1,2,3\ldots$

$$m_0=g'(y_0+\varepsilon_n)=-\sqrt{1-\alpha^2}\sum_{i=0}^{\infty}\frac{\varepsilon_n^{2i+1}}{(2i+1)!} \qquad -\sum_{i=1}^{\infty}\frac{\varepsilon_n^{2i}}{(2i)!}$$

$$m_0 = -2b\ \sinh(\varepsilon_n) - [\cosh\varepsilon_n - 1] \le -[\cosh(\varepsilon_n) - 1] \le -\frac{\varepsilon_n^{\,2}}{2} = \frac{-n^{\frac{-4}{5}}}{2}.$$

$$|G_5(\alpha,n)| \le \frac{e^{nd(\beta)-n^{1/5}b}}{\pi}\int_0^{\infty} e^{-(1/2)n^{\frac{1}{5}}t}\,dt[1 + O(n^{-1/5})].$$

If $\beta \ge 1+\varepsilon$, $\alpha \le (1+\varepsilon)^{-1}$ and $b \ge b_\varepsilon$, where $b_\varepsilon = \dfrac{\sqrt{2\varepsilon + \varepsilon^2}}{2(1+\varepsilon)}$. Thus, in the next chapter where $\beta \ge 1+\varepsilon$ we have uniform estimates by replacing b by b_ε.

$$|G_5(\alpha,n)| \le \frac{e^{nd(\beta)-b\,n^{1/5}}\,2[1 + O(n^{-1/5})]}{\pi n^{\frac{1}{5}}} \le \frac{e^{nd(\beta)-b_\varepsilon n^{1/5}}\,2[1 + O(n^{-1/5})]}{\pi n^{\frac{1}{5}}}.$$

Similarly we could show that

$$|K_5(\alpha,n)| \le \frac{e^{nd(\beta)-b\,n^{1/5}}\,2[1 + O(n^{-1/5})]}{\pi(n^{\frac{1}{5}} - 1)} \le \frac{e^{nd(\beta)-b_\varepsilon n^{1/5}}\,2[1 + O(n^{-1/5})]}{\pi(n^{\frac{1}{5}} - 1)}.$$

Lemma 3.5:

$$G_4(\alpha,n) = \frac{-ie^{nd(\beta)}\sqrt{2}[1 + O(n^{-1/5})]}{\sqrt{n\pi}(1-\alpha^2)^{\frac{1}{4}}} \tag{3.7}$$

$$K_4(\alpha,n) = \frac{-ie^{nd(\beta)}\beta\sqrt{2}(1-\alpha^2)^{\frac{1}{4}}[1 + O(n^{-1/5})][1 + O(n^{-1}b^{-1})]}{\sqrt{n\pi}} \tag{3.8}$$

The constants in the big 'O' notation are real and independent of n and k. $b = \frac{1}{2}\sqrt{1-\alpha^2}$. $G_4(\alpha,n)$ *and* $K_4(\alpha,n)$ *are purely imaginary. If* $\beta \ge 1+\varepsilon$, *then we have uniform estimates by replacing* b *by* b_ε *,where* $b_\varepsilon = \dfrac{\sqrt{2\varepsilon + \varepsilon^2}}{2(1+\varepsilon)}$.

Proof:

The main contribution of the $H_n(z)$ and $H_n'(z)$ will come from $z = it$

$-y_0 - \varepsilon_n \le t \le -y_0 + \varepsilon_n$, with the orientation of the curve is that it starts at $-y_0 + \varepsilon_n$ and ends at $-y_0 - \varepsilon_n$.

We expand $g(t)$ into a Taylor's expansion about the point $-y_0$. Recall,

$g(t) = -\alpha \sinh t + t$.

$$G_4(\alpha, n) = \frac{-i}{\pi} \int_{-y_0-\varepsilon_n}^{-y_0+\varepsilon_n} e^{-ng(t)} dt$$

$$g(t) = g(-y_0) + \frac{g''(-y_0)}{2}(t + y_0)^2 + \frac{g'''(\xi_2)}{6}(\varepsilon_n{}^3),$$

where $-y_0 - \varepsilon_n < \xi_2 < -y_0 + \varepsilon_n$.

We already showed in the proof of Lemma 3.4 that $|g'''(\xi_2)| \le e^2 + 1$. Thus, the implied constant in the remainder term is uniformly bounded.

$g(y_0) = d(\beta)$,

$$g(t) = g(-y_0) + \frac{\sqrt{1-\alpha^2}}{2}(t + y_0)^2 + O(n^{\frac{-6}{5}})$$

$$g(-y_0) = -d(\beta) = -[\cosh^{-1}(\beta) - \sqrt{1-\beta^{-2}}].$$

$$G_4(n) = \frac{-i}{\pi} \int_{-y_0-\varepsilon_n}^{-y_0+\varepsilon_n} e^{nd(\beta) - nb(t+y_0)^2} dt[1 + O(n^{-1/5})] \qquad \text{where } b = \tfrac{1}{2}\sqrt{1-\alpha^2}.$$

Perform a change of variables $s = t + y_0$

$$G_4(\alpha, n) = \frac{-ie^{nd(\beta)}}{\pi} \int_{-\varepsilon_n}^{\varepsilon_n} e^{-nbs^2} ds[1 + O(n^{-1/5})].$$

$$G_4(\alpha, n) = \frac{-ie^{nd(\beta)}}{\pi\sqrt{nb}} \int_{-\vartheta_n}^{\vartheta_n} e^{-u^2} du[1 + O(n^{-1/5})].$$

where $\vartheta_n = n^{1/10} b^{1/2}$.

Since $n \to \infty$ we can assume that $\vartheta_n \geq 1$. If $\beta \geq 1 + \varepsilon$, then $\vartheta_n \geq 1$ uniformly of k because $b \geq b_\varepsilon$.

Since $\int_{\vartheta_n}^{\infty} e^{-s^2} ds \leq \int_{\vartheta_n}^{\infty} e^{-s} ds \leq O(e^{-n^{\frac{1}{10}} b^{1/2}})$ and $\int_{-\infty}^{-\vartheta_n} e^{-s^2} ds \leq O(e^{-n^{\frac{1}{10}} b^{1/2}})$, we may replace the integral for $G_4(\alpha, n)$ with

$$G_4(\alpha, n) = \frac{-ie^{nd(\beta)}}{\pi\sqrt{nb}} \int_{-\infty}^{\infty} e^{-s^2} ds [1 + O(n^{-1/5})][1 + O(e^{-n^{\frac{1}{10}} b^{1/2}})]$$

$$G_4(\alpha, n) = \frac{-ie^{nd(\beta)}}{\sqrt{nb\pi}} [1 + O(n^{-1/5})][1 + O(e^{-n^{\frac{1}{10}} b^{1/2}})].$$

Thus,

$$G_4(\alpha, n) = \frac{-ie^{nd(\beta)}\sqrt{2}}{\sqrt[4]{1-\alpha^2}} \frac{1}{\sqrt{n\pi}} [1 + O(n^{-1/5})].$$

We now compute $R_4(\alpha, n)$ and $L_4(\alpha, n)$.

$$R_4(\alpha, n) = \frac{i}{2\pi} \int_{-y_0-\varepsilon_n}^{-y_0+\varepsilon_n} e^{-ng(t)+t} dt$$

$$R_4(\alpha, n) = \frac{e^{nd(\beta)} i}{2\pi} \int_{-y_0-\varepsilon_n}^{-y_0+\varepsilon_n} e^{-nb(t+y_0)^2+t} dt [1 + O(n^{-1/5})]$$

Again perform a change of variables $s = t + y_0$

$$R_4(\alpha, n) = \frac{e^{nd(\beta) - y_0} i}{2\pi} \int_{-\varepsilon_n}^{\varepsilon_n} e^{-nbs^2+s} ds [1 + O(n^{-1/5})]$$

We now complete the square in s.

$$R_4(\alpha, n) = \frac{e^{nd(\beta) - y_0 + \frac{1}{4nb}} i}{2\pi} \int_{-\varepsilon_n}^{\varepsilon_n} e^{-nb(s + \frac{1}{2nb})^2} ds [1 + O(n^{-1/5})]$$

$$R_4(\alpha, n) = \frac{e^{nd(\beta) - y_0 + \frac{1}{4nb}} i}{2\pi\sqrt{nb}} \int_{-(\varepsilon_n + \frac{1}{2nb})\sqrt{nb}}^{(\varepsilon_n + \frac{1}{2nb})\sqrt{nb}} e^{-s^2} ds [1 + O(n^{-1/5})].$$

Again we use the estimates $\int_{\varepsilon_n\sqrt{nb}}^{\infty} e^{-s^2} ds \leq O(e^{-n^{\frac{1}{10}} b^{1/2}})$ and $\int_{-\infty}^{-(\varepsilon_n + \frac{1}{2nb})\sqrt{nb}} e^{-s^2} ds \leq O(e^{-n^{\frac{1}{10}} b^{1/2}})$.

Thus we replace our integral for $R_4(\alpha,n)$ with $R_4(\alpha,n)=\dfrac{ie^{nd(\beta)-y_0+\frac{1}{4nb}}}{\sqrt{2\pi n}\sqrt[4]{1-\alpha^2}}[1+O(n^{-1/5})]$.

We are going to replace $e^{\frac{1}{4nb}}$ by $1+O(n^{-1}b^{-1})$. The implied constant in the big 'O' notation is uniform in n and k, and if $\beta\geq 1+\varepsilon$, $b\geq b_\varepsilon$.

In a similar way we can now compute $L_4(\alpha,n)$.

$$L_4(\alpha,n)=\frac{-e^{nd(\beta)+y_0}i}{\sqrt{2\pi n}\sqrt[4]{1-\alpha^2}}[1+O(n^{-1/5})][1+O(n^{-1}b^{-1})]$$

$$K_4(\alpha,n)=(R_4(\alpha,n)+L_4(\alpha,n))=\frac{-ie^{nd(\beta)}\sqrt{2}\sinh(y_0)}{\sqrt{\pi n}\sqrt[4]{1-\alpha^2}}[1+O(n^{-1/5})][1+O(n^{-1}b^{-1})].$$

$\sinh y_0=\beta\sqrt{1-\alpha^2}$. Hence,

$$K_4(\alpha,n)=\frac{-ie^{nd(\beta)}\sqrt{2}}{\sqrt{\pi n}}\beta\sqrt[4]{1-\alpha^2}[1+O(n^{-1/5})][1+O(n^{-1}b^{-1})]. \qquad (3.9)$$

Note, that the big 'O' notation means that the implied constants are real and independent of n and k.

We can easily see that the previous estimates are all of a smaller order than the estimates for $G_4(\alpha,n)$ and $K_4(\alpha,n)$. Thus, we summarize the results of this chapter in the case that k is fixed and $n\to\infty$.

$$H_n(k)=\frac{-ie^{nd(\beta)}\sqrt{2}}{\sqrt{n\pi}\sqrt[4]{1-\frac{k^2}{n^2}}}[1+O(n^{-1/5})]+\sigma_{R_1}(e^{-nd(\beta)})+\sigma_{I_1}(\ln(n/k)). \qquad (3.10)$$

$\sigma_{R_1}(e^{-nd(\beta)})$ is real valued and is bounded by a constant times the function in the parenthesis. $\sigma_{I_1}(\ln(n/k))$ is purely imaginary and is bounded by a constant times the function in the parenthesis. The bound for σ_{R_1}, σ_{I_1} and the implied constant in $O(n^{-1/5})$ are real and independent of n and k.

$$H_n^{'}(k) = \frac{-ie^{nd(\beta)}\sqrt{2n}}{k\sqrt{\pi}}\sqrt[4]{1-\frac{k^2}{n^2}}[1+O(n^{-1/5})][1+O(n^{-1}b^{-1})]+\sigma_{R_2}(\frac{e^{-nnd(\beta)}n}{k})+\sigma_{I_2}(\sqrt{\frac{n^2}{k^2}-1}). \tag{3.11}$$

where we used the fact that k is bounded, $1 \le |k| \le B$.

$\sigma_{R_2}(e^{-nd(\beta)}\beta)$ is real valued and is bounded by a constant times the function in the parenthesis. $\sigma_{I_2}(\sqrt{\beta^2-1})$ is purely imaginary and is bounded by a constant times the function in the parenthesis. The bound for σ_{R_1}, σ_{I_2} and the implied constant in the big 'O' notation are real and independent of n and k.

$$Y_n(k) = -(\frac{n}{ke})^n \frac{\sqrt{2}}{\sqrt{n\pi}}[1+O(n^{-1/5})]+\sigma_{I_1}[\ln(n/k)] \tag{3.12}$$

$$J_n(k) = \sigma_{R_1}(\frac{ke}{n})^n. \tag{3.13}$$

where we used the approximation $\frac{\cosh^{-1}(\frac{n}{k})}{\ln(\frac{n}{k})} \cong 1$ as $n \to \infty$.

$$Y_n'(k) = -(\frac{n}{ke})^n \frac{\sqrt{2n}}{k\sqrt{\pi}}[1+O(n^{-1/5})]+\sigma_{I_2}\sqrt{\frac{n^2}{k^2}-1} \tag{3.14}$$

$$J_n'(k) = \sigma_{R_2}((\frac{ke}{n})^n \frac{n}{k}). \tag{3.15}$$

$Y_n'(k)$ and $J_n'(k)$ are the derivatives of the Bessel functions.

IV. Uniform estimates of $H_n'(k)$ as n and k approach infinity with $\frac{n}{k}$ fixed and bounded away from one.

Now that we know that the power can be represented by the converging series given by (2.10), we would like to estimate the limit of this series as k goes to infinity. In order to do this, we need to get uniform estimates of $H_n'(k)$ as both n and k tend to infinity. In this chapter, we will use the results and method from the previous chapter to obtain the estimates in the case that k tends to infinity faster than n, and also the case that n tends to infinity faster than k. To be more precise, we obtain estimates as $n \to \infty$ and $k \to \infty$ with the ratio $\beta = \frac{k}{n}$ fixed. We will deal with the case $\beta > 1 + \varepsilon$ and the case $\beta < 1 - \varepsilon$, where ε is a fixed positive number. In the next two chapters we will deal with the case $1 - \varepsilon \le \beta \le 1 + \varepsilon$.

First, we will deal with $\beta > 1 + \varepsilon$.

$H_n(k) = \frac{1}{\pi} \int_\Gamma e^{i[-k\sin z + nz]} dz$ where Γ is the same curve that we described in Chapter III.

$$H_n'(k) = -\frac{i}{\pi} \int_\Gamma \sin(z) e^{i[-k\sin z + nz]} dz$$

Let $\beta = n/k$

$$H_{\beta k}'(k) = -\frac{i}{\pi} \int_\Gamma \sin(z) e^{ki[-\sin z + \beta z]} dz .$$

We look for the critical values of $w(z, \beta)$, where

$w(z, \beta) = i[-\sin z + \beta z]$, $w_z(z, \beta) = i[-\cos z + \beta]$, and

$w_z = 0 \quad \text{if} \quad \cos z = \beta$.

Since $\beta > 1 + \varepsilon$, the only solutions of $\cos z = \beta$ are $z = \pm i y_0$, where $y_0 = \cosh^{-1}(\beta) > 0$. Thus we can use the contours $\gamma_j \;\; 1 \le j \le 5$ that we described in Chapter III with the y_0 as defined above.

Let $0 < \varepsilon < 1/2$ be a fixed constant, and suppose that $\beta k = n$ and $\beta \geq 1 + \varepsilon$, $k \to \infty$□. Because our estimates in Chapter III are uniform in k and n in the case $\beta \geq 1 + \varepsilon$, we can use our results from that chapter.

$$H_{\beta k}'(k) = \frac{-ie^{nd(\beta)}\sqrt{2}\sqrt[4]{\beta^2 - 1}}{\sqrt{\pi k \beta}}[1 + O((\beta k)^{-1/5})] + \sigma_{R_2}(e^{-nd(\beta)}\beta) + \sigma_{I_2}(\sqrt{\beta^2 - 1})$$

$\sigma_{R_2}(e^{-nd(\beta)}\beta)$ is real valued and is bounded by a constant times the function in the parenthesis. $\sigma_{I_2}(\sqrt{\beta^2 - 1})$ is purely imaginary and is bounded by a constant times the function in the parenthesis. The constants for σ_{R_2}, σ_{I_2} and the implied constant in $O(k^{-1/5})$ are real and independent of n and k.

Let

$$\delta_1(\beta) = \beta d(\beta) = [\beta \cosh^{-1}(\beta) - \sqrt{\beta^2 - 1}]. \tag{4.1}$$

Thus, for $\beta > 1 + \varepsilon$, the formula above can be written as,

$$H_{\beta k}'(k) = \frac{-ie^{k\delta_1(\beta)}\sqrt{2}\sqrt[4]{\beta^2 - 1}}{\sqrt{\pi k \beta}}[1 + O((\beta k)^{-1/5})] + \sigma_{R_2}(e^{-k\delta_1(\beta)}\beta) + \sigma_{I_2}(\sqrt{\beta^2 - 1}). \tag{4.2}$$

□□□ We now estimate $H_{\beta k}'(k)$ for $\beta < 1 - \varepsilon$. By the theory of the steepest descent, the main contribution of $H_{\beta k}'(k)$ will come from the part of the path near the saddle point(s) of $w(z, \beta) = i[-\sin z + \beta z]$ on the path of integration. Since $\beta < 1 - \varepsilon$, $w_z(z, \beta) = 0$ gives rise to the saddle points $x = \pm \cos^{-1}\beta$. Let $x_1 = -\cos^{-1}\beta$, where we are using the branch of $\cos^{-1}\beta$ such that $0 < \cos^{-1}(\beta) < \pi$. We write $H_{k\beta}'(k) = \frac{-i}{\pi}\int_{\Omega} \sin z e^{kw(z,\beta)}dz$ where Ω can be written as a union of the curves $\Omega = \bigcup_{j=1}^{5} \tau_j$. The curves τ_j will be described below.

Since $\beta \le 1-\varepsilon$, we can choose k large enough such that

$\varepsilon_k = k^{-2/5} \le \frac{2}{\sqrt{2}}\cos^{-1}(1-\varepsilon) \le \frac{2}{\sqrt{2}}\cos^{-1}(\beta)$, or $\frac{\sqrt{2}}{2}\varepsilon_k \le \cos^{-1}(\beta)$. This inequality is necessary in order to make sure that the curve τ_3 (defined later) will be within the half-plane $\mathrm{Re}(z) \le 0$.

Let $\tau_3 = x_1 - te^{(3/4)\pi i}$ for $-\varepsilon_k \le t \le \varepsilon_k$ with the orientation given by increasing t.

Let τ_2 be the horizontal line segment which starts at $z = -\pi + i\frac{\sqrt{2}}{2}\varepsilon_k$ and ends at $z = x_1 + \frac{\sqrt{2}}{2}\varepsilon_k(-1+i)$.

Let $\tau_1 = -\pi + it$ for $\frac{\sqrt{2}}{2}\varepsilon_k \le t \le \infty$, with the orientation of the curve is that it starts $z = -\pi + i\infty$ and ends at $z = -\pi + i\frac{\sqrt{2}}{2}\varepsilon_k$.

Let τ_4 be a horizontal line segment which starts at $z = x_1 + \frac{\sqrt{2}}{2}\varepsilon_k(1-i)$ and ends at $z = -i\frac{\sqrt{2}}{2}\varepsilon_k$.

Let $\tau_5 = it$, $-\infty \le t \le -\frac{\sqrt{2}}{2}\varepsilon_k$, with the orientation of decreasing t.

Let $T_j(k,\beta) = \frac{-i}{\pi}\int_{\tau_j}\sin(z)e^{kw(z,\beta)}dz$, for j=1,...,5.

Note that $T_1(k,\beta) = O(e^{-k^{3/5}})$, where the estimate is independent of β for $\beta < 1-\varepsilon$.

In fact, $|T_1(k,\beta| \le \frac{1}{\pi}\int_{\frac{\sqrt{2}}{2}\varepsilon_k}^{\infty}\sinh(t)e^{-k(\sinh t+\beta t)} = O(e^{-k^{3/5}})$

since $\sinh t \ge t$ for t > 0.

We will show that $T_5(k,\beta) = O(e^{-k^{3/5}\varepsilon})$.

$$|T_5(k,\beta)| \le \frac{1}{\pi} \int_{\frac{\sqrt{2\varepsilon_k}}{2}}^{\infty} \sinh(t) e^{-k(\sinh t-\beta)} dt \le \frac{1}{\pi} \int_{\frac{\sqrt{2\varepsilon_k}}{2}}^{\infty} \sinh(t) e^{-k\varepsilon} dt = O(e^{-k^{3/5}\varepsilon}) \text{ ,}$$

using the above inequality for $\sinh(t)$ and the fact that $\beta \le 1-\varepsilon$.

Let $\quad u_R(x,y,\beta) = \operatorname{Re} w(z,\beta) = \cos x \sinh y - \beta y$

Let $\quad \psi_a(k,\beta) = u_R(x_1 + \varepsilon_k e^{(3/4)\pi i})$ and $\psi_b(k,\beta) = u_R(x_1 - \varepsilon_k e^{(3/4)\pi i})$.

To estimate $T_j(k,\beta)$ for j=2.4. it suffices to estimate $w(z,\beta)$ at $z = x_1 \pm \varepsilon_k e^{(3/4)\pi i}$.

because in the upper half plane $\dfrac{\partial u_R(x,y,\beta)}{\partial x} > 0$ if $-\pi < x < 0$ and therefore, $u_R(x,y,\beta)$ is not greater than $\psi_a(k,\beta)$ on τ_2. Similarly, in the lower half plane $\dfrac{\partial u_R(x,y,\beta)}{\partial x} < 0$ if $-\pi < x < 0$, and $u_R(x,y,\beta)$ is not greater than $\psi_b(k,\beta)$ on τ_4.

We expand $w(z,\beta)$ into a Taylor's expansion about the point x_1:

$$w(x_1 \pm \varepsilon_k e^{(3/4)\pi i},\beta) = w(x_1,\beta) - \frac{i}{2} k^{-4/5} w''(x_1,\beta) + R_1(\beta) k^{-6/5} \text{ .}$$

$|R_1(\beta)| \le M_1$, where M_1 is a constant independent of k and β.

Let

$$\delta_2(\beta) = [\sqrt{1-\beta^2} - \beta \cos^{-1}(\beta)] \tag{4.3}$$

Note that $w(x_1,\beta) = i\delta_2(\beta)$ and $w''(x_1,\beta) = -i\sqrt{1-\beta^2}$.

Let $c = \frac{1}{2}\sqrt{1-\beta^2}$

$w(x_1 \pm \varepsilon_k e^{(3/4)\pi i}) = i\delta_2(\beta) - ck^{-4/5} + O(k^{-6/5})$

Since $\beta \le 1-\varepsilon$, $c \ge c_\varepsilon = \frac{1}{2}\sqrt{2\varepsilon - \varepsilon^2}$.

Thus,

$T_j(k,\beta) = O(e^{-c_\varepsilon k^{1/5}})$ for j=2.4.

The big 'O' notation means that the implied constant is independent of k and β.

We have used the fact that lengths of the curves τ_2 and τ_4 are bounded by π.

We now want to estimate $T_3(k,\beta)$.

We expand $w(z,\beta)$ into a Taylor's expansion around the point x_1.

$w(z,\beta) = i[\delta_2(\beta) - c(z-x_1)^2 + R(z,\beta)(z-x_1)^3]$.

Since $z = x_1 - te^{\frac{\pi 3}{4}}$ for $-\varepsilon_k \le t \le \varepsilon_k$, we have $|z - x_1| \le \varepsilon_k$.

Also $|R(z,\beta)| \le M_2$, where M_2 is a constant which is independent of β and k, because $|w'''(z,\beta)|$ is uniformly bounded for $-\varepsilon_k \le |z-x_1| \le \varepsilon_k$. Therefore,

$kw(z,\beta) = ki[\delta_2(\beta) - c(z-x_1)^2 + O(k^{-1/5})]$ and

$$T_3(k,\beta) = \frac{ie^{(3/4)\pi}}{\pi} \int_{-\varepsilon_k}^{\varepsilon_k} \sin(z(t)) e^{ik[\delta_2(\beta) - c(z(t)-x_1)^2]} dt[1 + O(k^{-1/5})].$$

Use the identity $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ to write $T_3(k,\beta) = \dfrac{e^{(3/4)\pi + ik\delta_2(\beta)}}{2\pi}[Q_1 e^{ix_1} - e^{-ix_1} Q_2]$ where

$$Q_1 = \int_{-\varepsilon_k}^{\varepsilon_k} e^{[-ite^{\frac{1}{4}\pi} - ckt^2]} dt[1 + O(k^{-1/5})].$$

$$Q_2 = \int_{-\varepsilon_k}^{\varepsilon_k} e^{[ite^{\frac{1}{4}\pi} - ckt^2]} dt[1 + O(k^{-1/5})].$$

Let $m_0 = e^{\pi/4}$

Then the exponent appearing in the expression for Q_1 can be written as

$$m_0 t - kct^2 = -kc[t^2 - \frac{m_0}{kc}t].$$

Completing the square in t we get,

$$m_0 t - kct^2 = -kc[(t - \frac{m_0}{2kc})^2 - \frac{m_0^2}{4k^2c^2}] = -kc(t - \frac{m_0}{2kc})^2 + \frac{m_0^2}{4kc},$$

$m_0^2 = i$,

$$Q_1 = e^{\frac{i}{4kc}} \int_{-\varepsilon_k}^{\varepsilon_k} e^{-kc(t - (\frac{m_0}{2kc}))^2} dt[1 + O(k^{-1/5})].$$

We again add to this integral the two tails that could be estimated by $e^{-\sqrt{k}}$.

$$Q_1 = e^{\frac{i}{4kc}} \int_{-\infty}^{\infty} e^{-kc(t - (\frac{m_0}{2kc}))^2} dt[1 + O(k^{-1/5})].$$

By using the standard methods of a contour integration and the fact that the integrand is analytic and decays exponentially as $|t| \to \infty$ for t real. we can replace the above integral with a real valued integral.

Thus,
$$\int_{-\infty}^{\infty} e^{-kc(t-(\frac{m_0}{2kc}))^2} dt = \int_{-\infty}^{\infty} e^{-kc(t-(\frac{m_1}{2kc}))^2} dt$$

$$= \int_{-\infty}^{\infty} e^{-kct^2} dt$$

where $m_1 = \mathrm{Re}(m_0) = \frac{\sqrt{2}}{2}$. We have the estimate $e^{\frac{i}{4kc}} = 1 + O(\frac{1}{4kc})$.

Thus, $Q_1 = \int_{-\infty}^{\infty} e^{-kct^2} dt[1 + O(k^{-1/5})]$, where we included $O(\frac{1}{4kc})$ into the $O(k^{-1/5})$ notation.

Since $c \geq c_\varepsilon = \frac{1}{2}\sqrt{2\varepsilon - \varepsilon^2}$, the big 'O' estimate is real and independent of β and k.

Thus,

$$Q_1 = \sqrt{\frac{\pi}{kc}}[1 + O(k^{-1/5})].$$

Similarly,

$$Q_2 = e^{\frac{i}{4ck}} \int_{-\varepsilon_k}^{\varepsilon_k} e^{-ck(t+\frac{m_0}{2ck})^2} dt[1 + O(k^{-1/5})] = \frac{\sqrt{\pi}}{\sqrt{ck}}[1 + O(k^{-1/5})].$$

$$T_3(k,\beta) = \frac{ie^{(3/4)\pi i + ki\delta_2(\beta)}\sqrt{2}}{\sqrt{\pi k}\sqrt[4]{1-\beta^2}} \sin(x_1)[1 + O(k^{-1/5})]$$

$$\square \qquad T_3(k,\beta) = \frac{-ie^{(3/4)\pi i + ki\delta_2(\beta)}\sqrt{2}}{\sqrt{\pi k}} \sqrt[4]{1-\beta^2}[1 + O(k^{-1/5})].$$

Thus, we conclude that for $0 < \beta < 1-\varepsilon$, $k \to \infty$ and $n = \beta k$, with β fixed

$$H_{\beta k}'(k) = \frac{e^{\frac{1}{4}\pi i + ki\delta_2(\beta)}\sqrt{2}}{\sqrt{\pi k}} \sqrt[4]{1-\beta^2}[1 + O(k^{-1/5})] + \sigma\ (e^{-c_\varepsilon k^{1/5}}) \ . \qquad (4.4)$$

$c_\varepsilon = \frac{1}{2}\sqrt{2\varepsilon - \varepsilon^2}$. $\sigma\,(e^{-c_\varepsilon k^{1/5}})$ is complex valued and is bounded by a constant times the function in the parenthesis. The constant for σ and the implied constant in $O(k^{-1/5})$ are real and independent of β and k.

V. The conformal map

Let $w(z,\beta) = i(-\sin z + \beta z)$. For each $\beta \neq 1$, $w(z,\beta)$ has a saddle point of order 2. A saddle point is a value of z satisfying $w_z(z,\beta) = 0$. For $\beta = 1$, $w(z,\beta)$ has a saddle point of order 3. We cannot use the standard methods of Chapter IV in order to obtain uniform estimates of $H'_{k\beta}(k)$ as $k \longrightarrow \infty$, and for $|\beta - 1| < \epsilon$ because of the change of order of the saddle point. So we follow the approach of Bleistein and Handelsman [Bleistein, pp. 367-379], and of Chester, Friedman and Ursell [Chester, pp. 599-611], where we construct a family of maps indexed by β, defined implicitly near the origin, such that the image of the $w(z,\beta)$ is a polynomial given by (5.1).

$$\phi(t,\beta) = \gamma^2(\beta)t - \frac{t^3}{3}. \tag{5.1}$$

$\gamma^2(\beta)$ will be holomorphic and chosen in such a way that the map defined implicitly by

$$w(z(t,\beta),\beta) = \phi(t,\beta). \tag{5.2}$$

will be a conformal map; namely, $z(t,\beta)$ will be holomorphic for $|\beta - 1| < \epsilon$ and the critical points of $w(z,\beta)$ will correspond to the critical points of $\varphi(t,\beta)$ for $\beta \in B_0 = \{x : (\text{Im } x \geq 0) \text{ and } (|x-1| < \epsilon)\}$.

In fact, we will show that there exist three such maps corresponding to the three cube roots of unity. In the next chapter, we estimate the integral using the new transformation. Let Γ_β be curve of integration for the integral representation of $H'_{k\beta}(k)$ near the origin.

Let

$$F(k,\beta) = \frac{-i}{\pi} \int_{\Gamma_\beta} \sin z e^{kw(z,\beta)} dz. \tag{5.3}$$

We will need the following lemmas.

Lemma 5.1. *Let $F(z)$ and $G(t)$ be holomorphic functions defined in neighborhoods of z_0 and t_0 respectively. Suppose the $F(z_0) = G(t_0)$, and $F(z) - F(z_0)$ and $G(t) - G(t_0)$ have zeros of order n at z_0 and t_0. Then for some $\epsilon \geq 0$ and $\eta > 0$ there are exactly n holomorphic functions $z_j(t)$ mapping $|t - t_0| < \eta$ into $|z - z_0| < \epsilon$, $\epsilon > 0$, and satisfying $F(z_j(t)) = G(t)$ for $j = 1, 2, \ldots, n$. Moreover, if for some $|t_1 - t_0| \leq \eta$ and $|z_1 - z_0| < \epsilon$, $F(z_1) = G(t_1)$ then $z_1 = z_j(t_1)$ for* *some* $j = 1, 2, \ldots, n$.

The proof is based on the Weierstrass Preparation Theorem [Hille, pp. 265-268]. In fact, the theorem stated and proved on p. 268 is essentially the case $G(t) = t^n$ of the Lemma 5.1. The general result follows easily from this case. Note that $f(z)$ has a zero at z_0 of order n means that $\frac{d^j f(z_0)}{dz^j} = 0$ for $j = 1, 2, \ldots n-1$, but $\frac{d^n f(z_0)}{dz^n} \neq 0$.

Let

$$\delta(\beta) = i\{\sin(\cos^{-1}(\beta)) - \beta\cos^{-1}(\beta)\}. \tag{5.4}$$

Let $A_0 = \{\beta : \beta = 1 \text{ or } (\text{real } \beta \leq -1, \ \text{Im } \beta = 0)\}$. Let $A = \mathbf{C} - \mathbf{A_0}$. Then the cosine function maps the strip $\{|\text{Re}(z)| < \pi, \ z \neq 0\}$ as a 2-fold covering of A, so $\cos^{-1}(\beta)$ and δ are both well-defined on a 2-fold covering of the set A. In both cases, the two branches over the same point differ by the factor -1. It follows that the function $\delta(\beta)$ is defined and analytic on the same 2-fold covering of A. Therefore, $\delta^2(\beta)$ is single valued holomorphic function for $\beta \in A$. Since

$$\lim_{\beta \to 1} \cos^{-1}(\beta) = 0$$

$\cos^{-1}(\beta)$ is bounded near 1 as is $\delta^2(\beta)$. The Removable Singularity Theorem shows that $\delta^2(\beta)$ can be extended holomorphically to the entire complex plane. We will show now that $\delta^2(\beta)$ has a zero at $\beta = 1$ of order 3, and there exist three analytic roots of $\delta^2(\beta)$ for $|\beta - 1| < \epsilon$ for some ϵ. Hence, we will be able to prove the following lemma.

Lemma 5.2. *$\delta^2(\beta)$ is a holomorphic function of β. The three cube roots of $\delta^2(\beta)$ are holomorphic for $|\beta - 1| < \epsilon$.*

We expand $\cos^{-1}(\beta)$ into power series expansion in $\sqrt{\beta}$ using the integral representation of $\cos^{-1}(\beta)$ for $|\beta - 1| < \epsilon$. $\cos^{-1}(\beta) = \int_\beta^1 \frac{dt}{\sqrt{1-t^2}}$ with a branch cut along T_1 for the square root function, where $T_1 = \{x : x \text{ is real}, |x| \geq 1\}$ and with the branch of the square root function $\operatorname{Arg}(1 - t^2)^{1/2} = 0$ for t real. Hence, we must use the branch of $\cos^{-1}(\beta)$ such that $\cos^{-1}(1/2) = \pi/3$ with a branch cut on T_1.

Let $\alpha = 1 - \beta$ and let $s = 1 - t$

$$\cos^{-1}(1 - \alpha) = \int_0^\alpha \frac{ds}{\sqrt{2s}\sqrt{1 - \frac{s}{2}}},$$

where we have a branch cut along $S = \{s : \text{Img } s = 0, \text{ Re } \alpha < 0\}$ and the same branch of the square root function as above. For $|\alpha| < 1$ we can expand $(1 - \frac{s}{2})^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n-1)!!S^n}{2^{2n}n!}$ where

$$(1 - \tfrac{s}{2})^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n-1)!!S^n}{2^{2n}n!} \text{ where} \quad p!! = \begin{cases} & \text{if } p \leq 1 \\ p(p-2)(p-4)\ldots3(1) & \text{if } p \geq 3 \ \textit{and p is an odd integer} \end{cases}$$

$\cos^{-1}(1-\alpha) = \frac{1}{\sqrt{2}}\int_0^\alpha s^{-1/2}(1-s/2)^{-1/2}ds$

$$= \frac{1}{\sqrt{2}}\int_0^\alpha \sum_{n=0}^{\infty} \frac{(2n-1)!!s^{n-1/2}}{2^{2n}n!}ds$$

$$= \sqrt{2}\sum_{n=0}^{\infty} \frac{(2n-1)!!\alpha^{n+1/2}}{2^{2n}n!(2n+1)}$$

$$= \sqrt{2\alpha}\sum_{n=0}^{\infty} \frac{(2n-1)!!\alpha^{n}}{4^{n}n!(2n+1)}.$$

For $\beta \notin T_1$,

$$\sin(\cos^{-1}(\beta)) = \sqrt{1-\beta^2}$$

$$\sin(\cos^{-1}(1-\alpha)) = \sqrt{2\alpha}(1-\alpha/2)^{1/2} = \sqrt{2\alpha}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!\alpha^n}{4^n n!}.$$

$$\delta(1-\alpha) = i(\sin(\cos^{-1}(1-\alpha)) - (1-\alpha)\cos^{-1}(1-\alpha)).$$

$$\delta(1-\alpha) = i\sqrt{2\alpha}(\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!\alpha^n}{4^n n!} - (1-\alpha)\sum_{n=0}^{\infty}\frac{(2n-1)!!\alpha^n}{4^n(2n+1)n!}).$$

$$= i\sqrt{2\alpha}(\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!\alpha^n}{4^n n!} - \sum_{n=0}^{\infty}\frac{(2n-1)!!\alpha^n}{4^n(2n+1)n!} + \sum_{n=0}^{\infty}\frac{(2n-1)!!\alpha^{n+1}}{4^n(2n+1)n!}).$$

For the last sum let $m = n+1$ and the last sum becomes

$$4\sum_{m=1}^{\infty}\frac{(2m-3)!!\alpha^m}{(m-1)!4^m(2m-1)}$$

$$= 4\sum_{m=1}^{\infty}\frac{m(2m-1)!!\alpha^m}{(2m-1)^2 m!4^m}$$

$$= 4\sum_{n=1}^{\infty}\frac{n(2n-1)!!\alpha^n}{(2n-1)^2 n!4^n}.$$

For the first two sums the terms for $n = 0$ cancel. Thus,

$$\delta(1-\alpha) = i\sqrt{2\alpha}\sum_{n=1}^{\infty}\frac{(2n-1)!!\alpha^n}{n!4^n}((-1)^n - \frac{1}{2n+1} + \frac{4n}{(2n-1)^2}).$$

$$= i\sqrt{2}\alpha^{3/2}\frac{2}{3}\sum_{n=0}^{\infty}\frac{(2n+1)!!\alpha^n}{(n+1)!4^{n+1}}\frac{3}{2}((-1)^{n+1} - \frac{1}{2n+3} + \frac{4n+4}{(2n+1)^2}).$$

$$= i\sqrt{2}\frac{2}{3}\alpha^{3/2}q(\alpha)$$

where $q(\alpha)$ is a converging power series for $|\alpha| < 1$, $q(0) = 1$, and

$$q(\alpha) = \sum_{n=0}^{\infty}\frac{(2n+1)!!\alpha^n}{(n+1)!4^{n+1}}(\frac{3}{2})((-1)^{n+1} - \frac{1}{2n+3} + \frac{4n+4}{(2n+1)^2}).$$

Let $B_1 = \{x : x \text{ is real, } x \geq 1\}$ and $B_2 = \{x : x \text{ is real, } x \leq 1\}$. Let $p(\beta) = q(\alpha)$,

$$\delta(\beta) = \frac{2}{3}\sqrt{2}(1-\beta)^{3/2}p(\beta)i$$

with a branch cut along B_1 for the square root function with $\text{Arg}(1-\beta)^{1/2} = 0$ for β real.

We can use (5.4) to analytically continue $\delta(\beta)$ onto the set B_1 with a new branch for the square root function $\text{Arg}(1-\beta)^{1/2} = \pi$ for β real and a branch cut along B_2.

We are going to use (5.5) to compute the three roots of $\delta^2(\beta)$.

$$\delta^2(\beta) = -\frac{8}{9}(1-\beta)^3 p(\beta)^2. \tag{5.5}$$

We will now define the function $\gamma_j^2(\beta)$ for $|\beta - 1| < \epsilon$ such that the critical points of $w(z, \beta)$ will correspond to the critical points of $\phi(t, \beta)$ for $\beta \in B_0$ using (5.2) where $\gamma_j^2(\beta) = \gamma^2(\beta)$ in the notation of (5.1).

Define $\gamma_j^2(\beta)$ for $j = 0, 1, 2$ as follows

$$\gamma_j^2(\beta) \equiv e^{(\frac{\pi i}{3} + \frac{2\pi i j}{3})} 2^{1/3} (1 - \beta) p(\beta)^{2/3}. \tag{5.6}$$

with $p(\beta)^{1/3}$ real for β real.

We will need later on in the paper to know that $|p(\beta)|$ has a lower bound. There exists an ϵ such that $|q(\alpha)|$ has positive lower bound for $|\alpha| \leq \epsilon$ because, $q(\alpha)$ is continuous and $q(0) = 1$.

Thus, we have proved the lemma by constructing the three analytic cube roots of $\delta^2(\beta)$. We have defined a function $\gamma_j^2(\beta)$ that satisfies

$$(\gamma_j^2(\beta))^3 = (\frac{3}{2})^2 \delta^2(\beta)$$

where $p(\beta)^{1/3}$ is the positive real root for β real. For each $j = 0, 1, 2$, we will define a particular function $\gamma_j(\beta)$ such that $[\gamma_j(\beta)]^2 = \gamma_j^2(\beta)$ the functioned defined by (5.6) and satisfies the following equation for $\beta \in B_0$,

$$2/3(\gamma_j(\beta))^3 = (-1)^j \delta(\beta).$$

$\gamma_j(\beta)$ will be continuous for $\beta \in B_0$, while $\gamma_j^2(\beta)$ will be holomorphic for $|\beta - 1| < \epsilon$.

Let

$$\gamma_j(\beta) \equiv \begin{cases} 0 & \text{for } \beta = 1 \\ e^{(\frac{\pi i}{6}+\frac{\pi i j}{3})}2^{1/6}(1-\beta)^{1/2}p(\beta)^{1/3} & \text{for } \beta \notin B_1 \\ e^{(\frac{2\pi i}{3}+\frac{\pi i j}{3})}2^{1/6}(\beta-1)^{1/2}p(\beta)^{1/3} & \text{for } \beta \in B_1 \end{cases} \tag{5.7}$$

with a branch of the square root function with $\operatorname{Arg}(1-\beta)^{1/2} = 0$ for $\beta \leq 1$ and $\operatorname{Arg}(\beta - 1)^{1/2} = 0$ for $\beta \geq 1$. We defined $\gamma_j(\beta)$ in such a way, so that $\pm\cos^{-1}(\beta)$ correspond to $\pm\gamma_j(\beta)$ via the equation (5.2) $w(z) = \varphi(t)$. Let $z_- = -\cos^{-1}(\beta)$ and $z_+ = \cos^{-1}(\beta)$ for each branch of $cos^{-1}(\beta)$. If $j = 0, 2$ then for $\beta \in B_0$

$$w(z_-) = \varphi(\gamma_j(\beta)) \quad w(z_+) = \varphi(-\gamma_j(\beta)) \tag{5.8}$$

because $\varphi(\gamma_j(\beta)) = 2/3(\gamma_j(\beta))^3 = \delta(\beta)$ using equation (5.1) and

$$w(z_-, \beta) = i(\sin(\cos^{-1}(\beta)) - \beta\cos^{-1}(\beta)) = \delta(\beta)$$

using the definition (5.4).

$$\varphi(-\gamma_j(\beta)) = -\frac{2}{3}(\gamma_j(\beta))^3 = -\delta(\beta) = w(z_+, \beta)$$

If $j = 1$ then for $\beta \in B_0$

$$w(z_-) = \varphi(-\gamma_j(\beta)) \quad w(z_+) = \varphi(\gamma_j(\beta)) \tag{5.9}$$

because $\varphi(\gamma_j(\beta)) = -\delta(\beta) = w(z_+)$ and $\varphi(-\gamma_j(\beta)) = \delta(\beta) = w(z_-, \beta)$.

Thus the polynomial

$$\phi(t,\beta) = \gamma_j^2(\beta)t - \frac{t^3}{3}$$

is a holomorphic function of β for $|\beta - 1| < \epsilon$. $\phi(t,\beta)$ is also a holomorphic function of t for a fixed β, and it is a continuous function of both variables. By Osgood's Lemma, it is a holomorphic function of both variables (see, Gunning and Rossi, p.2).

Theorem 5.3 *Let $w(z,\beta)$ and $\phi(t,\beta)$ be as above. For each $j = 0,1,2$ and hence, for each choice of $\gamma_j^2(\beta)$, there exists a function $z_j(t,\beta)$ that is holomorphic in both variables for $|t| < \eta_1$, and $|\beta - 1| < \epsilon$. The $z_j(t,\beta)$ satisfies:*

i) $z_j(0,\beta) = 0$

ii) $w(z_j(t,\beta),\beta) = \phi(t,\beta)$

iii) $z_j(t,\beta)$ has a holomorphic inverse in the sence that there is a holomorphic function $T_j(z,\beta)$ that satisfies

$$T_j(z_j(t,\beta),\beta) \equiv t,$$

and

$$z_j(T_j(z,\beta),\beta) \equiv z \quad \text{for } |z| < \eta_2 \text{ and } |\beta| < \epsilon.$$

Proof: First suppose that $\beta_0 \neq 1$, so that $w_z(0,\beta_0) \neq 0$. The Implicit Function Theorem, (see, Gunning and Rossi, p.14), implies that there exists a function $z_j(t,\beta)$ defined for t near 0 and β near β_0 and satisfies condition i) of the Theoren 5.3.

We can continue $z_j(t,\beta)$ analytically along any path as long as we do not encounter critical points of $w(z,\beta)$. The critical points are solutions of $w_z(z,\beta) = 0$, $w_z(z,\beta) = i(-\cos z + \beta)$. For each branch of $\cos^{-1}(\beta)$, the critical points are $z_+ = \cos^{-1}(\beta)$ and $z_- = -\cos^{-1}(\beta)$. If

$$\lim_{t \to t_0} z_j(t,\beta) = z_+ \text{ or } z_-,$$

and t_0 is any given point, we will show that $t_0 = \pm\gamma_j(\beta)$. Let

$$h = \lim_{t \to t_0} z_{t,j}(t,\beta).$$

Along the path of analytic continuation we have

$$w(z_j(t,\beta),\beta) = \phi(t,\beta).$$

We differentiate both sides with respect to t.

$$w_z(z_j(t,\beta),\beta) z_{t,j}(t,\beta) = \gamma_j^2(\beta) - t^2. \tag{5.11}$$

Take the limit of (5.11) as $t \longrightarrow t_0$. Since $w_z(z_j(t),\beta)$ is a continuous function of z, we have

$$\lim_{z \longrightarrow z_\pm} w_z(z,\beta) = w_z(z_\pm,\beta) = 0.$$

Hence,

$$\gamma_j^2(\beta) - t_0^2 = w_z(z_k,\beta)h = 0$$

$$t_0^2 = \gamma_j^2(\beta).$$

Let

$$t_k(\beta) = (-1)^k \gamma_j(\beta) \tag{5.12}$$

for $k = 0, 1$.

Now, we will show that if $t \longrightarrow t_k$ then $z_j(t) \longrightarrow z_+(z_-)$, multiply (5.11) by dt and integrate from 0 to $\gamma_j(\beta)$. The right side becomes $\varphi(\gamma_j,\beta) = (-1)^j 2/3\gamma_j^3(\beta) = \delta(\beta)$. The left side becomes $w(z_0) - w(0) = w(z_0)$.

Since we know that the critical values $z_+(z_-)$ are roots, $z_0 = z_+$ or z_-. Since $w(z,\beta)$ is an odd function, the root must be unique. Picking $-\gamma_j(\beta)$ leads to the other critical point of w by the oddness of $z(t)$. The limit of $z_{t,j}(t,\beta)$ must exist and be nonzero, because both sides of (5.11) have to vanish to order exactly one. We know that $z(t)$ is an odd function of t, because $w(z,\beta) = \varphi(t,\beta)$, $\varphi(t,\beta)$ and $w(z,\beta)$ are odd functions of t and z respectively. $w(z(-t)) = \varphi(-t) = -\varphi(t) = -w(z) =$

$w(-z)$. By the uniqueness of the Implicit Function Theorem, $z(-t) = -z(t)$.

We already showed that continuation along any path is possible as long as it avoids $t = t_k$. We will now use Lemma 5.1 to show that we can even continue to t_k, for $\beta \neq 1$. Let $F(z) = w(z, \beta)$ with $\beta \neq 1$ fixed. Let $G(t) = \phi(t, \beta)$ with $\beta \neq 1$ fixed. From (5.8) and (5.9) we have

$$F(z_+) = G(t_{j+1})$$

$$F(z_-) = G(t_j)$$

for $j = 0, 1, 2$, and

$$w_z(z_+, \beta) = 0,$$

$$w_{zz}(z_+, \beta) = -i \sin(\cos^{-1}(\beta)) \neq 0$$

and similarly for z_-. Thus $F(z) - F(z_0)$ has a zero at z_+ and at z_- of order 2, with $z_0 = z_+$ or z_-.

$$G_t(t_j) = \gamma_j^2(\beta) - t_j^2 = 0,$$

$$G_{tt}(t_j) = -2t_j \neq 0.$$

Thus, $G(t) - G(t_0)$ has a zero at $t_0 = t_j$ or t_{j+1} of order 2. By Lemma 5.1, for each $j = 0, 1, 2$ there exist two holomorphic functions $g_i(t)$ for $i = 0, 1$, that satisfy $F(g_i(t)) \equiv G(t)$ for $|t - t_k| < \eta_{k,i}$. Each $g_i(t)$ maps the sets $|t - t_k| < \eta_{k,i}$ into the sets $|z - z_k| < \epsilon_{k,i}$ for $i = 0, 1$, and $z_k = z_+$ or z_-. By the last sentence of Lemma 5.1, only one $\phi_i(t)$ is a continuation of $z_j(t)$ near t_k for $k = 0, 1$ and $i = 0, 1$.

We have thus shown that for each β, we can extend $z_j(t, \beta)$ to t_k to be holomorphic in t for $k = 0, 1$. We will use the Riemann Removable Singularity Theorem in two variable to show that $z_j(t, \beta)$ is a holomorphic function of both variables for $\beta \neq 1$. (see, Gunning and Rossi, p.19).

We need the following definitions:

Definition 1. *Let D be a domain in $\mathbf{C}^2$. A subset $X \subset D$ is called thin if for every point $u \in D$ there are an open polydisc $\Delta(u, r) \subset D$, and a function f, which is holomorphic and not identically zero in $\Delta(u, r)$, such that f vanishes identically on $X \bigcap \Delta(u, r)$.*

Definition 2. *Let D be a domain in $\mathbf{C}^2$, and let X be a subset of* D. *A function defined on the set $D - X$ is said to be locally bounded in D if to every point $u \in D$, there is an open polydisc $\Delta(u, r) \subset D$ such that the function f is bounded on $\Delta(u, r) \bigcap (D - X)$.*

Theorem 5.4. *Let X be a thin subset of a domain D in $\mathbf{C}^2$, and let f be a holomorphic function on $D - X$, which is locally bounded on D. Then there exists a unique function $g(u)$ holomorphic on D and such that $g(u) = f(u)$ for $u \in D - X$. $g(u)$ is the analytic extension of $f(u)$.*

We want to apply the above theorem with $f(u) = z_j(t, \beta)$.

$$D = \{(t, \beta) : |t| < \eta \text{ and } 0 < |\beta - 1| < \epsilon\}$$

$$X_0 = \bigcup_{k=1,2} \{(t_k, \beta) : 0 < |\beta - 1| < \epsilon\}.$$

The set X_0 is clearly thin, and we know that $z_j(t, \beta)$ is a holomorphic function of both variables on $D - X_0$ by the use of the Implicit Function Theorem and Osgood's Lemma. Thus, it suffices to show that $z_j(t, \beta)$ is locally bounded in D.

We use the extension functions $g_m(t)$ for $m = 0$ or 1, which we constructed before using Lemma 5.1 with β fixed. Since $g_m(t)$ is a holomorphic function of t and satisfies (5.2), we can differentiate (5.2) with respect to t twice and we get,

$$z_{tt}(t, \beta) w_z(z, \beta) + w_{zz}(z, \beta) z_t^2(t, \beta) = -2t. \tag{5.13}$$

$g_m(t)$ satisfy this differential equation. Evaluate the above equation by taking the limit as $t \rightarrow t_k$, we get

$$\begin{aligned} |g_m'(\gamma_j(\beta))|^2 &= |\frac{2\gamma_j(\beta)}{\sin(\cos^{-1}(\beta))}| \\ &\leq \frac{2^{7/6}(1 - \beta)^{1/2} p(\beta)^{1/3}}{(1 - \beta)^{1/2}(1 + \beta)^{1/2}}. \end{aligned}$$

We showed earlier in this chapter that $p(\beta)$ is bounded. Thus, $|g_m'(t)|$ is bounded near $t = t_k$ uniformly of β for $|\beta - 1| < 1$. For $|t| < \eta_1$, $g_m(t) = \int_1^t g_m'(s) ds$. Thus

$|g_m(t)|$ is also uniformly bounded. By Theorem 5.4 we can extend $z_j(t,\beta)$ to be holomorphic of both variables for $\beta \neq 1$.

We will now use Lemma 5.1 to extend $z_j(t,\beta)$ for $\beta = 1$ and $t = 0$. Let $G(t) = -t^3/3$ and $F(z) = w(z,1)$. $G(t)$ has a zero of order 3 at $t = 0$. $F(z)$ also has a zero of order 3 at $z = 0$. By Lemma 5.1 there exist three holomorphic extensions of $z_j(t,\beta)$ for β fixed, $f_k(t)$ $k = 0,1,2$. By the last statement of Lemma 5.1, only one of the $f_k(t)$ will be the extension. We will later on describe a method how to define this extension. Differentiate (5.13) with respect to t. $f_k(t)$ satisfy this differential equation (5.14).

$$w_z(z,1)z_{ttt}(t,1) + 3w_{zz}(z,1)z_{tt}(t,1) + w_{zzz}(z,1)z_t^3(t,1) = -2 \tag{5.14}$$

take the limit as $t \longrightarrow 0$ of (5.14)

$$z_t^3(0,1) = \frac{2i}{\cos(z,(0,1))} = 2i \tag{5.15}$$

because $w_z(0,1) = w_{zz}(0,1) = 0$. Since $f_k(t)$ for $k = 0,1,2$ satisfy the above equation (5.15), we can conclude that $z_j(t,\beta)$ are bounded near $\beta = 1$, uniformally in β. Apply Theorem 5.4 with $X = \{(t,1) : |t| < \eta\}$. Thus, we can extend $z_j(t,\beta)$ to be holomorphic function of both variables for $|t| < \eta$ and $|\beta - 1| < \epsilon$.

To show the existence of the inverse map, it suffices to check that $z_{t_j}(0,\beta) \neq 0$. For $\beta \neq 1$, use equation (5.11) and set $t = 0$. By condition i) of Theorem 5.3,

$z_j(0,\beta) = 0$. For $\beta = 1$, use equation (5.13) to show $z_{t,j}(0,1) \neq 0$. By the Inverse Mapping Theorem for complex valued functions of several variable the inverse $T_j(z,\beta)$ exists locally. [Gunning and Rossi, p.17].

Once we know that the $z_j(t,\beta)$ can be extended, we describe a method how to determine the extension for each choice of $j = 0,1,2$. Differentiate (5.2) to show that $z_j(t,\beta)$ satisfy the following differential equation for $\beta \neq 1$ and $\beta \in B_0$.

$$u(0,\beta) = 0, \qquad u_t(t,\beta) = \begin{cases} \frac{\gamma_j^2(\beta)-t^2}{i(-\cos u(t)-\beta)} & \text{for } t \neq t_k \\ b_l(\beta) & \text{for } t = t_k \end{cases} \tag{5.16}$$

For $\beta \notin B_1$, we use the branch $-\pi < \text{Arg } z < \pi$ with a cut along T_1 for the $\cos^{-1}(\beta)$ function with $\cos^{-1}(1/2) = \pi/3$. For $\beta \in B_1$, we use the branch $0 < \text{Arg } z < 2\pi$ with a cut along B_2 with $\cos^{-1}(2) = i \cosh^{-1}(2)$ and $\cosh^{-1}(2) > 0$.

Let $b_l(\beta) = (-1)^l h(\beta)$, and

$h(\beta) = (\frac{(-1)^j 2\gamma_j(\beta))}{i \sin(\cos^{-1}(\beta))})^{1/2}$.

For $\beta \in B_0 - B_1$, we pick the branch of $h(\beta)$ such that for β real

$$\text{Arg } h(\beta) = \begin{cases} (4/6j - 1/6)\pi & \text{for } j = 0,1 \\ \pi/6 & \text{for } j = 2 \end{cases}$$

For $\beta \in B_1$, we pick the branch of $h(\beta)$ such that for β real

$$\text{Arg } h(\beta) = \begin{cases} (5/6)\pi & \text{for } j = 0 \\ \pi/2 & \text{for } j = 1 \\ \pi/6 & \text{for } j = 2 \end{cases}$$

We obtained these equations by evaluating (5.13) at $t = t_k$. The factor $(-1)^j$ is due to the matching of t_k with z_+ and z_-, which is described by (5.8) and (5.9). Evaluate (5.14) at $t = 0$, we get $z_{t,j}^3(0,1) = 2i$, and let $a_\alpha = z_{t,j}(0,1) = 2^{1/3}e^{\frac{\pi i}{6}+\frac{2\pi i\alpha}{3}}$ for $\alpha = 0, 1, 2$. Thus $z_{t,j}(t,1)$ satisfies the following differential equation

$$u(0,1) = 0 \quad u_t(t,1) = \begin{cases} \frac{-t^2}{i(-\cos u(t))-1)} & \mathit{for}\ t \neq 0 \\ a_\alpha & \mathit{for}\ t = 0 \end{cases} \tag{5.17}$$

For each given j, there is a unique choice of l and α such that

$$\lim_{\beta\to 1} b_l(\beta) = a_\alpha.$$

This choice determines how to extend $z_j(t,\beta)$ to $\beta = 1$. Pick $\beta < 1$

$$b_l(\beta) = (-1)^l\left(\frac{2e^{\pi(j-1/2)}\gamma_j(\beta)}{(1-\beta)^{1/2}(1+\beta)^{1/2}}\right)^{1/2}$$

use (5.7)

$$b_l(\beta) = (-1)^l\left(\frac{2^{7/6}e^{\pi i(4/3j-1/3)}p(\beta)^{1/3}}{(1+\beta)^{1/2}}\right)^{1/2}$$

$$\lim_{\beta\to 1} b_l(\beta) = 2^{1/3}e^{\pi i(1/6+2/3\alpha)}.$$

The solutions (j, l, α) are given by the following triple pairs $(0, 1, 1)$, $(1, 1, 2)$, $(2, 0, 0)$

For our purposes, it will suffice to set $j = 1$ and consider the integral

$$F(k, \beta) = \frac{-i}{\pi} \int_{D_\beta} \sin(z_1(t, \beta)) z_{t,1}(t, \beta) e^{k\varphi(t,\beta)} dt, \tag{5.18}$$

where D_β is the image of Γ_β under the transformation $T_1(z, \beta)$.

In chapter VI we will need to know the value of $z_t(z\pm, \beta)$.

$$z_t(z\pm, \beta) = \begin{cases} \frac{-\sqrt{2}\gamma_1^{1/2}(\beta)}{(\beta^2-1)^{1/4}} & for\ \beta > 1 \\ \frac{-\sqrt{2}\gamma_1^{1/2}(\beta)e^{\pi i/4}}{(1-\beta^2)^{1/4}} & for\ \beta < 1 \end{cases} \tag{5.19}$$

Note that we are using the following choice of $\gamma^{1/2}(\beta)$

$$\text{Arg}\ \gamma^{1/2}(\beta) = \begin{cases} \frac{\pi}{2} & for\ \beta > 1 \\ \frac{\pi}{4} & for\ \beta < 1 \end{cases}$$

VI. Uniform Estimates of $H'_{k\beta}(k)$

Recall the notation

$$w(z,\beta) = i[-\sin z + \beta z].$$

$H'_{k\beta}(k)$ can be expressed as a line integral

$$H'_{k\beta}(k) = \frac{-i}{\pi} \int_{\Gamma_z} \sin z e^{kw(z,\beta)} dz. \tag{6.1}$$

The curve Γ_z of integration, which lies in the strip $\{z : -\pi \leq \text{Re } z \leq 0\}$, is divided into 3 parts, which will be described more fully later:

1) Γ_β, a curve near a critical point of $w(z,\beta)$.

2) The two tails Γ_+, Γ_-.

3) The curve Γ_C that connects Γ_+ to Γ_β and the curves Γ_A, Γ_B, that connect Γ_- to Γ_β.

The curves Γ_β, depend on β but are all contained in a compact set $\sum_\epsilon$ which is independent of β. In fact $\sum_\epsilon$ is an ϵ neighborhood of 0 with $0 < \epsilon < 1/2$, where ϵ is determined by the conditions of Lemma 5.2 and Theorem 5.2 of chapter V.

Pick Γ_β to be a part of the steepest descent curve in the z-plane through the critical point z_- satisfying $|z| < \eta_2$, where η_2 was defined in Theorem 5.3. z_- will be defined below. Let

$$u_R(x, y, \beta) = \text{Re } w(z, \beta) = \cos x \sinh y - \beta y \tag{6.2}$$

and

$$u_I(x, y, \beta) = \text{Im } w(z, \beta) = -\sin x \cosh y + \beta x. \tag{6.3}$$

The curve of steepest descent satisfies

$$u_I(x, y) = u_I(x_-, y_-), \quad z_- = x_- + iy_- = \begin{cases} -\cos^{-1}(\beta) & \text{if } \beta \leq 1 \\ -i\cosh^{-1}(\beta) & \text{if } \beta \geq 1 \end{cases}$$

Here $\beta \geq 0$. $\cos^{-1}(\beta)$ and $\cosh^{-1}(\beta)$ are taken to be positive. One can show that as the point z moves along Γ_β away from the critical point z_- in the lower (upper) half plane y values approach $-\infty$ ($+\infty$). Thus, for all β such that $1 - \epsilon \leq \beta \leq 1 + \epsilon$, pick the points z_a, z_c, such that z_a and z_c are the first points of intersection of the steepest descent curve and the boundary of $\sum_\epsilon$ such that $y_a < 0$ and $y_c > 0$. Let Γ_A be the continuation of the steepest descent curve from z_a till the point $z = z_b = x_b + iy_b$, where z_b is the first point on the steepest descent curve such that $y_b \leq -3$. One can show that z_c and z_b depend continuously on β and that there exists a constant $m_{-1} < 0$ which is independent of β and k such that $u_R(z_c) \leq m_{-1}$ and $u_R(z_b) \leq m_{-1}$.

Let Γ_+ be the curve that starts at $z = -\pi + i\infty$ and goes down along the line $x = -\pi$ till the point $z = -\pi + iy_c$, and let Γ_- be the curve that starts at $z = iy_b$ and goes to $z = -i\infty$ along the line $x = 0$. Connect the points z_c, z_b to the end points of Γ_+, Γ_-, respectively by going on a horizontal line. Let Γ_C and Γ_B denote these horizontal lines respectively. By using the inequality sinh(t) $> t^3/6$ for $t > 0$ it is easy to see that the integral over the curves Γ_+ and Γ_- is $O(e^{-kb_2})$, where $b_2 > 0$ is a constant, which is independent of β and k. The estimates of the integral (6.1) restricted to the curves Γ_A, Γ_B, and Γ_C will be postponed till the end of the chapter.

Recall from Chapter V the following definition

$$\varphi(t,\beta) = \gamma_1^2(\beta)t - t^3/3. \tag{6.4}$$

If $\beta > 1$, then arg $\gamma_1(\beta) = \pi$. We showed in Chapter V the existence of the conformal map $z(t,\beta)$ that satisfies $w(z,\beta) = \phi(t,\beta)$. Let D_β denote the image of Γ_β under this map. It suffices to use the transformation corresponding to $j = 1$.

Let

$$G(t,\beta) = \sin z(t,\beta) z_t(t,\beta), \tag{6.5}$$

$$F(k,\beta) = \frac{-i}{\pi}\int_{D_\beta} G(t,\beta)e^{k\varphi(t,\beta)}dt, \tag{6.6}$$

We will estimate Re$(F(k,\beta))$ using the theory of steepest descent [Bleistein, pp.

262-265]. The main contribution of an integral of this type comes from the value of the integral near the saddle points. We first deal with the case $1 \leq \beta \leq 1+\epsilon$. We will approximate $G(t,\beta)$ by a constant term $P_0(k,\beta)$ (defined by (6.7)), and show that the remainder, $R_0(k,\beta)$ (defined by (6.8)), is of the smaller order of magnitude than the constant contribution as $k \longrightarrow \infty$.

Let $a_0 = G(-\gamma_1(\beta),\beta)$

$$P_0(k,\beta) = -\frac{a_0 i}{\pi}\int_{D_\beta} e^{k\varphi(t,\beta)}dt, \tag{6.7}$$

$$R_0(k,\beta) = \frac{-i}{\pi}\int_{D_\beta}(G(t,\beta)-a_0)e^{k\varphi(t,\beta)}dt, \tag{6.8}$$

$$\mathrm{Re}(F(k,\beta)) = \mathrm{Re}[P_0(k,\beta)+R_0(k,\beta)]. \tag{6.9}$$

For $1 < \beta \leq 1+\epsilon$, it will suffice to get an upper bound for $\mathrm{Re}F(k,\beta)$. Throughout this chapter let λ be any fixed constant such that, $0 < \lambda < 1/3$. Let $u_k = m_0^2 k^{-2/3+2\lambda}$, where m_0 is a constant that will be described in Lemma 6.3.

Lemma 6.1

$$|\mathrm{Re}[P_0(k,\beta)]| = \begin{cases} 0 & \text{if } \beta = 1 \\ O(k^{-2/3}) & \text{if } 1 < \beta \leq 1+\epsilon \\ \leq B_p(e^{(-2/3)k^\lambda}) & \text{if } 1+u_k \leq \beta \leq 1+\epsilon \end{cases} \tag{6.10}$$

B_p is a constant independent of β and k

Proof: For $\beta = 1$, $a_0 = 0$. Since the curves D_β are contained in a compact set, the integral is bounded. Thus, for $\beta = 1$, $P_0 = 0$.

Thus, it suffices to prove the lemma for $1 < \beta \leq 1 + \epsilon$. For such β perform the transformation $r = t/\gamma_1(\beta)$ in the integral (6.7). Thus,

$$P_0(k, \beta) = \frac{-a_0 i \gamma_1(\beta)}{\pi} \int_{D_{R_0}} e^{k|\gamma_1|^3 v(r)} dr, \tag{6.11}$$

where D_{R_0} is the image of D_β under the above transformation, and

$$v(r) = -(r - r^3/3). \tag{6.12}$$

We pick the curve of steepest descent using the following application of a well-known result [Bleistein, p.255].

Lemma 6.2 *1) The curves of steepest descent are given by* $\operatorname{Im} v(r) = \operatorname{Im} v(r_0)$ *where r_0 is the solution of $v_r(r) = 0$.*

2) If $v_{rr}(r_0) = ae^{i\alpha_0}$, $a > 0$, then the directions of the steepest descent curve are given by $\theta_1 = \frac{-\alpha_0+\pi}{2}$, $\theta_2 = \frac{-\alpha_0+3\pi}{2}$.

3) If $v_{rr}(r_0) = 0$ and $v_{rrr} = ae^{i\alpha_0}$ then the directions of the steepest descent curve are given by $\theta_j = -\frac{\alpha_0}{3} + (2j+1)\pi/3$ for $j = 0, 1, 2$.

The critical points of $v_r(r) = 0$ are $r_0 = \pm 1$.

$$v(-1) = \frac{2}{3} > 0, \quad \operatorname{Im} v(-1) = 0.$$

Thus, the curve of steepest descent satisfies the equation

$$\operatorname{Im} v(r) = 0. \tag{6.13}$$

Let $r = x+iy$, $\operatorname{Im} v(r) = -y(1-x^2+y^2/3) = 0$. $v_{rr}(r) = 2r$ and $v_{rr}(-1) = -2 < 0$.

Since the curves D_β are contained in a compact set, we can represent the end points of the curves by two functions of β: v_1, v_2. Let x_2 be the $x-$*coordinate* of v_2, which is the first point of the intersection of the hyperbola $(1-x^2+y^2/3=0)$ and the boundary of the image of $\sum_\epsilon$ (compact set defined in the beginning of this chapter) $1 \le -x_2 \le x_{max}$, where x_{max} is a constant independent of β and k, by the definition of x_2 and the fact that β is bounded. The corresponding points in the $r-$plane are $\frac{-v_1}{|\gamma_1|}, \frac{-v_2}{|\gamma_1|}$.

At $r = -1$, the angles of the tangent line of the steepest descent curve are $\theta_1 = 0$, and $\theta_2 = \pi$. Hence, near $r = -1$ we must use the branch of the curve to be $y = 0$ for $\frac{-v_1}{|\gamma_1|} \le x \le 1$. Let A_1 denote this curve.

At $r = 1$, the angles of the tangent line are $\theta_3 = \pi/2$, and $\theta_4 = 3/2\pi$ because $v_{rr}(1) = 2 > 0$. Thus, we must use the positive branch ($y \ge 0$) of the hyperbola $x^2 - \frac{y^2}{3} = 1$, for $1 \le x \le \frac{-x_2}{|\gamma_1|}$. Let A_2 denote this curve.

Let P_{0j} denote the integral P_0 where the curve of integration is the restriction of D_{R_0} to the curves A_j for $j = 1, 2$. We must now compute a_0 using (5.19)

$$z_{t,1}(-\gamma_1, \beta) = \frac{-\sqrt{2}\gamma_1^{1/2}}{(\beta^2 - 1)^{1/4}}. \tag{6.14}$$

Using equation (5.9) we see that $-\gamma_1$ corresponds to $z_- = -\cos^{-1}(\beta) = -i\cosh^{-1}(\beta)$, $\sin(z_-) = -i\sqrt{\beta^2 - 1}$. Thus,

$$a_0 = i(\beta^2 - 1)^{1/4}\sqrt{2}\gamma_1(\beta)^{1/2}.$$

$\arg\gamma_1(\beta) = \pi$ and therefore

$$-a_0(\beta)i\gamma_1(\beta) = -i|\gamma_1|^{3/2}(\beta^2 - 1)^{1/4}, \tag{6.15}$$

which is purely imaginary. On the curve A_1, the integral is real and the constant is purely imaginary. Therefore, P_{01} is purely imaginary. Since we are interested in estimates of the real part, it will suffice to estimate only the Re (P_{02}). Along the curve A_2, $f(x) = -\text{Re } v(r) = (8/3)x^3 - 2x$, for $1 \leq x \leq \frac{-x_2}{|\gamma_1|}$, $x_2 < 0$. Im $v(r) = 0 = 1 - x^2 + y^2/3$, so along A_2 we can express y as a function of x, $y = \sqrt{3}(x^2 - 1)^{1/2} > 0$. Thus, on A_2

$$dr = (1 + i\sqrt{3}x(x^2 - 1)^{-1/2})dx.$$

$$\text{Re } P_{02} = \text{Re } [\tfrac{-a_0 i\gamma_1}{\pi} \textstyle\int_1^{-x_2/|\gamma_1|} e^{-k|\gamma_1|^3 f(x)}[1 + i\sqrt{3}x(x^2 - 1)^{-1/2}]dx]$$

$$= \frac{|\gamma_1|^{3/2}(\beta^2 - 1)^{1/4}\sqrt{3}}{\pi} \int_1^{-x_2/|\gamma_1|} e^{-k|\gamma_1|^3 f(x)} x(x^2 - 1)^{-1/2} dx.$$

We divide the curve into two intervals: $A_I = [1,2]$ $A_{II} = [2, -x_2/|\gamma_1|]$, if $2|\gamma_1| \le -x_2$. Let P_I and P_{II} be the real parts of P_{02} corresponding to the intervals A_I and A_{II} respectively. For the curve A_I, we use the fact that $f(x)$ is an increasing function; hence $f(x) \ge 2/3$.

$$\begin{aligned} |P_I| &\le e^{-2/3k|\gamma_1|^3}|\gamma_1|^{3/2}\frac{(\beta^2-1)^{1/4}}{\pi}\sqrt{3}\int_1^2 x(x^2-1)^{-1/2}dx \\ &= \frac{3e^{-2/3k|\gamma_1|^3}|\gamma_1|^{3/2}(\beta^2-1)^{1/4}}{\pi}. \end{aligned} \tag{6.16}$$

To complete Lemma 6.1 and for future reference we need the next lemma.

Lemma 6.3 *There exist positive constants m_0 and m_1, which are independent of β and k, such that for $|1-\beta| < \epsilon$*

$$m_1|\gamma_1(\beta)| \le |\beta - 1|^{1/2} \le m_0|\gamma_1(\beta)|. \tag{6.17}$$

Proof: We showed in chapter V that

$$|\gamma_1(\beta)| = 2^{1/6}(\beta-1)^{1/2}p(\beta)^{1/3}, \tag{6.18}$$

and $0 < L_1 < |p(\beta)| < L_0$ where L_0 and L_1 are constants independent of β. The result of the lemma follows.

Thus Lemma 6.3 and (6.16) imply that

$$|P_I| \le e^{(-2/3)k|\gamma_1|^3}|\gamma_1|^2 m_3, \tag{6.19}$$

where $m_3 > 0$ is a constant independent of β and k.

To get an upper bound for P_{II}, let $g(x) = f(x) - 2x^3$, so $g'(x) = 2x^2 - 2 > 0$ for x on A_2 and $g(2) = 2/3$, thus $f(x) \geq 2x^3 + 2/3 \geq 2x^3$. Using the inequality $(x^2 - 1)^{-1/2} \leq 3^{-1/2}$, for x on A_2, we get

$|P_{II}| \leq (1/2)m_0|\gamma_1|^2\pi^{-1}\int_2^{-x_2/|\gamma_1|} e^{-k|\gamma_1|^3 2x^3} x dx.$

Let $u = k|\gamma_1|^3x^3$

$|P_{II}| \leq (1/2)m_0k^{-2/3}\pi^{-1}\int_{8k|\gamma_1|^3}^{-x_2^3k} e^{-u}u^{-1/3}du$. We can replace the above integral by $\int_0^\infty e^{-u}u^{-1/3}du = \Gamma(2/3)$. $|P_{II}| \leq \Gamma(2/3)k^{-2/3}\pi^{-1}m_0 = O(k^{-2/3})$.

For k such that $|\gamma_1| \geq k^{-1/3+\lambda}$ we replace the above integral by $O(\int_{8k|\gamma_1|^3}^{\infty} e^{-u}du) = O(e^{-2k|\gamma_1|^3})$. Thus $|P_{II}| \leq B_p(e^{-2k^\lambda})$. B_p is a constant independent of β and k. Thus, we combine the above estimate with (6.19) to conclude

$$|\mathrm{Re}[P_0(k,\beta)]| = \begin{cases} O(k^{-2/3}) \\ \leq B_p(e^{-2/3k^\lambda}) \quad \text{if } |\gamma_1| \geq k^{-1/3+\lambda} \end{cases} \qquad (6.20)$$

If $\beta \geq 1 + m_0^2k^{-2/3+2\lambda} = 1 + u_k$, we use Lemma 6.3 to conclude that $|\gamma_1| \geq k^{-1/3+\lambda}$, because

$$k^{-1/3+\lambda} \leq m_0^{-1}(\beta - 1)^{1/2} \leq |\gamma_1|.$$

Thus, we conclude the proof of Lemma 6.1

$$|\mathrm{Re}[P_0(k,\beta)]| = \begin{cases} 0 & \text{if } \beta = 1 \\ O(k^{-2/3}) & \text{if } 1 < \beta \leq 1+\epsilon \\ \leq B_p(e^{(-2/3)k^{\lambda}}) & \text{if } 1+u_k \leq \beta \leq 1+\epsilon \end{cases}$$

Lemma 6.4 *Let u_k be as in as in Lemma 6.1, then*

$$|\mathrm{Re}[R_0(k,\beta)]| \leq \begin{cases} B_V(k^{-2/3+\lambda}) & \text{for } 1 \leq \beta \leq u_k + 1 \\ B_R(e^{-2k^{\lambda}}) & \text{for } 1+u_k \leq \beta \leq 1+\epsilon \end{cases} \tag{6.21}$$

B_V and B_R are constants which are independent of β and k.

Proof: For $\beta = 1$, $\gamma_1(1) = 0$. $\phi(t,1) = -t^3/3$ has a critical point at $t = 0$ of order 3. The angles of the steepest descent curve are $\theta_5 = 0$, $\theta_6 = 2\pi/3$, $\theta_7 = 4\pi/3$ by Lemma 6.2.

The steepest descent curve Γ_β in the $z-$plane for $\beta = 1$, approaches the origin along the ray $\alpha_0 = 5\pi/6$ and emerges along the ray $\alpha_1 = 3\pi/2$. Since $z(t,\beta)$ is a conformal map which maps the positive real axis in the $t-$plane into the negative imaginary axis in the $z-$plane, the corresponding curve in the $t-$plane is a curve that approaches the origin along the ray with angle θ_7 and goes out along the line with angle θ_5. The curves of integration are straight lines and along these lines $\phi(t,1)$ is clearly real and negative.

$z(t,\beta)$ is an analytic function of both variables, hence we have the following estimate for $1 \leq \beta \leq 1+\epsilon$.

$$|G(t,\beta) - a_0| \leq C_0|t+\gamma_1|. \tag{6.22}$$

Thus,

$$|R_0(k,1)| \leq C_0 \int_0^\infty s e^{-ks^3/3} ds.$$

Let $u = ks^3/3$

$$R_0(k,1)| \leq C_0 3^{-1/3} k^{-2/3} \int_0^\infty e^{-u} u^{-1/3} du = m_2 k^{-2/3} \tag{6.23}$$

where m_2 is a constant independent of k and γ_1.

Thus, it suffices to consider the case $1 < \beta \leq 1+\epsilon$. Perform the change of variables, $r = t/\gamma_1(\beta)$ in (6.8),

$$R_0(k,\beta) = -i\gamma_1(\beta)\pi^{-1} \int_{D_{R_0}} (G(r\gamma_1,\beta) - a_0) e^{-k|\gamma_1|^3 v(r)} dr. \tag{6.24}$$

For t and β real, $z(t,\beta)$ is purely imaginary and $G(t,\beta)$ is real, where $G(t)$ was defined by (6.5). Let A_j be the curves that we defined in Lemma 6.1 for $j = 1,2$. The contribution of $H'_{k\beta}(k)$ to the curve A_1, is purely imaginary. Thus, for our purposes it suffices to estimate the integral for the curve A_2. We use the same notation as in Lemma 6.1. Using equation (6.22)

$$|G(r\gamma_1,\beta) - a_0| \leq C_0|\gamma_1||r+1| \tag{6.25}$$

$$|R_0(k,\beta)| \leq C_0|\gamma_1(\beta)|^2| \int_{A_2} (x+1+iy)e^{-k|\gamma_1|^3 f(x)} dr| \qquad (6.26)$$

where $y = \sqrt{3}(x^2-1)^{1/2}$ and $dr = (1 + i\sqrt{3}x(x^2-1)^{-1/2})dx$

$$|R_0(k,\beta)| \leq C_0|\gamma_1(\beta)|^2 \int_1^{\infty} h(x)e^{-k|\gamma_1|^3 f(x)} dx \qquad (6.27)$$

where $h(x) = |-2x+1+i[\sqrt{3}(x(x+1)(x^2-1)^{-1/2} + (x^2-1)^{1/2})]|$. We showed in the proof of Lemma 6.1, that $f(x) \geq 2x^3$ for x on A_2. By the same reasoning as we used in Lemma 6.1, it suffices to estimate the integral for $2 \leq x < \infty$. For such x,

$$|h(x)| \leq |-2x+1| + \sqrt{3}(x^2-1)^{1/2}[x(x+1)(x^2-1)^{-1}+1]$$

$$\leq |-2x+1| + \sqrt{3}(x^2-1)^{1/2}[2+(x-1)^{-1}] \leq m_3 x + m_4$$

where m_3 and m_4 are constants independent of β and k. Thus, we again can apply the method of Lemma 6.1 and complete the proof of Lemma 6.4.

We have so far estimated for $1 \leq \beta \leq 1+\epsilon$. Now, we will state a lemma for the case $1-\epsilon \leq \beta < 1$. In this case the point $z_- = -\cos^{-1}(\beta)$ is the only critical point on the curve Γ_z. The corresponding point in the t-plane is $-\gamma_1(\beta)$. Thus $-\gamma_1(\beta)$ is the only critical point on the curve D_β. Here $\cos^{-1}(\beta)$ is taken to be positive for $1-\epsilon \leq \beta \leq 1$.

Let $a_1 = G(-\gamma_1(\beta), \beta)$

$$P_1(k, \beta) = \frac{-a_1 i}{\pi} \int_{D_\beta} e^{k\varphi(t,\beta)} dt, \tag{6.28}$$

$$R_1(k, \beta) = \frac{-i}{\pi} \int_{D_\beta} (G(t, \beta) - a_1) e^{k\varphi(t,\beta)} dt, \tag{6.29}$$

$$F(k, \beta) = P_1(k, \beta) + R_1(k, \beta).$$

It will suffice for the proof of Lemma 6.5 and 6.7 to estimate $P_1(k, \beta)$ and just get an upper bound for $R_1(k, \beta)$, for $1 - \epsilon \leq \beta < 1$. Let

$$\delta_2(t) = \sqrt{1 - t^2} - t \cos^{-1}(t),$$

where $0 < \cos^{-1}(t) < \frac{\pi}{2}$.

The function $\delta(\beta)$ defined by (5.4) is equal to $i\delta_2(\beta)$ for $1 - \epsilon \leq \beta \leq 1$. From Chapter V it is clear that for $\beta < 1$, $\delta_2(\beta) = 2/3|\gamma_1|^3$.

Lemma 6.5

$$P_1(k, \beta) = \begin{cases} \dfrac{e^{ik\delta_2(\beta) + \frac{\pi i}{4}} (1-\beta^2)^{1/4}}{\sqrt{\pi k}} (1 + O(k^{-\lambda})) & for\ 1 - \epsilon \leq \beta \leq 1 - u_k \\ \leq B_Q(k^{-2/3+2\lambda}) & for\ 1 - u_k \leq \beta < 1 \end{cases} \tag{6.30}$$

B_Q is a constant independent of β and k.

Proof: Perform the change of variables in (6.28) $r = t/|\gamma_1(\beta)|$

$$P_1(k,\beta) = \frac{-ia_1|\gamma_1(\beta)|}{\pi}\int_{D_{R_1}} e^{k|\gamma_1|^3 v(r)}dr \tag{6.31}$$

$$v(r) = -(r + r^3/3). \tag{6.32}$$

D_{R_1} is the image of D_β. We have to evaluate the contribution to the integral for the part of the curve near the saddle point $r = -i$, which corresponds to $-\gamma_1(\beta)$ in the t-plane. $v(-i) = \frac{2i}{3}$ and $v''(-i) = 2i$. By Lemma 6.2 the angles of the tangents to the curves of steepest descent are given by $\theta_8 = \frac{\pi}{4}$ and $\theta_9 = \frac{5\pi}{4}$. Since our map $z(t,\beta)$ is conformal and maps the positive real line into the negative imaginary line, the incoming curve of D_{R_1} is tangent along $e^{i\theta_9}$ and the outgoing curve is along $e^{i\theta_8}$. The equation of the steepest descent curve is given by Im $v(r) = \frac{2}{3}$. Let $r = x + iy$, $v(x,y) = (u_1 + iu_2)$, where

$$u_1 = -x - \frac{x^3}{3} + xy^2,$$

$$u_2 = -y + \frac{y^3}{3} - x^2y.$$

Thus, $u_2 = 2/3$ defines implicitly the curve of steepest descent, which passes through the saddle point at $r_0 = -i$. We expand $v(r)$ about the point $r_0 = -i$. Perform a change of variable in (6.31) $\psi = r + i$, and let D_ψ denote the image of

D_{R_1}.

$$P_1(k,\beta) = \frac{-a_1 i e^{ik\delta_2(\beta)}|\gamma_1|}{\pi} \int_{D_\psi} e^{k|\gamma_1|^3(i\psi^2 - \frac{1}{3}\psi^3)} d\psi \tag{6.33}$$

The main contribution of this integral comes from the neighborhood of $\psi = 0$. For $j = 1, \ldots, 4$, let B_j be the restriction of D_ψ as described below. Our curve of integration is $D_\psi = \bigcup_{j=1}^{4} B_j$.

$$B_1 : \; |\psi| \leq \frac{1}{2} \;\text{ and } \mathrm{Re}\psi \geq 0$$

$$B_2 : \; |\psi| \leq \frac{1}{2} \;\text{ and } \mathrm{Re}\psi \leq 0$$

$$B_3 : \; |\psi| \geq \frac{1}{2} \;\text{ and } \mathrm{Re}\psi \geq 0$$

$$B_4 : \; |\psi| \geq \frac{1}{2} \;\text{ and } \mathrm{Re}\psi \leq 0.$$

For $j = 5, 6$, let ψ_j be the intersection of the arc $\frac{1}{2}e^{i\theta}$ and the curve B_{j-2}. Let α_j be the angle of the line that joins ψ_j to the origin. We will show later that $0 \leq \alpha_5 \leq \frac{\pi}{4}$ and $\frac{5\pi}{4} \leq \alpha_6 \leq \frac{4\pi}{3}$.

For $j = 5, 6$, define the curves B_j as follows:

$$B_5 : \psi = \frac{1}{2}e^{i\theta} \quad \alpha_5 \leq \theta \leq \frac{\pi}{4}$$

$$B_6 : \psi = \frac{1}{2}e^{i\theta} \quad \frac{5\pi}{4} \leq \theta \leq \alpha_6.$$

Since our integrand is analytic, we can replace the curves B_j by straight line segments L_j joined with B_{j+4} for $j = 1, 2$. L_j is the line segment given by $\psi = (-1)^{j+1}e^{\pi i/4}q$ for $0 \leq q \leq 1/2$. These lines segments are the segments of the tangent line to the curve D_ψ at $\psi = 0$.

Let $P_{1j}(k, \beta)$ be the integral $P_1(k, \beta)$ whose curve of integration is the restriction of D_ψ to L_j for $j = 1, 2$. Let $P_{1j}(k, \beta)$ be the integral $P_1(k, \beta)$ whose curve of integration is the restriction of D_ψ to B_j for $j = 3, 4, 5, 6$.

We will now estimate P_{1j} for $j = 1, 2$. We will get estimates in the case $|\gamma_1| \geq k^{-1/3+\lambda}$ and an upper bound in the case $|\gamma_1| < k^{-1/3+\lambda}$.

In the case $|\gamma_1| < k^{-1/3+\lambda}$, we need the inequality

$$\mathrm{Re}(i\psi^2 - 1/3\psi^3) = -q^2(1 \pm \sqrt{2}q/6) \leq 0,$$

for $|q| \leq 1/2$

$$e^{-k|\gamma_1|^3 q^2(1-\sqrt{2}q/6)} \leq 1. \tag{6.34}$$

Since the line segments have finite lengths, $|P_{1j}(k, \beta)| \leq |a_1|\,|\gamma_1| m_5$ where m_5 is a constant independent of k and β, for $j = 1, 2$. The calculation of a_1 is the same as the calculation of a_0 in (6.15).

$$a_1(\beta) = (1 - \beta^2)^{1/4}\sqrt{2}\gamma_1^{1/2}(\beta)e^{\pi i/4}. \tag{6.35}$$

From Lemma 6.3 and the assumption that $|\gamma_1| \leq k^{-1/3+\lambda}$, we can conclude $|P_{1j}(k,\beta)| \leq B_Q(k^{-2/3+2\lambda})$.

In the case $|\gamma_1| \geq k^{-1/3+\lambda}$, we substitute the relation $\psi = (-1)^{j+1}e^{\pi i/4}q$, into

$$i\psi^2 - 1/3\psi^3 = -q^2 + (-1)^{j+1}\sqrt{2}q^3/6 + (-1)^j i\sqrt{2}q^3/6.$$

Let $s = q^2$, and let $\sigma = (-1)^j\sqrt{2}(-1+i)/6$

$d\psi = (-1)^{j+1}e^{\pi i/4}dq$, and $dq = s^{-1/2}ds/2$

$$P_{1j}(k,\beta) = \frac{-ie^{i[k\delta_2(\beta)+\pi/4]}a_1(\beta)|\gamma_1|}{2\pi} \int_0^{1/4} e^{-k|\gamma_1|^3(s-\sigma s^{3/2})}s^{-1/2}ds \tag{6.36}$$

for $j = 1,2$. Let $u = k|\gamma_1|^3 s$

$$P_{1j}(k,\beta) = \frac{-ie^{i[k\delta_2(\beta)+\pi/4]}a_1(\beta)|\gamma_1|}{2\pi\sqrt{k}|\gamma_1|^{3/2}} \int_0^{k|\gamma_1|^3/4} e^{-u+k^{-1/2}|\gamma_1|^{-3/2}\sigma u^{3/2}}u^{-1/2}du \tag{6.37}$$

for $j = 1,2$. We expand

$$\begin{aligned} e^{k^{-1/2}|\gamma_1|^{-3/2}\sigma u^{3/2}} &= 1 + O(k^{-1/2}|\gamma_1|^{-3/2}u^{3/2}e^{k^{-1/2}|\gamma_1|^{-3/2}\sigma u_0^{3/2}}) \qquad (6.38) \\ &= 1 + O(k^{-\lambda}u^{3/2}e^{k^{-1/2}|\gamma_1|^{-3/2}\sigma u_0^{3/2}}), \end{aligned}$$

where $0 < u_0 < u$ and since we are assuming that $|\gamma_1| \geq k^{-1/3+\lambda}$. For $0 \leq u \leq \frac{k|\gamma_1|^3}{4}$, we have the inequalities for the exponent

$$-u + k^{-1/2}|\gamma_1|^{-3/2}|\sigma|u_0^{3/2} \leq -u(1 - k^{-1/2}|\gamma_1|^{-3/2}u^{1/2}/3) \leq -5u/6.$$

Thus,

$$P_{1j}(k,\beta) = \frac{-ie^{i[k\delta_2(\beta)+\pi/4]}a_1(\beta)}{2\pi\sqrt{k}|\gamma_1|^{1/2}}[1+O(e^{-k^{\lambda}/4})] \int_0^{\infty}[e^{-u}u^{-1/2}+O(e^{-5u/6}k^{-\lambda}u^{3/2})]du$$

for $j = 1, 2$. Note that

$$\int_0^{\infty} e^{-u}u^{-1/2} = \Gamma(1/2) = \sqrt{\pi}.$$

Thus,

$$P_{1j}(k,\beta) = \frac{-ie^{i[k\delta_2(\beta)+\pi/4]}a_1(\beta)}{2\sqrt{\pi k}|\gamma_1|^{1/2}}[1+O(k^{-\lambda})] \tag{6.39}$$

for $j = 1, 2$. Recall equation (6.35).

$$\operatorname{Arg} \gamma_1 = \pi/2,$$

$$\gamma_1(\beta)^{1/2}|\gamma_1(\beta)|^{-1/2} = e^{\pi i/4}.$$

Thus, we conclude

$$P_{1j}(k,\beta) = \frac{e^{\pi i/4+ki\delta_2(\beta)}(1-\beta^2)^{1/4}\sqrt{2}(1+O(k^{-\lambda})}{2\sqrt{\pi k}} \quad \text{if } |\gamma_1| \geq k^{-1/3+\lambda} \tag{6.40}$$

for $j = 1, 2$.

We will now estimate $P_{15}(k,\beta)$. We will use Lemma (6.3) to write

$$|P_{15}(k,\beta)| \leq |\gamma_1|^2 m_6 \int_0^{\pi/4} e^{-1/4k|\gamma_1|^3 p(\theta)} d\theta$$

where $p(\theta) = (\sin(2\theta) + \frac{1}{6}\cos(3\theta))$, and $m_6 > 0$ is a constant independent of β and k.

Lemma 6.6 *There exist a positive constant m_7 independent of β and k such that $p(\theta) \geq m_7$ for $0 \leq \theta \leq \pi/4$.*

Proof: Case a): for $0 \leq \theta \leq \pi/6$, $\sin(2\theta) \geq 0$ and $\cos(3\theta) \geq 0$. Since $\sin(2\theta)$ and $\cos(3\theta)$ are never zero simultaneously for any θ in the above interval and $p(\theta)$ is continuous on a compact set, the lemma is true in this case.

case b): for $\pi/6 \leq \theta \leq \pi/4$, $|\cos(3\theta)| \leq \frac{\sqrt{2}}{2}, \sin(2\theta) \geq \frac{\sqrt{3}}{2}$. $p(\theta) \geq \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{12} > 0$, which again proves the lemma in case b).

Thus,

$$|P_{15}(k,\beta)| \leq |\gamma_1|^2 m_6 \pi e^{-k|\gamma_1|^3 m_7/4}. \tag{6.41}$$

By Lemma 6.2, the angles of the tangent line to the curves Γ_β for $\beta = 1$ are $\theta_5 = 0$, $\theta_6 = \frac{2\pi}{3}$, $\theta_7 = \frac{4\pi}{3}$. As we explained in Lemma 6.4 the angle of the incoming curve is θ_7 and the outcoming curve is θ_5.

We write $\psi = |\psi|e^{i\theta}$. Let $f(|\psi|,\theta) = |\psi|^2\cos(2\theta) - (1/3)|\psi|^3\sin(3\theta) = \mathrm{Im}(i\psi^2 - 1/3\psi^3)$. The curve of steepest descent satisfies $f(|\psi|,\theta) = 0$. $f(|\psi|,0) = |\psi|^2 > 0$ and $f(|\psi|,\pi/4) < 0$. $\frac{\partial f}{\partial \theta}(|\psi|,\theta) < 0$ for $0 < \theta < \pi/4$ and $|\psi|$ sufficiently small. Thus, the steepest descent curve starts out in the interval $0 < \theta < \pi/4$ and by continuity must remain in that interval because $f(|\psi|,\theta) = 0$ on this curve. Similar argument can be used for the interval $5\pi/4 < \theta < 4\pi/3$. Thus, for B_5 $0 \le \theta \le \frac{\pi}{4}$, and for B_6, $\frac{5\pi}{4} \le \theta \le \frac{4\pi}{3}$. The estimates for $P_{16}(k,\beta)$ are done in a similar way as we did for $P_{15}(k,\beta)$.

Thus,

$$|P_{1j}(k,\beta)| \le |\gamma_1|^2 m_6 e^{-k|\gamma_1|^3 m_8} \quad \text{for } j = 5, 6 \tag{6.42}$$

where $m_8 > 0$ is a constant independent of k and β.

If $|\gamma_1| \ge k^{-1/3+\lambda}$ then

$$|P_{1j}(k,\beta)| = O(e^{-k|\gamma_1|^3 m_8}) = O(e^{-k^\lambda m_8}). \tag{6.43}$$

If $|\gamma_1| < k^{-1/3+\lambda}$ then

$$|P_{1j}(k,\beta)| \le m_6(k^{-2/3+2\lambda}) \tag{6.44}$$

for $j = 5, 6$

We now estimate $P_{1j}(k,\beta)$ for $j = 3, 4$. By using the fact that all our transformations are conformal, one can show that if we parameterize our curve as $\psi(t) = |\psi(t)|e^{i\theta(t)}$ for $0 \le t \le \infty$, then $|\psi(t)| \longrightarrow \infty$ as $t \longrightarrow \infty$. Since the steepest descent curve satisfies $f(|\psi|, \theta) = |\psi|^2\cos(2\theta) - (1/3)|\psi|^3\sin(3\theta) = \mathrm{Im}(i\psi^2 - 1/3\psi^3) = 0$, $|\psi(t)|\sin(3\theta(t)) = 3\cos(2\theta(t))$ for $t > 0$. Since $|\cos(2\theta(t))| \le 1$ and $|\psi(t)| \longrightarrow \infty$ for $t \longrightarrow \infty$, we conclude that $\sin(3\theta(t)) \longrightarrow 0$ as $t \longrightarrow \infty$ and $\theta(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, there exists a positive constant C_3 such that $0 \le \theta(t) \le \pi/9$ for $|\psi| \ge C_3$.

C_3 is independent of β and k.

We subdivide the curve B_j into two parts:

$$B_I : \text{for } 1/2 \leq |\psi| \leq C_3$$

$$B_{II} : \text{for } C_3 \leq |\psi|$$

For B_I, we can estimate the integral by the value of the integrand at $|\psi| = 1/2$, since the curve B_I is the steepest descent curve and is of finite length and the length of the curve has a uniform bound. Thus, this part of the integral is $O(e^{-k|\gamma_1|^3 m_8})$.

For B_{II}, we have

$$\text{Re } i\psi^2 - 1/3\psi^3 = -|\psi|^2((1/3)|\psi|\cos(3\theta) + \sin(2\theta)) \leq -|\psi|^2 C_3/6,$$

by the fact that $|\psi| \geq C_3$ for $0 \leq \theta(t) \leq \pi/9$.

Thus,

$$|P_{1j}(k,\beta)| \leq |a_1(\beta)||\gamma_1|\pi^{-1} \int_{C_3}^{\infty} e^{(-k|\gamma_1|^3|\psi|^2)C_3/6} d|\psi| \tag{6.45}$$

for $j = 3, 4$.

In the case $|\gamma_1| \leq k^{-1/3+\lambda}$, let $u = k^{1/2}|\gamma_1|^{3/2}|\psi|$

$$|P_{1j}(k,\beta)| \leq m_9 k^{-1/2}|\gamma_1|^{1/2} \int_0^{\infty} e^{(-C_3/6u^2)} du. \tag{6.46}$$

Using equation (6.35) and Lemma (6.3), we conclude that

$$|P_{1j}(k,\beta)| \leq m_{13}(k^{-2/3+\lambda}) \tag{6.47}$$

for $j = 3, 4$, where m_9 and m_{13} are positive constants independent of β and k.

In the case $|\gamma_1| \geq k^{-1/3+\lambda}$, let $u = k|\gamma_1|^3|\psi|^2C_3/6$

$$|P_{1j}(k,\beta)| \leq |\gamma_1(\beta)|^{1/2}k^{-1/2}m_{12}\int_{k|\gamma_1|^3}^{\infty} e^u u^{-1/2}du \leq m_{10}(k^{-2/3+\lambda}e^{-k|\gamma_1|^3}) \leq m_{10}(e^{-k^{\lambda}}), \tag{6.48}$$

where m_{10} and m_{12} are positive constants independent of β and k.

Thus, we conclude that the main contribution to our integral is given by (6.40) and the other estimates are of smaller order independent of β and this completes the proof of Lemma 6.5.

Lemma 6.7

$$R_1(k,\beta) \leq \begin{cases} B_S(k^{-2/3-\lambda}) & for\ 1-\epsilon \leq \beta \leq 1-u_k \\ B_T(k^{-2/3+2\lambda}) & for\ 1-u_k \leq \beta < 1 \end{cases} \tag{6.49}$$

B_S and B_T are constants independent of β and k

Proof: Let $R_{1j}(k,\beta)$ be defined in the same way that we defined $P_{1j}(k,\beta)$ in Lemma 6.5 for $j = 1,\ldots,6$. We will show that R_{1j} is of smaller order than P_{1j} for $j = 1, 2$, and is of the same order for the other values of j. By the same reasoning as we used in equation (6.25)

$$|G(t,\beta) - a_1(\beta)| \leq C_1|t + \gamma_1(\beta)| = C_1|\gamma_1||\psi|. \tag{6.50}$$

Thus,

$$|R_{1j}| \leq C_1|\gamma_1|^2\pi^{-1}\int_0^{1/2} e^{-q^2+\sigma q^3} q\,dq \tag{6.51}$$

where $\sigma = (-1)^j\sqrt{2}(-1+i)/6$. Let $s = q^2$ as we did in Lemma 6.5

$$|R_{1j}| \leq C_1|\gamma_1|^2\pi^{-1}\int_0^{1/2} e^{-k|\gamma_1|^3(s-\sigma s^{3/2})}ds \tag{6.52}$$

for $j = 1,2$. Let $u = k|\gamma_1|^3 s$

$$|R_{1j}| \leq C_1|\gamma_1|^{-1}k^{-1}\pi^{-1}\int_0^{\infty} e^{-u+k^{-1/2}|\gamma_1|^{3/2}\sigma u^{3/2}}du \tag{6.53}$$

for $j = 1,2$. In the case $|\gamma_1| \geq k^{-1/3+\lambda}$

$$|R_{1j}| \leq C_1 k^{-2/3-\lambda}\pi^{-1}\int_0^{\infty} e^{-u+k^{-1/2}|\gamma_1|^{3/2}\sigma u^{3/2}}du. \tag{6.54}$$

We use the expansion given by (6.38) to conclude

$$|R_{1j}| \leq C_1 k^{-2/3-\lambda}\pi^{-1}\int_0^{\infty}[e^{-u} + O(e^{-5u/6}k^{-\lambda}u^{3/2})]du \leq B_S(k^{-2/3-\lambda}) \tag{6.55}$$

for $j = 1,2$.

For the other cases the proof is the same as in lemma 6.5. This completes the proof of Lemma 6.7. Thus, we have concluded the estimates of the integral of (6.1) restricted to the curve Γ_β. We will summarize our results in the next chapter.

We will now estimate in the case $\beta \geq 1$, the real part of the integrals restricted to the curve Γ_A which was defined in the beginning of this chapter.

From Lemma 6.1 and Lemma 6.4, it is clear that the contribution from Γ_β to $H'_{k\beta}$ is decaying and hence the same is true for Γ_A, since Γ_A has finite length and the curve is of steepest descent.

We will now give the estimates of $H'_{k\beta}(k)$ for the curve Γ_A for the case $1 - \epsilon \leq \beta \leq 1$. For $1 - \epsilon \leq \beta < 1$, we obtained a uniform estimate of P_{1j} for $j = 3, 4$ given by equations (6.47) and (6.48) which correspond to the estimate for the curves B_j, for $j = 3, 4$. It is clear form this estimate that the contribution to our integral along the curve B_j for $j = 3, 4$ is of smaller order than the estimate from the curve near the critical point. Similar reasoning can be used for R_{1j} for $j = 3, 4$. Since Γ_A is part of the steepest descent curve, the values of the integrand restricted to this curve is bounded by the estimates of (6.47) and (6.48). For the case $\beta = 1$, we showed that $|H'_{k\beta}(k)| \leq m_2(k^{-2/3+\lambda})$ (see (6.23)). We can get a uniform bound on the length of this curve since it has finite length and β is in a compact set. Thus, we conclude that the estimates of $H'_{k\beta}(k)$ over Γ_A is dominated by the estimates of (6.47) and (6.48).

The estimates of the integral of (6.1) restricted to the curves Γ_C and Γ_B are the same as above because $\frac{\partial u_R}{\partial x}(x, y, \beta) > 0$ for $y > 0$ and $x < 0$, and hence, the values of the integrand over the curve Γ_C are bounded by the values of the integrand at z_c. Similarly, because $\frac{\partial u_R}{\partial x}(x, y, \beta) < 0$ for $y < 0$ and $x < 0$, the values of the integrand over the curve Γ_B are bounded by the values of the integrand at z_b.

VII Estimates of the power radiation E(k)

We now evaluate the power radiation $E(k)$ and the dissipation function $D(k)$ asymptotically as $k \longrightarrow \infty$.

$$E(k) = \frac{4k}{\pi}\left(2\sum_{n=0}^{\infty} \frac{J_n'(k)^2}{|H_n'(k)|^2} - \frac{J_0'(k)^2}{|H_0'(k)|^2}\right), \tag{7.1}$$

$$D(k) = k^2\left(2\sum_{n=0}^{\infty} J_n'(k)^2 - J_0'(k)^2\right), \tag{7.2}$$

We recall the notation established above.

$$\delta_1(t) = t\cosh^{-1}(t) - \sqrt{t^2-1}, \tag{7.3}$$

$$\delta_2(t) = \sqrt{1-t^2} - t\cos^{-1}(t), \tag{7.4}$$

where we choose the branches such that for $t > 0$, $\cosh^{-1}(t) > 0$, and $0 < \cos^{-1}(t) < \frac{\pi}{2}$. In this chapter we are concerned with $t \geq 1$ in (7.3) and $0 \leq t \leq 1$ in (7.4).

The function $\delta(\beta)$ defined by (5.4) is equal to $\delta_1(\beta)$ for $1 \leq \beta \leq 1+\epsilon$, and is equal to $i\delta_2(\beta)$ for $1-\epsilon \leq \beta \leq 1$. From Chapter V,we know that $|\delta_1(\beta)| = 2/3|\gamma_1|^3$

for $1 \leq \beta \leq 1 + \epsilon$, and $|\delta_2(\beta)| = 2/3|\gamma_1|^3$ for $1 - \epsilon \leq \beta \leq 1$.

By Lemma 6.3 $\delta_1(\beta) = \frac{2}{3}|\gamma_1|^3 \geq m_1^3|\beta - 1|^{3/2}(2/3)$ and $\delta_2(\beta) = \frac{2}{3}|\gamma_1|^3 \geq m_1^3|\beta - 1|^{3/2}(2/3)$.

We summarize the uniform estimates for $H'_{k\beta}(k)$ for each of the five cases with a fixed λ, $0 < \lambda < 1/3$. Later we will choose $\lambda = 1/10$. ϵ was chosen by the conditions of Lemma 5.2 and Theorem 5.3.

a) For $0 < \beta \leq 1 - \epsilon$

$$H'_{k\beta}(k) = \frac{e^{[ik\delta_1(\beta)+\frac{\pi i}{4}]}\sqrt{2}\sqrt[4]{1-\beta^2}(1+O(k^{-1/5}))}{\sqrt{\pi k}} + O(e^{-c_\epsilon k^{1/5}}) \qquad (7.5)$$

where $c_\epsilon > 0$ is a constant independent of β and k (see (4.4)).

Recall that $u_k = m_0^2 k^{-2/3+2\lambda}$.

b) for $1 - \epsilon \leq \beta \leq 1 - u_k$

$$H'_{k\beta}(k) = \frac{e^{[ik\delta_2(\beta)+\frac{\pi i}{4}]}\sqrt{2}\sqrt[4]{1-\beta^2}(1+O(k^{-\lambda}))}{\sqrt{\pi k}} + O(e^{-k^{\lambda}}) \qquad (7.6)$$

(See (6.28) (6.30), and (6.49)).

c) for $1 - u_k \leq \beta \leq 1 + u_k$

$$|H'_{k\beta}(k)| \leq m_{14}(k^{-2/3+2\lambda}) \qquad (7.7)$$

where m_{14} is a constant independent of β and k. (See (6.7) (6.10), (6.21), (6.28) (6.30), and (6.49)).

d) for $1 + u_k \le \beta \le 1 + \epsilon$

$$|J'_{k\beta}(k)| \le B_p(e^{(-2/3)k^{\lambda}}) \tag{7.8}$$

(See (6.7)–(6.10), and (6.21)). B_p is a constant independent of β and k.

e) for $1 + \epsilon < \beta$

$$H'_{k\beta}(k) = \frac{-ie^{k\delta_1(\beta)}\sqrt{2}\sqrt[4]{\beta^2 - 1}[1 + O(k^{-1/5})]}{\sqrt{\pi k\beta}} + \sigma_{R_2}(e^{-k\delta_1(\beta)}\beta) + \sigma_{I_2}(\sqrt{\beta^2 - 1}) \tag{7.9}$$

where $\sigma_{R_2}(e^{-k\delta_1(\beta)}\beta)$ is real valued: $\sigma_{I_2}(\sqrt{\beta^2 - 1})$ is purely imaginary: they are both bounded by a constant times the function in the parenthesis (See (4.2)). These constants and the implied constants in $O(k^{-1/5})$ and $O(k^{-\lambda})$ are real numbers which are independent of β and k.

For the intervals c and d the principal part is purely imaginary. Hence, the contribution to the estimate of $E(k)$ will be of smaller order than from the estimates of the intervals a and b. Let

$$\beta = n/k.$$

We divide $E(k)$ into five sums $S_j(k)$ for $j = 1, 2, 3, 4, 5$ corresponding in order to the five cases above a)–e). Let

$$f(n, k) = \frac{J'_n(k)^2}{|H'_n(k)|^2}. \tag{7.10}$$

First we estimate in case e)

$$S_5(k) = 2 \sum_{n=[(1+\epsilon)k]}^{\infty} f(n,k).$$

Using Lemma 6.3, we can estimate S_5 by $\int_{1+\epsilon}^{\infty} e^{-m_{11}k(t-1)^{3/2}}dt = O(e^{-k\epsilon^{3/2}m_{11}}k^{-2/3})$, where $m_{11} = (2/3)m_1^3$.

For the cases b), c), and d), it suffices to use the fact that $f(n,k) \leq 1$. Hence, $\sum_{j-2}^{4} S_j(k) \leq \sum_{[(1-\epsilon)k]}^{[(1+\epsilon)k]} 1 = 2\epsilon k$.

For the case a)

$$f(n,k) = \cos^2(\pi/4 + k\delta_2(\beta)) + O(k^{-1}).$$

Use the trigonometric identity for the double angle to write

$$f(n,k) = \frac{1}{2} + \frac{1}{2}\cos(\pi/2 + 2k\delta_2(\beta)) + O(k^{-1})$$

$$f(n,k) = \frac{1}{2} - \frac{1}{2}\sin(2k\delta_2(\beta)) + O(k^{-1}). \tag{7.11}$$

Let

$$S_6 = \sum_{n=0}^{[(1-\epsilon)k]-1} \sin(2k\delta_2(n/k)).$$

Lemma 7.1.

$$S_6 = O(k^{1/2}). \tag{7.12}$$

Proof: Note that

$$\sum_{n=0}^{[(1-\epsilon)k^{1/2}]} \sin(2k\delta_2(n/k)) = O(k^{1/2}).$$

We need to apply the next Theorem 7.2 [Neiderreiter, p. 17] and [Van der Corput, pp. 53-79].

Theorem 7.2. *Let a, b be integers with $a < b$ and let $f(x)$ be twice differentiable on $[a,b]$ with $f''(x) \geq \rho$ for $a \leq x \leq b$. Then*

$$|\sum_{n=a}^{b} e^{2\pi i f(n)}| \leq 2|f'(b) - f'(a) + 2|(4\rho^{-1/2} + 3). \tag{7.13}$$

Apply the above theorem with $f(x) = \frac{k}{2\pi}\delta_2(x/k)$, $a = [k^{1/2}(1-\epsilon)]$, $b = [k(1-\epsilon)] - 1$.

$f'(x) = \frac{1}{2\pi}\delta_2'(x/k) = \frac{-1}{2\pi}\cos^{-1}(x/k)$

$$f''(x) = \frac{1}{2\pi k\sqrt{1-(x/k)^2}} \geq \frac{1}{2k\pi}.$$

Thus we can take $\rho = k^{-1}(2\pi)^{-1}$ and $S_6 = O(k^{1/2})$. Therefore,

$$S_1 = 2\sum_{n=0}^{[(1-\epsilon)k]-1}(\frac{1}{2}) + O(k^{1/2}) - 1/2$$

$$= (1-\epsilon)k + O(k^{1/2}).$$

$|E(k)/k^2 - 4/\pi| \leq (2\epsilon) + O(k^{-1/2}).$

Thus, there exist constants B_1, B_2 independent of k, such that

$$\frac{4}{\pi} + B_2\epsilon \leq \underline{\lim}_{k\longrightarrow\infty}\frac{E(k)}{k^2} \leq \overline{\lim}_{k\longrightarrow\infty}\frac{E(k)}{k^2} \leq B_1\epsilon + \frac{4}{\pi}.$$

Since this is true for every $\epsilon > 0$, let $\epsilon \longrightarrow 0$ and we conclude that

$$\frac{E(k)}{k^2} = (\frac{4}{\pi}) + o(1). \tag{7.14}$$

We now estimate $D(k)$ by subdividing it into four sums. Let $g(n,k) = 2J_n'(k)^2$, for the case e), let

$$T_5(k) = \sum_{n=[(1+\epsilon)k]}^{\infty} g(n,k),$$

$$g(n,k) \leq O(\epsilon^{-2k\delta_1(\beta)}\beta^2).$$

We estimate T_5 the same way we estimated S_5 and we get $T_5 = O(k^{-2/3})$.

For the case d),

$$g(n,k) = O(e^{(-4/3)k^{3\lambda}})$$

$$T_4(k) = \sum_{n=[(1+u_k)k]}^{[(1+\epsilon k)]-1} g(n,k) \leq (ke^{-4k^{3\lambda}/3}).$$

For the case c),

$$g(n,k) = O(k^{-2/3+2\lambda})$$

$$T_3(k) = \sum_{n=[(1-u_k)k]}^{[(1+u_k)k]-1} g(n,k) \leq O(ku_kg(n,k)) = O(k^{-1+6\lambda}).$$

We pick $\lambda = \frac{1}{10}$ so $T_3(k) = O(k^{-2/5})$.

For the cases a), and b), let $\alpha_k = [k(1-u_k)]-1$. We follow the same procedure which we used before by using the trigonometric identity and we get

$$g(n,k) = \frac{1}{\pi k}[1/2 + 1/2\sin(2k\delta_2(n/k))]\sqrt{1-(n/k)^2}[1+O(k^{-1/5})].$$

Let $P_6 = \frac{1}{k}\sum_0^{\alpha_k - 1}\sin(2k\delta(n/k))\sqrt{1-(n/k)^2}$.

Lemma 7.3

$$P_6 = O(k^{-1/2}). \tag{7.15}$$

Proof: Perform summation by parts

$$P_6 = \frac{1}{k}\sum_{m=0}^{\alpha_k - 1} A_m[h(m) - h(m+1)] + \frac{1}{k}A_{\alpha_k}h(\alpha_k)$$

where $A_m = \sum_{j=0}^{m}\sin(2k\delta(n/k))$

$$h(x) = \sqrt{1-(x/k)^2}.$$

From Lemma 7.1 we know that $\frac{1}{k}|A_m| \le Bk^{-1/2}$ where B is a constant independent of m, n, and k.

$$|P_6| \le Bk^{-1/2}[\sum_{m=0}^{\alpha_k - 1}|h(m) - h(m+1)| + k^{-1/12}].$$

Since $h(x)$ is a decreasing function,

$$\sum_{m=0}^{\alpha_k - 1}|h(m) - h(m+1)| = -h(\alpha_k - 1) + h(0).$$

Therefore, $P_6 = O(k^{-1/2})$.

We will estimate $\frac{2}{\pi k}\sum_{n=0}^{\alpha_k} h(n)$. We estimate this sum with a Riemann integral with partition of length $\frac{1}{k}$.

$$\frac{2}{\pi k}\sum_{n=0}^{\alpha_k} h(n) = \frac{2}{\pi}\int_0^{(1-u_k)} \sqrt{1-t^2}dt + O(k^{-1})$$
$$= \frac{2}{\pi}\int_0^1 \sqrt{1-t^2}dt + O(k^{-1}) = \frac{1}{2} + O(k^{-1}).$$

Let

$$T_1(k) = \sum_{n=0}^{\alpha_k - 1} g(n,k) - J_0'(k)^2.$$

Thus,

$$T_1 = \frac{1}{2} + O(k^{-1/2})$$
$$\frac{D(k)}{k^2} = \frac{1}{2} + O(k^{-1/2}). \tag{7.16}$$

Thus, the ratio

$$R(k) = E(k)/D(k) = \frac{8}{\pi} + o(1). \tag{7.17}$$

Comments and Interpretation

The discussion in chapter I can be interpreted as showing how a plane wave of unit amplitude is scattered by a cylinder with its axis perpendicular to the direction of propagation. The result $\frac{E(k)}{k^2} \cong \frac{4}{\pi}$ obtained in chapter VII shows that the power in the scattered wave is for high frequencies (or large wave numbers) asymptotically proportional to the square of the frequency. Similarly $\frac{D(k)}{k^2} \cong \frac{1}{2}$ shows that the dissipation function is also proportional to the square of the frequency and therefore E(k) is asymptotically proportional to D(k). Since, in general, both the power and the dissipation are proportional to the square of the amplitude, the latter result holds for plane waves of any amplitude.

The quantities E(k) and D(k) are meaningful even if the impinging wave is not a plane wave. The fact that the ratio E(k)/D(k) is asymptotically constant regardless of amplitude suggests that this ratio can reasonably be used as a measure of the efficiency of transfer of the power from the impinging wave to the scattered wave in cases where the impinging wave is not necessarily a plane wave.

It is natural, in view of the Fourier expansion

$$u(r,\vartheta,k) = J_0'(k)\overline{H}_0(kr)/\overline{H}_0'(k) + 2\sum_{n=1}^{\infty}(i)^n \frac{\overline{H}_n(kr)J_n'(k)\cos(n\vartheta)}{\overline{H}_n'(k)}$$

of the scattered wave obtained in chapter I, to ask how much of the power (or the dissipation) of the scattered wave can be attributed to the various components in the expansion. The proofs in Chapter VII show that for the power, asymptotically as frequency increases, the contributions of the terms for $n > k(1-\varepsilon)$ are negligible while the terms for $n \leq k(1-\varepsilon)$ all contribute equally, for any constant ε which satisfies $0 < \varepsilon < 1/2$ and is independent of n and k. The proofs in Chapter VII also show that for the dissipation function, asymptotically as frequency increases, the contributions of the terms for $n > k - m_0^2 k^{8/15}$ are negligible while the terms where $n \leq k - m_0^2 k^{8/15}$ all contribute equally, where m_0^2 is a constant which is independent of n and k.

REFERENCES

[1] N. Bleistein and R.A. Handelsman. *Asymptotic Expansions of Integrals*. Dover Publications, New York, 1986.

[2] R. Courant and Hilbert. *Methods of Mathematical Physics*, vol.1. Interscience Publications, New York, 1953.

[3] C. Chester, B. Friedman, and F.Ursell. An Extension of the Method of Steepest Descent. *Proc. Camb. Phil. Soc.* **53**, p.p. 434-453, (1957).

[4] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, John Wiley and Sons, N.Y., p.p. 68-86, 1983.

[5] J.G. van der Corput. Zahlentheoretische Abschätzungen, *Math. Ann.* **84** (1921), pp. 53-79.

[6] R. Gunning and H. Rossi. *Analytic Functions of Several Complex Variables*. Prentice Hall, New Jersey, 1965.

[7] *Handbook of Mathematical Functions*. U. S. Dept. of Commerce- National Bureau of Standards, Applied Mathematics Series **55**, Washington D. C., 1964.

[8] E.Hille. *Analytic Function Theory*, vol. 1. Blaisdell Publishing Company, New York, 1959.

[9] L.Kuipers and H. Niederreiter. *Uniform Distribution of Sequences*. John Wiley and Sons, New York, p.p. 15-17, 1974.

[10] N. N. Lebedev. *Special Functions and Their Applications*. Dover Publications, New York, 1972.

[11] F. W. Olver. *Asymptotics and Special Functions*. Academic Press, New York, 1974.

[12] R. Sacksteder. On the Helmholtz Equation. *Contributions to Geometry and Analysis*, Johns Hopkins University Press, p.p. 293-305, (1981).

[13] G. N. Watson. *Theory of Bessel Functions*. University Press, Cambridge, 1944.