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On the Poincaré-Bertrand transformation formula

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by

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Abstract

On the Poincaré-Bertrand Transformation Formula

by

Randi Elizabeth Lerohl

Advisor: Professor Richard Sacksteder

This paper studies in what sense the Poincaré-Bertrand transformation formula holds for continuous functions which are not Hölder continuous. It is shown that the transformation formula holds, in the almost everywhere sense, for functions, $\Phi(x,y)$, which are products of $L^2[-1,+1]$ functions, continuous on $(-1,+1)$. We then study the case where $\Phi(x,y)$ is an infinite sum of such functions, $\Phi(x,y) = \sum_{i=1}^{\infty} a_i\phi_i(x)\phi_i(y)$, and certain conditions are imposed on Φ. These results are generalized to $L^2[\Gamma]$ functions where Γ is a smooth curve in the plane of class C^3. By applying the Poincaré-Bertrand formula to the continuous function $\Phi(x,y) = \sqrt{1-y^2}\, f(x)$, we derive an inversion formula for the finite Hilbert transform which can be used to solve the Neumann problem for Laplace's equation in the exterior of $[-1,+1]$ in the plane.

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I. Introduction

Singular integral equations with Cauchy type kernels have long been studied because of their relevance to problems arising in mathematical physics as well as in complex analysis. It is well known, that when working with singular integrals, one cannot assume in general that two singular integrals commute. For example, if $f(x) = \sqrt{1-x^2}$ and $g(y) = \dfrac{1}{\sqrt{1-y^2}}$, then

$$\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(x-y)(y-z_0)}\,dxdy = -\pi^2 \quad \text{for } z_0 \in (-1,+1),$$

whereas $\mathbf{P}\displaystyle\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(x-y)(y-z_0)}\,dydx = 0$, where **P** indicates that the integrals

are taken in the Cauchy Principal Value sense. (See [14], p. 180.)

However, as was first proved by Poincaré [9] and Bertrand [2] under quite restrictive hypotheses, we do have the following transformation or commutation formula for two singular integrals:

$$\mathbf{P}\int_{\Gamma}\mathbf{P}\int_{\Gamma} \frac{\Phi(x,y)}{(x-y)(y-z_0)}\,dxdy = -\pi^2\Phi(z_0,z_0) + \mathbf{P}\int_{\Gamma}\mathbf{P}\int_{\Gamma} \frac{\Phi(x,y)}{(x-y)(y-z_0)}\,dydx, \quad z_0 \in \Gamma.$$

In the classical literature on singular integral equations, the Poincaré-Bertrand formula is stated for functions, $\Phi(x,y)$, which are Hölder continuous on

Γ, which is assumed to be a simple, smooth arc or contour (see [7] and [8]). Tricomi proves the Poincaré-Bertrand formula for $\mathbf{L}^p$ functions, $1 < p < \infty$, when Γ is the real line (see [14]).

More recently, various generalizations of these classical results have been proved. One author extends the Poincaré-Bertrand formula by relaxing the smoothness assumptions of the contour of integration (see [6]). Other papers generalize the domain of integration to product manifolds (see [5] and [13]). Still other generalizations depend on adopting a modified principal value convention which can simplify or even eliminate the residual term, $-\pi^2\Phi(z_0,z_0)$ (see [1]). However, in each case, the authors maintain the assumption that the function, $\Phi(x,y)$, is Hölder continuous on the domain of integration.

In this paper, we generalize the Poincaré-Bertrand formula to include functions which are not necessarily Hölder continuous. We first prove that if Γ is the unit interval and $\Phi(x,y)$ is the product of $\mathbf{L}^2[-1,+1]$ functions which are continuous on $(-1,+1)$, then the Poincaré-Bertrand formula holds for almost all $z_0 \in (-1,+1)$ We then study the case where $\Phi(x,y)$ is an infinite sum of such functions, $\Phi(x,y) = \sum_{i=1}^{\infty} a_i\phi_i(x)\phi_i(y)$, and it is assumed that $\Phi(x,y)$ is a symmetric kernel of a positive definite integral operator. In chapter III, we generalize these results to arcs in the plane which are of class $\mathbf{C}^3$.

Note that when Γ is $[-1,+1]$, the Poincaré-Bertrand formula may be viewed as a statement about the finite Hilbert transform $\mathbf{H}$. The finite Hilbert transform of a suitable function ϕ is defined by $\mathbf{H}\phi(y) = \frac{1}{\pi}\, \mathbf{P} \int_{-1}^{+1} \frac{\phi(x)}{x-y}\, dx$. In the final chapter, we use the results of Chapter II to establish an inversion formula for the finite Hilbert transform. We then show how this inversion formula can be applied to solving the Neumann problem for Laplace's equation in the exterior of [-1,+1] in the complex plane.

II. The PBF for Continuous Functions on [-1,+1] × [-1,+1]

In this chapter we will prove that when Γ is the unit interval, the Poincaré-Bertrand formula holds in the almost everywhere sense for certain continuous functions, Φ, which are not Hölder continuous. To see that one cannot hope for much more, we first consider the following example.

<u>Construction of an Example</u>.

Let Γ be the unit interval [-1,+1] and let

$$\Phi_k(x,y) = \frac{x}{\sqrt{1-x^2}}\, f_k(y), \qquad k = 1,2,3,...$$

where
$$f_k(y) = \sum_{n=2}^{\infty} \frac{\sqrt{1-y^2}\; U_{nk-1}(y)}{n \ln(n)} \qquad \text{and}$$

$U_{nk-1}(y)$ is the Chebyshev polynomial defined by

$$U_{nk-1}(y) = \frac{\sin(nk\cos^{-1}y)}{\sin(\cos^{-1}y)} = \frac{\sin(nk\cos^{-1}y)}{\sqrt{1-y^2}} \qquad |y| < 1.$$

It is easily seen that $\Phi_k(x,y)$ is continuous on $(-1,+1) \times (-1,+1)$. Write $f_k(y)$ as a function of θ:

$$F_k(\theta) = \sum_{n=2}^{\infty} a_n \sin(nk\theta)\ , \quad \theta = \cos^{-1}y, \quad a_n = \frac{1}{n \ln(n)}\ .$$

It is well known that if $a_n \geq a_{n+1}$ and $na_n \to 0$, then $\sum_{n=2}^{\infty} a_n \sin(nk\theta)$ converges uniformly. (See, for example, [16], p. 182.) Clearly $a_n = \dfrac{1}{n \ln(n)}$ satisfies these conditions so that $F_k(\theta)$ is continuous on (-1,+1). It follows that

$$\Phi_k(x,y) = \frac{x}{\sqrt{1-x^2}} \; f_k(y) \text{ is continuous on } (-1,+1) \times (-1,+1).$$

We will show by direct calculation that the Poincaré-Bertrand formula does indeed hold for the continuous function $\Phi_k(x,y)$, k = 1,2,3,... except on a set S_k of measure zero. For this example, we can explicitly determine which $z_0 \in (-1,+1)$ are contained in S_k, k = 1,2,3 ... We then use this information to construct an example of a continuous function $\Phi(x,y)$ for which S, the set on which the Poincaré-Bertrand formula fails to hold, is a countable dense subset of [-1,+1].

For $\Phi(x,y) = \Phi_k(x,y)$, we can easily evaluate the singular integrals contained in the Poincaré-Bertrand formula because the finite Hilbert transform acts in a particularly simple way on the Chebyshev polynomials.

We will use the following definitions and identities for the Chebyshev polynomials:

(1) $\quad U_n(y) = \dfrac{\sin((n+1)\cos^{-1}y)}{\sin(\cos^{-1}y)}$ $\qquad$ In particular, $U_0(y) = 1$.

(2) $\quad T_n(y) = \cos(n\cos^{-1}y)$ $\qquad$ In particular, $T_1(y) = y$.

$$(3) \quad \frac{1}{\pi} \mathbf{P} \int_{-1}^{+1} \frac{T_n(y)}{\sqrt{1-y^2}\,(y-x)}\, dy = U_{n-1}(x)$$

$$(4) \quad \frac{1}{\pi} \mathbf{P} \int_{-1}^{+1} \frac{\sqrt{1-y^2}\;U_{n-1}(y)}{y-x}\, dy = -\,T_n(x)$$

$$(5) \quad yT_n(y) = \frac{1}{2}\left\{ T_{n+1}(y) + T_{n-1}(y) \right\}$$

$$(6) \quad yU_n(y) = \frac{1}{2}\left\{ U_{n+1}(y) + U_{n-1}(y) \right\}$$

For (3) and (4) see [14], p. 180. (5) and (6) are a consequence of the sum and difference formulas for the sine and cosine functions.

Using the above identities we have, for the left hand side of the Poincaré-Bertrand formula,

$$\mathbf{P}\int_\Gamma \mathbf{P}\int_\Gamma \frac{\Phi_k(x,y)}{(y-z_0)(x-y)}\, dxdy = \mathbf{P}\int_{-1}^{+1} \frac{f_k(y)}{(y-z_0)} \left[\mathbf{P}\int_{-1}^{+1} \frac{x}{\sqrt{1-x^2}\,(x-y)}\, dx \right] dy$$

$$= \mathbf{P}\int_{-1}^{+1} \frac{f_k(y)}{(y-z_0)} \left[\mathbf{P}\int_{-1}^{+1} \frac{T_1(x)}{\sqrt{1-x^2}\,(x-y)}\, dx \right] dy \qquad \text{by (2),}$$

$$= \pi\, \mathbf{P}\int_{-1}^{+1} \frac{f_k(y)}{(y-z_0)}\, dy \qquad \text{by (3) and (1),}$$

$$= \pi \, \mathrm{P}\int_{-1}^{+1} \sum_{n=2}^{\infty} \frac{\sqrt{1-y^2}\;\; U_{nk-1}(y)}{n \ln(n)} \; \frac{1}{yzz-z)} \, dy$$

$$= -\pi^2 \sum_{n=2}^{\infty} \frac{T_{nk}(z_0)}{n \ln(n)} \qquad \text{by (4).}$$

Note that $T_{nk}(z_0) = \cos(nk\cos^{-1}z_0)$ so that

$$\sum_{n=2}^{\infty} \frac{T_{nk}(z_0)}{n \ln(n)} = \sum_{n=2}^{\infty} \frac{\cos(nk\cos^{-1}z_0)}{n \ln(n)} = \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

whenever $z_0 = \cos\left(\frac{2m\pi}{k}\right)$, $0 \le m \le \left[\frac{k}{2}\right]$, where $[\;\;]$ means integer part.

Therefore, for each of these $\left[\frac{k+2}{2}\right]$ values of $z_0 \in [-1,+1]$,

$$\mathrm{P}\int_{\Gamma} \mathrm{P}\int_{\Gamma} \frac{\Phi_k(x,y)}{(y-z_0)(x-y)} \, dxdy = -\pi^2 \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \quad \text{diverges.}$$

On the other hand, if $z_0 \ne \cos\left(\frac{2m\pi}{k}\right)$, $0 \le m \le \left[\frac{k}{2}\right]$, the series $\sum_{n=2}^{\infty} \frac{\cos(nk\cos^{-1}z_0)}{n \ln(n)}$ converges to a continuous function, call it $g_k(z_0)$. Write $g_k(z_0)$ as a function of θ:

$$G_k(\theta) = \sum_{n=2}^{\infty} a_n b_n(\theta) \ , \quad a_n = \frac{1}{n\ \ln(n)} \ , \quad b_n(\theta) = \cos(nk\theta), \quad \theta = \cos^{-1}(z_0).$$

As $n \to \infty$, the sequence $a_n \longrightarrow 0$, the convergence being monotone and independent of θ. Furthermore, the partial sums of $\sum_{n=2}^{\infty} b_n(\theta)$ are uniformly bounded in every interval of the form $\left[\frac{2m\pi}{k} + \delta, \frac{2(m+1)\pi}{k} - \delta \right]$, $m = 0,1,\ldots\left[\frac{k}{2}\right]$, $\delta > 0$, since if

$$S_n(\theta) = \sum_{n=2}^{\infty} \cos(nk\theta), \text{ then}$$

$$\left| S_n(\theta) \right| \ \leq \ \left| \frac{\sin\left((3/2)k\theta\right) - \sin\left(k(n+1/2)\theta\right)}{2\sin(k\theta/2)} \right| \ \leq \ \frac{1}{|2\sin(k\theta/2)|} \ .$$

Hence, the series $\sum_{n=2}^{\infty} a_n b_n(\theta)$ satisfies the conditions of Dirichlet's test for uniform convergence in every interval of the form $\left[\frac{2m\pi}{k} + \delta, \frac{2(m+1)\pi}{k} - \delta\right]$. Every fixed $\theta \neq \frac{2m\pi}{k}$, (or equivalentally, every $z_0 \neq \cos(2m\pi/k)$), may be regarded as belonging to an interval of this form, since $\delta > 0$ is arbitrary. Hence, $g_k(z_0)$ is continuous for all $z_0 \neq \cos(2m\pi/k)$.

Reversing the order of integration and using partial fractions, we have

$$P\int_{\Gamma} P\int_{\Gamma} \frac{\Phi_k(x,y)}{(y-z_0)(x-y)}\,dydx = P\int_{-1}^{+1} \frac{x}{\sqrt{1-x^2}} \left[P\int_{-1}^{+1} \frac{f_k(y)}{(y-z_0)(x-y)}\,dy \right] dx$$

$$= P\int_{-1}^{+1} \frac{x}{\sqrt{1-x^2}} \frac{1}{(z_0-x)} \left[P\int_{-1}^{+1} \frac{f_k(y)}{(y-x)} - \frac{f_k(y)}{(y-z_0)}\,dy \right] dx$$

$$= \pi\, P\int_{-1}^{+1} \frac{x}{\sqrt{1-x^2}} \sum_{n=2}^{\infty} \frac{T_{nk}(x)}{n\ \ln(n)} \frac{1}{(x-z_0)}\,dx - \pi \sum_{n=2}^{\infty} \frac{T_{nk}(z_0)}{n\ \ln(n)} P\int_{-1}^{+1} \frac{x}{\sqrt{1-x^2}} \frac{1}{(x-z_0)}\,dx \text{ by (4),}$$

$$= \frac{\pi}{2} P\int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} \sum_{n=2}^{\infty} \frac{T_{nk+1}(x) + T_{nk-1}(x)}{n\ \ln(n)} \frac{1}{(x-z_0)}\,dx - \pi^2 \sum_{n=2}^{\infty} \frac{T_{nk}(z_0)}{n\ \ln(n)} \quad \text{by (5),}$$

$$= \frac{\pi^2}{2} \sum_{n=2}^{\infty} \frac{U_{nk}(z) + U_{nk-2}(z_0)}{n\ \ln(n)} - \pi^2 \sum_{n=2}^{\infty} \frac{T_{nk}(z_0)}{n\ \ln(n)} \quad \text{by (3),}$$

$$= \pi^2 \sum_{n=2}^{\infty} \frac{z_0 U_{nk-1}(z_0)}{n\ \ln(n)} - \pi^2 \sum_{n=2}^{\infty} \frac{T_{nk}(z_0)}{n\ \ln(n)} \quad \text{by (6),}$$

$$= \pi^2 \Phi(z_0,z_0) - \pi^2 \sum_{n=2}^{\infty} \frac{T_{nk}(z_0)}{n \ln(n)} .$$

We have shown that, for almost all $z_0 \in (-1,+1)$,

$$P\int_\Gamma P\int_\Gamma \frac{\Phi_k(x,y)}{(y-z_0)(x-y)} dxdy = -\pi^2 \Phi(z_0,z_0) + P\int_\Gamma P\int_\Gamma \frac{\Phi_k(x,y)}{(y-z_0)(x-y)} dydx.$$

Specifically, the formula holds for all $z_0 \notin S_k$,

$$S_k = \left\{ 1, \cos\left(\frac{2\pi}{k}\right), \cos\left(\frac{4\pi}{k}\right), \ldots \cos\left(\frac{2[k/2]\pi}{k}\right) \right\}.$$

Now consider the series $\sum_{j=1}^{\infty} a_j \Phi_{p_j}(x,y)$ where p_j is the j^{th} prime. The family of continuous functions, $\{\Phi_{p_j}\}$, is uniformly bounded on $[-1+\varepsilon,1-\varepsilon]$, $0 < \varepsilon < 1$:

$$|\Phi_{p_j}(x,y)| = \left| \frac{x}{\sqrt{1-x^2}} F_{p_j}(\theta) \right| = \left| \frac{x}{\sqrt{1-x^2}} F_1(p_j\theta) \right| \leq M, \text{ where M is a bound}$$

for the continuous function $\Phi_1(x,y)$ on $[-1+\varepsilon,1-\varepsilon]$. Therefore, if $\{a_j\}$ is any sequence such that $\sum_{j=1}^{\infty} |a_j|$ converges, then by the Weierstrass M-test, $\sum_{j=1}^{\infty} a_j \Phi_{p_j}(x,y)$ converges uniformly to a continuous function $\Phi(x,y)$ on $[-1+\varepsilon,1-\varepsilon]$ for any ε, $0 < \varepsilon < 1$.

We have seen that for each p_j, $g_{p_j}(z_0) = \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{\Phi_{p_j}(x,y)}{(y-z_0)(x-y)}\,dxdy =$

$\sum_{n=2}^{\infty}\frac{\cos(np_j\cos^{-1}z_0)}{n\ln(n)}$, diverges for a finite set of points, S_{p_j}, in [-1,+1]. Let

$\hat{S}_{p_j} = S_{p_j}/\{+1,-1\}$. The sets, $\{\hat{S}_{p_j}\}$, are pairwise disjoint: $z_0 \in \hat{S}_{p_j} \Leftrightarrow z_0 = \cos\left(\frac{2m\pi}{p_j}\right)$,

$1 \le m \le \frac{p_j - 1}{2}$, because if $p_j \ne p_k$, then $\cos\left(\frac{2m\pi}{p_j}\right) \ne \cos\left(\frac{2n\pi}{p_k}\right)$, since $1 \le m \le \frac{p_j - 1}{2}$

and $1 \le n \le \frac{p_k - 1}{2}$. Thus we have $\hat{S}_{p_j} \cap \hat{S}_{p_k} = \phi$, $j \ne k$.

For each j, let $\hat{I}_j$ be the union of ε_j-neighborhoods of each $z_0 \in \hat{S}_{p_j}$ where the ε_j's are chosen so that $\hat{I}_j \cap \left\{\bigcup_{k<j} \hat{S}_{p_k}\right\} = \phi$, (i.e., choose ε_j so that none of the singularities in $\hat{S}_{p_k}$, k<j, are contained in the ε_j neighborhoods). The ε_j's can be chosen to satisfy this condition, $\hat{I}_j \cap \left\{\bigcup_{k<j} \hat{S}_{p_k}\right\} = \phi$, because for any fixed j there are only finitely many elements in $\left\{\bigcup_{k<j} \hat{S}_{p_k}\right\}$ and $\hat{S}_{p_j} \cap \left\{\bigcup_{k<j} \hat{S}_{p_k}\right\} = \phi$. For example,

if $\{z_1^j, z_2^j, z_3^j, \ldots z_n^j\}$, $n = \left[\frac{p_j}{2}\right]$, denote the elements of $\hat{S}_{p_j}$, one can choose ε_j so that $\varepsilon_j < \frac{1}{2}\min\{d_1, d_2 \ldots d_{n-1}, d_n\}$ where $d_i = \min\limits_{z_0 \in \bigcup\limits_{k<j} S_{p_k}} |z_i^j - z_0|$. With this choice of ε_j, $\hat{I}_j \cap \left\{\bigcup\limits_{k<j} \hat{S}_{p_k}\right\} = \phi$.

Now let $I_j = [-1+\varepsilon, 1-\varepsilon]/\hat{I}_j$, where $\varepsilon > 0$ is arbitrarily small. Then $g_{p_j}(z_0)$ is uniformly continuous on I_j, for every j. Let $c_j = \max\{1, \max\limits_{z_0 \in I_j} |g_{p_j}(z_0)|\}$ and let $a_j = \dfrac{1}{c_j 2^j}$. Note that with this choice of a_j,

$$|\Phi(x,y)| \leq \sum_{j=1}^{\infty} a_j |\Phi_{p_j}(x,y)| \leq M\sum_{j=1}^{\infty} a_j \leq M\sum_{j=1}^{\infty}\frac{1}{2^j} \leq M,$$

so that $\Phi(x,y)$ is in fact continuous on $[-1+\varepsilon, 1-\varepsilon]$.

Again using the identities for the Chebyshev polynomials, we formally have

$$P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{\Phi(x,y)}{(y-z_0)(x-y)}\,dxdy = -\pi^2 \sum_{j=1}^{\infty} a_j g_{p_j}(z_0), \qquad g_{p_j}(z_0) = \sum_{j=1}^{\infty} \frac{\cos(np_j \cos^{-1} z_0)}{n\ln(n)}$$

while $$\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{\Phi(x,y)}{(y-z_0)(x-y)}\,dydx = -\pi^2\sum_{j=1}^{\infty}a_jg_{p_j}(z_0) + \pi^2\Phi(z_0,z_0).$$

However, fix $z_0 \in (-1,+1)$ and assume that $z_0 \in \hat{S}_{p_k}$, for some k. We have seen that $a_kg_{p_k}(z_0) = \sum_{n=2}^{\infty}\frac{a_k}{n\ln(n)}$ diverges. On the other hand, for $j > k$, $z_0 \in I_j$ and $|a_jg_{p_j}(z_0)| \leq \frac{1}{2^j}$. For $j < k$, $z_0 \notin \hat{S}_{p_j} \Rightarrow |a_jg_{p_j}(z_0)|$ is finite. Therefore, if $z_0 \in \hat{S}_{p_k}$, the first k-1 terms of the series $\sum_{j=1}^{\infty}a_jg_{p_j}(z_0)$ are finite while $\sum_{j=k+1}^{\infty}|a_jg_{p_j}(z_0)| \leq \sum_{j=k+1}^{\infty}\frac{1}{2^j} \leq 1$. Consequently, the divergent term $a_kg_{p_k}(z_0) = \sum_{n=2}^{\infty}\frac{a_k}{n\ln(n)}$, is not offset by the other terms and it is clear that $\sum_{j=1}^{\infty}a_jg_{p_j}(z_0) = \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{\Phi(x,y)}{(y-z_0)(x-y)}\,dxdy$ is unbounded if $z_0 \in \hat{S}_{p_k}$ for any k.

We note that if $z_0 \in \bigcap_j I_j$, then $\sum_{n=1}^{\infty}|a_jg_{p_j}(z_0)| \leq \sum_{j=1}^{\infty}\frac{1}{2^j} \leq 1$. Hence, for $z_0 \in \bigcap_j I_j$, $\sum_{n=1}^{\infty}a_jg_{p_j}(z_0)$ converges, and the Poincaré-Bertrand formula holds.

To summarize, we constructed a continuous function, $\Phi(x,y) = \sum_{j=1}^{\infty} a_j \Phi_{p_j}(x,y)$, and showed that, regardless of the order of integration, applying two singular integrals to this continuous function results in a function which becomes unbounded on at least a countable dense subset of $(-1,+1) \times (-1,+1)$. This example clearly demonstrates that if a function is not Hölder continuous, then one cannot be certain that the Poincaré-Bertrand formula holds for all $z_0 \in \Gamma$.

PBF for Products of Continuous Functions.

We will now prove that, when $\mathbf{L} = [-1,+1]$, the Poincaré-Bertrand formula holds in the almost everywhere sense if $\Phi(x,y) = f(x)g(y)$ where f and g are $L^2[-1,+1]$ functions which are continuous in $(-1,+1)$. We begin by showing that

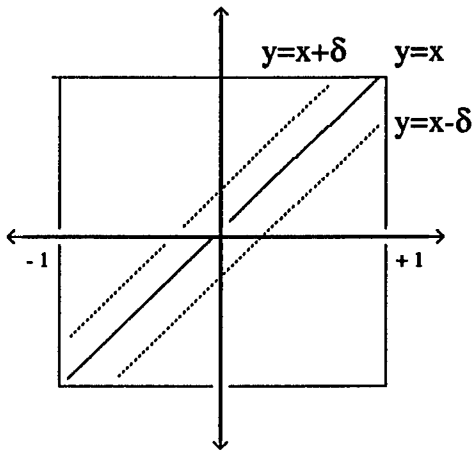
$$\int_{-1}^{+1} \mathbf{P} \int_{-1}^{+1} \frac{\Phi(x,y)}{(z-y)(y-x)}\, dxdy = \int_{-1}^{+1} \mathbf{P} \int_{-1}^{+1} \frac{\Phi(x,y)}{(z-y)(y-x)}\, dydx$$

where $z = z_0 + i\varepsilon$, $|z_0| < 1$, $\varepsilon > 0$, so that one of the singularities has been removed.

Lemma 1: Let f(x) and g(y) be $L^2[-1,+1]$ functions which are continuous on $(-1,+1)$. Let $\Phi(x,y) = f(x)g(y)$ and $z = z_0 + i\varepsilon$, $|z_0| < 1$, $\varepsilon > 0$. Then

$$\int_{-1}^{+1} \mathbf{P} \int_{-1}^{+1} \frac{\Phi(x,y)}{(z-y)(y-x)}\, dxdy = \int_{-1}^{+1} \mathbf{P} \int_{-1}^{+1} \frac{\Phi(x,y)}{(z-y)(y-x)}\, dydx.$$

Proof:



let **D** be the diagonal strip bounded by y = x + δ and y = x - δ as shown in the diagram. Let **R** = [-1,+1] × [-1,+1].

Then $\displaystyle\iint\limits_{\mathbf{R-D}} \frac{f(x)g(y)}{(z-y)(y-x)}\,dxdy = \iint\limits_{\mathbf{R-D}} \frac{f(x)g(y)}{(z-y)(y-x)}\,dydx$ because there are no singularities in this region. We are trying to show that

$$\iint\limits_{\mathbf{R-D}} \frac{f(x)g(y)}{(z-y)(y-x)}\,dxdy + \underset{\mathbf{D}}{\int \mathbf{P}\!\int} \frac{f(x)g(y)}{(z-y)(y-x)}\,dxdy =$$

$$\iint\limits_{\mathbf{R-D}} \frac{f(x)g(y)}{(z-y)(y-x)}\,dydx + \underset{\mathbf{D}}{\int \mathbf{P}\!\int} \frac{f(x)g(y)}{(z-y)(y-x)}\,dydx.$$

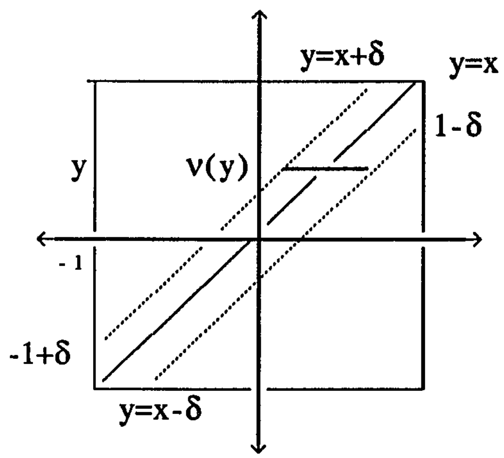
Hence, it suffices to show that

$$\lim_{\delta\to 0} \underset{\mathbf{D}}{\int \mathbf{P}\!\int} \frac{f(x)g(y)}{(z-y)(y-x)}\,dxdy = 0 = \lim_{\delta\to 0} \underset{\mathbf{D}}{\int \mathbf{P}\!\int} \frac{f(x)g(y)}{(z-y)(y-x)}\,dydx.$$

For the first integral, observe that

$$\int_{\mathbf{D}}\mathbf{P}\int \frac{f(x)g(y)}{(z-y)(y-x)}\,dxdy = \int_{-1}^{+1}\mathbf{P}\int_{V(y)} \frac{f(x)g(y)}{(z-y)(y-x)}\,dxdy$$

where $V(y)$ is a horizontal strip in **D,** y units from the x-axis as shown below.



Note that the value of $\mathbf{P}\int_{V(y)} \frac{f(x)}{y-x}\,dx$ depends on the position of y.

$$\mathbf{P}\int_{V(y)} \frac{f(x)}{y-x}\,dx = \begin{cases} \mathbf{P}\int_{-1}^{y+\delta} \frac{f(x)}{y-x}\,dx & \text{if } -1 \le y \le -1+\delta \\[2ex] \mathbf{P}\int_{y-\delta}^{y+\delta} \frac{f(x)}{y-x}\,dx & \text{if } -1+\delta < y < 1-\delta \\[2ex] \mathbf{P}\int_{y-\delta}^{1} \frac{f(x)}{y-x}\,dx & \text{if } 1-\delta \le y \le 1. \end{cases}$$

In each of the three cases we will show that

$$\lim_{\delta \to 0} \int_{-1}^{+1} P \int_{V(y)} \frac{f(x)g(y)}{(z-y)(y-x)} \, dxdy = \lim_{\delta \to 0} \int P \int_{D} \frac{f(x)g(y)}{(z-y)(y-x)} \, dxdy = 0.$$

Case 1: $-1 + \delta \le y \le 1 - \delta$.

$$P \int_{V(y)} \frac{f(x)}{y-x} \, dx = P \int_{y-\delta}^{y+\delta} \frac{f(x)}{y-x} \, dx = P \int_{-1}^{+1} \frac{F_{(\delta,y)}(x)}{y-x} \, dx$$

where $F_{(\delta,y)}(x) = \begin{cases} f(x) & |x-y| \le \delta \\ 0 & |x-y| > \delta \end{cases}$

For a fixed y and $\delta > 0$ arbitrary, $F_{(\delta,y)}(x) \in L^2[-1,+1]$ since

$$\|F_{(\delta,y)}\|_2 = \left\{ \int_{-1}^{+1} |F_{(\delta,y)}(x)|^2 dx \right\}^{1/2} = \left\{ \int_{y-\delta}^{y+\delta} |f(x)|^2 \, dx \right\}^{1/2} \le \|f\|_2.$$

Observe that, since the integral $\int_E |f(x)|^2 dx$ is continuous as a set function of $E \subseteq [-1,+1]$, $\lim_{\delta \to 0} \|F_{(\delta,y)}\|_2 = 0$. Set $F(E_\delta) = \|F_{(\delta,y)}\|_2 = \left\{ \int_{E_\delta} |f(x)|^2 dx \right\}^{1/2}$

where $E_\delta = [y-\delta, y+\delta]$. Since $m(E_\delta) = 2\delta \to 0$ as $\delta \to 0$, $\lim_{\delta\to 0} F(E_\delta) = \lim_{\delta\to 0} \|F_{(\delta,y)}\|_2 = 0$. Furthermore, it is well known that the Hilbert transform is bounded on L^p for any p such that $1 < p < \infty$, with $\|Hf\|_p \le c_p\|f\|_p$ where c_p is a constant independent of $f \in L^p$. Therefore, the finite Hilbert transform of $F_{(\delta,y)}$, $HF_{(\delta,y)}$, is also in $L^2[-1,+1]$ and $\|HF_{(\delta,y)}\|_2 \le c_2\|F_{(\delta,y)}\|_2$ implies that $\lim_{\delta\to 0} \|HF_{(\delta,y)}\|_2 = 0$ as well.

Now
$$\left| \int_{-1}^{+1} \frac{g(y)}{(z-y)} \left[P \int_{\nu(y)} \frac{f(x)}{(y-x)}\, dx \right] dy \right|$$

$$= \left| \int_{-1}^{+1} \frac{g(y)}{(z-y)} \left[P \int_{-1}^{+1} \frac{F_{(\delta,y)}(x)}{(y-x)}\, dx \right] dy \right|$$

$$\le \frac{1}{\varepsilon} \int_{-1}^{+1} \left| g(y)\, P \int_{-1}^{+1} \frac{F_{(\delta,y)}(x)}{(y-x)}\, dx \right| dy, \text{ since } |z-y| \ge \varepsilon,\ z = z_0 + i\varepsilon,$$

$$\le \frac{1}{\varepsilon} \int_{-1}^{+1} \left| g(y)\, HF_{(\delta,y)}(y) \right| dy \le \frac{1}{\varepsilon} \|g\|_2\, \|HF_{(\delta,y)}\|_2 .$$

So we have shown that, for the case $-1 + \delta \le y \le 1 - \delta$,

$$\lim_{\delta \to 0} \left| \int_D \!\! \mathbf{P}\!\!\int \frac{f(x)g(y)}{(z-y)(y-x)}\,dxdy \right| \le \lim_{\delta \to 0} \frac{1}{\varepsilon} \|g\|_2 \, \|\mathbf{H}F_{(\delta,y)}\|_2 = 0.$$

A similar argument may be used for the other two cases. For $-1 \le y \le -1 + \delta$, let

$$F_{(\delta,y)}(x) = \begin{cases} f(x) & -1 \le x \le y + \delta \\ 0 & y + \delta < x \le 1 \end{cases}$$

Again we have $\lim_{\delta \to 0} \|F_{(\delta,y)}\|_2 = 0.$

$$\|F_{(\delta,y)}\|_2 = \left\{ \int_{-1}^{y+\delta} |f(x)|^2\,dx \right\}^{1/2} = \left\{ \int_{E_\delta} |f(x)|^2\,dx \right\}^{1/2} \text{ where } \mathbf{E}_\delta = [-1,\ y+\delta].$$

$m(E_\delta) = y+1+\delta \le 2\delta$ since $-1 \le y \le -1 + \delta$, and the argument would carry through as before.

Similarly, for the case $1-\delta \le y \le 1$, we can write

$$\mathbf{P}\int_{\nu(y)} \frac{f(x)}{y-x}\,dx \text{ as } \mathbf{P}\int_{-1}^{+1} \frac{F_{(\delta,y)}(x)}{y-x}\,dx \quad \text{by setting}$$

$$F_{(\delta,y)}(x) = \begin{cases} f(x) & y - \delta \leq x \leq 1 \\ 0 & -1 \leq x < y - \delta \end{cases}$$

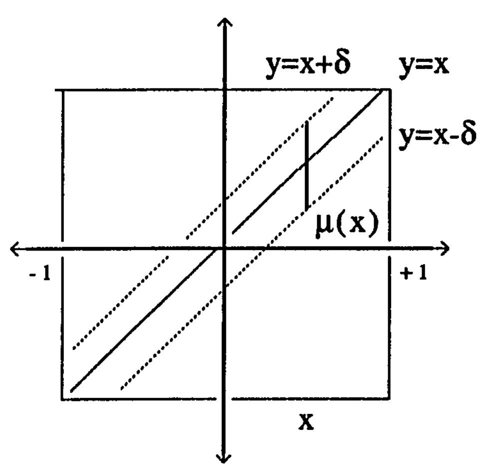
$$\|F_{(\delta,y)}\|_2 = \left\{ \int_{y-\delta}^{+1} |f(x)|^2 \, dx \right\}^{1/2} = \left\{ \int_{E_\delta} |f(x)|^2 dx \right\}^{1/2} \text{ with } m(E_\delta) = [y-\delta,1].$$

$mE_\delta = y-1+\delta \leq 2\delta$ since $1-\delta \leq y \leq 1$, and $\lim_{\delta \to 0} \|F_{(\delta,y)}\|_2 = 0$.

For the second integral, $\int_D \mathbf{P}\int \frac{f(x)g(y)}{(z-y)(y-x)} dydx$, we first use partial fractions to rewrite the integral as

$$\int_D \mathbf{P}\int \frac{f(x)g(y)}{(z-x)(y-x)} dydx + \iint_D \frac{f(x)g(y)}{(z-x)(z-y)} dydx.$$

Let $\mu(x)$ be the vertical line segment in **D** x units for the y-axis as shown in the diagram.



Then $\displaystyle\int\limits_{D} P\int \frac{f(x)g(y)}{(z-y)(y-x)}\,dydx = \int_{-1}^{+1} P\int\limits_{\mu(x)} \frac{f(x)g(y)}{(z-x)(y-x)}\,dydx + \int_{-1}^{+1}\int\limits_{\mu(x)} \frac{f(x)g(y)}{(z-x)(z-y)}\,dydx.$

For the first integral, $\displaystyle\lim_{\delta\to 0}\int_{-1}^{+1} P\int\limits_{\mu(x)} \frac{f(x)g(y)}{(z-y)(y-x)}\,dydx$ can be shown to be zero

by arguing as before but in the y variable: Represent $\displaystyle P\int\limits_{\mu(x)} \frac{g(y)}{(y-x)}\,dy$ as the

finite Hilbert transform of an L^2 function $F_{(\delta,x)}$, such that $\lim_{\delta\to 0}\|F_{(\delta,x)}\|_2 = 0$.

For the second integral, it is obvious that $\displaystyle\lim_{\delta\to 0}\int_{-1}^{+1}\int\limits_{\mu(x)} \frac{f(x)g(y)}{(z-x)(z-y)}\,dydx = 0,$

since $\displaystyle\left|\int_{-1}^{+1}\int\limits_{\mu(x)} \frac{f(x)g(y)}{(z-x)(z-y)}\,dydx\right| \le \frac{1}{\varepsilon^2}\int\int\limits_{D} |f(x)g(y)|\,dxdy < \infty$ and the area of the

region **D** goes to zero as $\delta \to 0$. This proves the Lemma.

Theorem 1: Let $\Phi(x,y) = f(x)g(x)$ where f and g are $L^2[-1,+1]$ functions which are continuous on (-1,+1). Then for almost all $z_0 \in (-1,+1)$,

$$P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{\Phi(x,y)}{(x-y)(y-z_0)}\,dxdy = -\pi^2\Phi(z_0,z_0) + P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{\Phi(x,y)}{(x-y)(y-z_0)}\,dydx.$$

Proof: By Lemma 1, we know that for $z = z_0 + i\varepsilon$, $\varepsilon > 0$,

$$\int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)}{(z-y)(y-x)}\, dxdy = \int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)}{(z-y)(y-x)}\, dydx \qquad \text{or equivalentally,}$$

$$\int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)}{(z-y)(y-x)}\, dxdy + \int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)}{(z-x)(x-y)}\, dydx - \int_{-1}^{+1}\int_{-1}^{+1} \frac{f(x)g(y)}{(z-x)(z-y)}\, dydx \equiv 0$$

We assume, without loss of generality, that f and g are real-valued. Hence, the Real part of the above equation must be zero, or

$$\int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)(z_0-y)}{[(z_0-y)^2+\varepsilon^2](y-x)}\, dxdy + \int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)(z_0-x)}{[(z_0-x)^2+\varepsilon^2](x-y)}\, dydx$$

$$- \int_{-1}^{+1}\int_{-1}^{+1} \frac{f(x)g(y)(z_0-x)(z_0-y)}{[(z_0x)^2)+\varepsilon^2][(z_0-y^2)+\varepsilon^2]}\, dydx + \int_{-1}^{+1}\int_{-1}^{+1} \frac{\varepsilon^2 f(x)g(y)}{[(z_0-x)^2+\varepsilon^2][(z_0-y)^2+\varepsilon^2]}\, dydx \equiv 0.$$

Let $I(z_0,\varepsilon)$, $J(z_0,\varepsilon)$, $K(z_0,\varepsilon)$, and $L(z_0,\varepsilon)$ denote the four above integrals. We will show that

$$\text{(a)} \lim_{\varepsilon\to 0} I(z_0,\varepsilon) = \lim_{\varepsilon\to 0} \int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)(z_0-y)}{[(z_0-y)^2+\varepsilon^2](y-x)}\, dxdy = P\int_{-1}^{+1} P \int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-y)(y-x)}\, dxdy$$

for almost every $z_0 \in (-1,+1)$.

$$\text{(b)} \lim_{\varepsilon\to 0} J(z_0,\varepsilon) = \lim_{\varepsilon\to 0} \int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)(z_0-x)}{[(z_0-x)^2+\varepsilon^2](x-y)}\, dydx = \mathbf{P}\int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-x)(x-y)}\, dydx$$

for almost every $z_0 \in (-1,+1)$.

$$\text{(c)} \lim_{\varepsilon\to 0} K(z_0,\varepsilon) = \lim_{\varepsilon\to 0} \int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)(z_0-x)(z_0-y)}{[(z_0-x)^2+\varepsilon^2][(z_0-y)^2+\varepsilon^2]}\, dydx = \mathbf{P}\int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-x)(z_0-y)}\, dydx$$

for almost every $z_0 \in (-1,+1)$.

$$\text{(d)} \lim_{\varepsilon\to 0} L(z_0,\varepsilon) = \lim_{\varepsilon\to 0} \int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{\varepsilon^2 f(x)g(y)}{[(z_0-x)^2+\varepsilon^2][(z_0-y)^2+\varepsilon^2]}\, dydx = \pi^2 f(z_0)g(z_0)$$

for almost every $z_0 \in (-1,+1)$.

Once we have shown that the above four limits hold, we will have shown that $\lim_{\varepsilon\to 0} [I(z_0,\varepsilon) + J(z_0,\varepsilon) + K(z_0,\varepsilon) + L(z_0,\varepsilon)] =$

$$\mathbf{P}\int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-y)(y-x)}\, dxdy + \mathbf{P}\int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-x)(x-y)}\, dydx$$

$$- \mathbf{P}\int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-x)(z_0-y)}\, dydx + \pi^2 f(z_0)g(z_0) \equiv 0,$$

Rewriting the above equation, we have

$$\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{\Phi(x,y)}{(z_0-y)(y-x)}\,dxdy = -\pi^2\Phi(z_0,z_0) + \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{\Phi(x,y)}{(z_0-y)(y-x)}\,dydx$$

for almost every $z_0 \in (-1,+1)$. This proves the theorem.

In order to prove (a)-(d), we will be using the following results on harmonic functions and their conjugates as presented in *Introduction to Fourier Analysis on Euclidean Spaces* by E. Stein and G. Weiss, which we list here for clarity. (See [12], pp. 47, 186, 218.)

(i) Let $P_\varepsilon(t) = \frac{1}{\pi}\left[\frac{\varepsilon}{t^2 + \varepsilon^2}\right]$ be the Poisson kernel for the upper half-plane of Euclidean space, $\mathbf{R}^2_+$. If $f(t) \in \mathbf{L}^p(-\infty,+\infty)$, $1 \leq p < \infty$, then

$$u(x,\varepsilon) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\,\frac{\varepsilon}{(x-t)^2 + \varepsilon^2}\,dt = (P_\varepsilon * f)(x)$$

is harmonic in $\mathbf{R}^2_+$ and $\lim_{\varepsilon\to 0} u(x,\varepsilon) = f(x)$ for almost every $x \in (-\infty,+\infty)$.

(ii) Let $Q_\varepsilon(t) = \frac{1}{\pi}\left[\frac{t}{t^2 + \varepsilon^2}\right]$ be the conjugate Poisson kernel. Then for

$f(t) \in \mathbf{L}^p(-\infty,+\infty)$, $1 \leq p < \infty$,

$$v(x,\varepsilon) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\,\frac{(x-t)}{(x-t)^2+\varepsilon^2}\,dt = (Q_\varepsilon * f)(x)$$

is harmonic in $\mathbf{R}^2_+$ and $\lim\limits_{\varepsilon\to 0} v(x,\varepsilon)$ exists for almost every $x \in (-\infty,+\infty)$.

Furthermore, $\lim\limits_{\varepsilon\to 0} v(x,\varepsilon) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{+\infty}\frac{f(t)}{(x-t)}\,dt$ for almost all x for which the

limit exists, (i.e., for almost all x).

$$\text{(a)} \lim_{\varepsilon\to 0} I(z_0,\varepsilon) = \lim_{\varepsilon\to 0}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{f(x)g(y)(z_0-y)}{[(z_0-y)^2+\varepsilon^2](y-x)}\,dxdy = \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{f(x)g(y)}{(z_0-y)(y-x)}\,dxdy$$

for almost every $z_0 \in (-1,+1)$.

<u>Proof</u> <u>of</u> <u>(a)</u>: $$I(z_0,\varepsilon) = \int_{-1}^{+1}\frac{g(y)(z_0-y)}{[(z_0-y)^2+\varepsilon^2]}\left\{\mathbf{P}\int_{-1}^{+1}\frac{f(x)}{y-x}\,dx\right\}dy = \int_{-\infty}^{+\infty}\frac{G(y)(z_0-y)}{[(z_0-y)^2+\varepsilon^2]}\,dy$$

where $$G(y) = \begin{cases} g(y)Hf(y) & |y| \leq 1 \\ 0 & |y| > 1 \end{cases}$$ and $$Hf(y) = \mathbf{P}\int_{-1}^{+1}\frac{f(x)}{(y-x)}\,dx.$$

We have seen that $Hf(y) = -\,\mathbf{H}f(y) \in \mathbf{L}^2[-1,+1]$, since the finite Hilbert transform is a bounded operator on $L^2[-1,+1]$. It follows that $G(y) \in \mathbf{L}^1(-\infty,+\infty)$ since $g(y)Hf(y) \in \mathbf{L}^1[-1,+1]$ as the product of two $\mathbf{L}^2$ functions. We have written $I(z_0,\varepsilon)$ as the convolution of an $\mathbf{L}^1(-\infty,+\infty)$ function with π times the conjugate Poisson kernel, $Q_\varepsilon(y) = \dfrac{y}{y^2 + \varepsilon^2}$. Applying Result (ii) from above, we have,

$$\lim_{\varepsilon\to 0} I(z_0,\varepsilon) = \lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \frac{G(y)(z_0-y)}{[(z_0-y)^2+\varepsilon^2]} dy = \mathbf{P}\int_{-\infty}^{+\infty} \frac{G(y)}{(z_0-y)} dy =$$

$$\mathbf{P}\int_{-1}^{+1} \frac{g(y)Hf(y)}{(z_0-y)} dy = \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-y)(y-x)} dxdy, \text{ for almost every } z_0 \in (-1,+1).$$

<u>Proof of (b)</u>: A similar argument applied to $J(z_0,\varepsilon) = \displaystyle\int_{-1}^{+1} \frac{f(x)Hg(x)(z_0-x)}{[(z_0-x)^2+\varepsilon^2]} dx$ can be used to show that $\displaystyle\lim_{\varepsilon\to 0} J(z_0,\varepsilon) = \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-x)(x-y)} dydx$ for almost every $z_0 \in (-1,+1)$.

<u>Proof of (c)</u>:

Let $F(x) = \begin{cases} f(x) & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$ and $G(y) = \begin{cases} g(y) & |y| \le 1 \\ 0 & |y| > 1 \end{cases}$. Then

$$\lim_{\varepsilon\to 0} K(z_0,\varepsilon) = \lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{F(x)G(y)(z_0-x)(z_0-y)}{[(z_0-x)^2+\varepsilon^2][(z_0-y)^2+\varepsilon^2]}\,dydx = \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \frac{f(x)g(y)}{(z_0-x)(z_0-y)}\,dydx$$

for almost every $z_0 \in (-1,+1)$, since F and G are both in $\mathbf{L}^2(-\infty,+\infty)$ and Result (ii) applies.

Proof of (d): This is a simple application of Result (i) regarding the Poisson kernel. $L(z_0,\varepsilon) = \left[g * P_\varepsilon(z_0)\right]\left[f * P_\varepsilon(z_0)\right]$ so that $\lim_{\varepsilon\to 0} L(z_0,\varepsilon) = \pi^2 f(z_0)g(z_0)$ for almost every $z_0 \in (-1,+1)$.

The PBF for Infinite Sums.

We have shown that the Poincaré-Bertrand formula holds for products of $L^2[-1,+1]$ functions which are continuous on $(-1,+1)$, $\Phi(x,y) = f(x)g(y)$. Clearly then, it also holds for finite sums, $\Phi(x,y) = \sum_{i=1}^{N} a_i f_i(x)g_i(y)$. We next consider what restrictions might be placed on an infinite sum, $\Phi(x,y) = \sum_{i=1}^{\infty} a_i f_i(x)g_i(y)$, in order for the Poincaré-Bertrand formula to hold. We will show that sufficient conditions for the $\lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \frac{a_i f_i(x)g_i(y)}{(x-y)(y-z_0)}\,dxdy$ to exist are provided by the hypotheses and conclusion of Mercer's Theorem which is stated below.

Theorem (Mercer): Assume that $\Phi(x,y)$ is symmetric and continuous on $[-1,+1] \times [-1,+1]$, and that the integral operator, $\Phi f(x) = \int_{-1}^{+1} \Phi(x,y)f(y)dy$, determined by $\Phi(x,y)$ is positive, i.e. $< \Phi f,f > \geq 0$ for all $f \in L^2[-1,+1]$. Then $\sum_{i=1}^{\infty} a_i\phi_i(x)\overline{\phi_i(y)}$ converges absolutely and uniformly to $\Phi(x,y)$, where the $\{\phi_i\text{'s}\}$ are the eigenfunctions of the operator, Φ, and the $\{a_i\text{'s}\}$ are the corresponding eigenvalues. (See [10], p. 245.)

We would like to include the case where $\Phi(x,y)$ is discontinuous at the endpoints. To this end, we state a modified version of Mercer's Theorem:

Modified Version of Mercer's Theorem: Let $\Phi(x,y)$ be a symmetric, square-summable function which is continuous on $(-1,+1) \times (-1,+1)$ and satisfies $\int_{-1}^{+1} |\Phi(x,x)|dx \leq M$. (By square-summable, we mean $\int_{-1}^{+1}\int_{-1}^{+1} |\Phi(x,y)|^2 dxdy \leq \infty$.) If the integral operator determined by $\Phi(x,y)$ is positive, then $\sum_{i=1}^{\infty} a_i\phi_i(x)\overline{\phi_i(y)}$ converges absolutely and uniformly to $\Phi(x,y)$ on compact subsets of $(-1,+1) \times (-1,+1)$.

It is easily verified that the proof of Mercer's Theorem remains valid under these assumptions. Note that since $\Phi(x,y)$ is a symmetric, square summable kernel, the eigenfunctions, $a_i\phi_i(x) = \int_{-1}^{+1}\Phi(x,y)\phi_i(y)dy$ are in $L^2[-1,+1]$. The continuity of $\Phi(x,y)$ on $(-1,+1) \times (-1,+1)$ guarantees that the eigenfunctions are continuous on $(-1,+1)$. Let $x \in (-1,+1)$ be arbitrary. For every $\varepsilon > 0$, there exists a positive $\delta = \delta(x)$ such that $|\Phi(x,y) - \Phi(t,y)| < \frac{\varepsilon}{\|\phi_i\|_1}$, if $|x-t| < \delta$. Hence,

$$|\phi_i(x) - \phi_i(t)| \le \int_{-1}^{+1}|\Phi(x,y) - \Phi(t,y)|\,|\phi_i(y)|dy \le \frac{\varepsilon}{\|\phi_i\|_1}\|\phi_i\|_1 \le \varepsilon, \text{ if } |x-t| < \delta.$$

<u>Theorem</u> <u>2</u>: Let $\Phi(x,y) = \sum_{i=1}^{\infty} a_i\phi_i(x)\phi_i(y)$ be a real, symmetric, square-summable function which is continuous on $(-1,+1) \times (-1,+1)$ and satisfies $\int_{-1}^{+1}|\Phi(x,x)|dx < M$. If the integral operator determined by $\Phi(x,y)$ is positive definite, then

$$\lim_{N\to\infty} 2\sum_{i=1}^{N} P\int_{-1}^{+1} P\int_{-1}^{+1} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-y)(y-z_0)}\,dxdy =$$

$$-\pi^2\sum_{i=1}^{\infty} a_i\phi_i(z_0)\phi_i(z_0) + \lim_{N\to\infty}\sum_{i=1}^{N} P\int_{-1}^{+1} P\int_{-1}^{+1} a_i\phi_i(x)\phi_i(y)\frac{1}{(z_0-y)(z_0-x)}\,dydx,$$

where both limits exist for almost all $z_0 \in (-1,+1)$. The $\{\phi_i\text{'s}\}$ are the eigenfunctions of the operator, Φ, and the $\{a_i\text{'s}\}$ are the corresponding eigenvalues.

Proof: It follows from Theorem 1 that, for every N and for almost every z_0,

$$P\int_{-1}^{+1} P\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-y)(y-z_0)}\,dxdy =$$

$$-\pi^2 \sum_{i=1}^{N} a_i\phi_i(z_0)\phi_i(z_0) + \int_{-1}^{+1} P\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-y)(y-z_0)}\,dydx,$$

or using partial fractions, that

$$\sum_{i=1}^{N} P\int_{-1}^{+1} P\int_{-1}^{+1} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-y)(y-z_0)}\,dxdy = -\pi^2 \sum_{i=1}^{N} a_i\phi_i(z_0)\phi_i(z_0) +$$

$$\sum_{i=1}^{N} P\int_{-1}^{+1} P\int_{-1}^{+1} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-z_0)(y-z_0)}\,dydx - \sum_{i=1}^{N} P\int_{-1}^{+1} P\int_{-1}^{+1} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-z_0)(y-x)}\,dydx.$$

Using the symmetry of $\phi_i(x)\phi_i(y)$,

$$P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{a_i\phi_i(x)\phi_i(y)}{(x-z_0)(y-x)}\,dydx = P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{a_i\phi_i(x)\phi_i(y)}{(y-z_0)(x-y)}\,dxdy.$$

It follows that for every N and almost every z_0,

$$2\sum_{i=1}^{N} \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-y)(y-z_0)}\,dxdy \;=$$

$$-\pi^2\sum_{i=1}^{N} a_i\phi_i(z_0)\phi_i(z_0) + \sum_{i=1}^{N}\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\sum_{i=1}^{N} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-z_0)(y-z_0)}\,dydx.$$

Hence, $\lim N\to\infty \left(2\sum_{i=1}^{N} \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-y)(y-z_0)}\,dxdy\right) \;=$

$$\lim N\to\infty \left(-\pi^2\sum_{i=1}^{N} a_i\phi_i(z_0)\phi_i(z_0) + \sum_{i=1}^{N}\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\sum_{i=1}^{N} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-z_0)(y-z_0)}\,dydx\right)$$

for almost all $z_0 \in (-1,+1)$. We will prove the theorem by showing that

$$\lim N\to\infty \left(-\pi^2\sum_{i=1}^{N} a_i\phi_i(z_0)\phi_i(z_0) + \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\sum_{i=1}^{N} a_i\phi_i(x)\phi_i(y)\frac{1}{(x-z_0)(y-z_0)}\,dydx\right)$$

exists for almost all z_0.

By Dini's Theorem we know that, for $z_0 \in (-1,+1)$,

$\lim N\to\infty \left(-\pi^2\sum_{i=1}^{N} a_i\phi_i(z_0)\phi_i(z_0)\right)$ converges uniformly to $\Phi(z_0,z_0)$. To show that

$$\lim_{N\to\infty} \left\{ P\int_{-1}^{+1} P\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(x)\phi_i(y)\frac{1}{(z_0-y)(z_0-x)}\,dydx \right\}$$ exists, we observe that

$$S_N(z_0) = P\int_{-1}^{+1} P\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(x)\phi_i(y)\frac{1}{(z_0-y)(z_0-x)}\,dydx = \pi^2\sum_{i=1}^{N} a_i\left[H\phi_i(z_0)\right]^2,$$ where **H**

again denotes the finite Hilbert transform. We will show that $\left\{S_N(z_0)\right\}$ is a monotonic, increasing sequence of positive $L^1[-1,+1]$ functions such that

$$\int_{-1}^{+1} S_N(z_0)dz_0 \le A,$$ for every N. Hence, the $\lim_{N\to\infty} S_N(z_0)$ must be finite for almost every $z_0 \in (-1,+1)$.

Each ϕ_i is in $L^2[-1,+1]$, so that $H\phi_i(z_0) \in L^2[-1,+1]$ for each i, and $[H\phi_i(z_0)]^2 \in L^1[-1,+1]$. $S_N(z_0) \in L^1[-1,+1]$ as well, as the finite sum of $L^1[-1,+1]$ functions.

Secondly, we know that $< \Phi\phi_i,\phi_i > \ge 0$ for all ϕ_i , since Φ is positive definite by assumption. But $< \Phi\phi_i,\phi_i > = < a_i\phi_i,\phi_i > = a_i$ because the ϕ_i's are orthonormal. Therefore, we have $a_i \ge 0$ for all i and we have shown that $\{S_N(z_0)\}$ is a monotonic increasing sequence of positive $L^1[-1,+1]$ functions. Furthermore, using the boundedness of the Hilbert transform on L^2,

$$\int_{-1}^{+1} S_N(z_0)dz_0 = \int_{-1}^{+1} \sum_{i=1}^{N} a_i\left[H\phi_i(z_0)\right]^2 dz_0 = \sum_{i=1}^{N} a_i\left\|H\phi_i(z_0)\right\|_2^2 \le c_2\sum_{i=1}^{N} a_i\left\|\phi_i(z_0)\right\|_2^2 = c_2\sum_{i=1}^{N} a_i .$$

But $c_2\sum_{i=1}^{N} a_i \le c_2\sum_{i=1}^{\infty} a_i = c_2\int_{-1}^{+1} \sum_{i=1}^{\infty} a_i\, \phi_i^2(x)dx = c_2\int_{-1}^{+1}\Phi(x,x)dx \le 2c_2M.$

Letting $A = 2c_2M$, we have shown that for each N, $\int_{-1}^{+1} S_N(z_0)dz_0 \le A$. Therefore, $\lim_{N\to\infty} S_N(z_0)$ must be finite for almost every $z_0 \in (-1,+1)$, which was to be proved.

Next we will show that if $T(x,y)$ is real, continuous on $(-1,1) \times (-1,+1)$, and skew-symmetric, i.e., $T(x,y) = -T(y,x)$, then the Poincaré-Bertrand formula holds for all $z_0 \in (-1,+1)$. To see this, note that the residual term, $-\pi^2 T(z_0,z_0)$, is zero because of the skew-symmetry of T. Hence, it suffices to show that

$$P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(x,y)}{(x-y)(y-z_0)}\, dxdy = P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(x,y)}{(x-y)(y-z_0)}\, dydx,$$ or using partial fractions,

$$P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(x,y)}{(x-y)(y-z_0)}\, dxdy = P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(x,y)}{(x-z_0)(x-y)}\, dydx + P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(x,y)}{(x-z_0)(y-z_0)}\, dydx.$$

By making the change of variables x=y and then using the skew-symmetry of T, we first observe that

$$\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(x,y)}{(x-y)(y-z_0)}\, dxdy = P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(y,x)}{(y-x)(x-z_0)}\, dydx = P\int_{-1}^{+1} P\int_{-1}^{+1} \frac{T(x,y)}{(x-y)(x-z_0)}\, dydx.$$

It remains to show that $\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{T(x,y)}{(x-z_0)(y-z_0)}\,dydx = 0$. Again making the change of variables x=y and using the skew-symmetry of **T**,

$$\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{T(x,y)}{(x-z_0)(y-z_0)}\,dydx = \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{T(y,x)}{(y-z_0)(x-z_0)}\,dxdy = -\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{T(x,y)}{(x-z_0)(y-z_0)}\,dxdy.$$

Therefore, $\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{T(x,y)}{(x-z_0)(y-z_0)}\,dydx = 0$, which was to be shown.

Note that if $\Phi(x,y)$ is a Hermitian complex kernel, then the real and imaginary parts of $\Phi(x,y)$ are real symmetric and real skew-symmetric kernels, respectively:

$$\Phi(x,y) = \mathrm{Re}\{\Phi(x,y)\} + i\mathrm{Im}\{\Phi(x,y)\} = \overline{\Phi(y,x)} = \mathrm{Re}\{\Phi(y,x)\} - i\{\mathrm{Im}\ \Phi(y,x)\}.$$

Therefore, Theorem 2 holds if $\Phi(x,y)$ is a complex kernel satisfying the hypotheses of the modified version of Mercer's Theorem.

Similarly, if the symmetric part of an arbitrary real continuous kernel, $\Phi(x,y)$, defined on $(-1,+1) \times (-1,+1)$, satisfies the hypotheses of the modified version of Mercer's Theorem, then Theorem 2 holds. To see this, write $\Phi(x,y)$ as $\frac{1}{2}\left\{S(x,y) + T(x,y)\right\}$ where $S(x,y) = \Phi(x,y) + \Phi(y,x)$ is symmetric and $T(x,y) = \Phi(x,y) - \Phi(y,x)$ is skew symmetric. Clearly, if $S(x,y)$ satisfies the hypotheses of Theorem 2, then the Poincaré-Bertrand formula holds for $\Phi(x,y)$, for almost all $z_0 \in (-1,+1)$.

III. Generalization to an Arc in the Plane

In this chapter we prove that Theorems 1 and 2 remain valid if we generalize to the case where Γ is a curve in the plane. $\Phi(x,y)$ becomes a function of two complex variables w and z. By writing $\mathbf{P}\int_\Gamma \mathbf{P}\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dwdz$ in terms of arclength, we transform the integral into a sum of singular and non-singular integrals over the region $[-1,+1] \times [-1,+1]$ where we know that the Poincaré-Bertrand formula holds almost everywhere for products of continuous functions.

Theorem 3: Let Γ be a curve of class $\mathbf{C}^3$ and assume that $\Phi(w,z) = f(w)g(z)$ is the product of two real $\mathbf{L}^2(\Gamma)$ functions, continuous on the interior of Γ. Then for almost every $z_0 \in \Gamma$,

$$\mathbf{P}\int_\Gamma \mathbf{P}\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dwdz = -\pi^2\Phi(z_0,z_0) + \mathbf{P}\int_\Gamma \mathbf{P}\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dzdw.$$

Proof: Assume that Γ is parameterized by arc length, s = t. For $w, z \in \Gamma$,

$$w = w(s) = x(s) + iy(s), \quad dw = [\dot{x}(s) + i\dot{y}(s)]ds,$$

$$z = z(t) = x(t) + iy(t), \quad dz = [\dot{x}(t) + i\dot{y}(t)]dt, \quad -1 \le s,\ t \le +1.$$

In terms of arc length, $\mathbf{P}\int_L \frac{\Phi(w,z)}{(w-z)}\,dw = \mathbf{P}\int_{-1}^{+1} f(s)g(t)\,\frac{\dot{x}(s)+i\dot{y}(s)}{[x(s)-x(t)] + i[y(s)-y(t)]}\,ds,$

so that the real and imaginary parts of $\mathbf{P}\int_{\mathbf{L}} \frac{\Phi(w,z)}{(w-z)}\,dw$ are given by

$$\mathbf{P}\int_{-1}^{+1} f(s)g(t)\,\frac{\dot{x}(s)[x(s)-x(t)] + \dot{y}(s)[y(s)-y(t)]}{[x(s)-x(t)]^2 + [y(s)-y(t)]^2}\,ds$$

$$+\, i\,\mathbf{P}\int_{-1}^{+1} f(s)g(t)\,\frac{\dot{y}(s)[x(s)-x(t)] - \dot{x}(s)[y(s)-y(t)]}{[x(s)-x(t)]^2 + [y(s)-y(t)]^2}\,ds.$$

In order to reduce $\mathbf{P}\int_{\mathbf{L}} \frac{\Phi(w,z)}{(w-z)}\,dw$ to the sum of a singular integral and two ordinary integrals, we rewrite the above as

$$= \mathbf{P}\int_{-1}^{+1} f(s)g(t)\left(\frac{1}{s-t} + \frac{\dot{x}(s)[x(s)-x(t)] + \dot{y}(s)[y(s)-y(t)]}{[x(s)-x(t)]^2 + [y(s)-y(t)]^2} - \frac{1}{s-t}\right)ds$$

$$+\, i\,\mathbf{P}\int_{-1}^{+1} f(s)g(t)\,\frac{\dot{y}(s)[x(s)-x(t)] - \dot{x}(s)[y(s)-y(t)]}{[x(s)-x(t)]^2 + [y(s)-y(t)]^2}\,ds.$$

Let $K(s,t) = \dfrac{\dot{x}(s)[x(s)-x(t)] + \dot{y}(s)[y(s)-y(t)]}{[x(s)-x(t)]^2 + [y(s)-y(t)]^2} - \dfrac{1}{s-t}$

and $L(s,t) = \dfrac{\dot{y}(s)[x(s)-x(t)] - \dot{x}(s)[y(s)-y(t)]}{[x(s)-x(t)]^2 + [y(s)-y(t)]^2}$.

By assumption, $x(s)$ and $y(s)$ are C^3 functions. Therefore, we can write

$x(s) - x(t) = a_1(s-t) + a_2(s-t)^2 + o[(s-t)^2]$ where $a_1 = \dot{x}(t)$, $a_2 = \ddot{x}(t)/2$.

$\dot{x}(s) = a_1 + 2a_2(s-t) + o(s-t)$,

$y(s) - y(t) = b_1(s-t) + b_2(s-t)^2 + o[(s-t)^2]$ where $b_1 = \dot{y}(t)$, $a_2 = \ddot{y}(t)/2$.

$\dot{y}(s) = b_1 + 2b_2(s-t) + o(s-t)$.

Using the fact that $a_1^2 + b_1^2 = 1$, $a_1a_2 + b_1b_2 = 0$ and $a_2^2 + b_2^2 = k^2(t)/4 = c(t)$ where $k(t)$ is the curvature of Γ at t, we have

$$K(s,t) \cong p(s,t) = \frac{c(t)(s-t) + o(s-t)}{1 + c(t)(s-t)^2 + o[(s-t)^2]} \quad \text{and}$$

$$L(s,t) \cong q(s,t) = \frac{a(t) + o(s-t)}{1 + c(t)(s-t)^2 + o[(s-t)]^2]} \quad \text{where}$$

$$a(t) = 2(a_1b_2 - a_2b_1) = \dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t).$$

Note that $p(s,t)$ and $q(s,t)$ are continuous on $[-1,+1] \times [-1,+1]$. Thus we have written $\mathbf{P}\int_\Gamma \frac{\Phi(w,z)}{(w-z)}\,dw$ as the sum of a singular and two nonsingular integrals:

$$\mathbf{P}\int_\Gamma \frac{\Phi(w,z)}{(w-z)}\,dw = \mathbf{P}\int_{-1}^{+1}\frac{f(s)g(t)}{s-t}\,ds + \int_{-1}^{+1} f(s)g(t)p(s,t)ds + i\int_{-1}^{+1} f(s)g(t)q(s,t)ds.$$

Similarly, letting $z_0 = x(t_0) + iy(t_0)$, z_0 a fixed point on Γ, then

$$\mathbf{P}\int_{\Gamma}\frac{dz}{(z-z_0)} = \mathbf{P}\int_{-1}^{+1}\frac{dt}{t-t_0} + \int_{-1}^{+1}p(t,t_0)dt + i\int_{-1}^{+1}q(t,t_0)dt$$

Putting these expressions together we have $\mathbf{P}\int_{\Gamma}\mathbf{P}\int_{\Gamma}\frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dwdz =$

$$\left[\underbrace{\mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{f(s)g(t)}{(s-t)(t-t_0)}\,dsdt}_{\mathbf{I}_1} + \underbrace{\mathbf{P}\int_{-1}^{+1}\int_{-1}^{+1}\frac{f(s)g(t)p(s,t)}{(t-t_0)}\,dsdt}_{\mathbf{I}_2} + \underbrace{\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{f(s)g(t)p(t,t_0)}{(s-t)}\,dsdt}_{\mathbf{I}_3}\right.$$

$$\left. + \underbrace{\int_{-1}^{+1}\int_{-1}^{+1}f(s)g(t)p(s,t)p(t,t_0)dsdt}_{\mathbf{I}_4} - \underbrace{\int_{-1}^{+1}\int_{-1}^{+1}f(s)g(t)q(t,t_0)q(s,t)dsdt}_{\mathbf{I}_5}\right] -$$

$$i\left[\underbrace{\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{f(s)g(t)q(t,t_0)}{(s-t)}\,dsdt}_{\mathbf{I}_6} + \underbrace{\int_{-1}^{+1}\int_{-1}^{+1}f(s)g(t)q(t,t_0)p(s,t)dsdt}_{\mathbf{I}_7} - \right.$$

$$\left.\underbrace{\mathbf{P}\int_{-1}^{+1}\int_{-1}^{+1}\frac{f(s)g(t)q(s,t)}{(t-t_0)}\,dsdt}_{\mathbf{I}_8} + \underbrace{\int_{-1}^{+1}\int_{-1}^{+1}f(s)g(t)p(t,t_0)q(s,t)dsdt}_{\mathbf{I}_9}\right].$$

By Theorem 1, $\mathbf{I}_1 = -\pi^2 f(t_0)g(t_0) + \int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1}\frac{f(s)g(t)}{(s-t)(t-t_0)}\,dtds$ since f and g are $L^2[-1,+1]$ functions which are continuous on (-1,+1). For the remaining integrals, we will show that the order of integration may be reversed without changing the value of the integrals. We first show that $I_2 = \hat{I}_2$ for almost every $t_0 \in [-1,+1]$, where $\hat{I}_j$ is I_j with the order of integration reversed.

$$\text{Let } g_n(t) = \begin{cases} g(t) & |t-t_0| \geq 1/n \\ 0 & |t-t_0| \leq 1/n\,. \end{cases}$$

Then $g_n \in L^2[-1,+1]$ for each n and $\|g_n - g\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. We first observe that

$$I_2 = \int_{-1}^{+1} \frac{g(t)}{(t-t_0)} \left\{ \int_{-1}^{+1} f(s)p(s,t)ds \right\} dt = \lim_{n\to\infty} \int_{-1}^{+1} \frac{g_n(t)}{(t-t_0)} \left\{ \int_{-1}^{+1} f(s)p(s,t)ds \right\} dt,$$ in the

$L^2[-1,+1]$ norm. To see this, note that $Pf(t) = \int_{-1}^{+1} f(s)p(s,t)ds$ is continuous and therefore is in $L^\infty[-1,+1]$. Considering $\{Pf(t)g_n(t)\}$ as elements of $L^2[-1,+1]$,

$\|Pf(t)g(t) - Pf(t)g_n(t)\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, by the boundedness of the Hilbert transform, $\|H[Pf(t)g(t)] - H[Pf(t)g_n(t)]\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore,

$$I_2 = P\int_{-1}^{+1} \frac{g(t)}{(t-t_0)} \left\{ \int_{-1}^{+1} f(s)p(s,t)ds \right\} dt = P\int_{-1}^{+1} \frac{Pf(t)g(t)}{(t-t_0)} dt = \lim_{n\to\infty} P\int_{-1}^{+1} \frac{Pf(t)g_n(t)}{(t-t_0)} dt =$$

$$\lim_{n\to\infty} P\int_{-1}^{+1} \frac{g_n(t)}{(t-t_0)} \left\{ \int_{-1}^{+1} f(s)p(s,t)ds \right\} dt,$$ as asserted. Hence, there is a subsequence

g_{n_k} such that $I_2 = \mathbf{P}\int_{-1}^{+1} \frac{g(t)}{(t-t_0)} \left\{ \int_{-1}^{+1} f(s)p(s,t)ds \right\} dt =$

$$\lim_{n_k \to \infty} \mathbf{P}\int_{-1}^{+1} \frac{g_{n_k}(t)}{(t-t_0)} \left\{ \int_{-1}^{+1} f(s)p(s,t)ds \right\} dt,$$ for almost every t_0. By

Fubini's Theorem, we can reverse the order of integration in this last integral

and $$\lim_{n_k \to \infty} \int_{-1}^{+1} \frac{g_{n_k}(t)}{(t-t_0)} \left\{ \int_{-1}^{+1} f(s)p(s,t)ds \right\} dt = \lim_{n_k \to \infty} \int_{-1}^{+1} f(s) \left\{ \int_{-1}^{+1} \frac{g_{n_k}(t)p(s,t)}{(t-t_0)} dt \right\} ds.$$

Thus it remains to show that

$$\lim_{n_k \to \infty} \int_{-1}^{+1} f(s) \left\{ \int_{-1}^{+1} \frac{g_{n_k}(t)p(s,t)}{(t-t_0)} dt \right\} ds = \int_{-1}^{+1} f(s) \left\{ \int_{-1}^{+1} \frac{g(t)p(s,t)}{(t-t_0)} dt \right\} ds = \hat{I}_2.$$

It suffices to show that $\lim_{n_k \to \infty} \left\{ \int_{-1}^{+1} f(s) \left\{ \int_{-1}^{+1} \frac{[g(t)-g_{n_k}(t)]p(s,t)}{(t-t_0)} dt \right\} ds \right\} = 0$ almost everywhere, or that

$$\lim_{n_k \to \infty} \left\{ \int_{-1}^{+1} f(s) \left\{ \int_{-1}^{+1} \frac{[g(t)-g_{n_k}(t)][p(s,t)-p(s,t_0)]}{(t-t_0)} dt \right\} ds + \int_{-1}^{+1} f(s)p(s,t_0) \left\{ \int_{-1}^{+1} \frac{g(t)-g_{n_k}(t)}{(t-t_0)} dt \right\} ds \right\} = 0.$$

We will show that each integral goes to zero separately. For the first integral, observe that, as a function of t, $p(s,t) = \dfrac{c(t)(s-t) + o(s-t)}{1 + c(t)(s-t)^2 + o[(s-t)^2]}$ has bounded derivative in [-1,+1] since $2c(t) = [\ddot{x}(t)]^2 + [\ddot{y}(t)]^2$ and $\ddot{x}(t)$ and $\ddot{y}(t)$ are assumed to have continuous derivatives. Therefore, $\left|\dfrac{p(s,t)-p(s,t_0)}{t-t_0}\right| \leq A$, A a positive constant. Therefore,

$$\left|\int_{-1}^{+1} f(s)\left\{P\int_{-1}^{+1}\frac{[g(t)-g_{n_k}(t)][p(s,t)-p(s,t_0)]}{(t-t_0)}\,dt\right\}ds\right| \leq A\int_{-1}^{+1}\int_{-1}^{+1}|f(s)|\,|g(t)-g_{nk}(t)|\,dt\,ds$$

$$\leq A\|f\|_1\|g-g_{n_k}\|_1 \rightarrow 0 \text{ as } n_k \rightarrow \infty.$$

For the second integral, we use the convergence of $Hg_{n_k} \rightarrow Hg$ in the L^2 norm.

$$\left|\int_{-1}^{+1} f(s)p(s,t_0)\left\{P\int_{-1}^{+1}\frac{g(t)-g_n(t)}{(t-t_0)}\,dt\right\}ds\right| \leq \|fp_{t_0}\|_1\,|Hg(t_0) - Hg_{n_k}(t_0)| \text{ where}$$

$p_{t_0} = p(\cdot,t_0)$. $\|g - g_{n_k}\|_2 \rightarrow 0 \Rightarrow \|Hg - Hg_{n_k}\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Since the sequence of L^2 functions, Hg_{n_k}, converges to Hg in the L^2 norm, there is a subsequence, $Hg_{n_{k_j}}$, which converges to Hg almost everywhere, i.e., $|Hg(t_0) - Hg_{n_{k_j}}(t_0)| \rightarrow 0$ as $j \rightarrow \infty$ for almost every $t_0 \in [-1,+1]$. Convergence of the subsequence is sufficient because the previous limits would hold for any subsequence of the g_n's.

To summarize, we have shown that, for almost every t_0,

$$\int_{-1}^{+1} P\int_{-1}^{+1} \frac{f(s)g(t)p(s,t)}{(t-t_0)}\,dsdt = \lim_{n_{k_j}\to\infty} \int_{-1}^{+1} P\int_{-1}^{+1} \frac{f(s)g_{n_{k_j}}(t)p(s,t)}{(t-t_0)}\,dsdt =$$

$$\lim_{n_{k_j}\to\infty} P\int_{-1}^{+1}\int_{-1}^{+1} \frac{f(s)g_{n_{k_j}}(t)p(s,t)}{(t-t_0)}\,dtds = P\int_{-1}^{+1}\int_{-1}^{+1} \frac{f(s)g(t)p(s,t)}{(t-t_0)}\,dtds, \text{ so that } I_2 = \hat{I}_2.$$

One can use a similar argument to show that $I_8 = \hat{I}_8$ for almost every $t_0 \in (-1,+1)$ because $\left| \frac{q(s,t)-q(s,t_0)}{t-t_0} \right|$ is bounded and $q(s,t)$ is continuous on $[-1,+1] \times [-1,+1]$.

It is well known that under certain conditions, the singular integral commutes with the ordinary integral. In particular, if $F \in L^p[-1,+1]$, $G \in L^q[-1,+1]$, $1 < p < \infty$, $1/p + 1/q = 1$, then

$$\int_{-1}^{+1} P\int_{-1}^{+1} \frac{G(t)F(s)}{(s-t)}\,dsdt = P\int_{-1}^{+1}\int_{-1}^{+1} \frac{G(t)F(s)}{(s-t)}\,dtds.$$

(See, for example, [7], p. 59.)

We use this result to show that $I_3 = \hat{I}_3$. Let $f(s) = F(s)$ and $g(t)p(t,t_0) = G_{t_0}(t)$. For any fixed t_0, $G_{t_0}(t) \in L^2[-1,+1]$ as a function of t. The hypotheses of the above are satisfied and $I_3 = \hat{I}_3$ for all $t_0 \in (-1,+1)$.

Similarly, letting $f(s) = F(s)$ and $g(t)q(t,t_0) = G_{t_0}(t)$, then the hypotheses of the above are satisfied and $I_6 = \hat{I}_6$ for all $t_0 \in (-1,+1)$.

The integrand of I_4 is the product of continuous functions of s and t. Fubini's Theorem applies and $I_4 = \hat{I}_4$. The same is true for I_5, I_7, and I_9. Thus, we have shown that, for almost every $t_0 \in (-1,+1)$,

$$P\int_\Gamma P\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dwdz = \sum_{j=1}^{5} I_j + i\sum_{j=6}^{9} I_j = -\pi^2 f(t_0)g(t_0) + \sum_{j=1}^{5}\hat{I}_j + i\sum_{j=6}^{9}\hat{I}_j.$$

Going back through the change of variables, $z = z(t) = x(t) + iy(t)$, $z_0 = z(t_0) = x(t_0) + iy(t_0)$, $w = w(s) = x(s) + iy(s)$, we have for almost all $z_0 \in \Gamma$,

$$P\int_\Gamma P\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dwdz = -\pi^2\Phi(z_0,z_0) + P\int_\Gamma P\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dzdw.$$

We next prove the corresponding generalization for Theorem 2, which follows almost immediately from Theorems 2 and 3.

<u>Theorem</u> <u>4</u>: Let Γ be a curve of class C^3. Let $\Phi(w,z) = \sum_{i=1}^{\infty} a_i\phi_i(w)\phi_i(z)$ be a real, symmetric,square-summable kernel on $\Gamma \times \Gamma$, continuous on the interior of $\Gamma \times \Gamma$ and satisfying $\int_\Gamma |\Phi(w,w)|\,|dw| < M$. If the integral operator determined by Φ is positive definite, then

$$\lim_{N\to\infty} 2\sum_{i=1}^{N} \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} a_i\phi_i(w)\phi_i(z)\frac{1}{(w-z)(z-z_0)}\, dwdz =$$

$$-\pi^2\sum_{i=1}^{\infty} a_i\phi_i(z_0)\phi_i(z_0) + \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{P}\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} a_i\phi_i(w)\phi_i(z)\frac{1}{(w-z)(z-z_0)}\, dzdw,$$

where both limits exist for almost all $z_0 \in \Gamma$. The $\{\phi_i\text{'s}\}$ are the eigenfunctions of the operator, Φ, and the $\{a_i\text{'s}\}$ are the corresponding eigenvalues.

Proof: Since $\Phi(w,z)$ satisfies the Modified Version of Mercer's Theorem on compact subsets of $\Gamma \times \Gamma$, we know that $\Phi(w,z) = \sum_{i=1}^{\infty} a_i\phi_i(w)\phi_i(z)$, where the $\{\phi_i\text{'s}\}$ are the eigenfunctions of the integral operator determined by $\Phi(w,z)$ and the $\{a_i\text{'s}\}$ are the corresponding eigenvalues. Convergence of the series is absolute and uniform on compact subsets of $\Gamma \times \Gamma$. Furthermore, we know that the a_i's are positive and that $\sum_{i=1}^{\infty} a_i$ converges.

It follows from Theorem 3, that for any N and for almost every $z_0 \in \Gamma$,

$$(*) \quad \mathbf{P}\int_{\Gamma}\mathbf{P}\int_{\Gamma} \sum_{i=1}^{N} a_i\phi_i(w)\phi_i(z)\frac{1}{(w-z)(z-z_0)}\, dwdz = -\pi^2\sum_{i=1}^{N} a_i\phi_i(z_0)\phi_i(z_0) +$$

$$\mathbf{P}\int_{\Gamma}\mathbf{P}\int_{\Gamma} \sum_{i=1}^{N} a_i\phi_i(w)\phi_i(z)\frac{1}{(w-z)(z-z_0)}\, dzdw.$$

Using the arc length change of variables, w = w(s) = x(s) + iy(s), z = z(t) = x(t) + iy(t) as in Theorem 3, we know that the left hand side of the equation can be written as $\sum_{j=1}^{9} I_{j_N}$ where the integrals, I_{j_N}, are defined below:

$$\mathbf{P}\int_{\Gamma} \mathbf{P}\int_{\Gamma} \sum_{i=1}^{N} a_i\phi_i(w)\phi_i(z)\frac{1}{(w-z)(z-z_0)}\,dwdz = \Bigg[\underbrace{\mathbf{P}\int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)\frac{1}{(s-t)(t-t_0)}\,dsdt}_{I_{1_N}}$$

$$+ \underbrace{\mathbf{P}\int_{-1}^{+1} \int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)\frac{p(s,t)}{(t-t_0)}\,dsdt}_{I_{2_N}} + \underbrace{\int_{-1}^{+1} \mathbf{P}\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)\frac{p(t,t_0)}{(s-t)}\,dsdt}_{I_{3_N}}$$

$$+ \underbrace{\int_{-1}^{+1} \int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)p(s,t)p(t,t_0)dsdt}_{I_{4_N}} - \underbrace{\int_{-1}^{+1}\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)q(s,t)q(t,t_0)dsdt}_{I_{5_N}}\Bigg]$$

$$+ i \Bigg(\underbrace{\mathbf{P}\int_{-1}^{+1} \int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)\frac{q(t,t_0)}{(s-t)}\,dsdt}_{I_{6_N}} + \underbrace{\int_{-1}^{+1} \int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)p(s,t)q(t,t_0)dsdt}_{I_{7_N}}$$

$$- \underbrace{\mathbf{P}\int_{-1}^{+1} \int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)\frac{q(s,t)}{(t-t_0)}\,dsdt}_{I_{8_N}} + \underbrace{\int_{-1}^{+1} \int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)q(s,t)p(t,t_0)dsdt}_{I_{9_N}}\Bigg].$$

It also follows from Theorem 3 that for each $j \neq 1$, $I_{j_N} = \hat{I}_{j_N}$, while $I_{1_N} = \hat{I}_{1_N} + \sum_{i=1}^{N} a_i\phi_i(t_0)\phi_i(t_0)$, where $\hat{I}_{j_N} = I_{j_N}$ with the order of integration reversed. Hence, another way of writing (*) is $\sum_{j=1}^{9} I_{j_N} = \sum_{i=1}^{N} a_i\phi_i(t_0)\phi_i(t_0) + \sum_{j=1}^{9} \hat{I}_{j_N}$. We will show that, for almost all $t_0 \in (-1,+1)$ and for each j, $\lim_{N\to\infty} I_{j_N}$ exists. Going back through the arc length change of variables and using the uniform convergence of $\sum_{i=1}^{\infty} a_i\phi_i(t_0)\phi_i(t_0)$, we have

$$P\int_\Gamma P\int_\Gamma \sum_{i=1}^{\infty} a_i\phi_i(w)\phi_i(z)\frac{1}{(w-z)(z-z_0)}\,dwdz = -\pi^2\sum_{i=1}^{\infty} a_i\phi_i(z_0)\phi_i(z_0) + P\int_\Gamma P\int_\Gamma \sum_{i=1}^{\infty} a_i\phi_i(w)\phi_i(z)\frac{1}{(w-z)(z-z_0)}\,dzdw, \text{ or}$$

$$P\int_\Gamma P\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dwdz = -\pi^2\Phi(z_0,z_0) + P\int_\Gamma P\int_\Gamma \frac{\Phi(w,z)}{(w-z)(z-z_0)}\,dzdw,$$ which proves the theorem.

The existence of $\lim_{N\to\infty} I_{1_N}$ follows from Theorem 2. We only need observe that the ϕ_i's are orthonormal in $\mathbf{L}^2[-1,+1]$ since

$$\int_{-1}^{+1} |\phi_i(s)\phi_j(s)|\,ds = \int_{-1}^{+1} |\phi_i(s)\phi_j(s)|\,|\dot{x}(s) + i\dot{y}(s)|\,ds = \int_{\Gamma} |\phi_i(w)\phi_j(w)|\,|dw| = \delta_{ij}.$$

Convergence of $\sum_{i=1}^{\infty} a_i \|H\phi_i\|_2^2 \le c_2^2 \sum_{i=1}^{\infty} a_i < \infty$ in the $L^2[-1,+1]$ norm follows, which is sufficient to show that $P\int_{-1}^{+1} P\int_{-1}^{+1} \sum_{i=1}^{\infty} a_i\phi_i(s)\phi_i(t)\frac{1}{(s-t)(t-t_0)}\,dsdt$ exists and is equal to $-\pi^2\sum_{i=1}^{\infty} a_i\phi_i^2(t_0) + P\int_{-1}^{+1} P\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)\frac{1}{(s-t)(t-t_0)}\,dtds$, as in the proof of Theorem 2.

To show that $\lim_{N\to\infty} I_{2_N}$ exists, let $G_i(t_0) = P\int_{-1}^{+1} g_i(t)\frac{\phi_i(t)}{t-t_0}\,dt$ where

$g_i(t) = \int_{-1}^{+1} p(s,t)\phi_i(s)ds.$ Note that since $p(s,t)$ is continuous on $[-1,+1]\times[-1,+1]$, it follows that $g_i(t)$ is continuous on $[-1,+1]$. Let $\omega(\delta)$ be the modulus of continuity of $p(s,t)$ as a function of t. Then

$$|g_i(t) - g_i(t_0)| \le \int_{-1}^{+1} |p(s,t) - p(s,t_0)|\,|\phi_i(s)|\,ds \le \omega(|t-t_0|)\|\phi_i\|,$$

$\lim_{t\to t_0} |g_i(t) - g_i(t_0)| \le \lim_{t\to t_0} \omega(|t-t_0|)\|\phi_i\| = 0$ and $g_i(t)$ is continuous on $[-1,+1]$.

Furthermore, since the ϕ_i's are orthornormal and $\omega(|t|)$ is bounded, it follows that $g_i(t)$ is bounded, independent of i. Consequently,

$|G_i(t_0)| = P\int_{-1}^{+1} g_i(t)\frac{\phi_i(t)}{t-t_0}\,dt \le C\|H\phi_i\|_1$ where C is a bound for $g_i(t)$. But we have seen that $\|H\phi_i\|_1 \leq 2c_2\|\phi_i\|_2$ so that $|G_i(t_0)| \leq \hat{C}$, $\hat{C} = 2c_2c$. Therefore,

$$\sum_{i=1}^{\infty} a_i|G_i(t_0)| \leq \hat{C}\sum_{i=1}^{\infty} a_i \text{ converges, and } \lim_{N\to\infty} I_{2_N} = \mathbf{P}\int_{-1}^{+1}\int_{-1}^{+1} \sum_{i=1}^{\infty} a_i\phi_i(s)\phi_i(t)\frac{p(s,t)}{(t-t_0)}\,dsdt$$

$= \sum_{i=1}^{\infty} a_iG_i(t_0)$ exists for every $t_0 \in (-1,+1)$. A similar argument could be used to

show that $\lim_{N\to\infty} I_{8_N}$ exists.

To show that $\lim_{N\to\infty} I_{3_N}$ exists, we simply use the fact that $p(t,t_0)$ is continuous and then apply Schwarz' Inequality: $\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)\frac{p(t,t_0)}{(s-t)}\,dsdt$

$= \sum_{i=1}^{N} a_iG_i(t_0)$ where $G_i(t_0) = \int_{-1}^{+1} p(t,t_0)\phi_i(t)H\phi_i(t)dt$. $|G_i(t_0)| \leq a_iM\|\phi_i\|_2\|H\phi_i\|_2$

$= Ca_i$. Therefore, $\sum_{i=1}^{\infty} a_iG_i(t_0)$ converges absolutely which proves that

$\int_{-1}^{+1}\mathbf{P}\int_{-1}^{+1} \sum_{i=1}^{\infty} a_i\phi_i(s)\phi_i(t)\frac{p(t,t_0)}{(s-t)}\,dsdt$ exists. A similar argument can be used to show that $\lim_{N\to\infty} I_{6_N}$ exists.

The convergence of $\lim_{N\to\infty} I_{j_N}$, $j = 4,5,7,9$, is an easy consequence of the

continuity of the kernels $p(s,t)$ and $q(s,t)$ together with the convergence of

$\sum_{i=1}^{\infty} a_i$. For example, $|I_{4_N}| \leq \left| \int_{-1}^{+1}\int_{-1}^{+1} \sum_{i=1}^{N} a_i\phi_i(s)\phi_i(t)p(s,t)p(t,t_0)dsdt \right| \leq$

$\leq M^2 \sum_{i=1}^{N} a_i \|\phi_i\|_1^2 \leq 2M^2 \sum_{i=1}^{N} a_i$, where $M = \max|p(s,t)|$, $-1 < s,t < +1$.

The convergence of $\sum_{i=1}^{\infty} a_i$ guarantees that $\lim_{N\to\infty} I_{4_N}$ exists.

IV. Applications

We first derive an inversion formula for the finite Hilbert transform by applying the Poincaré-Bertrand formula to the function $\Phi(x,y) = \sqrt{1-y^2}\, f(x)$, where $f(x)$ is an $L^2[-1,+1]$ function which is continuous on $(-1,+1)$. $\Phi(x,y)$ satisfies the hypotheses of Theorem 1 and the Poincaré-Bertrand formula holds almost everywhere:

$$\frac{1}{\pi^2} P\int_{-1}^{+1} \frac{\sqrt{1-y^2}}{y-z_0} \left\{ P\int_{-1}^{+1} \frac{f(x)}{x-y}\, dx \right\} dy = -\sqrt{1-z_0^2}\, f(z_0) + \frac{1}{\pi^2} P\int_{-1}^{+1} f(x) \left\{ P\int_{-1}^{+1} \frac{\sqrt{1-y^2}}{(y-z_0)(x-y)}\, dy \right\} dx,$$

for almost all $z_0 \in (-1,+1)$. Note that the left hand side of the equation can be written as $\frac{1}{\pi} P\int_{-1}^{+1} \frac{\sqrt{1-y^2}\, \mathbf{H}f(y)}{y-z_0}\, dy$ where **H** denotes the finite Hilbert transform.

For the right hand side, using partial fractions we have

$$\frac{1}{\pi^2} P\int_{-1}^{+1} f(x) \left\{ P\int_{-1}^{+1} \frac{\sqrt{1-y^2}}{(y-z_0)(x-y)}\, dy \right\} dx = \frac{1}{\pi^2} P\int_{-1}^{+1} \frac{f(x)}{x-z_0} \left\{ P\int_{-1}^{+1} \frac{\sqrt{1-y^2}}{y-x} - \frac{\sqrt{1-y^2}}{y-z_0}\, dy \right\} dx.$$

$$= \frac{1}{\pi} P\int_{-1}^{+1} \frac{f(x)}{x-z_0} \left\{ -x + z_0 \right\} dx = -\frac{1}{\pi} \int_{-1}^{+1} f(x) dx,$$

where we have used the identity, given in Chapter II,

$\frac{1}{\pi} P \int_{-1}^{+1} \frac{\sqrt{1-y^2}\; U_{n-1}(y)}{y-x} dy = -T_n(x)$ with $n = 1$. Combining these results, it follows that

$$\frac{1}{\pi} P \int_{-1}^{+1} \frac{\sqrt{1-y^2}\; Hf(y)}{y-z_0} dy = -\sqrt{1-z_0^2}\; f(z_0) - \frac{1}{\pi}\int_{-1}^{+1} f(x)dx, \text{ or more explicitly,}$$

$$f(z_0) = -\frac{1}{\pi^2} P \int_{-1}^{+1} \frac{\sqrt{1-y^2}\; Hf(y)}{\sqrt{1-z_0^2}\,(y-z_0)} dy - \frac{C}{\sqrt{1-z_0^2}}, \text{ where } C = \frac{1}{\pi}\int_{-1}^{+1} f(x)dx.$$

Assuming that $\langle f,1 \rangle = 0$ or $\int_{-1}^{+1} f(x)dx = 0$, we have an inversion formula for $L^2[-1,+1]$ functions, f, which are continuous on (-1,+1) and orthogonal to 1:

If $g(y) = Hf(y) = \frac{1}{\pi} P \int_{-1}^{+1} \frac{f(x)}{x-y} dx$, then

$$f(z_0) = -\frac{1}{\pi} P \int_{-1}^{+1} \frac{\sqrt{1-y^2}\; g(y)}{\sqrt{1-z_0^2}\,(y-z_0)} dy .$$

Next we apply the inversion formula to solve the Neumann problem for Laplace's equation in the exterior of the interval [-1,+1] in the plane. We will construct a function, $u(x,y)$, is harmonic in $\mathbb{C} - [-1,+1]$, such that the limiting value of the normal derivative of u, $\lim_{y \to 0} \frac{\partial u}{\partial y}(x,y) = g(x)$, where $g(x)$ is a prescribed function, continuous on $[-1,+1]$.

Let $u(x,y) = \frac{1}{\pi} \int_{-1}^{+1} h(\xi) \frac{\partial}{\partial \eta} \ln\sqrt{(\xi-x)^2 + (\eta-y)^2}\,\Big|_{\eta=0} d\xi$, for $(x,y) \in \mathbb{C} - [-1,+1]$, where $h(\xi)$ is differentiable on $(-1,+1)$ and satifies $h(-1) = 0 = h(+1)$. Note that $u(x,y)$ is a double layer potential with kernel $\mathbf{K}(p,q) = \frac{1}{\pi} \frac{\partial}{\partial \eta} \ln|p-q|\,\Big|_{\eta=0}$, where $p = (\xi,\eta)$, $q = (x,y)$, $|x| \le 1$.

Differentiating with respect to η and evaluating at 0, we see that

$$u(x,y) = \frac{1}{2\pi} \int_{-1}^{+1} h(\xi) \frac{-y}{(\xi-x)^2+y^2}\, d\xi.$$

Integrating by parts and using the assumption that $h(-1) = 0 = h(1)$,

$$u(x,y) = -\frac{1}{2\pi} h(\xi) \tan^{-1}\left(\frac{\xi-x}{y}\right)\Bigg|_{-1}^{+1} + \frac{1}{2\pi} \int_{-1}^{+1} \dot{h}(\xi) \tan^{-1}\left(\frac{\xi-x}{y}\right) d\xi$$

$$= \frac{1}{2\pi} \int_{-1}^{+1} \dot{h}(\xi) \tan^{-1}\left(\frac{\xi-x}{y}\right) d\xi.$$

We assume that $\dot{h}$ exists and is in $\mathbf{L}^2[-1,+1]$.

For $|x| < 1$ we can compute the normal derivative, u_N, of $u(x,y)$ by differentiating under the integral sign (See [4], p. 218).

$$u_N(x,y) = \frac{\partial}{\partial y} u(x,y) = \frac{1}{2\pi}\frac{\partial}{\partial y}\int_{-1}^{+1} \dot{h}(\xi) \tan^{-1}\left(\frac{\xi-x}{y}\right) d\xi = -\frac{1}{2\pi}\int_{-1}^{+1} \dot{h}(\xi) \frac{(\xi-x)}{(\xi-x)^2 + y^2} d\xi.$$

Applying (ii), pp. 24 - 25, it follows that, for almost every $x \in (-1,+1)$,

$$\lim_{y \to 0} u_N(x,y) = \lim_{y \to 0} -\frac{1}{2\pi}\int_{-1}^{+1} \dot{h}(\xi) \frac{(\xi-x)}{(\xi-x)^2 + y^2} d\xi = -\frac{1}{2\pi}\int_{-1}^{+1} \frac{\dot{h}(\xi)}{\xi-x} d\xi.$$

We have constructed a function, $u(x,y) = \frac{1}{2\pi}\int_{-1}^{+1} h(\xi) \frac{\partial}{\partial \eta} \ln\sqrt{(\xi-x)^2 + (\eta-y)^2}\,\Big|_{\eta=0} d\xi$,

harmonic on $\mathbb{C}$ - [-1,+1], with $\lim_{y \to 0} u_N(x,y) = -\frac{1}{2\pi}\int_{-1}^{+1} \frac{\dot{h}(\xi)}{\xi-x} d\xi$, almost everywhere.

Let $\dot{h}(\xi) = \frac{2}{\pi} P\int_{-1}^{+1} \frac{\sqrt{1-x^2}\ \ g(x)}{\sqrt{1-\xi^2}\ \ (x-\xi)} dx$. Note that $\int_{-1}^{+1} \dot{h}(\xi) d\xi = 0$. Therefore, there is

a continuous functions $h(\xi)$ such that $\dot{h}(\xi) = \frac{d}{d\xi} h(\xi)$ and $h(+1) = 0 = h(-1)$. For

almost every $x \in (-1,+1)$, $\lim_{y \to 0} u_N(x,y) = g(x)$ and $u(x,y)$ solves the stated Neumann problem.

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