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COMPUTABILITY OF HOMOTOPY GROUPS

OF NILPOTENT COMPLEXES

by

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INTRODUCTION

In 1957, Brown [1] proved that the higher homotopy groups of a connected, simply connected, finite simplicial complex are effectively computable in the sense that, for each $n \geqslant 2$, there is an effective procedure for obtaining a finite presentation of $\pi_n X$ as an abelian group.

Brown's method relies on the fact that if X is simply connected, the nth fibration of the Postnikov tower of X is induced from a principal fibration over a $K(\pi_n X;n+1)$ with fiber $K(\pi_n X;n)$.

Now let

$$\Gamma_1\pi_n X = \pi_n X \geqslant \Gamma_2\pi_n X \geqslant \ \ldots \ \geqslant \Gamma_i\pi_n X \geqslant \ \ldots$$

denote the lower central series of the action of $\pi_1 X$ on $\pi_n X$. When X is nilpotent, the Postnikov tower of X can be refined in such a way that the nth fibration is written as a composition of principal fibrations with fibers $K(\Gamma_i\pi_n X/\Gamma_{i+1}\pi_n X;n)$ over the complexes $K(\Gamma_i\pi_n X/\Gamma_{i+1}\pi_n X;n+1)$. In light of this fact, it is natural to ask whether Brown's result can be generalized to nilpotent complexes.

In this paper we answer the question affirmatively for connected, nilpotent simplicial sets which are finite in each dimension. Precisely, we show that the Postnikov tower of X is computable with the groups $\Gamma_i\pi_n X/\Gamma_{i+1}\pi_n X$ given effectively as finitely generated abelian groups, and that for $n \geqslant 2$, $\pi_n X$ can be recursively enumerably presented as an abelian group.

We work, as Brown did, in the category of simplicial sets. Familiarity with this category is assumed. Some basic constructions and properties are set forth in Chapter I. More complete expositions can be found in May [4], or in Gabriel and Zisman [2].

In Chapter II we construct, inductively, the Postnikov tower of a connected, nilpotent simplicial set.

We define recursive simplicial sets and maps in Chapter III. We then show that the Postnikov tower of a connected, nilpotent simplicial set X, which is finite in each dimension, is effectively computable. That is to say, all complexes and maps are recursive. As a corollary we deduce that $\pi_n X$ is finitely generated for $n \geqslant 2$.

The crux of the argument involves showing that there is an effective way of presenting the groups $\Gamma_i \pi_n X / \Gamma_{i+1} \pi_n X$ as finitely generated abelian groups. The proof is long and technical and forms the body of Chapter IV.

Finally, in Chapter V, we prove the following theorem. Let X be a connected, nilpotent simplicial set which is finite in each dimension. Then $\pi_n X$ has an r.e. abelian group presentation.

Chapter I

PRELIMINARIES

Let SS denote the category of simplicial sets and top the category of topological spaces.

Definition 1.1. A simplicial set X is a graded set together with face operators

$$d_i : X^n \to X^{n-1}, \quad 0 \leqslant i \leqslant n$$

and degeneracy operators

$$s_i : X^n \to X^{n+1}, \quad 0 \leqslant i \leqslant n$$

which satisfy the commutativity relationships

$$d_i d_j = d_{j-1} d_i \text{ if } i < j,$$
$$s_i s_j = s_{j+1} s_i \text{ if } i \leqslant j,$$
$$d_i s_j = s_{j-1} d_i \text{ if } i < j,$$
$$d_j s_j = \text{identity} = d_{j+1} s_j,$$
$$d_i s_j = s_j d_{i-1} \text{ if } i > j+1.$$

We shall often refer to a simplicial set as a complex.

Definition 1.2. A simplicial map $f : X \to Y$ is a map of degree zero of graded sets which commutes with face and degeneracy operators.

Definition 1.3. A simplicial set X is called locally finite if the set of n-simplices X^n is finite for all $n \geqslant 0$.

$\Delta[q]$ is the simplicial q-simplex. The n-simplices of $\Delta[q]$ are sequences of integers $(a_0, a_1, \ldots, a_n)$ with

$$0 \leqslant a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant q,$$

$$d_i(a_0, \ldots, a_n) = (a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

$$s_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i, a_i, \ldots, a_n).$$

The single nondegenerate q-simplex of $\Delta[q]$, $(0, 1, 2, \ldots, q)$, will be written Δq.

$\Delta[1]$ will often be written as I.

Let X be a simplicial set. Let $C_n(X)$ be the free abelian group on the set of n-simplices of X, X^n. Let

$$\partial : C_n(X) \to C_{n-1}(X)$$

be given by

$$\partial(x) = \sum_{i=o}^{n} (-1)^i d_i x.$$

The resulting chain complex is $C_*(X)$, whose nth homology group is $H_n(X)$. We shall assume that all cochains are normalized, i.e., that they are zero on degenerate simplices.

Let $*_X$ be a 0-simplex of X. Then $*_X$ generates a subcomplex of X whose sole n-simplex is $s_0 s_0 \ldots s_0(*_X)$, taking the composition of s_0 $n-1$ times. By a pointed complex we mean a pair $(X, *_X)$ where we abuse notation by allowing $*_X$ to refer both to the 0-simplex and to the subcomplex it generates. We adopt the convention that the base point of $\Delta[n]$ is (0), for all $n \geqslant 0$.

Let Y be a pointed complex. $\mathrm{Hom}_\bullet(I,Y)$ is the simplicial function complex:

$$\mathrm{Hom}_\bullet(I,Y)^n = \{u{:}I \times \Delta[n] \to Y \mid u \text{ is a pointed simplicial map}\}.$$

Let u be an n-simplex of $\mathrm{Hom}_\bullet(I,Y)$; $d_i u$ is defined to be the composite

$$d_i u{:}I \times \Delta[n-1] \xrightarrow{1 \times \delta_i} I \times \Delta[n] \xrightarrow{u} Y,$$

where $\delta_i{:}\Delta[n-1] \to \Delta[n]$ is given by

$$\delta_i(a_0, \ldots, a_m) = (a'_0, \ldots, a'_m),$$
$$a'_j = a_j \text{ if } j < i, \quad a'_j = a_j + 1 \text{ if } j \geqslant i.$$

Similarly, $s_i u$ is given by the composite

$$s_i u{:}I \times \Delta[n+1] \xrightarrow{I \times \sigma_i} I \times \Delta[n] \xrightarrow{u} Y,$$

where $\sigma_i{:}\Delta[n+1] \to \Delta[n]$, and

$$\sigma_i(a_0, \ldots, a_m) = (a'_0, \ldots, a'_m)$$
$$a'_j = a_j \text{ if } j \leqslant 1, \; a'_j = a_j - 1 \text{ if } j > i.$$

Let $t{:}\mathrm{Hom}_\bullet(I,Y) \to Y$ be given by the restriction

$$i^{\#}{:}\mathrm{Hom}_\bullet(I,Y) \to \mathrm{Hom}((1),Y),$$

induced from the inclusion $i{:}(1) \to I$, followed by the natural identification of $\mathrm{Hom}((1),Y)$ with Y.

Let G be an abelian group. The simplicial $K(G;n)$ can be described as the complex whose q-simplices are cochains $u \in Z^n(\Delta[q];G)$, with face operator

$$d_i = \delta_i^{\#}$$

and degeneracy operator

$$s_i = \sigma_i^{\#}$$

K(G;n) is in fact a subcomplex of the complex E(G;n) given by $E(G;n)^q = C^n(\Delta[q];G)$ and face and degeneracy operators given as above. The coboundary operator δ induces a simplicial map

$$\delta: E(G;n) \to K(G;n+1).$$

Proposition 1.4. The complex K(G;n+1) described above is in fact an Eilenberg-MacLane complex of type (G;n+1). $\delta: E(G;n) \to K(G;n+1)$ is a principal fibration with fiber K(G;n).

Proof. See May [4]. □

$| \ |: SS \to Top$ will denote Milnor's geometric realization functor. It is well known that $|X|$ is a CW complex having one n-cell for each nondegenerate n-simplex of X.

By $\pi_n X$ we mean $\pi_n|X|$ and by the action of $\pi_1 X$ on $\pi_n X$ the action of $\pi_1|X|$ on $\pi_n|X|$.

Definition 1.5. The lower central series of the action of $\pi_1 X$ on $\pi_n X$,

$$\Gamma_1\pi_n X \geqslant \Gamma_2\pi_n X \geqslant \ldots \geqslant \Gamma_i\pi_n X \geqslant \ldots$$

is defined as follows.

Let $\Gamma_1\pi_n X = \pi_n X$. Let $\Gamma_{i+1}\pi_n X$ be the subgroup generated by the set $\{\alpha\gamma - \gamma | \alpha \in \pi_1 X, \gamma \in \Gamma_i\pi_n X\}$. We say that $\pi_1 X$ operates nilpotently on $\pi_n X$ if $\Gamma_j \pi_n X = 0$ for some j. If c is the largest integer such that $\Gamma_c \pi_n X \neq 0$, then we call c the nilpotency class of the action of $\pi_1 X$ on $\pi_n X$.

<u>Definition 1.6</u>. A simplicial set X is said to be nilpotent if $\pi_1 X$ is a nilpotent group and acts nilpotently on $\pi_n X$, for all $n > 1$.

Examples of nilpotent simplical sets are most easily arrived at from examples of nilpotent spaces. For example, any triangulation of a nilpotent space gives a CW complex whose CW structure can be obtained from the realization of a simplicial set. Since the category of nilpotent spaces contains all simple spaces, and in particular all connected topological groups, we can obtain many examples of nilpotent simplicial sets from triangulations of such spaces.

Moreover, nilpotent spaces are particularly nice in that they respect the function space operation. Given a nilpotent CW complex X and a finite CW complex W, it can be shown that the function space X^W is a nilpotent CW complex. See, for example, Hilton, Mislin, and Roitberg [3].

It then follows that given a nilpotent complex X and a finite complex W, that Hom(W,X) is nilpotent.

Furthermore, if X is also locally finite, it is easy to see that Hom(W,X) is also locally finite.

Let $f: X \to Y$ be a pointed simplicial map. Define, Tf, the mapping fiber of f, to be the pullback in SS of the following diagram:

$$\begin{array}{ccc} Tf & \longrightarrow & \mathrm{Hom}_{\bullet}(I,Y) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

Since SS is complete, Tf is a simplicial set.

<u>Proposition 1.6</u>. Let Y be a Kan complex, $f:X \to Y$ a pointed map. There is a long exact sequence,

$$\cdots \to \pi_n Tf \to \pi_n X \xrightarrow{f_*} \pi_n Y \to \pi_{n-1} Tf \to \cdots$$

<u>Proof</u>. In Gabriel and Zisman [2], it is proved that t is a Kan fibration. Let ΩY be the fiber of t. t gives rise to the long exact sequence,

$$\cdots \to \pi_n \Omega Y \to \pi_n \mathrm{Hom}_{\bullet}(I,Y) \to \pi_n Y \xrightarrow{\cong} \pi_{n-1} \Omega Y \to 0 \to \cdots,$$

in which $\pi_n \mathrm{Hom}_{\bullet}(I,Y) = 0$, and $\pi_n Y$ is naturally identified with $\pi_{n-1}\Omega Y$.

Quillen [5] has proved that the geometric realization of a Kan fibration is a Serre fibration. Since $|\ |$ is adjoint to the total singular complex functor $S:\mathrm{Top} \to S$, it preserves limits.

Therefore,

$$\begin{array}{ccc} |Tf| & \longrightarrow & |\mathrm{Hom}_{\bullet}(I,Y)| \\ \Big\downarrow |p| & & \Big\downarrow |t| \\ |X| & \xrightarrow{\ |f|\ } & |Y| \end{array}$$

is a pullback in Top in which $|p|$ is a Serre fibration induced from $|t|$, and the long exact sequence of the fibration $|p|$ is the desired sequence. This completes the proof. □

We define $\pi_{n+1} f$ to be $\pi_n Tf$. Let $\mathrm{cyl}|f|$ be the mapping cylinder of $|f|$. By $H_{n+1} f$ we mean $H_{n+1}(\mathrm{cyl}|f|, |X|)$. It is easy to see that $H_{n+1} f$ is the $(n+1)$th homology group of the mapping cone $C_* f$ given by $C_{n+1} f = C_n X \oplus C_{n+1} Y$ and $\partial(x,y) = (-\partial x,\ fx + \partial y)$.

We are now ready to begin the Postnikov construction.

Chapter II

THE POSTNIKOV TOWER

§1. The Construction of the Postnikov Tower

Throughout this chapter we assume X to be a connected nilpotent simplicial set.

Let $c(n)$, $n \geqslant 2$, be the nilpotency class of the action of $\pi_1 X$ on $\pi_n X$. We construct the Postnikov tower of X by induction of the pair n,i $(1 \leqslant i \leqslant c(n)+1)$. We shall identify $n, c(n)+1$ with $n+1, 1$.

For each pair n,i we construct a group $G_{n,i}$, a complex $Y_{n,i}$, and a map $f_{n,i}:X \to Y_{n,i}$. Let $G_{1,1}$ be the trivial group, $Y_{1,1}$ the trivial complex $*$, and $f_{1,1}$ the obvious simplicial map.

Assume $G_{n,i}$, $Y_{n,i}$, and $f_{n,i}:X \to Y_{n,i}$ have been defined. If $H_{n+1}f_{n,i} = 0$, set $n,i = n+1, 1$; $f_{n,i} = f_{n+1,i}$, and let $G_{n,i+1} = G_{n+1,2} = H_{n+2}f_{n+1,1}$. Let $G_{n,i+1} = H_{n+1}f_{n,i}$ if $H_{n+1}f_{n,i} \neq 0$.

Lemma 2.1. Let $C_* f$ be a chain complex. Let $p:Z_{n+1}f \to H_{n+1}f$ be the quotient map. p can be extended to a cocycle $E \in Z^{n+1}(f;H_{n+1}f)$.

Proof. By the Universal Coefficient Theorem, there exists an epimorphism,

$$\phi:H^{n+1}(f,H_{n+1}f) \to \mathrm{Hom}(H_{n+1}f,H_{n+1}f).$$

p gives rise to the identity homomorphism $1:H_{n+1}f \to H_{n+1}f$. Let $[E]$ be any element of the kernel of ϕ. Any representative of $[E]$ extends p. □

Now choose any extension of the quotient map $Z_{n+1}f_{n,i} \to G_{n,i+1}$ to a cocycle $E_{n,i+1}$. Define $\hat{A}_{n,i+1} \in Z^{n+1}(Y_{n,i};G_{n,i+1})$ by $\hat{A}_{n,i+1}(y) =$

$E_{n,i+1}(0,y)$, and $\hat{B}_{n,i+1} \in C^n(X,G_{n,i+1})$ by $\hat{B}_{n,i+1}(x) = E_{n,i+1}(x,0)$.

Then $\hat{A}_{n,i+1}$ and $\hat{B}_{n,i+1}$ induce explicit maps

$$A_{n,i+1}:Y_{n,i} \to K(G_{n,i+1};n+1)$$

and

$$B_{n,i+1}:X \to E(G_{n,i+1};n)$$

as follows. Let y be a q-simplex of $Y_{n,i}$, and let $r_y:\Delta[q] \to Y_{n,i}$ be defined by $r_y(\Delta q) = y$. Similarly, given $x \in X^q$ we can define r_x.

$$r_y^{\#}:Z^{n+1}(Y_{n,i};G_{n,i+1}) \to Z^{n+1}(\Delta[q];G_{n,i+1}).$$

Then $A_{n,i+1}(y) = r_y^{\#}(\hat{A}_{n,i+1})$. Similarly, $B_{n,i+1}(x) = r_x^{\#}(\hat{B}_{n,i+1})$.

<u>Lemma 2.2</u>. The square

$$\begin{array}{ccc} X & \xrightarrow{B_{n,i+1}} & E(G_{n,i+1};n) \\ {\scriptstyle f_{n,i}} \Big| & & \Big| {\scriptstyle \delta} \\ Y_{n,i} & \underset{A_{n,i+1}}{\longrightarrow} & K(G_{n,i+1};n+1) \end{array}$$

commutes.

<u>Proof</u>. First, observe that

$$\begin{aligned} A(fx) &= E(0,fx) \\ &= E((\partial x,0) + (-\partial x, fx + \partial 0)) \\ &= E((\partial x,0) + \partial(x,0)) \\ &= E(\partial x,0) + E(\partial(x,0)) \\ &= E(\partial x,0) + \partial E(x,0) \\ &= E(\partial x,0). \end{aligned}$$

But $\partial(B(x)) = B(\partial x) = E(\partial x,0)$. □

Now take $Y_{n,i+1}$ to be the pullback

$$\begin{array}{ccc} Y_{n,i+1} & \longrightarrow & E(G_{n,i+1};n) \\ \downarrow & & \downarrow \delta \\ Y_{n,i} & \xrightarrow{A_{n,i+1}} & K(G_{n,i+1};n+1) \end{array}$$

and $f_{n,i+1}$ to be the map induced from $f_{n,i}$ and $B_{n,i+1}$. This completes the Postnikov construction.

§2. Properties of the Postnikov Construction

Definition 2.3. A simplicial set Y is called n-trivial if, given any two q-simplices y and y' of Y, $q \geqslant n$, such that $d_i y = d_i y'$, $0 \leqslant i \leqslant q$, then $y = y'$.

Remark. If Y is a Kan complex, then Y is n-trivial implies $\pi_q Y = 0$, $q \geqslant n$. This is obvious using the usual definition of the homotopy groups for Kan complexes. See May [4] for details.

Proposition 2.4. $Y_{n,i}$ is n+1-trivial for all n,i; $1 \leqslant i \leqslant c(n)+1$.

Proof. $Y_{1,1}$ is obviously 2-trivial. We claim that E(G;n) is n+1-trivial, for all groups G. Let u_1 and $u_2 \in E(G,n)^q$, and suppose for all i, $d_i u_1 = d_i u_2$, i.e.,

$$d_i u_1 = d_i u_2 : C_n\Delta[q-1] \to C_n\Delta[q] \xrightarrow{u_j} G, \quad j = 1, 2.$$

Now, as long as $q \geqslant n+1$, the n simplices of $\Delta[q-1]$ are nondegenerate, and the set $\{d_i u_j\}_{i=0}^{q}$ completely determines u_j $(j = 1, 2)$. Hence, $u_1 = u_2$.

Now suppose $Y_{n,i}$ is n+1-trivial. Two q-simplices of $Y_{n,i+1}$, (y_1, u_1) and (y_2, u_2) clearly have the property that if

$$d_i(y_1,u_1) = d_i(y_2,u_2), \quad 0 \leqslant i \leqslant q,$$

then $(y_1,u_1) = (y_2,u_2)$. Thus $Y_{n,i+1}$ is n+1-trivial.

Now let $i = c(n)+1$. Setting $Y_{n,c(n)+1} = Y_{n+1,1}$, we see that we need to show that $Y_{n+1,1}$ is n+2-trivial. But $Y_{n+1,1}$ is n+1-trivial, and n+1-triviality implies n+2-triviality. This completes the inductive step. □

Remark. Since $Y_{1,1}$, $E(G;n)$, and $K(G;n+1)$ are Kan complexes, it is clear that $Y_{n,i}$ is a Kan complex for all pairs n,i. Hence, we have proved that $\pi_q Y_{n,i} = 0$, $q \geqslant n+1$.

Theorem 2.5. The map $f_{n,i}$ is n-connected for all n,i, $0 \leqslant i \leqslant c(n)+1$.

We will need the following lemmas.

Lemma 2.6. $p: Y_{n,i+1} \to Y_{n,i}$ is n-connected.

Proof. p is a Kan fibration induced from the principal fibration δ. The long exact sequence of p reduces to

$$G_{n,i+1} \rightarrowtail \pi_n Y_{n,i+1} \xrightarrow{p_*} \pi_n Y_{n,i}$$

and

$$\pi_q Y_{n,i+1} \xrightarrow[\cong]{p_*} \pi_q Y_{n,i}, \quad q \leqslant n. \ \square$$

Lemma 2.7. If $f_{n,i}$ is n-connected, then

$$(f_{n,i+1},1)_*: \pi_{n+1} f_{n,i} \to \pi_{n+1} p$$

is onto with the same kernel as the relative Hurewicz homomorphism, h.

Proof. We will show that there exists a commuting diagram,

$$(1)\qquad \begin{array}{ccc} \pi_{n+1}f_{n,i} & \xrightarrow{(f_{n,i+1},1)_*} & \pi_{n+1,P} \\ h\big\downarrow & \nearrow_{\cong} & \\ H_{n+1}f_{n,i} & & \end{array}$$

and from this the result if obvious.

First, observe the commuting diagram

$$(2)\qquad \begin{array}{ccccc} & & H^{n+1}(f_{n,i};G_{n,i+1}) & \xrightarrow[\cong]{\phi} & \mathrm{Hom}(H_{n+1}f_{n,i};G_{n,i+1}) \\ & & j^{\#}\big\downarrow & (A) & \big\downarrow j^{\#} \\ [Y_{n,i};K(G_{n,i+1};n+1) & \xrightarrow[\cong]{\psi} & H^{n+1}(Y_{n,i};G_{n,i+1}) & \xrightarrow[\phi]{} & \mathrm{Hom}(H_{n+1}Y_{n,i},G_{n,i+1}) \end{array}$$

Notice that $A_{n,i+1}:Y_{n,i} \to K(G_{n,i+1};n+1)$ is a specific choice of the homotopy class of maps $\psi^{-1}j^{\#}\phi^{-1}(1) = [A_{n,i+1}]$, where 1 is the identity map on $H_{n+1}f_{n,i}$. Let ι^{n+1} denote the fundamental class of $H^{n+1}(K(\pi,n+1);\pi)$; namely, $\iota^{n+1} = \phi^{-1}(h_0^{-1})$, where h_0 is the Hurewicz homomorphism. Then the map ψ is defined by

$$\psi[A_{n,i+1}] = A^{\#}_{n,i+1}(\iota^{n+1}).$$

Now, $\phi(A^{\#}_{n,i+1}(\iota^{n+1})) = j^{\#}1$ by commutativity of (A). Equivalently, write

$$\begin{aligned} \phi(A^{\#}_{n,i+1}\iota^{n+1}))(z) &= \langle A^{\#}_{n,i+1}(\iota^{n+1};z\rangle \\ &= \langle\iota^{n+1},A_{n,i+1_{\#}}z\rangle \\ &= \phi(\iota^{n+1})(A_{n,i+1_{\#}}z\rangle \\ &= h_0^{-1}(A_{n,i+1_{\#}}z\rangle. \end{aligned}$$

Therefore, $j^{\#}(1)(z) = h_0^{-1}(A_{n,i+1_{\#}}z)$ for all z, i.e.,

$$(3)\qquad \begin{array}{ccc} H_{n+1}Y_{n,i} & \xrightarrow{A_{n,i+1_{\#}}} & H_{n+1}(K(G_{n,i+1};n+1)) \\ & \searrow^{j^{\#}1} & \downarrow h_0^{-1} \\ & & \pi_{n+1}(K(G_{n,i+1};n+1) = H_{n+1}f_{n,i} \\ & & = G_{n,i+1} \end{array}$$

commutes.

Now consider

$$(4)\qquad \begin{array}{ccccc} \pi_{n+1}f_{n,i} & \xrightarrow{(f_{n,i+1},1)_*} & \pi_{n+1}p & \xrightarrow{(\tilde{A}_{n,i+1},A_{n,i+1})_*} & \pi_{n+1}\delta \\ \downarrow h & & \downarrow h & & \downarrow h \\ H_{n+1}f_{n,i} & \xrightarrow[(f_{n,i+1},1)_{\#}]{} & H_{n+1}p & \xrightarrow[(\tilde{A}_{n,i+1},A_{n,i+1})_{\#}]{} & H_{n+1}\delta \end{array}$$

where h is the relative Hurewicz homomorphism, and $\hat{A}_{n,i+1}$ is the map $\tilde{A}_{n,i+1}:Y_{n,i+1} \to E(G_{n,i+1};n)$ in the pullback diagram by which $Y_{n,i+1}$ was defined. It is evident that diagram (4) commutes.

We claim that the diagram

$$(5)\qquad \begin{array}{ccccc} H_{n+1}f_{n,i} & \xrightarrow{(f_{n,i+1};1)_{\#}} & H_{n+1}p & \xrightarrow{(\tilde{A}_{n,i+1},A_{n,i+1})_{\#}} & H_{n+1}\delta \\ \uparrow j^{\#}(1) & & & & \uparrow j'_{\#} \\ H_{n+1}Y_{n,i} & & \xrightarrow{A_{n,i+1_{\#}}} & & H_{n+1}K(G_{n,i+1};n+1) \end{array}$$

also commutes, where $j'_{\#}$ is obtained from the long exact homology sequence of the map δ. To see this, observe that from the diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f_{n,i+1}} & Y_{n,i+1} & \xrightarrow{\hat{A}_{n,i+1}} & E(G_{n,i+1};n) \\
\downarrow{\scriptstyle f_{n,i}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \delta} \\
Y_{n,i} & \xrightarrow{1} & Y_{n,i} & \longrightarrow & K(G_{n,i+1};n+1)
\end{array}$$

upon passing to homology we obtain

$$\begin{array}{ccccc}
H_{n+1}f_{n,i} & \xrightarrow{(f_{n,i+1},1)_\#} & H_{n+1}p & \xrightarrow{(\hat{A}_{n,i+1},A_{n,i+1})_\#} & H_{n+1}\delta \\
\uparrow{\scriptstyle j^\#} & & \uparrow & & \uparrow{\scriptstyle j'_\#} \\
H_{n+1}Y_{n,i} & \xrightarrow{1} & H_{n+1}Y_{n,i} & \xrightarrow[A_{n,i+1\#}]{} & H_{n+1}K(G_{n,i+1};n+1)
\end{array}$$

and $j_\# = j^\# \circ 1$.

Assembling (3), (4) and (5), we have

(6)
$$\begin{array}{cccccc}
 & \pi_{n+1}f_{n,i} & \xrightarrow{(f_{n,i+1},1)_*} & \pi_{n+1}p & \xrightarrow{(\tilde{A}_{n,i+1},A_{n,i+1})_*} & \pi_{n+1}\delta \\
 & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\
\nearrow{\scriptstyle \text{identity}} & H_{n+1}f_{n,i} & \longrightarrow & H_{n+1}p & \longrightarrow & H_{n+1}\delta \\
 & \uparrow{\scriptstyle j_\#} & & & & \uparrow{\scriptstyle j'_\# \cong} \\
 & H_{n+1}Y_{n,i} & & \xrightarrow{A_{n,i+1\#}} & & H_{n+1}K(G_{n,i+1};n+1) \\
 & & & & \swarrow{\scriptstyle h_0^{-1} \cong} & \\
\pi_{n+1}K(G_{n,i+1};n+1) & & & & &
\end{array}$$

The connectedness of δ implies that $j'_\#$ is an isomorphism.

Finally, we claim $(\tilde{A}_{n,i+1};A_{n,i+1})_*$ is an isomorphism. We omit the proof, which follows from general facts about maps induced from fibrations, but refer the reader to Whitehead [6, p. 191]. With this fact we have completed the construction of diagram (1), and this completes the proof. □

Proof of Theorem 2.5

We prove the theorem by induction on n,i. Because of the convention that $f_{n,c(n)+1} = f_{n+1,i}$, the induction step is in two parts: (i) if $f_{n,i}$ is n-connected, then $f_{n,i+1}$ is n-connected, and (ii) if $f_{n,c(n)}$ is n-connected, then $f_{n,c(n)+1} = f_{n+1,1}$ is $n+1$-connected.

Since X is pathconnected and $Y_{1,1}$ is the trivial complex, clearly, $f_{1,1}$ is 1-connected.

Proof. Now suppose that $f_{n,i}$ is n-connected. We know from Lemma 2.6 that $p:Y_{n,i+1} \to Y_{n,i}$ is n-connected. From the commutative diagram, for $q < n$,

$$\begin{array}{ccc} \pi_q X & \xrightarrow{f_{n,i+1_*}} & \pi_q Y_{n,i+1} \\ {\scriptstyle f_{n,i}\,\cong}\downarrow & & \downarrow{\scriptstyle p_*\,\cong} \\ \pi_q Y_{n,i} & \xrightarrow[\cong]{1_*} & \pi_q Y_{n,i} \end{array}$$

we see that $f_{n,i+1_*}:\pi_q X \cong \pi_q Y_{n,i+1}$, $q < n$.

From the long exact sequence

$$0 \to \pi_{n+1} f_{n,i+1} \to \pi_{n+1} f_{n,i} \xrightarrow{(f_{n,i+1},1)_*} \pi_{n+1} p \xrightarrow{\partial} \pi_n f_{n,i} \to 0,$$

and the fact that $(f_{n,i+1},1)_*$ is onto, it is clear that

$$\pi_{n+1}p = \ker \sigma \twoheadrightarrow \pi_n f_{n,i+1},$$

and hence that $\pi_n f_{n,i+1} = 0$. This completes (i). For part (ii) we will need the following lemma.

Lemma 2.9. $\pi_{n+1} f_{n,i} \cong \Gamma_i \pi_n X$ when $f_{n,i}$ is n-connected.

Proof. By induction. Since $f_{1,1}$ is 1-connected and $Y_{1,1}$ is trivial, the long exact sequence of $f_{1,1}$ reduces to

$$0 \to \pi_2 f_{1,1} \underset{\cong}{\to} \pi_1 X \to 0.$$

Hence, $\pi_2 f_{1,1} \cong \Gamma_1 \pi_1 X$.

Now suppose $f_{n,i}$ is n-connected implies $\pi_{n+1} f_{n,i} \to \Gamma_i \pi_n X$, and suppose that $f_{n,i+1}$ is n-connected. First note that the fact that $f_{n,i+1}$ and p are n-connected implies that $f_{n,i}$ is, and hence by the induction hypothesis that $\pi_{n+1} f_{n,i} \cong \Gamma_i \pi_n X$. By the relative Hurewicz theorem, we see that $H_{n+1} f_{n,i} \cong \Gamma_i \pi_n X / \Gamma_{i+1} \pi_n X$, and $H_{n+1} f_{n,i} = G_{n,i+1} \cong \pi_{n+1} p \cong \Gamma_i \pi_n X / \Gamma_{i+1} \pi_n X$. Then the long exact sequence of the triple $(f_{n,i+1}, f_{n,i}, p)$ reduces to

$$0 \to \pi_{n+1} f_{n,i+1} \rightarrowtail \Gamma_i \pi_n X \xrightarrow{(f_{n,i+1},1)_*} \Gamma_i \pi_n X / \Gamma_{i+1} \pi_n X \to 0.$$

Lemma 2.7 implies $\ker(f_{n,i+1};1)_* \cong \Gamma_{i+1} \pi_n X$; hence, $\pi_{n+1} f_{n,i+1} \cong \Gamma_{i+1} \pi_n X$.

Proof of part (ii). Suppose $f_{n,c(n)}$ is n-connected. By (i) we know $f_{n,c(n)+1}$ is also n-connected. Therefore, it suffices to show that $\pi_{n+1} f_{n,c(n)+1} = 0$. But by the preceding lemma, we know that

$$\pi_{n+1} f_{n,c(n)+1} \cong \Gamma_{c(n)+1} \pi_n X = 0. \quad \square$$

Corollary 2.10.

$$\pi_q Y_{n,i} \cong \begin{cases} \pi_q X, & q < n \text{ and } i \leqslant c(n) \\ \pi_q X/\Gamma_i \pi_n X, & q = n \\ 0, & q > n. \end{cases}$$

Proof. Immediate. □

Chapter III

RECURSIVE SIMPLICIAL SETS

§1. A Special Case.

When $\pi_1 X$ is a finite nilpotent group, the groups $\Gamma_i\pi_1 X/\Gamma_{i+1}\pi_1 X$ will also be finite. If, in addition, X is a connected locally finite pointed simplicial set, the Postnikov tower may be used to construct, effectively, a 1-connected cover of X which is finite in each dimension. Then Brown's theorem applies, and there is an effective procedure for constructing finite abelian group presentations of $\pi_n X$, $n \geqslant 2$.

Theorem 3.1. Let X be a connected locally finite pointed simplicial set. Suppose additionally that $\pi_1 X$ is a finite nilpotent group. Then $\pi_n X$ is effectively computable, $n \geqslant 2$.

Proof. Observe first that $Y_{1,i}$ must be finite in each dimension. This is obvious for $Y_{1,1}$, and by induction on i, if $Y_{1,i-1}$ is finite in each dimension, then $H_2 f_{1,i-1}$ must be finitely generated, since $C_2 f_{1,i-1}$ is, and we know $H_2 f_{1,i-1} = G_{1,i} \cong \Gamma_{i-1}\pi_1 X/\Gamma_i\pi_1 X$, which is a finite abelian group. Then, evidently, $E(G_{1,i};1)$ is finite in each dimension, and so is the pullback

$$Y_{1,i} = Y_{1,i-1} \underset{A_{1,i}\ \delta}{\times} E(G_{1,i};1).$$

Let c denote the nilpotency class of $\pi_1 X$. We have established that $Y_{1,c+1}$ is effectively computable and finite in each dimension. For ease of notation, let $f = f_{1,c+1}: X \to Y = Y_{1,c+1}$. Construct the mapping fiber Tf. Clearly, if Y is finite in each dimension, then, since $\mathrm{Hom}_\bullet(I,Y)^n$ is the set of pointed simplicial maps $u:I \times \Delta[n] \to Y$, and the number of nondegenerate simplices of $I \times \Delta[n]$ is finite, then $\mathrm{Hom}_\bullet(I,Y)$ is finite in each dimension, and so is Tf. It is easy to see that Tf is simply connected with $\pi_q Tf \cong \pi_q X$, $q > 1$. An application of Brown's theorem to Tf completes the proof. □

§2. Recursive Simplicial Sets.

In general, the groups $G_{n,i}$ will not be finite groups, nor will the complexes $Y_{n,i}$ be finite in each dimension. In this chapter we definite recursive simplicial sets and maps and give some examples. In the next section we will see that complexes of this type make up the Postnikov tower.

Definition 3.2. A simplicial set Y is called recursive if the following conditions are met: (i) there is a recursive enumeration, possibly with repetitions, of Y^n, $n \geq 0$; (ii) the operators d_i and s_i are recursive; and (iii) given n-simplices y_i and y_j, there is an effective procedure for deciding whether or not $y_i = y_j$.

Definition 3.3. A simplicial map $f:X \to Y$ of recursive simplicial sets is called recursive if $f:X^n \to Y^n$ is recursive for $n \geq 0$.

By way of motivation for this definition, observe that given a recursive complex Y, there is an effective procedure for constructing, for $n \geq 0$, a free presentation of C_nY as an abelian group, and that the boundary operator $\partial:C_nY \to C_{n-1}Y$ is recursive. The generators of C_nY will be selected from the list of n-simplices of Y, and at each step we can check to see whether the next one on the list is equal to any of the preceding generators. If so, it can be discarded; otherwise, it can be added to the list of generators. Since $\partial = \sum_{i=o}^{n}(-1)^i d_i$, and each d_i is recursive, so is ∂.

Proposition 3.4. Let $C_*(K,\partial)$ be a chain complex in which C_nK is given via a free abelian presentation on a countable set of generators, and ∂ is recursive. Then the groups H_nK have r.e. abelian group presentations.

Proof. Since C_nK is countably generated, so is the group Z_nK. Furthermore, since $C_{n-1}K$ is freely presented, it has a solvable word problem and so $Z_nK = \{c \in C_nK | \partial c = 0\}$ is a recursive subset of C_nK. Take Z_nK as the set of generators of H_nK. For relators, use the set of generators of B_nK. We have a recursive homomorphism $\partial: C_{n+1}K \to C_nK$, whose image is B_nK. Using the set of generators of $C_{n+1}K$ and applying ∂ we obtain an r.e. subset of Z_nK which generates B_nK. □

2.1. Examples of Recursive Simplicial Sets

Let K be a simplicial set which is derived from a countable simplicial complex. It is easy to see that K is recursive.

Proposition 3.5. Let Y be a pointed recursive simplicial set. Then $\mathrm{Hom}_\bullet(I,Y)$ is recursive, as is the map $t:\mathrm{Hom}_\bullet(I;Y) \to Y$. The proof will make use of the following fact.

Lemma 3.6. The simplices of $I \times \Delta[n]$ which are neither degenerate nor faces of a nondegenerate simplex are the elements of the set $A \cup B$, where

$$A = \{((0, 0, \ldots, 0)\ \Delta_n), (1, 1, \ldots, 1))\} \subset (I \times \Delta[n])^n$$

$$B = \{(a_j, s_{j-1}\Delta_n) | j \neq 0, n+2\} \subset (I \times \Delta[n])^{n+1}$$

$$a_j = (\underbrace{0, 0, \ldots, 0}_{j}, \underbrace{1, 1, \ldots, 1}_{n+2-j}) \in I^{n+1}.$$

a_j can also be written

$$a_j = s_n s_{n-1} \cdots \hat{s}_{j-1} \cdots s_0(0,1).$$

Proof. For a discussion of the nondegenerate simplices of a product, see May [4]. The specific details of this case are then obvious. □

Proof of Proposition 3.5. Recall that $\mathrm{Hom}_{\bullet}(I,Y)^n$ is the set of pointed simplicial maps $u:I \times \Delta[n] \to Y$. Observe that u is determined by its value on the simplices of $I \times \Delta[n]$ which are neither degenerate nor the faces of a nondegenerate simplex. u is pointed if and only if $u((0, 0, \ldots, 0), \Delta n) = *_Y$. Thus, u may be uniquely described as an ordered n+3-triple,

$$(y_0, \ldots, y_{n+2}) \in Y^n \times Y^{n+1} \times \ldots \times Y^{n+1} \times Y^n,$$

where $y_0 = *_Y$

$$y_i = u(a_j, s_{j-1}\Delta_n) \in Y^{n+1} \quad (i \neq 0, n+2 \quad \text{and} \quad j = n+2-i),$$

$$y_{n+2} = u(a_0, \Delta_n) \in Y^n.$$

Conversely, any n+3-triple

$$(*_Y, y_1, \ldots, y_{n+2}), \quad y_i \in Y^{n+1}, \quad i \leqslant i \leqslant n+1, \quad y_{n+2} \in Y^n$$

determines an element u of $\mathrm{Hom}_{\bullet}(I,Y)^n$. It follows that to enumerate all such n+3-triples, and these are enumerable since Y is recursive. Given any two elements u_i and u_j in $Hom_{\bullet}(I,Y)^n$, $u_i = u_j$ if and only if the corresponding n+3-triples are equal, and this is decidable when Y is recursive.

We now show that d_i is recursive. Given $u \in Hom_{\bullet}(I,Y)^n$, $d_i u$ may be viewed as an n+2-triple of $*_Y \times Y^{n+1} \times \dots \times Y^{n+1} \times Y^n$. d_i is defined by

$$d_i u : I \times \Delta[n-1] \xrightarrow{I \times \delta_i} I \times \Delta[n] \xrightarrow{u} Y.$$

Denote the nondegenerate simplices of $(I \times \Delta[n-1])^n$ which are not faces by $(\overline{a}_{j+1}, s_j \Delta_n)$, where $\overline{a}_j$ =

$$\overline{a}_j = (\underbrace{0, \dots, 0}_{j}, 1, \dots, 1) \in I^n.$$

Then $d_i u = (*_Y, \overline{y}_1, \overline{y}_2, \dots, \overline{y}_{n+1})$, where

$$\overline{y}_k = d_i u(\overline{a}_j, s_{j-1}\Delta_n), \qquad j = n+2-k.$$

We wish to compute $\overline{y}_k$ in terms of the n+3-triple

$$u = (*_Y, y_1, y_2, \dots, y_{n+2}).$$

First note that if $i > j+1$, say, $j+1 = i+k$, $k > 0$, so that $j = i-k-1$,

$$\begin{aligned} \delta_i s_j \Delta_{n+1} &= \delta_i s_{i-k-1} \Delta_{n-1} \\ &= \delta_i s_{i-k-1}(0,1,\dots,n-1) \\ &= \delta_i(0,1,\dots,i-k-1,i-k-1,\dots,n-1) \\ &= \ (0,1,\dots,i-k-1,i-k-1,\dots,i-2,i,i+1,\dots,n) = \end{aligned}$$

$$= s_{i+k-1}(d_{i-1}\Delta_n)$$

$$= s_j d_{i-1}\Delta_n$$

$$= d_i s_j \Delta_n, \quad i > j+1,$$

while if $i \leqslant j+1$, say, $j+1 = i+k$, $k \geqslant 0$, so that $j = i+k-1$, we have

$$\delta_i s_j\ {}_{n-1} = \delta_i s_{i+k-1}\Delta_{n-1}$$

$$= \delta_i s_{i+k-i}(0,1,\ldots,n-1)$$

$$= \delta_i(0,1,\ldots,i+k-1,i+k-1,\ldots,n-1)$$

$$= (0,1,\ldots,i-1,i+1,\ldots,i+k,i+k,\ldots,n)$$

$$= d_i s_{i+k}\Delta_n$$

$$= d_i s_{j+1}\Delta_n.$$

Suppose $i > j+1$. Then,

$$d_i u(\overline{a}_{j+1}, s_j\Delta_n) = u(1 \times \delta_i(\overline{a}_{j+1}, s_j\Delta_n))$$

$$= u(\overline{a}_{j+1}, \delta_i s_j \Delta_n)$$

$$= u(\overline{a}_{j+1}, d_i s_j \Delta_n)$$

$$= u(d_i(a_{j+1}, s_j\Delta_n)) \quad (\text{since } d_i a_{j+1} = \overline{a}_{j+1},\ i > j+1)$$

$$= d_i[u(a_{j+1}, s_j\Delta_n)]$$

$$= d_i y_k, \text{ where } k = n+3-(j+1).$$

Now suppose $i \leqslant j+1$. Then

$$d_i u(\overline{a}_{j+1}, s_j\Delta_n) = u(\overline{a}_{j+1}, \delta_i s_j \Delta_n)$$

$$= u(\overline{a}_{j+1}, d_i s_{j+1}\Delta_n)$$

$$= u(d_i(a_{j+2}, s_{j+1}\Delta_n) \quad (\text{since } d_i a_{j+2} = \overline{a}_{j+1},\ i \leqslant j+1)$$

$$= d_i y_k, \text{ where } k = n+3-(j+2).$$

This gives $\overline{y}_j$ in terms of y_k, $j < n+1$. The case $j = n+1$ is handled similarly, and so is the proof for the operator s_i.

Regarding $u \in \mathrm{Hom}_\bullet(I,Y)^n$ as $u = (*_Y, y_1, \ldots, y_{n+2})$, where

$$y_{n+2} = u((1),\Delta_n) \in Y^n,$$

it is clear that $t(u) = y_{n+2}$ is a projection onto the last factor, and therefore t is a recursive map. □

Proposition 3.7. Let G be a group with an r.e. presentation as an abelian group, $G = \langle X;R\rangle_{ab}$. Suppose, in addition, that the word problem is solvable for the given presentation. Then $E(G,n)$ is recursive, for $n \geqslant 0$.

Proof. $E(G,n)^q = C^n(\Delta[q];G)$. If $q < n$, then $E(G,n)^q = 0$, since we assume all cochains are zero on degenerate simplices.

Let $q = n$. $C^n(\Delta[q],G) \cong G$ is enumerable, since G is countably generated. For $q > n$, observe that $C^n(\Delta[q];G)$ may be identified with $\binom{q+1}{n+1}$-tuples of $G \times \ldots \times G$, where $\binom{q+1}{n+1}$ is the number of nondegenerate n-simplices of $\Delta[q]$. An enumeration of $E(G,n)^q$ is thus equivalent to an enumeration of elements of $G \times \ldots \times G$ ($\binom{q+1}{n+1}$ factors). Two elements u_i and u_j are equal if and only if the $\binom{q+1}{n+1}$-tuples are equal, and this is decidable when G has a solvable word problem.

That the face and degenerating operators are recursive follows almost immediately from the definitions. $d_i u(a_j) = u(\delta_i a_j)$. We need only observe that if a_j is nondegenerate, so is $\delta_i a_j$, and that δ_i is clearly recursive. s_i may be seen to be recursive in the same way, since $s_i u(a_j) = u(\sigma_i a_j)$. Here, however, $\sigma_i a_j$ may be degenerate, in which case $u(\sigma_i a_j) = 0$, since we assume the cochains are normalized.

Corollary 3.8. Let G be as above. Then for $n \geqslant 0$, $K(G,n)$ is a recursive complex, and $\delta : E(G,n-1) \to K(G,n)$ is a recursive map, for all $n > 1$.

Proof. $\delta u = u \circ \sum_{i=0}^{q}(-1)^i d_i$, where d_i is recursive. Hence, δ is recursive, and $K(G,n)^q$ must be a recursive subset of $E(G,n)^q$, since G has a solvable word problem. The rest is obvious. □

Proposition 3.9. Let

$$\begin{array}{ccc} & & C \\ & & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

be a diagram in which A, B, and C are recursive complexes, and f and g are recursive maps. Then the pullback

$$A \underset{f\,g}{\times} C$$

is recursive.

Proof. $A^n \times C^n$ is obviously enumerable. We claim

$$A^n \underset{f\,g}{\times} C^n \qquad A^n \times C^n$$

is a recursive subset.

$$(a,c) \in A^n \underset{f\,g}{\times} C^n$$

if $fa = gc$ in B^n, and this is clearly decidable. Then

$$A^n \underset{f\,g}{\times} C^n$$

is enumerable. The rest of the conditions are easily seen to be satisfied. □

§3. The Computability of the Postnikov Tower

This section is devoted to the proof of the following theorem.

Theorem 3.10. Let X be a connected nilpotent locally finite simplicial set. Then the Postnikov tower of X is effectively computable in the following sense. For each n,i: (i) $Y_{n,i}$ is a recursive complex; (ii) $G_{n,i}$ has an r.e. abelian group presentation with solvable word problem; and (iii) $f_{n,i}$ is recursive.

The proof will require the following lemmas.

Lemma 3.11. Let X and Y be recursive simplicial sets. Let $f:X \to Y$ be a recursive map. Let $p:Z_n f \twoheadrightarrow H_n f$ be the natural quotient map. Then an extension of p to a cocycle $E \in Z^n(f;H_n f)$ can be constructed effectively.

Proof. Since X, Y, and f are recursive, the chain groups

$$C_n f = C_{n-1}X \oplus C_n Y$$

can be effectively presented freely on countable sets of generators, with recursive boundary operator. Let $c_1, c_2, c_3, \ldots$ be an enumeration of the generators of $C_n f$. There is a short exact sequence,

$$Z_n f \rightarrowtail C_n f \xrightarrow{\partial} B_{n-1} f,$$

in which $E_{n-1}f$ has solvable word problem which we claim splits effectively. For

$$\{\partial c_i\}_{i=0}^{\infty}$$

generates $B_{n-1}f$. Let $s: B_{n-1}f \to C_n f$ be defined by

$$s(\partial c_1) = c_1$$

$$s(\partial c_j) = \begin{cases} s(\partial c_i) & \text{where } i \text{ is the least integer less than } j \text{ such that } c_i = c_j \text{ if such an } i \text{ exists} \\ c_j & \text{otherwise.} \end{cases}$$

Clearly, $\partial \circ s = 1$.

Every element of $C_n f$ can then be written uniquely as $c = z + s\partial c$, where $z = c - s\partial c$.

Define $E(c) = p[z]$. Since $\delta E(c) = E(\partial c) = p(\partial c) = 0$, it is evident that E is a cocycle. □

From now on, our choice of cocycle $E_{n,i} \in Z^{n+1}(f_{n,i}; H_{n+1}f_{n,i})$ in the Postnikov construction will be assumed to be this one.

Corollary 3.12. Let X be a connected nilpotent locally finite simplicial set. Let $f_{n,i}: X \to Y_{n,i}$ be the (n,i)th map in the Postnikov tower of X. Suppose $Y_{n,i}$ and $f_{n,i}$ are recursive. Then the maps

$$A_{n,i+1}: Y_{n,i} \to K(H_{n+1}f_{n,i}; n+1)$$

and

$$B_{n,i+1}: X \to E(H_{n+1}f_{n,i}; n)$$

are recursive maps.

Proof. This is immediate on recollection of the definitions of $A_{n,i+1}$ and $B_{n,i+1}$--namely, that $A_{n,i+1}(y) = E_{n,i+1}(0,y)$ and $B_{n,i+1}(x) = E_{n,i+1}(x,0)$. □

Lemma 3.13. Let X be connected nilpotent locally finite complex. Suppose theorem 3.10 holds for all pairs $p,q < n,i$, i.e., for $p < n$ and $1 \leqslant q \leqslant c(n)$ or $p = n$ and $i \leqslant q < i \leqslant c(n)$. Then there is an effective procedure for modifying the given r.e. presentation of $G_{n,i}$ to obtain a finite presentation of $G_{n,i}$ as an abelian group.

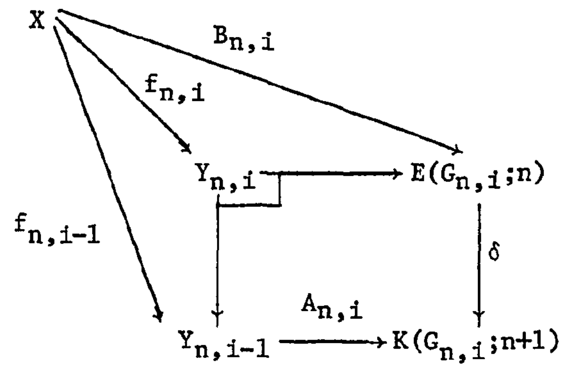
Remark. Observe that lemma 3.13 implies the decidability of the question "$H_{n+1}f_{n,i-1} = 0$?" as well as the solvability of the word problem for $G_{n,i}$.

Proof of Theorem 3.10. The proof is by induction on n,i. The theorem is obvious for the case $n,i = 1,1$.

Suppose $Y_{n,i-1}$ is recursive, $f_{n,i-1}:X \to Y_{n,i-1}$ is recursive, and that $G_{n,i-1}$ has an r.e. abelian group presentation with solvable word problem. We claim that $G_{n,i} = H_{n+1}f_{n,i-1}$ has an r.e. presentation as an abelian group with a solvable word problem. The induction hypothesis implies that $C_*f_{n,i-1}$ is recursive and hence that $H_{n+1}f_{n,i-1}$ has an r.e. presentation. Then lemma 3.13 states the existence of an algorithm for reducing the r.e. presentation to a finite presentation, thereby providing a solution for the word problem. It is then apparent that

$$\delta:E(G_{n,i};n) \to K(G_{n,i};n+1)$$

is a recursive map of the recursive complexes, and we have shown that the maps $A_{n,i}$ and $B_{n,i}$ are also recursive. Then the pullback diagram below



is a diagram in the category of recursive simplicial sets.

To complete the proof, recall that in the induction process, when $H_{n+1}f_{n,i-1} = 0$ we set $Y_{n,i} = Y_{n+1,1}$, $f_{n,i} = f_{n+1,1}$ and $G_{n+1,1} = H_{n+2}f_{n,i-1}$. Lemma 3.13 establishes the decidability of whether $H_{n+1}f_{n,i-1} = 0$, and this completes the proof.

Corollary 3.14. Let X be a connected nilpotent locally finite complex. Then $\pi_n X$ is a finitely generated abelian group, $n > 1$.

Proof. We have shown that the groups $G_{n,i}$ are finitely generated abelian groups, for all n,i, so that $\Gamma_i\pi_n X/\Gamma_{i+1}\pi_n X$ is f.g. abelian. Let $i = c$, where c is the nilpotency class of the action of $\pi_1 X$ or $\pi_n X$. Then $\Gamma_{c+1}\pi_n X = 0$ and $\Gamma_c\pi_n X$ is f.g. abelian.

$\Gamma_0\pi_n X \rightarrowtail \Gamma_{c-1}\pi_n X \twoheadrightarrow \Gamma_{c-1}\pi_n X/\Gamma_c\pi_n X$ is a short exact sequence, and the class of f.g. abelian groups is closed with respect to extensions, so we see that $\Gamma_{c-1}\pi_n X$ is an f.g. abelian group.

Then an easy induction argument shows that

$$\Gamma_2\pi_n \rightarrowtail \Gamma_n X \twoheadrightarrow \pi_n X/\Gamma_2\pi_n X$$

is an extension of one f.g. abelian group by another, and the result follows. □

Note, however, that we have not shown that these extensions are computable.

Chapter IV

A LOCALLY FINITE SUBTOWER

§1. q-Deformation Retracts

The objective of Chapter IV is to show that the groups $G_{n,i}$, defined via the Postnikov tower, have a solvable word problem. This will complete the proof of the effective computability of the Postnikov tower. The method of the proof is to give an effective procedure for extracting from the given r.e. presentation of $G_{n,i}$, a finite presentation. The set of generators of the new presentation will be a finite subset of the original generating set.

To do this we show that for each complex $Y_{n,i}$ in the Postnikov tower, and for each $q \geqslant n+1$, there exists a subcomplex $Y_{n,i,q} \subset Y_{n,i}$, which is locally finite, and for which

$$H_k Y_{n,i,q} \cong H_k Y_{n,i}, \quad k \leqslant q.$$

These subcomplexes are called q-deformation retracts of the complex $Y_{n,i}$. The definition is due to Brown [1], and proofs of all the properties of q-deformation retracts which we cite in this section will be found in his paper.

Definition 4.1. Let G be an abelian group. Let S be a subset of G. $E(G,S;n)$ denotes the subcomplex of $E(G;n)$ whose q-simplices n are cochains which take the generator of $C_n(\Delta[q])$ to elements of S.

Observe that if S is a finite subset of G, then $E(G,S;n)$ is locally finite.

Definition 4.2. Let M be a subcomplex of a complex N. M is called a q-deformation retract of N if and only if: (i) $N^0 \subset M$; and (ii) given a simplicial pair (K,L) such that $\dim K \leq q$ and $\dim L < q$, and a simplicial map $f:(K,L) \to (N,M)$, then there exists a pair $(K',L') \subset (K,L)$ and a map $f':(K',L') \to (N,M)$ extending f, such that $\dim K' \leq q+1$ and $\dim L' \leq q$, and the pairs (K',K) and (K',L') are acyclic.

Brown [1] has proved that q-deformation retracts have the following properties.

Proposition 4.3. (Transitivity.) Let P be a q-deformation retract of m, and M is a q-deformation retract of N. Then P is a q-deformation retract of N.

Proposition 4.4. Let M be a q-deformation retract of N; let $i:M \to N$ be the inclusion. Let N(q) be the subcomplex of N generated by nondegenerate simplices of dimension less than or equal to q. Then there exists a chain map,

$$\alpha:C_*(N(q)) \to C_*(n)$$

and a chain homotopy

$$D:C_*(N) \to C_*(N)$$

such that

$$\alpha i_{\#}:C_p(N) \to C_p(M), \quad p \leq q,$$

is the identity map, and

$$\partial D(x) + D(\partial x) = x - i_{\#}\alpha(x)$$

for $x \in C_pN$, $p \leqslant q$. Hence, $H_kM \cong H_kN$, $k \leqslant q$.

Proposition 4.5. Let M be a q-deformation retract of N. $A:N \to K(G,n)$. Then the pullback

$$M \underset{i^{\#}A\ \delta}{\times} E(G;n-1)$$

is a q-deformation retract of the pullback

$$N \underset{A\ \delta}{\times} E(G;n-1).$$

Definition 4.6. Let G be an f.g. abelian group, say,

$$G \cong \bigoplus_{i=1}^{r} Z_i \oplus T,$$

where Z_i is infinite cyclic and T a finite group. For $i = 1, \ldots, r$, let $p_i : G \to Z_i \cong Z$ be the projection onto the ith factor followed by an isomorphism of Z_i with Z. Then $P = \{p_i\}$ is called a projective decomposition of the infinite part of G.

Definition 4.7. Let $V \in C^n(X;Z)$. $|V| = \max\{|V(x)| \,\big|\, x \in X\}$ or ∞ if no maximum exists.

Let $A:Y \to K(G,n)$, where G is an f.g. abelian group with a projective decomposition $P = \{p_i\}_{i=1}^{r}$ of the infinite part of G. For each j, $1 \leqslant j \leqslant r$, choose $a_j \geqslant (q+1)|p_jA| + 1$.

Let $A = \{a_1, a_2, \ldots, a_r\}$.

Let $S = S(P,a) = \{g \in G \,\big|\, |p_jg| \leqslant a_j,\ 1 \leqslant j \leqslant r\}$.

We are now in a position to state the most important proposition concerning q-deformation retracts.

Proposition 4.8. Let $\delta_S = \delta$ restricted to $E(G,S;n-1)$. With S defined as above, the pullback

$$Y \underset{A\ \delta_S}{\times} E(G,S;n-1)$$

is a q-deformation retract of the pullback

$$Y \underset{A\ \delta}{\times} E(G;n-1).$$

§2. q-Subtowers.

We now construct, inductively, for $q > n+1$, a finite tower of subcomplexes $Y_{n,i,q}$, where $Y_{n,i,q}$ is a q-deformation retract of $Y_{n,i}$ which is locally finite, and of maps $f_{n,i,q}:X \to Y_{n,i,q}$.

As before, we identify $n, c(n)+1, q = n+1, 1, q$, where $c(n)$ is the nilpotency class of the action of $\pi_1 X$ on $\pi_n X$. However, here we stop when, upon setting $n, c(n)+1, q = n+1, 1, q$, we find that $n+1 = q$.

Let X be a connected nilpotent locally finite complex.

Define

$$Y_{1,1,q} = Y_{1,1} = * \qquad \text{(where } * = \text{the trivial complex)}$$

$$G_{1,1,q} = G_{1,1} = 0,$$

$$f_{1,1,q} = f_{1,1}.$$

Now assume that for suitably large q we have defined $Y_{n,i,q}$, $f_{n,i,q}$, and $G_{n,i,q}$ in such a way that $Y_{n,i,q}$ is a locally finite q-deformation retract of $Y_{n,i}$, and if $i:Y_{n,i,q} \to Y_{n,i}$ is the inclusion map, then $i \circ f_{n,i,q} = f_{n,i}$.

If $H_{n+1}f_{n,i,q} = 0$, set $n, i, q = n+1, 1, q$, and $G_{n+1,2} = H_{n+2}f_{n+1,1,q}$. Otherwise, $G_{n,i+1,q} = H_{n+1}f_{n,i,q}$. Observe that the local finiteness conditions on X and $Y_{n,i,q}$ imply that $H_{n+1}f_{n,i,q}$ is given effectively as a finitely generated abelian group.

Lemma 4.9. $H_{n+1}f_{n,i,q} \cong H_{n+1}f_{n,i}$.

Proof. $Y_{n,i,q} \subset Y_{n,i}$ is a q-deformation retract. Thus, the inclusion map i induces an isomorphism $i_{\#}: H_jY_{n,i,q} \cong H_jY_{n,i}$, $j < q$, by 4.4. From the long exact homology sequences of the maps $f_{n,i,q}$ and $f_{n,i}$ we obtain the following commutative diagram:

$$\begin{array}{ccccccccccc}
\rightarrow H_{n+1}X & \rightarrow & H_{n+1}Y_{n,i,q} & \rightarrow & H_{n+1}f_{n,i,q} & \rightarrow & H_nX & \rightarrow & H_nY_{n,i,q} & \rightarrow \\
\cong \downarrow 1_{\#} & & \downarrow i_{n+1} & & \downarrow (1,i)_{\#} & & \downarrow 1_{\#} \cong & & \downarrow i_n & \\
\rightarrow H_{n+1}X & \rightarrow & H_{n+1}Y_{n,i} & \rightarrow & H_{n+1}f_{n,i} & \rightarrow & H_nX & \rightarrow & H_nY_{n,i} & \rightarrow
\end{array}$$

in which, for $n+1 < q$, i_{n+1} and i_n are isomorphisms. The Five Lemma implies $(1,i)_{\#}$ is an isomorphism as well. In fact, since

$$(1,i)_{\#}: Z_{n+1}f_{n,i,q} \rightarrow Z_{n+1}f_{n,i}$$

is inclusion, it is clear that it induces the identity of $G_{n,i}$. From now on we consider $G_{n,i,q}$ as equal to $G_{n,i}$. Observe that given any word in the generators of $G_{n,i}$, i.e.,

$$z = (\Sigma n_jx_j, \Sigma m_jy_j) \in Z_{n+1}f_{n,i,q}$$

that $z = 0$ in $H_{n+1}f_{n,i}$ if and only if $(1,i)_*^{-1}(\Sigma n_jx_j, \Sigma m_jy_j) = 0$ in $H_{n+1}g_{n,i,q}$, the finite presentation of $G_{n,i}$. But $(1,i)_{\#}^{-1}$ may be described as the map that takes

$$(x,y) \longmapsto \begin{cases} (x,y) & \text{if } y \in Y_{n,i,q}, \\ (x,0) & \text{if } y \notin Y_{n,i,q}. \end{cases}$$

Since $Y_{n,i,q}$ is finite in each dimension, $Y^k_{n,i,q}$ is a recursive subset of $Y^k_{n,i}$ for all $k \geq 0$. This is an effective procedure yielding a finite presentation of the abelian group $G_{n,i}$.

We now continue the construction. Since in our inductive hypothesis we assumed $Y_{n,i}$ to be recursive, recall that we have an effective construction of the cocycle $E_{n,i+1} \in Z^{n+1}(f_{n,i};G_{n,i+1})$ from which the maps $A_{n,i+1}:Y_{n,i} \to K(G_{n,i+1};n+1)$ and $B_{n,i+1}:X \to E(G_{n,i+1};n)$ are defined. An application of the cochain map

$$(1,i)^{\#}:C^*(f_{n,i};G_{n,i+1}) \to C^*(f_{n,i,q};G_{n,i+1})$$

to $E_{n,i+1}$ defines $E_{n,i+1,q} \in Z^{n+1}(f_{n,i,q};G_{n,i+1})$. Then let

$$\hat{A}_{n,i+1,q}(y) = E_{n,i+1,q}(0,y)$$

and

$$\hat{B}_{n,i+1,q}(x) = E_{n,i+1,q}(x,0).$$

$\hat{A}_{n,i+1,q}$ defines a map

$$A_{n,i+1,q}:Y_{n,i,q} \to K(G_{n,i+1};n+1)$$

and $\hat{B}_{n,i+1,q}$ defines

$$B_{n,i+1,q}:X \to E(G_{n,i+1};n).$$

Let Q be the pullback

$$\begin{array}{ccc} Q & \longrightarrow & E(G_{n,i+1};n) \\ \downarrow & & \downarrow \delta \\ Y_{n,i,q} & \xrightarrow{A_{n,i+1,q}} & K(G_{n,i+1};n+1) \end{array}$$

Since $Y_{n,i,q}$ is a q-deformation retract of $Y_{n,i}$, Q is a q-deformation retract of $Y_{n,i+1}$ by 4.6.

Now $G_{n,i+1}$ has been given effectively as an f.g. abelian group. Choose a projective decomposition P of the infinite part of $G_{n,i+1}$. Observe that since $Y_{n,i,q}$ is finite in each dimension that $|p_j A_{n,i,q}| < \infty$, for all j. Then we may choose

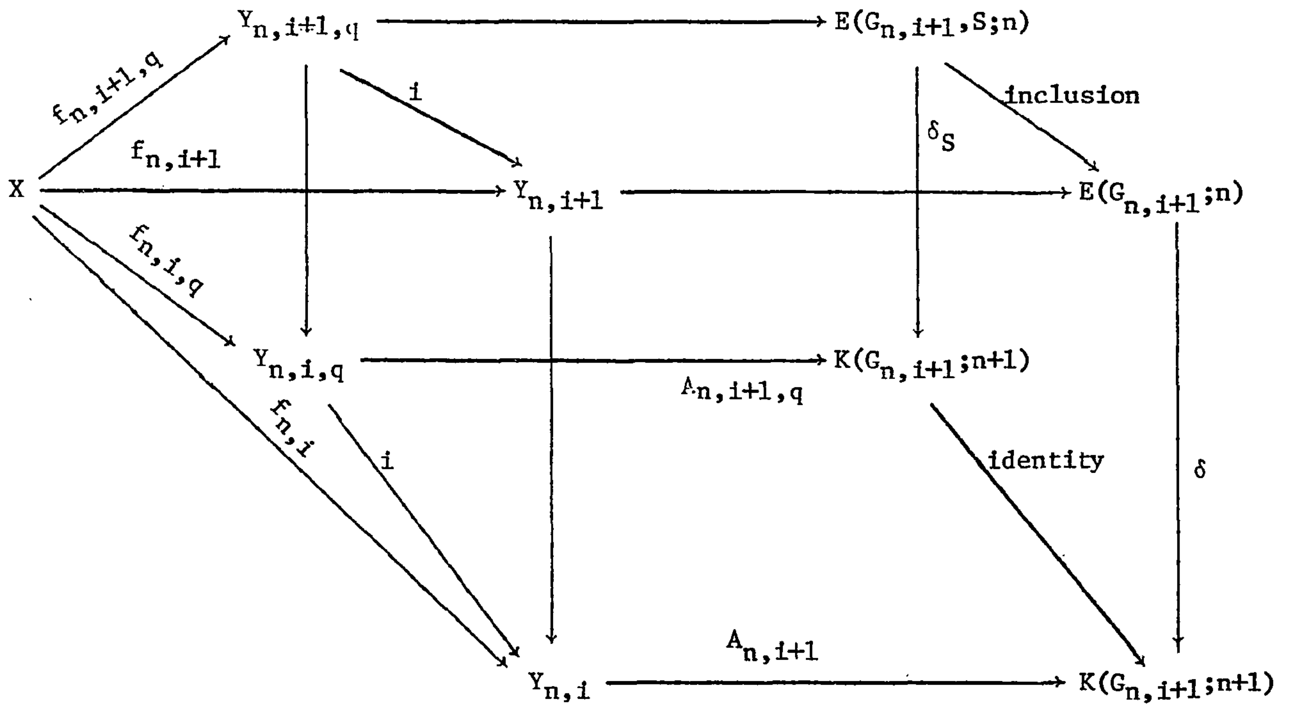
$$A = \{a_1, \ldots, a_r \mid a_j = (g+1)|p_j A_{n,i+1,q}| + 2\}.$$

Letting $S = S(P,A)$, we define $Y_{n,i+1,q}$ to be the pullback

$$\begin{array}{ccc} Y_{n,i+1,q} & \longrightarrow & E(G_{n,i+1},S;n) \\ \downarrow & & \downarrow \delta_S \\ Y_{n,i,q} & \xrightarrow{A_{n,i+1,q}} & K(G_{n,i+1};n+1) \end{array}$$

By property 4.7 of q-deformation retracts, $Y_{n,i+1,q}$ is a q-deformation retract of Q, and hence, by trasitivity, of $Y_{n,i+1}$. Since S is finite, $Y_{n,i+1,q}$ is locally finite.

It remains to define $f_{n,i+1,q}$. But S has been chosen large enough to ensure that $B_{n,i+1,q}: X \to E(G_{n,i+1},S;n)$, and therefore a map is induced by $f_{n,i,q}$ and $B_{n,i+1,q}$. Let $(f_{n,i,q}, B_{n,i+1,q}) = f_{n,i+1,q}$. We now have commuting



Y n,i+1,q
E(G n,i+1,S;n)
f n,i+1,q
i
inclusion
δ S
f n,i+1
X
Y n,i+1
E(G n,i+1;n)
f n,i,q
Y n,i,q
K(G n,i+1;n+1)
A n,i+1,q
f n,i
i
identity
δ
A n,i+1
Y n,i
K(G n,i+1;n+1)

Finally, to complete the induction step, we need to show that $Y_{n,i+1,q}$ is recursive. But by Proposition 3.9, $Y_{n,i+1,q}$ is recursive if $G_{n,i+1}$ has a solvable word problem. This completes the proof. □

Chapter V

COMPUTABILITY OF $\pi_n X$

We have now established the computability of the Potnikov tower of a connected nilpotent locally finite simplicial set X. However, we still need a presentation of $\pi_n X$, $n > 1$.

Theorem 5.1. Let X be a pointed connected nilpotent locally finite simplicial set. Then there is an r.e. abelian group presentation of $\pi_n X$, $n > 1$.

Proof. $f_{n,1}: X \to Y_{n,1}$ is a pointed recursive map with recursive source and target. The the mapping fiber $Tf = Tf_{n,1}$ is also recursive, and hence $H_k Tf$ has an r.e. abelian group presentation, for all $k \geqslant 0$.

Consider the long exact sequence of the fibration $p: Tf \to X$. It is clear that

$$\pi_k Tf \cong \pi_k X, \quad k \geqslant n,$$

and that

$$\pi_k Tf = 0, \quad k < n.$$

Then Tf is n-1 connected, and the Hurewicz homomorphism $h: \pi_n Tf \to H_n Tf$ is an isomorphism.

Since $\pi_n X \cong \pi_n Tf \cong H_n Tf$, this completes the proof. □

BIBLIOGRAPHY

1. Brown, E.H. Jr. "Finite Computability of Postnikov Complexes," *Annals of Mathematics* 65 (1957): 1-20.

2. Gabriel, P., and Zisman, M. *Calculus of Fractions and Homotopy Theory*. New York: Springer-Verlag, 1967.

3. Hilton, P.; Mislin, G.; and Roitberg, J. *Localization of Nilpotent Groups and Spaces*. Amsterdam: North Holland Publishing Co., 1975.

4. May, J.P. *Simplicial Objects in Algebraic Topology*. Princeton, NJ: Van Nostrant, 1967.

5. Quillen, D.G. "The Geometric Realization of a Kan Fibration is a Serré Fibration," *Proceedings of the American Mathematics Society* 19 (1968): 1499-1500.

6. Whitehead, G.W. *Elements of Homotopy Theory*. New York: Springer-Verlag, 1978.