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THE ASYMPTOTIC DISTRIBUTION OF LATTICE POINTS IN HYPERBOLIC SPACE

by

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Introduction

A. Lattice Points in the Plane.

The ordinary lattice in the plane is the set of points (m,n) where m and n are integers. Since these points correspond to those integers which can be written as the sum of squares of two integers, studying this lattice has applications in number theory. (See Hilbert-Cohn-Vossen [4] and Landau [8]).

From one standpoint, a torus (compact Riemann surface of genus 1) is the plane factored by this lattice. A fundamental domain D for the torus is an open unit square whose vertices are lattice points and furthermore the group Γ of translations, $(x,y) \to (x+m,y+n)$ $m,n \in \mathbb{Z}$, tesselates the plane with copies of D, i.e.: $\mathbb{R}^2 = \bigcup_{\gamma\in\Gamma} \gamma \bar{D}$ where $\bar{D}$ is the closure of D, and $\gamma_1 D \cap \gamma_2 D = \emptyset$ for all $\gamma_1 \neq \gamma_2$ in Γ. In this way the fundamental group of the torus, $\mathbb{Z} \times \mathbb{Z}$, is identified with a discrete subgroup Γ of the group of all conformal isometries G of $\mathbb{R}^2$. Note that G is the group generated by translation and rotations whereas reflections (anti-conformal) must be added to obtain all isometries of $\mathbb{R}^2$.

B. A Lattice Point Problem in the Plane.

Letting N_T be the number of lattice points in or on a circle of radius T leads to an interesting problem,

Let $N_T = \#\{(x,y):\ x^2 + y^2 \leq T \quad x,y \in \mathbb{Z}\}$. That $N_T \sim \pi T^2$ is intuitively clear and easy to show (see Hilbert [4]). Thus we have a method for approximating π via counting lattice points, the quantative accuracy of which is discussed in Kendall [6].

Setting $R_T = N_T - \pi T^2$, Landau [8] shows $R_T = O(T^\theta)$ for some $1/2 < \theta < 2/3$. The infimum of the set of θ's having this property is yet to be found. This question has a long history not yet over.

However, it is known that $R_T = O(T^\theta)$ holds for some $1/2 < \theta < 13/20$ but the conjecture that this statement is valid for $\theta = 1/2 + \epsilon$ for any $\epsilon > 0$ remains a conjecture.

Along these lines, Kendall [6] gets an average result by letting the center of the circle vary in a unit square:

$$N_T(x,y) = \#\{(u,v):\ (x-u)^2 + (y-v)^2 \le T\} .$$

Treating $N_T(x,y)$ as a random variable, whose value is $N_T(x,y)$ when $(x,y) \in [0,1] \times [0,1]$ and T are chosen, Kendall shows:

$$\text{(1)} \qquad \text{Mean } (N_T) = \int_0^1\int_0^1 N_T(x,y)\, dx\, dy = \pi T^2$$

$$\text{(2)} \qquad \text{Variance } (N_T) = \int_0^1\int_0^1 \{N_T(x,y) - \pi T^2\}^2\, dx\, dy = O(T) .$$

It is the purpose of this paper to obtain results analogous to (1) and (2) for the Poincare upper half plane H^+.

1. The Isometries of Hyperbolic 2-space.

When the upper half plane $H^+ = \{x + iy \in \mathbb{C}:\ y > 0\}$ is given the Riemannian structure determined by the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ we have what is called the Poincare upper half plane or hyperbolic 2-space. H^+ is then a model for non-Euclidean hyperbolic geometry with constant Gaussian curvature -1.

It is well known that the group G of all conformal isometries of H^+ can be identified with $SL(2,\mathbb{R})/\pm I$, via the action $z \to \frac{az+b}{cz+d}$, and that the geodesics in H^+ consist of straight lines and semi-circles perpendicular to the real axis (see Stoker [13] and Auslander [1]).

Moreover, using the formula $\operatorname{Im}(Tz) = \frac{\operatorname{Im}(z)}{|cz+d|^2}$ $(T \in G)$, it is easy to show that the quantity $\frac{|z_1-z_2|^2}{y_1\, y_2}$ $(z_1, z_2 \in H^+,\ z_1 = x_1 + iy_1,\ z_2 = x_2 + iy$ is invariant under G and hence may be regarded as a function of the hyperbolic distance $\delta(z_1,z_2)$ between z_1 and z_2. Taking $z_1 = i$ and letting z_2 vary along the imaginary axis, we find that this expression is a smooth strictly increasing function of the distance, and so, conversely $\delta(z_1,z_2)$ itself may be regarded as a smooth strictly increasing function of $\frac{|z_1-z_2|^2}{y_1\ y_2}$. Again setting $z_1 = i$ and letting z_2 vary along the imaginary axis, so that $z_2 = iy_2$, we see that $\frac{|y_2-1|^2}{y_2} = 4\sinh^2(\frac{\log y_2}{2})$, and inasmuch as the desired function of this quantity must equal $\delta(i,iy_2) = \log y_2$, we easily find that $\delta(z_1,z_2) = \cosh^{-1}(1 + \frac{|z_1-z_2|^2}{2y_1\ y_2})$.

2. Discrete groups Γ and Compact Fundamental Domains D.

A subgroup Γ of G is discrete if for each $x \in H^+$ the set $\{\gamma x:\ \gamma \in \Gamma\}$ has no accumulation point in H^+. An open subset D of H^+ is a fundamental domain for Γ if (i) $\bigcup_{\gamma\in\Gamma} \gamma\,\bar{D} = H^+$ where $\bar{D}$ is the closure of D and (ii) $\gamma_1 D \cap \gamma_2 D = \emptyset$ for $\gamma_1 \neq \gamma_2$ in Γ. Clearly a fundamental domain is not unique since γD for any $\gamma \in \Gamma$ is also a fundamental domain.

Two points $x,y \in H^+$ are equivalent under Γ if $\gamma x = y$ for some $\gamma \in \Gamma$. Hence from (ii) no two distinct points interior to D are equivalent and from (i) each $y \in H^+$ is equivalent to some $x \in \bar{D}$.

When $x_0 \in H^+$ is not a fixed point of any $\gamma \in \Gamma$, $D = \{x \in H^+:\ \delta(x_0,x) < \delta(\gamma x_0,x)$ for all $\gamma \neq I$ in $\Gamma\}$ is clearly a fundamental domain and since $\delta(x_0,x) = \delta(\gamma x_0,x)$ defines a geodesic, we conclude that D is bounded by a system of geodesic arcs.

We are primarily concerned here with discrete groups Γ having compact fundamental domains D (i.e.: $\bar{D}$ is compact). In this case D is simply a hyperbolic polygon. Furthermore, in [2] Gel'fand shows that when Γ has compact fundamental domain the set of elements conjugate in G to a given element of Γ is closed. Thus Γ can have no parabolic elements since a sequence a_n in G, each conjugate in G to a given parabolic element, always exists for which $a_n \to I$ but no parabolic element is conjugate to I .

3. The Laplacian on H^+ .

The Laplacian derived from the hyperbolic metric on H^+ is the second order differential operator $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ (see Helgason [3]).

It is well known that when Γ has compact fundamental domain D there is a complete orthonormal set of real analytic eigenfunctions $\{\varphi_n\}_{n=0}^{\infty}$, each automorphic under Γ (ie: $\varphi_n(\gamma x) = \varphi_n(x)$ for all $x \in H^+$ and $\gamma \in \Gamma$) with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$, where the λ_n's have finite multiplicities and form a discrete nonpositive set: $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots \downarrow -\infty$. Furthermore, it is known that $\sum_{n=1}^{\infty} \left(\frac{1}{|\lambda_n|}\right)^{1+\epsilon} < \infty$ for any $\epsilon > 0$.

4. A Lattice Point Problem in H^+ .

A lattice in H^+ is a set of points γy as γ runs over a discrete group Γ and y is a given point in H^+ .

A "circle problem" is obtained by letting $N_T(x,y)$ denote the number of lattice points inside or on a hyperbolic disk of radius T centered at x ; i.e.

$$N_T(x,y) = \#\{\gamma y:\ x,y \in D,\ \delta(x,\gamma y) \leq T\} .$$

By investigating the Dirichlet series $\sum_{\gamma\in\Gamma} \cos h^{-s}(\delta(x,\gamma y))$, and employing a theorem of Wiener-Ikehara, Huber [5] was able to show that for fixed x and y : $N_T(x,y) \sim \frac{V(T)}{A}$ where $V(T) = 4\pi(\sinh \frac{T}{2})^2$ = the area of a hyperbolic disk of radius T , and A is the hyperbolic area of D . Note that the limiting behavior does not depend on x or y .

Treating $N_T(x,y)$ as a random variable on D×D it will be shown later (§ 8 and 9) using methods developed in [11] by Selberg that the

$$\text{Mean} = \iint_{D\times D} N_T(x,y)\,dx\,dy = \frac{V(T)}{A}, \text{ and the}$$

$$\text{Variance} = \iint_{D\times D} \left\{N_T(x,y) - \frac{V(T)}{A}\right\}^2 dx\,dy$$ depends on the "small"‡

(i.e. $-1/4 < \lambda < 0$) eigenvalues of the laplacian Δ acting on functions automorphic under Γ.

5. Some Results of Selberg.

In [11] Selberg introduces integral operators L defined by $Lf(x) = \int_{H^+} k(x,y)f(y)dy$ where k is a "point-pair" invariant; i.e., $k(x,y) = k(gx,gy)$ for all $g \in G$ and in particular $k(x,y) = k(y,x)$. L is invariant in the sense that $(Lf_g)(x) = (Lf)_g(x)$ for all $g \in G$ where f_g is the function defined by $f_g(x) = f(gx)$. Add to these the laplacian Δ which is also an invariant operator and we have:

Proposition 1. Invariant operators commute: $LMf(x) = MLf(x)$.

Proof: The proof is in four steps.

Step 1. Invariant operators applied to point pair invariants as a function of the first variable give the same result as when applied to the second: $N_x k(x,y) = N_y k(x,y)$.

Proof of Step 1. An invariant operator applied to a point pair invariant as a function of the first variable is again a point pair invariant. Hence $N_x k(x,y) = N_y k(y,x) = N_y k(x,y)$.

Step 2. Invariant operators commute when applied to a point pair invariant as a function of the first variable:

$$L_x M_x k(x,y) = M_x L_x k(x,y).$$

$\ddagger$ Footnote: Of course, this has little meaning if "small" eigenvalues never occur; however, in [10] Randol shows that they can occur and points out that they are of considerable interest.

Proof of Step 2. $L_x M_x k(x,y) = L_x M_y k(x,y)$ by Step 1 and $L_x M_y k(x,y) = M_y L_x k(x,y)$ since L and M are now operating on independent variables, and finally $M_y L_x k(x,y) = M_x L_x k(x,y)$ by Step 1.

Step 3. f may be radialized about x_0 via $f(x;x_0) = \int_{G_{x_0}} f(Tx)dT$ where G_{x_0} is the stabilizer of x_0 ; i.e., $G_{x_0} = \{T \in G: Tx_0 = x_0\}$, and dT is normalized Haar measure on G_{x_0} . $f(x;x_0)$ is radialized about x_0 in the sense that $f(Tx;x_0) = f(x;x_0)$ for all $T \in G_{x_0}$. Furthermore $f(x;x_0)$ defines a point pair invariant $k(x,y) = f(mx;x_0)$ where $my = x_0$.

Step 3 is the observation that $L_x f(x;y)$ evaluated at $x = y$ is the same as $Lf(y)$:

$$L_x f(x;y)\Big|_{x=y} = L_x \int_{G_y} f(Tx)dT\Big|_{x=y} = \int_{G_y} Lf_T(x)dt\Big|_{x=y}$$

$$= \int_{G_y} Lf(Tx)dT\Big|_{x=y} = Lf(y) .$$

Step 4. Now invariant operators commute:

$$L\,M\,f(x)\Big|_{x=y} = L_x M_x f(x;y)\Big|_{x=y} \quad \text{(by Step 3)}$$

$$= M_x L_x f(x;y)\Big|_{x=y} \quad \text{(by Step 2)}$$

$$= ML\,f(x)\Big|_{x=y} \quad \text{(by Step 3).}$$

Thus, Proposition 1 is proved.

Proposition 2. If φ is an eigenfunction of Δ , i.e., $\Delta\varphi = \lambda\varphi$, then φ is an eigenfunction of all our invariant operators, i.e., $L\varphi = h(\lambda)\varphi$, furthermore the eigenvalue $h(\lambda)$ depends only on L and λ and not on the eigenfunction φ .

Proof: We need the fact that there is a unique normalized eigenfunction $w_\lambda(x,y)$ of Δ, radialized about y, and having eigenvalue λ, i.e.,

(i) $\Delta_x w_\lambda(x,y) = \lambda w_\lambda(x,y)$

(ii) $w_\lambda(Tx,y) = w_\lambda(x,y)$ for all $T \in G_y$

(iii) $w_\lambda(y,y) = 1$.

Since Δ has a set of "representative" eigenfunctions defined by $f_s(x) = [\mathrm{Im}(x)]^s$, $s \in \mathbb{C}$, having eigenvalues $s(s-1)$, we obtain the existence of $w_\lambda(x,y)$ by radializing $w_\lambda(x,y) = \frac{f_s(x;y)}{f_s(y)}$.

To see that $w_\lambda(x,y)$ is unique, it suffices to take y = the origin in the disk model of hyperbolic 2-space. Now by rotational invariance, $w_\lambda(x,0)$ is an even function of r , the Euclidean distance of x from the origin. Accordingly, $w_\lambda(x,0)$, which is real-analytic, can be expanded in powers of r^2 about the origin, with constant term 1 . It is then a simple matter to show that one can recover the coefficients in this expansion by repeatedly applying the hyperbolic laplacian to $w_\lambda(x,0)$ and setting $x = 0$. In other words, $w_\lambda(x,0)$ is completely determined by λ .

Next we show for any invariant operator L that $L_x w_\lambda(x,y)$ is an eigenfunction of Δ : $\Delta_x L_x w_\lambda(x,y) = L_x \Delta_x w_\lambda(x,y)$ since invariant operators commute, and $L_x \Delta_x w_\lambda(x,y) = \lambda L_x w_\lambda(x,y)$. Now $L_x w_\lambda(x,y)$ is a point pair invariant so by the uniqueness of $w_\lambda(x,y)$ we have: $L_x w_\lambda(x,y) = h(\lambda) w_\lambda(x,y)$. At first it appears that $h(\lambda)$ depends on y but the roles of x and y may be exchanged since both $L_x w_\lambda(x,y)$ and $w_\lambda(x,y)$ are point pair invariants, thus $h(\lambda)$ depends only on λ and L .

Now we can prove the proposition: If φ is such that $\Delta\varphi = \lambda\varphi$ and L any invariant operator then $L\varphi = h(\lambda)\varphi$ where $h(\lambda)$ does not depend on φ . Namely, $L\varphi(x)\big|_{x=y} = L_x\varphi(x;y)\big|_{x=y} = \varphi(y)\, L_x w_\lambda(x,y)\big|_{x=y}$

$$= h(\lambda)\,\varphi(y)\; w_\lambda(x,y)\Big|_{x=y} = h(\lambda)\,\varphi(x;y)\Big|_{x=y} = h(\lambda)\,\varphi(x)\Big|_{x=y} .$$

6. <u>An Eigenfunction Expansion for $N_T(x,y)$</u> .

Now let Γ be a discrete subgroup of G having compact fundamental domain D . Let $\{\varphi_n\}_{n=0}^{\infty}$ be a complete orthonormal set of eigenfunction of Δ , automorphic under Γ and having eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$.

Thus, φ_n is an eigenfunction of any invariant operator and in particular:

$$\int_{H^+} k(x,y)\; \varphi_n(y)\; dy = h(\lambda_n)\; \varphi_n(x) .$$

Furthermore, since φ_n is automorphic under Γ we have:

$$h(\lambda_n)\; \varphi_n(x) = \int_{H^+} k(x,y)\; \varphi_n(y)\; dy = \int_D \Big\{ \sum_{\gamma\in\Gamma} k(x,\gamma y) \Big\}\; \varphi_n(y)\; dy .$$

Letting $K(x,y) = \sum_{\gamma\in\Gamma} k(x,\gamma y)$ gives:

$$\int_D K(x,y)\; \varphi_n(y)\; dy = h(\lambda_n)\; \varphi_n(x) .$$

Now since D is compact and $K(x,y) = K(y,x)$, we have a Hilbert-Schmidt kernel K for which we have the standard L^2 expansion:

$$K(x,y) = \sum_{n=0}^{\infty} h(\lambda_n)\; \varphi_n(x)\; \varphi_n(y)$$

or

$$(3) \qquad \sum_{\gamma\in\Gamma} k(x,\gamma y) = \sum_{n=0}^{\infty} h(\lambda_n)\; \varphi_n(x)\; \varphi_n(y) .$$

For the correct choice for k , the left hand side of (3) will be identically $N_T(x,y)$. Now as we have seen, we can identify k with a function, called $\tilde{k}$, on the right half line:

$$k(z_1,z_2) = \tilde{k}\left(\frac{|z_1-z_2|^2}{y_1\; y_2} \right)$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in H^+$.

Now for

$$\tilde{k}(t) = \begin{cases} 1 & t \leq 2(\cosh T) - 2 \\ 0 & t > 2(\cosh T) - 2 \end{cases}$$

we have

$$k_T(x,y) = \begin{cases} 1 & \delta(x,y) \leq T \\ 0 & \delta(x,y) > T \end{cases}$$

Thus, $N_T(x,y) = \sum_{\gamma \in \Gamma} k_T(x,\gamma y)$ and equation (3) becomes:

$$N_T(x,y) = \sum_{n=0}^{\infty} h_T(\lambda_n) \; \varphi_n(x) \; \varphi_n(y)$$

again in the L^2-sense.

7. <u>An Explicit Expression for h_T</u> .

In general, since the eigenvalue $h(\lambda)$ is independent of the eigenfunction, we can employ the "representative" set of eigenfunctions: $f_s(x+iy) = y^s$, $x+iy \in H^+$, $s \in \mathbb{C}$, to obtain an expression for h in terms of λ and k :

$$h(s) \; f_s(w) = \int_{H^+} k(w,z) \; f_s(z) \; dz$$

where $w = w_1 + iw_2$, $z = x + iy \in H^+$, $\lambda = s(s-1)$. That is,

$$h(s)w_2^s = \int_0^{\infty}\int_{-\infty}^{\infty} \tilde{k}\left(\frac{\|w-z\|^2}{w_2\, y}\right) y^s \, \frac{dx\, dy}{y^2} ,$$

and letting $w = i$:

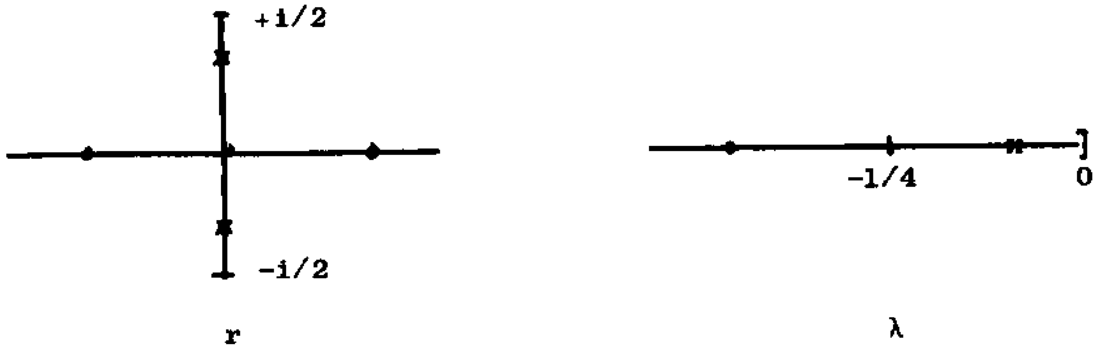
$$h(s) = \int_0^{\infty} \int_{-\infty}^{\infty} \tilde{k}\left(\frac{x^2+(1-y)^2}{y}\right) y^s \, \frac{dx\, dy}{y^2} ,$$

a change of parameter $s = 1/2 + ir$, and a change of variable leads to

(4) $$h(r) = \int_0^\infty y^{ir}\,\frac{dy}{y}\int_{c(y)}^\infty \tilde{k}(x)\{x - c(y)\}^{-1/2}\,dx$$

where $c(y) = \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2$.

Note that since $\lambda = -(1/4 + r^2)$ there are two r's for each λ except when $\lambda = -1/4$ which corresponds to $r = 0$. In fact, for each $\lambda < -1/4$ there are two r's symmetrically placed on the real axis and for each $\lambda > -1/4$ there are two r's symmetrically placed on the imaginary axis between $+i/2$ and $-i/2$:



Now we compute h_T , regarded as a function of r . From equation (4), and the fact that $c(y) \le 2(\cosh T) - 2$ if and only if $e^{-T} \le y \le e^{T}$, we have:

$$h_T(r) = \int_{e^{-T}}^{e^{T}} y^{ir}\,\frac{dy}{y}\int_{c(y)}^{2(\cosh T)-2}\{x - c(y)\}^{-1/2}\,dx .$$

Thus,

$$h_T(r) = 2\int_{e^{-T}}^{e^{T}} y^{ir}\,\{2(\cosh T) - 2 - c(y)\}^{1/2}\,\frac{dy}{y} .$$

By the change of variable $y = e^x$ and using the identities $2(\cosh y) - 2 = c(e^y) = 4(\sinh y/2)^2$ we obtain

$$h_T(r) = 4 \int_{-T}^{T} e^{irx}\{(\sinh T/2)^2 - (\sinh x/2)^2\}^{1/2}\, dx .$$

Another change of variable gives:

$$h_T(r) = 4T \int_{-1}^{1} e^{irTx}\{(\sinh T/2)^2 - (\sinh Tx/2)^2\}^{1/2}\, dx ,$$

so

$$(5) \qquad h_T(r) = 8T \int_0^1 \cos r\, Tx\{(\sinh T/2)^2 - (\sinh Tx/2)^2\}^{1/2}\, dx ,$$

and finally:

$$(6) \qquad h_T(r) = 8T \sinh T/2 \int_0^1 \cos r\, Tx\left\{1 - \left(\frac{\sinh Tx/2}{\sinh T/2}\right)^2\right\}^{1/2} dx .$$

8. <u>The Mean of $N_T(x,y)$</u> .

Going back to equation (5) we see that $h_T(\pm i/2) = V(T)$ since:

$$h_T(\pm i/2) = 8T \int_0^1 \cos iTx/2\{(\sinh T/2)^2 - (\sinh Tx/2)^2\}^{1/2}\, dx$$

gives

$$h_T(\pm i/2) = 8T \int_0^1 \cosh Tx/2\{(\sinh T/2)^2 - (\sinh Tx/2)^2\}^{1/2}\, dx$$

and by the change of variable $y = \sinh Tx/2$ we get:

$$h_T(\pm i/2) = 16 \int_0^{\sinh T/2} \{(\sinh T/2)^2 - y^2\}^{1/2}\, dy .$$

Thus,

$$h_T(\pm i/2) = 16 \int_0^a \sqrt{a^2 - y^2}\, dy$$

where $a = \sinh T/2$.

So:

$$h_T(\pm i/2) = 16\left(\frac{\pi a^2}{4}\right) = 4\pi a^2 = 4\pi(\sinh T/2)^2 = V(T) .$$

Combining this result with the fact that $\varphi_0 = \frac{1}{\sqrt{A}}$, equation (3) becomes:

$$N_T(x,y) = \frac{V(T)}{A} + \sum_{n=1}^{\infty} h_T(\lambda_n)\ \varphi_n(x)\ \varphi_n(y) . \tag{7}$$

Hence, we have:

$$\text{Mean } (N_T) = \iint_{D\times D} N_T(x,y)\,dx\,dy = \frac{V(T)}{A} .$$

9. The Asymptotic behavior of Variance (N_T) .

From equation (7) we obtain an expression for Variance (N_T) :

$$\text{Variance } N_T = \iint_{D\times D} \left\{N_T(x,y) - \frac{V(T)}{A}\right\}^2 dx\,dy = \sum_{n=1}^{\infty} \{h_T(\lambda_n)\}^2$$

Theorem. If $\lambda_1 < -1/4$, then

$$\text{Variance } (N_T) = O(T^{\delta} e^{T}) \text{ for any } \delta > 0 .$$

If $-1/4 \le \lambda_1 < 0$ then

$$\text{Variance } (N_T) \sim c_1 e^{(1+2\beta_1)T}$$

where $c_1 = \left(2 \sum_{n=0}^{\infty} \frac{a_n}{n+\beta_1}\right)^2$, $\beta_1 = \sqrt{\lambda_1 + 1/4}$, and the a_n's are defined by

$$\sum_{n=0}^{\infty} a_n x^n = \sqrt{1-x} .$$

Assuming $\beta_1 > \frac{1}{4}$, if $-1/4 < \lambda_N \le \lambda_{N-1} \le \dots \le \lambda_1 < 0$ and $\beta_k > \beta_1 - 1/4$ for $k = 2,3,\dots,N$ then Variance $(N_T) = c_1\, e^{(1+2\beta_1)T} + c_2\, e^{(1+2\beta_2)T}$

$+ \cdots\cdots + c_N e^{(1+2\beta_N)T} + O(e^{(1/2+2\beta_1)T})$ where $c_k = \left(2 \sum_{n=0}^{\infty} \frac{a_n}{n+\beta_k}\right)^2$,

$\beta_k = \sqrt{\lambda_k + 1/4}$ and the a_n's are as above. If $\beta_1 \le 1/4$, all is the same except the error term is $O(T^\delta e^T)$ if there is no $\lambda = -\frac{1}{4}$ and $O(T^2 e^T)$ if there is.

Proof: The theorem is proved via two lemmas:

Lemma 1. For any $\delta > 0$, $\sum_{\lambda_n < -1/4} \{h_T(\lambda_n)\}^2 = O(T^\delta e^T)$.

Proof of Lemma 1. Recall that for $\lambda \le -1/4$ a corresponding r is real. Letting, for convenience of notation, $f(T,x) = \left(1 - \left(\frac{\sinh Tx/2}{\sinh T/2}\right)^2\right)^{1/2}$ equation (6) becomes

$$h_T(r) = 8T \sinh T/2 \int_0^1 \cos rTx\, f(T,x)\, dx .$$

Integration by parts yields:

$$h_T(r) = \frac{4T}{r \sinh T/2} \int_0^1 \frac{\sin rTx \sinh Tx/2 \cosh Tx/2}{f(T,x)}\, dx$$

so:

$$h_T(r) = \frac{4T}{r \sinh T/2} \left(\int_0^{1-\epsilon} \frac{\sin rTx \sinh Tx/2 \cosh Tx/2}{f(T,x)}\, dx \right.$$

(8)
$$\left. + \int_{1-\epsilon}^{1} \frac{\sin rTx \sinh Tx/2 \cosh Tx/2}{f(T,x)}\, dx\right)$$

where $0 < \epsilon < 1$ is to be chosen. Applying the second mean value theorem to the first integral gives, for some $\xi \in [0,1-\epsilon]$:

$$\int_0^{1-\epsilon} \frac{\sin rTx \sinh Tx/2 \cosh Tx/2}{f(T,x)}\, dx,$$

$$= \frac{\sinh T(1-\epsilon)/2 \cosh T(1-\epsilon)/2}{f(T,1-\epsilon)} \int_\xi^{1-\epsilon} \sin rTx\, dx$$

$$= \frac{\sinh T(1-\epsilon)/2 \cosh T(1-\epsilon)/2}{f(T,1-\epsilon)} \cdot \frac{\cos rT\xi - \cos rT(1-\epsilon)}{rT}$$

$$= 0\left(\frac{\sinh T(1-\epsilon)/2 \cosh T(1-\epsilon)/2}{r\ T\ f(T,1-\epsilon)} \right)$$

Letting $\epsilon = \frac{1}{rT^{1+\delta}}$ $(\delta > 0)$ gives $\epsilon T \to 0$ as $T \to \infty$ thus:

$$\sinh \frac{T(1-\epsilon)}{2} \cosh \frac{T(1-\epsilon)}{2} = 0(e^T)$$

and

$$f(T,1-\epsilon) = \Omega(\sqrt{1-e^{-\epsilon T}}) = \Omega(\sqrt{\epsilon}\, T) = \Omega\left(\frac{1}{\sqrt{r}\ T^{\delta/2}} \right)$$

so

$$\int_0^{1-\epsilon} \frac{\sin rTx \sinh Tx/2 \cosh Tx/2}{f(T,1-\epsilon)} dx = 0\left(\frac{e^T}{\sqrt{r}\ T^{1-1/2\,\delta}} \right).$$

Now consider the second integral in equation (8) :

$$\left| \int_{1-\epsilon}^{1} \frac{\sin rTx \sinh Tx/2 \cosh Tx/2}{f(T,x)} dx \right|$$

$$\leq \int_{1-\epsilon}^{1} \frac{\sinh Tx/2 \cosh Tx/2}{f(T,x)} dx$$

$$= \frac{2(\sinh T/2)^2}{T} \cdot f(T,1-\epsilon) = 0\left(\frac{e^T}{\sqrt{r}\ T^{1+1/2\,\delta}} \right).$$

Finally we obtain

$$h_T(r) = 0\left(\frac{T}{re^{T/2}} \left(\frac{e^T}{\sqrt{r}\ T^{1-1/2\,\delta}} + \frac{e^T}{\sqrt{r}\ T^{1+1/2\,\delta}} \right) \right)$$

$$= 0\left(\frac{T^{\delta/2}\, e^{T/2}}{r^{3/2}} \right).$$

Combine this result with the fact that $\sum_{r_n \text{ real}>0} \frac{1}{r_n} < \infty$ and

Lemma 1 is proved.

<u>Lemma 2</u>. If $\beta = \sqrt{\lambda+1/4}$ where r corresponds to a λ in $(-1/4,0)$

then $h_T(r) \sim \left(2 \sum_{n=0}^{\infty} \frac{a_n}{n+\beta}\right) e^{(1/2+\beta)T}$ where the a_n's are defined by

$\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n$.

<u>Proof</u>: Again letting $f(T,x) = \left(1 - \left(\frac{\sinh Tx/2}{\sinh T/2}\right)^2\right)^{1/2}$ we have

$$h_T(r) = 8T \sinh T/2 \int_0^1 \cos rTx\, f(T,x)\, dx$$

$$= 8T \sinh T/2 \int_0^1 \cos rTx \sqrt{1 - e^{(x-1)T}}\, dx + O(e^{\beta T})$$

since

$$\left| \int_0^1 e^{\beta Tx}\left(\sqrt{1-e^{(x-1)T}} - f(T,x)\right)dx \right| = O\left(\frac{e^{\beta T}}{T\sinh T/2}\right) .$$

(Here we are using the fact that $\left|\sqrt{A} - \sqrt{B}\right| \le \sqrt{|A-B|}$). Now

$$\int_0^1 \cos rTx \sqrt{1 - e^{(x-1)T}}\, dx = \int_0^1 \cos rTx\left(\sum_{n=0}^{\infty} a_n e^{(x-1)nT}\right) dx$$

where the a_n's are the binomial coefficients of $\sqrt{1-x}$, i.e.,

$\sqrt{1-x} = \sum_{n=0}^{\infty} a_n x^n$ which converges uniformly on the closed interval

$[0,1]$. Thus

$$\int_0^1 \cos rTx\left(\sum_{n=0}^{\infty} a_n e^{(x-1)nT}\right) dx = \sum_{n=0}^{\infty} a_n e^{-nT} \int_0^1 \cos rTx\, e^{nTx}\, dx$$

$$= \sum_{n=0}^{\infty} a_n e^{-nT}\left(\frac{e^{nT}(n \cos rT + r \sin rT) - n}{(n^2 + r^2)T}\right)$$

From this, using $r = i\beta$ for $0 < \beta < 1/2$, it is readily shown that

$$h_T(r) \sim \left(2 \sum_{n=0}^{\infty} \frac{a_n}{n+\beta}\right) e^{(1/2+\beta)T} .$$

Thus Lemma 2 is proved.

Recalling that Variance $(N_T) = \sum_{n=1}^{\infty} \left\{h_T(r_n)\right\}^2$, we see that Lemma 1 and Lemma 2 prove the Theorem.

10. Some Concluding Remarks concerning Noncompact D.

If the fundamental domain D for Γ has finite hyperbolic area but is noncompact the analysis is complicated by the presence of a continuous spectrum in addition to the discrete spectrum $h(\lambda_n)$. Selberg ([11],[12]) shows how to obtain the L^2 spectral decomposition:

$$\sum_{\gamma\in\Gamma} k(x,\gamma y) = \sum_{n=0}^{\infty} h(\lambda_n)\ \varphi_n(x)\ \varphi_n(y) \tag{9}$$

$$+ \sum_{p\in P} \frac{1}{4\pi} \int_{-\infty}^{\infty} E_p(x,1/2+ir)\ E_p(y,1/2-ir) h(r)\ dr$$

where P is a complete finite set of inequivalent cusps lying on the boundary of D and $E_p(x,s)$ is the Eisentein series associated with P (see Kubota [7]).

In a recent paper Patterson [9] applied to (9) the point pair invariant corresponding to

$$\tilde{k}_T(t) = \begin{cases} 1 - \dfrac{t}{2(\cosh T)-2} & t \le 2(\cosh T)-2 \\ 0 & t > 2(\cosh T)-2 \end{cases}$$

for which he shows equation (9) to be absolutely convergent and furthermore obtains:

$$N_T(x,y) = \frac{V(T)}{A} + \sum_{1/4<\beta_n<1/2} \sqrt{\pi}\ \frac{\Gamma(\beta_n)}{\Gamma(\beta_n+3/2)}\ e^{(1/2+\beta_n)T}\ \varphi_n(x)\ \varphi_n(y)$$

$$+\ O(e^{3/4T})\ .$$

This generalizes Huber's result to the noncompact (finite hyperbolic area) case.

BIBLIOGRAPHY

[1] Auslander, L., Differential Geometry, (1967), Harper and Row.

[2] Gel'fand, I., Representation Theory and Automorphic Functions, (1968-1969), Saunders Mathematics Books.

[3] Helgason, S., Differential Geometry and Symmetric Spaces, (1962), Academic Press.

[4] Hilbert, D., and Cohn-Vossen S., Geometry and the Imagination, (1952), Chelsea Publishing Company.

[5] Huber, H., Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen, Math. Ann. 138 (1959), 1-26.

[6] Kendall, D., On the Number of Lattice Points inside a Random Oval, Quart. J. Math., Oxford Ser. 19, 1-26 (1948).

[7] Kubota, T., Elementary Theory of Eisenstein Series, (1973), Kodansha Ltd., Tokyo.

[8] Landau, E., Vorlesungen uber Zahlentheorie, (1927), Chelsea Publishing Company.

[9] Patterson, S., A Lattice-Point Problem in Hyperbolic Space, Mathematika 22 (1975), 81-88.

[10] Randol, B., Small Eigenvalues of the Laplace Operator on Compact Riemann Surfaces, Bull. Amer. Math. Soc., Vol. 80, No. 5, Sept. 1974.

[11] Selberg, A., Harmonic Analysis and Discontinuous Groups in Weakly Symmetric Riemannian Spaces with Applications to Dirichlet Series, J. Indian Math. Soc., 20 (1956), 47-87.

[12] ____________ Discontinuous Groups and Harmonic Analysis, Proc. I.C.M., Stockholm, 1962.

[13] Stoker, J., Differential Geometry (1969), Wiley-Interscience.

AUTOBIOGRAPHICAL STATEMENT

William Wolfe was born in New York City on September 30, 1949. He was graduated from William C. Bryant High School in 1966, Queensborough Community College in 1969, and Queens College in 1972. In June 1972 he married the former Altagracia Garcia. He entered The City University Graduate School in the Fall of 1972 to study mathematics. He began teaching at Herbert H. Lehman College under the title of Graduate Fellow A in the Fall of 1973. He is currently an Adjunct Instructor at New York Institute of Technology.

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