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by

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Abstract

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The properties of the conjugacies between infinitely renormalizable maps of an interval are studied here. In the main part of the work we deal with the renormalization with uniformly bounded return time. For such maps we show the conjugacies, that a priori are arbitrary homeomorphisms, to be quasisymmetric on the whole domain intervals.

Subsequently, we examine the properties of these conjugacies reduced to smaller domains, namely the closures of the orbits of critical points. Here we show that the classes of Lipschitz continuous equivalence coincide with the classes of C^1–smooth equivalence. The proof is based on a more general argument asserting that bilipschitz continuous conjugacies between two ω–limit sets containing dense subsets of preimages of respective critical points are differentiable with nonvanishing derivative at either of the critical points.

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Contents

1 Introduction

This work is devoted to the study of the properties of conjugacies of infinitely renormalizable unimodal maps of an interval.

Iterated maps of an interval have been a subject of vigorous research for quite a few recent years, the unexpected richness of the patterns of their behavior attracting the interests of physicists as well as pure mathematicians. Their joint efforts gave rise to what is commonly called the nonlinear science. The thorough survey of classical (up to 1980) results in this field is given in the renowned book by Collet and Eckmann [1].

However, a great deal of development, predictably enough creating a good many of new problems, has been done since the book came into print. This is also true for a particularly intensely researched class of maps called *infinitely renormalizable*.

Consider a continuous mapping f of an interval $I = [a, b]$ into itself that takes ∂I into itself and has exactly one turning point c in (a, b), a maximum, so that f is strictly increasing on $[a, c]$ and strictly decreasing on $[c, b]$.

One of the possible ways to describe the property that f is *renormalizable* is to request that there exists a neighborhood of c, contained with its closure in (a, b) such that some finite iterate of f takes this neighborhood back into itself. For an f that has this property the the renormalized map Rf is defined, up to an affine change of coordinates, by choosing the maximal neighborhood of c and the minimal iterate of f. We usually rescale the renormalized map back onto the original interval and fix the orientation so that the new turning point is again a maximum. Infinite renormalizability of f is tantamount to the existence of an infinite nested family of neighborhoods of c with diameter decreasing to zero, such that on each of them f is renormalizable in the above sense with the number of iterates growing with the nesting.

In our work, except for the result of Chapter 6, where we work in more general setting, we focus on renormalization with uniformly bounded return times (see Chapter 4 for precise definition).

The main objective of the former and ongoing research is to classify these objects. First, we have the notion of *topological* equivalence. Here we say that two maps of an interval f and g are *topologically equivalent* (or *conjugate*), if there exists a homeomorphism h between the domains of the two maps such that $g \circ h = h \circ f$.

We first examine the properties of the conjugacies on the whole domain intervals. We prove that under mild smoothness requirements (discussed in Chapter 3) classes of topological equivalence of our maps coincide with the classes of quasisymmetric equivalence (a homeomorphism h is called quasisymmetric if, for some finite $M > 0$ it satisfies the condition

$$\frac{1}{M} < \frac{h(x+t) - h(x)}{h(x) - h(x-t)} < M$$

uniformly, for all triples of points $\{x - t, x, x + t\}$ in the domain).

Other results of that kind involve, chronologically, those of Sullivan ([6], Chapter 15), Jakobson and Swiatek [2] and Jiang [3]. Their proofs require stronger hypotheses, however we note that the work of Jiang is based on quite an akin concept. We also would like to emphasize that our proof admits the situation where the maps shown to be quasisymmetrically conjugate do not have the same type of singularity at their turning point. Bounded return time of the renormalization is almost certainly essential for this to be true, though it is believed that this assumption may not be of significance for maps sharing the same type of singularity.

The prerequisite for our approach to the problem of quasisymmetricity of the conjugacies is Theorem 1 of Chapter 3 of [6], which provides the tool for controlling the shapes of renormalized maps. We derive our theorem (Theorem 2) in the case of infinitely renormalizable folding maps of bounded type from an analogous statement for Markov maps of an interval (Theorem 1) which are conjugate by the same homeomorphism and are piecewise expanding in a metric (also constructed piecewise) which is smoothly equivalent to the Euclidean. Those Markov maps are more general objects than the infinitely renormalizable maps they arise from and in particular their distinguished point (the one that belongs to the ranges of all the branches) 'forgets' the singularity so that we can compare maps with different singularities.

A known important consequence of the quasisymmetricity of the conjugacies of our maps is the uniqueness of their representation in one-parameter quadratic family $\{-x^2+a\}$, which in other words means that the corresponding component in the Mandelbrot set actually reduces to a single point.

However, no technique in known so far that would allow to derive such a conclusion in general for families of the form $\{-|x|^\alpha + a\}$ with α real, $\alpha > 1$.

Having dealt with the quasisymmetric properties of h on the whole domain we then turn our focus to the more challenging question of the properties of h on the Cantor-set closure of the post-critical orbit. The stunning discovery of Coulet-Tresser and Feigenbaum of mid-70's, known as 'renormalization convergence', in today's language describes as Holder continuity of the derivative of h ($C^{1+\gamma}$–smoothness) on that Cantor set. In that direction, apart from the numerical computer-generated evidence, first there came the result of Collet and Eckmann (see for example [1]), giving, in case of analytic maps with a singularity of the type $|x|^\alpha$ with α close to 1, a local version of the desired statement for the simplest example of infinite renormalizability (so-called period doubling situation). More precisely, they proved the existence of a fixed point of the renormalization operator and that in some vicinity of that point renormalization acting on maps conjugate to that fixpoint is a contraction.

In 1982 Lanford ([4]) has performed a computer-assisted proof of the similar result for maps with quadratic singularity. However the long expected *global* result (the eventually universal structure of the post-critical Cantor set within the class of topological equivalence of infinitely renormalizable maps sharing the same singularity) had not been proven until the work of Sullivan [6] that gives the 'pure thought' insight into the inducing cause of renormalization convergence through the theory of quasiconformal mappings and the Teichmuller theory. His arguments work for all infinitely renormalizable real-analytic maps with universally bounded return times, but the quest for solution to the case of singularities r real, $r > 1$ which do not admit holomorphic extension about the domain interval remains open. Here we were able to make the following contribution: we can prove (Theorem 5 and Corollary 3) that conjugacies that are Lipschitz continuous on the respective Cantor sets are automatically C^1–smooth. This way we get a global result, for all singularities $r > 1$, at the cost of a much stronger starting hypotheses.

The techniques we use are purely real-variable and although the logic of the proofs may seem a bit intricate (or perhaps weird), the arguments are very elementary.

We do not want to suggest that closing the gap between quasisymmetric (that translates into bounded geometry in case of Cantor sets we are dealing with) and Lipschitz is possibly easy. Actually, even an expectedly simpler task of getting $C^{1+\gamma}$ rather than merely C^1 in the above setting is not yet achieved and almost certainly requires a more sensitive version of the tool used in Chapter 7. However, as far as we know, the argument we give is the only known so far to give even a partial result toward eventual universality of the Cantor set that works for *all* singularity types.

Finally, we outline the structure of this work.

We begin (Chapter 2) by formulating two Theorems, 1 and 2, firstly asserting the property of quasisymmetricity of conjugacies between Markov maps subject to several conditions listed as topological (T1-T5) and analytical (A1-A4) 'axioms' about those maps; secondly stating that the situation of infinitely renormalizable maps of bounded type can be reduced to the Markov case. In the remainder of this chapter we prove Theorem 1.

Chapter 3 is a study of smoothness required for the proof of Theorem 2 to be carried out. The main result there (Theorem 3 followed by Remark 1) is that for homeomorphisms of the interval in the class $C^{1+Zygmund}$ (see that chapter for the definition) the distortion of the Poincare metric on subintervals of the (standard) length t is a Lipschitz continuous function in t. This result will further (Chapter 5) allow us to work outside the Cantor set.

In Chapter 4 we provide the definitions and basic properties relevant in our situation and then explain (Proposition 3) why the tip of a renormalizable map cannot fall too close to the diagonal and how (Theorem 4) this impiies (after sufficiently many renormalizations) the definite repelling of fixed points.

In Chapter 5 we construct a sequence of dynamical partitions of the interval with geometrically decreasing diameter. The fundamental analytical step (Lemma 1) in the proof of Theorem 2 is done there. This lemma lets us verify axiom A3 for the generated Markov map.

We conclude this chapter (and the first part of our work) with the construction of the Markov map related to the starting infinitely renormalizable map and satisfying the axioms of Chapter 2. This way the reduction of Theorem 2 to Theorem 1 is complete.

We point out that the proof of Theorem 2 is very 'intrinsic' in the sense that we check some properties that *any* map in our class must have and by virtue of those properties shared by all of the maps in the class we derive the quasisymmetricity of their conjugacies.

Quite a different approach is employed in the last two chapters (Chapters 6 and 7). There we start with the conjugacy between post-critical orbits of the two maps and introduce the notion of distortion of the *conjugacy* itself from being smooth. The power-law singularity makes this distortion decrease every time the itinerary passes through small enough vicinity of the critical point c. For we are close to c infinitely many times we conclude that, in the infinitesimally small scale, the distortion is non-existent and so is the conjugacy differentiable (Theorem 5 and Corollary 3). Technically the argument turns out more complicated, but the underlying idea is exactly as told. We emphasize again that unlike in the first part we work with *a pair of maps* (and their conjugacy) all the way through. While the original argument can be carried out also in the setting of Holder bounds, the problem is that only with Lipschitz bounds in place it is true that what we get from this argument is more than we put into it.

Why the Lipschitz bounds should hold for all singularities $r > 1$ remains a mystery.

2 Markov maps

In this chapter we state and prove a theorem about quasisymmetricity of conjugacies of Markov maps of the kind that arises from infinitely renormalizable unimodal maps of an interval under mild hypothesis about smoothness discussed in detail in Chapter 3. The benefit of this approach over the direct proof is twofold: the argument is clearer and it frees us from the worry about the condition that makes the two branches of the reconstructed unimodal map meet smoothly at the critical point. Markov maps like those discussed here may possibly fail to correspond to *smooth* at the critical point unimodal maps, and so we get a more general statement. Here we give a theorem.

Theorem 1 *If f_1 and f_2 are two topologically conjugate maps satisfying conditions (T1-T5) and (A1-A4) below, then the conjugacy is quasisymmetric. The quasisymmetric constant depends only on constants and bounds in the (T's) and (A's).*

Axioms for the Markov maps Set $J_0 = \langle 0,1\rangle$, $c \in (0,1)$ and let $\{J_n\}_{n=0}^{\infty}$ be a nested family of closed neighborhoods of c, $J_{n+1} \subset J_n$, $\partial J_{n+1} \subset Int J_n$, $\bigcap_{n=0}^{\infty} J_n = \{c\}$.

Let $f : \langle 0,1\rangle \to \langle 0,1\rangle$ satisfy the following:

T1. $f(J_n) \subset J_n$.

T2. $f(\partial J_n) \subset \partial J_n$, $f(\partial J_n) \neq \partial J_n$.

T3. f is continuous on each of the intervals L_n, R_n constituting $J_n \setminus J_{n+1}$, strictly increasing on one of them and strictly decreasing on the other. $f(L_n) \supset J_{n+1}$, $f(R_n) \supset J_{n+1}$.

T4. any two preimages (taking into account both branches) of $Int J_{n+1}$ under $f|_{J_n}$ are either disjoint or coincide. The union of the preimages of all orders fills in J_n, except for a nowhere dense set.

T5. if $\hat{f}$ is a mapping such that $\hat{f} \equiv f$ on the interior of each L_n, R_n and $\hat{f}$(endpoint of J_{n+1}) = continuous extension of $\hat{f}$ in $J_n \setminus J_{n+1}$ then one of the endpoints of J_{n+1} is periodic under $\hat{f}$ and $\hat{f}$(one endpoint of J_{n+1}) = $\hat{f}$(the other endpoint of J_{n+1}).

Note 1 *Let $\hat{P}_n$ be the family of preimages of J_n within J_{n-1} and $P_n = \hat{P}_n \setminus \{J_n\}$. Then f is Markov with respect to the partition $P = \bigcup P_n$.*

A1. For each n there exists a metric on the set $L_n \cup R_n$, smoothly equivalent to the standard Euclidean metric on that set such that:

1. the bounds on the metric equivalence are uniform in n,

2. the length distortion on the two preimages of J_{n+1} of the first degree is bounded, and uniformly so, in n,

3. on the rest of $L_n \cup R_n$ the absolute value of the derivative of f with

respect to that metric is between two constants C_1, C_2, $1 < C_1 < C_2 < \infty$. The constants can be chosen independently of n.

A2. There exist universal $\alpha > 0$ and γ, $0 < \gamma < 1$, such that for any preimage by f of J_l contained in J_k for some $k < l$ we have

$$\frac{\text{length of the preimage}}{\text{length of } J_k} \leq \alpha\gamma^{(l-k)}.$$

A3. The ratio $\frac{|J_{n+1}|}{|J_n|}$ is (uniformly in n) bounded away from 0 and 1.

A4. Except for the case (related to the period doubling dynamics) when an endpoint of J_{n+1} is a fixed point of $\hat{f}$, the absolute value of the eigenvalue of $\hat{f}$ along the periodic orbit of an endpoint of J_{n+1} coincides with the absolute value of the eigenvalue of $\hat{f}$ at the fixed point, being an endpoint of J_{n+1}. In the exceptional case the former is the square root of the latter.

Now consider an infinitely renormalizable map of the interval. When we start to renormalize some kind of 'pathological' behavior *is* possible for a time. For instance we may have the dangerous situation of 'renormalization of degree 1', described in the point b) of the Lemma of §4 of [6], namely that the basic renormalization interval is sort of too large and admits 'extra' fixed points inside it other than the natural two fixed points, one at the boundary and the other between the critical point and the second boundary point. We can have no control over the eigenvalues of such 'interior' fixed points, and those points can be, for instance, neutral. This gives no ground for the distortion estimates and maps with these 'pathologies' can truly fail to be quasisymmetrically conjugate even if they have the same sequence of appearance of pathological fixed points. This is one of the major technical difficulties compared to the negative-Schwartzian case. However, after renormalizing sufficiently many times all those pathologies must disappear. This will be proved later in Chapter 4; here we formulate the following

Theorem 2 *Sufficiently high renormalizations of an infinitely renormalizable unimodal map with bounded combinatorics of type* $\leq T$ *generate, in a natural way, Markov maps satisfying (T1-T5) and (A1-A4) with constants depending only on the bound* T *on combinatorics and the exponent of the power law at the singularity.*

By the above theorems we shall immediately have

Corollary 1 *Sufficiently high renormalizations of two infinitely renormalizable unimodal maps of bounded type with the same kneading sequence are quasisymmetrically conjugate with quasisymmetric constants depending only on the bound on combinatorics and the singularity types of those two maps.*

While the following chapters will be devoted to reducing Theorem 2 to Theorem 1, we now show how to prove Theorem 1.

Proof of Theorem 1. First we describe the structure of the proof. The general idea is to take an interval in the domain of f (think of a short one) with its midpoint and iterate it forward until some dynamically defined stopping moment. Axioms guarantee that the quasisymmetric distortion on the way was universally bounded so the image of the midpoint divides the image of the interval into two quasi-halves. Within each quasi-half we find a dynamically defined subinterval whose length, solely on the grounds of the dynamics and the axioms, can be shown to be comparable to the length of the respective quasi-half. We go to the conjugate picture for g and there, by the above, the conjugates to the dynamically defined quasi-equal subintervals are quasi-equal and their lengths are comparable to the length of the whole conjugate interval, which was divided into two quasi-halves. Now recall that the image under the conjugacy of the quasi-midpoint in the first picture is trapped in between the images of those two dynamically defined subintervals, so it is bounded away from the endpoints. Thus the image of the quasi-midpoint in the first picture is still quasi-centered within the conjugate interval in the other picture.

Again we use pullback in the picture for g, keeping this information (distorted only boundedly) until reaching the conjugate of the original short interval, and we get the desired statement.

Now we go to a more detailed account.

Let Λ be an interval contained in J_0. First we go along the itinerary of Λ under f so long as the images of Λ do not contain any of the endpoints of the intervals of the family $\{J_n\}$. By Axiom T4 preimages of J_l's within J_0 form, when l goes to infinity, a nested family of partitions and by Axiom A2 the diameter of the largest element tends to zero when $l \to \infty$. So, there are finite smallest nonnegative integers p and l such that $f^p(\Lambda)$ istersects both $J_l \setminus J_{l+1}$ and J_{l+1}. In these first p steps the quasisymmetric distortion is bounded because by Axiom A1 we can, uniformly in i, extend each of the branches of f over a definite proportional neighborhoods of L_i and R_i, perhaps slightly worsening the bound. Poincare metric on such a definite neighborhood of L_i (or R_i), restricted to L_i (or R_i), is equivalent, with uniform bounds, to the standard metric. So by Axioms A2 and A1 we have the bound on the sum of the $P-$lengths of the first $(p-1)$ images of Λ. This, given sufficient smoothness (see Chapter 3) is enough to yield bounded quasisymmetric distortion by the argument in [6].

The moment the image of Λ contains a monotonicity point we can no longer use the above argument, but periodicity of the endpoints shall provide for the necessary replacement.

Perhaps adding one more circuit we may assume (T2) that we hit a periodic point. Let M' be the part of $f^p(\Lambda)$ contained in $J_l \setminus J_{l+1}$ and N' be the part of $f^p(\Lambda)$ contained in J_{l+1}. We shall further act inside J_l (T1) by f on N' and by f^q on M', where q is the smallest positive integer such that $\hat{f}^q(\partial J_{l+1}) \subset \partial J_{l+1}$, except that in case we got $q = 1$ we change it and put $q = 2$ (this is justified dynamically in Chapters 4 and 5 and fits with axiom A4). By axioms T5, A4, A3, A1 q is universally finite. The two branches of this piecewise defined map glue together into a smooth expanding about the periodic endpoint of J_{l+1} with the absolute value of the derivative bounded. We use this map until the last step before image of N' contains an endpoint of J_{l+2} or image of M' contains an endpoint of J_{l+1}. In that last step before this happens, the length of the image of the starting interval $f^p(\Lambda)$ is comparable to the length of J_l, because we could only gain finitely in the last step we dropped (axiom A1). Also, the quasisymmetric distortion added in this part is bounded, by the argument analogous to the one used in the first p iterates. The image of the original midpoint of Λ is still quasi-centered, so both quasi-halves have lengths comparable to the length of J_l and also the supremum of the distance from c, taken over the two quasi-halves,. is comparable to the length of J_l. Thus, by axioms T3, A1, A3 each quasi-half contains a preimage of finite bounded order of $J_{l+l'} \setminus J_{l+l'+1}$, with l' bounded as well. Both bounds above are given in the terms of the bounds in the analytic axioms A1-A4. These preimages are the dynamically defined intervals contained in the two quasi-halves with the lengths comparable to the length of J_l. The same is true in the conjugate picture for g, so substituting the $g-$picture for the $f-$picture at this moment perturbs the quasicentricity of the image of the original midpoint only boundedly. It remains to pull back along the conjugate itinerary and we are done. □

3 One plus Zygmund class of smoothness and control over short intervals

This chapter is aimed to be an addendum to the work done in Chapters 1 and 2 of [6] where the following question is studied:

What is the optimal class of smoothness under which the 'real Koebe distortion argument' for long compositions of diffeomorphisms of an interval can be carried out.

It is found there that smoothness $C^{1+Zygmund}$, by which we mean that the logarithm of the derivative $\varphi = \log h'$, satisfies the Zygmund condition

$$\varphi(x) + \varphi(y) - 2\varphi(\frac{x+y}{2}) = O(|x-y|)$$

, is equivalent to distortion of cross-ratios for all equally distributed 4-tuples of points (so-called standard 4-tuples) being *O(size of the 4-tuple)*.

Then a theorem is proven there asserting that smoothness $C^{1+Zygmund}$ guarantees the additive change of Poincare length of 4-tuples of P-length ≈ 1 contained in an interval T to be of the order $O(T)$. This way we get multiplicative control with the factor of the order $1+O(T)$ over the distortion of all subintervals of T of any P-length ≥ 1, simply by partitioning them into a union of intervals of the P-length ≈ 1. Control over "Poincare-long" intervals is enough to yield the real Koebe distortion principle.

To work with the Markov-type expansion argument of Chapter 5, and in particular to derive the property stated in axiom A2 of the previous chapter, we will need to get some sort of control over the multiplicative distortion of small P-lengths. However, the proof of the theorem above fails in the case of small P-lengths. Thus the question of optimal class of smoothness to perform this kind of argument arises. We are able to show that exactly the same class of smoothness i.e. $C^{1+Zygmund}$ is the right one for our purposes. What we prove here is:

Theorem 3 *Let h be a $C^{1+Zygmund}$ diffeomorphism of an interval I, $T = \langle a, d \rangle$ be an interval in I, and $M = \langle b, c \rangle$ where $a < b < c < d$ be a proper subinterval of T. Then the length of M in Poincare metric on T, $P_T(M)$, is distorted by h only by a multiplicative factor of the order $1 + O(\text{length of } T)$, i.e. in the notation identifying an interval with its standard length we have*

$$(1 + O(T))\, P_T(M) \geq P_{hT}(hM) \geq (1 - O(T))\, P_T(M)$$

The coefficient in $O(T)$ is controlled by the Zygmund norm and $\frac{1}{2}$-Hölder norm of $\log h'$.

Proof: *Step 1. Estimate of the cross-ratio distortion.*

Let us set the notation $T = \langle a, d \rangle$, $L = \langle a, b \rangle$, $M = \langle b, c \rangle$, $R = \langle c, d \rangle$, $T' = h(T)$, $L' = h(L)$, $M' = h(M)$, $R' = h(R)$. The cross-ratio of the four points a, b, c, d, denoted $[a, b, c, d]$ is defined by the following formula:

$$-\log [a, b, c, d] \stackrel{def}{=} \int_c^d \int_a^b \frac{dx\, dy}{(y - x)^2}$$

By the cross-ratio distortion under h we mean

$$D_h(a,b,c,d) \stackrel{def}{=} \log \frac{[ha,hb,hc,hd]}{[a,b,c,d]}.$$

The distortion may be computed through an integral formula

$$\log \frac{[ha,hb,hc,hd]}{[a,b,c,d]} = \int_c^d \int_a^b \left[1 - \frac{h'x\, h'y}{[h']_{xy}^2}\right] \frac{dx\, dy}{(y-x)^2}$$

where $[h']_{xy}$ means the average of h' over the interval $\langle x,y\rangle$. Thus, for the cross- ratio distortion we have

$$|D_h(a,b,c,d)| = \left|\int_c^d \int_a^b \frac{O(|x-y|)}{(y-x)^2} dx\, dy\right| \leq$$

$$Const \cdot \int_c^d \int_a^b \frac{dx\, dy}{y-x} = Const \cdot \log \frac{T^T M^M}{(L+M)^{L+M}(M+R)^{M+R}} \qquad (1)$$

First we need to check if the cross-ratio distortion is of the order $O(T)$.

Since our formula is symmetric in L and R we may assume that $L \leq R$. Taking into account $T = L + M + R$ we obtain

$$\left|\frac{1}{T}\log \frac{T^T M^M}{(M+R)^{M+R}(L+M)^{L+M}}\right| = q$$

$$\left|\frac{1}{T}\log \frac{T^T}{(T-L)^T} \cdot \frac{(T-L)^L M^M}{(L+M)^{L+M}}\right| =$$

$$\left|\log \frac{1}{1-\frac{L}{T}} + \frac{1}{T}\log \frac{(T-L)^L}{(L+M)^L} \cdot \frac{M^M}{(L+M)^M}\right| \leq$$

$$\left|\log \frac{1}{1-\frac{L}{T}}\right| + \left|\frac{L}{T}\log \frac{T-L}{L+M}\right| + \left|\left|\frac{M}{T}\log \frac{M}{L+M}\right|\right.$$

The first term in (2) is certainly bounded, since $L \leq R \leq T/2$. To estimate the second one we recall that $1 < \frac{T-R}{T-L} < \frac{L}{T}$, so it is bounded by the maximal value of $-x \log x$ in the interval$(0, 1)$. We bound the third term in the same way since $\frac{M}{T} < \frac{M}{M+L} < 1$. This shows that the cross-ratio distortion actually is of the order $O(T)$ as we wanted it. The constant bounding (2) is universal and may be chosen to be $(\log 2 + \frac{2}{e}$ and the constant in (1) depends only on the Zygmund and $\frac{1}{2}$-Hölder norms of $\log h'$ by the referred theorem from [6] This concludes step 1.

Step 2. Distortion of the Poincare metric.

The length of an interval $M \subset T$ with respect to the Poincare metric on T is given by the formula

$$P_T(M) = \log\left(1 + \frac{MT}{LR}\right)$$

and the $P-$metric form on T by

$$\frac{dt}{t(|T| - t)}.$$

Under a diffeomorphism h the element of the $P-$length transforms as

$$\frac{h'(t)}{h(t)(||T'| - h(t)|)}dt.$$

To estimate this for a $C^{1+Zygmund}$ diffeomorphism h we notice that, (for M very small relative to both L and R), the $P-$length, up to the terms of higher order, is nothing but our cross-ratio, so a multiplicative bound on the cross-ratio distortion bounds the distortion of the $P-$length element as well. The former has been just found to be of the order $(1 + O(T))$, and so must be the latter.

Integrating the form we can have the same estimate for the multiplicative distortion of the P-length of any interval $M \subset T$, independently of its P-length. This extends the statement of the Lemma of Chapter 2 in [6] over to all subintervals of T, with finite P-length, no matter how small. □

Remark 1 We may view Theorem 1 as stating a kind of Lipschitz property for multiplicative P-length distortion considered as a real variable function. Denote

$$\mathcal{D}_h(y) \stackrel{def}{=} \sup_{|T|=y} \sup_{M \in \mathcal{M}(T)} \left| \log \frac{P_{T'} M'}{P_T M} \right|, \qquad (2)$$

where $\mathcal{M}(T)$ is the collection of subsets $M \subset T$ such that $T \setminus M$ is 2-connected. Theorem 1 asserts that $\mathcal{D}_h(y)$ is a Lipschitz continuous function in y with the Lipschitz constant controlled by the Zygmund and $\frac{1}{2}$-Hölder norm of h.

4 Renormalizable maps.

In this chapter we set some notations and definitions to establish the language we could refer to later on. Simple propositions are named *facts* here and their verification is left to the reader. By Theorem 4 we shall see that our definitions fit well with the *intuitive picture* of what the 'renormalization box' should be.

Definition 1 *Renormalizable map.*

Suppose $f: P \to P$ is a continuous map of an interval $P = \langle 0, 1\rangle$ into itself, $f(0) = f(1) = 0$, f has a unique turning point $c \in (0, 1)$ and is strictly increasing on $\langle 0, c\rangle$ and strictly decreasing on on $\langle c, 1\rangle$. For an integer $\ell > 1$ we will say that f is ℓ-renormalizable (on the interval Q) if there exists an interval $Q \subset P$ such that $c \in Q$, $c \in f^\ell(Q) \subset Q$ and Q, $f(Q)$, ... $f^{\ell-1}(Q)$ are pairwise interior disjoint. We shall call f renormalizable if it is ℓ-renormalizable with some $\ell > 1$. We say f is properly ℓ-renormalizable if ℓ is the minimal integer for which f is ℓ-renormalizable [cf. *Fact 2*, Remark 2 at the end of this chapter and Proposition 5 in the next chapter].

Definition 2 *Renormalization interval.*

Let f be ℓ-renormalizable. The smallest Q satisfying Definition 1 will be called the small ℓ-renormalization interval; the largest such Q will be referred to as the large ℓ- renormalization interval. For ℓ minimal respective intervals will simply be called the small renormalization interval and the large renormalization interval, and denoted $I(f)$ and $J(f)$ respectively.

Fact 1 If f is ℓ-renormalizable its large and small ℓ-renormalization intervals are well defined. The endpoints of the small ℓ-renormalization interval are $f^{\ell}(c)$ and $f^{2\ell}(c)$. If p is one of the endpoints of the large ℓ-renormalization interval then the other endpoint is the dynamically symmetric counterpart of p i.e. the unique point p such that $f(p) = f(p)$.

Fact 2 If f is l-renormalizable and properly ℓ-renormalizable then l/ℓ is a positive integer.

Proposition 1 *If f is l-renormalizable then one of the endpoints of its large l-renormalization interval, say p, satisfies $f^l p = p$.*

Proof: Suppose that $f^{\ell}(\partial J) \subset int\, J$. If the images of J $J, f(J), \ldots f^{\ell-1}(J)$ were actually *disjoint*, i.e. there were spaces in between of them, one might enlarge J by continuity. So there are two intervals $f^l(J)$, $f^{l'}(J)$ $0 \le l < l' \le \ell$ sharing a common endpoint. But $f^{\ell-l'}\left(f^{l'}(J)\right) \subset int\, J$, so $f^{\ell-l'}\left(f^{l}(J)\right) \cap int\, J \neq \emptyset$, which contradicts interior disjointness of the family $\left\{J, f(J), \ldots f^{\ell-1}(J)\right\}$. □

Note 2 The only situation when p of the Proposition 1 can also satisfy $f^{l'}p = p$ with some $l' < l$ is the case when some $R^k f$ is 2-renormalizable [cf. Proposition 5 in the next chapter].

Fact 3 c is the unique turning point of $f^\ell|_J$.

Definition 3 *Renormalization operator.*

By the renormalization of f we mean $Rf = A \circ \left(f^\ell|_J\right) \circ A^{-1}$, where ℓ is minimal and A is the affine mapping (preserving or reversing the orientation, as needed) taking J onto $\langle 0, 1\rangle$ in such a way that the turning point of $f^\ell|_J$ becomes a maximum. We write $f = R^0 f$, $Rf = R^1(f)$ and, inductively, if $R^k f$ is further renormalizable we can define $R(R^k f) = R^{k+1} f$.

We shall continue to denote by c the turning point of the renormalized map whenever it is not confusing.

Definition 4 *Infinitely renormalizable map*

We say f is infinitely renormalizable if $R^k f$ is defined for all nonnegative integers k. If there exists a finite T such that each $R^k f$ is properly ℓ-renormalizable with some $\ell \leq T$ we say f is infinitely renormalizable of bounded type T (or, with bounded combinatorics).

Let now f be a properly ℓ-renormalizable map and $J'(f)$ be the *largest* "renormalization interval in the preimage sense" i.e. the largest interval about the critical point c such that following the critical orbit backwards of the family $\left\{J'(f), f^{-1}(J'(f)), \ldots, f^{-l+1}(J'(f))\right\}$ all the intervals along the way are interior disjoint and $f^{-l}((J'(f)) = J'(f)$.

Proposition 2 $J(f) = J'(f)$.

Proof: Clearly $J'(f) \subset J(f)$. We must rule out the possibility that the preimages of $J(f)$ following the critical orbit backwards are not interior disjoint. Suppose $int\, f^{-i}(J(f)) \cap int\, f^{-j}(J(f)) \neq \emptyset$ for some $0 \leq i < j < l$. Then $f^{i+1}(f^{-j}(J(f)))$ contains in the interior an element of the orbit of l-periodic boundary point of $J(f)$. Contradiction. □

By virtue of Proposition 2 we can now copy the proof of Theorem 1 of [6] with J in place of I', obtaining that sufficiently high renormalizations in our sense are of the form Qh with the Zygmund and $\frac{1}{2}$-Hölder norm of $\log h'$ universally bounded by a positive constant $\mathcal{B}$ and with Q a quadratic polynomial universally bounded away from zero quadratic polynomial, provided we start with a map of the form Qh with $\log h'$ Zygmund. Having established that we move to the next

Proposition 3 *Let $\hat{f}$ be infinitely renormalizable of the form $\hat{f} = Qh$, where $\log h'$ belongs to the Zygmund class and Q is a non-zero quadratic polynomial. There exists a universal $\mu > 0$ such that for high enough renormalization f of $\hat{f}$ the interval $(0, c)$ undergoes definite stretching under f by a factor of at least $(1 + \mu)$.*

Proof: First notice that the universal $\mathcal{B}$ bound on the $\frac{1}{2}$-Hölder norm of $\log h'$ yields universal upper and lower bounds on h' since the domain of h is a compact interval $\langle 0, 1\rangle$. Call them $u(\mathcal{B})$ and $l(\mathcal{B})$ respectively. Quadratic part Q of the composition $f = Qh$ is universally bounded away from zero quadratic polynomial (or otherwise 0 would be a sink attracting the whole interval $\langle 0, 1\rangle$ under iterations of f) and also bounded from the above, so there are universal upper and lower bounds on the second derivative of Q. Call them $udp(\mathcal{B})$ and $ldp(\mathcal{B})$ respectively. Let π be the orientation reversing fixed point of f.

First we claim that there is a universal $\epsilon > 0$ such that $|f'(\pi)| > \epsilon$. If not, for ϵ sufficiently small the absolute value of the derivative of f on the whole interval (c, π) is less than 1 because if for some $y \in (c, \pi)$ we had $|Q'(h(y)) \cdot h'(y) > 1|$ then

$$|Q'(h(y)) \cdot h'(\pi)| > \frac{Q'(h(y)) \cdot h'(\pi)}{Q'(h(y)) \cdot h'(y)}$$

and

$$\epsilon > |Q'(h(\pi)) \cdot h'(\pi)| \geq |Q'(h(y)) \cdot h'(\pi)| > \frac{h'(\pi)}{h'(y)}$$

but $h'(\pi)/h'(y)$ cannot be smaller than $\epsilon^{-\mathcal{B}}$. Thus $\epsilon < \epsilon^{-\mathcal{B}}$ guarantees $f(c) - \pi < \pi - c$. Similarly, because $h(f(c)) - h(\pi) < u(\mathcal{B})(\pi - c)$ we get for $y \in \langle \pi, f(c)\rangle$

$$|Q'(h(y)) - Q'(h(\pi))| \leq u(\mathcal{B})(y - \pi) \cdot udp(\mathcal{B}) < u(\mathcal{B})(\pi - c) \cdot udp(\mathcal{B})$$

and since $|Q'(h(\pi))| \geq l(\mathcal{B})(\pi - c) \cdot ldp(\mathcal{B})$ we have

$$\frac{Q'(h(y))}{Q'(h(\pi))} \leq 1 + \frac{u(\mathcal{B})}{l(\mathcal{B})} \cdot \frac{udp(\mathcal{B})}{ldp(\mathcal{B})} = Const(\mathcal{B}).$$

Thus, if $\epsilon < (Const(\mathcal{B})\epsilon^{\mathcal{B}})^{-1}$ and at some point $y \in \langle \pi, f(c) \rangle$ the absolute value of the derivative of f were larger or equal to 1 we would obtain

$$\frac{1}{Const(\mathcal{B})\epsilon^{\mathcal{B}}} > \epsilon > |Q'(h(\pi))\, h'(\pi)| \geq \frac{1}{Const(\mathcal{B})} |Q'(h(y))\, h'(\pi)| \geq$$
$$\frac{1}{Const(\mathcal{B})} \cdot \frac{Q'(h(y))\, h'(\pi)}{Q'(h(y))\, h'(y)} > \frac{h'(\pi)}{h'(y)}$$

This contradicts the $\mathcal{B}$ bound on Hölder norm of h and so we see that if ϵ were smaller than $(Const(\mathcal{B})\epsilon^{\mathcal{B}})^{-1}$ the derivative of f would have to be less than 1 on the whole interval $\langle c, f(c) \rangle$ forcing the fixed point π to attract the post-critical orbit. Thus f could not be further renormalizable and the existence of universal ϵ in question is established with $\epsilon = (Const(\mathcal{B})\epsilon^{\mathcal{B}})^{-1}$.

From the above we easily derive the existence of universal $\mu > 0$ such that $f(c) - c > \mu$. Actually, since $f'(c) = 0$ and $-f'(\pi) > \epsilon$ we see that $\pi - c > \mu = \epsilon \cdot udp(\mathcal{B})^{-1} \cdot u(\mathcal{B})^{-2}$ because

$$\epsilon < |Q'(h(\pi))h'(\pi)| \leq |Q'(h(\pi))| \cdot u(\mathcal{B}) \leq$$
$$udp(\mathcal{B})(h(\pi) - h(c))u(\mathcal{B}) \leq (\pi - c) \cdot udp(\mathcal{B}) \cdot (u(\mathcal{B}))^2$$

and since $f(c) - c > \pi - c$ we have our proposition proven. □

Before we can go on to proving the main result of this chapter we need one more tool.

Proposition 4 *If f is a high renormalization of $\hat{f}$ and f takes an interval M homeomorphically onto its image fM then for any $K \subset M$ we can have $P_{fM}fK \geq (1-\nu)P_M K$ with $\nu > 0$ arbitrarily small if we renormalized sufficiently many times.*

Proof: By the proof of Theorem 1 of [6] carried out with the renormalization interval in our sense, which is possible due to Proposition 2, we see that the total length of the orbit of $J(f)$ goes down exponentially fast with the depth of renormalization and the proposition follows immediately from Theorem 3 of the previous chapter because Q increases P-lengths and h nearly does not perturb them along the orbit. □

We are now in a position to use the definite stretching of the interval $\langle 0, c\rangle$ to prove

Theorem 4 *There exist universal $\varepsilon, \varepsilon' > 0$ such that if f is sufficiently high renormalization of $\hat{f}$ satisfying the hypothesis of Proposition 3 then $(1+\varepsilon') > f'(0) > (1+\varepsilon)$ and also f has exactly one fixed point in $(0,1)$, with negative slope, universally definitely repelling.*

Proof: Denote by $\hat{f}$ the "unrenormalization of f" i.e. $f = R\hat{f}$, and let $\hat{f}$ be properly ℓ-renormalizable. Let $\hat{a}$ be the critical point of $\hat{f}^{\ell}$ closest to the ℓ-periodic endpoint p of $J(\hat{f})$ lying on the opposite side of p as the dynamically symmetric counterpart p of p. Let $a < 0$ be the image of $\hat{a}$ under the same affine change of coordinates that takes $J(\hat{f})$ onto $(0,1)$ in such a way that p goes onto 0.

First notice that 0 has to be a topologically repelling fixed point of f, or otherwise a would be inside the basin of attraction of 0 whereas it is supposed to be a preimage of the critical point of an infinitely renormalizable map. So if we think of f as of a mapping defined by rescaling of the appropriate piece of of the graph of $\hat{f}^{\ell}$ onto the interval $\langle a, 1\rangle$ we see that $f(a) < a$. By $\mathcal{B}$-boundedness of f on some definite neighborhood of $\langle 0, 1\rangle$ a has to be definitely smaller than 0, i.e. there exists a universal $\nu > 0$ such that $-a > \nu$, since $f'(0) \geq 1$ and $f'(a) = 0$. By Proposition 3 and a straightforward calculation this yields $f'(0) > \frac{1+\mu+\nu+\frac{1}{2}\mu\nu}{1+\mu+\nu}$, or otherwise the P-lengths of tiny intervals about 0 would undergo definite contraction by a factor of at least $\frac{1+\mu+\nu+\frac{1}{2}\mu\nu}{(1+\mu)(1+\nu)}$ in violation of Proposition 4.

Here is the calculation. Infinitesimally, we have for the Poincare metric forms

$$dP_{\langle a,c\rangle}(0) = \frac{dt}{-a} + \frac{dt}{c}$$

$$dP_{\langle fa,fc\rangle}(0) = \frac{f'0\,dt}{-fa} + \frac{f'0\,dt}{fc} \leq f'0\left(\frac{dt}{-a} + \frac{dt}{(1+\mu)c}\right)\text{, so}$$

$$\frac{dP_{\langle a,c\rangle}(0)}{dP_{\langle fa,fc\rangle}(0)} \leq f'0\left(\frac{(1+\mu)c+a}{(a+c)(1+\mu)}\right) < f'0\left(1 - \frac{\mu\nu}{(1+\mu)(1+\nu)}\right). \qquad (3)$$

We have proved the definite repelling of the ℓ-periodic point on the boundary of $J(\hat{f})$, which goes onto 0 under renormalization. The existence of a universal upper bound on $f'0$ follows from $\mathcal{B}$-boundedness of f.

If there were another fixed point inside $(0, c)$, again by $\mathcal{B}$-boundedness of f, there has to be well defined the one which is closest to 0, call it ω, and ω has to attract topologically at least on its left hand side, so $f'\omega \leq 1$; since $fc - c > \mu$ by a calculation alike (3) this would also violate Proposition 4.

On the interval $\langle c, 1 \rangle$ f is decreasing so there exists exactly one fixed point π inside, with the slope strictly negative bounded away from 0 by the proof of Proposition 3. The proof of definite repelling of π is given in the Remark 6 below. We are done. □

Remark 2 Theorem 4 shows, in particular, that sufficiently deep in the renormalization process the large renormalization interval in the sense of Definition 2 coincides with the the 'bigger renormalization interval' of $\hat{f}$ as defined in [6] i.e. the *smallest* interval $I'(\hat{f})$ about the critical point of $\hat{f}$ such that its inverse orbit following post-critical orbit backwards is a collection of intervals $\left\{\hat{f}^{-1}I'(\hat{f}), \hat{f}^{-2}I'(\hat{f}), \ldots, \hat{f}^{-\ell}I'(\hat{f}) = I'(\hat{f})\right\}$ permuted by $\hat{f}$ and ℓ is minimal. As a matter of fact by this we also know that deep in the renormalization the case described in point b) of the Lemma in §4 of [6] cannot actually happen, i.e. there is no 'renormalization of degree 1' possible there.

Remark 2 does not matter for the structure of the attracting Cantor set but is relevant for understanding what happens outside the attractor, as we are going to need in the next chapter.

Remark 3 The proof of Theorem 4 goes the same also for non-quadratic singularities of any type $|x|^\alpha$ with $\alpha > 1$.

Remark 4 Due to the exponential decrease of the dependence on h (cf. the proof of the Theorem in §4 of [6]), renormalizing many times we can make c as close to $\frac{1}{2}$ as we wish so that the interval $\langle \pi, 1 \rangle$ would undergo definite stretching under f and repeating the argument from the proof above we see that sufficiently deep in the renormalization $|f'1| > 1 + \varepsilon$.

Remark 5 Consider the mapping f^2 on the interval $\langle \pi, \bar{\pi} \rangle$, where the 'bar' stands for dynamically symmetric counterpart, as usual. By the Remark 4 c is nearly centered in $\langle \pi, \bar{\pi} \rangle$ for high renormalizations so we see that either $f^2 c \notin \langle \pi, \bar{\pi} \rangle$ or we can proceed as in the proof of Proposition 3 , in either case obtaining that the interval $\langle f^2 c, c \rangle$ comprises a definite proportion of $\langle \pi, \bar{\pi} \rangle$, thus a definite proportion of $\langle 0, 1 \rangle$.

Remark 6 From the previous remark it follows, by copying the proof of Proposition 3 and Theorem 4 that the fixed point π is actually definitely repelling, i.e. by a factor bounded away from 1.

5 Dynamical partition

So far we have not achieved the full understanding of irreducible orbits that can occur in the renormalization process. Nevertheless, we are able to carry out a dynamical construction leading, under mild assumptions, to a partition of the interval which is general and good enough for managing an estimate of the distortion. Unfortunately, but at no surprise, we cannot help restricting ourselves at some point to the the bounded combinatorics situation in the sense of Definition 4. The construction is very much like that in §3 of [5] in case of 2-renormalizable maps and generalizes it for other infinitely renormalizable cases.

Let $\hat{f}$ be an infinitely renormalizable map of $\langle 0, 1\rangle$ into itself. Set $f = f_n = R^n \hat{f}$, $J = J_n = J(f_n)$, and let f_n be properly $\ell = \ell_n$−renormalizable. Let $J'' = J''_n$ and $J' = J'_n$ be the $(\ell_n - 2)$-th and $(\ell_n - 1)$-th preimages of J_n following the critical orbit backwards, $f_n^{\ell_n - 1} J'_n = J_n$, $f_n J_n \subset J'_n$, $J''_n = f_n J'_n$.

Proposition 5 *Either $J'' = J$ or all distinct preimages of J are pairwise disjoint.*

Proof: If $\ell = 2$ we have $fJ' = J'' = J$ and the the right hand fixed point p has to be the endpoint of J, otherwise, i.e. if $f^2 \langle p, p\rangle$ were not contained in $\langle p, p\rangle$, the return time of the critical point c to $\langle p, p\rangle$, thus also to any interval contained in $\langle p, p\rangle$ would have to be larger than 2. On the other hand $\langle p, p\rangle$ cannot be a proper subinterval of J because in that case the images of J would not be interior disjoint. Thus $J + \langle p, p \rangle$ and the intervals J, J' abut at the point p.

If $\ell > 2$ no two distinct preimages of J by f abut. The proof of that goes as follows. First notice that every preimage of J (save for J') is disjoint from J' because the left endpoint of J' is a periodic point which orbit comprise one of the two endpoints of each interval in the forward orbit of J' up to J; preimages of J are either interior disjoint or coincide (see Proposition 2); if there is a preimage of J ,say L, of the order of i adjacent to J' at the right hand endpoint then either $i < l - 1$ and $f^{i+1}J'$ is a preimage of J attached to the left hand endpoint of J' or

$$i > l - 1$$

and then $f^{\ell}L$ is a preimage of J attached to J' at its left hand endpoint. At any rate, if there is a preimage of J adjacent to J' on either side there is one attached to J' at its left hand endpoint and the ℓ-th iterate of f preserves orientation in the neighborhood of that (fixed) point. Thus the preimage of J adjacent to J' at its left hand endpoint, call it for the moment J''' if there exists one, actually has to be of the order $j < l-1$, for $f^{\ell}J''' \cap J''' \neq \emptyset$ and J' is a preimage of J of the $l-1-$th order. Since we have to have then $f^{j+1}J''' \subset J'$ and $f^{j+1}J' \subset J'''$ we get $l = 2(j-1)$, intervals $(J' \cup J''')$, $f(J' \cup J''')$, ... $f^{j}(J' \cup J''')$ are pairwise interior disjoint and $f^{j+1}(J \cup f^{j}J') \subset (J \cup f^{j}J')$, so the map f is $(j+1)$-renormalizable which contradicts minimality of ℓ in the definition of renormalization except for the case $\ell = 2$ and $j = 0$ discussed before. Now if any two preimages of J are adjacent, under the appropriate iterate of f the one of the the smaller order goes homeomorphically onto J whereas the other goes homeomorphically onto a preimage of J adjacent to J which after one more iterate gives the situation contradicting our consideration above. □

Let now $\Delta = \Delta_n$ denotes the interval spanned by $J' \cup J''$ and let $\mathcal{R} = \mathcal{R}_n$ be a collection of intervals defined as follows:

$$J^\alpha = \{I \subset \Delta \mid f^\alpha I = J\} \quad \text{and} \quad \bigcup_{\alpha \in \Theta} J^\alpha = \mathcal{R}$$

where $\Theta = \{0, 1\}$ if $\ell = 2$ and $\Theta = \{0, 1, 2, \ldots\}$ if $\ell > 2$.

Put $\mathcal{R}^\bullet = \mathcal{R}_n^\bullet = \{I \in \mathcal{R} \mid I \subset span(\overline{J''} \cup J')\}$ and $\hat{\mathcal{R}} = \widetilde{\mathcal{R}_n} = \overline{\mathcal{R}_n^\bullet}$; where the 'bar' means the dynamic symmetricity, as before. Finally we define

$$\mathcal{P} = \mathcal{P}(f_n) = \mathcal{P}_n = \mathcal{R} \cup \bigcup_{i=0}^{\infty} f^{-i}\hat{\mathcal{R}}.$$

We say that an element K of $\mathcal{P}$ is of β-th order if $f^\beta K = J$. We now prove a fundamental

Lemma 1 *If f is a sufficiently high renormalization of an infinitely renormalizable map $\hat{f} = Qh$ of bounded type T, with the assumptions on Q and h as in the previous chapter, then for every interval $K \subset \langle 0, 1\rangle$, if $\left(\bigcup_{i=0}^{j} f^i K\right) \cap J = \emptyset$ for some integer $j \geq 0$, then*

$$\frac{|K|}{|f^j K|} \leq A\gamma^{-j}$$

with constants $A = A(T) > 0$ and $\gamma = \gamma(T)$, $0 < \gamma < 1$ which depend only on T.

Proof: By Theorem 4, Remark 4 and boundedness there exists a universal κ such that the absolute value of the derivative of f is larger than $1 + \frac{\epsilon}{2}$ in $\langle 0, \kappa\rangle$ and $\langle 1 - \kappa, 1\rangle$. There also exists a universal $\theta > 0$ such that for every point x in the interval $\langle \kappa, c\rangle$ we have $fx > x + \theta$, because if for some $x \in \langle \kappa, c\rangle$ we had $fx = x + \vartheta$ with ϑ very small then, by $\frac{1}{2}$–Hölder $\mathcal{B}$-boundedness, in some neighborhood of x of the radius ϑ' there would exist a point x' such that $fx' < 1 + 2\vartheta$ and $f'x' < 1 + \vartheta''$, where

$$\vartheta' = \vartheta'(\vartheta)\,, \quad \vartheta' \xrightarrow[\vartheta \to 0]{} 0 \qquad \text{and} \qquad \vartheta'' = \vartheta''(\vartheta)\,, \quad \vartheta' \xrightarrow[\vartheta \to 0]{} 0$$

and since $fc > c + \mu$ and x' is universally definitely bounded away from 0, the Poincare metric in a minute vicinity of x' would have to be contracted by a definite factor due to a calculation analogous to inequality (3) in the proof of Theorem 4, thus contradicting Proposition 4. By the Theorem in §4 of [6] we see an 'almost symmetric' situation on the decreasing branch of f, so there is only finitely many preimages of $\hat{\mathcal{R}}$ outside $\langle 0, \kappa\rangle \cup \langle 1 - \kappa\rangle$. Also, the derivative of f is bounded away from 0 on each of these preimages including $\hat{\mathcal{R}}$ itself, due to the 'almost symmetricity', Remark 5 and boundedness of f. The above discussion was the proof that *it suffices to show the lemma when $K \subset \Delta$.*

From now on we suppose we are in that situation. If $\ell = 2$ then $fJ' = J$

and there is nothing to prove. We shall now be assuming $\ell > 2$, so that by Propositon 5 we have gaps between the preimages of J and we shall be using the T-bound od ℓ in an essential way.

By T-boundedness of ℓ and the upper bound on f' there is a lower bound on the length of J and so also on the absolute value of the derivative of f, $|f'p|, |f'p|$, at the endpoints of J. There is only bounded number of of final steps where the image of K could possibly intersect the forward orbit of J', so we may assume that the whole j-step long orbit of K is disjoint with the orbit of J', i.e. falls into the gaps between the intervals of the orbit of J'. Let $\mathcal{G}(f^i K)$ be the gap containing $f^i K$. We shall also define the notion of the increased gap. Recall I was the small renormalization interval which, by Definition 2, was not a dynamically symmetric object, so we rather consider the interval $I \cup I$ instead, and let τ, τ be the endpoints beside p and p respectively. Inside each preimage of J belonging to the orbit of J' there is an appropriate preimage of $I \cup I'$. By the increased gap containing $f^i K$, $\mathcal{G}'(f^i K)$ we understand the gap between those preimages of $I \cup I'$ contained in the preimages of J delimiting the 'regular' gap $\mathcal{G}(f^i K)$. By the T-boundedness and 'almost symmetricity' of f the intervals $\langle p, \tau\rangle$ and $\langle \tau, p\rangle$ comprise a definite, bounded away from 0 fraction of J, so by bounded quasi-symmetric distortion each preimage, up to the $(\ell - 1)$-th, of the interval $\langle \tau, \tau\rangle$ following the critical orbit backwards has a definite space on both sides within the appropriate preimage of J. Let k be the first time such that $f^k\mathcal{G}(K)$ contains an element of the orbit of J. We must have $k \leq l$; otherwise $f^l\mathcal{G}(K)$ would not contain any element of the orbit of J', which implies $f^l\mathcal{G}(K) = \mathcal{G}(K)$, endpoints of $\mathcal{G}(K)$ are $l-$ periodic repelling points due to Theorem 4 and so there exists at least one additional l-periodic point inside the gap. If χ is such an l-periodic point closest to, say, the left endpoint of the gap and that left endpoint goes onto the periodic endpoint of J under, say, m-th iterate of f then J is a proper subset of $\left\langle \overline{f^m\chi}, f^m\chi \right\rangle$ and the latter interval has the preimages following the critical orbit backwards pairwise interior disjoint until it falls back onto itself, thus violating the definition of the renormalization interval J. f^k takes the increased gap $\mathcal{G}'(K)$ homeomorphically onto its image containing, though maybe larger larger than $\mathcal{G}'(f^kK)$. Provided f^kK remains in a gap we see that f^kK stays in a definite distance away from the endpoints of $\mathcal{G}'(f^kK)$. By T-boundedness the element of the orbit of J' contained in $f^k\mathcal{G}'(K)$ and, say, closest to f^kK on either side if there were more than one, cuts off a definite fraction from the interval $\langle 0, 1\rangle$ and so from $f^k\mathcal{G}'(K)$. By Proposition 4 the loss of the P-length of K in $\mathcal{G}'(K)$ along the orbit of the increased gap up to $f^k\mathcal{G}'(K)$ can be arbitrarily small if we are deep in the renormalization; by the argument above the P-length of f^kK in $\mathcal{G}'(f^kK)$ is by a definite factor larger than the P-length of f^kK in $f^k\mathcal{G}'(K)$ and so it is also by a definite factor larger than the P-length of K in $\mathcal{G}'(K)$. This definite increase follows from what we already know about the Poincare metric from calculation (3) in the proof of Theorem 4: at any point definitely bounded away from the endpoints of an interval containing that point the infinitesimal P-length form is definitely decreased if we do not move the point and increase the interval (on either side) by gluing a piece whose length is a definite fraction of the initial length. So the other way the P-length form is definitely increased if if we cut a definite fraction of the interval off. For there is a definite space on both sides of the interval $\mathcal{G}(f^k K)$ embedded in $\mathcal{G}'(f^k K)$ the Poincare and standard metrics are equivalent on $\mathcal{G}(K)$; this perhaps going through the circuit of the P-length increase several more times, concludes the proof of the lemma. □

Now we derive an immediate

Corollary 2 *$\mathcal{P}$ is a partition of* $\langle 0, 1\rangle$.

Proof: Preimages of J are interior disjoint or coincide; if there were an interval disjoint with all the preimages by the first part of the proof of Lemma 1 after a finite time it would have to fall into one of the gaps in Δ and stay forever in the gaps, so its length would grow to infinity. □

Construction of the generated Markov map. As expected we choose the consecutive large renormalization intervals to form the nested family of neighborhoods of c, referred to in the axioms for the Markov maps of Chapter 2. The iterate of f that makes the map (properly) renormalizable on that large renormalization interval,after removing the middle part overlapping the renormalization interval of the next order gives the two branches of the generated Markov map at this level.

Topological axioms T1-T3 and T5 follow immediately from the construction. Axiom T4 comes from Corollary 2 above.

Analytical axioms A4 and A3 result from the construction (A3 uses bounded combinatorics). A1 follows from Theorem 1 of Chapter 1 and A2 from the Lemma 1 in this chapter.

This way the proof of the main theorem (Theorem 2 of Chapter 2) is complete.

6 Upgrading of Lipschitz continuous conjugacies

In this part, we shall be concerned with dynamical systems arising from C^1−smooth unimodal maps of the interval I with the singularity at the critical point c which, perhaps after smooth change of coordinates take the form of the power law $|x|^\alpha$ in some open neighborhood $U \ni c$, with $\alpha > 1$.

Fix the power α and denote by $\mathcal{A}$ the class of maps f satisfying all of the above and subject to the Conditions 1 and 2 below:

Condition 1

$$c \in cl\{f^n(c)\}_{n=1}^{\infty}. \tag{4}$$

Denote $cl\{f^n(c)\}_{n=0}^{\infty}$ by $\mathcal{C}_f$. This is a closed metric subspace of I. Action of f restricted to $\mathcal{C}_f$, $\check{f} = f|_{\mathcal{C}_f}$, makes the pair $\{\check{f}, \mathcal{C}_f\}$ into a topological dynamical system.

The second condition we shall be assuming about $\{\check{f}, \mathcal{C}_f\}$ is:

Condition 2 *There exists a dense subset of $\mathcal{C}_f$ made of preimages of c.*

Now take another map in $\mathcal{A}$, say g, and suppose that the systems $\{\check{f}, \mathcal{C}_f\}$, $\{\check{g}, \mathcal{C}_g\}$ are topologically conjugate by a homeomorphism $h : \mathcal{C}_f \mapsto \mathcal{C}_g$.

From now on we will use the notation

$$\hat{x} \stackrel{def}{=} h(x).$$

It makes sense to say that h has a derivative at a point x in C_f if the limit

$$\lim_{y \to x, y \in C_f} \frac{y - x}{y - x}$$

exists. Here we have a theorem.

Theorem 5 *If h is bilipschitz then there exist (non-vanishing) derivatives $h'(c)$ and $(h^{-1})'(h(c))$.*

Proof. By virtue of Condition 2 we may assume, without any loss of generality, that all the points in the following consideration are actually preimages of c.

For all $t \in C_f$, we will define two quantities which are kind of local measure of 'distortion from smoothness' (or more precisely 'distortion from symmetricity', but we work under the hypothesis of Lipschitz condition). They will be referred to as Badness and badness. Here they are:

$$B_\delta(t) = \left(\sup_{z_1, z_2 \in (x-\delta, x+\delta) \cap C_f} \left| \frac{z_1 - z_2}{z_1 - z_2} \right| \right) : \left(\inf_{z_1, z_2 \in (x-\delta, x+\delta) \cap C_f} \left| \frac{z_1 - z_2}{z_1 - z_2} \right| \right).$$

$$b_\delta(t) = \left(\sup_{z \in (x-\delta, x+\delta) \cap C_f} \left| \frac{t - z}{t - z} \right| \right) : \left(\inf_{z \in (x-\delta, x+\delta) \cap C_f} \left| \frac{t - z}{t - z} \right| \right)$$

$$b(t) = \limsup_{\delta \to 0} b_\delta(t).$$

Of course we always have $\frac{M}{K} \geq B \geq b \geq 1$, where M and K are the upper and lower Lipschitz bounds on h respectively. Also notice that Badness (and badness) are C^1-smoothly invariant.

Differentiability of h at c is tantamount to $b(c) = 1$, since we have the Lipschitz condition.

We assume that c is not periodic; otherwise the Theorem holds trivially.

Suppose then that the derivative does not exist, so there is a positive ε such that arbitrarily close to c one can find a pair of points $x, y \in \mathcal{C}_f$ satisfying

$$\frac{y}{y} : \frac{x}{x} > 1 + \varepsilon. \tag{5}$$

Step 1. *The 'four points' argument.* We find x, y satisfying (5) in U. On the tiny neighborhoods of x and y f acts almost linearly as $f'(x)$ and $f'(y)$ respectively, with arbitrarily small error. On the conjugate picture g' acts almost linearly in the small (conjugate) neighborhoods of x and y, thus we have

$$B(f(x)) = B(x) \leq \frac{M}{K} e^{-\left|\log \frac{f'(x)}{g'(x)}\right|} \tag{6}$$

$$B(f(y)) = B(y) \leq \frac{M}{K} e^{-\left|\log \frac{f'(y)}{g'(y)}\right|}. \tag{7}$$

Because of inequality (5) we have

$$\frac{f'(y)}{g'(y)} < \frac{f'(x)}{g'(x)} \cdot \frac{1}{(1+\varepsilon)^{\alpha-1}}.$$

So, at least one of the absolute values of the logarithms in (6) or (7), say the one in (6), has to be larger than

$$\frac{\alpha - 1}{2} \log(1+\varepsilon) \overset{def}{=} -\log \beta.$$

Same is true for $\delta-$Badness if δ is small enough. So we have found a point, namely x, such that in its sufficiently small neighborhood the Badness does not reach the upper bound of $\frac{M}{K}$ but is smaller than $\frac{M}{K}$ by a definite multiplicative factor of $\beta < 1$.

Step 2. *Recurrence.* Now, recall we assumed x was a preimage of c, say $f^m(x) = c$. By the smooth invariance of Badness, $B(f^l(x)) = B(x)$ for any finite l so $B(c) < \beta\frac{M}{K}$. (By the way, notice that by this remark also $B(y) < \beta\frac{M}{K}$.)

On tiny neighborhoods of x action of f^m is virtually linear with arbitrarily small error, so we also have $B_{\delta_1}(c) < \beta\frac{M}{K}$ provided δ_1 is sufficiently small. Put $U_0 = U$ and let U_1 be a symmetric neighborhood with diameter equal to such a δ_1. Of course for any point $t \in \mathcal{C}_f \cap U_1$ it is also true that $B(t) < \beta\frac{M}{K}$, and same for $\delta-$Badness with sufficiently small δ. Let ℓ be the moment of first return of the orbit of c into U_1.

Now consider a new x and a new y (we keep the same notation, hopefully with no confusion) which satisfy (5) and are both extremely close to c compared to the diameter of U_1. On the neighborhoods of those new x and y with the diameter minute compared to $\min(|x-c|, |y-c|)$ f acts almost linearly by f'. The same is true on the conjugate picture for g.

$f^{\ell-1}$ along the orbits of $f(x)$ and $f(y)$ acts almost linearly on $f($*minute neighborhood of* $x)$ and $f($*minute neighborhood of* $y)$ respectively, and the analogue for g, but this time we return into U_1 where we know a priori that

Badness is bounded by $\beta\frac{M}{K}$, so we have a modified version of (6), (7), namely

$$B(f^{\ell}(x)) = B(x) \leq \beta\frac{M}{K}e^{-\left|\log\frac{f'(x)}{g'(x)}\cdot\frac{(f^{(\ell-1)})'(f(x))}{(g^{(\ell-1)})'(g(x))}\right|}, \tag{8}$$

$$B(f^{\ell}(y)) = B(y) \leq \beta\frac{M}{K}e^{-\left|\log\frac{f'(y)}{g'(y)}\cdot\frac{(f^{(\ell-1)})'(f(y))}{(g^{(\ell-1)})'(g(y))}\right|}. \tag{9}$$

Now notice that choosing the new x and y sufficiently close to c we can have the distance $|f(x) - f(y)|$ ($|g(x) - g(y)|$ respectively) so small that $\frac{(f^{(\ell-1)})'(f(x))}{(f^{(\ell-1)})'(f(y))}$ and $\frac{(g^{(\ell-1)})'(g(x))}{(g^{(\ell-1)})'(g(y))}$ will be both arbitrarily close to 1 or in other words the quantities

$$\frac{(f^{(\ell-1)})'(f(x))}{(g^{(\ell-1)})'(g(x))}, \quad \frac{(f^{(\ell-1)})'(f(y))}{(g^{(\ell-1)})'(g(y))} \tag{10}$$

can be made nearly identical with arbitrarily small error.

Thus by the argument as in Step 1 at least one of the absolute values of the logarithms in (8) or (9) has to be larger than $-\log\beta$. This tells that $B(x)$ (and/or equivalently $B(y)$) does not exceed $\beta^2\frac{M}{K}$. Neither does $B(c)$ and so also $\delta-$Badness for sufficiently minute δ's on a appropriately small symmetric neighborhood U_2 of c. Repeat this argumentation recurrently with U_2, U_3, e.t.c. $\left[\log\frac{K}{M} / \log\beta\right] + 1$ times to get $B(c) < 1$, a contradiction.

Along the same lines we prove the existence of $(h^{-1})'(h(c))$. □

Remark 7 Notice that although Badness rather than badness was our main tool in the proof we cannot prove in this general setting that $B(c) = 1$. Actually we have only proved that $b(c) = 1$.

Remark 8 Of course for all positive integers i and all inverse branches s.t. $f^{-i}(c) \in \mathcal{C}_f$ we also have $b(f^{-i}(c) = 1$ and so the derivative of h exists at all the preimages of c. However in our general setting there is no way to carry this differentiability property over onto the whole set $\mathcal{C}_f$, despite density of the preimages of c. Compare the next section.

7 Upgrading of Lipschitz continuous conjugacies – infinitely renormalizable case

In this chapter we once again restrict our attention to the case of infinitely renormalizable unimodal maps of the interval I of bounded combinatorial type. Since we are going to make use of the theorems from the previous chapters, we shall assume that their hypotheses are satisfied in the sequel.

From now on we consider a pair of conjugate maps f and g like above; we will keep 'the pure' notation for objects arising in the picture for f and 'the bar' notation for the conjugate objects in the picture for g. We assume that the conjugating homeomorphism h, which is now defined on the whole interval, $h : I \to I$, when restricted to the closure of the post-critical orbit $\mathcal{C}_f$ satisfies the bi-Lipschitz condition there with upper and lower bounds M and K respectively.

Let $L_n \ni c$, $n = 0, 1, 2, \ldots$ be the basic small renormalization interval (i.e. bounded by the appropriate points of the post-critical orbit) at the depth n in the process of renormalization, $L_0 = \langle f(c), f^2(c)\rangle$.

By the definition of L_n there exists a positive integer $r = r(f, n)$ such that $f^r L_n = L_n$ and $f^i L_n \cap f^j L_n = \emptyset$ for $0 \leq i < j < r$. Let's denote by C_n the collection of intervals $C_n = \{f^i L_n\}_{i=0}^{r-1}$. Of course

$$\mathcal{C}_f = \bigcap_{n \geq 0} \bigcup_{\mathcal{L} \in C_n} \mathcal{L}.$$

Bounded combinatorics with the bound T means that at most T elements of C_{n+1} are contained in L_n for all $n \geq 0$.

Now in place of standard Lipschitz bounds K and M on $h|_{C_f}$ we consider new bounds $\mathcal{K}$, $\mathcal{M}$ related to the way the Cantor set C_f is obtained through the nested family of collections of intervals. Set

$$\mathcal{K}_n = \inf_{\mathcal{L}\in\bigcup_{i\geq n} C_i} \frac{|\mathcal{L}|}{|\overline{\mathcal{L}}|} \qquad \mathcal{M}_n = \sup_{\mathcal{L}\in\bigcup_{i\geq n} C_i} \frac{|\mathcal{L}|}{|\overline{\mathcal{L}}|}$$

and

$$\mathcal{K} = \lim_{n\to\infty} \mathcal{K}_n \qquad \mathcal{M} = \lim_{n\to\infty} \mathcal{M}_n.$$

Clearly $K \leq \mathcal{K} \leq h'(c) \leq \mathcal{M} \leq M$ where $h'(c)$ is the derivative of the conjugacy at the critical point c in the sense of the previous chapter.

As a technical tool we will introduce one more notion of distortion. For $x \in C_f$ let us set

$$\mathcal{B}_\delta(x) = \left(\sup_{\mathcal{L}\subset(x-\delta,x+\delta),\mathcal{L}\in\bigcup C_i} \frac{|\mathcal{L}|}{|\overline{\mathcal{L}}|}\right) : \left(\inf_{\mathcal{L}\subset(x-\delta,x+\delta),\mathcal{L}\in\bigcup C_i} \frac{|\mathcal{L}|}{|\overline{\mathcal{L}}|}\right), \qquad (11)$$

$$\mathcal{B}_\delta(x) = \lim_{\delta\to 0} \mathcal{B}_\delta(x).$$

Here we prove the following theorem.

Theorem 6 *Under the above hypotheses on f and g $\mathcal{B}(x) = 1$ at every point $x \in C_f$.*

Proof If there exists a point $x \in C_f$ with the property that $\mathcal{B}(x) = \gamma > 1$ then by the argument that the action of a finite (though perhaps very long) composition of f's (or g's, respectively) is virtually linear on sufficiently small neighborhoods of x we could push this property forward into arbitrarily small neighborhood of the critical point c, so it is enough to show that $\mathcal{B}(c) = 1$.

Suppose then that $\mathcal{B}(c) = \gamma > 1$. First we point out two facts:

1. *Fact 1.* $\sum_{\Gamma \in C_l} |\Gamma|$ decreases exponentially in l for every infinitely renormalizable map (cf. §3 of [6]).

2. *Fact 2.*

$$\frac{\sum_{\Gamma \subset L_n \setminus L_{n+1}, \Gamma \in C_l} |\Gamma|}{|L_{n+1}|}$$

   is exponentially small in $(l - n)$ for infinitely renormalizable maps of bounded type $\leq T$, because of 'bounded geometry property' (cf. §15 of [6]).

Now consider an interval $\Lambda \in C_l$ (l will vary in the considerations below) and another interval $\Omega \subset \Lambda$, $\Omega \in \bigcup C_i$ (we think of Ω very short relative to Λ or $\Omega \in C_{l+p}$, $p >> 1$). Let s be the first moment when the itinerary of Λ under f intersects L_n, $f^s\Lambda \subset L_n$, and consider the quantity

$$\exp\left\{\left|\log\left(\frac{|\Lambda|}{|f^s\Lambda|} : \frac{|\Omega|}{|f^s\Omega|}\right)\right|\right\}. \tag{12}$$

By Facts 1 and 2 this quantity can be made arbitrarily close to 1 provided $(l - n)$ is large enough. This is true because in the part of the itinerary that goes through V (see the notation at the beginning of the previous chapter) f' is Lipschitz, bounded away from zero and we can use Fact 1 there. On U,

f' can be small, but the part of the itinerary that goes through U adds to the distortion of the quantity (12) from 1 only the tail of a geometric series, by Fact 2. So there exists n_0 such that if $(l-n) \geq n_0$ the logarithm of the quantity (12) is much smaller than $\log\gamma$. We can also claim the same for g, maybe increasing n_0.

Thus, if we had $\frac{|\Lambda|}{|\overline{\Lambda}|} = \tau$ and $\frac{|\Omega|}{|\overline{\Omega}|} = \sigma$ we still can have

$$\frac{|f^s\Lambda|}{|g^s\overline{\Lambda}|} : \frac{|f^s\Omega|}{|g^s\overline{\Omega}|} \approx \frac{\tau}{\sigma}$$

with arbitrarily small error.

Now, if we are very close to c, i.e. n is very large, we can have

$$\frac{|f^s\Lambda|}{|g^s\overline{\Lambda}|} \approx h'(c)$$

and also

$$\frac{dist(c,\Omega)}{dist(\overline{c},\overline{\Omega})} \approx h'(c)$$

with arbitrarily small error (recall we assumed $\frac{|\Omega|}{|\Lambda|} << 1$). In the set U f and g act as the power law $|x|^\alpha$, so after one more step we have (with arbitrarily small error)

$$\frac{|f^{s+1}\Lambda|}{g^{s+1}\overline{\Lambda}|} \approx \frac{|f^s\Lambda|}{|g^s\overline{\Lambda}|} \cdot \alpha \cdot (h'(c))^{\alpha-1}$$

and if Ω was very short compared to $dist(c,\Omega)$ we also have (with arbitrarily small error)

$$\frac{|f^{s+1}\Lambda|}{|g^{s+1}\overline{\Lambda}|} : \frac{|f^{s+1}\Omega|}{g^{s+1}\overline{\Omega}|} \approx \frac{\tau}{\sigma}.$$

Notice that we first choose and fix n_0 to make the change of $\frac{\tau}{\sigma}$ along the circuit 'far from the critical point' very small and then we move that information toward the critical point by increasing n to have the change of $\frac{\tau}{\sigma}$ arbitrarily small in the step 'close to the critical point'. Now we go through the next circuit until the moment when the image of $f^{s+1}\Lambda$ falls again into L_n. Because the estimate of the change in $\frac{\tau}{\sigma}$ we gave above by Facts 1 and 2 was an upper estimate using all the intervals of C_l not contained in L_n, we only used a fraction of those in the first circuit and we can keep the same estimate for the itinerary containing the first and second circuits. Then we are again inside L_n and use the argument about $|x|^\alpha$ for the second time; then we go through the next circuit keeping the original estimate for the whole part of the itinerary outside L_n and so forth till after no more than $(n_0)^T$ visits in L_n Λ falls onto L_l. But if n was large enough (recall again we choose n after n_0 was fixed), the change of $\frac{\tau}{\sigma}$ by the $(n_0)^T$ of '$|x|^\alpha$' steps can be made arbitrarily small. Conclusion of all the above is the following

Claim 1 *For any $\Lambda \in C_l$, with l large enough, and $\Omega \subset \Lambda$, $\Omega \in C_{l+p}$, $p >> 1$, if we have*

$$\frac{|\Lambda|}{|\overline{\Lambda}|} : \frac{|\Omega|}{|\overline{\Omega}|} = \frac{\tau}{\sigma}$$

and t is the smallest number such that

$$f^t\Lambda = L_l$$

then

$$\frac{|f^t\Lambda|}{|g^t\overline{\Lambda}|} : \frac{|f^t\Omega|}{|g^t\overline{\Omega}|} \approx \frac{\tau}{\sigma}$$

with arbitrarily small error, provided t and p are large enough.

□

We are now in a position to proceed towards the completion of the proof. Let's pick an interval $\Lambda \in C_q$ such that for this interval the bound $\mathcal{M}$ is almost assumed, i.e. $\frac{|\Lambda|}{|\overline{\Lambda}|} \approx \mathcal{M}$ vith very small error and q is very large so that the considerations above and the Claim hold.

Set

$$\kappa = \lim_{j\to\infty} \inf_{\Gamma\subset\Lambda, \Gamma\in\bigcup_{i\geq j} C_i} \frac{|\Gamma|}{|\overline{\Gamma}|}$$

and let $\Omega \subset \Lambda$, $\Omega \in \bigcup C_i$ be an interval that is very short with respect to Λ (i.e. $\Omega \in C_{q+q'}$ with q' very large) such that the κ bound is almost assumed on Ω, i.e. $\frac{|\Omega|}{|\overline{\Omega}|} \approx \kappa$. For such Λ and Ω we have (with arbitrarily small error)

$$\frac{|\Lambda|}{|\overline{\Lambda}|} : \frac{|\Omega|}{|\overline{\Omega}|} \approx \frac{\mathcal{M}}{\kappa}.$$

By the Claim above this 'almost equality' is preserved along the forward itinerary of Λ until some image of Λ, say $f^t\Lambda$, falls onto L_q for the first time. But $\frac{|f^t\Lambda|}{|g^t\Lambda|} \approx h'(c)$, so

$$\frac{|f^t\Omega|}{|g^t\overline{\Omega}|} \approx h'(c) : \frac{\mathcal{M}}{\kappa} \tag{13}$$

and for any $\Gamma \subset f^t\Omega$, $\Gamma \in \bigcup_{i \geq q+q'} C_i$ we have

$$\frac{|\Gamma|}{|\overline{\Gamma}|} > h'(c) : \frac{\mathcal{M}}{\kappa} - (\text{very small error}). \tag{14}$$

Notice that $\frac{\mathcal{M}}{\kappa} \geq\approx \gamma$, since if $\frac{\mathcal{M}}{\kappa}$ were less than γ, so would be $\mathcal{B}(c)$ (recall the remark at the beginning of the proof about pushing that property forward into an arbitrarily small neighborhood of c).

Put

$$\mu = \lim_{j \to \infty} \sup_{\Gamma \subset \Omega, \Gamma \in \bigcup_{i \geq j} C_i} \frac{|\Gamma|}{|\overline{\Gamma}|}.$$

By the same argument as before, $\frac{\mu}{\kappa}$ is about γ or larger (perhaps with an error minute with respect to γ). Now within $f^t\Omega$ there exists an interval Ω', $\Omega' \in \bigcup C_i$, very short relative to Ω (or $\Omega' \in C_{q+q'+q''}$ with q'' very large) such that $\frac{|\Omega'|}{|\overline{\Omega'}|} \approx \mu$.

We now go forward along the itinerary of $f^t\Omega$ until its image, say $f^u(f^t\Omega)$ becomes for the first time a basic renormalization interval about c, namely $L_{q+q'}$. Then, by the Claim, the image of Ω' satisfies

$$\frac{|f^u\Omega'|}{|g^u\overline{\Omega'}|} \approx \frac{\mu}{\kappa} \cdot h'(c)$$

because

$$\frac{|L_{q+q'}|}{|\overline{L_{q+q'}}|} \approx h'(c).$$

Also, by the Claim and inequality (14), for any $\Gamma \subset f^u\Omega'$, $\Gamma \in \bigcup_{i \geq q+q'+q''} C_i$

$$\frac{|\Gamma|}{|\overline{\Gamma}|} \geq h'(c) - (\text{very small error}). \tag{15}$$

Now we set

$$\kappa' = \lim_{j\to\infty} \inf_{\Gamma\subset f^n\Omega', \Gamma\in\bigcup_{i\geq j} C_i} \frac{|\Gamma|}{|\overline{\Gamma}|}.$$

Again, $\frac{\mu}{\kappa'}$ is about γ (perhaps minus a minute error) or larger. Within $f^n\Omega'$ there must be an interval $\Omega'' \in \bigcup C_i$ such that Ω'' is very short relative to Ω' (i.e. $\Omega'' \in C_{q+q'+q''+q'''}$ with very large q''') and

$$\frac{|\Omega''|}{|\overline{\Omega''}|} \approx \kappa'.$$

If r is the smallest number such that $f^{n+r}\Omega' = L_{q+q'+q''}$ then of course

$$\frac{|f^{n+r}\Omega'|}{|g^{n+r}\overline{\Omega'}|} = \frac{|L_{q+q'+q''}|}{|\overline{L_{q+q'+q''}}|} \approx h'(c)$$

and, by the Claim,

$$\frac{|f^r\Omega''|}{|g^r\overline{\Omega''}|} \approx h'(c) : \frac{\mu}{\kappa'},$$

which contradicts (15).

Thus the assumption $\gamma > 1$ leads to a contradiction, so $\mathcal{B}(c) = 1$ and the proof is complete. □

Considering the derivative $h'(x)$ for $x \in \mathcal{C}_f$ in the sense of the previous chapter the above theorem, by virtue of the Lipschitz condition, immediately yields

Corollary 3 *For every $x \in \mathcal{C}_f$ $h'(x)$ exists and the derivative h' is continuous in the metric on $\mathcal{C}_f$ inherited from I.*

References

[1] P. Collet, J.P. Eckmann, *Iterated maps of the interval as dynamical systems*, Boston, Birkhauser 1980.

[2] M. Jakobson, G. Swiatek, Quasisymmetric conjugacies between unimodal maps, preprint IMS Stony Brook (1991).

[3] Y. Jiang, On the Quasisymmetrical Classification of Infinitely Renormalizable Maps: I. Maps with Feigenbaum Topology, II. Maps with a Bounded Type Topology.

[4] O.E. Lanford III, A computer-assisted proof of the Feigenbaum conjectures, Bull. Amer. Math. Soc, New Series, **6**, 127 (1982).

[5] W. Paluba, The Lipschitz condition for the conjugacies of Feighenbaum-like mappings, Fund. Math. **132**, 227-258 (1989).

[6] D. Sullivan, Bounds, quadratic differentials and renormalization conjectures, to appear in AMS Centennial Publications, vol. **2**, AMS, Providence, R.I. (1991)