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VARIOUS ASPECTS OF MAX-ALGEBRA.

CITY UNIVERSITY OF NEW YORK, PH.D., 1978

VARIOUS ASPECTS OF MAX-ALGEBRA

by

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A dissertation submitted to the Graduate faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1978

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

May 11, 1978
date

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date

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ACKNOWLEDGEMENTS

I would like to thank Dr. Alan J. Hoffman for his unfailing encouragement, valuable suggestions, and immeasurable assistance when I encountered apparent unsurmountable obstacles. Many of his suggestions helped me a great deal in bringing this dissertation to its final stage. I am amazed how each word of advice tremendously increased my sense of direction towards my ultimate goal.

I would also like to thank all the faculty, staff, and students in the Mathematics Department at The Graduate Center for their help and friendship.

I would like to thank my wife, Beverly, and son, Lennox, Jr., for their patience, consideration, and tolerance of my sometimes intolerable behavior.

I would like to say a special thanks to my deceased father who was determined that I receive a good education. May he rest in peace.

Finally, I would like to say thanks to my "lifelong source" of inspiration: my brothers - Andréw and Bobby; my uncle - Isiah; my aunts - Robertina, Josephine, and Kate; my cousins - Stephen, Lincoln, Daphne, Fletcher, and Lawrence; my nephews, nieces and the rest of the gang - too numerous to mention.

I must give special thanks to Ione Hutson for doing a splendid job of typing this dissertation.

This research was partially supported by an Intramural Faculty Award I obtained through Hunter College.

Chapter 1: Introduction.

Let S be an algebra of real numbers, extended by the symbol $-\infty$, in which the regular multiplication and addition of two numbers are replaced by arithmetical addition and the selection of the greater of two numbers respectively. Thus, if $S = \mathbb{R} \cup -\infty$, we define for each $x,y \in S$

(1.1) (i) $x \otimes y$ to be $(x+y)$

(ii) $x \oplus y$ to be $\max(x,y)$.

In this algebra, which we will refer to as "max-algebra", 0 is represented by $-\infty$ and 1 by 0 . Note that most of the familiar laws of ordinary algebra hold in this new notation (see [1],[2],[4]). In the past, this type of algebra (in some instances extended by ∞ , and minimum used instead of maximum in (1.1))has been widely used in a variety of operational research problems (see [1],[2],[3],[4], and [14]). Also, our above mentioned algebra was utilized in linear programming duality [12].

We shall be concerned in this thesis with various aspects of our max-algebra. Topics to be treated include the following:

(1.2) Matrix Inversion and Systems of Linear Equations in Chapter 2;

(1.3) Linear Programming Duality in Chapter 3;

(1.4) Eigenvector-value Problem in Chapter 4;

(1.5) Blocking Theory in Chapter 5.

Matrix Inversion. In our max-algebra, our treatment - the question of existence and uniqueness for inversion of matrices - is entirely new. We shall prove the following results:

(1.6) If A is a square matrix, and if the real elements of A are in one-to-one correspondence with the 1's in a permutation matrix, then the negative of the transpose of A is a unique left inverse and also a unique right inverse of A ;

(1.7) If A (no row consisting entirely of $-\infty$) has n rows and p columns, $n > p$ say, then A has no right inverse and the existence and uniqueness of a left inverse of A rely on the existence and uniqueness of a submatrix of order p whose real entries are in one-to-one correspondence with the 1's in a permutation matrix.

Systems of Linear Equations. Here, we treat the questions of existence and uniqueness for solutions of systems of linear equations:

(1.8) $A \otimes x = b$

That is,

(1.8a) $\max_j \{a_{ij} + x_j\} = b_i$, for all i .

The existence of a solution requires that equality be attained in each row, which means for all i there exists a j such that $a_{ij} + x_j = b_i$. In the past, the existence question was treated in [2], but only when all variables and constants were real. We shall extend the existence question to include elements in $S = \mathbb{R} \cup -\infty$. Our uniqueness treatment, which is included in Chapter 2, is entirely new. We shall show that if a solution exists, but the deletion of any column of $A \otimes x = b$ implies that equality is not attained in some row, then the solution is unique.

Linear Programming Duality. This is an extension of an article published by Professor A.J. Hoffman in which he examined the duality theorem of linear programming in the context of a general algebraic setting [12]. In his treatment, he was mainly concerned in proving the duality theorem (max = min) when all variables and constants were real. In this thesis we shall consider the role of $-\infty$ with respect to the established results in [12].

Blocking Theory (see [5] - [9]). This is one of the more significant aspects of our max-algebra. Here, we define a concept of "blocking" in our max-algebra that generalizes the concept of blocking for sets described by Edmund and Fulkerson in [5]. They showed in [5] that blocking is a dual notion. We generalize their blocking notion and show that the dual notion also holds in our generalization.

Eigenvector-value Problem. This topic constitutes the main aspect of this thesis. We shall be concerned with the following problem:

Given the n-square irreducible (to be defined in Chapter 4) matrix A of quantities a_{ij}, the problem is to determine $x_1, x_2, \ldots, x_n$ and λ such that

$$\max_j \{a_{ij} + x_j\} = \lambda + x_i \quad (i = 1,2,\ldots,n) \ . \tag{1.12}$$

In the past, if all elements in A were assumed real, it was shown [3] that the only possible value of λ was given by the greatest loop mean (to be defined in Chapter 4). Also, if there was an n-termed loop (to be defined in Chapter 4) of maximum mean λ , then the corresponding eigenvector X was demonstrated [3].

We shll first extend the uniqueness question to include elements in $S = \mathbb{R} \cup -\infty$. Then, we shall establish through ordinary linear programming the existence of λ (the existence of λ was not even done in the real case before).

The Operational Research (OR) problem modeled by the eigenvalue-eigenvector question in this max-algebra is extremely interesting, and will now be described. In [3] it was observed that Cranes and other handling equipment in a steelworks often have a cycle of duties and a common feature of these industrial processes is that these machines do not act independently, for a typical machine cannot begin a fresh cycle of activity until certain other machines have all completed their current cycles. We need to study the through-put of such a system. To explain the technique evolved by Cunningham-Green [2] for dealing with this situation, we label the machines $1,2,\ldots,n$ and describe the interferences by recurrence relations such as:

$$(1.13) \qquad x_3^{(r+1)} = \max\,(x_1^{(r)} + t_1^{(r)},\; x_2^{(r)} + t_2^{(r)}) \;.$$

Equation (1.13) means that machine 3 must wait to begin its $(r+1)^{st}$ cycle until machines 1 and 2 have both finished their r^{th} cycle, the symbol $x_i^{(r)}$ denoting the starting times of the r^{th} cycle of machine i, and $t_i^{(r)}$ denoting the corresponding activity duration. This type of analysis gives rise, after simplification, to a formidable-looking system of recurrence relations:

$$(1.14) \qquad x_i^{(r+1)} = \max\,(x_1^{(r)} + a_{i1},\ldots,x_n^{(r)} + a_{in}) \quad i = 1,2,\ldots,n \;.$$

where, for notational uniformity, all terms a_{ij} and $x_j^{(r)}$, for $j = 1,2,\ldots,n$, are made to occur for each i by introducing where necessary quantities $a_{ij} = -\infty$ for each (ij) which has no physical significance (see [13]). Now, if we write

$$(1.15) \qquad x \oplus y \text{ instead of } \max(x,y)$$

$$x \otimes y \text{ instead of } (x+y)$$

and introduce the obvious vector-matrix notation

$$(1.16) \qquad A = (a_{ij}) \,,\; x^{(r)} = x_j^{(r)}$$

(1.14) becomes

$$(1.17) \qquad x^{(r+1)} = A \otimes x^r \;.$$

An interesting operational question is this: "How must the system be set in motion to ensure that it moves forward in regular steps; that is, so that for some constant λ, the interval between the beginnings of consecutive cycles on every machine is λ? And what are the possible values of λ?" To answer the preceding questions, it suffices to observe that if the system is to progress by regular steps, the $(r+1)^{st}$ value of each variable is to exceed the r^{th} by the same constant - in our notation,

(1.18) $$x^{(r+1)} = \lambda \otimes x^{(r)} .$$

Replacing $x^{(r+1)}$ in (1.17) by the equivalent expression in (1.18), we obtain

(1.19) $$\lambda \otimes x^{(r)} = A \otimes x^{r} ,$$

that is,

(1.19a) $$\lambda \otimes x = A \otimes x .$$

Therefore, if we determine λ, and $x_1, x_2, \ldots, x_n$ such that (1.19a) is satisfied, then our "interesting" operational research question is answered.

Chapter 2: Matrix Inversion and Systems of Linear Equations

In this chapter, we consider (in our max-algebra) the questions of existence and uniqueness for inverses of matrices and for systems of linear equations.

Clearly, the multiplicative identity for matrices in our algebra, I_n , is given by

$$(2.1) \qquad (I_n)_{ij} = \begin{Bmatrix} 0 & \text{if} & i = j \\ -\infty & \text{if} & i \neq j \end{Bmatrix} .$$

If A is a matrix with n rows and p columns, B a matrix with p rows and n columns, then

$$(2.2) \qquad A \otimes B = I_n$$

means

$$(2.2a) \qquad \max_k \left(a_{ik} + b_{kj} \right) = \begin{Bmatrix} 0 & \text{if} & i = j \\ -\infty & \text{if} & i \neq j \end{Bmatrix} .$$

If these stipulations occur, then B is said to be a right inverse of A and A a left inverse of B . We shall first treat the case $n = p$, then $n \neq p$.

Next, we treat the solution of systems of equations, that is,

$$(2.3) \qquad A \otimes x = b$$

which means

$$(2.3a) \qquad \max \left(a_{ij} + x_j \right) = b_i \text{ , for all } i .$$

Our analysis depends on a partitioning of A and b according to the pattern of the location of $-\infty$ in b and A .

We first consider (2.2). Recall that a permutation matrix is a square (0,1) matrix in which each row and column contains exactly one 1 . Let A be any matrix with entries in $\{\mathbb{R} \cup -\infty\}$, define the indicator of A (written $\hat{A} = (\hat{a}_{ij})$) by the rule

$$\hat{a}_{ij} = \begin{cases} 1 & \text{if } a_{ij} > -\infty \\ 0 & \text{if } a_{ij} = -\infty \end{cases} . \tag{2.4}$$

Also define $\theta A = (\theta a_{ij})$ by

$$\theta a_{ij} = \begin{cases} -a_{ij} , & \text{if } a_{ij} > -\infty \\ -\infty , & \text{if } a_{ij} = -\infty \end{cases} . \tag{2.5}$$

Theorem 2.5. Let A be a square matrix over the max-algebra. Then the following statement about A are equivalent

(i) A has at least one left inverse

(ii) A has at least one right inverse

(iii) A has at most one left inverse

(iv) A has at most one right inverse

(v) $\hat{A}$ is a permutation matrix

(vi) θA^T is a unique left inverse and also a unique right inverse of A .

Proof: It is clear that (v) ⇒ (vi) ⇒ (i),(ii),(iii),(iv). So it is sufficient to prove that (ii) implies (v). Suppose A and B are square and (2.2a) holds. This implies that, for each i , there exists at least one index k such that

$$a_{ik} > -\infty, b_{ki} > -\infty, \; a_{ik} + b_{ki} = 0 . \tag{2.6}$$

For each i , let k(i) be a choice of k satisfying (2.6). Suppose there are two indices i and j such that

$$a_{jk(i)} > -\infty . \tag{2.7}$$

Then (2.6) and (2.7) imply that

$$\max_{\ell} \{ a_{j\ell} + b_{\ell i} \} \geqq a_{jk(i)} + b_{k(i)i} > -\infty \ , \tag{2.8}$$

violating (2.2a). Therefore, (2.7) is false. But this means $k(i)$ are distinct, and, since A is square, $a_{ij} > -\infty$ implies $j = k(i)$. Thus $\hat{A}$ is a permutation matrix, establishing (v) and completing our proof.

We now consider the case where A has n rows and p columns, $n \neq p$.

<u>Theorem 2.2.</u> If A has n rows and p columns, $n > p$, then (i) A has no right inverse; (ii) A has a left inverse if and only if $\hat{A}$ contains a submatrix of order p which is a permutation matrix; (iii) A (a row of all $-\infty$ is excluded) has a unique left inverse if and only if $\hat{A}$ contains exactly one submatrix of order p which is a permutation matrix.

<u>Proof:</u> (i) Suppose A has n rows and p columns, B has p rows and n columns, $n > p$, and

$$A \otimes B = I_n \ . \tag{2.9}$$

Reasoning analogous to (2.6) and (2.7) indicates that if C is a p-square submatrix of A and D a p-square submatrix of B such that

$$C \otimes D = I_p \ , \tag{2.10}$$

then $\hat{C}$ is a permutation matrix by Theorem 2.1 (ii) and (v), but the remaining entries in A are all equal to $-\infty$, thus contradicting (2.9). Therefore A has no right inverse.

(ii) Suppose A has a left inverse. Then

$$B \otimes A = I_p \ ; \ B = p \times n \ \text{matrix}. \tag{2.11}$$

This implies that by arguments similar to (2.6), (2.7), (2.8), there exists at least one p-square submatrix, C say, of A and D of B such that

(2.12) $\hat{D} \otimes \hat{C} = I_p$.

But by Theorem 2.1, (i) and (v), $\hat{C}$ is a permutation matrix. Therefore $\hat{A}$ contains a submatrix of order p which is a permutation matrix.

Conversely, let $\hat{A}$ contain a submatrix $\hat{C}$ of order p which is is a permutation matrix. Then by Theorem 2.1 (v) and (vi) $\hat{C}$ has a unique left inverse $\hat{D}$ say, (which is also a permutation matrix) satisfying (2.12). What we need is a p×n matrix $\hat{B}$, constructed from $\hat{D}$, such that $\hat{B}$ is a left inverse of $\hat{A}$. So, firstly, without loss of generality, let the p rows of $\hat{C}$ be the first p rows of $\hat{A}$ and the matching p columns of $\hat{D}$ be the first p columns of $\hat{B}$. Secondly, considering the remaining (n-p) rows of $\hat{A}$, we construct the matrix $\hat{B}$ by adjoining (n-p) matching columns to $\hat{D}$ in the following way:

If any of these (n-p) rows of $\hat{A}$

(2.13) has only zero entries, then the matching column in $\hat{B}$ may have either one's or zeroes as entries;

(2.14) has more than one 1 then the matching column in $\hat{B}$ must have only zero entries;

(2.15) has exactly one 1 , say row $i' = (1,00 \ldots\ldots\ldots.0)$ with $a_{i'1} = 1$ then the matching column in $\hat{B}$ may have all zero entries or exactly one 1 , with $b_{1i'} = 1$.

Thus $\hat{A}$, with a p-square permutation submatrix $\hat{C}$ has a left inverse $\hat{B}$ which is comprised of $\hat{D}$ and one or more of the conditions stated in (2.13) through (2.15) on its remaining (n-p) columns.

(iii) Suppose A has a unique left inverse B . Without loss of generality:

(2.16) let the first p columns of $\hat{B}$ correspond to the columns of $\hat{D}$ and

(2.17) the remaining $(n-p)$ columns of $\hat{B}$ have only zero entries.

I claim that $\hat{B}$ is the unique left inverse of $\hat{A}$, that is, $\hat{D}$ in $\hat{B}$ depends only on the unique $\hat{C}$ in $\hat{A}$, so that $\hat{B} \otimes \hat{A} = I_p$.

Suppose not. Then one or more of the $(n-p)$ columns in $\hat{B}$ either

(2.18) has at least two 1's

or

(2.19) has exactly one 1 .

If (2.18) or (2.19) occurs, then the existence or uniqueness of $\hat{B}$ respectively would have to depend on the location of 1's and 0's among the $(n-p)$ matching columns in $\hat{A}$. Hence (2.16) and (2.17) define the unique left inverse $\hat{B}$. Now, without loss of generality

(*) Let the first p rows of $\hat{A}$ correspond to the rows of $\hat{C}$.

Then (2.17) implies that each of the remaining $(n-p)$ rows of $\hat{A}$ has at least two 1's . So $\hat{A}$ has exactly one submatrix of order p which is a permutation matrix.

Conversely, suppose $\hat{A}$ contains exactly one submatrix, $\hat{C}$ of order p which is a permutation matrix. Assume (*) holds , then

(**) Each of the remaining $(n-p)$ rows of $\hat{A}$ has at least two 1's .

From Theorem (v) and (vi), (*) implies that $\hat{C}$ has a unique left inverse $\hat{D}$ say, (which is a p-square permutation submatrix of $\hat{B}$). Due to (**) we see that the (2.17) immediately follows, or else $\hat{B}$ is not a left inverse of $\hat{A}$. Therefore, $\hat{B}$ depends only on $\hat{D}$ so that $\hat{B} \otimes \hat{A} = \hat{I}_p$.

By assumption $\hat{C}$ is the only p-square submatrix of $\hat{A}$ that is a permutation matrix, and since $\hat{D}$ is the unique left inverse of $\hat{C}$, it follows that $\hat{B}$ is the unique left inverse of $\hat{A}$. From Theorem (i),(ii) and (v), B is the unique left inverse of A, completing the proof of our theorem.

We now consider (2.3), where A is any p×n matrix with entries in $\{\mathbb{R} \cup -\infty\}$, and b any p-vector, also with entries in $\{\mathbb{R} \cup -\infty\}$. Clearly, from (2.3a)

(2.20) $$a_{ij} + x_n \le b_i \quad \text{for all } i ,$$

which implies

(2.20a) $$x_j \le b_i - a_{ij} , \text{ for all } a_{ij} \ne -\infty .$$

From (2.20) it is obvious that a solution for (2.3) exists if for all i there exists a j such that

(2.21) $$a_{ij} + x_j = b_i .$$

For each such j in (2.21) we define

(2.22) $$Z_j = b_i - a_{ij} , \ a_{ij} \ne -\infty$$

which means

(2.22a) $$Z_j = \min_{r | a_{rj} \ne -\infty} \{b_r - a_{rj}\} .$$

Otherwise we let

(2.23) Z_j be arbitrary.

Observe that from (2.20a) and (2.22) a solution for (2.3) exists if and only if $x_j \le z_j$ for all j .

Now, setting $x_j = z_j$, we define a position (ij) in matrix A to be marked if

(2.24) $$a_{ij} + z_j = b_i \; .$$

It follows that (2.24) also means that row i is marked, since z_j occurs in row i .

Before proceeding to answer our question on existence and uniqueness of a solution for (2.3), we define the following sets which illustrate our partitioning of A and b , on which our analysis depends.

Given (2.3) we let

(2.25) $$W = \{i \mid b_i = -\infty\}$$

(2.26) $$T = \{i \mid b_i \neq -\infty\} \quad \text{that is, all other rows}$$

(2.27) $$U = \{j \mid a_{ij} \neq -\infty \quad \text{amd} \quad i \in W\}$$

(2.28) $$V = \{j \mid j \in V\} \quad \text{that is, all other columns}$$

(2.29) $$W \times U = \{(ij) \mid i \in W, \; j \in U\}$$

(2.30) $$T \times U = \{(ij) \mid i \in T, \; j \in U\}$$

(2.31) $$W \times V = \{(ij) \mid i \in W, \; j \in V\}$$

(2.32) $$T \times V = \{(ij) \mid i \in T, \; j \in V\} \; .$$

<u>Theorem 2.3</u>. Let $A \otimes x = b$ be a system of equations over the max-algebra. Then

(2.33) $A \otimes x = b$ has a solution if and only if each row is marked at least once.

(2.34) If $A \otimes x = b$ has a solution, then its solution is unique if and only if deleting a column of $A \otimes x$ implies that at least one row is not marked.

Proof: Referring to (2.25) through (2.32) we dispose of a few cases in the preliminary stages. Observe that reference to any of (2.29) through (2.32) implies that (2.3) is restricted to that section in our partitioning of A and b respectively.

Clearly, (2.29) has a solution (which is its only solution), that is

(2.35) $$z_j = -\infty \quad \text{all} \quad j \in V ,$$

since $b_i = -\infty$, $i \in W$. Furthermore, because $b_i \neq -\infty$, $i \in T$, (2.35) implies that (2.30) has no solution. Also, if $z_j = \min_{i \mid a_{rj} \neq -\infty} \{b_r - a_{rj}\}$, $j \in V$ is a solution to (2.32), then it follows immediately, that since (2.30) is comprised of only $-\infty$'s, that z_j, $j \in V$ is a solution for (2.30).

Therefore, it is sufficient to prove our theorem for (2.32).

We first prove (2.33).

Suppose (2.32) has a solution, then by (2.21), (2.22), and (2.24), each row is marked at least once.

Conversely, suppose each row is marked at least once, then by (2.24) we have equality for each row i, that is

(2.36) $$a_{ij} + z_j = b_i , \quad \text{all} \quad i .$$

Now, from (2.20a) and (2.22)

(2.37) $$x_j \leq z_j \quad \text{all} \quad j$$

But from (2.22), $x_j = z_j$ in (2.36). We then obtain

(2.38) $$a_{ij} + x_j = b_i \quad \text{all} \quad i .$$

Recall that $a_{ij} + x_j \leq b_i$ all i. Then (2.38) means that

(2.39) $$\max_j \{ a_{ij} + x_j\} = b_i \quad \text{all} \quad i ,$$

that is, (2.32) has a solution.

We now prove (2.34). To begin with, we assume that no column has all $-\infty$'s ; otherwise, no unique solution is possible. Suppose (2.32) has a unique solution, then for all i , there exists a j such that

$$(2.40) \qquad a_{ij} + x_j = b_i$$

but, decreasing the value of any x_j in (2.32) implies that there is an i' , say, such that

$$(2.41) \qquad \max_{j \in V} \left\{ a_{ij} + x_j \right\} < b_{i'} \; .$$

Set $x_j = z_j$ in (2.32). Therefore, for all i there exists a j such that

$$(2.42) \qquad a_{ij} + z_j = b_i$$

which implies that every row is marked. But (2.41) implies that deleting a column j of (2.32) leaves at least one row unmarked.

Conversely, suppose that (2.32) has a unique solution and that the deletion of a column leaves each row still marked. Say column j' is deleted. Since $x_{j'} \leq z_{j'}$, then $x_{j'}$ can be assigned arbitrary values $\leq z_{j'} = \min_{i \mid a_{ij'} \neq -\infty} \{b_r - a_{ij'}\}$ which means that (2.32) has infinitely many solutions - yielding a contradiction. This completes the proof of our theorem.

Chapter 3: Linear Programming Duality.

In this chapter we investigate, in our max-algebra, an analogue of the duality theorem of linear programming. One version of the latter is the following: Consider the two linear programming problems:

$$\text{I}': \quad \text{Maximize} \quad \sum_{j=1}^{n} c_j x_j \text{ , where } (x_1, x_2, \ldots, x_n)$$

satisfies

(3.1) $$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i \quad (i = 1,2,\ldots,m)$$

and

(3.2) $$x_j \geqq 0 \quad (j = 1,2,\ldots,n)$$

$$\text{II}': \quad \text{Minimize} \quad \sum_{i=1}^{m} b_i y_i \text{ , where } (y_1, y_2, \ldots, y_m)$$

satisfies

(3.3) $$\sum_{i=1}^{m} a_{ij} y_i \geqq c_j \quad (j = 1,2,\ldots,n)$$

and

(3.4) $$y_i \geqq 0 \quad (i = 1,2,\ldots,m) \text{ .}$$

The duality theorem asserts

(3.5) if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ satisfy (3.1) - (3.4), then

$$\Sigma\, c_j x_j \leqq \Sigma\, b_i y_i \text{ ;}$$

(3.6) if x and y exist satisfying (3.1) - (3.4), then there exist $\bar{x}$ and $\bar{y}$ satisfying (3.1) - (3.4), with

$$\Sigma\, c_j \bar{x}_j = \Sigma\, b_i \bar{y}_i \text{ ;}$$

(3.7) if there exists $\bar{x}$ satisfying (3.1) and (3.2) such that

$$\Sigma c_j \bar{x}_j \geqq \Sigma c_j x_j$$

for all x satisfying (3.1) and (3.2), then there exists y satisfying (3.3) and (3.4);

(3.8) if there exists $\bar{y}$ satisfying (3.3) and (3.4) such that

$$\Sigma b_i \bar{y}_i \leqq \Sigma b_i y_i$$

for all y satisfying (3.3) and (3.4), then there exists x satisfying (3.1) and (3.2).

We shall see that the analogous result holds in our max-algebra (except that there is no role played by any analogue of (3.2) or (3.4).).

Consider the following pair of problems, where all constants and variables lie in $R \cup \{-\infty\}$.

I: Maximize $\max_j \{c_j + x_j\}$, where $x = (x_1, x_2, \ldots, x_n)$

satisfies

(3.9) $$\max_j \left\{a_{ij} + x_j\right\} \leqq b_i \quad (i = 1,2,\ldots,m)$$

II: Minimize $\max_j \{b_i + y_i\}$, where $y = (y_1, y_2, \ldots, y_n)$

satisfies

(3.10) $$\max_i \left\{a_{ij} + y_i\right\} \geqq c_j \quad (j = 1,2,\ldots,n) .$$

We shall call an x satisfying (3.9) feasible; similarly, a y satisfying (3.10) is feasible.

Theorem 3.1. Given problems I and II,

(3.11) There is always at least one feasible x . There is a maximum value of $\max_j \{c_j + x_j\}$ among feasible x if and only if

(*) For every j such that $c_j > -\infty$, there is at least one i such that $a_{ij} > -\infty$;

(3.12) there is a feasible y if and only if (*) holds;

(3.13) if x and y are feasible, $\max_j \{c_j + x_j\} \leqq \max_i \{b_i + y_i\}$;

(3.14) if (*) holds, there exist feasible $\bar{x}$ and $\bar{y}$ such that

$$\max_j \left\{c_j + \bar{x}_j\right\} = \max_i \left\{b_i + \bar{y}_i\right\} .$$

Proof: To show (3.11), we first remark that $x = (-\infty, -\infty, \ldots, -\infty)$ satisfies (3.9). Suppose $c_1 > -\infty$, but $a_{i1} = -\infty$ for all $i = 1, 2, \ldots, m$. Then the $x^a = (a, -\infty, \ldots, -\infty)$ satisfies (3.9) for all a, and $\max_j \{c_j + x_j^a\} = c_1 + a$ which can be made arbitrarily large by making a large.

So if (*) does not hold, our objective function has no maximum. Suppose (*) holds. If $c_j = -\infty$, the value of x_j in ay feasible x does not affect $\max_j \{c_j + x_j\}$, so assume $x_j = -\infty$. If $c_j > -\infty$, note that (3.9) implies $a_{ij} + x_j \leqq b_i$ for every i, hence $x_j \leqq \min_{i | a_{ij} > -\infty} \{b_i - a_{ij}\}$. (In particular, if there exists at least one i such that $b_i = -\infty$, $a_{ij} > -\infty$, $x_j \leqq -\infty$, so $x_j = -\infty$).

It follows that, when (*) holds, and we define $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, by the rule:

(3.15)
$$\bar{x}_j = \left\{ \begin{array}{ll} -\infty & \text{if } c_j = -\infty \\ \min\limits_{i | a_{ij} > -\infty} b_i - a_{ij}, & \text{if } c_j > -\infty \end{array} \right\},$$

then $\max_j \{c_j + \bar{x}_j\}$ solves problem I.

Proof of (3.12): Recall that we call a y feasible if it satisfies (3.10). Suppose $c_1 > -\infty$, but $a_{i1} = -\infty$ for all $i = 1,2,\ldots,m$. Then, for any y

$$\max_i \{a_{i1} + y_i\} < c_1 .$$

So, if (*) does not hold, there is no feasible y.

Suppose (*) holds. If $c_j = -\infty$

(3.16) $$\max_i \{a_{ij} + y_i\} \geq c_j$$

for $y_i \geq -\infty$, that is, (3.16) places no restriction on y when $c_j = -\infty$.

If $c_j > -\infty$, note that (3.10) implies that there exists i' such that $a_{i'j} + y_{i'} \geqq c_j$ for every j, hence $y_{i'} \geqq c_j - a_{i'j}$, $a_{i'j} > -\infty$. In particular, if there exists at least one j such that $c_j = -\infty$, $a_{ij} > -\infty$, then $y_i \geqq -\infty$ (repeating (3.16)). It follows that, when (*) holds, and we define $y = (y_1, y_2, \ldots, y_n)$, by the rule:

(3.17) $$y_i = \max_{i \,|\, a_{ij} \neq -\infty} \{c_j - a_{ij}\} \quad \text{if } c_j \geq -\infty$$

then $\{y_i\}$, $i = 1,2,\ldots,m$ is feasible.

To prove (3.13), let $x_1, x_2, \ldots, x_n$ satisfy (3.9) and $y_1, y_2, \ldots, y_m$ satisfy (3.10), and consider

$$\max_j \left[x_j + \max_i \{y_i + a_{ij}\}\right] = \max_j \left(\max_i \{y_i + a_{ij}\} + x_j\right)$$

$$= \max_i \max_j \left(y_i + \{a_{ij} + x_j\}\right)$$

$$= \max_i \left[y_i + \max_j \{a_{ij} + x_j\}\right] ,$$

that is,

(3.18) $$\max_j \left[x_j + \max_i \{y_i + a_{ij}\} \right] = \max_i \left[y_i + \max_j \{a_{ij} + x_j\} \right] .$$

Since $\max_j \{a_{ij} + x_j\} \le b_i \quad (i = 1,2,\ldots,m)$, we have

$$y_i + \max_j \{a_{ij} + x_j\} \le y_i + b_i \quad (i = 1,2,\ldots,m)$$

and

(3.19) $$\max_i \left[y_i + \max_j \{a_{ij} + x_j\} \right] \le \max_i \{y_i + b_i\} .$$

Similarly, the left hand side of (3.18) is

(3.20) $$\max_j \left[x_j + \max_i \{y_i + a_{ij}\} \right] \ge \max_j \{c_j + x_j\} .$$

By (3.18), (3.19), and (3.20), we have

(3.21) $$\max_j \{c_j + x_j\} \le \max_i \{y_i + b_i\} .$$

Proof of (3.14). Suppose (*) holds. From (3.11) and (3.12) we know that (3.9) and (3.10) have a solution respectively. In problem I we seek $\{x_j\}$ in order to maximize

(3.22) $$\max_j \{c_j + x_j\}$$

where

(3.23) $$\max_j \{a_{ij} + x_j\} \le b_i \quad (i = 1,2,\ldots,m) .$$

If $\{\bar{x}_j\}$ is defined as in (3.15), then clearly $\{\bar{x}_j\}$ satisfy (3.23), and (3.22) becomes

(3.24) $$\max \left\{ \begin{array}{ll} -\infty & \text{if } c_j = -\infty \\ \min\limits_{i \mid a_{ij} > -\infty} (c_j + b_i - a_{ij}) & \text{if } c_j > -\infty \end{array} \right\} .$$

That is,

(3.24a) $$\max_j \min_i (c_j + b_i - a_{ij}) \, , \; a_{ij} \neq -\infty \, .$$

A seemingly more convenient way of stating this value of (3.22) is as follows: Consider the matrix

(3.25) $$M = (m_{ij}) = (c_j + b_i - a_{ij}) \, , \; a_{ij} \neq -\infty$$

(in which no column consists entirely of $-\infty$), and let a position (ij) in M be marked if

(3.26) $$\min_{i \,|\, a_{ij} \neq -\infty} (b_i + c_j - a_{ij}) = \text{minimum} \, (M_{ij}) \, , \; (j = 1, 2, \ldots, n)$$

occurs in position (ij) .

(3.26) implies that row i in M is marked, since (ij) lies in the i^{th} row of M .

Let $\hat{S}$ be the set of marked positions (ij) defined in (3.26). Now for all j there exist at least one i^* such that

(3.27) $$(i^* j) \in \hat{S} \, ,$$

which implies that for all j

(3.28) $$\min_{i \,|\, a_{ij} \neq -\infty} (b_i + c_j - a_{ij}) = b_{i^*} + c_j - a_{i^* j}$$

(3.28a) $$= M_{i^* j} \, .$$

Therefore, for each row i^* , if there exists k such

$(i^* j)$'s $\in \hat{S}$, then let

(3.29) $$\left(i^* j^*\right) \in \hat{S}$$

be the marked position in the i^{*th} row, such that

(3.30) $$M_{i^* j^*} = \max_{j \,|\, (i^* j) \in S} \left(M_{i^* j} \right) \, ,$$

that is, for a given j^*,

(3.31) $$\min_{i \mid a_{ij^*} \neq -\infty} (c_{j^*} + b_i - a_{ij^*})$$

(3.32) $$= c_{j^*} + b_{i^*} - a_{i^*j^*} .$$

Note that (3.32) is the maximum of the column minima in row i^*.

Now, if row i is not marked, let

(3.33) $$\overline{y_i} = -\infty ;$$

if row i is marked, let

(3.34) $$\overline{y_i} = \max_{j \mid (ij) \in \hat{S}} (c_j - a_{ij})$$

and consider the

(3.35) $$\text{maximum } (M_{i^*j^*})$$

defined in (3.30).

Claim 1: $\overline{y_i}$ is feasible.

Claim 2: $\max_j \min_i (b_i + c_j - a_{ij}) = \max_i \{\overline{y_i} + b_i\}$

(from which it follows that $\overline{y_i}$ is optimal).

Proof of Claim 1: For each marked row i^*

$$\overline{y_{i^*}} = \max_{j \mid (i^*j) \in \hat{S}} (c_j - a_{i^*j})$$

$$\geq c_{j_{\hat{s}}} - a_{i^*j_{\hat{s}}}$$

$$\overline{y_{i^*}} + a_{i^*j_{\hat{s}}} \geq c_{j_{\hat{s}}} ,$$

and since for all j there exists i^* such that $(i^* j) \in \hat{S}$, it follows that $\overline{y_{i^*}} + a_{i^* j_{\hat{s}}} \geq c_{j_{\hat{s}}}$ $(j_{\hat{s}} = 1, 2, \ldots, n)$. Hence,

$$\max_i \left\{ y_i + a_{ij} \right\} \geq c_j \quad (j = 1, 2, \ldots, n) ,$$

that is, $\overline{y_i}$ is feasible.

To prove Claim II, it is sufficient to show that the maximum $(b_{i^*} + c_{j^*} - a_{i^* j^*})$

$$= b'_{i^*} + c'_{j^*} - a'_{i^* j^*} = \max_i \left\{ \bar{y}_i + b_i \right\} .$$

For each row i^* in M ,

(3.36) $\qquad b_{i^*}$ is constant

and

(3.37) $$\bar{y}_{i^*} = c_{j^*} - a_{i^* j^*} .$$

(3.36) and (3.37) imply that, for each i^*

(3.38) $$b_{i^*} + \bar{y}_{i^*} = b_{i^*} + c_{j^*} - a_{i^* j^*} .$$

Then

(3.39) $$\max_{i^*} \left\{ b_{i^*} + \overline{y_{i^*}} \right\} = \max_{i^*} \left\{ b_{i^*} + c_{j^*} - a_{i^* j^*} \right\} ,$$

that is

(3.39a) $$\max_i \left\{ b_i + \overline{y_i} \right\} = b'_{i^*} + c'_{j^*} - a'_{i^* j^*} .$$

Hence, $\max_i \left\{ b_i + \overline{y_i} \right\} = \max_j \min_i (b_i + c_j - a_{ij})$

which means

$$\max_i \left\{ b_i + \overline{y_i} \right\} = \max_j \left\{ c_j + \bar{x}_j \right\} ,$$

completing the proof of our theorem.

Chapter 4: The Eigenvector-value Problem.

In this chapter, we examine, in our max-algebra, an analogue of the eigenvector-value problem of linear algebra. The ordinary linear algebra version of this problem goes as follows: If $A = (a_{ij})$ is an n-square matrix, we seek a scalar λ and a non-zero vector $X = (x_1, x_2, \ldots, x_n)$ such that

$$\sum_j a_{ij} x_j = \lambda x_i \qquad (i = 1,2,\ldots,n) \text{ .}$$

We refer to λ as an eigenvalue and X as the associated eigenvector belonging to the eigenvalue λ. We shall see that, if A is irreducible (to be defined later), we can formulate an analogous problem in our max-algebra. As a point of information, an eigenvector in our max-algebra, should not have each component $-\infty$; in the irreducible case, no component will be $-\infty$.

We consider the following problem where all constants and variables lie in $S = (\mathbb{R} \cup -\infty)$: Given the n-square matrix $A = (a_{ij})$, then determine $x_1, x_2, \ldots, x_n$ and λ such that

(4.1) $\quad A \otimes x = \lambda \otimes x$,

that is,

(4.1a) $\quad \max\limits_j \{a_{ij} + x_j\} = \lambda + x_i \quad (i = 1,2,\ldots,n)$.

Clearly, (4.1a) implies that

(4.2) $\quad a_{ij} + x_j \leq \lambda + x_i$ for all ij

which can be rewritten as

(4.2a) $\quad \lambda + x_i - x_j \geqq a_{ij}$ for all ij .

Moreover, we require that at least one $x_j \neq -\infty$ or else any λ would satisfy (4.1).

We first define an irreducible matrix, then proceed to treat the case where A in (4.1) is irreducible. A matrix $A = (a_{ij})$, $a_{ij} \in S = \mathbb{R} \cup -\infty$ is reducible if there is a permutation matrix P such that

$$(4.3) \qquad P\,A\,P^T = \begin{pmatrix} A_1 & -\infty \\ A_2 & A_3 \end{pmatrix}$$

where A_1 and A_3 are square matrices. Otherwise A is <u>irreducible</u>, (see [10]). We assume henceforth that A is irreducible.

Now, to study (4.1) we first consider (4.2) and examine a linear programming problem and its dual problem (we mean here ordinary linear programming, not what was mentioned in Chapter 3).

<u>Problem I</u>: Minimize λ, where $\lambda, x_1, x_2, \ldots, x_n$ satisfy

$$(4.4) \qquad \lambda + x_i \geqq a_{ij} + x_j \quad \text{for all } ij$$

or

$$(4.4a) \qquad \lambda + x_i - x_j \geqq a_{ij} \quad \text{for all } ij .$$

<u>Problem II</u>: Maximize $\sum_{ij} y_{ij}\, a_{ij}$ where $y_{ij} \geqq 0$, all ij, satisfy

$$(4.5) \qquad \left\{ \begin{array}{l} \sum_{ij} y_{ij} = 1 \\ \sum_{j \neq i} (y_{ij} - y_{ji}) = 0 \quad \text{for each } i \end{array} \right\}.$$

We will let

$$(4.6) \qquad Q = \{\, y \mid y \geqq 0,\ By = P_0 \}$$

represent the convex polyhedron that defines the region of Problem II. Note that B is an $(n+1) \times n^2$ matrix, defined by the left hand side of (4.5), whereas P_0 is the vector with $n+1$ components on the right hand side of (4.5).

We now list a few definitions that will allow us to give proper consideration to (4.1):

Let G be the directed complete graph on n points (see Harary [11] for discussion on a variety of graphs). It has three properties:

(4.7) Each point i in G is joined to itself by a line $\{ii\}$;

(4.8) Two points i and j in G are joined by two distinct lines $\{ij\}$ and $\{ji\}$.

(4.9) If $A = (a_{ij})$ is an n-square matrix, define the length of a line $\{ij\}$ in G to be a_{ij} .

Remark 4.1. *It is easy to see that* $A = (a_{ij})$ *is irreducible (in (4.3)) if and only if for all* i *and* j *there exists a path* $i = i_1, i_2, \ldots, i_k = j$ *such that* $a_{i_r i_{r+1}} \neq -\infty$ *for all* r .

Then by a loop in G we mean:

(4.10) any line $\{ii\}$, for some i, $1 \leq i \leq n$

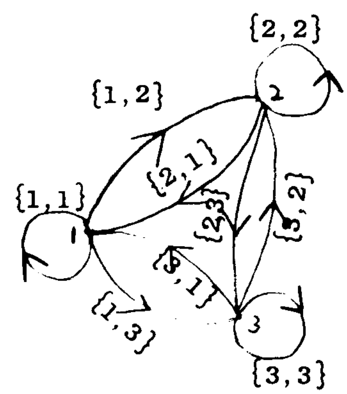
(4.11) any sequence of lines $\{j_1j_2\}, \{j_2j_3\} \ldots \{j_{r-1}i_r\}, \{j_rj_1\}$ where $j_1j_2 \ldots j_r$ $2 \leq r \leq n$ is a permutation of a subset of $1, 2, \ldots, n$.

(4.12) A loop mean in A is obtained by dividing the sum of the lengths of the lines in a loop by the number of lines in the loop.

Example (Illustration of (4.7) - (4.12))

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

G =



loops	loop means
$\{1,1\}$	a_{11}
$\{1,2\}$, $\{2,1\}$	$(a_{12} + a_{21})/2$
$\{1,2\}$, $\{2,3\}$, $\{3,1\}$	$(a_{12} + a_{23} + a_{31})/3$

Let b_{ij} be a column vector of B (in (4.6)), then the following is true:

(4.13) the 0^{th} component of b_{ij} is +1 ;

if $i \neq j$

(4.13a) the i^{th} component of b_{ij} is +1 $(i = 1,2,\ldots,n)$,

(4.13b) the j^{th} component of b_{ij} is -1 $(j = 1,2,\ldots,n)$

(4.13c) Everything besides (4.13) - (4.13b) in b_{ij} is 0 ; and if $i = j$, all components are 0 except the 0^{th} .

Then by a loop in B we mean

(4.14) any column vector b_{ii} $1 \leq i \leq n$

(4.15) any set of column vectors $b_{j_1j_2}, b_{j_2j}, \ldots, b_{j_rj_1}$ where the subscripts are defined as in (4.11). Note (considering (4.7) - (4.15)) that lines in G define column vectors in B . In particular, loops in G define loops in B . Therefore, we can say that loops in B correspond to loops in G and vice-versa.

Recall that, in P_0 ,

(4.16) all components are 0 except the o^{th} (which is 1).

We now consider (4.1). (Note that if A has only real entries, then A is, of course, irreducible). In Theorem 4.1 below, we prove the uniqueness of λ , and in Theorem 4.3 we prove existence.

Theorem 4.1. The only possible value of λ is given by the greatest loop mean in A .

Lemma 4.1. Let A be irreducible in (4.1a). If $c = \{i \mid x_i > -\infty\}$, then $\bar{c} = \{i \mid x_i = -\infty\} = \emptyset$;

Proof: Suppose there exists an

(4.17) $\quad i' \in \bar{c}$,

then for any λ

(4.18) $$\lambda + x_{i'} = -\infty = \max_j \{a_{ij} + x_j\} .$$

Also, if there exists an $a_{i'j} > -\infty$, then $x_j = -\infty$, that is,

(4.19) $\quad x_j \in \bar{c}$.

Now, if A is irreducible, then (4.17) and (4.18) imply that $c = \emptyset$. But we require at least one $x_j \neq -\infty$ in (4.1a).

Hence, $\bar{c} = \emptyset$.

It follows that if $\bar{c} \neq \emptyset$, then A is reducible # .

Lemma 4.2. If A is irreducible in (4.1a), then $\lambda \neq -\infty$.

Proof: Suppose $\lambda = -\infty$, then from Lemma 4.1, we assume $x_1 > -\infty$. Since A is irreducible, there exists an $a_{1j} > -\infty$. Consider

(4.20) $$\lambda + x_1 = \max_j \{a_{1j} + x_j\} .$$

It follows easily from (4.20) that $x_j = -\infty$, a contradiction by Lemma 4.1. #

Proof of Theorem 4.1:

It is clear from (4.2) that $x_i + \lambda \geq x_i + a_{ii}$, $i = 1,2,\ldots,n$ so that $\lambda \geq a_{ii}$. Moreover, from (4.2), choose any $r \leq n$ subset of inequalities such that the subscripts of the a_{ij}'s form a loop (there exist at least one real loop $k > 1$ in A, since A is irreducible; see Remark 4.1). Say, we obtain

$$(4.21)\qquad \begin{aligned} a_{12} + x_2 &\le x_1 + \lambda \\ a_{23} + x_3 &\le x_2 + \lambda \\ \cdots \quad \cdots &\quad \cdots \quad \cdots \\ a_{r-1r} + x_r &\le x_{r-1} + \lambda \\ a_{r1} + x_1 &\le x_r + \lambda . \end{aligned}$$

By addition, we obtain

$$\lambda \ge \frac{a_{12} + a_{23} + \dots + a_{r1}}{r} .$$

Hence λ is not less than any loop mean in A. On the other hand, from (4.1a), we see that the equality

$$(4.22)\qquad x_i + \lambda = a_{ij} + x_j$$

holds at least once for each i.

We want to show that not only is λ not less than any loop mean, but it is always equal to at least one loop mean.

Now, suppose $\lambda \neq -\infty$. If (4.22) holds for a given $i = j$, then $\lambda = a_{ii}$. Otherwise, consider the sequence

$$(4.23)\qquad \begin{aligned} a_{12} + x_2 &= \lambda + x_1 \\ a_{23} + x_3 &= \lambda + x_2 \\ \cdots \quad \cdots &\quad \cdot \quad \cdots \\ a_{p-1p} + x_p &= \lambda + x_{p-1} \\ a_{p\cdot p+1} + x_{p+1} &= \lambda + x_p \end{aligned}$$

(in which we assumed that (4.23) referred to the first p equations in (4.22), $p \leq n$) until we come to a loop with r terms, $r \leq p$. It is always possible to obtain a loop, by means of 4.23, since A is irreducible, (see Remark 4.1). Therefore, we derive a set of expressions similar to (4.21) but equality in all cases.

Hence, by addition of (4.23)

$$r\ \lambda = a_{12} + a_{23} + \dots + a_{r1}$$

and

$$\lambda = \frac{a_{12} + a_{23} + \dots + a_{r1}}{r} ,$$

so that λ is not less than any loop mean, and is always equal to at least one loop mean. Therefore, λ is given by the greatest loop mean in A , completing the proof of Theorem 4.1. #

Theorem 4.2. Given Problem I, the minimum λ which satisfies

(4.24) $\lambda + x_i \geqq a_{ij} + x_j$ for all ij

is given the maximum loop mean in A .

Before proving this Theorem, we will rephrase Problems I and II and list some facts.

Problem I: Minimize (P_0,w) where $w = (\lambda_1, x_1, x_2, \dots, x_n)$ satisfies

$$wB^T \geqq \mathcal{a} ;$$

Problem II: Maximize (a,y) where $y \geqq 0$ satisfies

$$By = P_0 .$$

From I and II we shall have need for the following:

(4.25) $\max\ (a,y) = \min\ (P_0,w)$;

(4.26) max (a,y) on Q occurs at a vertex of Q ;

(4.27) max $(a,y\ )$ on Q exists, since Q is closed and bounded;

(4.28) If $\bar{y} \in Q$, $\bar{y}$ is a vertex of Q if and only if

$$\{b_{ij} | (ij) \in \text{Support } \bar{y}\}$$

is a linearly independent set of columns (by the support of a vector we mean the components different from 0);

(4.29) If a linear function defined on a polyhedron is bounded from below, then the minimum exists.

Now, we want to show that the minimum in Problem I is the maximum loop mean. To do that, it is sufficient to show, from (4.25), that the maximum in Problem II is the maximum loop mean. By (4.26) and (4.27), it is sufficient to show that:

if Z is a vertex of Q , $I_Z = \{(ij) | Z_{ij} > 0\}$
and $k = |I_Z|$, then

(4.30) I_Z is a loop of order $k \geqq 1$;

and

(4.31) $\{Z_{ij} | (ij) \in \text{Support of } Z\}$

are all $1/k$; and if L is a loop $(j_1 j_2), (j_2 j_3) \ldots (j_r j_1)$, then setting $Z_{ij} = 1/k$ for all $(ij) \in L$ and $Z_{ij} = 0$ for all other $(ij) \notin L$ gives an n^2 vector

(4.32) $$Z = (\underbrace{1/k, 1/k, \ldots, i/k}_{\text{k-times}}, 0, 0, \ldots, 0)$$

which is a vertex of Q . (In this notation, we assumed that the loop referred to the first k coordinates of Z).

To prove (4.30) and (4.31) we first consider $k=1$, then $k > 1$.

<u>Case 1</u>: $k = 1$.

Suppose Z is a vertex of Q . If

(4.33) $$I_Z = (ij)$$

then

(4.34) $$b_{ij}Z_{ij} = P_0 .$$

Clearly $i \neq j$ and (4.34) imply P_0 has a negative coordinate, a contradiction.

Thus, $i = j$, implies b_{ii} is a loop of order 1 and (from the 0^{th} row of P_0) , we see that $Z_{ii} = 1$.

<u>Case 2</u>: $k > 1$.

Suppose Z is a vertex of Q . If

(4.35) $$I_Z = \{(ij) | Z_{ij} > 0\}$$

then

(4.36) $$\sum_{ij \in I_Z} b_{ij}Z_{ij} = P_0$$

and from (4.28), the b_{ij}'s in (4.36) is a linearly independent set of column vectors. From the discussion following (4.15), it is sufficient to show that $\{b_{ij} | (ij) \in \text{Support } Z\}$ in (4.36) is a loop of order k to prove (4.30).

(*) Consider, from (4.38), the location of the 1's and -1's in $\{b_{ij} | (ij) \in \text{Support } Z\}$ as compared to the +1 in P_0 . (See (4.13) - (4.13c) and (4.16)). Also consider

(4.37) $\bar{B} = (n+1) \times k$ submatrix of B. Then $\bar{B}$ is a loop of order k if the

i) 0^{th} row are all 1,

ii) other rows either have exactly one 1 and one -1 and 0's elsewhere or consist of all 0,

iii) moreover, each $b_{ij} \in \bar{B}$, $i \neq j$ satisfies (4.13) - (4.13c).

We now prove that this is the case. Suppose

(4.38) $\{b_{ij} \mid (ij) \in \text{Support } Z\}$

in (4.36) is not a loop, that is, it does not satisfy (4.37) i),ii), iii). Then consider a subset of (4.38) satisfying (4.37) i),ii),iii), say,

(4.39) $\bar{B}' = (n+1) \times k'$ submatrix of B,

and correspondingly from (4.35), choose

(4.40) $I_{Z'} \subset I_Z$.

Now, $\bar{B}'$ is a loop of order $k' < k$. By relating (*) to $\bar{B}'$ and P_0, and first considering (4.37) ii), then (4.37) i), we see that 1's and -1 in $\bar{B}'$ cancel each other among row i, $(i = 1,2,\ldots,n)$. But the o^{th} row of $\bar{B}'$ has k' 1's, whereas the 0^{th} row of P_0 has 1. It follows easily that

(4.41) $Z'_{ij} = 1/k$, for all $(ij) \in I_{Z'}$

and we can set

(4.42) $Z_{ij} = 0$ for all $(ij) \notin I_{Z'}$ but belonging to I_Z

and obtain

(4.43) $$\sum_{ij \in I_{Z'}} b'_{ij} Z'_{ij} = P_0$$

Therefore (4.42) implies that the set of b'_{ij}'s in (4.36) was not in the support of Z , a contradiction. Hence, (4.38) is a loop of order k , proving (4.30). By reasoning analogous to that used for deriving (4.41) and (4.43), we see that $\{Z_{ij} | (ij) \in \text{Support } Z\}$ are all $1/k$, thus proving (4.31).

Proof of (4.32): Suppose L is a loop of order $k \geqq 1$, (defined as in (4.10) and (4.11)). We first consider the case $k = 1$, then $k > 1$.

If $k = 1$, then for a fixed i , $1 \leq i \leq n$,

$$L = (ii) .$$

Consider the corresponding

(4.44) $\quad b_{ii} \qquad (ii) = L$

which is a non-zero (linearly independent) vector.

Now, for $(ii) = L$, set

(4.45) $\quad Z_{ii} = 1$

and for all $ij \notin L$ set

(4.46) $\quad Z_{ij} = 0$.

From (4.45) and (4.46), we obtain an n^2 vector

(4.47) $\quad Z = (1,00,\ldots,0)$

(the meaning of this notation is explained in (4.32)).

Now (4.45) implies that

(4.48) $\{Z_{ii} | (ii) = \text{Support } Z\} = 1$

and from (4.44),

(4.49) $\{b_{ii} | (ii) = \text{Support } Z\}$

is linearly independent. Hence, from (4.28),(4.48), and (4.49)

Z in (4.47) is a vertex of Q .

If $k > 1$, then from (4.11)

$$L = (j_1 j_2), (j_2 j_3), \ldots, (j_k j_1) .$$

We then consider the obvious, by now, corresponding

(4.50) $\bar{B} = (b_{j_1 j_2}), (b_{j_2 j_3}), \ldots, (b_{j_k j_1})$

which satisfies (4.37), i),ii), and iii). Consider

(4.51) $\sum_{ij \in L} b_{ij} Z_{ij} = 0 .$

It follows that in row i , $(i = 1,2,\ldots,n)$, the Z_{ij}'s are all equals, since $+1$ and -1 balance out. But the o^{th} row has all 1's , which implies that

(4.52) Z_{ij}'s are all 0 .

Hence,

(4.53) $\bar{B}$ is a linearly independent set of column vectors.

Now, for all $ij \in L$, set

(4.54) $Z_{ij} = 1/k$,

and for all $ij \notin L$, set

(4.55) $Z_{ij} = 0$.

From (4.54) and (4.55), we obtain an n^2 vector

(4.56) $Z = (\underbrace{1/k, 1/k, \ldots, 1/k}_{\text{k-times}}, 0, 0, \ldots, 0)$

(see (4.32) for clarification of notation).

From (4.54)

(4.57) $\{Z_{ij} \mid (ij) \in \text{Support } Z\}$

are all $1/k$ and from (4.53)

(4.58) $\{b_{ij} \mid (ij) \in \text{Support } Z\}$

is a linearly independent set of column vectors. Hence, from (4.28), (4.57), (4.58),

$$Z \text{ in (4.56) is a vertex of } Q .$$

This completes the proof of Theorem 4.2.

We now seek an eigenvector $\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ belonging to $\bar{\lambda}$ - the minimum λ in Problem I.

Theorem 4.3. If $\bar{\lambda}$ satisfies (4.4), then there is an eigenvector $\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, $\bar{x}_i > -\infty$ $(i = 1,2,\ldots,n)$ such that

$$\max_j \{a_{ij} + \bar{x}_j\} = \bar{\lambda} + \bar{x}_i \quad (i = 1,2,\ldots,n) .$$

Proof: From (4.26), we know that $\bar{\lambda} + x_i \geqq a_{ij} + x_j$, for all ij, has at least one solution. (Note that there is a loop in A by Remark 4.1). Now, let L be a loop of order $k \geqq 1$ whose lines identify the subscripts of the a_{ij}'s in the maximum loop mean, $\bar{\lambda}$, in the matrix A. So if

(4.59) $\bar{\lambda} + x_i \geqq a_{ij} + x_j$ all ij,

we may then assume that

(4.60) $\bar{\lambda} + x_i \geqq a_{ij} + x_j$ all $(ij) \in L$.

Summing the inequalities in (4.60), we observe that

(4.61) $$k\,\bar{\lambda} + \sum_{i=1}^{k} x_i \geqq \sum_{ij \in L} a_{ij} + \sum_{j=1}^{k} x_j .$$

(In this notation, we assumed that the loop referred to the first k coordinates of $\bar{X}$). But since

$$\bar{\lambda} = \sum_{(ij)\in L/k} a_{ij} \quad \text{and} \quad \sum_{i=1}^{k} x_i = \sum_{j=1}^{k} x_j ,$$

it follows that (4.61) is a set of equalities, which, in turn, means that (4.60) is also a set of equalities, that is,

$$(4.62) \qquad \bar{\lambda} + x_i = a_{ij} + x_j \quad \text{all} \quad (ij) \in L .$$

From (4.62) it follows that

$$\bar{\lambda} + x_i = \max_j \{a_{ij} + x_j\} \quad (i= 1,2,\ldots,k) .$$

Note that adding or subtracting the same quantity from both sides of (4.59) does not have any effect on the inequalities, so we can set one of the x_i's in (4.62), say $x_1 = 0$, (this can be done by subtracting $(-x_1)$ throughout), and obtain

$$\begin{aligned} x_1 &= 0 \\ x_2 &= \bar{\lambda} - a_{12} \\ x_3 &= 2\bar{\lambda} - a_{12} - a_{23} \\ &\ldots \ldots \ldots \ldots \ldots \\ x_k &= (k-1)\lambda - a_{12} - a_{23} - \ldots - a_{(k-1)(k)} . \end{aligned}$$

If the loop L in (4.62) is of order $k = n$, then we are finished, that is,

$$\bar{\lambda} + \bar{x}_i = \max_j \{a_{ij} + \bar{x}_j\} \quad (i = 1,2,\ldots,n) .$$

If not, that is, the loop in (4.62) is of order $k < n$, then we need $\bar{x}_{k+1}, \bar{x}_{k+2}, \ldots, \bar{x}_n$ such that

$$(4.63) \qquad \bar{\lambda} + \bar{x}_i = \max_j \{a_{ij} + x_j\} \qquad (i = k+1,\ldots,n) .$$

To obtain (4.63) we formulate the following linear programming problem ($x_1, x_2, \ldots, x_k$ will not be affected):

$$\text{minimize} \quad \sum_{i=k+1}^{n} x_i \ , \ x_1 = 0$$

subject to

(4.64) $\bar{\lambda} + x_i \geqq a_{ij} + x_j$ for all ij .

Let $\bar{L}$ be the complement of L . By Remark 4.1, if A is irreducible, then there is a path with one point in L and the next in $\bar{L}$. Therefore, there exists

$$i^* \in \bar{L},\ j \in L \ ,$$

such that

(4.65) $a_{i^*j} \neq -\infty$.

Also, by definition of irreducibility, we know that if and only if $j \in L$, then

$$x_j \neq -\infty \ .$$

Now from (4.64), we consider for a_{i^*j} in (4.65) and $x_j \in L$

$$x_{i^*} \geqq a_{i^*j} + x_j - \bar{\lambda} \ ,$$

that is, x_{i^*} is bounded from below.

Now, let $L \cup i^* = L'$. Again, since A is irreducible, there exists

$$i' \in \bar{L}', j \in L'$$

such that

(4.66) $a_{i'j} \neq -\infty$.

Also by definition of irreducibility, we know that if $j \in L'$, then either $x_j \neq -\infty$ or x_j is bounded from below.

Again, from (4.64), we consider, for $a_{i'j}$ in (4.66) and $x_j \in L'$,

$$x_{i'} \geqq a_{i'j} + x_j - \bar{\lambda} ,$$

that is, $x_{i'}$ is bounded from below.

This process can be continued, (since A is irreducible) until we obtain that each

$$x_i \quad i = k+1, k+2, \ldots, n$$

is bounded from below. Hence the objective function $\sum_{i=k+1}^{n} x_i$ exists.

Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ consists of $x_{k+1}, x_{k+2}, \ldots, x_n$ giving the minimum together with the coordinates in the loop - $x_1, x_2, \ldots, x_k$.

We contend that

(4.67) $$\bar{\lambda} + \bar{x}_i = \max_j \{a_{ij} + \bar{x}_j\} \quad (i = 1, 2, \ldots, n) .$$

Suppose (4.67) is false, that is, for at least one row i

(4.68) $$\bar{\lambda} + \bar{x}_i > \max_j \{a_{ij} + \bar{x}_j\} .$$

Then replace $\bar{x}_i$ by $\bar{x}_i - \epsilon$ so that (4.68) still holds.

Call the new vector $\bar{X}(\epsilon)$. For all other $\bar{\lambda} + \bar{x}_i(\epsilon) > \max_j \{a_{ij} + \bar{x}_j(\epsilon)\}$.

Then $\bar{X}$ did not yield a minimum, a contradiction.#

Chapter 5: Blocking Theory

Our purpose in this chapter is to define a concept of "blocking" in our max-algebra that generalizes the concept of blocking for sets described by Edmonds and Fulkerson in [5]. And we shall see from their observation that blocking is a dual notion holds in the generalization.

Let us first describe the main idea of [5]. Let E be a finite set, $\mathcal{A}$ a clutter of subsets of E (that is, $\mathcal{A}$ is a non-empty family of distinct subsets of E such that

(5.1) $$A_i, A_j \in \mathcal{A},\ i \neq j \Rightarrow A_i \not\subset A_j$$

A subset $B \subset E$

such that for all $A_i \in \mathcal{A}$,

(5.2) $$B \cap A_i \neq \emptyset$$

is said to block $\mathcal{A}$. If B blocks $\mathcal{A}$, but, for every $e \in B$, $B - \{e\}$ does not block $\mathcal{A}$, then B is said to be a minimal set blocking $\mathcal{A}$. Let $\mathcal{B}$ be the collection of all minimal sets blocking $\mathcal{A}$. Then $\mathcal{B}$ is called the blocker of $\mathcal{A}$, written $\mathcal{B} = b(\mathcal{A})$. Clearly, $b(\mathcal{A})$ is also a clutter.

Theorem 5.1 [5]. If $\mathcal{A}$ is a clutter, $b(b(\mathcal{A})) = \mathcal{A}$.

We now proceed to generalize the above. Let $\mathcal{A}$ be a matrix with entries in $R \cup \{-\infty\}$, no row of which has all entries $-\infty$, such that for any two rows A_i, A_j of $\mathcal{A}$, with

$$A_i = (a_{i1}, a_{i2}, \ldots, a_{in}),\ A_j = (a_{j1}, a_{j2}, \ldots, a_{jn}),$$

there exists

(5.3) $$k, \ell \text{ with } a_{ij} > a_{jk},\ a_{i\ell} < a_{j\ell}.$$

We will use the phrase "rows A_i and A_j are incomparable" if (5.3) holds, and we call a matrix $\mathcal{A}$ with no row consisting entirely of $-\infty$ and any pair of rows incomparable a tower. Note that if the entries in $\mathcal{A}$ are 0 or $-\infty$, then the indicator of $\mathcal{A}$, $\hat{\mathcal{A}}$ (see (2.4)) is the incidence matrix of sets versus elements of a clutter.

A row vector $B = (b_1, b_2, \ldots, b_n)$ satisfying

(5.4) for all row $A = (a_1, a_2, \ldots, a_n)$ of $\mathcal{A}$, $\max_k (a_k + b_k) \geqq 0$

is said to block $\mathcal{A}$. If B blocks $\mathcal{A}$, but, for every row vector

(5.5) $C = (c_1, c_2, \ldots, c_n)$

such that $c_{ik} \leqq b_{ik}$ for all k, and $c_{ik} < b_{ik}$ for at least one k, (5.4) is false, then B is said to be a minimal vector blocking $\mathcal{A}$. Clearly, the set of all minimal vectors blocking $\mathcal{A}$ are pairwise incomparable.

<u>Lemma 5.1</u>. If $\mathcal{A}$ is a tower, the set of all minimal vectors blocking $\mathcal{A}$ (that is, $b(\mathcal{A}) = \mathcal{B}$) is finite.

<u>Proof:</u> Suppose $\mathcal{B}$ be the set of all minimal vectors blocking $\mathcal{A}$. If $B \in \mathcal{B}$ is a row minimal vector blocking $\mathcal{A}$, then by (5.5)

(5.6) the real components of B are equal to the negative of a subset of the real elements in $\mathcal{A}$.

But since $\mathcal{A}$ is a tower,

(5.7) the real elements in $\mathcal{A}$ is finite.

It follows from (5.6) and (5.7) that in order to preserve the minimality condition on $\mathcal{B}$, $\mathcal{B}$ must be finite and hence $\mathcal{B}$ is a tower.

<u>Lemma 5.2</u>. Let $\mathcal{A}$ be an $m \times n$ tower and consider $b(\mathcal{A}) = \mathcal{B}$. If there exist an n-vector

$$X = (x_1, x_2, \ldots, x_k, \ldots, x_n)$$

such that

(5.8) $$\max_k \{x_k + b_{rk}\} \geqq 0$$

for all $B_r = (b_{r1}, b_{r2}, \ldots, b_{rn})$ of $\mathcal{B}$, then there exists an $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ of $\mathcal{A}$ such that

(5.9) $$x_k \geqq a_{ik}$$

for all k.

Proof: Let

(5.10) $$X_k = \{ i \mid a_{ik} > x_k \} .$$

If the Lemma is false, then

(5.11) $$\bigcup_k X_k = \{A_1, A_2, \ldots, A_m\}$$

where $\mathcal{A}$ has m rows. Suppose that for all $B_r \in b(\mathcal{A})$ and $A_i \in \mathcal{A}$

$$\max_k \{a_{ik} + b_{rk}\} \geqq 0 .$$

We want to construct a $B = (b_1, b_2, \ldots, b_n)$ such that for all $A_i \in \mathcal{A}$

(5.12) $$\max_k \{a_{ik} + b_k\} \geqq 0 ,$$

that is,

(5.12a) B blocks $\mathcal{A}$,

but for the given X in our hypothesis and B

(5.13) $$\max_k \{x_k + b_k\} < 0 .$$

We now define $B = (b_1, b_2, \ldots, b_n)$ by the rule:

$$(5.14) \qquad b_k = \left\{ \begin{array}{l} -\min_{i \in X_k} \{a_{ik}\} , \\ -\infty \quad \text{otherwise} \end{array} \right\}.$$

Claim: B (in (5.14) satisfies (5.12) - (5.13) (yielding a contradiction, since if (5.8) is true and B is at least a minimal blocking vector then (5.13) should be false.)

Proof of Claim: To show that B satisfies (5.12) - (5.12a), it is sufficient to point out that each $b_k = \left(-\min_{i \in X_k} \{a_{ik}\}\right)$ blocks a subset of the rows of $\mathcal{A}$, but the union of these b_k's block all rows of $\mathcal{A}$. Hence, B blocks $\mathcal{A}$.

Proof that B satisfies (5.13): From (5.10) and (5.14) we observe that for each k such that $b_k = -\min_{i \in X_k} \{a_{ik}\}$ that (at best)

$$x_k + b_k < 0 \ ;$$

otherwise,

$$x_k + b_k = -\infty \ .$$

Hence, it follows from the preceding that for our given X and B (in (5.14)

$$\max_k \{x_k + b_k\} < 0 \ .$$

Note that, if $\mathcal{A}$ is a tower, then $b(\mathcal{A})$ is a tower. Also, if $b(\mathcal{A})$ is a tower, so is $b(b(\mathcal{A}))$.

Theorem 5.2. Let $\mathcal{A}$ be a tower with m rows and n columns, then $b(b(\mathcal{A})) = \mathcal{A}$.

Proof: Suppose that $b(b(\mathcal{A})) = \mathcal{A}^* \neq \mathcal{A}$. Let $A^* = (a^*_1, a^*_2, \ldots, a^*_n)$ be a row of $\mathcal{A}^*$. Then for all $B_r \in b(\mathcal{A})$ and A^* ,

$$(5.17) \qquad \max_k \{a^*_k + b_{rk}\} \geqq 0 \ ,$$

since $\mathcal{A}^*$ is a minimal blocker of $b(\mathcal{A})$.

Again, let $A_i = (a_{i1}, a_{12}, \ldots, a_{in})$ be a row of $\mathcal{A}$. Then for all $B_r \in b(\mathcal{A})$ and $A_i \in \mathcal{A}$

(5.18) $$\max_k \{a_{ik} + b_{rk}\} \geqq 0 ,$$

since $\mathcal{A}$ is a blocker of $b(\mathcal{A})$. Note that both $\mathcal{A}$ and $\mathcal{A}^*$ block $b(\mathcal{A})$.

Now, if A^* is a fixed row of $\mathcal{A}^*$, then by (5.17) and Lemma 5.2

(5.19) $$A^* \geqq A_i$$

for some $A_i \in \mathcal{A}$. Also, if A_i in (5.19) is a row of $\mathcal{A}$, then by (5.18) and Lemma 5.2

(5.20) $$A_i \geqq A^{**}$$

for some $A^{**} \in \mathcal{A}^*$.

From (5.19) and (5.20), it follows that

(5.21) $$A^* \geqq A_i \geqq A^{**} , \ A^*; A^{**} \in \mathcal{A}^* ,$$

which implies that if $A^* > A_i > A^{**}$, then

$$A^* \text{ and } A^{**}$$

are comparable, (which is false - $\mathcal{A}^*$ is a tower). Therefore,

$$A^* = A_i = A^{**} .$$

Hence, it follows that if $A^* \in \mathcal{A}^*$, then $A^* \in \mathcal{A}$, that is

(5.22) $$\mathcal{A}^* \subseteq \mathcal{A} .$$

Again, suppose A_i is a fixed row of $\mathcal{A}$. Then by (5.18) and Lemma 5.2

(5.23) $$A_i \geqq A^* \text{ for some } A^* \in \mathcal{A}^* .$$

Also, if A^* in (5.23) is a row of $\mathcal{A}^*$, then by (5.17) and Lemma 5.2)

(5.24) $$A^* \geqq \bar{A}_i \text{ for some } \bar{A}_i \in \mathcal{A} .$$

From (5.23) and (5.24)

$$A_i \geqq A^* \geqq \bar{A}_i \ , \ A_i, \bar{A}_i \in \mathcal{A} ,$$

which implies that if $A_i > A^* > \bar{A}_i$, then

$$A_i \text{ and } \bar{A}_i$$

are comparable, (contradicting the fact that $\mathcal{A}$ is a tower). So, $A_i = A^* = \bar{A}_i$. Hence, from reasoning analogous to (5.22), we have

(5.25) $$\mathcal{A} \subseteq \mathcal{A}^* .$$

From (5.22) and (5.25), we obtain

$$\mathcal{A}^* = \mathcal{A} ,$$

that is,

(5.26) $$b(b(\mathcal{A})) = \mathcal{A} ,$$

completing the proof of Theorem 5.1.

Note that starting with $b(\mathcal{A}) = \mathcal{B}$, we can, similarly, prove Theorem 5.1, that is $b(b(\mathcal{B}) = \mathcal{B}$.

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