MATHEMATICAL AND PHYSICAL ANALYSIS OF PRICING MODELS FOR STRUCTURED FINANCIAL SECURITIES

by

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Abstract

Mathematical and Physical Analysis of Pricing Models for Structured Financial Securities

by

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In this thesis, we present an extension of the one-factor Gaussian copula model for pricing collateralized debt obligations (CDOs): Instead of using flat default correlation and rate parameters across the whole portfolio, we use individual correlation coefficients between each reference entity and the market (S&P 500 index) based on 5-year daily stock prices, and we use specific rate parameter for each entity by curve-fitting the default probability term structure. Spreads from this improved model are compared to those obtained from the one-factor Gaussian copula model with flat correlation. Results show that uniform correlation and rate parameters fail to capture that a few or even one single asset can substantially impact the credit quality of the whole portfolio. Heterogeneity of correlations and rate parameters of different reference entities is indispensable for constructing reliable and realistic models for pricing synthetic CDOs.

We also introduce analytical solutions to the pricing of both homogeneous and heterogeneous CDOs. We compare these analytical solutions with results obtained from simulation models. Results show very good consistency.

At the end, we introduce the analysis of another financial derivative - Securitized Life Settlements (SLSs) and present an analytical solution to the pricing of homogeneous SLSs.

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Preface

Traditionally, physics is a natural science studying matter and its motion and all other related concepts. If we take stocks or bonds or any other financial products as objects, principles and methods in physics can also be used to study their motions and other related concepts, only in a more complicated and less predictable way due to the involvement of human activities, just as Alexander Lipton-Lifschitz, the new york head of foreign exchange product development of Deutsche Bank and professor of the University of Illinois at Chicago said [1]:

> I certainly use my physics experience to the largest degree possible in the financial context. The problems I have to face are very physical in nature as well as financial. It's not quite so much transplanting physics ideas to finance as a general flow of ideas in both directions. There are many cases

> when there are direct analogues; for example, there is a concept called the Kelvin wave, which amazingly enough has a direct analogue in a financial model for the evaluation of interest derivatives and derivatives and assets with stochastic volatility, called the Affine model.

As more and more complicated financial derivatives created and more and more financial strategies depend on numerical algorithms, like the high-frequency strategies, Wall Street are in demand of talents who are experts in mathematics, physics and computer sciences. The term "quantitative analysts" or simply "quants" refers to this group of people.

In this dissertation, we will combine techniques and ideas in physics, mathematics and computer sciences to solve some financial problems.

Chapter 1

Introduction

Collateralized debt obligation (CDO) has played an important role in the recent financial crisis started in 2007. This financial crisis highlighted the need for better understanding and better valuation models for CDOs.

Simply speaking, a collateralized debt obligation is a pool of bonds or loans. Issuers of these bonds or loans are usually companies or corporations, even governments. We call them obligors, borrowers, or specifically in the CDO world, "reference entities" or simply "names". Investors bear credit risk of this pool of bonds or loans. Credit risk means failing to repay principal and interest in a timely manner, also referred to as "default". Therefore collateralized debt obligations, including credit default swaps (CDSs), the building blocks of CDOs, are also called credit derivatives because their values are derived from the credit risk on the underlying bonds or loans.

There have been two main kinds of pricing approaches for CDOs: "bottom-up" approach and "top-down" approach. In the "bottom-up" approach, CDO prices depend on the individual credit risk and correlation between default times of reference entities in a CDO portfolio. The "top-down" approach, on the other hand, starts from modeling the cumulative loss of the whole portfolio, then tries to derive individual reference entity dynamics. Typical "top-down" approach models are discussed by Errais [2], Arnsdorf [3], Cont [4], Giesecke [5], Longstaff [6], Bielecki [7], Halperin [8] and Giesecke [5], among others.

The central problem for any "Bottom-up" method is how to incorporate correlation between different reference entities into the pricing model. Andersen [9] mentioned three lines of thinking existing in dealing with co-dependence between reference entities: In one of them, default of each reference entity is modeled as a Poisson process with stochastic intensity and the intensity process of one reference entity may afferct the intensity of another. Papers following this approach include [10, 11, 12], and [13], among others.

Another line of "Bottom-up" methods is also called structural models, which involve a micro-economic representation of the capital structure of a firm, with default being triggered by company assets falling below some threshold representing a fraction of company debt. Papers following this approach include [14, 15, 16, 17], as well as [18].

The third line of "Bottom-up" methods involve the usage of the so-called copula function, which is first introduced to the credit field by Li [19]. A detailed discussion on copula follows in Section 3.2. Outside of the credit field, copula functions are common in actuarial science and are surveyed by Embrechts [20] and Schönbucher [21]. The Gaussian copula model soon became an industry standard and developed into one-factor or multi-factor Gausisan copula models, in which default loss distribution by a certain time is conditional on the factor values. By integrating over the factor values, one can get the unconditional loss distribution. Early papers on this development of copula model include: [22, 23] and [24]. Laurent and Gregory use fast Fourier transform techniques, while Andersen, Hull and Basu apply an iterative numerical procedure to avoid Monte Carlo simulation to build up the loss distribution of a tranche of an index or a CDO.

Other recent extensions of the factor copula model include: a stochastic correlation extension by Burtschell [25]; a random recovery and random factor extension by

Andersen [9]; a double-t extension by Hull [24], to name a few. For more references on this topic, please refer to [26].

Gibson [27] drew on the innovations in CDO pricing and presented a pricing model for CDO tranches that does not require Monte Carlo simulation. In that paper, Gibson also showed how CDO tranches are sensitive to the correlation of defaults among the reference entities in a CDO portfolio by changing the uniform correlation coefficient across reference entities from 0 to 0.9. However, assuming uniform correlation across the whole CDO portfolio may not be appropriate if tranche spreads show excessive sensitivity to correlation coefficient. As Whitehill [28] states:

> Specifying one number to quantify the default-time correlation across a pool of typically 100 or more unique companies is an unrealistic simplification.

Since the central problem of pricing synthetic CDOs is to calculate the loss distributions of a portfolio of reference entities over different time horizons, we must estimate the default probability of each reference entity. Gibson [27] assumes exponential distributions with uniform rate parameter for the default probabilities of all reference entities in a CDO portfolio. We will show that the exponential distribution assumption is well-founded, but assuming uniform default rate parameter across the whole

CDO portfolio is an unrealistic and inaccurate simplification because spreads of CDO tranches are extremely sensitive to the default probability of each single reference entity in the portfolio.

The rest of the dissertation is organized as follows: In chapter 2, We will review the structure and properties of collateralized debt obligations (CDO), including the building blocks of collateralized debt obligations - credit default swap (CDS). In chapter 3, we will talk about the pricing of homogeneous CDO tranches. First we discuss the attributes of homogeneous CDOs, where "homogeneous" means zero correlation and uniform default probability. Then we provide a detailed introduction to the concept of copula and the one-factor Gaussian copula model for CDOs. In this chapter, we also talk about the numerical techniques we use in our research. At the end of chapter 3, we will introduce an analytical solution to the pricing of homogeneous CDO tranches. In chapter 4, we will talk about the pricing of heterogeneous CDO tranches. Heterogeneous CDO means reference entities in a CDO portflio are not independent anymore, they are correlated to each other with different correlation coefficients and will default with different default probabilities. In chapter 5, we will introduce a tentative analytical solution to the pricing of heterogeneous CDOs. In the last chapter, we will talk about the pricing of another financial derivative similar to CDOs - securitized life settlements (SLS).

Following is a list of financial terms and their explanations in the environment of this disseration. We want to make sure terminology is not a barrier between physics or mathematics and fiance.

Tranche: an issue of bonds derived from a pooling of like obligations that is differentiated from otehr issues especially in risks and returns.

Reference Entities: also called “names” in the Collateralized Debt Obligations (CDOs) world, meaning the companies or entities which are included or refered in the CDO portfolio.

Rate Parameter: specifically mean the rate parameter in exponential distribution, which we assume some companies’ default probabilities follow.

There are some other terms which may also be new to physicists, such as CDO itself, CDS, Copula, One-Factor Gaussian Copula model, etc. We will define and explain these terms in detail in the corresponding sections.

Notional Amount: the nominal or face amount that is used to calculate payments made on that financial instrument, also referred to as notional principal amount, notional value or face value.

Viatical Settlement: an arrangement in which someone with a terminal disease sells his or her life insurance policy at a discount from its face value for ready cash. The buyer cashes in the full amount of the policy when the original owner dies. Also considered as a type of Life Settlement.

Chapter 2

Colaterallized Debt Obligations

2.1 Credit Default Swap

To talk about CDOs, we need to introduce credit default swaps (CDSs) first, since they are the building blocks of CDOs. A CDS, similar to an insurance policy, is a bilateral contract under which two parties (protection sellers and buyers) agree to trade the credit risk of a reference entity (usually a bond or a loan). Under such an agreement, the protection buyer pays a periodic premium to a protection seller in exchange for a contingent payment by the seller upon a credit event (such as a default or failure to pay) happening in the reference entity. When a credit event is

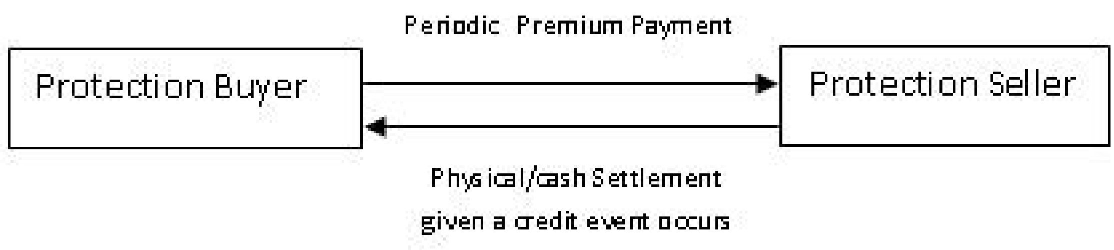


Figure 2.1: Cashflows of a CDS

triggered, the protection seller either takes delivery of the defaulted bond for the par value (physical settlement) or pays the protection buyer the difference between the par value and recovery value of the bond (cash settlement). Figure 2.1 illustrates the cash flows of a CDS. A CDS contract can be used as both a hedge strategy and a speculation tool. No matter what the purpose is, the market size for credit default swaps more than doubled in size each year from $3.7 trillion in 2003 [29]. By the end of 2007, the CDS market had a notional value of $62.2 trillion [29]. But notional amount fell during 2008 as a result of dealer "portfolio compression" efforts (replacing offsetting redundant contracts), and by the end of 2008 notional amount outstanding had fallen 38 percent to $38.6 trillion [30].

Risks involved in CDS contracts are mainly "counterparty risk". A classic example for this kind of risk is the Goldman Sachs-American International Group (AIG) scandal during the most recent financial crisis. Goldman Sachs bought protection on various toxic assets from AIG and then bought protection on AIG from a variety of highly rated banks. By doing this, a crisis at AIG became a crisis of all the banks Goldman Sachs had used to hedge its AIG exposure. In this situation, the government had to take steps to avoid a systemic risk threatening the entire financial market.

2.2 Colaterallized Debt Obligation

A colaterallized debt obligation is a type of structured asset-backed security. Underlying assets include bonds, loans, mortgages, etc. What makes a CDO special is its capital structure. There is a variety of ways to split a CDO into different tranches. Each tranche is consistent with a desired risk level. Tranches with higher risk have higher priority in getting paid and tranches with lower risk has lower priority in getting paid. In this sense, tranches are a reflection of risk allocation. Figure 2.2 illustrates different tranches in a typical CDO capital structure (with six tranches). From the bottom to the top, tranches are taking less and less risks and correspondingly, the returns are also becoming less and less. For example, the Equity tranche takes the most risks (the first 3% losses of the whole portfolio will be absorbed by this

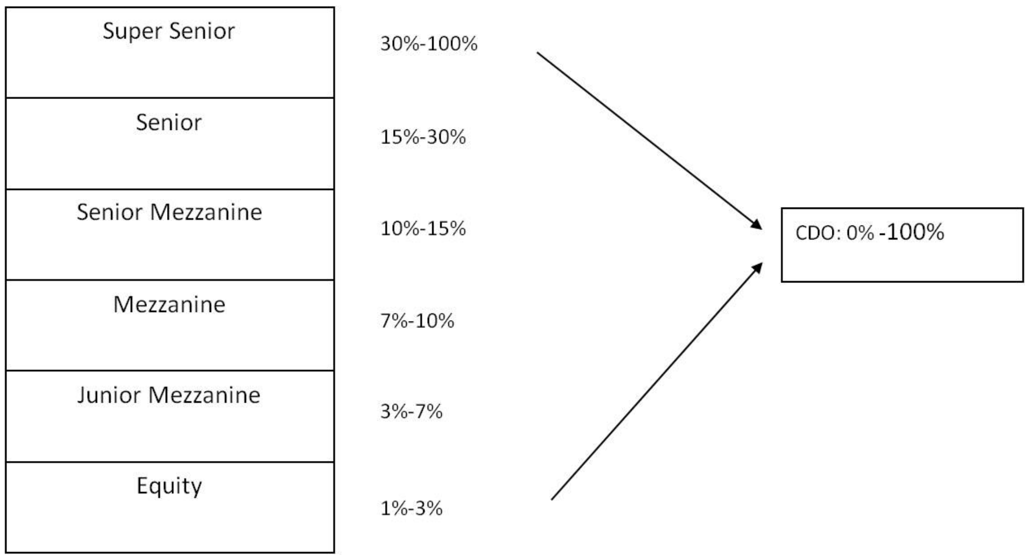


Figure 2.2: A CDO capital structure

tranche) and at the same time receives the highest return rate (say 5% or 500bps). In contrast, the junior Mezzanine tranche takes fewer risks and receives a lower return (say 107bps). The super senior tranche is even safer than a mezzanine tranche and the return rate is also lower (say 7bps). In other CDO capital structures, there may be different number of tranches and each tranche may absorb different percents of losses. As we will see in the following chapters, we will base our reserach and analysis on a simple 3-tranche CDO (junior tranche, mezzanine tranche and senior tranche).

There are two categories of CDOs: Synthetic CDOs and cash CDOs. A synthetic CDO is an investment where the underlying collateral is a portfolio of credits, such as

CDX index and iTraxx index. CDX indices contain North American and Emerging Market companies and are administered by CDS Index Company (CDSIndexCo) and marketed by Markit Group Limited, and iTraxx contain companies from the rest of the world and are managed by the International Index Company (IIC), also owned by Markit. As opposed to just a single bond or loan underlying a single CDS contract, there is a collection of bonds or loans issued by different reference entities underlying a synthetic CDO. Take Markit CDX North America Investment Grade (CDX NA IG) for example, there are usually 125 reference entities included in this index. It is essentially a collection of single-name credit default swaps (CDSs). In contrast, traditional cash CDO structures have funded physical portfolios of bonds or loans as the underlying collateral pool, while the CDS assets of a synthetic CDO are typically not funded.

CDOs were first introduced in 1987 by bankers at the now-defunct Drexel Burnham Lambert Inc. 10 years after their invention, CDOs emerged as the fastest growing sector of the asset backed securities market. High returns attracted a growing number of asset managers and investors from all kinds of companies, including investment banks, insurance companies, mutual fund companies, etc. This significant growth peaked in 2007, when global investors began to stop funding CDOs in the wake of the subprime mortgage crisis.

CDOs have played an extremely dangerous role in spreading risks and uncertainty about the value of the underlying assets more widely, expecially when regulations were disappointingly poor and information assymetry existed between investors and bankers. Even now, a better understanding of this kind of compliated financial derivatives is necessary.

Chapter 3

Valuation of Homogeneous CDOs

3.1 Uncorrelated Assets with Uniform Default Probability

If all reference entities in a CDO portfolio are uncorrelated with each other, the number of entities defaulting by time t_i will follow a binomial distribution. This may not true for a CDO portfolio with fixed reference entities because any default will reduce the total number of trials in a binomial distribution. However, for a CDX index, there is a re-examination (which is usually called "roll") every six months to replace some reference entities which are not existing any more or not in the risk category any more with other qualified ones so as to keep the total number of entities in the index unchanged (usually 125 entities for a CDX NA IG index). So using a binomial distribution to model the number of defaults by time t_i is valid for CDX index, which is our case. Let N be the total number of reference entities in the CDO portfolio, p be the uniform default probability of all entities by time t_i, k be the number of defaults by time t_i, we have the probability that k out of N entities default by time t_i defined as:

$$P(k,t) = \binom{N}{k} p^k (1-p)^{N-k} \tag{3.1}$$

3.2 Copula

A copula is a statistical measure that represents a multivariate uniform distribution, which examines the association or dependence between many variables. In other words, a copula links multiple one-dimensional probability distributions to a joint multivariate distribution using probability as a bridge. Although the statistical calculation of a copula was invented as early as 1957 [31], it was not applied to financial markets and finance until 2000 [19]. Researchers usually use Gaussian copulas because of its analytical tractability and the fact that only a small number of parameters are required to characterize the distribution. It was also the market standard in modeling portfolio credit risk. In order to illustrate this rather abstract concept, the following example by Hull [32] is helpful: suppose we have two triangularly distributed functions. Please refer to Figure 3.1 and Figure 3.2 for a data table and line plot of function V_1 and Figure 3.3 and Figure 3.4 for a data table and line plot of function V_2. Note that the line plots of V_1 and V_2 are not perfect triangles due to limited points used for plotting.

$$V_1(x) = \begin{cases} 10x & 0 < x \leq 0.2 \\ -2.5x + 2.5 & 0.2 < x \leq 1 \end{cases} \tag{3.2}$$

$$V_2(x) = \begin{cases} 4x & 0 < x \leq 0.5 \\ -4x + 4 & 0.5 < x \leq 1 \end{cases} \tag{3.3}$$

The table of the V_1 function shows variable x, the distribution density function $V_1(x)$, the cumulative area and the corresponding Z-score in a Gaussian distribution. The same for function V_2 is also shown in table of the V_2 function.

Assume medium correlation between V_1 and V_2, which is 0.5. Using John Hull's

x	V1(x)	Cumulative Area	Z1 Value
0	0	0	
0.1	1	0.05	-1.64
0.2	2	0.2	-0.84
0.3	1.75	0.3875	-0.29
0.4	1.5	0.55	0.13
0.5	1.25	0.6875	0.49
0.6	1	0.8	0.84
0.7	0.75	0.8875	1.21
0.8	0.5	0.95	1.64
0.9	0.25	0.9875	2.24
1	0	1	

Figure 3.1: Table of the V_1 function

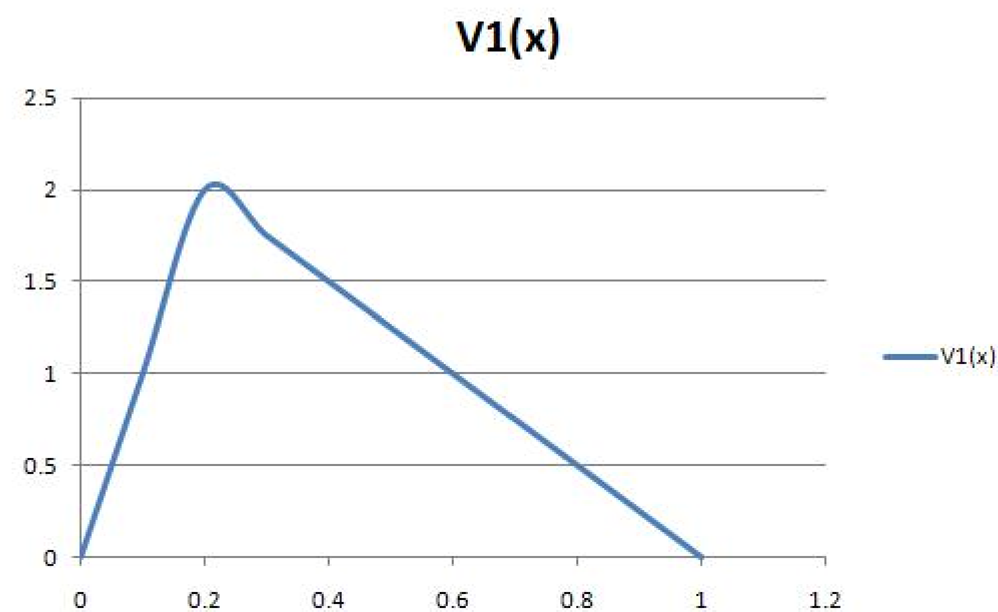


Figure 3.2: Distribution of the V_1 function

x	V2(x)	Cumulative Area	Z2 Value
0	0	0	
0.1	0.4	0.02	-2.05
0.2	0.8	0.08	-1.41
0.3	1.2	0.18	-0.92
0.4	1.6	0.32	-0.47
0.5	2	0.5	0
0.6	1.6	0.68	0.47
0.7	1.2	0.82	0.92
0.8	0.8	0.92	1.41
0.9	0.4	0.98	2.05
1	0	1	

Figure 3.3: Table of the V_2 function

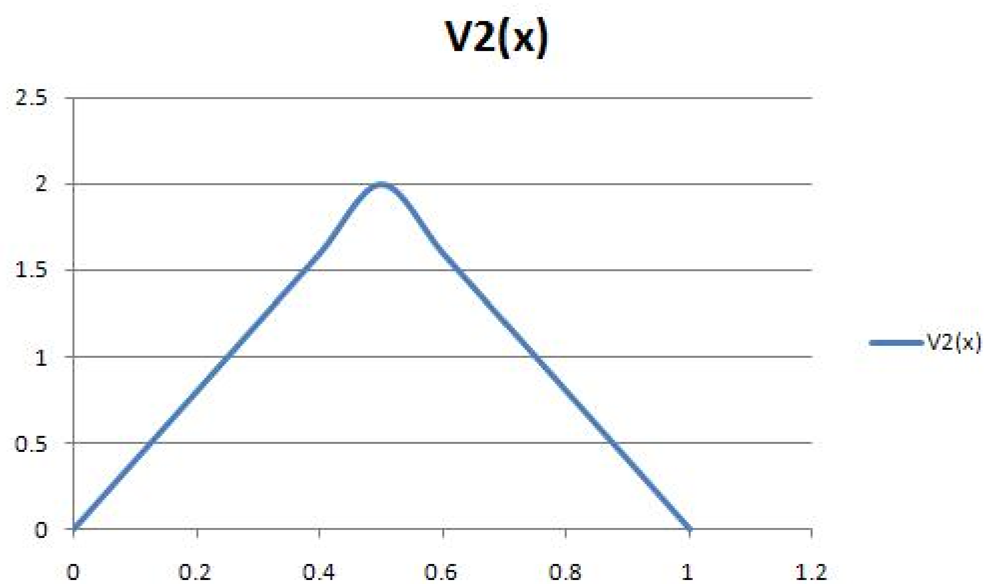


Figure 3.4: Distribution of the V_2 function

software for calculating Bivariate Gaussian Copula, we got a joint probability distribution as shown in the table of the joint distribution (3.5), where the first column and row represent the variables of V_1 and V_2 respectively; the second column and row represent $N^{-1}(V_1 \leq x)$ and $N^{-1}(V_2 \leq x)$ respectively; the matrix from the third row and third column gives the joint Gaussian distribution of V_1 and V_2, with the correlation considered.

Take 0.006, the value in the third row and third column for example, it is the joint cumulative normal probability of -1.64 and -2.05. It is calculated through the following bivariate cummulative probability formula:

$$\Phi(x,y) = \int_{-2.05}^{-\infty} \int_{-1.64}^{-\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)\right] dxdy \qquad (3.4)$$

where $x = N^{-1}(V_1)$ and $y = N^{-1}(V_2)$ and $\Phi(N^{-1}(V_1), N^{-1}(V_2))$ is a Gaussian copula.

3.3 One-Factor Gaussian Copula Model

Assume a reference entity i of a portfolio of N entities will default when its asset value is below a threshold, if the creditworthiness of each reference entity is denoted

x		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	V(1,2)	-2.05	-1.41	-0.92	-0.47	0	0.47	0.92	1.41	2.05
0.1	-1.64	0.006	0.017	0.028	0.037	0.044	0.048	0.049	0.05	0.05
0.2	-0.84	0.013	0.043	0.081	0.12	0.157	0.181	0.193	0.198	0.2
0.3	-0.29	0.017	0.061	0.124	0.197	0.274	0.331	0.364	0.381	0.387
0.4	0.13	0.019	0.071	0.149	0.248	0.358	0.449	0.505	0.535	0.548
0.5	0.49	0.019	0.076	0.164	0.281	0.417	0.537	0.616	0.663	0.683
0.6	0.84	0.02	0.078	0.173	0.301	0.456	0.6	0.701	0.763	0.793
0.7	1.21	0.02	0.079	0.177	0.312	0.481	0.642	0.76	0.837	0.877
0.8	1.64	0.02	0.08	0.179	0.318	0.494	0.667	0.798	0.887	0.936
0.9	2.24	0.02	0.08	0.18	0.32	0.499	0.678	0.816	0.913	0.97

Figure 3.5: Table of the ioint distribution

by X_i, where:

$$X_i = a_i M + \sqrt{1 - a_i^2} Z_i \tag{3.5}$$

Here X_i, M, and Z_i are mean-zero, unit-variance random variables with distribution functions F_i, G, and H_i, which are in the following assumed to be normal. Other distribution functions, however, can be easily incorporated in the model. By introducing independent random variables M and Z_i we characterize a reference entity's creditworthiness by two independent factors: the common (market) factor M and the individual factor Z_i. The total default probability by time t of entity i is $q_i(t)$. We will first fix the common factor M and let the individual factor vary to get conditional default probability $q_i(t|M)$. Later, we integrate over all possible common factors M

with appropriate weights given by probability density of the common factor's distribution. The correlation between a reference entity and the market is a_i and the correlation between two reference entities i and j is $a_i a_j$ accordingly, which means that the pair-wise correlation of two assets is fully determined by their individual correlation to the common factor M. This is the typical set-up for a one-factor model, which can be generalized to a model including several factors in a straightforward way, but at a cost of greater computational time as discussed by Andersen, Sidenius and Basu(2003) and Hull and White (2004).

Under a certain market condition (M fixed), each entity's independent default probability by time t given by $q_i(t|M) = Prob(X_i < \bar{x}_i\,|M\,)$ can be expressed as

$$q_i(t|M) = H_i\left(\frac{\bar{x}_i - a_i M}{\sqrt{1 - a_i^2}}\right) \tag{3.6}$$

As part of the copula concept, the default threshold of a given reference entity is the inverse Gaussian distribution function of its default probability, i.e. $\bar{x}_i = F_i^{-1}(q_i(t))$.

It is appropriate to assume that the default probability of each reference entity follows an exponential distribution:

$$q_i(t) = 1 - \exp(-\alpha_i t) \tag{3.7}$$

where α_i is the rate parameter of the exponential distribution.

In order to find the distribution of the number of defaults by time t under certain market condition, we can use a recursion method introduced by Andersen, Sidenius, and Basu [22].

Denote the probability of having one default by time t in a portfolio with N entities as:

$$p^N(l, t\,|M\,), \text{ where } l = 0, 1, 2 \ldots N$$

The basic idea of the recursion is to compute the default vectors for a portfolio with k+1 assets from the default vectors of the portfolio with k assets by adding the $(k+1)^{th}$ asset to the portfolio.

Starting with $p^0(0, t|M) = 1$ for $k = 0$, the default probabilities for a portfolio of $k + 1$ credits is:

$$p^{k+1}(0, t|M) = p^k(0, t|M)(1 - q_{k+1}(t|M)) \tag{3.8}$$

for $l = 1, ..., k$,

$$p^{k+1}(l,t|M) = p^k(l,t|M)(1 - q_{k+1}(t|M)) + p^k(l-1,t|M)q_{k+1}(t|M) \tag{3.9}$$

then

$$p^{k+1}(k+1,t|M) = p^k(k,t|M)q_{k+1}(t|M) \tag{3.10}$$

With the conditional probabilities at hand, we can easily find the unconditional default probability by integrating the common factor M out, i.e.

$$p(l,t) = \int_{-\infty}^{\infty} p^N(l,t\,|M\,)g(M)dM \tag{3.11}$$

where g(M) is the probability density function of M.

With the default probability of number of defaults in the CDO portfolio, the spread of a CDO tranche is found by equating the premium leg and the contingent leg. The expected loss (EL) can be calculated using the following formula:

$$EL_i = \sum_{i=0}^{N} p(l,t_i) \max(\min(lA(1-R), H) - L, 0) \tag{3.12}$$

assuming defaults only occur at the payment dates. Discounting the possible losses from every payment date and the total contingent loss is:

$$Contingent = \sum_{i=1}^{N} D_i(EL_i - EL_{i-1}) \tag{3.13}$$

here D_i is the risk-free discount factor at time i.

The expected present value of the payment leg is much more straightforward as shown below:

$$Annuity = s\sum_{i=1}^{N} D_i\Delta_i[(H - L) - EL_i] \tag{3.14}$$

where Δ_i is the accrual factor for payment date $i(\Delta_i \approx T_i - T_{i-1})$ and s is the spread per annum paid to the tranche buyer.

Assuming no arbitrage, these two legs should be equal. By equating 3.13 and 3.14, we find the par spread of the chosen tranche of the CDO portfolio:

$$S_{par} = \frac{contingent}{\sum_{i=1}^{N} D_i\Delta_i[(H - L) - EL_i]} \tag{3.15}$$

Obviously, this spread is heavily dependent on the underlying probability distributions characterizing the possibility of defaults which themselves depend on the rate parameter and correlation parameter. It is tempting, for a portfolio of many assets, to simply use average values for both the correlation and the rate parameters. We will show later that this averaged approach is likely to produce incorrect prices and therefore, careful modeling of the underlying probability distributions is essential for the computation of the spread of CDO tranches. For this purpose, in Chapter 4, we will first estimate individual correlation and rate parameters of default probability distibutions from actual market data and then compare the pricing of tranches to the pricing obtained using averaged parameters.

3.4 Numerical Techniques

A wide range of numerical techniques have been used in our research: Some are used to speed up computations; some are used to increase accuracy or to substitute other more complicated approaches. Below we will take several topics as examples.

3.4.1 Numerical Integration

Numerical integration is using numerical techniques to compute definite integrals, which are difficult or impossible to be computed analytically. The basic idea is to evaluate the integrand at a finite set of points and then use a weighted sum of these values to approximate the integral. The number of integration points and weights depend on the specific method used and the accuracy required from the approximation. The simplest method is using a rectangle to approximate the area under the curve of the integrand function $f(x)$. In this case, the interpolating function is a constant function. For an definite integral $f(x)$ within limits a to b, the constant interpolating function is just $f(\frac{a+b}{2})$. This is called midpoint rule or rectangle rule. Two most frequently used integration methods for continuous integrands include Trapezoidal rule and Simpson's rule. Both evaluate the integrand with small increments, but they have different interpolating functions.

Trapezoidal Rule

As the title explains itself, the trapezoidal rule approximates the region under the curve of function $f(x)$ as a trapezoid and calculates its area as follows:

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2} \tag{3.16}$$

To calculate an integral accurately, we split the interation interval $[a, b]$ into N uniform subintervals and apply the trapezoidal rule on each of them. Then the composite

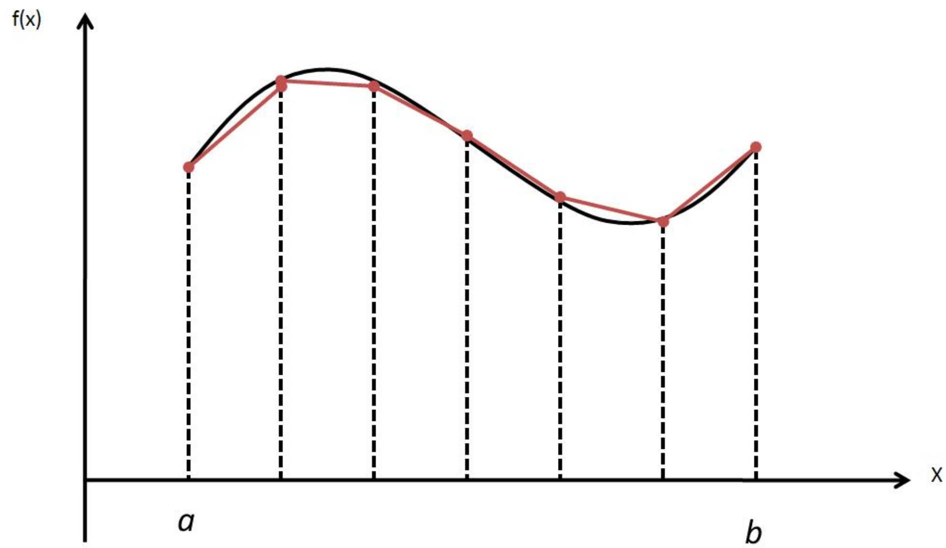


Figure 3.6: Illustration of the composite Trapezoidal rule.

trapezoidal rule is as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{N}\left[\frac{f(a)+f(b)}{2} + \sum_{k=1}^{N-1} f(a+k\frac{b-a}{N})\right] \tag{3.17}$$

Simpson's Rule

Different from the Trapezoidal rule using an affine function (a polynomial of degree 1) as the interpolating function, Simpson's rule use a Lagrange polynomial as the

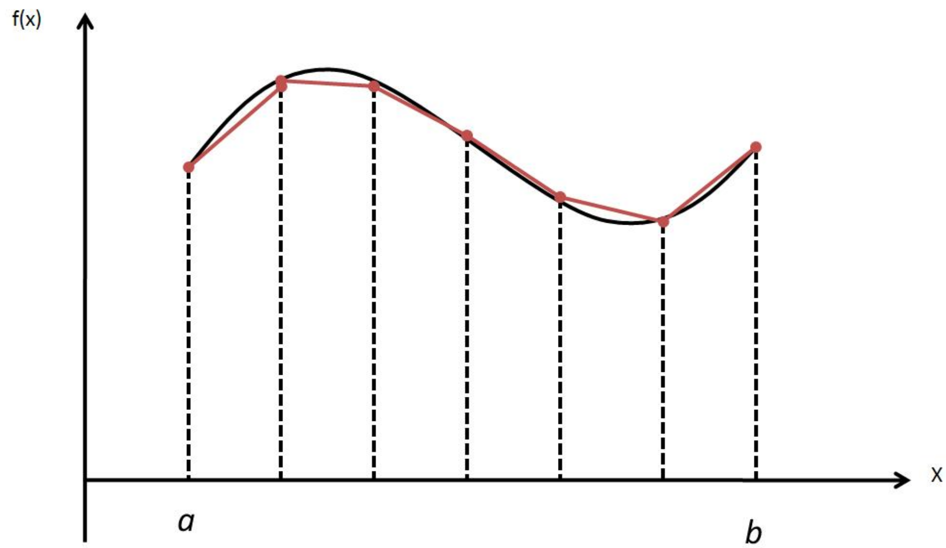


Figure 3.7: Illustration of the Simpson's rule.

interpolating funciton. Let m denote the middle point of the integration intervel $[a, b]$, the Lagrange polynomial is just as follows:

$$P(x) = f(a)\frac{(x-m)(x-b)}{(a-m)(a-b)} + f(m)\frac{(x-a)(x-b)}{(m-a)(m-b)} + f(b)\frac{(x-a)(x-m)}{(b-a)(b-m)} \quad (3.18)$$

As illustrated in the following graph, Simpson's rule uses the Lagrange polynomial to approximate the integrand function $f(x)$ to get a more accurate computation of the area under the curve than the Trapezoidal rule. Integration of the Lagrange polynomial is as follows:

$$\int_a^b P(x)dx = \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right] \tag{3.19}$$

For even more accurate computation using Simpson's rule, we break up the integration interval $[a, b]$ into N small subintervals, just as we did in the composite Trapezoidal rule. And then apply Simpon's rule to each of the subinterval, with the results being summed to get an approximation of the integral over the whole interval $[a, b]$. This is called composite Simpson's rule and is expressed as follows:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{\frac{N}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{N}{2}} f(x_{2j-1}) + f(x_N)\right] \tag{3.20}$$

where $h = \frac{b-a}{N}$ is the "step length".

In the last chapter of this part, we are going to use both the Trapezoidal rule and Simpson's rule for integration and compare the results from these two different integration approaches.

3.4.2 Recursion Algorithm

As the name suggests, recursion defines a problem in terms of itself. In programming language, a recursion function can call itself during its own execution. Parts of a simple recursive algorithm include one or more base cases and one or more recursive cases.

One classical example is to compute a factorial ($n!$) using recursion. Here, the base case is $0! = 1$ and $n!$ for $n \geq 1$ are the recursive cases. The following short program shows how to compute a factorial using recursion in Matlab:

```
function y = fact(n)

y = n

if n == 0

    y = 1

else

    y = y * fact(n-1)

end
```

As we can see from the codes, once the base case $0! = 1$ is defined, function $y = fact(n)$ can call itself during its execution.

3.4.3 Parallelization

When programming in Matlab, we used different techniques to optimize our codes to speed up computations. One approach is to vectorize loops, which is a branch of parallel computing. We use a simple example to introduce the benefits of using vectors instead of loops:

We want to calculate the sine of 1001 values ranging from 0 to 10 with increments of 0.01, that is sin(0), sin(0.01), sin(0.02), sin(0.03), etc. We can either use a loop or use vector operations instead.

Using Loop:

```
i = 0;
for t = 0:.01:10
    i = i+1;
    y(i) = sin(t);
end
```

A vectorized version of the same code is:

```
t = 0:.01:10;
y = sin(t);
```

Time required to finish computing using the first program is 0.08 seconds. In contrast, time required to finish computing using the vectorized alternative is only 0.0025 seconds, which is 32 times faster than the loop operations.

Another approach we tried to speed up computations is parallel computing using a computer cluster. A computer cluster, sometimes called as a super computer, is a group of linked computers, working together closely thus in many respects forming a single computer. The components of a cluster are commonly, but not always, connected to each other through fast local area networks as proposed by [33]. We used Athena, the newly established super computer system at the College of Staten Island, The City University of New York. This computer cluster is capable of completing research projects 10 times faster than it used to take a single personal computer. Computations in our research is not very demanding, but the advantages of using Athena is still notable.

3.5 Analytical Solution

In an uncorrelated homogeneous CDO portfolio, default probabilities of different reference entities are independent. Assuming exponential distribution, the uniform default probablity across the whole portfolio is just:

$$q(t) = 1 - \exp(-\alpha t) \tag{3.21}$$

When reference entities in a homogeneous CDO portfolio are not independent of each other, factor models (Equation 3.5) are used to reflect the correlation among them. For a given market condition, the correlated default probability is:

$$p(M,t) = \phi\left(\frac{C - aM}{\sqrt{1-a^2}}\right) \tag{3.22}$$

where $C = \phi^{-1}(q(t))$ and $q(t) = 1 - e^{-\alpha t}$. Different reference entities are correlated through M, the common factor or market factor. Conditional on M, under a certain market condition, the probability of k reference entities default by time t follows a binomial distribution:

$$P(l = k|M) = \begin{pmatrix} N \\ k \end{pmatrix} p(M,t)^k (1 - p(M,t))^{N-k} \tag{3.23}$$

3.5.1 Large Portfolio Approximation

As the size of the CDO portfolio getting larger, $N \to \infty$, cumulative binomial distribution (Equation 3.23) becomes approximately normal distribution with mean $Np(M,t)$ and variance $Np(M,t)(1-p(M,t))$:

$$F(x|M) = \sum_{k=0}^{Nx} P(l = k|M) = \phi\left(\frac{Nx - Np(M,t)}{\sqrt{Np(M,t)(1-p(M,t))}}\right) \tag{3.24}$$

where $x = \frac{k}{N}$ is the percent loss of the whole portfolio. As $N \to \infty$, $x \to 0$. Also

$$\phi\left(\frac{Nx - Np(M,t)}{\sqrt{Np(M,t)(1-p(M,t))}}\right) = \phi\left(\frac{x - p(M,t)}{\sqrt{p(M,t)(1-p(M,t))/N}}\right) \tag{3.25}$$

Figure 3.8 compares the default probability distribution assuming binomial distribution and large portfolio approximation using heaviside function for a portfolio with 2000 assets. It verified that as the size of the CDO portfolio getting larger, binomial distribution becomes approximately normal distribution, and further becomes approximately Dirac delta function, which is the derivative of heaviside function.

As N goes to infinity, $\sqrt{p(M,t)(1-p(M,t))/N}$ goes to zero, and cumulative normal distribution can be approximated as a Heaviside step function as discussed in

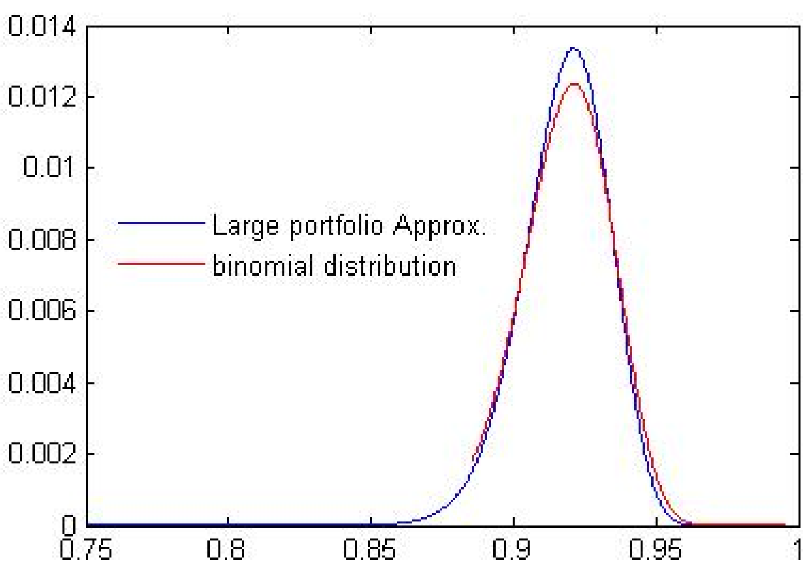


Figure 3.8: Large portfolio approximation using heaviside function

[34, 35, 36, 37, 38], and [39]:

$$H(x - p(M,t)) = \begin{cases} 1 & \text{for} \quad x \geq p(M,t) \\ \\ 0 & \text{for} \quad x < p(M,t) \end{cases} \tag{3.26}$$

3.5.2 Analytical Solution to Homogeneous CDO Pricing

Equation (3.15) shows a general analytical solution to the spread of a tranche with attachment point L and detachment point H. To be consistent with the previous section of Large Portfolio Approximation, We convert dollar loss to percent loss x and the solution to spread of a CDO tranche becomes:

$$s = \frac{\sum_{i=1}^{n} (EL_{(L,H)}(t_i) - EL_{(L,H)}(t_{i-1})B(t_0, t_i)}{\sum_{i=1}^{n} \Delta_i (1 - EL_{(L,H)}(t_i))B(t_0, t_i)} \tag{3.27}$$

Assume a continuous portfolio loss distribution function $F(t, x)$ is known, which is:

$$\begin{aligned} EL_{(L,H)}(t) &= \frac{1}{H-L} \int_L^1 (min(x, H) - L) dF(t, x) \\ &= \frac{1}{H-L} (\int_L^1 (x - L) dF(t, x) - \int_H^1 (x - H) dF(t, x)) \end{aligned} \tag{3.28}$$

so the premium leg is:

$$\int_0^t s(1 - E(t')) \exp(-rt')dt' \tag{3.29}$$

and the protection leg is:

$$\int_0^t \frac{dE(t')}{dt'} \exp(-rt')dt' = \int_0^t \exp(-rt')dE(t') \tag{3.30}$$

With large portfolio approximation,

$$F(t, x) = H(x - P(M, t)) \tag{3.31}$$

where P(M,t) is the uniform default probability of a reference entity in a homogeneous CDO portfolio and $H(x - p(M, t))$ is the Heaviside step function.

The Simplest Case: Single Tranche CDO

In the simplest scenario, assume the whole portfolio as one big tranche, which implies L=0 and H=1. So:

$$EL_{(L,H)}(t) = \int_0^1 x dF(t,x) - \int_1^1 (x-1) dF(t,x) = \int_0^1 x\delta(t, x - p(M,t)) dx = p(M,t) \tag{3.32}$$

Further assume there is no correlation among different reference entities (a=0) and no interest rate (r=0), then $EL(t) = q(t) = 1 - \exp(-\alpha t)$ and the premium leg is:

$$s \int_0^t \exp(-\alpha t') dt' \tag{3.33}$$

and the protection leg is:

$$\int_0^t \alpha \exp(-\alpha t') dt' \tag{3.34}$$

so that the spread in this simplest special case (L=0, H=1, a=0 and r=0) is:

$$s = \alpha \tag{3.35}$$

This result can be understood as that in the simplest case, when the whole CDO tranche is a single big tranche and there is neither correlation nor interest rate, the spread or the price of a CDO in percentage is just the rate parameter of the default probability distribution, which is assumed as an exponential distribution.

A More General Case

Now we relax some of the assumptions and consider a more general case, in which the percent attachment point L and detachment point H can be any non-zero values in between 0 and 1 and interest rate is non-zero ($r \neq 0$), but we still assume the correlation coefficient a=0. In this case, $P(M, t) = q(t)$ still holds due to the non-correlation assumption. Using the Heaviside step function approximation, Equation 3.28 changes to:

$$EL_{(L,H)}(t) = \begin{cases} 0 & 0 \leq x < t_1 \\ \frac{q(t)-L}{H-L} & t_1 \leq t \leq T \end{cases} \tag{3.36}$$

where T is the maturity of the CDO portfolio and assume the total loss will keep not exceeding H until maturity. Since L and H here are attachment point and dettachment point in percentages instead of dollar values, they are corresponding to percent losses of the whole CDO portfolio. A CDO tranche between [L,H] will not be affected until the total loss, and so the uniform default probability $q(t)$ for a homogeneous portfolio, is greater than L. Let t_1 be the time the total loss reaches L, and t_1 can be decided by the following equation:

$$q(t_1) = 1 - \exp(-\alpha t_1) = L \tag{3.37}$$

so that

$$t_1 = -\frac{\ln(1-L)}{\alpha} \tag{3.38}$$

Just because expected loss EL(t) is not continuous on time interval [0,T], we should treat the integral of EL(t) carefully. Firstly, the premium leg now is:

$$s\int_0^T (1 - EL(t'))\exp(-rt')dt' = s\left(\int_0^T \exp(-rt')dt' - \int_0^T EL(t')\exp(-rt')dt'\right) \tag{3.39}$$

and the protection leg is:

$$\int_0^T \frac{dEL(t')}{dt'}\exp(-rt')dt' = \int_0^T \exp(-rt')dEL(t') = EL(T)\exp(-rT) + r\int_0^T EL(t')\exp(-rt')dt'$$

Let $I = \int_0^T EL(t') \exp(-rt') dt'$,then:

$$I = \int_0^T EL(t') \exp(-rt') dt' = \int_0^{t_1} EL(t') \exp(-rt') dt' + \int_{t_1}^T EL(t') \exp(-rt') dt' \tag{3.40}$$

where EL(t) is given by equation 3.36 and $q(t) = 1 - \exp(-\alpha t)$, so:

$$\begin{aligned} I &= \int_{t_1}^T \frac{1 - \exp(-\alpha t) - L}{H - L} \exp(-rt) dt \\ &= \frac{1 - L}{r(H - L)} (\exp(-rt_1) - \exp(-rT)) \\ &\quad - \frac{1}{(H - L)(r + \alpha)} (\exp(-(r + \alpha)t_1) - \exp(-(r + \alpha)T)) \end{aligned} \tag{3.41}$$

Equate the premium leg and protection leg, we have:

$$s \frac{1 - \exp(-rT)}{r} - I = EL(T) \exp(-rT) + rI \tag{3.42}$$

So the analytical solution of the spread of a homogeneous CDO tranche between $[L, H]$ with no correlation is given by:

$$s = \frac{EL(T)\exp(-rT) + rI}{\frac{1-\exp(-rT)}{r} - I} \tag{3.43}$$

We compared the spreads calculated using this analytical solution with those calculated using the one-factor Gaussian copula model for $r = 0.3$, $\alpha = 0.2$, $L = 0.2$ and $H = 0.7$. As shown in Figure 3.9, this analytical solution matches well with the numerical computation. The small discrepancy in the starting points on the time line is due to the difference in steepness between the heaviside function and normal distribution.

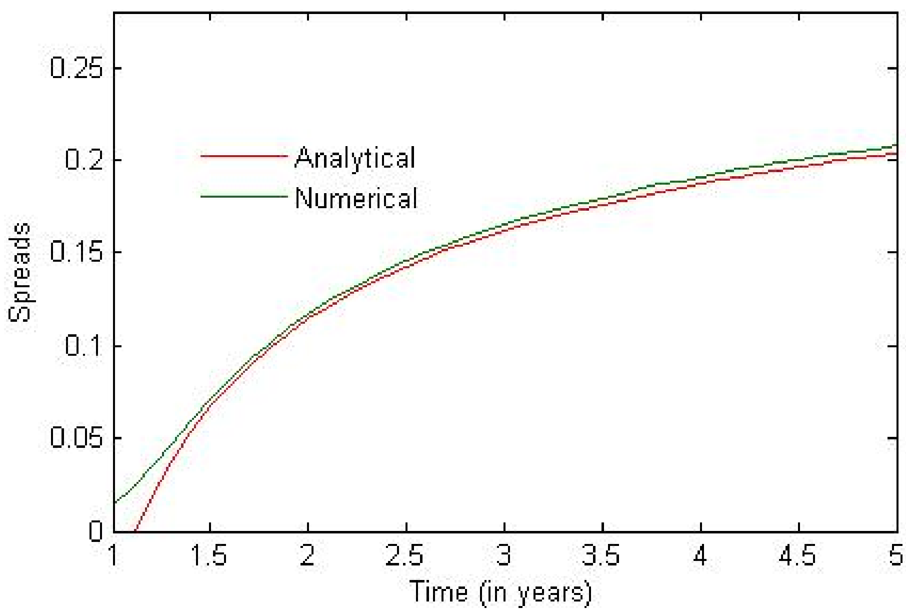


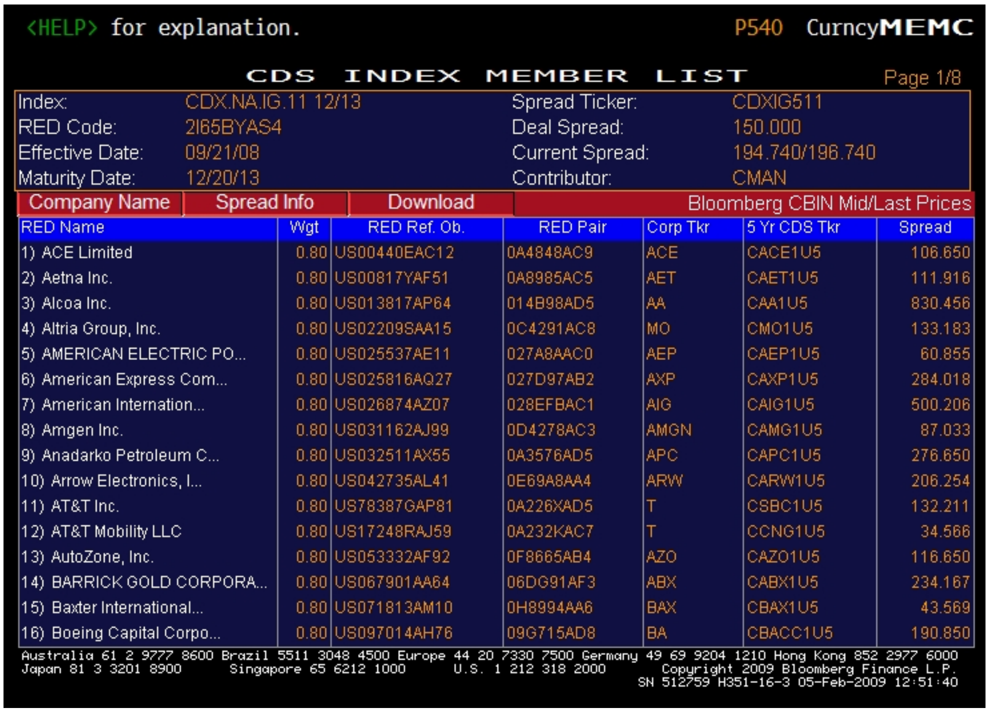
Figure 3.9: Spreads calculated using the analytical solution and simulated from the one-factor Gaussian copula model.

Chapter 4

Valuation of Heterogeneous CDOs

4.1 Market Data

In contrast to homogeneous CDOs, the term "heterogeneous" for this chapter means that the correlation coefficients (a_i) and rate parameters(α_i) of the default probability distributions of different reference entities in the CDX index are not uniform anymore. Each reference entity has a unique correlation coefficient with the market (S&P500) and a unique rate parameter. In this section, we will talk about how to get individual correlation coefficients and rate parameters for all reference entities in a CDO portfolio. Here we still use the CDX.NA IG.11 index as an example.



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CDS INDEX MEMBER LIST Page 1/8

Index:	CDX.NA.IG.11 12/13	Spread Ticker:	CDXIG511
RED Code:	2I65BYAS4	Deal Spread:	150.000
Effective Date:	09/21/08	Current Spread:	194.740/196.740
Maturity Date:	12/20/13	Contributor:	CMAN

Company Name | Spread Info | Download | Bloomberg CBIN Mid/Last Prices

RED Name	Wgt	RED Ref. Ob.	RED Pair	Corp Tkr	5 Yr CDS Tkr	Spread
1) ACE Limited	0.80	US00440EAC12	0A4848AC9	ACE	CACE1U5	106.650
2) Aetna Inc.	0.80	US00817YAF51	0A8985AC5	AET	CAET1U5	111.916
3) Alcoa Inc.	0.80	US013817AP64	014B98AD5	AA	CAA1U5	830.456
4) Altria Group, Inc.	0.80	US02209SAA15	0C4291AC8	MO	CMO1U5	133.183
5) AMERICAN ELECTRIC PO...	0.80	US025537AE11	027A8AAC0	AEP	CAEP1U5	60.855
6) American Express Com...	0.80	US025816AQ27	027D97AB2	AXP	CAXP1U5	284.018
7) American Internation...	0.80	US026874AZ07	028EFBAC1	AIG	CAIG1U5	500.206
8) Amgen Inc.	0.80	US031162AJ99	0D4278AC3	AMGN	CAMG1U5	87.033
9) Anadarko Petroleum C...	0.80	US032511AX55	0A3576AD5	APC	CAPC1U5	276.650
10) Arrow Electronics, I...	0.80	US042735AL41	0E69A8AA4	ARW	CARW1U5	206.254
11) AT&T Inc.	0.80	US78387GAP81	0A226XAD5	T	CSBC1U5	132.211
12) AT&T Mobility LLC	0.80	US17248RAJ59	0A232KAC7	T	CCNG1U5	34.566
13) AutoZone, Inc.	0.80	US053332AF92	0F8665AB4	AZO	CAZO1U5	116.650
14) BARRICK GOLD CORPORA...	0.80	US067901AA64	06DG91AF3	ABX	CABX1U5	234.167
15) Baxter International...	0.80	US071813AM10	0H8994AA6	BAX	CBAX1U5	43.569
16) Boeing Capital Corpo...	0.80	US097014AH76	09G715AD8	BA	CBACC1U5	190.850

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2009 Bloomberg Finance L.P.
SN 512759 H351-16-3 05-Feb-2009 12:51:40

Figure 4.1: Bloomberg screen shot of reference entities in CDX.NA.IG.11 index

4.1.1 Data for Default Probability Rate Parameter

We started our research by collecting data from BLOOMBERG PROFESSIONAL, one of the most powerful financial data service terminals, for the credit default swap index CDX.NA.IG.11 (as shown in Figure 4.1), which took effect on 09/21/2008 and matures on 12/20/2013. There are totally 125 reference entities in this investment grade CDS index. We will use this “roll” of CDS index as actual market data through our following research. In order to obtain the default probability rate parameter for

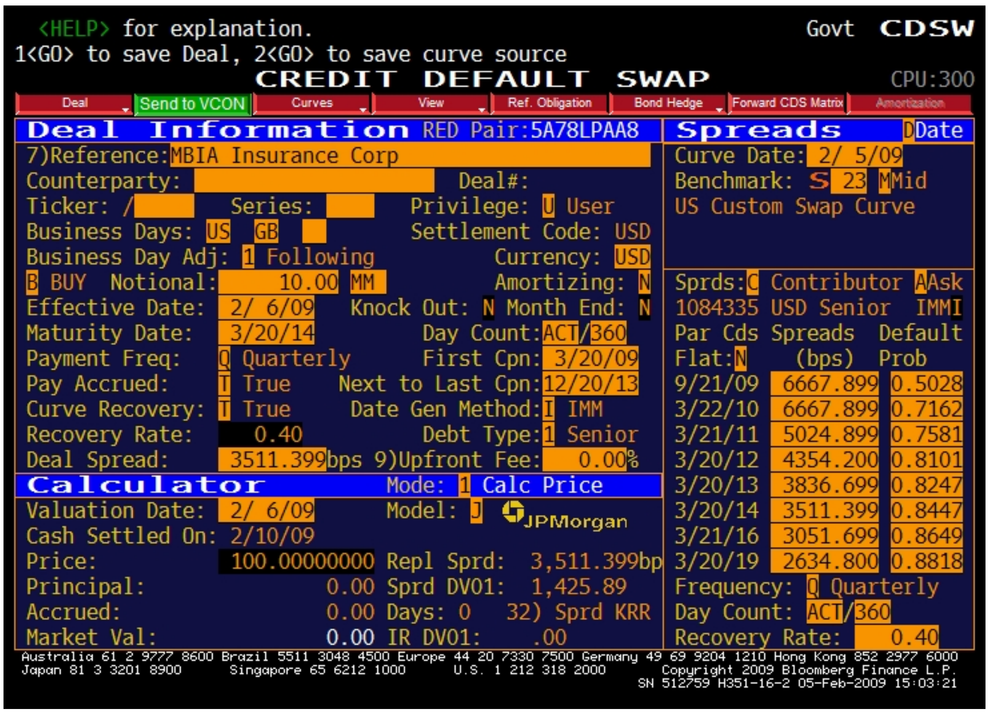


Figure 4.2: Bloomberg screen shot of CDSW page of a reference entity of the CDX.NA.IG.11 index

each reference entity, we use the $\langle CDSW \rangle$ function in BLOOMBERG PROFESSIONAL and do curve fittings for all 125 reference entities to get the desired rate parameters.

Take one reference as example (as shown in Figure 4.2): There are three parts in the $\langle CDSW \rangle$ function in BLOOMBERG PROFESSIONAL: Deal Information, Calculator and Spreads. We use the spread term structure on the right hand side and do a curve fitting to exponential functions in Matlab to find the desired rate parameter.

We collected these data in February 2009. At that time, the market, especially the financial sector, was still in a highly volatile situation. As shown in 4.3, most entities in this 125 index fit well to an exponential fuction:

$$q_i(t) = 1 - exp(-\alpha_i t) \tag{4.1}$$

where α is the rate parameter of the exponential distribution. We simply use "rate parameter" to refer to it throughout the whole dissertation.

Only two reference entities, iStar Financial Inc. and MBIA Insurance Corp., both in the financial sector, seriously deviate from exponential curves. For them, it was necessary to use a more general model to fit them. We decided to use the following:

$$q_i(t) = -a_i \exp(-b_i t) + (a_i - 1)\exp(-c_i t) + 1 \tag{4.2}$$

As shown in Figure 4.4, these two reference entities which did not fit well to a regular exponential do fit well to this more general exponential model. We will treat these two entities in our research in a special way and compare the effect of the

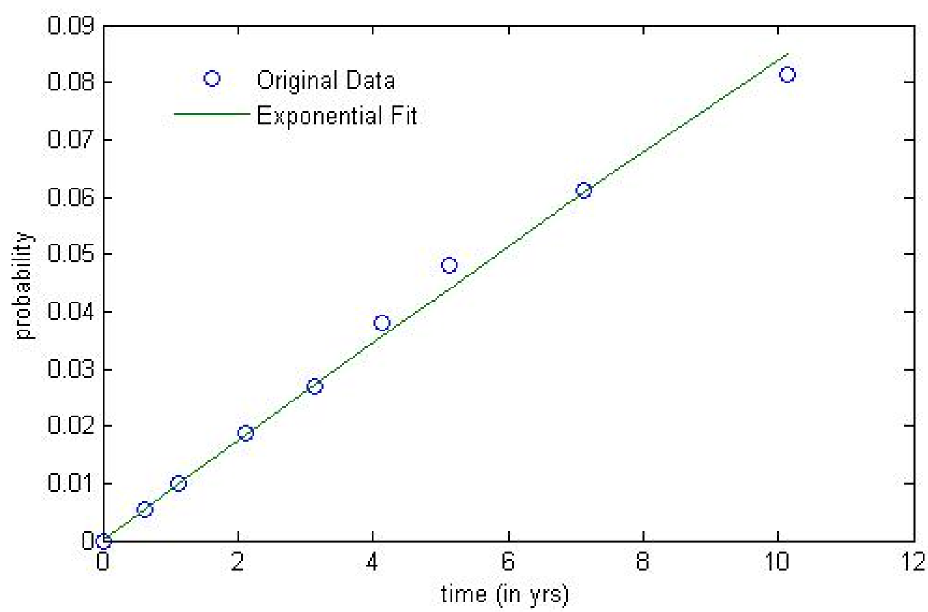


Figure 4.3: Good fitting to exponential curve of a randomly selected entity

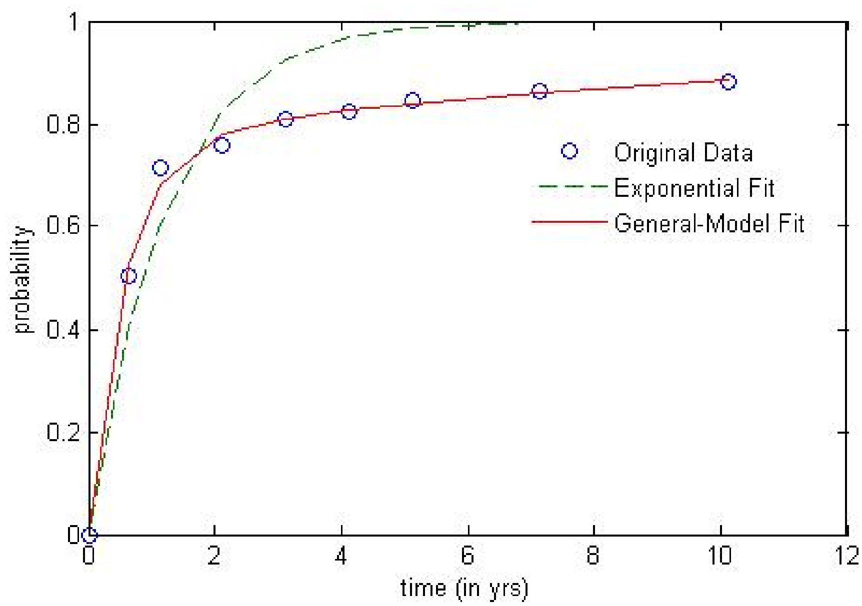


Figure 4.4: Bad fitting to exponential curve of a randomly selected entity

existence of this kind of "odd" names in a synthetic CDO portfolio.

4.1.2 Data for Correlation Parameter

To study correlations between different entities and the financial market, we downloaded actual stock prices of all 125 reference entities and prices of the $S\&P500$ index from Yahoo! Finance for a 5-year period from 2/9/2004 to 2/6/2009, put them together and acquired the correlation matrix using MicroSoft Excel (as shown in Figure 4.5).

4.2 Default Probability Parameter

As mentioned in the section 4.1.1, default probability term structures of most (123 out of 125) reference entities fit well to exponential distributions except two reference entities which were experiencing high volatility in February 2009 when the data from BLOOMBERG PROFESSIONAL was collected. Average rate parameter of the 123 reference entities was 0.04148, with a standard deviation of 0.03758, while the rate parameters of the two singular reference entities were 0.7939 and 0.8303, respectively, when curve-fitting to exponential distributions. The huge differences between the rate

	ACE	AET	Alcoa I	Altria	Ameri	Ameri	Ameri	Amge	Anadc	Arrow	AT&T	AT&T	Autoz	Barric	Baxte	Boein	Bristo	Burlin	Campl	Capita	Cardin
ACE	1.00																				
AET	0.63	1.00																			
Alcoa Inc	0.60	0.88	1.00																		
Altria Grou	0.58	0.89	0.79	1.00																	
American	0.74	0.91	0.89	0.84	1.00																
American	0.75	0.88	0.84	0.89	0.94	1.00															
American	0.65	0.84	0.83	0.78	0.91	0.90	1.00														
Amgen Inc	-0.49	-0.46	-0.47	-0.42	-0.57	-0.57	-0.73	1.00													
Anadarko	0.73	0.85	0.87	0.78	0.94	0.91	0.86	-0.60	1.00												
Arrow Ele	0.56	0.93	0.86	0.88	0.88	0.87	0.80	-0.33	0.84	1.00											
AT&T Inc	0.04	0.13	0.25	-0.06	0.15	-0.07	0.07	0.18	0.14	0.10	1.00										
AT&T Mob	0.79	0.78	0.80	0.70	0.91	0.89	0.93	-0.72	0.89	0.72	0.12	1.00									
Autozone	0.12	0.24	0.24	0.03	0.18	0.02	-0.05	0.43	0.11	0.25	0.67	0.05	1.00								
Barrick Go	0.36	0.67	0.64	0.56	0.71	0.60	0.72	-0.55	0.71	0.69	0.12	0.62	0.12	1.00							
Baxter Int	0.20	0.55	0.46	0.63	0.47	0.50	0.23	0.28	0.40	0.64	0.04	0.18	0.37	0.15	1.00						
Boeing Ca	0.76	0.90	0.86	0.86	0.96	0.97	0.94	-0.65	0.92	0.84	0.04	0.93	0.05	0.66	0.40	1.00					
Bristol-My	0.02	0.03	0.07	-0.22	0.09	-0.11	0.00	0.22	0.01	0.01	0.67	0.10	0.74	0.22	-0.06	-0.02	1.00				
Burlington	0.71	0.80	0.75	0.81	0.84	0.90	0.70	-0.31	0.84	0.81	-0.01	0.76	0.16	0.38	0.65	0.84	-0.14	1.00			
Campbell	0.12	0.19	0.09	0.44	0.14	0.29	-0.01	0.29	0.06	0.25	-0.33	-0.04	-0.07	-0.30	0.74	0.16	-0.40	0.45	1.00		
Capital On	0.76	0.72	0.63	0.77	0.75	0.89	0.71	-0.42	0.68	0.72	-0.28	0.74	-0.05	0.35	0.44	0.81	-0.22	0.81	0.43	1.00	
Cardinal H	0.62	0.95	0.86	0.93	0.91	0.92	0.81	-0.37	0.86	0.95	0.04	0.75	0.20	0.63	0.69	0.90	-0.04	0.87	0.36	0.77	1.00
Carnival C	0.66	0.90	0.81	0.90	0.88	0.93	0.80	-0.38	0.78	0.91	-0.08	0.77	0.16	0.54	0.60	0.88	-0.08	0.86	0.37	0.89	0.93
Caterpillar	0.73	0.89	0.87	0.84	0.96	0.96	0.90	-0.56	0.95	0.90	0.02	0.90	0.12	0.69	0.47	0.96	-0.02	0.89	0.16	0.81	0.91
Wal-Mart	0.43	0.27	0.29	0.28	0.35	0.41	0.09	0.28	0.35	0.38	-0.03	0.25	0.36	-0.11	0.62	0.27	0.01	0.65	0.50	0.50	0.40
Wells Farg	0.33	0.14	0.00	0.26	0.08	0.29	0.05	0.17	-0.03	0.17	-0.46	0.09	-0.06	-0.24	0.30	0.15	-0.26	0.36	0.56	0.64	0.21
Weyerhae	0.72	0.86	0.78	0.85	0.87	0.95	0.83	-0.54	0.81	0.84	-0.22	0.83	0.00	0.58	0.45	0.90	-0.15	0.83	0.30	0.92	0.88
Whirlpool	0.55	0.85	0.75	0.87	0.77	0.86	0.71	-0.29	0.67	0.89	-0.19	0.64	0.10	0.50	0.61	0.78	-0.15	0.77	0.42	0.87	0.89
Wyeth	0.41	0.71	0.74	0.57	0.80	0.64	0.67	-0.38	0.81	0.72	0.42	0.66	0.38	0.75	0.46	0.70	0.34	0.57	-0.07	0.30	0.71
Xerox Corp	0.65	0.93	0.89	0.89	0.94	0.95	0.88	-0.44	0.89	0.97	0.07	0.82	0.18	0.69	0.57	0.92	0.01	0.85	0.23	0.81	0.95
XL Capital	0.77	0.89	0.83	0.85	0.93	0.96	0.94	-0.71	0.90	0.82	-0.04	0.93	-0.05	0.64	0.33	0.98	-0.12	0.82	0.14	0.83	0.88
XTO Energ	0.75	0.82	0.85	0.76	0.93	0.90	0.89	-0.70	0.97	0.80	0.06	0.90	0.01	0.75	0.30	0.92	-0.04	0.77	0.00	0.68	0.82
Yum! Brar	0.75	0.89	0.87	0.79	0.94	0.90	0.82	-0.43	0.88	0.87	0.21	0.86	0.37	0.61	0.51	0.91	0.19	0.85	0.15	0.78	0.88
SP500	0.47	0.52	0.54	0.41	0.60	0.54	0.79	-0.70	0.50	0.45	0.14	0.71	-0.08	0.61	-0.22	0.65	0.21	0.21	-0.32	0.44	0.42

Figure 4.5: Correlation matrix of the 125 reference entities in the CDX index

parameters of the 123 reference entities and those of the two singular entities, which are almost doomed to default due to high default probabilities, justify the necessity of special treatments for them using a different, more general exponential model as indicated in section 4.1.1.

Further more, the heterogeneousness in rate parameter, especially the existence of the two singular entities, motivated us to use different rate parameters for different reference entities and compare the difference between spread from uniform-parameter model and our heterogeneous model.

4.3 Correlation Parameter

Correlation coefficients between different reference entities and the market (S&P500 index) range from -0.70 to 0.79, with mean 0.39 and standard deviation 0.30. The large range and standard deviation indicate high heterogeneousness in correlation coefficients, so using uniform correlation coefficient for all 125 reference entities is highly inconsistent with the facts and will lead to unexpected catastrophic consequences.

4.4 Comparative Analysis

4.4.1 Comparison of Spreads from the Improved One-Factor Model and Homogeneous One-Factor Model

Assume a 5-year CDX portfolio of 125 reference entities, with constant notional amount (A=20 million dollars) and recovery rate (R=0.4) across the portfolio. And simply assume there are only three tranches in the portfolio: attachment (L) and detachment (H) points of mezzanine tranche are 3% (75 million dollars) and 7% (175 million dollars), respectively. Correspondingly, 0% to 3% is the junior tranche and 7% to 100% the senior tranche.

The following provides a comparative analysis of prices (spreads) of different tranches calculated from the one-factor Gaussian copula model and the improved model with individual correlation and rate parameter for each reference entity. For the one-factor Gaussian copula model with constant correlation and constant default probability across the whole CDX portfolio, we use the average correlation coefficient and average rate parameter of the 125 reference entities, which are $a_i = 0.39$ and $\alpha_i = 0.0538$, respectively. Then use another arbitrarily-selected set of parameters for comparison, $a_i = 0.7$ and $\alpha_i = 0.01$ for example.

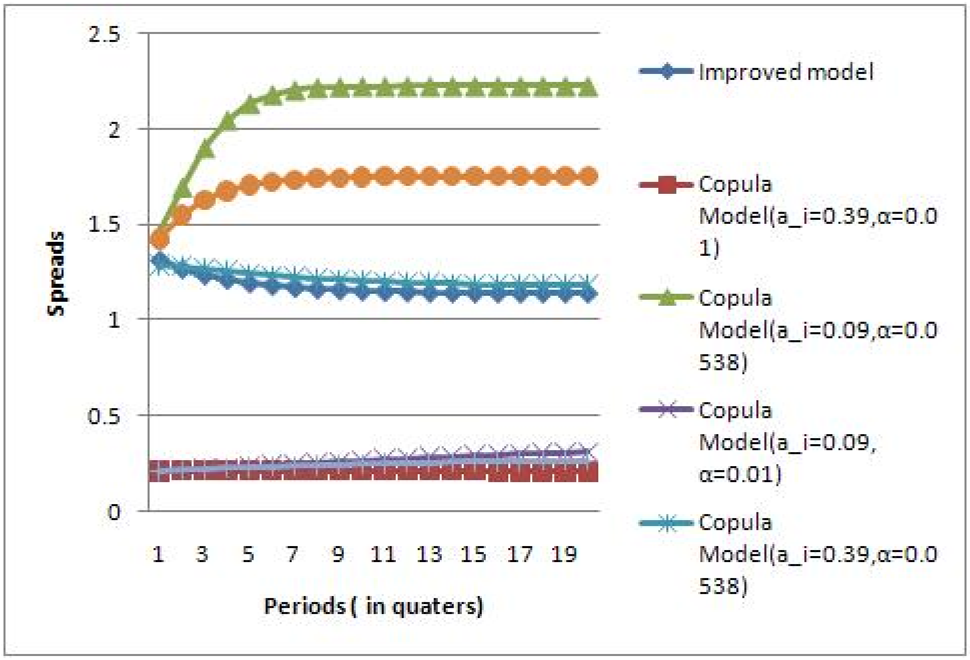


Figure 4.6: Spreads comparison of junior tranche.

Junior Tranche

Figure 4.6 shows spreads of junior tranche calculated using different models. As shown, the one-factor Gaussian copula model is very sensitive to correlation coefficients (a_i) and rate parameters (α_i). Even small changes lead to observable differences in spreads, not to mention combined effect of both factors. Figure 4.6 is also a first indication that differences due to default probability have a larger impact on pricing than differences due to the correlation coefficient. A more thorough analysis will be done in the next section.

Specifically for junior tranche, spread falls as correlation rises. For fixed rate parameters, 0.01 and 0.0538, respectively, a correlation coefficient of 0.42 leads to a lower spread than a correlation coefficient of 0.03 does. On the other hand, spread increases substantially as the rate parameter increases: for fixed correlation coefficients, 0.3 and 0.42, respectively, a rate parameter of 0.0538 leads to much higher spreads than a rate parameter of 0.01 does. And the higher the spread, the riskier the tranche becomes.

When combining the effects of the two factors, we see that differences due to correlation increase as default rate parameter increases. For example, differences between spreads for correlation coefficient of 0.3 and 0.42 increase as rate parameter increase from 0.01 to 0.0538. On the other hand, effects due to rate parameter decrease as correlation increases. For example, differences between spreads for rate parameters of 0.0538 and 0.01 decrease as the correlation coefficient increases from 0.3 to 0.42.

Mezannine Tranche

Figure 4.7 is a comparative analysis for spreads of the mezzanine tranche. Same as for junior tranche, we studied the effects on tranche price of correlation coefficient and rate parameter. As known from above, spreads of the junior tranche are proportional to correlation. As will be known from next section, spreads of senior tranche are inversely proportional to correlation. Therefore, the value of the mezzanine tranche, which is subject to both effects, is expected to be less sensitive to correlation as shown in Figure 4.7.

When rate parameter is fixed at $\alpha = 0.01$, there is little difference between spreads of mezzanine tranche for correlation coefficient 0.3 and 0.42. However, when rate parameter increases from $\alpha = 0.01$ to $\alpha = 0.0538$, difference between spreads of mezzanine tranche of different correlation coefficient $\alpha = 0.01$ and $\alpha = 0.0538$ increases. The lower the correlation is, the higher the spread. That is, spread of mezzanine tranche increases as correlation decreases. And as maturity increases, the discrepancy expands. All these make sense, because as correlation increases, it becomes more likely to have either few or many defaults. In both cases, junior tranche or senior tranche are affected most and the mezzanine tranche is the safest. As correlation decreases from that level, few but enough entities will default to affect the mezzanine

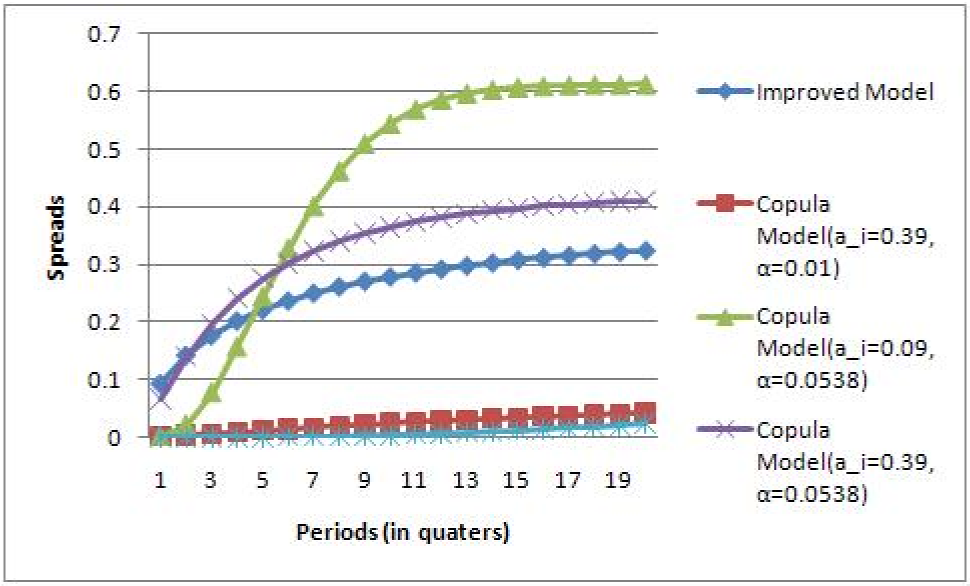


Figure 4.7: Spreads comparison of mezzanine tranche.

tranche and spread of the mezzanine tranche will increase correspondingly. As maturity increases, the total number of defaults is likely to increase, so spreads increases with maturity. Compared with the junior tranche, when the default rate parameter is low ($\alpha = 0.01$), effects due to correlation on spread of mezzanine tranche is even weaker than when default rate parameter is relatively high ($\alpha = 0.0538$).

Senior Tranche

Figure 4.8 provides a comparative analysis for spreads of the senior tranche. As maturity increases, spreads increases because more entities are likely to default and

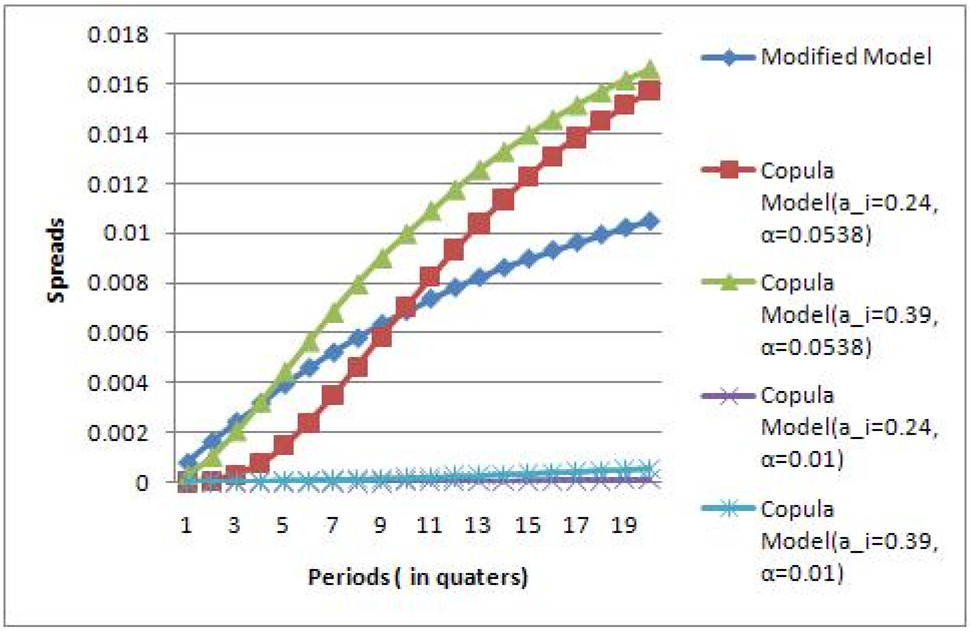


Figure 4.8: Spreads comparison of senior tranche.

loss will spread to the senior tranche. Spreads of the senior tranche also increase as either correlation or rate parameter increase. Combination of high correlation and high rate parameter pushes the spreads of senior tranche to an even higher level. This is in accordance with what we learned from junior and mezzanine tranches.

To summarize, when few defaults happen, junior and mezzanine tranches will be affected most and the senior tranche stays safe. As more and more defaults happen, both junior and mezzanine tranches will be wiped out and the senior tranche starts to suffer. High correlation means high probability of a large number of defaults to happen, therefore high correlation is good news for junior tranche, but bad news for senior tranche. The mezzanine tranche is in the middle and the effect due to correlation depends on the exact value of correlation coefficient, as well as the attachment and detachment points, which decides the size of junior tranche - the buffer tranche.

4.4.2 Comparative Analysis of the Improved Model With and Without Special Treatment to the Default Rate Parameters of the Two Singular Reference Entities

From the above analysis, we find that when the default rate parameter of the one-factor Gaussian copula model is $\alpha = 0.0538$, differences between spreads of improved model and the one-factor Gaussian copula model is modest. When default rate parameter of the one- factor Gaussian copula model takes a lower value which is $\alpha = 0.01$, differences between spreads of improved model and the one-factor Gaussian copula model substantially increased. In other words, our improved model fits better to the one-factor Gaussian copula model with higher rate parameter. The reason for this may lie in that we specially treated the default probability of two singular reference entities by using a general model 4.2 rather than the regular exponential model 4.1

to fit them.

Now we compare the spreads from our improved model with and without special treatment to the two singular reference entities. For the improved model without special treatment to the two singular reference entities, we apply the exponential model 4.1 to all 125 reference entities and replace rate parameters of the two singular reference entities with the average rate parameter ($\alpha = 0.0538$) of the CDX index portfolio. Figure 4.9 show big differences in spreads of the improved model with and without special treatment to the two singular reference entities, which indicates that including two singular entities in the CDO portfolio significantly affects the spreads of the CDO tranches. Even if we stay with exponential model for all 125 reference entities, but assign high default probabilities ($\alpha = 0.7939$ and 0.8308, respectively) to those two reference entities, spreads of this semi-specially-treated model are still higher than those from the improved model with average rate parameter ($\alpha = 0.0538$) for these two singular reference entities. This says that as long as the two reference entities are very likely to fail, no matter what probability distribution is being used, they will increase the spreads of the whole portfolio substantially.

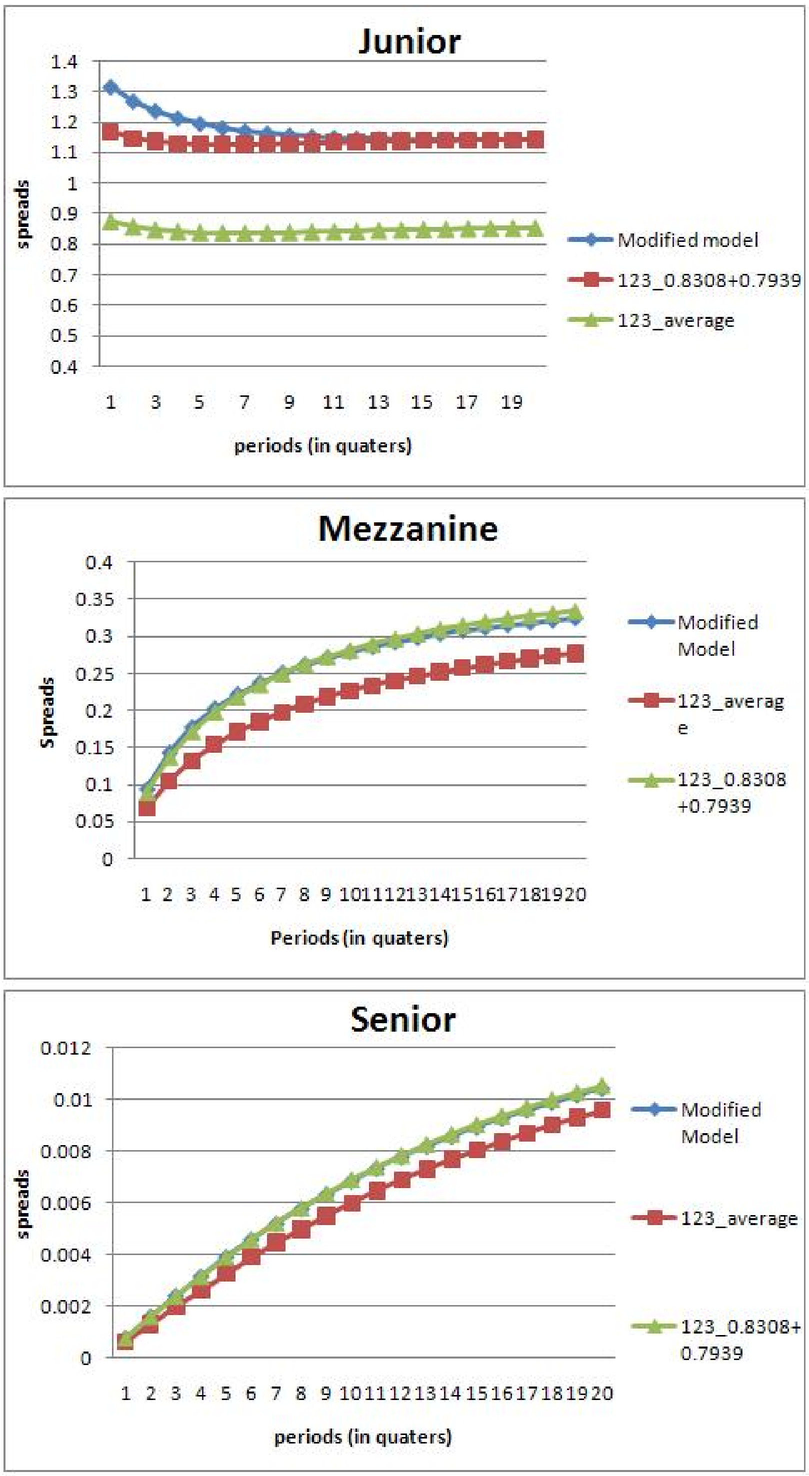


Figure 4.9: comparative analysis of spreads calculated from the improved model with and without special treatments to the two singular names in the CDX portfolio

4.4.3 Spread's Sensitivity to Correlation and Rate Parameter

This section provides a detailed analysis of the sensitivity to changes in correlation and default rate parameter of the one-factor Gaussian copula model. Let correlation parameter (a_i) and default rate parameter (α_i) vary around their average level of the 125 reference entities in the CDX index ($a_i = 0.65$ and $\alpha = 0.0538$). The variation is uniformly distributed between -0.5 and 0.5:

$$a_i = \bar{a}_i + \beta(rand() - 0.5); \tag{4.3}$$

$$\alpha = \bar{\alpha} + \gamma(rand() - 0.5) \tag{4.4}$$

where β, γ are weights of the variation part and $rand()$ is a random number between 0 and 1 generated by the computer. Also, a_i should be between 0 and 1 and α should be greater than or equal to 0. They impose limits on the range of β (varies between -0.7 and 1.3) and γ (≤ 0.107).

Exhibit 4.10, 4.11 and 4.12 show that spread of each tranche varies little when correlation coefficient changes from 0 to 0.03 to 0.1, but varies visibly when default rate

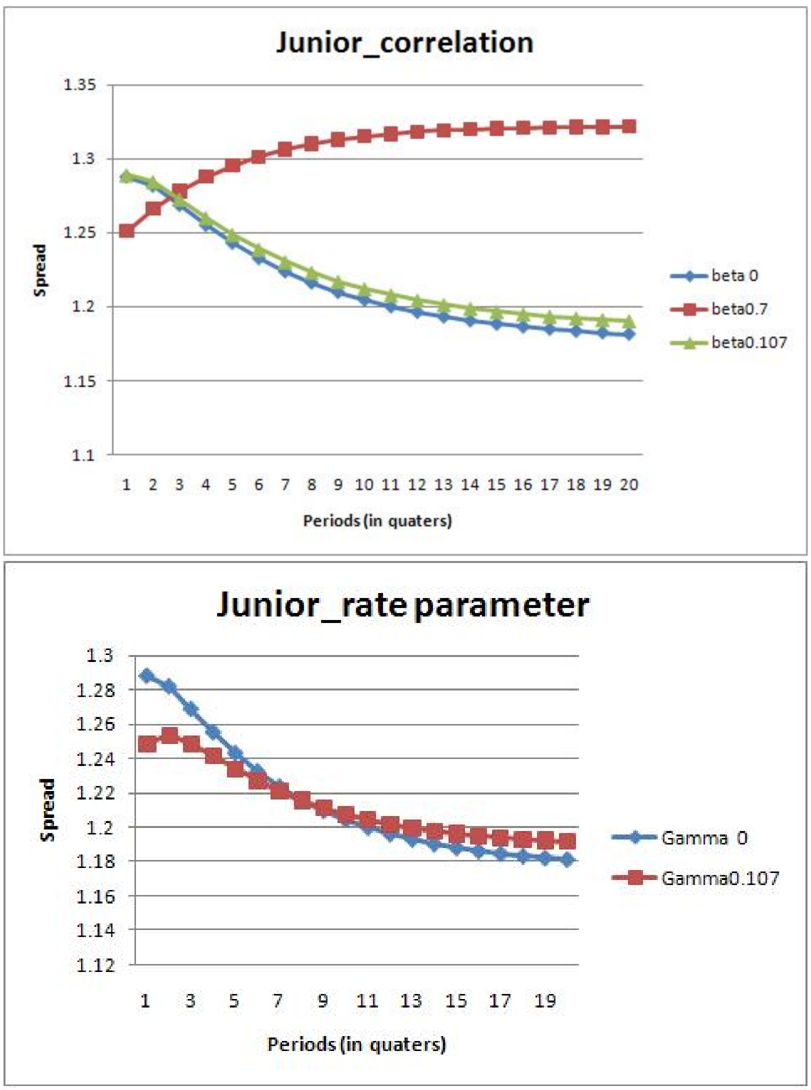
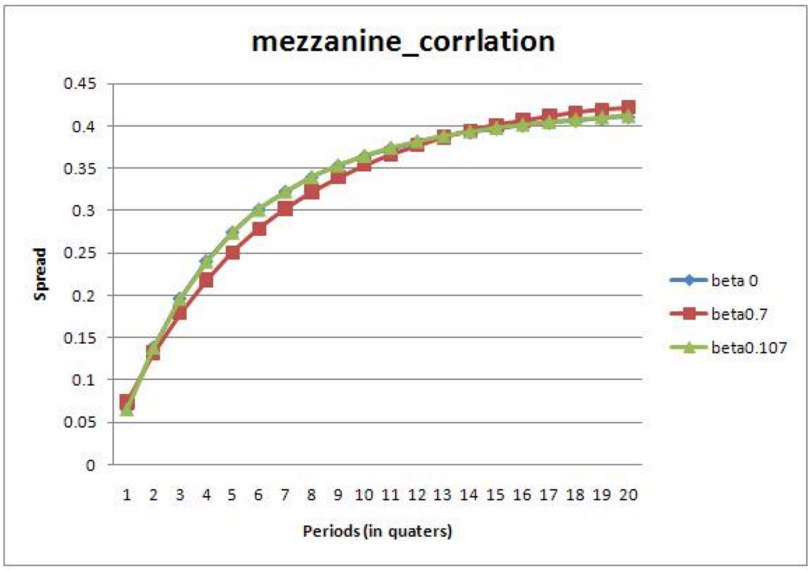


Figure 4.10: sensitivity to correlation and rate parameter of the one factor model-junior tranche.



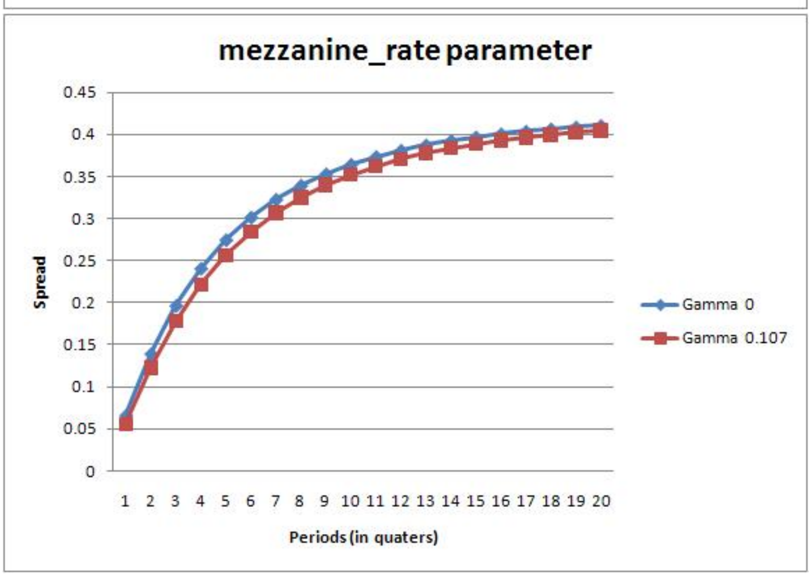


Figure 4.11: sensitivity to correlation and rate parameter of the one factor model-mezzainine tranche.

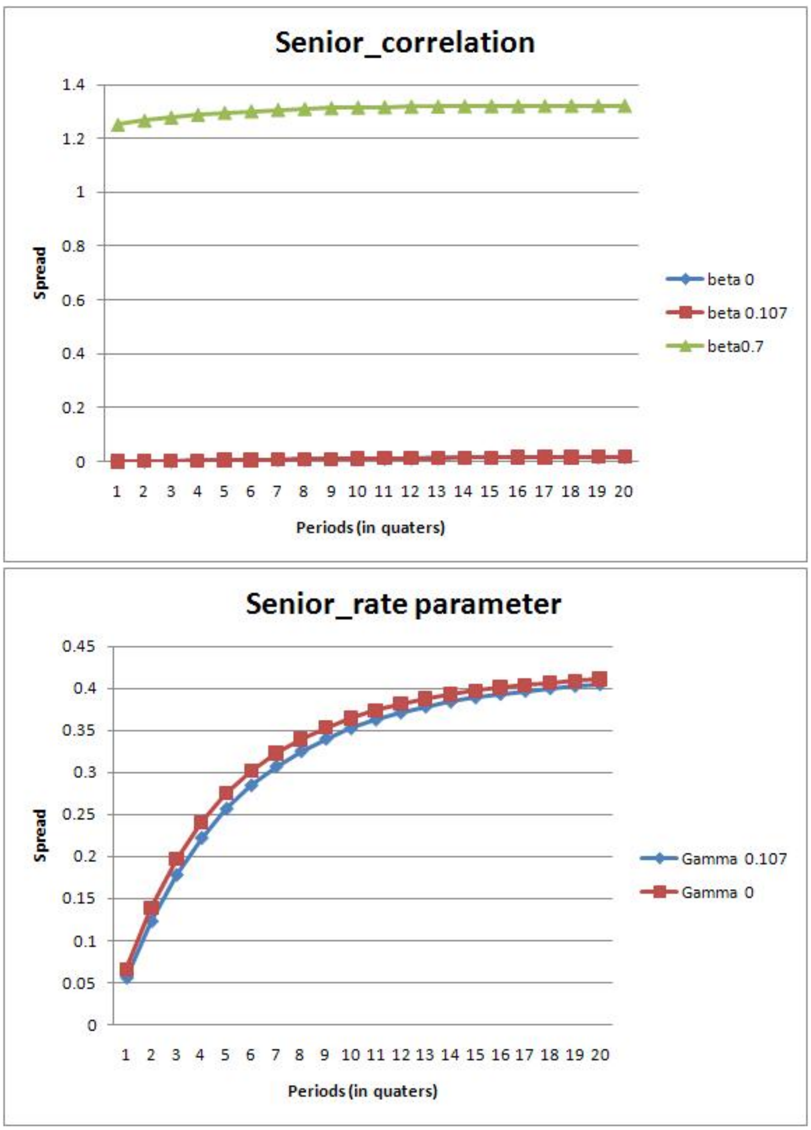


Figure 4.12: sensitivity to correlation and rate parameter of the one factor model-senior tranche.

parameter changes from 0 to 0.03 to 0.1. If we increase the variation of the correlation coefficient from 0 to 0.7, as shown in Figure 4.10, 4.11, and 4.12, the spread of each tranche increases substantially. It confirms that using constant correlation coefficient and default rate parameter is an unrealistic and inaccurate simplification. When comparing tranches, we find that sensitivity to default correlation or rate parameter varies considerably across different tranches of the CDX index, which reinforces the view that using flat correlation and default rate parameter for all 125 reference entities of the CDX index is not appropriate because it is almost impossible to find one single number as correlation coefficient or default rate parameter for all three tranches.

4.5 Calibration

In section 4.4.1, we compared the one-factor Gaussian copula model to our improved model. Spread term structures of the improved model are in between high and low levels of spread term structures of the one-factor Gaussian copula model with various sets of default correlation and rate parameter. Since none of the sets of flat default correlation and rate parameter, including average parameters of the 125 reference entities, makes spreads of CDX tranches fit the spreads from our improved model, we are interested to calibrate spreads from the one-factor Gaussian copula model to spreads from our improved model and see if there is a set of flat default correlation and rate parameter existing to make the one-factor Gaussian copula model fit well with our improved model. The Calibration of the one-factor Gaussian copula model is

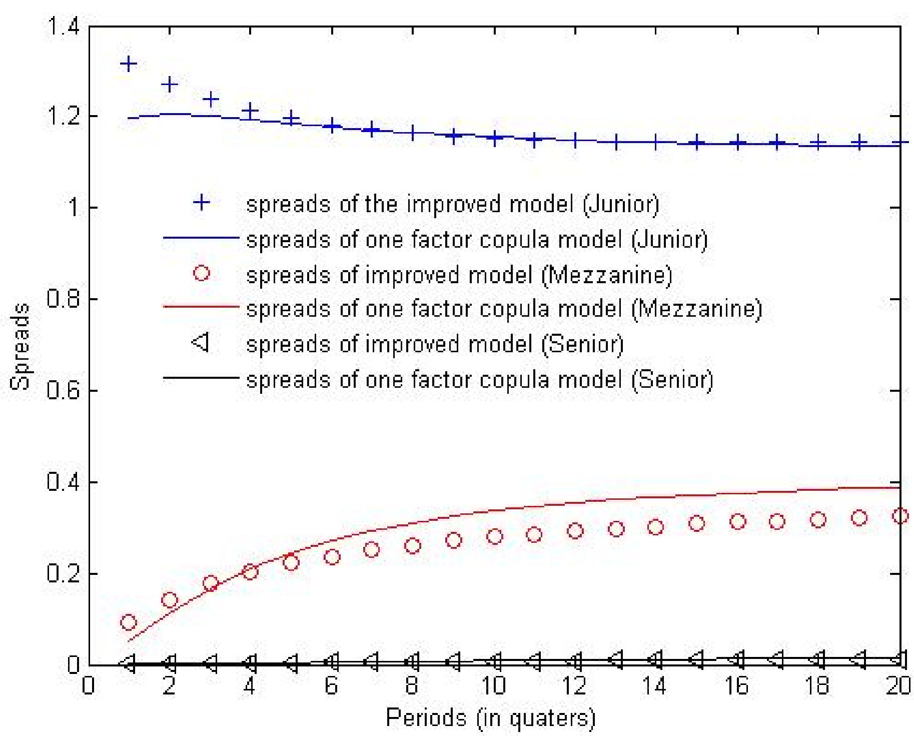


Figure 4.13: Calibration of the one factor Gaussian copula model to the improved model.

shown in Figure 4.13. The corresponding parameters for junior tranche are correlation coefficient $a = 0.8320$ and rate parameter $\alpha = 0.2434$. Same fitting procedures have also been done to the mezzanine and senior tranche. For the mezzanine tranche, the fitted parameters are $a = 0.39$ and $\alpha = 0.036$. For the senior tranche, the fitted parameters are $a = 0.86$ and $\alpha = 0.00266$.

Fitting curves for three tranches are not all smooth and show big discrepancy from the term structure of the improved model. They once again verify that it is impossible to find one single set of parameters to capture the diversity in default correlation and default probability. On the other hand, by using individual correlation and default rate parameter for each reference entity in the CDX index, our improved model provides a convenient and more accurate way to price CDOs.

4.6 Spread's Sensitivity to Different Integration Methods

When we integrated the market factor out of the conditional default probability to get unconditional default probability, we used the Trapezoidal Rule. All results discussed above have been done using this method. We also tried the Monte-Carlo method to do the integrations, assuming both normal and student t distribution of the market factor. Figure 4.14 is a comparison graph of the term structures obtained from

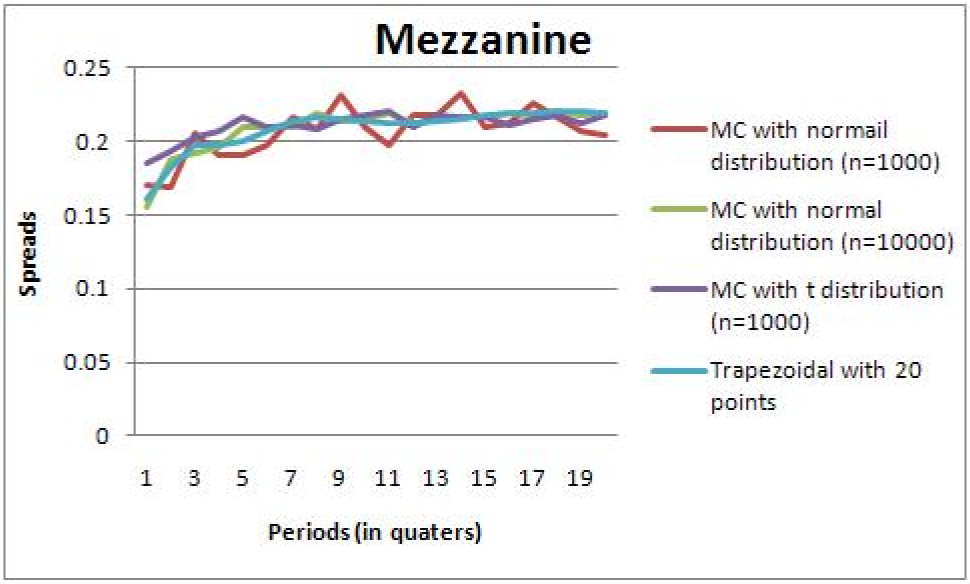


Figure 4.14: Comparison between different integration methods

different integration methods for the Mezzanine tranche.

From the Comparison graph, we can clearly see that the term structure obtained using the Trapezoidal Rule is much smoother than that from the Monte-Carlo simulation with 1000 paths. However,as the number of paths increase, the term structures from the Monte-Carlo method become smoother and smoother, but the running times also increase considerably. To balance the tradeoff between speed and accuracy, we chose to use the Trapezoidal rule in our programs.

To determin how many subintervals and how wide range are enough for an accurate integration, we changed them for both the one-factor copula model and our improved

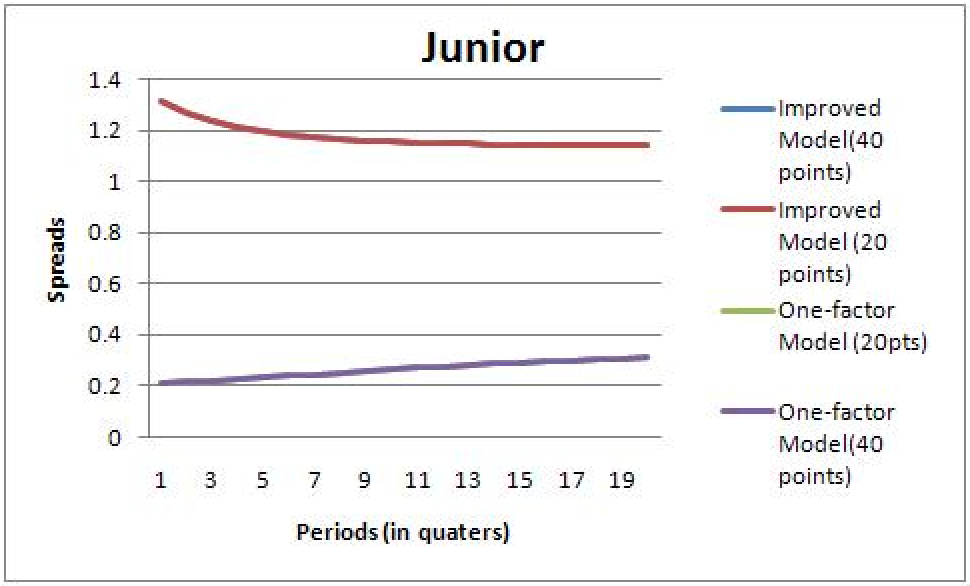


Figure 4.15: Comparison between different number of subintervals for integration.

one-factor copula model. Results show that it does not make any significant difference between using 40 subintervals and 20 subintervals and there is no significant difference between using a range of $[-14, 14]$ and a range of $[-7, 7]$. Figure 4.15 shows the difference between using different subintervals for both the one-factor copula model and the improved one-factor copula model for the junior tranche.

4.7 Improved Model Under a Different Market Condition

In the previous sections, we discussed the CDX spreads calculated from our improved model using data collected in February 2009. Back then, the market was still in high volatility under the impact of the then ongoing financial crisis. To study the sensitivity of our improved model to different maket conditions, we collected another set of CDS spreads of the 125 reference entities of the CDX index in a more current period from 10/14/2010 to 11/03/2010. We did curve-fitting and obtained another set of default probability rate parameters.

In other words, we used our improved model to calculate the spread of a CDX index with two sets of parameters obtained from data of different market condisitons. One is in the middle of the 2007-2010 financial crisis and there was high volatiliy in the market; another is right after the financial crisis and the market was in a stable condition.

The average default probability parameter was 0.05381 in the middle of the crisis and 0.02329 after the crisis. Spreads of various tranches of the CDX index for different sets of data collected under different market conditions are compared in Figure 4.16.

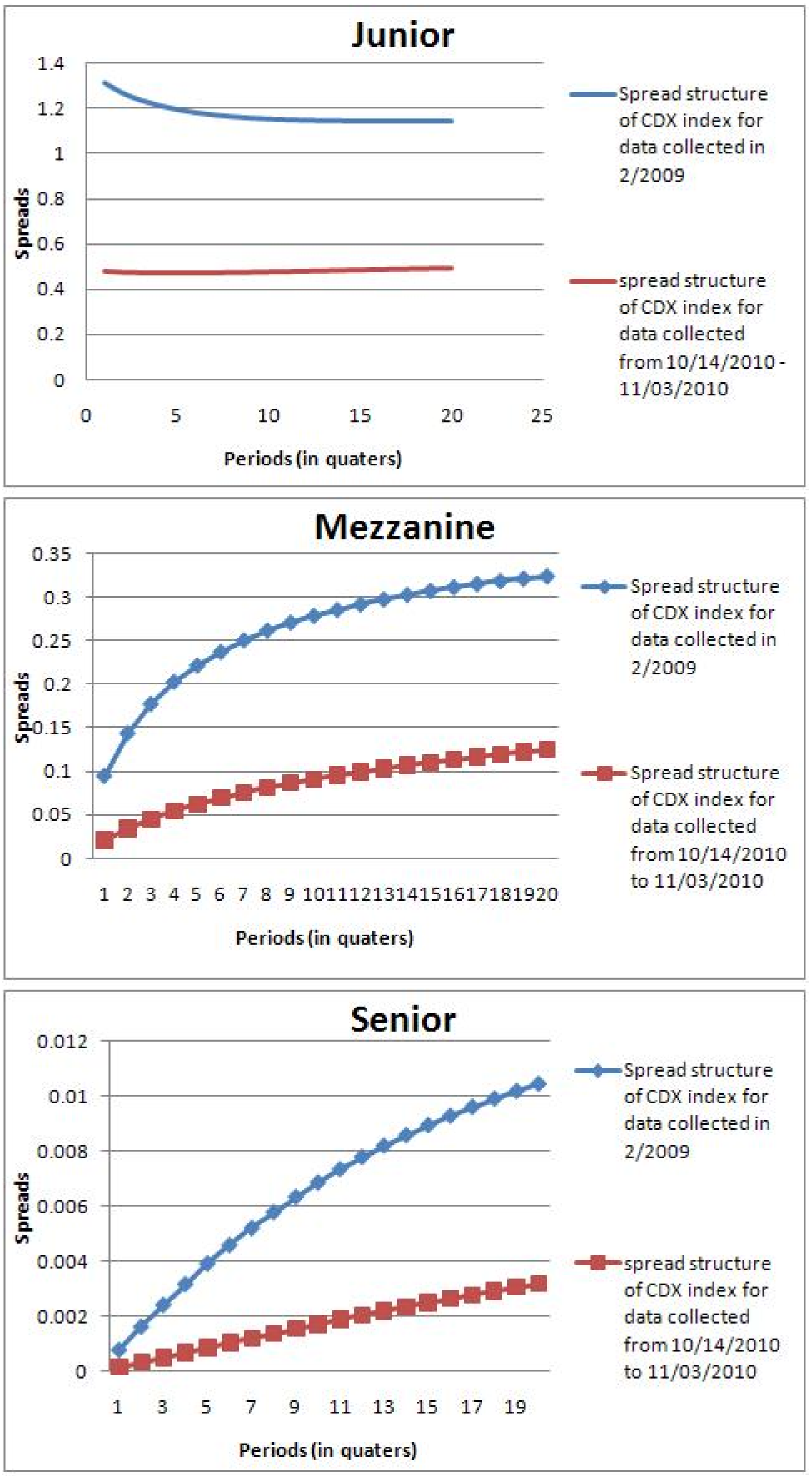


Figure 4.16: sensitivity to different market conditions of the improved one factor model.

As we can see from the graphs, the improved one factor copula model is very sensitive to different market conditions. Under severe market conditions, there were 2 or 3 reference entities seemed to be doomed to default, so that the spreads of a CDX index in such a market condition is much higher than those of data collected in a relative stable market, which confirmed from another aspect that even small number of "bad" reference entities would lead to the collapse of the whole CDX index. In other words, spreads of CDOs are very sensitive to default probabilities of single reference entities out of the 125 reference entities in the portfolio.

Chapter 5

Analytical Solutions for Heterogeneous CDOs

In section 3.5 we introduced an analytical solution to the pricing of homogeneous CDO portfolio. To expand the Heaviside step function approximation to the heterogeneous case, we assume the total number of reference entities N in a CDO portfolio can be categorized into two groups of entities: n_1 and n_2, with local uniform default probability p_1 and p_2 respectively. Both n_1 and $(n-2)$ are large enough, but $n_1 \gg n_2$. The number of defaults of each category follows a binomial distribution. When n_1, n_2 are large enough, binomial distributions can be approximated as normal distributions.

Then the distribution of the total number of defaults of the whole portfolio is just the sum of two normal distributions, which is $N_{(n_1p_1+n_2p_2,\sigma_1^2+\sigma_2^2)}$. Let x be the percent number of defaults by time t, Φ the normal cumulative distribution function, and M still the market or common factor in the factor model, then the distribution of conditional percent loss is as follows:

$$F(x|M) = \Phi\left(\frac{Nx - n_1p_1 - n_2p_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) = \phi\left(\frac{\sqrt{n_1}[(x-p_1) + \frac{n_2}{n_1}(x-p_2)]}{\sqrt{p_1(1-p_1) + \frac{n_2}{n_1}p_2(1-p_2)}}\right) \tag{5.1}$$

with $N = n_1 + n_2$, $\sigma_1 = \sqrt{n_1p_1(1-p_1)}$, and $\sigma_2 = \sqrt{n_2p_2(1-p_2)}$. Let $\epsilon = \frac{n_2}{n_1}$. Since $n_1 \gg n_2$, $\epsilon \ll 1$, equation (5.1) becomes:

$$\phi\left(\frac{(1+\epsilon)x - (p_1+\epsilon p_2)}{\sqrt{\frac{p_1(1-p_1)+\epsilon p_2(1-p_2)}{n_1}}}\right) \approx \phi\left(\frac{(1+\epsilon)x - (p_1+\epsilon p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1}}}\right) \tag{5.2}$$

For $n_1 \gg 1$, $\frac{p_1(1-p_1)}{n_1} \ll 1$, equation 5.2 can be approximated as a Heaviside function:

$$H[(1+\epsilon)x - (p_1+\epsilon p_2)] \tag{5.3}$$

The Heaviside function approximation makes it easier to find the unconditional cumulative distribution function of x as follows:

$$F(x) = \int_{-\infty}^{\infty} H[(1+\epsilon)x - (p_1 + \epsilon p_2)] f(M) dM \tag{5.4}$$

where $p_1 = p_1(M) = \Phi(\frac{C_1 - a_1 M}{\sqrt{1-a_1^2}})$ and $p_2 = p_2(M) = \Phi(\frac{C_2 - a_2 M}{\sqrt{1-a_2^2}})$ with $C_1 = \phi^{-1}(q_1)$ and $C_2 = \phi^{-1}(q_2)$. Since

$$H[(1+\epsilon)x - (p_1 + \epsilon p_2)] = \begin{cases} 1 & (1+\epsilon)x \geq p_1 + \epsilon p_2 \\ 0 & (1+\epsilon)x < p_1 + \epsilon p_2 \end{cases} \tag{5.5}$$

the unconditional cumulative distribution:

$$F(x) = \begin{cases} \phi(-M_c) & (1+\epsilon)x \geq p_1 + \epsilon p_2 \\ 0 & (1+\epsilon)x < p_1 + \epsilon p_2 \end{cases} \tag{5.6}$$

where $\phi(-M_c)$ is the normal cumulative distribution function when $M = M_c$, which is given by:

$$(1+\epsilon)x = p_1(M_c) + \epsilon p_2(M_c) = \Phi(\frac{C_1 - a_1 M_c}{\sqrt{1-a_1^2}}) + \epsilon\phi(\frac{C_2 - a_2 M_c}{\sqrt{1-a_2^2}}) \tag{5.7}$$

For small ϵ, let $M_c \approx M_c^{(0)} + \epsilon M_c^{(1)}$ with:

$$M_c^{(0)} = \frac{\Phi^{-1}(x)\sqrt{1-a_1^2} - C_1}{-a_1} \tag{5.8}$$

then

$$\phi[\frac{C_1 - a_1(M_c^{(0)} + \epsilon M_c^{(1)})}{\sqrt{1-a_1^2}}] \approx x - \frac{\epsilon a_1}{\sqrt{1-a_1^2}}\phi'(\phi^{-1}(x))M_c^{(1)} \tag{5.9}$$

Comparing (5.7) and (5.9), the second order of ϵ yields:

$$\epsilon x = \epsilon\phi(\frac{C_2 - a_2 M_c^{(0)}}{\sqrt{1-a_2^2}}) - \frac{\epsilon a_1}{\sqrt{1-a_1^2}}\phi'(\phi^{-1}(x))M_c^{(1)} \tag{5.10}$$

so that

$$M_c^{(1)} = \frac{\phi(\frac{C_2 - a_2 M_c^{(0)}}{\sqrt{1-a_2^2}}) - x}{\phi'(\phi^{-1}(x))}\frac{\sqrt{1-a_1^2}}{a_1} \tag{5.11}$$

substitute (5.11) into (5.6), we obtain the cumulative probability distribution of percent loss x for a heterogeneous CDO portfolio as follows:

$$F(x) = \phi(-M_c^{(0)} - \epsilon\frac{\sqrt{1-a_1^2}}{a_1}\frac{\phi(\frac{C_2 - a_2 M_c^{(0)}}{\sqrt{1-a_2^2}}) - x}{\phi'(x)}) \tag{5.12}$$

where $\phi'(x)$ is the first derivative or the probability density function of the normal distribution $\phi(x)$ and $M_c^{(0)}$ is given by (5.8) and $\epsilon = \frac{n_2}{n_1}$. With Equation 5.12, we can finally calculate the spread of a CDO using Equation (3.28), (3.29) and (3.30) as we did for the homogeneous case. Future research is expected on this topic.

Chapter 6

Pricing Model for Securitized Life Settlement (SLS)

6.1 Introduction

Securitized life settlement (SLS) in the insurance industry is another type of financial derivative based on periodic premium payments and contingent payment at maturity. They have been actively discussed resently on Wall Street Journal (see [40], [41], [42], [43]) and other Financial media. Considering the high similarity between securitized life settlements and CDOs, we are interested to find a solution to the pricing of securitized life settlements after having studied the pricing of CDOs.

A life settlement is an insurance policy sold by the insured or a trust at a price higher than the surrender value but lower than the face value of the policy. The difference in between is the profit an investor expects to make by purchasing this insurance policy. An immediate predecessor of life settlements are viatical settlements, in which the insured with terminal diseases (especially AIDS) sell their life insurance policies to a third party at a price higher than the surrender value, but lower than the policy's face value. After a new class of drugs developed and lives of AIDS patients have been extended substantially, investors instead focused their attention on senior life settlements and the securitization of them.

The first securitized senior life settlements were issued by Tarrytown Second, LLC in January 2004. It was a $64 Million of class A senior life settlement-securitization backed by $195 million in face value of life insurance policies. The term "senior" refers to the age of the insured from whom life settlemtn companies will buy insurance policies. In the current market the life expectancy of the insured is no more than twelve years. As of February 2005 there was approximately $12.7 trillion of life policies in the U.S, of which approximately 10% of the insured are at least 70 years old [44].

Life settlements were once considered promising diversification investments and even the possible creating of a bubble, but their growth has met with a lot of resistance: Critics have called them “death bonds”, “blood pools” and “collateralized death obligations” because they pay off when the insured dies; Standard & Poor’s refused to rate the securities and emphasized these securities’ “unique risks” in their March report.

The major risk of life settlement investment is longevity risk. If the insured lives longer than expected, investors will have to pay premiums for longer time and get death benefits paid at a later time, which lead to poor returns or even loss. Mortality risk, which is similar to the prepayment risk of collateralized mortgage obligations, on the other hand, is not a problem for securitized life settlements because the shorter the insured live, the sooner the investor will get paid and the less premium the investor has to pay out.

Despite their bad reputation and “unique risks”, life settlements do have a positive side. From the policy seller’s point of view, life settlements allow older people who no longer need or can afford life-insurance policies to sell them and get immediate cash, which is higher than the surrender value they would get from the insurance company. From the investor’s point of view, life settlements will diversify their investments and generate pretty good profit [45].

This seems like a perfect "win-win" deal, but there does exist a big loser - the insurance company. As discussed in [45], insurance companies can make a bunch of bad underwriting bets but still be profitable if lapse rates are high enough. "Lapse" means that the insurance company receives premiums but does not have to pay out on the policy. Anytime a policyholder can not keep a policy up, he or she stops paying premiums and gives up what he or she has paid before, then there is a lapse. There are even some industry analysts who suggest that some life insurance companies are only profitable because of their lapse rates.

Securitized Life Settlements can theoretically reduce lapse rates, because investors of life settlements will buy the insurance policy, which will otherwise be a lapse, and keep paying premiums until it pays off. If enough people sell their policies before they lapse, the lapse rate will decrease and finally the life insurance companies would be force to raise rates and insurance prices will be driven up high. However, situations in practice are way more complicated. There are many factors affecting the sales and profits of a insurance company. So how many sell their policies is enough is not a easy question to answer.

The A.M.Best rating company has published a methodology [46] on the securitization of life settlements. The methodology details the rating policy on securitized life settlements, as well as requirements on the securitization of life settlements.

Unlike CDOs, research on the pricing of securitized life settlements are scarce and less sophisticated. Gupta [47] mentioned that currently the life settlements are priced from an investor's point of view. The offering price from an investor is usually based on the purchaser's desired internal rate of return (IRR) and the status of the insured's mortality. He further pointed out that so far, the risk analysis on life settlement is insufficient.

In this chapter, we are going to introduce a binomial model to price homogeneous securitized life settlements and further study the risk of a securitized life settlement portfolio.

6.2 Binomial Model for Homogeneous Portfolio

Consider a homogeneous securitized life settlements portfolio of n policies. Assume monthly premium payment and all insured live no longer than the maturity of the policies, which is 100 months. Define the following notations:

n: total number of policies in the SLS portfolio;

m: total number of periods (100 months);

V: the value of a SLS portfolio;

Var: variance of the SLS portfolio;

B: face value or benefit payment contigent on the death of an insured;

P: amount of monthly premium payment;

r: discount rate;

Δt: payment period, 1 month = $\frac{1}{12}$ year;

k or k': the k^{th} or the k'^{th} period;

X_k or X_l: cumulative number of death up to the k^{th} or l^{th} period, with boundary conditions $X_0 = 0$ and $X_m = n$;

$\tilde{X_k}$: number of death in the k^{th} period, $\tilde{X_k} = X_k - X_{k-1}$, and $\sum_{k=1}^{m} \tilde{X_k} = n$.

The probability that one insured dies in any of the periods is $\frac{1}{m}$. The probability that one insured dies in the first k periods is $p_k = \frac{k}{m}$. Then out of n insured, the probability that s insured will die by the end of period k is:

$$p_k(s) = \begin{pmatrix} n \\ s \end{pmatrix} p_k^s (1 - p_k)^{n-s} = \begin{pmatrix} n \\ s \end{pmatrix} (\frac{k}{m})^s (1 - \frac{k}{m})^{n-s} \tag{6.1}$$

Expected value of X_k is:

$$E_{(X_k)} = np_k = n\frac{k}{m} \tag{6.2}$$

The value of an SLS portfolio equals the difference between the premium leg and the benefit leg, which is:

$$V = \sum_{k=1}^{m} B(X_k - X_{k-1})e^{-r\Delta tk} - \sum_{k=1}^{m} P(n - X_{k-1})e^{-r\Delta tk} \tag{6.3}$$

Since X_k's are random binomial variables, the expected value of a SLS portfolio is:

$$\begin{aligned} E(V) &= \sum_{k=1}^{m} B[E(X_k) - E(X_{k-1})]e^{-rk\Delta t} - \sum_{k=1}^{m} P[n - E(X - k - 1)]e^{-rk\Delta t} \\ &= \sum_{k=1}^{m} B\frac{n}{m}e^{-rk\Delta t} - \sum_{k=1}^{m} P[n - \frac{n(k-1)}{m}]e^{-rk\Delta t} \end{aligned} \tag{6.4}$$

And the variance of V is:

$$\begin{aligned} Var(V) &= C\left(\sum_{k=1}^{m-1} e^{-2kr\Delta t}[E(X_k^2) - E^2(X_k)]\right. \\ &\quad \left. + 2\sum_{k=1}^{m-1} e^{-(k+l)r\Delta t}[E(X_k X_l) - E(X_k)E(X_l)]\right) \\ &= C\left(\sum_{k=1}^{m-1} e^{-2kr\Delta t}[\frac{nk}{m}(1 - \frac{k}{m})]\right. \end{aligned}$$

$$+\frac{2n}{m}\sum_{k<l}^{m-1}\sum_{l}^{m-1}e^{-(k+l)r\Delta t}[k(1-\frac{k}{m})+\frac{kk'(n-1)}{m}-\frac{nk}{m}(l-k)]\Bigg) \quad (6.5)$$

where

$$C = [B-(B-P)e^{-r\Delta t}]^2 \quad (6.6)$$

From variance, we can easily get standard deviation of the value of SLS, which is the volatility or risk of the securitized life settlements.

All above formulas can also be expressed in $\tilde{X}_k$:

$$V(\tilde{X}_k) = \sum_{k=1}^{m} B\tilde{X}_k e^{-rk\Delta t} - \sum_{k=1}^{m} P(n - \sum_{k=1}^{m}\tilde{X}_k)e^{-rk\Delta t} \quad (6.7)$$

$$E(V) = \sum_{\text{all paths}} PV(\tilde{X}_k) = \sum_{\text{all paths}} \frac{n!}{n_1!n_2!...n_m!}(\frac{1}{m})^n V(\tilde{X}_k) \quad (6.8)$$

$$Var(V) = \sum_{\text{all paths}} P[V(\tilde{X}_k) - E(V)]^2 \quad (6.9)$$

The above analytical solution to homogeneous securitized life settlements(SLS) serves as a good start in pricing SLSs. In our future research, we will take correlation among different life insurance contracts into consideration and borrow either the “tranche” or “roll”(add or remove single reference entity periodically) ideas for CDOs to distribute risk or maintain the portfolio at a certain risk level.

Chapter 7

Conclusion

This dissertation presents an extension of the one-factor Gaussian copula model for pricing collateralized debt obligations (CDOs): Instead of using flat default correlation and rate parameters across the whole portfolio, we use individual correlation coefficients between each reference entity and the market (S&P 500 index) based on 5-year daily stock prices, and we use specific rate parameter for each entity's default probability distribution by curve-fitting the default probability term structure. Various comparative analyses have been performed and results show that uniform correlation and rate parameters fail to capture that a few or even one single asset can substantially impact the credit quality of the whole portfolio. Heterogeneity of correlations and rate parameters of different reference entities is indispensable for constructing reliable and realistic models for pricing synthetic CDOs.

We also introduced analytical solutions to the pricing of both homogeneous and heterogeneous CDOs. We compared these analytical solutions with results obtained from simulation models. Results show very good consistency.

At the end, we introduced the analysis of another financial derivative - Securitized Life Settlements(SLSs) and presented an analytical solution to the pricing of homogeneous SLSs. Further work is expected on heterogeneous securitized life settlements, which will be of great interest to the insurance industry, the market and the investors.

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