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**Coherent homotopical algebras: Special gamma categories**

**Arroyo, Fangjun Hsu, Ph.D.**

**City University of New York, 1993**

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**COHERENT HOMOTOPICAL ALGEBRAS:  
"SPECIAL GAMMA-CATEGORIES"**

**b y**

**Fangjun H. Arroyo**

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

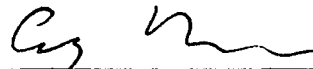
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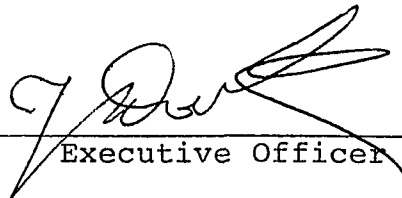
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## ABSTRACT

### COHERENT HOMOTOPICAL ALGEBRAS: "SPECIAL GAMMA-CATEGORIES"

by

Fangjun H. Arroyo

Advisor: Professor Alex Heller

We introduce a theory of coherence for symmetric monoidal categories in the spirit of Segal and show that it is equivalent, in an appropriate sense, to MacLane's original notion. More precisely, we prove that "special  $\Gamma$  categories", the analogue of special  $\Gamma$  spaces, and coherently symmetric monoidal categories are one and the same. This is analogous to the situation in topology where special  $\Gamma$  spaces are precisely homotopical commutative monoids. In light of the observation that the category of small categories  $\text{Cat}$  bears a functorial Quillen model structure with respect to the class of categorical equivalences: in fact, is a homotopy theory in the sense of Alex Heller, we may reinterpret the theorem as stating that coherently symmetric monoidal categories are precisely the homotopical commutative monoids within this new homotopy theory.

## ACKNOWLEDGEMENT

I would like to dedicate this paper to my daughter, Aisha, as partial amends for our long separation. I would like to express my gratitude and respect to my advisor Professor Alex Heller, who gave up much of his precious time freely and generously. The valuable advice and mathematic insights he shared with me has made this thesis possible. Last but not least, I would like to thank my husband for his advice encouragement throughout the years and for the excellent job he did in typing the thesis.

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## INTRODUCTION

Classically, coherence theorems are theorems which state that a large class of diagrams always commute, or describe conditions sufficient to insure commutativity. The idea of homotopical coherence theory on H-spaces is to specify enough higher homotopies for the tensor product in an H-space  $Y$  to ensure that  $Y$  is an  $n$ -fold loop space,  $1 \leq n \leq \infty$ . It would be an awesome task to specify the required homotopies explicitly, so one adduces an abstract framework. There exist two main ways of doing this: either by the use of contractible parameter spaces for  $n$ -fold products (May's operads [M] or Adams and MacLane's PROP [A] as topologized by Boardman and Vogt [B]), or by the use of sequences of spaces (Segal's  $\Gamma$  spaces [S]) which look formally and homotopically as if they were powers of a based space.

The former approach originated with Stasheff's early work on infinite loop spaces. Essentially, he proved that an H-space  $X$  is a loop space if and only if it has all possible higher coherence homotopies for associativity. This was taken up, in the categorical setting, by MacLane [M1], who in his seminal 1963 paper proved a coherence theorem for symmetric monoidal categories. (In fact, it implicitly includes the more elementary cases of semigroupoidal, symmetric semigroupoidal and monoidal categories.) By analogy to the situation in spaces, he parametrized the powers of the given monoidal category by contractible categories  $W_n$ , where  $W = \coprod_n W_n$  is the free monoidal category on one generator [M2]. For the most part, the subsequent work on categorical coherence (Eilenberg-Kelly [EK], Kelly

[K], Kelly-Laplaza [KL], Joyal-Street [JS], Lewis [Le], etc.) has followed and extended MacLane's work.

In the later approach to (homotopical) coherence for H-spaces due to Segal [S] and Anderson [A] one starts with the category of simplicial sets  $K$  rather than with the internal structure on a space. More precisely, Segal proves that an H-space  $\mathcal{A}$  is an infinite loop space iff it has a homotopy inverse and  $X_1 = \mathcal{A}$ , where  $X: \Gamma^{op} \rightarrow K$  is a special  $\Gamma$  space, i.e. a functor which preserves products "up to equivalence". In Heller's language [H], this says that the homotopical groups (monoids) in the usual (e.g. simplicial) homotopy theory are precisely the infinite loop ( $\Gamma$ ) spaces. In this paper, I introduce the analogous notion for categorical coherence. In particular, I show that coherent symmetric monoidal categories are precisely  $\Gamma$  categories, where categorical equivalence takes the place of nerve equivalence. Joyal [J&T] has observed that the category of small categories  $\text{Cat}(\mathbb{E})$  in an arbitrary Grothendieck topos bears a Quillen model structure with respect to the class of (internal) categorical equivalences. Letting  $\mathbb{E}$  be the topos of sets, we obtain a (functorial) Quillen model structure on  $\text{Cat}$  in which the cofibrations are functors injective on objects, the fibrations are functors satisfying the "isomorphism lifting" property and weak equivalences are the equivalences of categories. This is in fact a homotopy theory in the sense of Heller and our theorem may then be reinterpreted as stating that within this new homotopy theory coherently symmetric monoidal categories and homotopical commutative monoids are one and the same. Actually, our theorem seems to apply to any algebraic theory provided one gives suitable definitions for the corresponding algebraic categories. In particular,

we may show that the coherent monoidal categories with suitably defined coherent inverses are precisely the homotopical groups.

In section one, I will recall the definition of an algebraic theory in the sense of Lawvere and show that the theories of semigroups, commutative semigroups, monoids and commutative monoids each has a "mini-theory" imbedded in it.

In section two, we give the definition of symmetric monoidal, symmetric semigroupoidal, monoidal and semigroupoidal categories, collectively called homtopical algebraic categories.

In section three, we state our main theorem, namely, that a functor from the mini-theory of monoids (commutative monoids, semigroups or commutative semigroups) to the category of small categories which preserves product up to categorical equivalence induces, upon choosing inverse equivalence together with units and counits, a monoidal (symmetric monoidal, semigroupoidal or symmetric semigroupoidal) category. Conversely, given a monoidal (symmetric monoidal, semigroupoidal or symmetric semigroupoidal) category, we construct a functor from the mini-theory of monoids (commutative monoids, semigroups or commutative semigroups) to the category of small categories which preserves product up to categorical equivalence. In fact, what we have actually constructed is an equivalence of categories between the homotopy category of functors from the mini-theory of monoids (commutative monoids, semigroups or commutative semigroups) to the category of small categories which preserves product up to categorical equivalence and the homotopy category of monoidal (symmetric monoidal, semigroupoidal or symmetric semigroupoidal) categories.

In section four, we state and prove those lemmas which we need for the proof of our theorem and its converse. The theorem is proved in section five, while the converse is proved in section six.

## 1. Theories.

We recall the definition of an algebraic theory in the sense of Lawvere and make some elementary observations [H] .

**Definition.** Denote by  $\Phi$  the full subcategory of *Sets* whose objects are the natural numbers. An *algebraic theory* is a category  $\mathfrak{H}$  with the same objects, containing  $\Phi$  as a subcategory in such a way that the inclusion preserves coproducts, so that  $m+n$  is the coproduct in either category.

**Definition.** An  $\mathfrak{H}$ -*algebra* in a category with finite products  $C$  is a product-preserving functor  $X:\mathfrak{H}^{op} \rightarrow C$ , i.e.  $X_n \xrightarrow{\langle p_{n,1}, \dots, p_{n,n} \rangle} X_1^n$  is an isomorphism for each  $n \in \mathfrak{H}$ , where  $\langle X_{p_{n,1}}, \dots, X_{p_{n,n}} \rangle \triangleq \langle p_{n,1}, \dots, p_{n,n} \rangle$  and  $p_{n,i}:n \rightarrow 1, 1 \leq i \leq n$ , are defined by  $p_{n,i}^{-1}(1) = \{i\}$ .

**FACT.** Let  $\text{Alg}(\mathfrak{H}, \text{Sets}) \subset \text{Sets}^{\mathfrak{H}^{op}}$  be the full subcategory of  $\mathfrak{H}$ -algebras in *Sets*. Evaluation at 1 yields a faithful functor  $\text{Alg}(\mathfrak{H}, \text{Sets}) \rightarrow \text{Sets}$  which has a left adjoint  $F$ , the *free algebra* functor. The category  $\mathfrak{H}$  is isomorphic to the full subcategory of  $\text{Alg}(\mathfrak{H}, \text{Sets})$  containing the objects  $F0, F1, F2, \dots$ , the imbedding of  $\Phi$  in  $\mathfrak{H}$  being given by the restriction of  $F$  to  $\Phi$ .

In particular, the theory of semigroups **SG** (commutative semigroups **CSG**, monoids **M** or commutative monoids **CM**) imbeds onto the category whose objects are the natural numbers and whose morphisms from  $m$  to  $n$  are the homomorphisms from the free

semigroup (commutative semigroup, monoid or commutative monoid) on  $m$  elements to the free semigroup (commutative semigroup, monoid or commutative monoid) on  $n$  elements.

Denote by  $\Gamma_{\text{CM}} (\Gamma_{\text{M}}, \Gamma_{\text{CSG}}, \Gamma_{\text{SG}})$  the category whose objects are the natural numbers and whose arrows are the (order preserving, surjective, order preserving and surjective) partial maps.

Observe that  $\Gamma_{\text{CM}}^{\text{op}} (\Gamma_{\text{M}}^{\text{op}}, \Gamma_{\text{CSG}}^{\text{op}}, \Gamma_{\text{SG}}^{\text{op}})$  is isomorphic to a subcategory of  $\text{CM} (\text{M}, \text{CSG}, \text{SG})$ . For example, the embedding of  $\Gamma_{\text{CM}}^{\text{op}}$  into  $\text{CM}$  is given as follows. First observe that  $\Gamma_{\text{CM}}^{\text{op}}$  is the category whose objects are the same as  $\Gamma_{\text{CM}}$ 's and whose morphisms from  $m$  to  $n$  are the maps  $\theta: m \rightarrow P(n)$ , where  $P(n)$  denotes the subsets of  $n$ , such that  $\theta(i) \cap \theta(j) = \emptyset$  whenever  $i \neq j$ . In fact our  $\Gamma_{\text{CM}}^{\text{op}}$  is exactly the  $\Gamma$  defined by Segal in [S]. For any morphism  $m \xrightarrow{f} n$  in  $\Gamma_{\text{CM}}^{\text{op}}$  with

$$i \mapsto S_i = \{n_1 < \dots < n_{k_i}\}, \quad 0 \leq i \leq m-1, \quad k_i > 0,$$

where  $S_i \in P(n)$  and  $S_i \cap S_j = \emptyset$ , if  $i \neq j$ , let

$$(m \xrightarrow{f} n) \mapsto (m \xrightarrow{f_{\text{CM}}} n)$$

where  $m \xrightarrow{f_{\text{CM}}} n$  in  $\text{CM}$  is given by  $i \mapsto n_1 \dots n_{k_i}$ ,  $0 \leq i \leq m-1$ , if  $k_i > 0$ , and  $i \mapsto I$ , the unit object of  $\text{CM}$ , if  $k_i = 0$ . The other embeddings are constructed in a similar fashion.

## 2. Homotopical Algebraic Categories

**Definition.** A *semigroupoidal category*  $\mathcal{A} = (\mathcal{A}, \otimes, a)$  consists of a category  $\mathcal{A}$ , a functor  $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  (called the *tensor product*) together with a natural isomorphism

$$a = a_{A,B,C}: (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

(called the *associativity*) s.t. for all objects  $A, B, C, D \in \mathcal{A}$ , the following diagram (called the *associativity pentagon*) commutes.

$$\begin{array}{ccccc}
 & & (A \otimes B) \otimes (C \otimes D) & & \\
 & \nearrow a & & \searrow a & \\
 ((A \otimes B) \otimes C) \otimes D & & & & A \otimes (B \otimes (C \otimes D)) \\
 \searrow a \otimes 1 & & & & \nearrow 1 \otimes a \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a} & A \otimes ((B \otimes C) \otimes D) & & 
 \end{array}$$

**Definition.** A *monoidal category*  $\mathcal{A} = (\mathcal{A}, \otimes, I, a, l, r)$  consists of a semigroupoid  $\mathcal{A} = (\mathcal{A}, \otimes, a)$  together with an object  $I \in \mathcal{A}$  (called the *unit object*) and two additional natural isomorphisms

$$l = l_A: I \otimes A \xrightarrow{\cong} A \quad \text{and} \quad r = r_A: A \otimes I \xrightarrow{\cong} A$$

(called the *left unit* and *right unit*, respectively), s.t. for all objects  $A \in \mathcal{A}$  the following additional diagram (called the *triangle for unit*) commutes.

$$\begin{array}{ccc}
 & a & \\
 (A \otimes I) \otimes B & \xrightarrow{\quad} & A \otimes (I \otimes B) \\
 r \otimes 1 \searrow & & \swarrow 1 \otimes l \\
 & A \otimes B &
 \end{array}$$

**Definition.** A *symmetric semigroupoidal category*  $\mathcal{A} = (\mathcal{A}, \otimes, a, c)$  consists of a semigroupoidal category  $\mathcal{A} = (\mathcal{A}, \otimes, a)$  together with natural isomorphisms  $c = c_{A,B}: A \otimes B \xrightarrow{\cong} B \otimes A$  s.t.  $c^2 = id$  and for all objects  $A, B, C \in \mathcal{A}$ , the following diagram (called the *symmetry hexagon*) commutes.

$$\begin{array}{ccccc}
 & & c & & \\
 & & (A \otimes B) \otimes C & \xrightarrow{\quad} & C \otimes (A \otimes B) \\
 a^{-1} \nearrow & & & & \searrow a^{-1} \\
 A \otimes (B \otimes C) & & & & (C \otimes A) \otimes B \\
 1 \otimes c \searrow & & & & \nearrow c \otimes 1 \\
 & & a^{-1} & & \\
 A \otimes (C \otimes B) & \xrightarrow{\quad} & (A \otimes C) \otimes B & &
 \end{array}$$

**Definition.** We call  $\mathcal{A} = (\mathcal{A}, \otimes, I, a, c, l, r)$  a *symmetric monoidal category* if  $\mathcal{A} = (\mathcal{A}, \otimes, a, c)$  is a symmetric semigroupoidal category and  $\mathcal{A} = (\mathcal{A}, \otimes, I, a, l, r)$  is a monoidal category.

### 3. Definitions And Statement Of The Theorem.

Let  $X: \mathbf{H}^{\text{op}} \rightarrow \text{Cat}$  be a homotopical  $\mathbf{H}$ -functor, i.e.  $X: \mathbf{H}^{\text{op}} \rightarrow \text{Cat}$  is a functor s.t.  $X_n \xrightarrow{\langle p_{n,1}, \dots, p_{n,n} \rangle} X_1^n$  is an equivalence of categories ( $\mathbf{H} = \text{SG}, \text{M}, \text{CSG}$  or  $\text{CM}$ ), where  $\langle X_{p_{n,1}}, \dots, X_{p_{n,n}} \rangle \triangleq \langle p_{n,1}, \dots, p_{n,n} \rangle$  and  $p_{n,i}: n \rightarrow 1, 1 \leq i \leq n$ , are defined by  $p_{n,i}^{-1}(1) = \{i\}$ . Moreover, we assume  $X_0 = X_1^0 = \mathbf{1}$ . We say that the functor  $X: \mathbf{H}^{\text{op}} \rightarrow \text{Cat}$  preserves products up to equivalence.

Introduce the following notation: For each pair  $(i, j)$  of non-negative integers, let  $i+0: i+j \rightarrow i$  and  $0+j: i+j \rightarrow j$  in  $\Gamma_{\mathbf{H}}$  denote the projections on the first  $i$  and last  $j$  components, resp., and let  $F_{i,j}$  be a fixed inverse equivalence of  $\langle i+0, 0+j \rangle \triangleq \langle X_{i+0}, X_{0+j} \rangle: X_{i+j} \rightarrow X_i \times X_j$  in  $\text{Cat}$ .

We define the *tensor functor*  $\otimes: X_1^2 \rightarrow X_1$  by the following composition

$$X_1^2 \xrightarrow{F_{1,1}} X_2 \xrightarrow{b_1} X_1,$$

where  $b_1$  is induced by the total map  $b: 2 \rightarrow 1$  in  $\Gamma_{\mathbf{H}}$ .

Let  $\mathbf{1} \xrightarrow{\star} X_0$  be some inverse equivalence to  $X_0 = X^0 = \mathbf{1}$ , where  $\mathbf{1}$  is the trivial category. For  $\mathbf{H} = \text{M}$  or  $\text{CM}$  define the *unit object*  $I$  be the image of the composite functor  $e: \mathbf{1} \xrightarrow{\star} X_0 \xrightarrow{\phi} X_1$ , where  $X_0 \xrightarrow{\phi} X_1$  is induced by the unique map  $\phi: 0 \rightarrow 1$  in  $\Gamma_{\mathbf{H}}$ .

The *associativity isomorphism*  $a: \otimes(\otimes \times 1) \xrightarrow{\cdot} \otimes(1 \times \otimes)$  is then defined as the composition of the natural isomorphisms obtained from the following diagram. (In the sequel, we will denote  $X_m \times X_n$  by  $X_m X_n$ .)

$$\begin{array}{ccccc}
 X_1^3 & \xrightarrow{F_{1,1} \times 1} & X_2 X_1 & \xrightarrow{b_* \times 1} & X_1^2 \\
 \downarrow 1 \times F_{1,1} & & \downarrow F_{2,1} & & \downarrow F_{1,1} \\
 X_1 X_2 & \xrightarrow{F_{1,2}} & X_3 & \xrightarrow{(b+1)_*} & X_2 \\
 \downarrow 1 \times b_* & & \downarrow (1+b)_* & & \downarrow b_* \\
 X_1^2 & \xrightarrow{F_{1,1}} & X_2 & \xrightarrow{b_*} & X_1
 \end{array}$$

The *unit isomorphisms*

$$\begin{aligned}
 l: \otimes(e \times 1) &\xrightarrow{\cdot} P_2 \\
 r: \otimes(1 \times e) &\xrightarrow{\cdot} P_1
 \end{aligned}$$

are defined to be the natural isomorphisms obtained from the following two diagrams, respectively:

$$\begin{array}{ccccc}
 1 X_1 & \xrightarrow{* \times 1} & X_0 X_1 & \xrightarrow{\phi_* \times 1} & X_1^2 \\
 P_2 \downarrow & & P_2 \downarrow & & \downarrow F \\
 X_1 & \xrightarrow{id} & X_1 & \xrightarrow{i_{2*}} & X_2 \\
 id \downarrow & & id \downarrow & & \downarrow b_* \\
 X_1 & \xrightarrow{id} & X_1 & \xrightarrow{id} & X_1
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 X_1 1 & \xrightarrow{1 \times *} & X_1 X_0 & \xrightarrow{1 \times \phi_*} & X_1^2 \\
 P_1 \downarrow & & P_1 \downarrow & & \downarrow F \\
 X_1 & \xrightarrow{id} & X_1 & \xrightarrow{i_{1*}} & X_2 \\
 id \downarrow & & id \downarrow & & \downarrow b_* \\
 X_1 & \xrightarrow{id} & X_1 & \xrightarrow{id} & X_1
 \end{array}$$

where the functors  $i_{1*}: X_1 \rightarrow X_2$  and  $i_{2*}: X_1 \rightarrow X_2$  are induced by the injections in  $\Gamma_H$  onto the first and second components,  $i_1: 1 \rightarrow 2$  and  $i_2: 1 \rightarrow 2$ , respectively, the functors  $P_2: 1 X_1 \rightarrow X_1$  and  $P'_2: X_0 X_1 \rightarrow X_1$  are the second coordinate projections and the functors  $P_1: X_1 1 \rightarrow X_1$  and  $P'_1: X_1 X_0 \rightarrow X_1$  are the first coordinate projections.

Observe that  $P'_1$  and  $P'_2$  have inverse equivalences  $\langle 1, \phi'_1 \rangle$  and  $\langle \phi'_2, 1 \rangle$ , respectively, where  $\phi'_1: X_1 \rightarrow X_0$  is induced by the empty map  $1 \xrightarrow{\varphi} 0$ .

We define the *commutativity isomorphism*  $c: \otimes \xrightarrow{\cdot} \otimes (12)$  to be the natural isomorphism obtained from the following diagram:

$$\begin{array}{ccc}
 X_1^2 & \xrightarrow{(12)} & X_1^2 \\
 \downarrow F_{1,1} & & \downarrow F_{1,1} \\
 X_2 & \xrightarrow{\langle 12 \rangle} & X_2 \\
 \downarrow b_* & & \downarrow b_* \\
 X_1 & \xrightarrow{id} & X_1
 \end{array}$$

where the functor  $(12): X_1^2 \rightarrow X_1^2$  maps

$$\begin{array}{ccc}
 (A, B) & & (B, A) \\
 \downarrow (f, g) & \mapsto & \downarrow (g, f) \\
 (A', B') & & (B', A')
 \end{array}$$

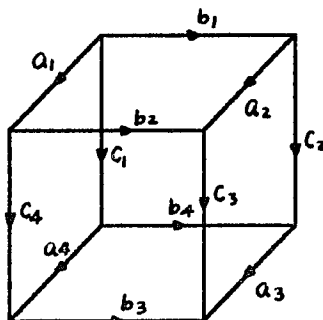
and the functor  $\langle 12 \rangle: X_2 \rightarrow X_2$  is induced by the transposition map  $2 \rightarrow 2$ .

**Theorem.** Let  $\mathbf{H} = \mathbf{SG}$  (resp.  $\mathbf{M}$ ,  $\mathbf{CSG}$ ,  $\mathbf{CM}$ ). If  $X: \mathbf{H}^{\text{op}} \rightarrow \mathbf{Cat}$  is a functor which preserves products up to equivalences, then  $X_1 = (X_1, \otimes, a)$  (resp.  $X_1 = (X_1, \otimes, l, a, l, r)$ ,  $X_1 = (X_1, \otimes, a, c)$ ,  $X_1 = (X_1, \otimes, l, a, c, l, r)$ ), as defined above, is a semigroupoidal (resp. monoidal, symmetric semigroupoidal, symmetric monoidal) category.

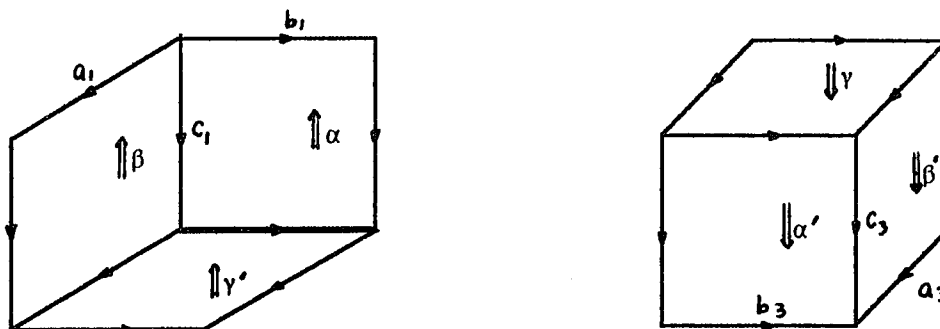
Conversely, given any semigroupoidal (resp. monoidal, symmetric semigroupoidal, symmetric monoidal) category  $\mathcal{A}$ , there exists a functor  $X: \Gamma_{\mathbb{H}} \rightarrow \mathit{Cat}$  such that  $X_n = \mathcal{A}^n$ , for all  $n$ .

#### 4. Cube Lemmas.

Let  $[\delta]$  represent the following cube



in which the vertices are categories, the edges are functors denoted by  $a_i, b_i, c_i, 1 \leq i \leq 4$ , as above and the faces are natural isomorphisms denoted  $\alpha, \alpha', \beta, \beta', \gamma$  and  $\gamma'$  for the back, front, left, right, top and bottom faces, respectively, as shown below.



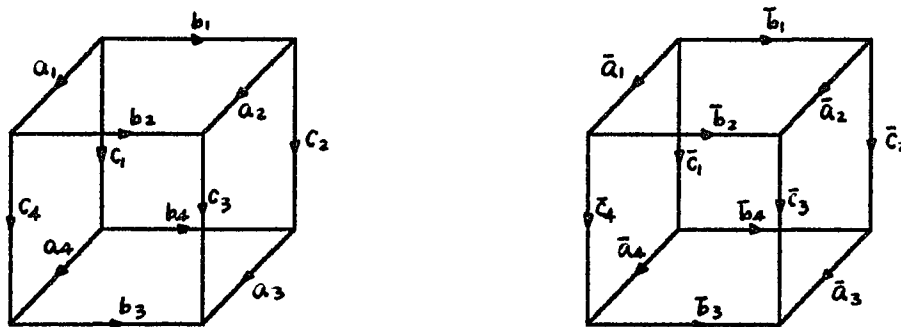
**Definition.** We say that the cube  $[\delta]$  is a *homotopy coherently commutative cube* if the composition of the sequence of six natural isomorphisms connecting the six composite functors from the initial to terminal vertex, namely

$$a_3c_2b_1 \xrightarrow{\beta b_2} c_3a_2b_1 \xrightarrow{c_3\gamma} c_3b_2a_1 \xrightarrow{\alpha'a} b_3c_4a_1 \xrightarrow{b_3\beta} b_3a_4c_1 \xrightarrow{\gamma c_1} a_3b_4c_1 \xrightarrow{a_3\alpha} a_3c_2b_1,$$

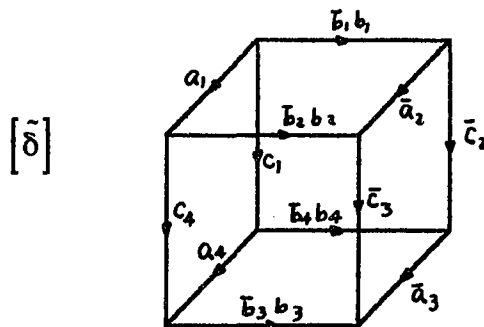
equals the identity .

**Note.** Identifying the domain and codomain of the above composition results in a hexagon diagram which commutes iff the cube  $[\delta]$  is homotopy coherently commutative.

Let us say we are provided with two cubes  $[\delta]$  and  $[\bar{\delta}]$ .



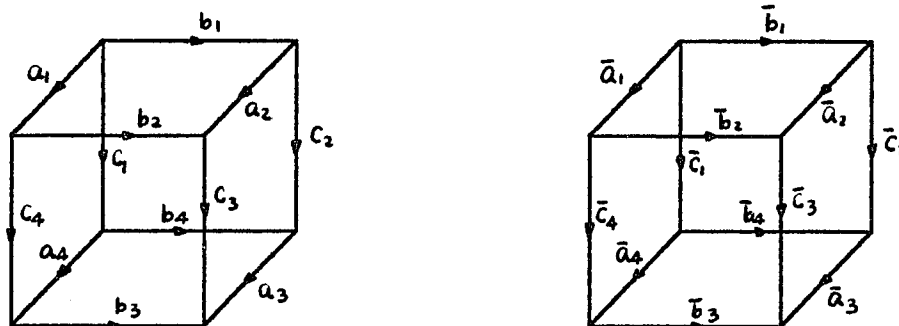
If  $a_2 = \bar{a}_1$ ,  $a_3 = \bar{a}_4$ ,  $c_2 = \bar{c}_1$  and  $c_3 = \bar{c}_4$ , we can paste the right face of  $[\delta]$  and the left face of  $[\bar{\delta}]$  together. We call the resulting cube  $[\tilde{\delta}]$  below the *(left-right) composition* of the two cubes  $[\delta]$  and  $[\bar{\delta}]$ .



Similarly, we can form their *(vertical, front-back) composition* if their bottom and top faces or their front and back faces, resp., are identical.

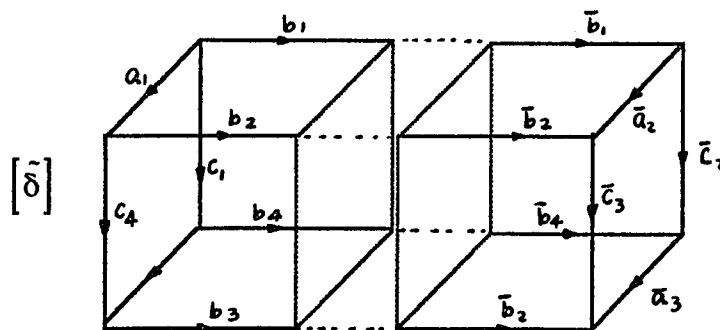
**Lemma 1. (The Pasting Lemma)** *The composition of two homotopy coherently commutative cubes is again a homotopy coherently commutative cube.*

**Proof:** Let



be two homotopy coherently commutative cubes.

Let  $\alpha, \alpha', \beta, \beta', \gamma$  and  $\gamma'$  and  $\bar{\alpha}, \bar{\alpha}', \bar{\beta}, \bar{\beta}', \bar{\gamma}$  and  $\bar{\gamma}'$  represent separately the back, front, left, right, top and bottom faces of  $[\delta]$  and  $[\bar{\delta}]$ , respectively. Assuming  $a_2 = \bar{a}_1, a_3 = \bar{a}_4, c_2 = \bar{c}_1, c_3 = \bar{c}_4$  and  $\beta' = \bar{\beta}^{-1}$ , we can form their left-right horizontal composition.



We show  $[\tilde{\delta}]$  is homotopy coherently commutative as follows.

Since  $[\delta]$  and  $[\bar{\delta}]$  are homotopy coherently commutative cubes, we then have

$$\begin{aligned}\delta &= (a_3\alpha)(\gamma'c_1)(b_3\beta)(\alpha'a_1)(c_3\gamma)(\beta'b_1) = id. \\ \bar{\delta} &= (\bar{a}_3\bar{\alpha})(\bar{\gamma}c_1)(\bar{b}_3\bar{\beta})(\bar{\alpha}'\bar{a}_1)(\bar{c}_3\bar{\gamma})(\bar{\beta}'\bar{b}_1) = id.\end{aligned}$$

We wish to show

$$\begin{aligned}\tilde{\delta} &= (\bar{a}_3\bar{\alpha}b_1)(\bar{a}_3\bar{b}_4\alpha)(\bar{\gamma}b_4c_1)(\bar{b}_3\gamma'c_1)(\bar{b}_3b_3\beta) \\ &\quad (\bar{b}_3\alpha'a_1)(\bar{\alpha}'b_2a_1)(\bar{c}_3\bar{b}_2\gamma)(\bar{c}_3\bar{\gamma}b_1)(\bar{\beta}'\bar{b}_1b_1) = id.\end{aligned}$$

By the interchange law ([M2], p.42-44), we have

$$\text{Fact. } (\bar{\alpha}'b_2a_1)(\bar{c}_3\bar{b}_2\gamma) = (\bar{b}_3c_3\gamma)(\bar{\alpha}'\bar{a}_1b_1) \quad \text{and} \quad (\bar{a}_3\bar{b}_4\alpha)(\bar{\gamma}b_4c_1) = (\bar{\gamma}\bar{c}_1b_1)(\bar{b}_3a_3\alpha).$$

Consequently,

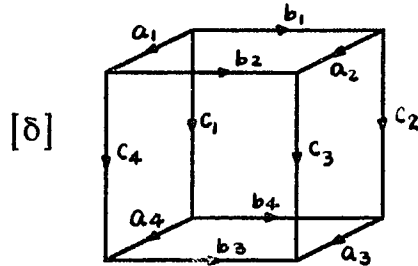
$$\begin{aligned}\tilde{\delta} &= \{[(\bar{a}_3\bar{\alpha})(\bar{\gamma}c_1)]b_1\} \{ \bar{b}_3[(a_3\alpha)(\gamma'c_1)(b_3\beta)(\alpha'a_1)(c_3\gamma)] \} \\ &\quad \{ [(\bar{\alpha}'\bar{a}_1)(\bar{c}_3\bar{\gamma})(\bar{\beta}'\bar{b}_1)]b_1 \} \\ &= \{[(\bar{a}_3\bar{\alpha})(\bar{\gamma}c_1)]b_1\} \{ \bar{b}_3\bar{\beta}b_1 \} \{ [(\bar{\alpha}'\bar{a}_1)(\bar{c}_3\bar{\gamma})(\bar{\beta}'\bar{b}_1)]b_1 \} \\ &= [(\bar{a}_3\bar{\alpha})(\bar{\gamma}c_1)(\bar{b}_3\bar{\beta})(\bar{\alpha}'\bar{a}_1)(\bar{c}_3\bar{\gamma})(\bar{\beta}'\bar{b}_1)]b_1 \\ &= \bar{\delta}b_1 \\ &= id.\end{aligned}$$

Similarly, we can prove that the front-back horizontal composition or the top-bottom vertical composition of two homotopy coherently commutative cubes is homotopy coherently commutative.

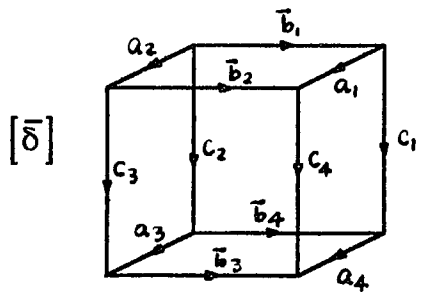
**Q.E.D.**

**Corollary.** *If  $[\delta]$  is a homotopy coherently commutative cube, then  $[\tilde{\delta}]$  is a homotopy coherently commutative cube if and only if  $[\bar{\delta}]$  is a homotopy coherently commutative cube.*

**Lemma 2. (The Cancellation Lemma).** *If a cube*



*is homotopy coherently commutative and  $b_i, i=1,2,3,4,$  are homotopy equivalences with inverse equivalences  $\bar{b}_i,$  units  $\eta_i$  and counits  $\epsilon_i, 1 \leq i \leq 6,$  resp., then the cube*



*is homotopy coherently commutative.*

**Proof.** By the corollary of Lemma 1, we only need to show that the left-right horizontal composition  $[\tilde{\delta}]$  is homotopy coherently commutative, i.e.  $\tilde{\delta} = id$ . We have

$$\begin{aligned}\tilde{\delta} &= (a_4 \bar{\alpha} b_1) (a_4 \bar{b}_4 \alpha) (\bar{\gamma} b_4 c_1) (\bar{b}_3 \gamma' c_1) (\bar{b}_3 b_3 \beta) \\ &\quad (\bar{b}_3 \alpha' a_1) (\bar{\alpha} b_2 a_1) (c_4 \bar{b}_2 \gamma) (c_4 \bar{\gamma} b_1) (\bar{\beta}' \bar{b}_1 b_1)\end{aligned}$$

The following two commutative diagrams

$$\begin{array}{ccc} \bar{b}_2 b_2 a_1 \bar{b}_1 & \xrightarrow{\varepsilon_2 \gamma^{-1} \bar{\varepsilon}_1} & \bar{b}_2 a_2 b_1 \bar{b}_1 \\ \uparrow \varepsilon_2^{-1} a_1 \bar{b}_1 & & \uparrow \bar{b}_2 a_2 \eta_1 \\ a_1 \bar{b}_1 & \xrightarrow{\bar{\gamma}} & \bar{b}_2 a_2 \end{array} \quad \begin{array}{ccc} b_1 & \xrightarrow{\eta_1 b_1} & b_1 \bar{b}_1 b_1 \\ \swarrow id & & \downarrow b_1 \varepsilon_1 \\ & & b_1 \end{array}$$

and the interchange law, yield

$$\begin{aligned}(\bar{b}_2 \gamma) (\bar{\gamma} b_1) &= (\bar{b}_2 \gamma) (\bar{b}_2 a_2 \eta_1^{-1} b_1) (\bar{b}_2 \gamma^{-1} \bar{b}_1 b_1) (\varepsilon_2^{-1} a_1 \bar{b}_1 b_1) \\ &= (\bar{b}_2 \gamma) (\bar{b}_2 a_2 b_1 \varepsilon_1) (\bar{b}_2 \gamma^{-1} \bar{b}_1 b_1) (\varepsilon_2^{-1} a_1 \bar{b}_1 b_1) \\ &= (\bar{b}_2 b_2 a_1 \varepsilon_1) (\bar{b}_2 \gamma \bar{b}_1 b_1) (\bar{b}_2 \gamma^{-1} \bar{b}_1 b_1) (\varepsilon_2^{-1} a_1 \bar{b}_1 b_1) \\ &= (\bar{b}_2 b_2 a_1 \varepsilon_1) (\varepsilon_2^{-1} a_1 \bar{b}_1 b_1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(\bar{b}_3 \alpha') (\bar{\alpha}' b_2) &= (\bar{b}_3 b_3 c_4 \varepsilon_2) (\varepsilon_3^{-1} c_4 \bar{b}_2 b_2), \\ (\bar{\gamma} b_4) (\bar{b}_3 \gamma') &= (\varepsilon_3 a_4 \bar{b}_4 b_4) (\bar{b}_3 b_3 a_4 \varepsilon_4^{-1}), \\ (\bar{\alpha} b_1) (\bar{b}_4 \alpha) &= (\varepsilon_4 c_1 \bar{b}_1 b_1) (\bar{b}_4 b_4 c_1 \varepsilon_1^{-1}).\end{aligned}$$

Hence,

$$\begin{aligned}\tilde{\delta} &= (a_4 \varepsilon_4 c_1 \bar{b}_1 b_1) (a_4 \bar{b}_4 b_4 c_1 \varepsilon_1^{-1}) (\varepsilon_3 a_4 \bar{b}_4 b_4 a_1) (\bar{b}_3 b_3 a_4 \varepsilon_4^{-1} c_1) (\bar{b}_3 b_3 \beta) \\ &\quad (\bar{b}_3 b_3 c_4 \varepsilon_2 a_1) (\varepsilon_2^{-1} c_4 \bar{b}_2 b_2 a_1) (c_4 \bar{b}_2 b_2 a_1 \varepsilon_1) (c_4 \varepsilon_2^{-1} a_1 \bar{b}_1 b_1) (\bar{\beta}' \bar{b}_1 b_1).\end{aligned}$$

By the interchange law

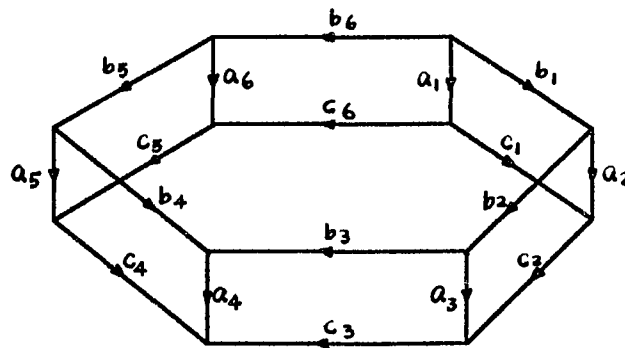
$$\begin{aligned}
(c_4 \bar{b}_2 b_2 a_1 \epsilon_1)(c_4 \epsilon_2^{-1} a_1 \bar{b}_1 b_1) &= (c_4 \epsilon_2^{-1} a_1)(c_4 a_1 \epsilon_1), \\
(\bar{b}_3 b_3 c_4 \epsilon_2 a_1)(\epsilon_3^{-1} c_4 \bar{b}_2 b_2 a_1) &= (\epsilon_3^{-1} c_4 a_1)(c_4 \epsilon_2 a_1), \\
(\epsilon_3 a_4 \bar{b}_4 b_4 c_1)(\bar{b}_3 b_3 a_4 \epsilon_4^{-1} c_1) &= (a_4 \epsilon_4^{-1} c_1)(\epsilon_3 a_4 c_1), \\
(a_4 \epsilon_4 c_1 \bar{b}_1 b_1)(a_4 \bar{b}_4 b_4 c_1 \epsilon_1^{-1}) &= (a_4 c_1 \epsilon_1^{-1})(a_4 \epsilon_4 c_1), \\
(\bar{b}_3 b_3 \beta)(\epsilon_3^{-1} c_4 a_1) &= (\epsilon_3^{-1} a_4 c_1) \beta \\
\text{and } (c_4 a_1 \epsilon_1)(\bar{\beta}' \bar{b}_1 b_1) &= \bar{\beta}'(a_4 c_1 \epsilon_1).
\end{aligned}$$

From this it follows that

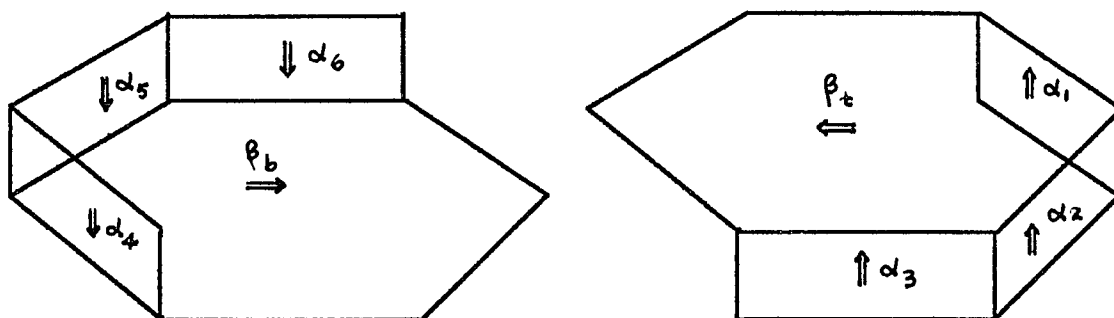
$$\begin{aligned}
\tilde{\delta} &= (a_4 c_1 \epsilon_1^{-1})(a_4 \epsilon_4 c_1)(a_4 \epsilon_4^{-1} c_1)(\epsilon_4 a_4 c_1)(\bar{b}_3 b_3 \beta) \\
&\quad (\epsilon_3^{-1} c_4 a_1)(c_4 \epsilon_2 a_1)(c_4 \epsilon_2^{-1} a_1)(c_4 a_1 \epsilon_1)(\bar{\beta}' \bar{b}_1 b_1) \\
&= (a_4 c_1 \epsilon_1^{-1})(\epsilon_3 a_4 c_1)(\bar{b}_3 b_3 \beta)(\epsilon_3^{-1} c_4 a_1)(c_4 a_1 \epsilon_1)(\bar{\beta}' \bar{b}_1 b_1) \\
&= (a_4 c_1 \epsilon_1^{-1})(\epsilon_3 a_4 c_1)(\epsilon_3^{-1} a_4 c_1) \beta \bar{\beta}'(a_4 c_1 \epsilon_1) \\
&= id.
\end{aligned}$$

Q.E.D.

**Note.** In a similar manner we may define the concept of a *homotopy coherently commutative prism*. More precisely, the prism  $\langle \delta \rangle$



in which the vertices are categories, the edges  $a_i, b_i, c_i$ ,  $1 \leq i \leq 6$  are functors, and the faces  $\alpha_i$ ,  $1 \leq i \leq 6$ ,  $\beta_i$  and  $\beta_b$  are natural isomorphisms, as shown below



is *homotopy coherently commutative* if the composition of the eight natural isomorphisms

$$\begin{aligned}
 & c_3 c_2 c_1 a_1 \xrightarrow{c_3 c_2 a_1} c_3 c_2 a_2 b_1 \xrightarrow{c_3 a_2 b_1} c_3 a_3 b_2 b_1 \xrightarrow{a_3 b_2 b_1} a_4 b_3 b_2 b_1 \xrightarrow{a_4 b_1} a_4 b_4 b_5 b_6 \\
 & \xrightarrow{a_4 b_5 b_6} c_4 a_5 b_5 b_6 \xrightarrow{c_4 a_5 b_6} c_4 c_5 a_6 b_6 \xrightarrow{c_4 c_5 a_6} c_4 c_5 c_6 a_1 \xrightarrow{\beta_b a_1} c_3 c_2 c_1 a_1
 \end{aligned}$$

equals the identity.

**Note.** The prism  $\langle \delta \rangle$  is homotopy coherently commutative iff the associated octagon diagram commutes.

**Lemma 1'.** (a) *If we are given two homotopy coherently commutative prisms (fig. 1.1) with  $\bar{b}_i = c_i$ ,  $1 \leq i \leq 6$ , then the top-bottom vertical composition  $\langle \tilde{\delta} \rangle$  (fig. 1.2) of  $\langle \delta \rangle$  and  $\langle \bar{\delta} \rangle$  is again a homotopy coherently commutative prism.*

(b) *If instead of  $\bar{b}_i = c_i$ ,  $1 \leq i \leq 6$  we had  $\bar{a}_1 = a_3$ ,  $\bar{a}_6 = a_4$ ,  $\bar{b}_6 = b_3$ ,  $\bar{c}_6 = c_3$ , then the front-back horizontal composition  $\langle \bar{\delta} \rangle$  of  $\langle \delta \rangle$  and  $\langle \bar{\delta} \rangle$  (fig.*

2.1 & 2.2) is homotopy coherently commutative, i.e. the composition of the twelve natural isomorphisms.

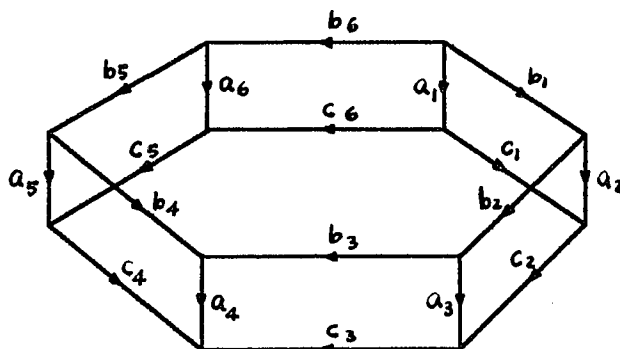
$$\begin{aligned}
 & \tau_3 \tau_2 \tau_1 c_2 c_1 a_1 \xrightarrow{\tau_3 \tau_2 \tau_1 c_2 a_1} \tau_3 \tau_2 \tau_1 c_2 a_2 b_1 \xrightarrow{\tau_3 \tau_2 \tau_1 a_2 b_1} \tau_3 \tau_2 \tau_1 a_3 b_2 b_1 \xrightarrow{\tau_3 \tau_2 \tau_1 b_2 b_1} \tau_3 \tau_2 \bar{a}_1 \bar{b}_1 b_2 b_1 \\
 & \xrightarrow{\tau_3 \tau_2 \bar{b}_1 b_2 b_1} \tau_3 \bar{a}_3 \bar{b}_2 \bar{b}_1 b_2 b_1 \xrightarrow{\tau_3 \tau_2 \bar{b}_1 b_2 b_1} \bar{a}_4 \bar{b}_3 \bar{b}_2 \bar{b}_1 b_2 b_1 \xrightarrow{\tau_4 \beta_1} \bar{a}_4 \bar{b}_4 \bar{b}_5 b_4 b_5 b_6 \\
 & \xrightarrow{\tau_4 \bar{b}_4 b_5 b_6} \tau_4 \bar{a}_5 \bar{b}_5 b_4 b_5 b_6 \xrightarrow{\tau_4 \tau_3 \bar{a}_4 b_5 b_6} \tau_4 \tau_3 \bar{a}_4 b_4 b_5 b_6 \xrightarrow{\tau_4 \tau_3 \tau_2 \bar{a}_4 b_5 b_6} \tau_4 \tau_3 \tau_2 c_4 a_5 b_5 b_6 \\
 & \xrightarrow{\tau_4 \tau_3 \tau_2 c_4 a_5 b_6} \tau_4 \tau_3 \tau_2 c_4 c_5 a_6 b_6 \xrightarrow{\tau_4 \tau_3 \tau_2 c_4 c_5 a_6} \tau_4 \tau_3 \tau_2 c_4 c_5 c_6 a_1 \xrightarrow{\beta_2 \tau_1} \tau_3 \tau_2 \tau_1 c_2 c_1 a_1
 \end{aligned}$$

is equal to the identity.

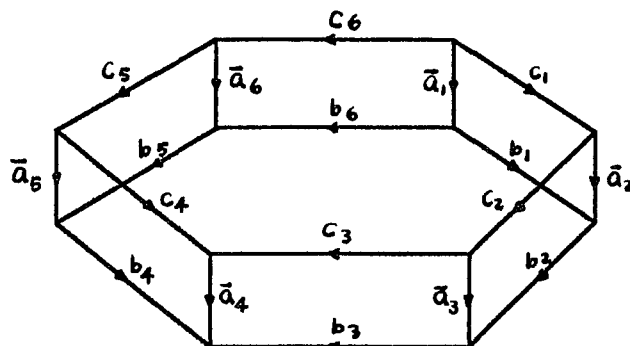
(c) Given a homotopy coherently commutative polyhedron  $\langle \tilde{\delta} \rangle$ , as in part (b), and two homotopy coherently commutative cubes  $[\delta_l]$  and  $[\delta_r]$  (fig. 3.1), the composition of  $[\delta_l]$ ,  $\langle \tilde{\delta} \rangle$  and  $[\delta_r]$  (fig. 3.2) (along their identical faces) is again homotopy coherently commutative.

**Proof.** The proofs of parts (a) and (b) are straightforward. The proof of part (c) is in the appendix.

**Lemma 2' (Cancellation Lemma).** Suppose that the prism



is homotopy coherently commutative and  $a_i, 1 \leq i \leq 6$  are homotopy equivalences with inverse equivalences  $\bar{a}_i$ , units  $\eta_i$  and counits  $\varepsilon_i$ ,  $1 \leq i \leq 6$ . Then the prism



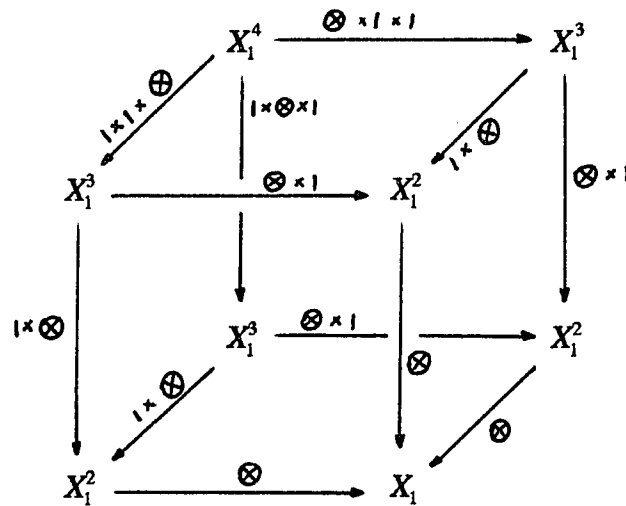
is also homotopy coherently commutative.

The proof of the cancellation lemma for the prism will be omitted since it is analogous to the proof of the cancellation lemma for the cube.

## 5. Proof Of The Theorem.

We first show that the associativity pentagon diagram commutes.

The cube



is homotopy coherently commutative if and only if the composition

$$\begin{aligned}
 & \otimes(\otimes \times 1)(\otimes \times 1 \times 1) \xrightarrow{\alpha(\otimes \times 1 \times 1)} \otimes(1 \times \otimes)(\otimes \times 1 \times 1) \xrightarrow{id} \otimes(\otimes \times 1)(1 \times 1 \times \otimes) \\
 & \xrightarrow{\alpha(1 \times 1 \times \otimes)} \otimes(1 \times \otimes)(1 \times 1 \times \otimes) \xrightarrow{\otimes(1 \times \alpha^{-1})} \otimes(1 \times \otimes)(1 \times \otimes \times 1) \\
 & \xrightarrow{\alpha^{-1}(1 \times \otimes \times 1)} \otimes(\otimes \times 1)(1 \times \otimes \times 1) \xrightarrow{\otimes(\alpha^{-1} \times 1)} \otimes(\otimes \times 1)(\otimes \times 1 \times 1)
 \end{aligned}$$

is equal to the identity. By the note following the definition of homotopy coherently commutative cubes, this holds precisely when the associativity pentagon commutes.

Replace the cube edges  $\otimes, \otimes \times 1, 1 \times \otimes, \otimes \times 1 \times 1, 1 \times \otimes \times 1$  and  $1 \times 1 \times \otimes$  by  $b_* F_{1,1}, (b_* \times 1)(F_{1,1} \times 1), (1 \times b_*)(1 \times F_{1,1}), (b_* \times 1 \times 1)(F_{1,1} \times 1 \times 1), (1 \times b_* \times 1)(1 \times F_{1,1} \times 1)$  and  $(1 \times 1 \times b_*)(1 \times 1 \times F_{1,1})$ , resp.

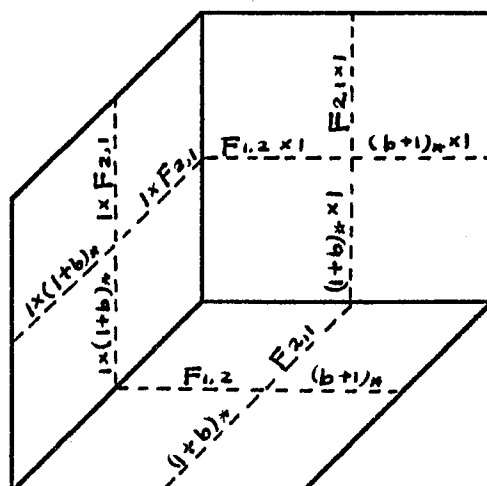
On the top face of the cube paste the diagram:

$$\begin{array}{ccccc}
 X_1^4 & \xrightarrow{F_{1,1} \times 1} & X_2 X_1^2 & \xrightarrow{b_* \times 1} & X_1^3 \\
 \downarrow 1 \times F_{1,1} & & \downarrow 1 \times F_{1,1} & & \downarrow 1 \times F_{1,1} \\
 X_1^2 X_2 & \xrightarrow{F_{1,1} \times 1} & X_2^2 & \xrightarrow{b_* \times 1} & X_1 X_2 \\
 \downarrow 1 \times b_* & & \downarrow 1 \times b_* & & \downarrow 1 \times b_* \\
 X_1^3 & \xrightarrow{F_{1,1} \times 1} & X_2 X_1 & \xrightarrow{b_* \times 1} & X_1^2
 \end{array}$$

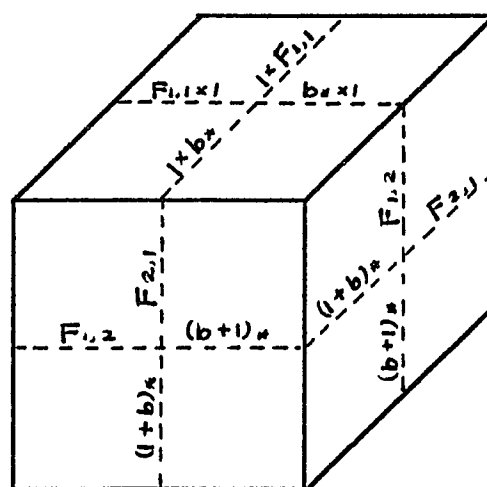
On the right, front and bottom faces, paste the defining diagram for associativity:

$$\begin{array}{ccccc}
 X_1^3 & \xrightarrow{F_{1,1} \times 1} & X_2 X_1 & \xrightarrow{b_* \times 1} & X_1^2 \\
 \downarrow 1 \times F_{1,1} & & \downarrow F_{2,1} & & \downarrow F_{1,1} \\
 X_1 X_2 & \xrightarrow{F_{1,2}} & X_3 & \xrightarrow{(b+1)_*} & X_2 \\
 \downarrow 1 \times b_* & & \downarrow (1+b)_* & & \downarrow b_* \\
 X_1^2 & \xrightarrow{F_{1,1}} & X_2 & \xrightarrow{b_*} & X_1
 \end{array}$$

Finally, on the left and back faces, paste the diagrams corresponding to  $1 \times \alpha$  and  $\alpha \times 1$ , respectively. The reader may refer to the following two figures:



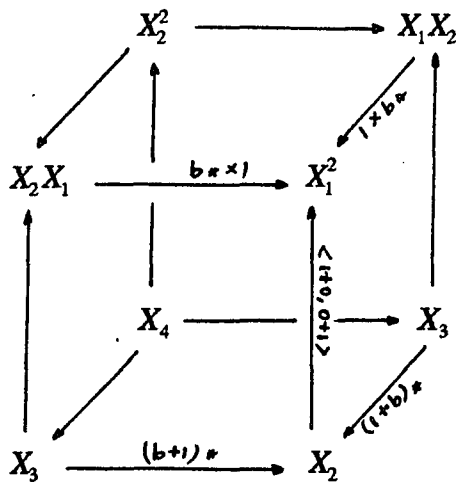
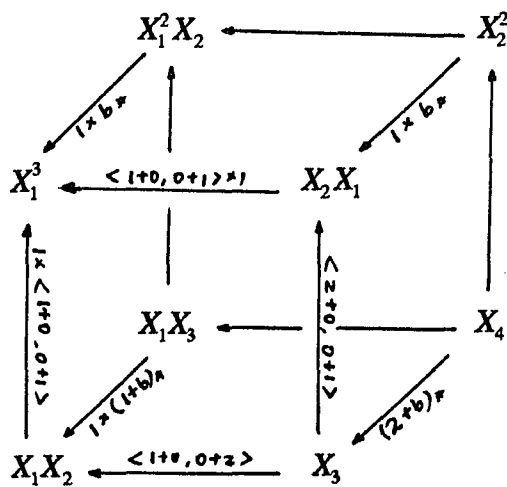
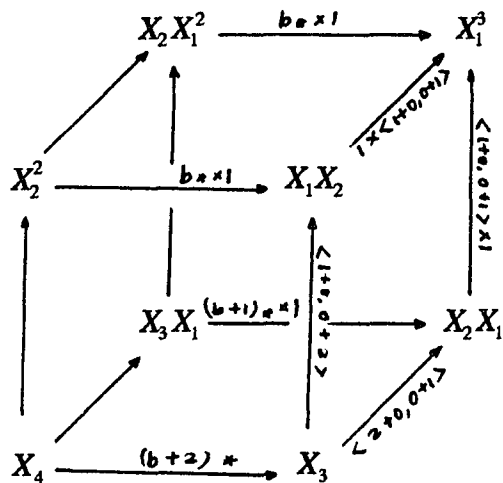
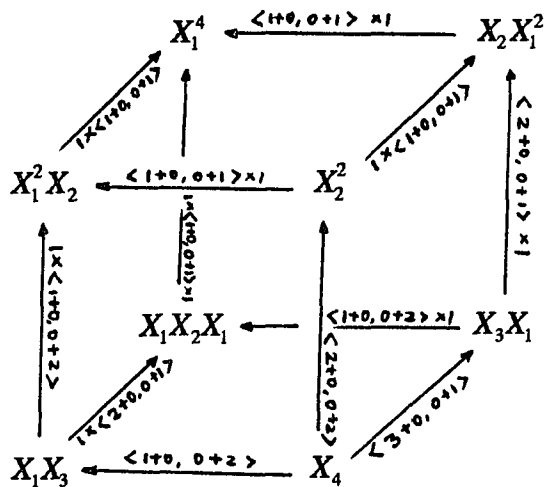
The subdivision of the back three faces.



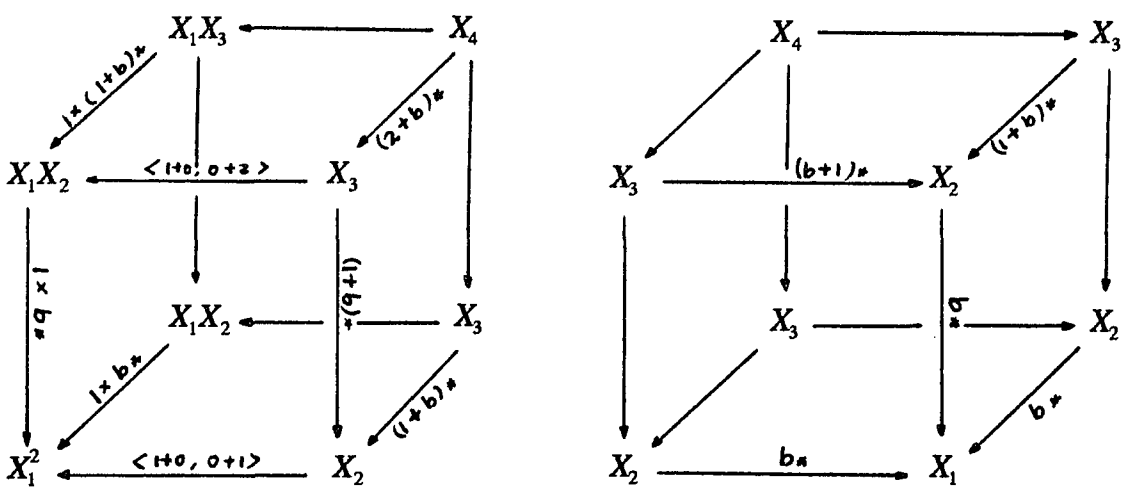
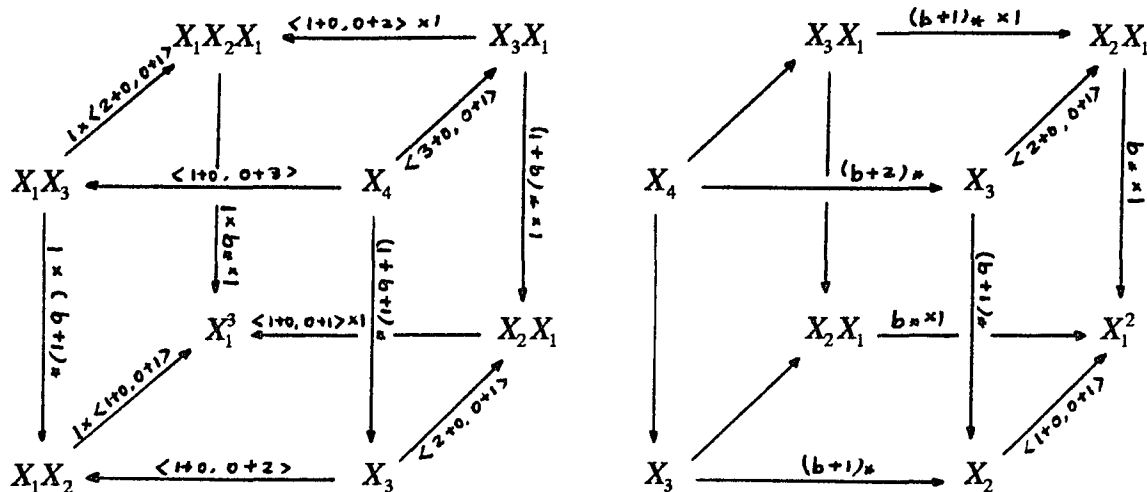
The subdivision of the front three faces.

Replace each  $F_{i,j}$  by  $\langle i+0, 0+j \rangle$  and then extend the subdivision of the faces into a subdivision of the cube. We then get eight subcubes as follows (see the next two pages).

The following four subcubes combine to form the top half of the cube:



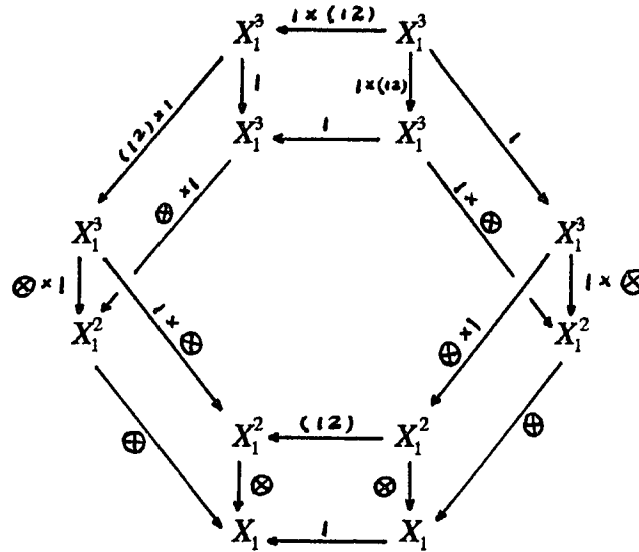
The next four form the bottom half:



Each of the eight subcubes is homotopy coherently commutative since all of its faces literally commute. Replace each  $\langle i+0, 0+j \rangle$  by  $F_{i,j}$ . By the cancellation lemma, the resulting subcubes are still homotopy coherently commutative. Pasting these together gives us

back the original cube. But, by the pasting lemma, it must be homotopy coherently commutative as well.

The prism



is homotopy coherently commutative iff the composition

$$\begin{aligned} & \otimes(1 \times \otimes)(1 \times (12)) \xrightarrow{\otimes(1 \times c^{-1})} \otimes(1 \times \otimes) \xrightarrow{a^{-1}} \otimes(\otimes \times 1) \xrightarrow{a(\otimes \times 1)} \otimes(12)(\otimes \times 1) \\ & \xrightarrow{id} \otimes(1 \times \otimes)((12) \times 1)(1 \times (12)) \xrightarrow{a^{-1}((12) \times 1)(1 \times (12))} \otimes(\otimes \times 1)((12) \times 1)(1 \times (12)) \\ & \xrightarrow{\otimes(c^{-1} \times 1)(1 \times (12))} \otimes(\otimes \times 1)(1 \times (12)) \xrightarrow{id} \otimes(\otimes \times 1)(1 \times (12)) \xrightarrow{a(1 \times (12))} \otimes(1 \times \otimes)(1 \times (12)) \end{aligned}$$

is equal to the identity. This holds precisely when the symmetry hexagon commutes.

As in the case of the cube, we replace all occurrences of  $\otimes$  on the edges of the prism by  $b.F_{1,1}$ . We then paste the defining diagram for associativity

$$\begin{array}{ccccc}
 X_1^3 & \xrightarrow{F_{1,1} \times 1} & X_2 X_1 & \xrightarrow{b \times 1} & X_1^2 \\
 \downarrow 1 \times F_{1,1} & & \downarrow F_{2,1} & & \downarrow F_{1,1} \\
 X_1 X_2 & \xrightarrow{F_{1,2}} & X_3 & \xrightarrow{(b+1)b} & X_2 \\
 \downarrow 1 \times b_* & & \downarrow (1+b)_* & & \downarrow b_* \\
 X_1^2 & \xrightarrow{F_{1,1}} & X_2 & \xrightarrow{b_*} & X_1
 \end{array}$$

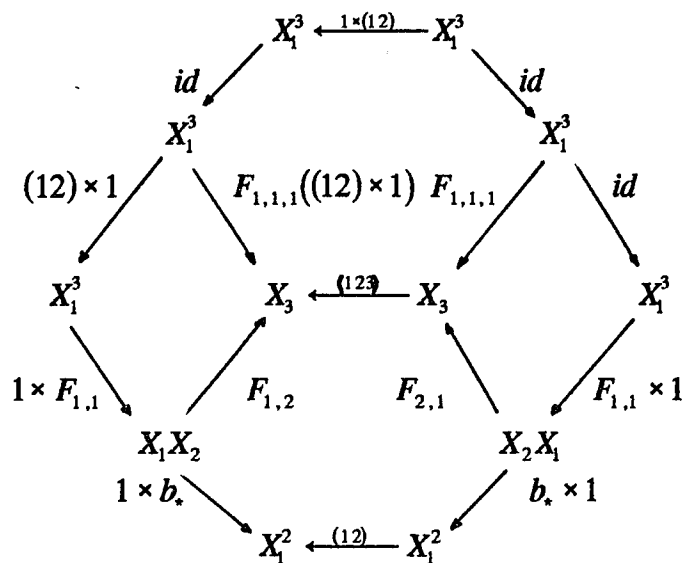
on the right-front, left-front and bottom faces each; the defining diagram for commutativity

$$\begin{array}{ccc}
 X_1^2 & \xleftarrow{(12)} & X_1^2 \\
 \downarrow F_{1,1} & & \downarrow F_{1,1} \\
 X_2 & \xleftarrow{(12)} & X_2 \\
 \downarrow b_* & & \downarrow b_* \\
 X_1 & \xleftarrow{id} & X_1
 \end{array}$$

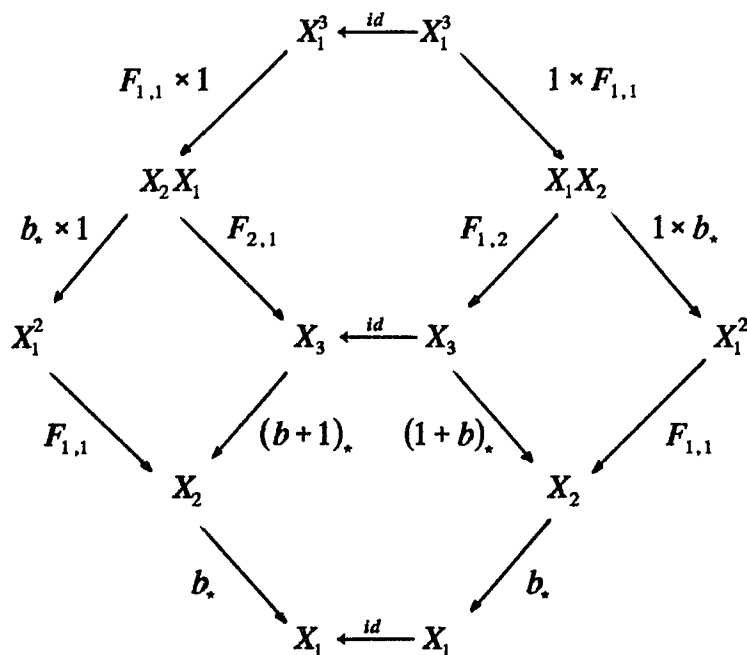
on the front face; the diagrams corresponding to  $1 \times c$  and  $c \times 1$  on the right-back and left-back faces, respectively; the following diagram

$$\begin{array}{ccc}
 X_1^3 & \xleftarrow{1 \times (12)} & X_1^3 \\
 \downarrow id & & \downarrow id \\
 X_1^3 & \xleftarrow{1 \times (12)} & X_1^3 \\
 \downarrow id & & \downarrow 1 \times (12) \\
 X_1^3 & \xleftarrow{id} & X_1^3
 \end{array}$$

on the back face; the following diagram



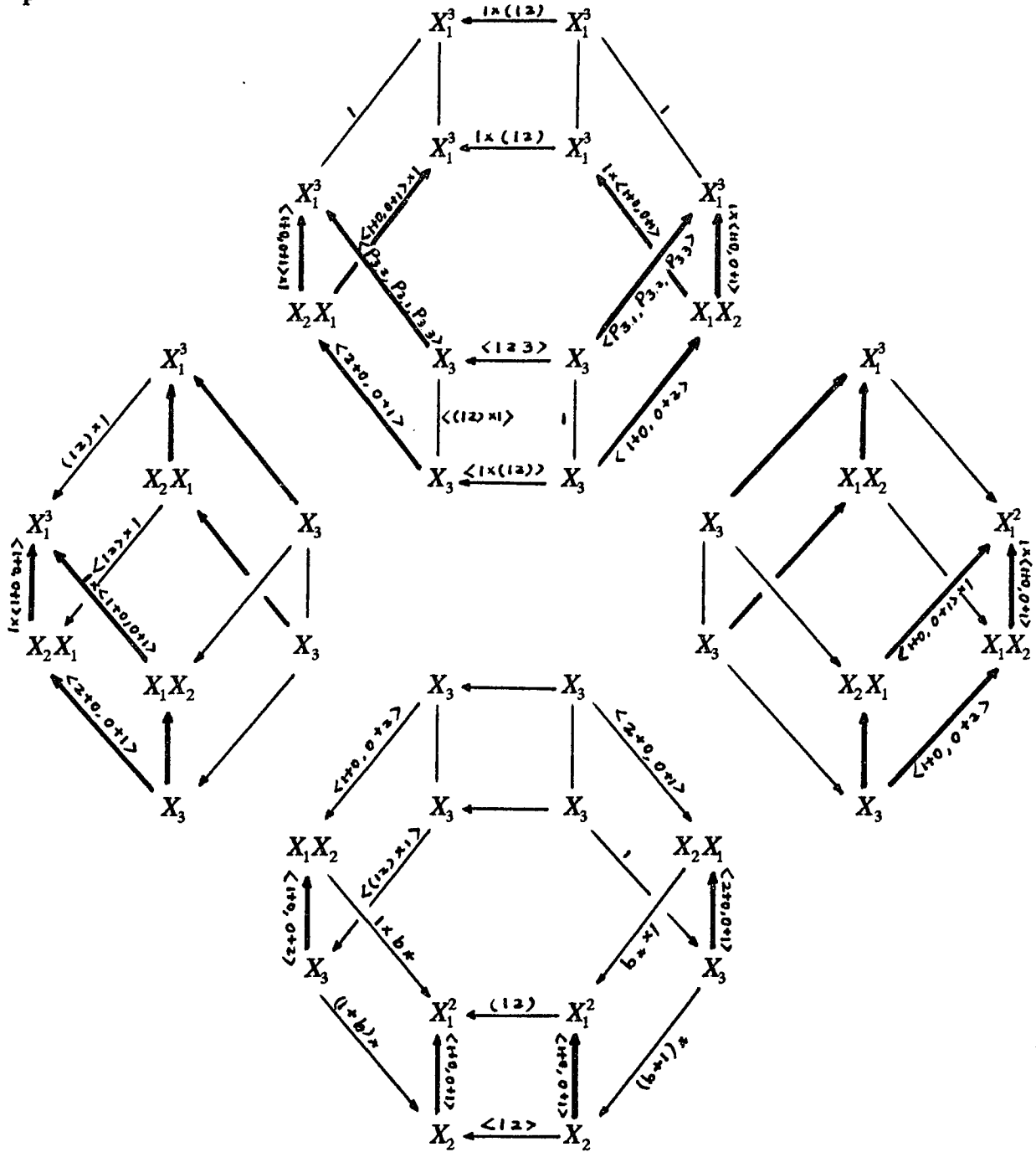
on the top face, where  $\langle 123 \rangle$  is induced by the 3 cycle  $3 \rightarrow 3$  in  $\Gamma_H$ , and  $F_{1,1,1}$  is some inverse equivalence of  $\langle p_{3,1}, p_{3,2}, p_{3,3} \rangle$  in  $Cat$ ; and the following diagram



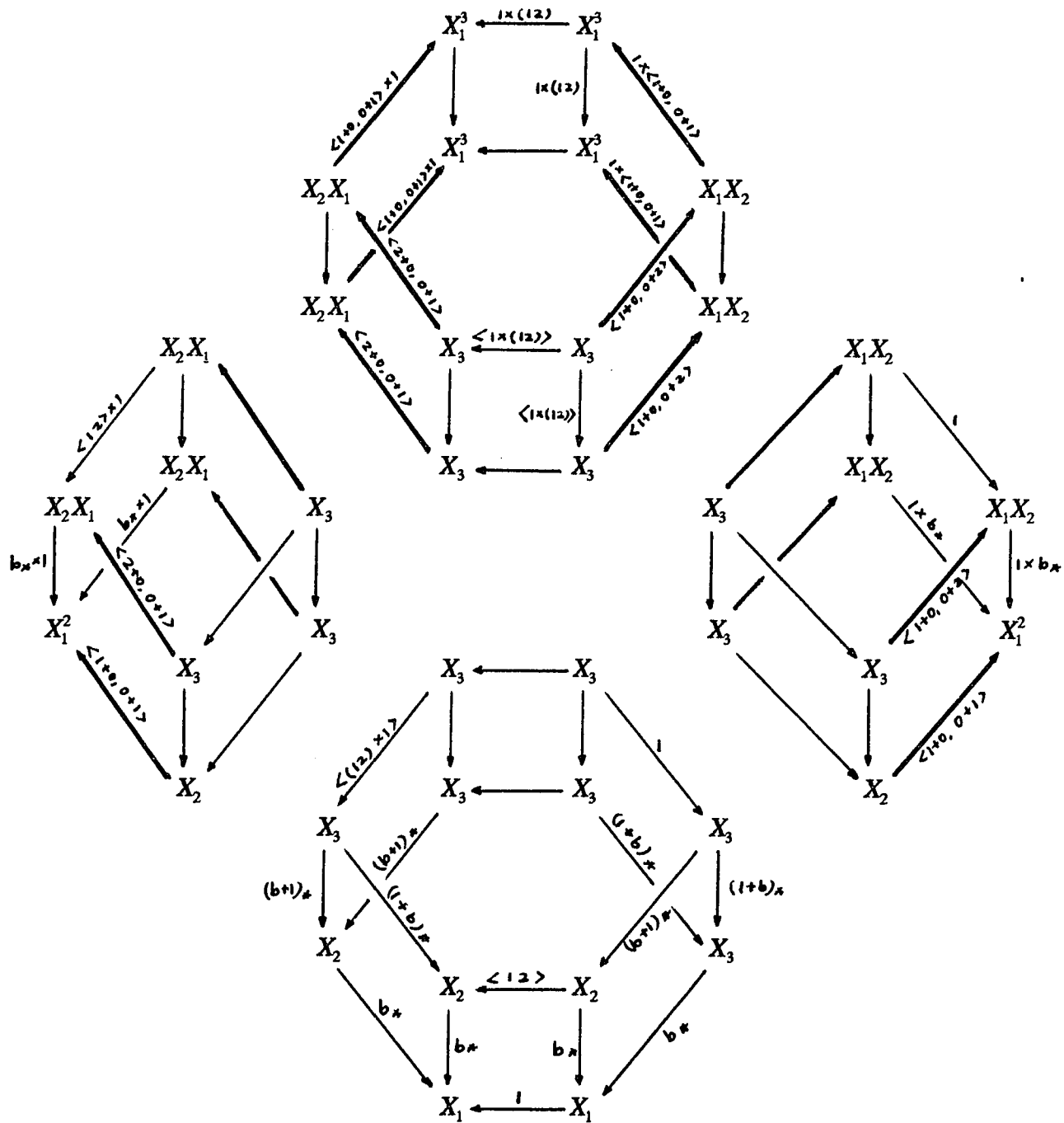
on the bottom face. Next we replace each  $F_{i,j}$  by  $\langle i+0, 0+j \rangle$  and each

$F_{1,1,1}$  by  $\langle p_{3,1}, p_{3,2}, p_{3,3} \rangle$  and extend the subdivision of the faces into the following subdivision of the prism.

The following four polyhedra combine to form the top half of the prism:



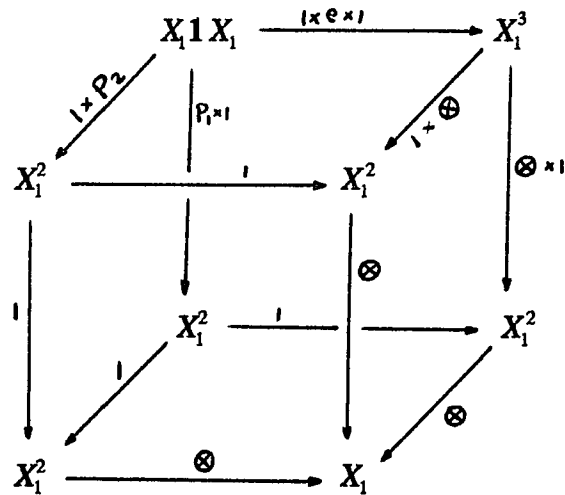
The next four combine to form the bottom half:



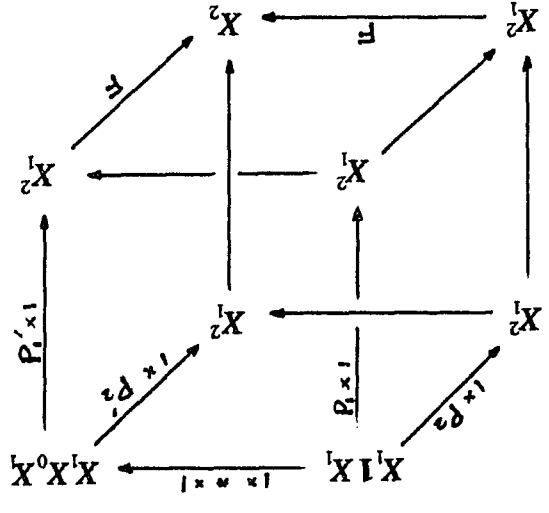
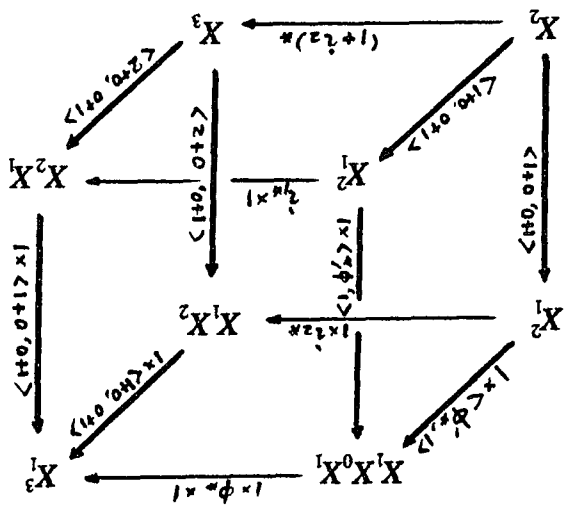
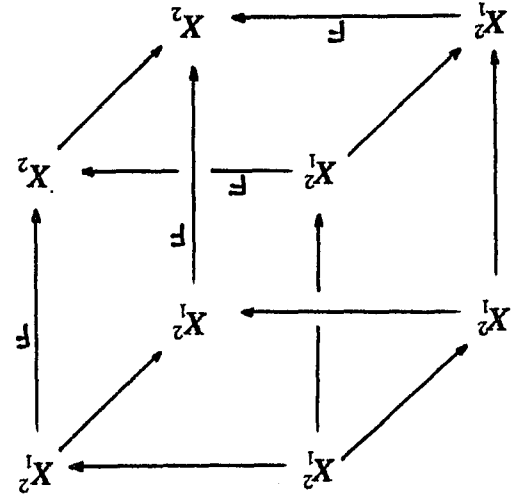
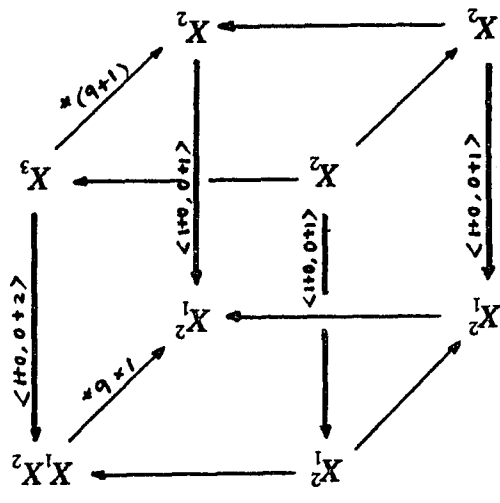


$$\begin{aligned}
 \alpha(12) \circ \langle 12 \rangle \alpha &= (F(12)\eta^{-1}(12))(\epsilon^{-1}\langle 12 \rangle F(12)) \\
 &\quad \left( \langle 12 \rangle F(12)\eta^{-1} \right) \left( \langle 12 \rangle \epsilon^{-1}\langle 12 \rangle F \right) \\
 &= (F(12)(12)\eta^{-1})(F(12)\eta^{-1} \langle 1+0, 0+1 \rangle \langle 12 \rangle F) \\
 &\quad (F \langle 1+0, 0+1 \rangle \langle 12 \rangle \epsilon^{-1}\langle 12 \rangle F) (\epsilon^{-1}\langle 12 \rangle \langle 12 \rangle F) \\
 &= (F\eta^{-1}) \left( F(12)\eta^{-1} \langle 1+0, 0+1 \rangle \langle 12 \rangle F \right) \\
 &\quad (F(12) \langle 1+0, 0+1 \rangle \epsilon^{-1}\langle 12 \rangle F) (\epsilon^{-1}F) \\
 &= (F\eta^{-1}) \left( F(12)\eta^{-1} \langle 1+0, 0+1 \rangle \langle 12 \rangle F \right) \\
 &\quad (F(12)\eta \langle 1+0, 0+1 \rangle \langle 12 \rangle F) (F\eta) \\
 &= id.
 \end{aligned}$$

Similarly, the triangle for unit may be placed on a cube:

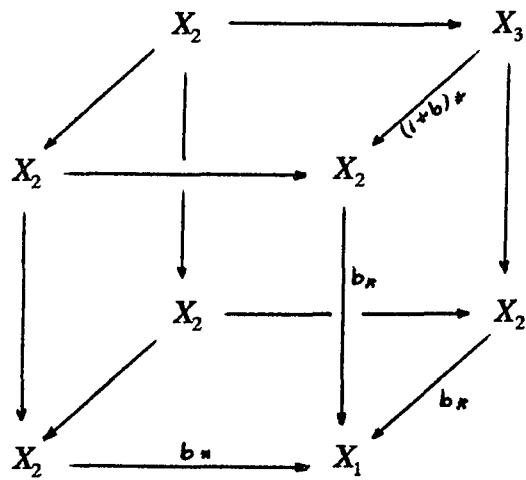
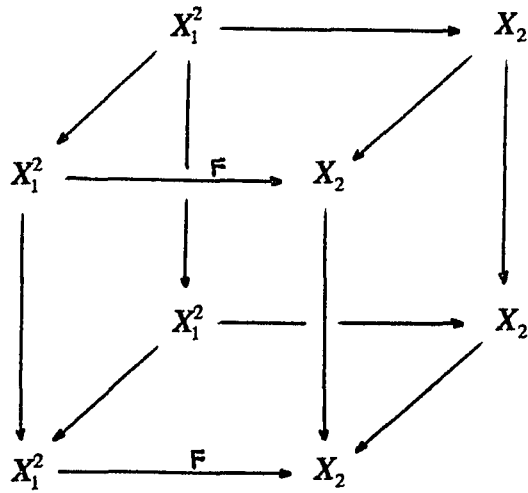
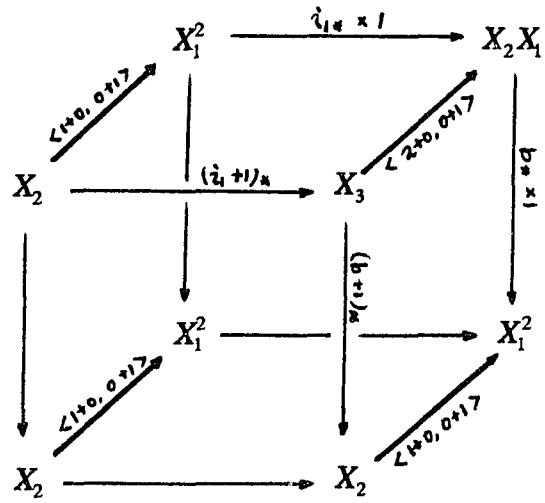
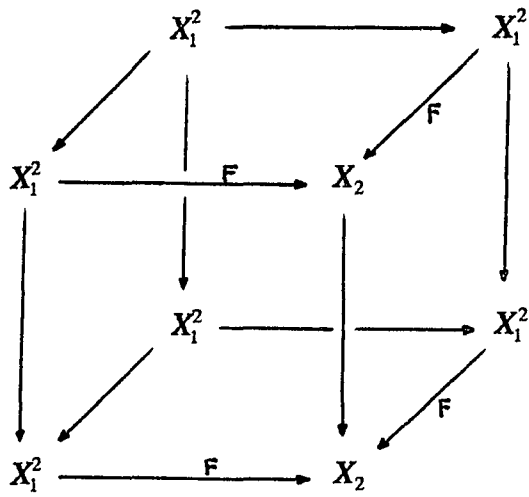


Subdividing yields eight homotopy coherently commutative subcubes.



The top four subcases:

The bottom four subcubes:



The rest of the details are like in the previous cases. This concludes the proof of our theorem.

**Q.E.D.**

## 6. Proof of the Converse.

We will prove the converse only for the symmetric monoidal case, the proof for the other cases being similar. For a given symmetric monoidal category  $\mathcal{A}$ , we construct a functor  $\bar{X} : \Gamma_{\text{CM}} \rightarrow \text{Cat}$  s.t.  $\bar{X}_n = \mathcal{A}^n$  for all  $n$ . For simplicity we write  $\Gamma$  instead of  $\Gamma_{\text{CM}}$ .

6.1. Briefly, an operad  $\zeta$  (in the category of topological spaces) is a sequence of spaces  $\zeta(j)$  such that  $\zeta(0) = \{*\}$ , there is a unit  $1 \in \zeta(1)$ , the symmetric group  $\Sigma_j$  acts from the right on  $\zeta(j)$  and there is a suitably associative, unital and equivariant family of maps:

$$\gamma : \zeta(k) \times \zeta(j_1) \times \dots \times \zeta(j_k) \rightarrow \zeta(j_1 + \dots + j_k).$$

Let  $\text{Set}_{\text{f.l.o.}}$  be the set of finite linearly ordered sets. We construct an (categorical) operad  $\{C_{1,X}\}_{X \in \text{Set}_{\text{f.l.o.}}}$  as follows.

Set  $C_{1,\emptyset} = \{*\}$ , the trivial category and, for  $X \neq \emptyset$ , let  $C_{1,X}$  be the indiscrete category whose objects are all the associations and permutations of the ordered set  $X$ .

The operation of the operad  $\gamma : C_{1,X} \times C_{1,X_1} \times \dots \times C_{1,X_n} \rightarrow C_{1,\bigcup_{1 \leq k \leq n} X_k}$ , where  $X_k \in \text{Set}_{\text{f.l.o.}}$  and  $X$  is a linearly ordered set with  $n$  elements, is defined as follows.

Given  $(d, c_1, \dots, c_n) \in C_{1,X} \times C_{1,X_1} \times \dots \times C_{1,X_n}$ ,  $\gamma$  combines  $(c_1, \dots, c_n)$  according to the permutation-association  $d$ . It is then

straightforward to show that  $\gamma$  is a functor which satisfies the appropriate axioms [M].

6.2. From the operad  $\{C_{1,X}\}_{X \in \text{Set}_{t.i.o.}}$ , we may construct a groupoid-enriched category  $\hat{C}$  as follows.

The objects of  $\hat{C}$  are to be the natural numbers. Let  $\hat{C}(m, 0) = \hat{C}(0, n) = \{*\}$  for all natural numbers  $m$  and  $n$  and for  $m, n > 0$  let  $\hat{C}(m, n) = \prod_{\substack{\phi: m \rightarrow n \\ \text{in } \Gamma}} \prod_{i \in \Gamma} C_{1, \phi^{-1}(i)}$ .

The composite  $(\phi; c) \circ (\varphi; d) \triangleq (\phi \circ \varphi; e)$  of  $(\phi; c) \triangleq (\phi; c_1, \dots, c_n) \in \hat{C}(m, n)$  and  $(\varphi; d) \triangleq (\varphi; d_1, \dots, d_m) \in \hat{C}(l, m)$  is defined by setting  $e = \prod_{i \in \Gamma} \gamma\left(c_i; \prod_{\phi(j)=i} d_j\right)$ , where

$$\gamma: C_{1, \phi^{-1}(i)} \times C_{1, \varphi^{-1}(j_1)} \times \dots \times C_{1, \varphi^{-1}(j_\alpha)} \rightarrow C_{1, \bigcup_{1 \leq k \leq \alpha} \varphi^{-1}(j_k)} = C_{1, (\phi \circ \varphi)^{-1}(i)}$$

and

$$\phi^{-1}(i) = \{j_1 < \dots < j_\alpha\}.$$

Then the associativity of composition follows from the definition of an operad.

6.3 . Suppose that  $\mathcal{A} = (\mathcal{A}, \otimes, l, a, c, l, r)$  is a given symmetric monoidal category. For any finite set  $X$  we have a bifunctor  $W_X: C_{1,X} \times \mathcal{A}^{|X|} \rightarrow \mathcal{A}$ , where  $|X|$  is the cardinality of  $X$  [Ma]. We use these to construct a 2-functor  $\hat{X}: \hat{C} \rightarrow \text{Cat}$  as follows.

Set  $\hat{X}_n = \mathcal{A}^n$  for all  $n$ . Define  $\hat{X}: \hat{C}(m,n) \rightarrow \text{Cat}(\mathcal{A}^m, \mathcal{A}^n)$  to be the enriched adjoint of the functor  $W_{m,n}: \hat{C}(m,n) \times \mathcal{A}^m \rightarrow \mathcal{A}^n$  which is given on components  $\varphi \in \Gamma(m,n)$  by the following composite, where the  $P_{\varphi^{-1}(i)}$  are projections:

$$\prod_{i \in n} (C_{1, \varphi^{-1}(i)} \times \mathcal{A}^m) \xrightarrow{\prod_{i \in n} P_{\varphi^{-1}(i)}} \prod_{i \in n} (C_{1, \varphi^{-1}(i)} \times \mathcal{A}^{|\varphi^{-1}(i)|}) \xrightarrow{\prod_{i \in n} W_{\varphi^{-1}(i)}} \prod_{i \in n} \mathcal{A}$$

6.4. In any category we shall say that morphisms  $u: A \rightarrow X$ ,  $p: E \rightarrow B$  are *transverse* if for any  $f: A \rightarrow E$ ,  $g: X \rightarrow B$  such that  $pf = gu$  there is an  $h: X \rightarrow E$  with  $hu = f$ ,  $ph = g$ . If  $C$  is a finitely complete and cocomplete category, by a *class of weak equivalences* in  $C$ , we mean a class  $\mathcal{E}$  of morphisms closed under retraction (in  $C^{\rightarrow}$ ) and under composition and cancellation. By a *closed Quillen model structure* in  $C$ , relative to a class  $\mathcal{E}$  of weak equivalences, we mean a pair of classes of morphisms,  $\text{Cof}$ ,  $\text{Fib}$ , whose members are called, respectively, *cofibrations* and *fibrations*, each closed under retraction in  $C^{\rightarrow}$ , and such that

- (Q1) Every cofibration is left transverse to every trivial fibration, i.e. fibrations which also happen to be weak equivalences. Dually, every trivial cofibration is left transverse to every fibration.
- (Q2) Any morphism in  $C$  has a *left factorization*  $f''f'$ , with  $f'' \in \text{Fib}$ ,  $f' \in \text{Cof} \cap \mathcal{E}$  and a *right factorization*  $g''g'$  with  $g'' \in \text{Fib} \cap \mathcal{E}$ ,  $g' \in \text{Cof}$ .

A category  $C$  supplied with such a structures is a *closed Quillen model category*. In particular,  $Cat$  is a closed Quillen model category with

$$\begin{aligned}\mathcal{E} &= \{f \mid f \text{ is an equivalence of categories}\}, \\ \text{Fib} &= \{f \mid f \text{ has the isomorphism lifting property}\} \text{ \&} \\ \text{Cof} &= \{f \mid f \text{ is injective on objects}\}.\end{aligned}$$

Moreover,  $Cat$  satisfies the following property which Baues called '*the axiom of properness*'. Given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ p \downarrow & & \downarrow q \\ A' & \longrightarrow & X' \end{array}$$

where  $p \in \mathcal{E}$  and  $\varphi \in \text{cof}$ , then  $q \in \mathcal{E}$ .

To show this we first factor  $p$  as follows:

$$A \xrightarrow{\text{Fib} \cap \mathcal{E}} B \xrightarrow{\text{Cof} \cap \mathcal{E}} A'.$$

We then obtain the following pushout diagrams.

$$\begin{array}{ccc} A & \xrightarrow{\text{Cof}} & X \\ \text{fib} \cap \mathcal{E} \downarrow & & \downarrow \\ B & \xrightarrow{\text{Cof}} & Y \\ \text{Cof} \cap \mathcal{E} \downarrow & & \downarrow \text{Cof} \cap \mathcal{E} \\ A' & \xrightarrow{\text{Cof}} & X' \end{array}$$

We may then assume that  $p \in \text{Fib} \cap \mathcal{E}$ . Let  $\Gamma = \{f \in A_1 \mid pf = id.\}$ . We have  $A_0 \subset \Gamma \subset A$  and  $\Gamma$  is a groupoid. In fact, since  $p \in \text{Fib} \cap \mathcal{E}$ ,  $\Gamma$  is a

preordered groupoid, i.e. for any  $a, b \in A_0$ , there exists at most one arrow in  $\Gamma$  from  $a$  to  $b$ . We call such a  $\Gamma$  a "web". For any web  $\theta \subset X$  define a category  $X|\theta$  as follows.

$$(X|\theta)_0 = \pi_0\theta = \{[a] \mid [a] = [a'] \text{ iff } \exists! f: a \rightarrow a' \text{ in } \Gamma\};$$

$$(X|\theta)([a], [b]) = X(a, b).$$

Note that if  $[a] = [a']$  and  $[b] = [b']$ , then  $X(a, b)$  is canonically isomorphic to  $X(a', b')$ . The quotient functor  $X \rightarrow X|\theta$  is clearly a trivial fibration. Since  $\varphi$  is a cofibration  $\varphi\Gamma$  is a preordered groupoid, and hence  $\varphi\Gamma \cup X_0$  is a web. We therefore have the following pushout:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ p \downarrow & & \downarrow q \\ A' & \longrightarrow & X|\theta(\varphi\Gamma \cup X_0) \end{array}$$

with  $q$  a trivial fibration.

Baues shows that for any Quillen model category with this property the following "glueing lemma" holds [B].

Consider the following diagram in which the two squares are pushouts and the four horizontal arrows are cofibrations.

If  $\alpha, \beta$  and  $\gamma$  are weak equivalences, then so is  $\delta$ .

In [H], Alex Heller shows that for any small category  $C$  there exists two model structure on  $\mathcal{K}^C$ , where  $\mathcal{K}$  is the category of simplicial sets. In fact, his arguments still hold if we replace  $\mathcal{K}$  by  $Cat$ . In particular, letting

$$\begin{aligned} \mathcal{E}_w &= \{f \mid f_c \in \mathcal{E}, \text{ for all objects } c \text{ in } C\}, \\ \text{Fib}_w &= \{f \mid f_c \in \text{Fib}, \text{ for all objects } c \text{ in } C\} \text{ \& } \\ \text{Cof}_s &= \left\{f \mid \text{all } g \in \text{Fib}_w \cap \mathcal{E}_w \text{ are right transverse to } f\right\}. \end{aligned}$$

yields a functorial closed Quillen model structure on  $Cat^C$ , for any small category  $C$ . Define  $T: Cat^{\hat{C}} \rightarrow Cat^{\hat{C}}$  using the right factorization of  $\emptyset \xrightarrow{f} X$  into  $\emptyset \xrightarrow{\text{Cof}_s} TX \xrightarrow{\text{Fib}_w \cap \mathcal{E}_w} X$  in  $Cat^{\hat{C}}$ , where  $\emptyset$  is the initial object.

6.5. The forgetful 2-functor  $v: \hat{C} \rightarrow \Gamma$ , defined by  $n \mapsto n$  and  $(\phi; c) \mapsto \phi$ , induces a 2-functor  $v^*: Cat^\Gamma \rightarrow Cat^{\hat{C}}$ . Since  $Cat$  is cocomplete,  $v^*$  has a left adjoint  $Lan_v$ , the left Kan extension along  $v$ . Moreover it satisfies the following property.

**Prop.** For any  $X$  in  $Cat^{\hat{C}}$ , the unit  $\eta_{TX}: TX \rightarrow v^* Lan_v TX$  is in  $\mathcal{E}_w \subseteq Cat^{\hat{C}}$ , i.e. for all objects  $n$  in  $\hat{C}$ , we have an equivalence of categories  $(\eta_{TX})_n: (TX)_n \simeq (v^* Lan_v TX)_n$ .

Since  $(TX)_n \simeq X_n$  is an equivalence of categories for all  $n$ ,  $(Lan_v TX)_n \simeq X_n$  is also an equivalence of categories for all  $n$ .

Let  $\bar{X} \triangleq \text{Lan}_{\gamma} \text{TX} : \Gamma \rightarrow \text{Cat}$ , where  $\hat{X} : \hat{C} \rightarrow \text{Cat}$  is the 2-functor we have previously constructed. By the above proposition, we have that  $X$  satisfies the hypothesis in our theorem, i.e.  $\bar{X}_n \cong \hat{X}_n = \mathcal{A}^n$ , for all objects  $n$  in  $\Gamma$ .

All that remains to be done is to prove the proposition. We will only sketch the proof. For more details see [H2].

Define the functor

$$\otimes_{\hat{C}} : \text{Cat}^{\Gamma \times \hat{C}^{op}} \times \text{Cat}^{\hat{C}} \rightarrow \text{Cat}^{\Gamma}$$

by  $(F \otimes_{\hat{C}} G)_n = \int^m F_{n,m} \times G_m$ , for all objects  $n$  in  $\Gamma$ . Similarly, we may define the functor

$$\text{Hom}_{\Gamma} : \text{Cat}^{\Gamma^{op} \times \hat{C}} \times \text{Cat}^{\Gamma} \rightarrow \text{Cat}^{\hat{C}}$$

by  $\text{Hom}_{\Gamma}(F, G)_m = \int_n G_n^{F_{n,m}}$  for all  $m$  in  $\hat{C}$ . We then have

$$v^* Y = \Gamma \circ (1 \times v) \otimes_{\Gamma} Y, \quad Y \in \text{Cat}^{\Gamma};$$

$$\text{Lan}_{\gamma} X = \Gamma \circ (v \times 1) \otimes_{\hat{C}} X, \quad X \in \text{Cat}^{\hat{C}}.$$

The Yoneda Lemma applied to  $\text{TX} : \hat{C} \rightarrow \text{Cat}$  with  $\hat{C} : \hat{C}^{op} \times \hat{C} \rightarrow \text{Cat}$  yields a natural isomorphism  $\text{Hom}_{\hat{C}}(\hat{C}, \text{TX}) \cong \text{TX}$ . This corresponds via the adjointness relation

$$\text{Cat}^{\hat{C}}(\text{TX} \otimes_{\hat{C}^{op}} \hat{C}, \text{TX}) \cong \text{Cat}^{\hat{C}}(\text{TX}, \text{Hom}_{\hat{C}}(\hat{C}, \text{TX}))$$

to a natural isomorphism  $\hat{C} \otimes_{\hat{c}} TX \cong TX$ . Since

$$\begin{aligned} v^* Lan_v TX &= v^*(\Gamma \circ (v \times 1) \otimes_{\hat{c}} TX) \\ &= \Gamma \circ (1 \times v) \otimes_{\Gamma} [\Gamma \circ (v \times 1) \otimes_{\hat{c}} TX] \\ &\cong [\Gamma \circ (1 \times v) \otimes_{\Gamma} \Gamma \circ (v \times 1)] \otimes_{\hat{c}} TX \\ &\cong \Gamma \circ (v \times v) \otimes_{\hat{c}} TX, \end{aligned}$$

it suffices to show  $\hat{C} \otimes_{\hat{c}} TX \cong \Gamma \circ (v \times v) \otimes_{\hat{c}} TX$  is an equivalence of categories. But this is a consequence of the following observation and lemma.

The morphism  $\hat{C} \rightarrow \Gamma \circ (v \times v)$  in  $Cat^{\hat{c}^{op} \times \hat{c}}$  defined by  $v$  is a weak equivalence, i.e. for any  $(m, n) \in \hat{C}^{op} \times \hat{C}$ ,  $\hat{C}(m, n) \rightarrow \Gamma(m, n)$  is an equivalence of categories in  $Cat$ .

**Lemma.** If  $X$  is  $Cat^{\hat{c}}$  and  $f: Y \rightarrow Y'$  is a weak equivalence in  $Cat^{\hat{c}^{op} \times \hat{c}}$ , then  $f \otimes_{\hat{c}} TX$  is a weak equivalence in  $Cat^{\hat{c}}$ .

We adapt Heller's proof of the analogous result with  $Cat$  replaced by  $\mathcal{K}$  [H2] to the present case.

Let  $J: \hat{C}_0 \rightarrow \hat{C}$  and  $X_0 = \emptyset$ . Now form the following pushout where  $J^* \emptyset \rightarrow D_0$  is a weak cofibration in  $Cat^{\hat{c}_0}$ .

$$\begin{array}{ccc} Lan_J J^* \emptyset & \xrightarrow{Cof_S} & Lan_J D_0 \\ \downarrow & & \downarrow \\ \emptyset & \longrightarrow & X_1 \end{array}$$

Since  $\hat{C}_0$  is a discrete category, then, for any  $X \in \text{Cat}^{\hat{C}_0}$ ,  $(\text{Lan}_j X)_c = \coprod_{\varphi: d \rightarrow c} X_d$ .

Therefore, the top arrow above is a weak cofibration since it can be expressed as a coproduct of weak cofibrations. It is also a strong cofibration because  $\text{Lan}_j(\text{Cof}_w) \subset \text{Cof}_s$ . Therefore,  $\emptyset \rightarrow X_1$  is a strong cofibration in  $\text{Cat}^{\hat{C}}$ . Iterating this construction, we then have a sequence of strong cofibrations in  $\text{Cat}^{\hat{C}}$ .

$$\emptyset = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots$$

with  $\text{TX} = \text{colim} X_n$ . So  $f \otimes_{\hat{C}} \text{TX}$  is the colimit of the following diagram.

$$\begin{array}{ccccc} Y \otimes_{\hat{C}} X_0 & \longrightarrow & Y \otimes_{\hat{C}} X_1 & \longrightarrow & \dots \\ f \otimes_{\hat{C}} X_0 \downarrow & & f \otimes_{\hat{C}} X_1 \downarrow & & \\ Y' \otimes_{\hat{C}} X_0 & \longrightarrow & Y' \otimes_{\hat{C}} X_1 & \longrightarrow & \dots \end{array}$$

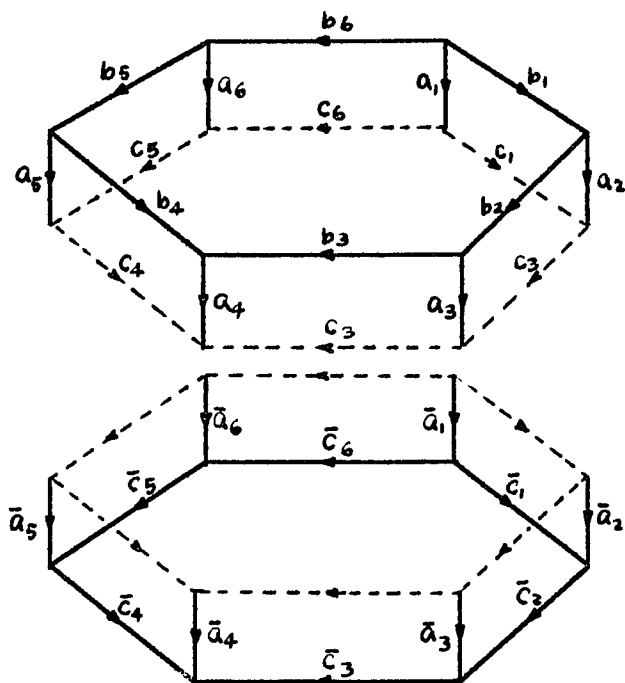
It remains to show that all the  $f \otimes_{\hat{C}} X_n$  are weak equivalences. This will be done inductively. First of all,  $f \otimes_{\hat{C}} X_0$  is a weak equivalence since  $f$  is and  $X_0 = \emptyset$ . Assume then that  $f \otimes_{\hat{C}} X_n$  is a weak equivalence. We show that  $f \otimes_{\hat{C}} X_{n+1}$  is as well. Consider the following commutative diagrams.

$$\begin{array}{ccc}
Y \otimes_{\hat{c}} \text{Lan}_J J^* X_n & \xrightarrow{\quad} & Y \otimes_{\hat{c}} \text{Lan}_J J^* D_n \\
\downarrow & \searrow u & \swarrow v \\
& Y' \otimes_{\hat{c}} \text{Lan}_J J^* X_n & \longrightarrow Y' \otimes_{\hat{c}} \text{Lan}_J J^* D_n \\
& \downarrow & \downarrow \\
& Y \otimes_{\hat{c}} X_n & \longrightarrow Y' \otimes_{\hat{c}} X_{n+1} \\
\downarrow f \otimes_{\hat{c}} X_n & \nearrow & \nwarrow f \otimes_{\hat{c}} X_{n+1} \\
Y \otimes_{\hat{c}} X_n & \xrightarrow{\quad} & Y \otimes_{\hat{c}} X_{n+1}
\end{array}$$

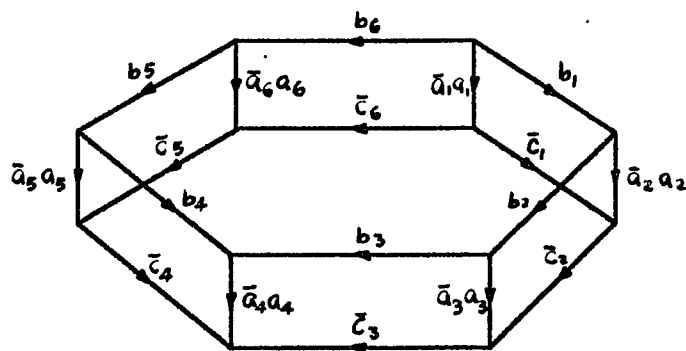
in which the top two horizontal arrows are weak and strong cofibrations and the two squares are pushouts. From the inductive hypothesis we have that  $(J \times \hat{C})^* Y \otimes_{\hat{c}_0} J^* X_n \cong (J \times \hat{C})^* Y' \otimes_{\hat{c}_0} J^* X_n$ . But, for all  $Z$  in  $\text{Cat}^{\hat{c} \times \hat{c}}$ ,  $Z \otimes_{\hat{c}} \text{Lan}_J A \cong (J \times \hat{C})^* Z \otimes_{\hat{c}_0} A$ . It follows that  $Y \otimes_{\hat{c}} \text{Lan}_J J^* X_n \cong (J \times C)^* Y \otimes_{\hat{c}_0} J^* X_n$  and  $Y' \otimes_{\hat{c}} \text{Lan}_J J^* X_n \cong (J \times C)^* Y' \otimes_{\hat{c}_0} J^* X_n$ . Therefore,  $u$  is a weak equivalence. Similarly we may show that  $v$  is a weak equivalence. The gluing lemma then implies that  $f \otimes_{\hat{c}} X_{n+1}$  is a weak equivalence and that completes the inductive step.

**Q.E.D.**

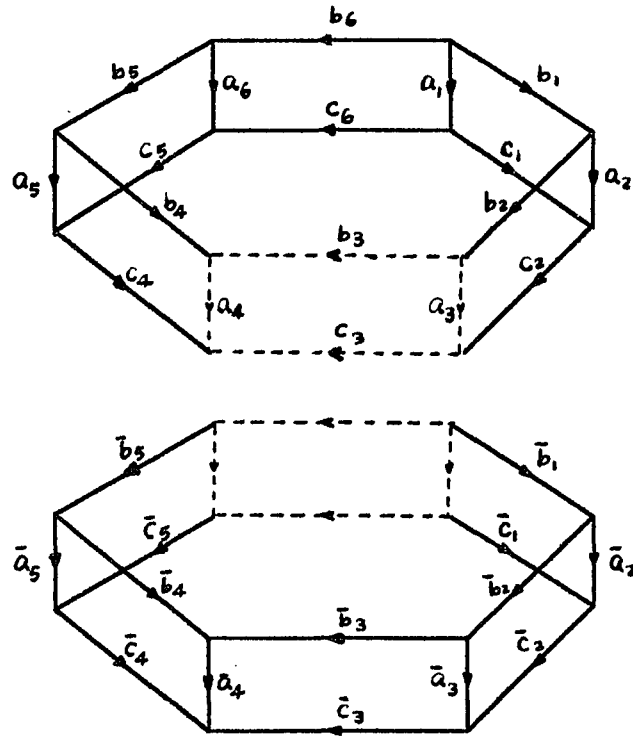
## 7. List Of Figures.



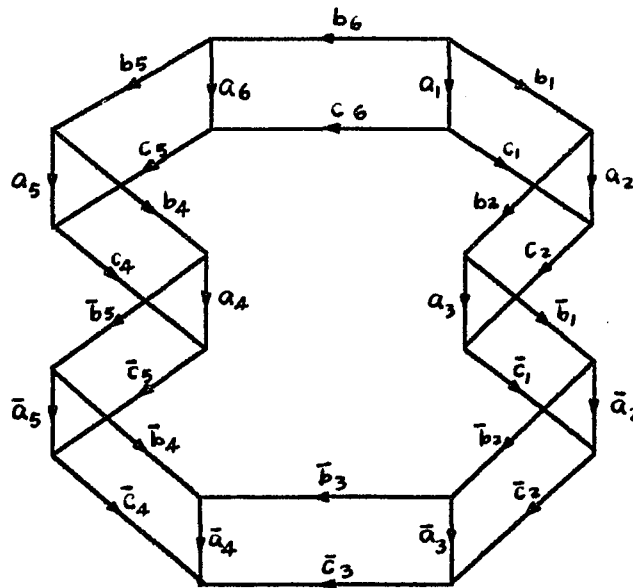
(Fig. 1.1)



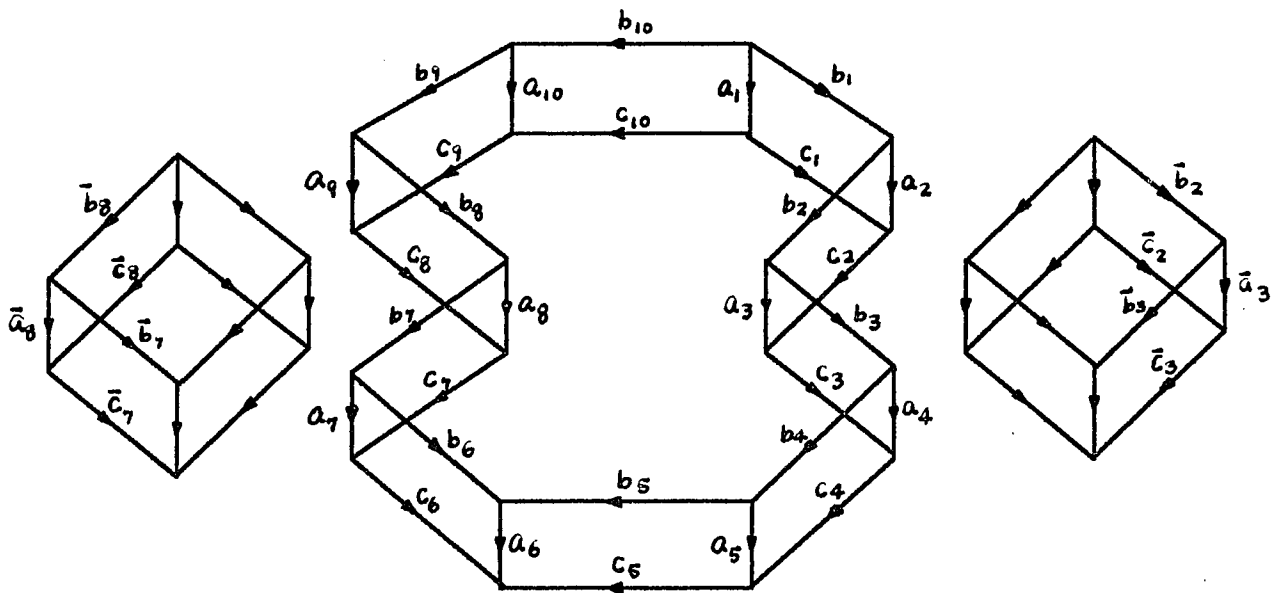
(Fig. 1.2)



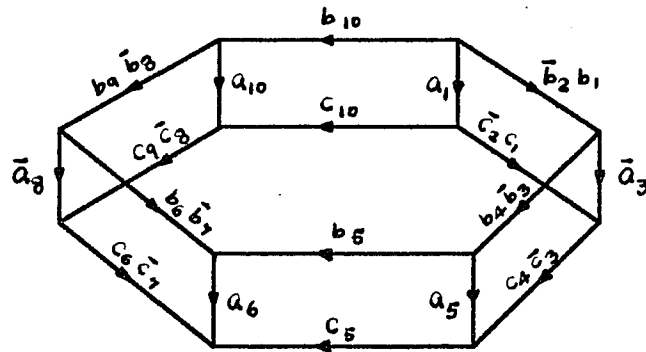
(Fig. 2.1)



(Fig. 2.2)



(Fig. 3.1)



(Fig. 3.2)

## APPENDIX

**Proof of Lemma 1' (c).** Let  $\alpha_i, 1 \leq i \leq 10$  be the natural isomorphisms of  $\langle \delta \rangle$  from the right face to the back face. Let  $\beta_t$  and  $\beta_b$  be the natural isomorphisms of the top face and bottom face of  $\langle \delta \rangle$  and let  $\bar{\alpha}_2, \bar{\alpha}_3, \bar{\beta}_{rt}$  and  $\bar{\beta}_{rb}$  be the natural isomorphisms of the right back face, the right front face, the top face and bottom face of  $[\delta_r]$  and let  $\bar{\alpha}_8, \bar{\alpha}_7, \bar{\beta}_{lt}$  and  $\bar{\beta}_{lb}$  be the natural isomorphisms of the left back face, the left front face, the top face and the bottom face of  $[\delta_l]$ . We know that

$$\begin{aligned}\delta_r &= (\bar{\beta}_{rb} a_2)(c_3 \alpha_2^{-1})(\alpha_3^{-1} b_2)(a_4 \bar{\beta}_{rt})(\bar{\alpha}_3 \bar{b}_2)(c_3 \bar{\alpha}_2) = id, \\ \delta_l &= (\bar{\beta}_{lb} a_9)(c_7 \bar{\alpha}_8)(\bar{\alpha}_7 \bar{b}_8)(a_7 \bar{\beta}_{lt})(\alpha_7^{-1} b_8)(c_7 \alpha_8^{-1}) = id,\end{aligned}$$

and

$$\begin{aligned}\tilde{\delta} &= (\beta_b a_1)(c_6 c_7 c_8 c_9 \alpha_{10})(c_6 c_7 c_8 \alpha_9 b_{10})(c_6 c_7 \alpha_8 b_9 b_{10})(c_6 \alpha_7 b_8 b_9 b_{10}) \\ &\quad (\alpha_6 b_7 b_8 b_9 b_{10})(a_6 \beta_t)(\alpha_5 b_4 b_3 b_2 b_1)(c_5 \alpha_4 b_3 b_2 b_1)(c_5 c_4 \alpha_3 b_2 b_1) \\ &\quad (c_5 c_4 c_3 \alpha_2 b_1)(c_5 c_4 c_3 c_2 \alpha_1) \\ &= id.\end{aligned}$$

We want to show  $\tilde{\delta} = id$ . By using the interchange law and three identities, we have

$$\begin{aligned}
\bar{\delta} &= (c_5 c_4 \bar{\beta}_{r_8} c_1 a_1) (\beta_6 a_1) [(c_6 \bar{\beta}_{10} c_9 c_{10} a_1) (c_6 \bar{c}_7 \bar{c}_8 c_9 \alpha_{10}) (c_6 \bar{c}_7 \bar{c}_8 \alpha_9 a_{10}) (c_6 \bar{c}_7 \bar{c}_8 \alpha_9 b_{10})] \\
&\quad [(c_6 \bar{c}_7 \bar{\alpha}_8 b_9 b_{10}) (c_6 \bar{\alpha}_7 \bar{b}_8 b_9 b_{10})] [(c_6 \bar{c}_7 \bar{b}_8 b_9 b_{10}) (c_6 \bar{c}_7 \bar{b}_8 b_9 b_{10})] [(c_6 \bar{c}_7 \bar{c}_8 b_9 b_{10})] [(c_6 \bar{c}_7 \bar{c}_8 \alpha_9 a_{10})] \\
&\quad [(c_6 \bar{c}_7 \bar{c}_8 \alpha_9 b_{10})] [(c_6 \bar{c}_7 \bar{c}_8 \alpha_9 a_{10})] [(c_6 \bar{c}_7 \bar{c}_8 \alpha_9 b_{10})] [(c_6 \bar{c}_7 \bar{c}_8 \alpha_9 a_{10})] [(c_6 \bar{c}_7 \bar{c}_8 \alpha_9 b_{10})] \\
&= (c_5 c_4 \bar{c}_3 \bar{\alpha}_2 b_1) (c_5 c_4 \bar{c}_3 \bar{c}_2 \alpha_1) \\
&= (c_5 c_4 \bar{\beta}_{r_8} c_1 a_1) (\beta_6 a_1) (c_6 c_7 c_8 c_9 \alpha_{10}) (c_6 c_7 c_8 \alpha_9 a_{10}) (c_6 \bar{\beta}_{10} a_9 b_9 b_{10}) \\
&\quad [(c_6 \bar{c}_7 \bar{\alpha}_8 b_9 b_{10}) (c_6 \bar{\alpha}_7 \bar{b}_8 b_9 b_{10}) (c_6 a_7 \bar{b}_8 b_9 b_{10}) (\alpha_6 b_7 b_8 b_9 b_{10}) (\alpha_6 \beta_7)] \\
&\quad [(c_6 \bar{c}_7 \bar{c}_8 b_9 b_{10}) (c_6 \bar{c}_7 \bar{c}_8 \alpha_9 a_{10}) (c_6 \bar{c}_7 \bar{c}_8 \alpha_9 b_{10}) (c_6 \bar{c}_7 \bar{c}_8 \alpha_9 a_{10}) (c_6 \bar{c}_7 \bar{c}_8 \alpha_9 b_{10})] \\
&= (c_5 c_4 \bar{c}_3 \bar{\alpha}_2 b_1) (c_5 c_4 \bar{c}_3 \bar{c}_2 \alpha_1) \\
&\quad \{c_6 [(c_7 \bar{b}_8 a_9) (c_7 \bar{\alpha}_8) (\bar{\alpha}_7 \bar{b}_8) (a_7 \bar{\beta}_{11})] b_9 b_{10}\} \\
&\quad [(c_6 b_7 b_8 b_9 b_{10}) (\alpha_6 \beta_7) (\alpha_5 b_4 b_3 b_2 b_1) (c_5 \alpha_4 b_3 b_2 b_1)] \\
&\quad \{c_5 c_4 [(a_2 \bar{\beta}_{r_7}) (\bar{\alpha}_3 \bar{b}_2) (c_3 \bar{\alpha}_2)] b_1\} (c_5 c_4 \bar{c}_3 \bar{c}_2 \alpha_1) \\
&= (c_5 c_4 \bar{\beta}_{r_8} c_1 a_1) (\beta_6 a_1) (c_6 c_7 c_8 c_9 \alpha_{10}) (c_6 c_7 c_8 \alpha_9 a_{10}) \\
&\quad \{c_6 [(c_7 \alpha_8) (\alpha_7 b_8)] b_9 b_{10}\} (\alpha_6 b_7 b_8 b_9 b_{10}) (\alpha_6 \beta_7) (\alpha_5 b_4 b_3 b_2 b_1) \\
&\quad [(c_5 \alpha_4 b_3 b_2 b_1) \{c_5 c_4 [(a_3 b_2) (c_3 \alpha_2) (\bar{\beta}_{r_8}^{-1} a_2)] b_1\} (c_5 c_4 \bar{c}_3 \bar{c}_2 \alpha_1) \\
&= (c_5 c_4 \bar{\beta}_{r_8} c_1 a_1) (\beta_6 a_1) (c_6 c_7 c_8 c_9 \alpha_{10}) (c_6 c_7 c_8 \alpha_9 b_{10}) (c_6 c_7 c_8 \alpha_9 b_{10}) (c_6 c_7 c_8 \alpha_9 b_9 b_{10}) \\
&\quad [(c_6 \alpha_7 b_8 b_9 b_{10}) (\alpha_6 b_7 b_8 b_9 b_{10}) (\alpha_6 \beta_7) (\alpha_5 b_4 b_3 b_2 b_1) (c_5 \alpha_4 b_3 b_2 b_1) \\
&\quad [(c_5 c_4 \alpha_3 b_2 b_1) (c_5 c_4 c_3 \alpha_2 b_1)] (c_5 c_4 \bar{\beta}_{r_8}^{-1} a_2 b_1) (c_5 c_4 \bar{c}_3 \bar{c}_2 \alpha_1)] \\
&= (c_5 c_4 \bar{\beta}_{r_8} c_1 a_1) \delta (c_5 c_4 \bar{\beta}_{r_8}^{-1} c_1 a_1) \\
&= id.
\end{aligned}$$

Q.E.D.

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