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THE SINGLET FIELDS METHOD AND THE LARGE-N EXPANSION OF FIELD THEORIES

by

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Abstract

THE SINGLET FIELDS METHOD AND THE LARGE-N EXPANSION OF FIELD THEORIES

by

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We develop the quadratic singlet fields method to provide a general procedure for the large-N expansion of quantum field theories. The first part consists of the application of the singlet fields method to Yukawa interactions. In this case, the singlet field method provides a thorough picture of the large-N expansion. As an important example, the formalism is applied to QCD in two-dimensions. It is well-known that for two-dimensional QCD the integral equations of the large-N expansion are solvable, and that all the physical consequences of the model can be calculated . The second part focuses on the extension of the singlet fields method, to treat the large-N expansion of the fermion sector of field theories in four dimensions, such as the matrix model and QCD. The argument is carried out at the formal level of the path integral representation of quantum field theory.

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1. INTRODUCTION AND SUMMARY

Particle physics theories of today are dominated by quantum gauge theories. The weak and electromagnetic interactions have been comprehensively formulated by the $SU(2) \times U(1)$ broken local gauge theory[1]. So far, all the experimental tests confirm its predictions, and it is presently accepted as the standard model for the unified description of the electro-weak interactions.

The quantum field theory, widely accepted as a candidate for a theory of strong interactions, is the local SU(3) gauge theory also designated as Quantum Chromodynamics (QCD).

The very high-energy, or very short distance, behavior of non-abelian local gauge theories is generically described by the property known as asymptotic freedom. In an asymptotically free theory, the effective coupling constant, which is the running coupling constant defined by the renormalization group theory, goes to zero for large momenta (small distances). Therefore the strength of the interaction falls off and a perturbative expansion in the effective small coupling constant is justified.[2]

This picture is consistent with the properties of hadrons in the high-energy region which behave as if made out of a set of point-like, almost free particles. These quasi-free particles are the first phenomenological evidence of the concept of quarks interacting through local gauge fields (gluons) at very short distances.

However, the most common properties of nuclear matter such as masses, lifetimes, and decay ratios of hadrons are low-energy phenomena. This low-energy hadron physics is characterized by a large coupling constant, and is therefore not accessible to a quantum field theory treated with the conventional perturbation theory for small coupling constants.

New alternative expansions of quantum field theories have been devised. These new methods, sometimes designated as non-perturbative methods, are used to study quantum field theories in regions out of the reach of small coupling perturbation theory. Some of these new methods also provide new perturbative expansion schemes.

A general classification of these new methods is given by the following main lines of research:

a) Lattice field theories.

b) Classical and semi-classical models.

c) The large-N expansion.

It is well-known that these fields of interest have enriched each other, and several proposed schemes have developed non-perturbative approaches by combining these points of view. As each of these areas of research represents a very large body of knowledge, this introduction is limited to an overview of the relations of the large-N expansion to lattice field theories and semi-classical models. In this thesis, we shall concentrate on the large-N expansion for quantum field theories formulated in the continuum.

The large-N expansion arises from the observation that QCD can be generalized to an arbitrary gauge group SU(N). The theory then contains two arbitrary parameters, the coupling constant g and the group integer parameter N.

A new expansion can be formulated in terms of the parameter N. A way to introduce this expansion is by a re-arrangement of the small coupling constant perturbation series. The starting point is to redefine the coupling constant g^2 as a new parameter given by Ng^2. Then it can be shown that the conventional diagrams can be re-arranged according to positive powers of $N^{-1/2}$ which appear as overall factors of the diagrams. This is so because the diagrams contain different group combinatorial factors.

The new coupling constant in terms of which the expansion is performed is now $N^{-1/2}$. Each term of this new series contains a subset of an infinite number of the initial diagrams. The group combinatorial factors of the conventional diagrams are directly related to the topological structure of the diagrams.

The leading term of the re-arranged series is made out of all the planar diagrams[3]. This constitutes the large-N limit of the theory, since all other diagrams are dropped by the powers of $N^{-1/2}$ as N goes to infinity. In the large-N limit, g^2N is held finite. The characterization of the large-N limit in terms of planar diagrams is common to all field theories that admit generalizations to large symmetry groups.

From the diagrammatic point of view of the large-N expansion of QCD, the task of making the required diagrams summations seems formidable, even for the first term of the expansion which is given by the planar diagrams. Much preliminary work has been necessary to study the numerous difficulties related to the limit of a large internal symmetry. The problems arising from the large-N expansion are directly correlated to the types of fields and their interactions.

In general, the research has focused mainly on theories containing fields transforming as vectors, and fields transforming as matrices under the application of the global or local symmetry group. Regarding the difficulties encountered in the large-N expansion, two major types of models can be distinguished:

a) Models containing matter fields that transform as vectors, and mediating fields that transform as matrices or vectors. The interactions are restricted to Yukawa couplings between the matter fields and the mediating fields. For this class of models, the matrix fields interactions should not be higher than a quadratic power. We shall call these models vector models. On the whole, the large-N expansion of vector models has been understood, although the explicit resolution of the integral equations that specifically sum the planar diagrams in four dimensions remains an important obstacle.

b) Models where there are also matrix interactions with powers equal or higher than cubic. These we shall call matrix models. It is in this class of models that we find the four-dimensional QCD. Most of the difficulties lie in this type of models.

The first model where the large-N limit was calculated is the two-dimensional QCD[4]. Remarkably enough at the time, this was done by straight analysis of the small coupling constant diagrams. The key point that allows such a summation of planar diagrams in two-dimensional QCD is the application of the light cone gauge that reduces all interactions to a Yukawa coupling between the quarks and the gluons. The light cone gauge also simplifies the equations for the summation of planar diagrams.

A simple integral equation gives the masses of quark-antiquark bound states which are the mesons predicted by the model. Also, the elastic scattering of mesons is mediated by the interchange of other mesons, and not of quarks[5].

Subsequently, some vector models were solved in the large-N limit by the application of various procedures. A number of two-dimensional models were considered too[6-9]. The principal physical motivation to study these models in the large-N expansion was to address the question of dynamical symmetry breaking. The method here employed consists of the introduction of composite singlet fields to evaluate, up to 1/N order, the functional effective potential .

The planar approximation to the matrix model with quartic interactions is known only in zero dimension and in one dimension including fermion couplings [10-12]. Unfortunately these are, so far, the only explicit solutions known for matrix models as examples of continuum field theory models.

Behind some of the non-perturbative procedures utilized to derive the planar approximation is the major idea of performing a suitable change of variables to new fields that are invariant under the application of group transformations. These new invariant fields have also been designated as collective fields.

The method of the collective fields has been developed in the hamiltonian formalism[13]. In this case, a general change of variables is possible, and the formulation of the collective variables in quantum mechanical models can thus be thoroughly accomplished. The difficulties of this method lies in its extension to field theories[14]. Here the chosen invariant fields are given by the loop-field variables which are the gauge invariant path-ordered phase factors. Although the change of field variables can be done in the hamiltonian formalism, the resulting field theory in loop space is not a simple model.

Neither is the treatment of the resulting large-N functional equations in loop space.[15]

An interesting correspondence between the large-N limit and classical models has also been explored. It was observed that the calculation of certain group-invariant Green's functions in the large-N limit is equivalent to solving classical equations of motion with particular, imposed, boundary conditions [16].

The idea of the master field[17] configuration also gave a way of finding the large-N limit. The argument is the following: the vacuum expectation value of symmetry invariant operators factorizes in the large-N limit; the quantum fluctuations that violate the factorization are of the order of $1/N^2$; therefore there must be in the large-N limit a dominant field configuration that saturates the functional integral, and which is called the master field. Investigations of this topic have been carried out, leading to formulations of equations for the master field.[18]

All these methods have had great success in vectors models, two-dimensional models, and quantum mechanical matrix models. Though there have been many proposals of definite schemes to solve the problems stemming from the large-N limit of matrix and gauge models in four-dimensions, the explicit solution remains unknown.

There is also qualitative evidence that the large-N expansion of QCD is a good approximation of strong interaction processes. Many characteristics of the hadronic phenomenology have been found to be consistent[19] with an analysis based on the counting of $N^{-1/2}$ powers in the large-N expansion of QCD. Recently there have been efforts to improve the qualitative nature of the analysis of planar diagrams.[20]

Another line of approach has considered scalar and fermion models that are locally invariant under symmetry groups. There were both formal and phenomenological attempts made to build approximations to large-N gauge fields theories based on these models.[21] The formal approach revealed the gluon fields as the composite object of scalar fields. From a more phenomenological point of view, the soliton solutions of these scalar models have been used to derive some physical properties of mesons and baryons.

A more direct mathematical language for quantum field theories is found in the generating functional formalism. After changing field variables in the generating functional, it becomes clear that there exists in vector models an invariant field that can be used to formulate the whole large-N expansion of the matter fields Green's functions. This invariant field is simply a quadratic singlet product of matter fields. The singlet field is generally a bilocal function of the coordinates.

It is the objective of Chapter 2 to introduce the general formulation of the quadratic singlet field . Firstly, all original local fields are replaced by the singlet field in the generating functional of fermion Green's functions. The large-N expansion of the fermion Green's functions can be, in principle, formulated to any order by means of the singlet field. To actually perform such expansion, we find that it is necessary to solve two integral equations. These are the simplified Dyson-Schwinger equations for the large-N limit. This approximation has been known in the past as a heuristic self-consistent approximation, used for example in dynamical symmetry-breaking problems.[22]

The first equation is for the summation of planar diagrams contributing to the self-energy of the fermion propagator. It is derived from the saddle point equation for the singlet field. This equation is analogous to the previously known gap equation.

The second equation is the inhomogeneous Bethe-Salpeter equation which represents the summation of planar diagrams for the four-fermion scattering amplitudes. The homogeneous part yields the equation for the fermion-antifermions bound states. These are the mesons states predicted by the theory.

The presentation is sufficiently general to show the applicability of the method to many other vectors models in any dimension. We want to stress that once the integral equations are solved, the large-N expansion could be performed to all orders. Unfortunately, little is known about these equations in four-dimensions. Two-dimensional QCD is an important special case because these equations are solvable for this particular model.

Chapter 3 is devoted to the derivation of the equations that represent the sum of planar diagrams contributing to the meson states equation of two-dimensional QCD. This derivation, as originally done[23], depended very much on comparisons with the first derivation based on diagram analysis. All derivations carried out in Chapter 3 are now justified by the formalism developed in Chapter 2 which provides a more independent foundation for the singlet field method.

Also physical processes as meson scattering are described by the formalism of higher functional derivatives with respect to the singlet field.

The large-N limit produces a simplification of the space-time structure of field theories. In fact, the large-N limit can be found from a reduced model where there is no explicit space-time dependence. This property emerged from ideas developed in lattice gauge theories.

During the last decade, lattice gauge theories[24] have played a major role in our understanding of strong-interaction dynamics. Many of these advances have been obtained by Monte-Carlo computer simulations[25]. From the synthesis of the large-N approach and lattice formalism [26-30] two notions have played an important role: the lattice Dyson-Schwinger equations and the invariant variables.

These procedures, for example, led to the calculation of the one link integral and to the formulation of the reduced model.[31] This original reduced model turn out to be valid only for large coupling. The model had to be somewhat modified to make it applicable to small coupling constant. To evaluate the momentum dependence of the Green's functions, the 'quenching' procedure was introduced. [32-34]

Chapters 2 and 3 contain a section illustrating another application of the singlet field method. We show how to obtain the large-N limit of the fermion Green's functions from the reduced model.[35] This derivation addresses the physical implications of the reduced model which were difficult to establish. The application of the singlet field method to the reduced model yields an extension of the reduced model that avoids the quenched momentum prescription[33].

There is a possibility of extending the quadratic singlet field method, or QSF, to the case of QCD. The first application of the QSF method to the gluon sector of QCD led to a difficulty.[36] The action could not be completely reformulated in terms of singlet fields. This obstacle can be removed if we apply the QSF method to the quark sector of QCD. This application adds more significance to the method, since most of the physical states of QCD are in the quark sector.

The objective of Chapters 4 and 5 is to demonstrate at the formal level of the path integral framework that the matrix model and QCD can be reformulated in terms of the singlet fields if our attention is focused on the calculation of the fermion Green's functions.

The QSF method is used to investigate the large-N behavior of the quark Green's functions at the general level of the generating functional formalism.

The purpose of developing the singlet field method for the matrix model is to make available a simpler model to illustrate and develop the QSF method. The solution of one example would have more relevance for this formulation, since the equations are basically the same in any number of space-time dimensions. The hope in this case lies in the possibility of solving exactly in one dimension the large-N equations generated by the QSF method, and therefore attain an explicit verification of the application to matrix models of the singlet fields method.

2. THE SINGLET FIELD FORMULATION

I. INTRODUCTION

The large-N approach to field theories has provided a good deal of information about models analyzed in less than four dimensions. These models have been used as training grounds for non-conventional perturbative methods, and as analogous models of QCD.

The construction of a systematic expansion in power series of 1/N is known in few cases of field theories. Examples in this area are some models in one and two dimensions[9-12], and the O(N) scalar model in four dimensions with quartic interaction[7].

In this chapter we present the generalization of a path integral change of field variables to the singlet field that initially was applied some time ago to find the large-N limit[23] of two dimensional QCD.

We apply this technique to a many-component Yukawa interaction model. The model contains a multicomponent fermion field coupled to a matrix of scalar fields. The lagrangian is invariant under the global U(N) group transformations. The model is considered in four dimensions, though it is clear that the whole procedure is independent of the number of space-time dimensions.

The singlet field method applied to the Yukawa interaction model provides a way to obtain in principle the complete large-N expansion of the fermion Green's functions.

The success of the self-energy expansion in providing the whole large-N expansion for the Yukawa interaction model is due to the fact that the singlet field eliminates all the group indices from the action.

The path integral functional manipulations can be carried out either in coordinate or momentum space. We choose the momentum representation, since the integral equations for the summation of planar diagrams are directly derived in momentum space where they are usually analyzed.

Section II is devoted to the introduction of the singlet field method using the proposed many-component Yukawa interaction model. The generating functional for the large-N expansion is derived.

The large-N limit of the fermion Green's functions is shown in Section III. The factorization of the expectation value of group invariant operators becomes a simple consequence of the large-N limit in this formulation.

The bound states equation and its relation to the large-N expansion is presented in Section IV.

A reduced model is discussed in Section V. We show that it provides the correct leading large-N contribution to the fermion Green's functions. The N-independent contributions required for the meson states equation are not correctly predicted by the reduced model. However, the correct equation can be deduced from the complete model. The application of the self-energy expansion to the reduced model has also been done for two-dimensional QCD, where the same conclusions were reached[35]. Additional discussion regarding this point can be found in the last section of Chapter 3.

Basic definitions and formulas of the path integral formalism are included in the appendix.

II. THE SINGLET FIELD ACTION OF THE YUKAWA INTERACTION

The model we are going to consider is given by the following action

$$S = i\int\left((1/2)\,\mathrm{tr}(\partial_\mu A\partial^\mu A - m_1^2A^2) + \bar{f}(i\gamma^\mu\partial_\mu - m_2)f + ig\bar{f}_a\gamma_5 A_{ab}f_b\right)d^4x + \int(\bar{\eta}f + \bar{f}\eta)d^4x \qquad (1)$$

where A is a NxN hermitian matrix of scalar fields and η, $\bar{\eta}$ are sources for the fermion fields $\bar{f}$ and f .

Introducing the Fourier transforms of the fields and sources

$$A(x) = \int e^{ikx}A(k)d^4k, \quad f(x) = \int e^{iqx}f(q)d^4q,$$
$$\eta(x) = \int(2\pi)^{-4}\cdot e^{ipx}\eta(p)d^4p\ , \qquad (2)$$

we get the following form of the action

$$S = i(2\pi)^4\int((1/2)(k^2-m_1^2)\,\mathrm{tr}A(k)A(-k) - \qquad (3)$$
$$\bar{f}(k)(\gamma k+m_2)f(k) + ig\int\bar{f}(k)\gamma_5A(k-q)f(q)dq)dk + \int(\bar{\eta}f + \bar{f}\eta)dk.$$

The generating functional is given by, $Z(\bar{\eta},\eta) = \int DAD\bar{f}Df\ \exp(S)$.

The integration of the scalar fields leaves the action with a non-local four-fermion interaction

$$S^{(1)} = i(2\pi)^4\Big(\int(-\bar{f}(k)(\gamma k+m_2)f(k)dk + (g^2/2)\int \bar{f}_a(q+p)\gamma_5 f_b(q)\bar{f}_b(k)\gamma_5 f_a(k+p)V(p)dpdqdk + \int(\bar{\eta}f + \bar{f}\eta)dk\ , \qquad (4)$$

where $V(p)=1/(p^2-m_1^2)$. We introduce the self-energy field P to substitute the four-fermion interaction as

$$S^{(2)} = i(2\pi)^4\Big(\int(-\bar{f}_a(k)(\gamma k+m_2)f_a(k)dk + \int(\bar{f}_a(k)\gamma_5)^{\alpha} P^{\alpha\beta}(k,q)f_a^{\beta}(q)dkdq + (1/2g^2)\int P^{\alpha\beta}(p,k)Q(p-k_1,k-p_1) P^{\beta\alpha}(p_1,k_1)dpdkdp_1dk_1) + \int(\bar{\eta}f + \bar{f}\eta)dk, \qquad (5)$$

where Q is a function such that two functions E and F are related as follows: given $\int E(k+q,p+q)V(q)dq = F(p,k)$, then $\int F(r,s)Q(k-s,p-r)drds = E(k,p)$. Explicitly in this case, the function Q is given by $Q(k,p) = \delta(k-p)(2\pi)^{-4}\int(e^{-ixp}/v(x))dx = \delta(k-p)Q_0(p)$, where $v(x) = \int e^{-iqx}V(q)dq$. Some useful properties of Q are: $Q(k,p) = Q(p,k) = Q(-k,-p)$.

The function Q has been introduced to build the quadratic form of the field P. Later on Q disappears, leaving only the propagator V(q) in the action.

From $S^{(2)}$ we can derive the field equation for P

$$P^{\beta\alpha}(k,p) = -g^2\int(\bar{f}_b(k+q)\gamma_5)^{\alpha}\ f_b^{\beta}(p+q)V(q)dq\ . \qquad (6)$$

Note that P does not have indeces of the symmetry group.

The fermion variables can be integrated from action $S^{(2)}$. The generating functional will depend only on the field P, and on the sources of the fermion fields. Using the following formula for the integration of the fermion variables

$$\det(I\delta(x-y)-M(x,y)) = \exp(-\mathrm{tr}(\int M(x,x)dx+ \qquad (7)$$
$$(1/2)\int M(x,y)M(y,x)dxdy+(1/3)\int M(x-y)M(y-z)M(z-x)dxdydz+\ldots))$$

we find that the action is given by

$$S^{(3)}(\bar{\eta},\eta,P) = -i(2\pi)^{-4}\int\bar{\eta}_a(k)S_P(k,p)\eta_a(p)dkdp +$$
$$(i(2\pi)^4/2g^2)\mathrm{tr}\int P(p,k)P(r,s)Q(p-s,k-r)dpdkdrds +$$
$$-N\mathrm{tr}(\int S(k)\gamma_5 P(k,k)dk + (1/2)\int S(k)\gamma_5 P(k,q)S(q)$$
$$\gamma_5 P(q,k)dkdq + \ldots) = -i(2\pi)^{-4}\int\bar{\eta}S_P\eta + NS_0(P) \qquad (8)$$

where $S(k)(\gamma k+m_2)=I$ and

$$S_P(k,p) = S(k)(\delta(k-p) + \gamma_5 P(k,p)S(p) + \int dp_1 \gamma_5 P(k,p_1)S(p_1)\gamma_5 P(p_1,p)S(p) + \ldots) \quad (9)$$

The generating functional for the fermion Green's functions is given by $Z(\bar{\eta},\eta) = \int DP\exp(S^{(3)})$.

From the source term of $S^{(3)}$ we can see that the field P plays the role of the self-energy contribution to the fermion propagator.

III. LARGE-N LIMIT OF FERMION GREEN'S FUNCTIONS

Any fermion Green's function could be given in terms of field P only. For example the four fermion function is

$$\langle \bar{f}_a(p_1) f_b(p_2) \bar{f}_c(p_3) f_d(p_4) \rangle = \qquad (10)$$
$$(-i(2\pi)^{-4})^2 \int DP (S_P(p_4,p_3) S_P(p_2,p_1) \delta_{ab} \delta_{cd}$$
$$+ S_P(p_4,p_1) S_P(p_2,p_3) \delta_{ad} \delta_{bc}) \exp(NS_0(P)),$$

and similar expressions can be found for higher fermion Green's functions. The action S_0 does not contain any other dependence on the group besides the overall factor N. Therefore the large-N limit of the Green's functions is given by the solution of the saddle point equation derived from the action. The equation is given by the condition, $\delta S_0(P)/\delta P^{\alpha\beta}(p,q)=0$. The first functional derivative is

$$\delta S_0(P)/\delta P^{\alpha\beta}(q_1,q_2) = (i(2\pi)^4/g^2 N) \qquad (11)$$
$$\int Q(q_1-k,q_2-p)\ P^{\beta\alpha}(p,k)\,dp\,dk - (S_P(q_2,q_1)\gamma_5)^{\beta\alpha}.$$

Therefore the saddle point equation is explicitly given by

$$P(q_2,q_1) = -ig_0 \int V(q) S_P(q_2+q,q_1+q) \gamma_5 dq, \qquad (12)$$

where $g_0 = g^2 N/(2\pi)^4$.

This last equation can be simplified if we introduce the following form for the solution $P(k,p) = R(k)\delta(k-p)$. We find that $S_P(k,p) = S_R(k)\delta(k-p)$ where

$$S_R(k) = S(k)(1 + \gamma_5 R(k)S(k) + (\gamma_5 R(k)S(k))^2 + \ldots).$$
$$= (\gamma k + m_2 - \gamma_5 R(k))^{-1}. \qquad (13)$$

The equation for $R(k)$ is

$$R(k) = -ig_0\int V(q)S_R(k+q)\gamma_5 dq. \qquad (14)$$

This last equation represents precisely the leading large-N contribution for the self-energy of the fermion propagator. The iteration of the integral equation for $R(q)$ reproduces all the planar graphs that contribute to the self-energy.

The action could be expanded around the solution of the saddle point equation. In this way a systematic expansion for the Green's functions would be generated to all orders. Perturbations would be calculated around the quadratic piece of the fluctuations of P around the saddle point solution.

IV. THE BOUND STATES EQUATION AND THE LARGE-N EXPANSION

To expand the action, it is convenient to introduce in the action given by (5) the shift of variables: $P(k,p) = R(k)\delta(k-p) + N^{-1/2}P'(k,p)$. The field P' represents the new quantum field. Action $S^{(2)}$ acquires the following form

$$\begin{aligned} S^{(2)} = i(2\pi)^4 \Big(&\int(-\bar{f}_a(k)(\gamma k+m_2-\gamma_5 R(k))f_a(k)dk + \\ &N^{-1/2}\int\bar{f}_a(k)\gamma_5 P'(k,q)f_a(q)dkdq + \\ &g^{-2}N^{-1/2}\mathrm{tr}\int P'(p,k)R(q)Q(p-q,k-q)dkdpdq + \\ (1/2g^2N)\,\mathrm{tr}&\int P'(p,k)P'(r,s)Q(p-s,k-r)dpdkdrds\Big) + \\ &\int(\bar{\eta}f + \bar{f}\eta)dk. \end{aligned} \qquad (15)$$

Now we integrate the fermion variables. We obtain an action equivalent to $S^{(3)}$

$$\begin{aligned} S^{(3)} = -i(2\pi)^{-4}&\int\bar{\eta}_a(k)R_{P'}(k,p)\eta_a(p)dkdp + \\ &(i/2g_0)\,\mathrm{tr}\int P'(p,k)P'(r,s)Q(p-s,k-r)dpdkdrds + \\ -(1/2)\,\mathrm{tr}&\int S_R(k)\gamma_5 P'(k,q)S_R(q)\gamma_5 P'(q,k)dkdq + S_I, \end{aligned} \qquad (16)$$

where we define an interaction term as

$$\begin{aligned} S_I(P') = -\mathrm{tr}\big(&(1/3)N^{-1/2}\int S_R\gamma_5 P' S_R\gamma_5 P' S_R\gamma_5 P' \\ &+ (1/4)N^{-1}\int \quad +\ldots\big). \end{aligned}$$

The linear term in P' has vanished because $\bar{R}(k)$ satisfies eq. (14). This last form of $S^{(3)}$ could have been also derived from expression (8). For example,

$$S_P(k,p)\Big|_{P=R+N^{-1/2}P'} = R_{P'}(k,p) = \tag{17}$$
$$S_R(k)(\delta(k-p) + N^{-1/2}\gamma_5 P'(k,p)S_R(p) +$$
$$N^{-1}\int dq\gamma_5 P'(k,q)S_R(q)\gamma_5 P'(q,p)S_R(p) + \ldots).$$

To obtain a systematic expansion using the field P', the terms higher than quadratic contained in S_I are treated as perturbations. A source term is added to $S^{(3)}$ of the following type $\mathrm{tr}\int J(p,k)P'(p,k)dpdk$. To invert the quadratic form of P' we use the field equation for P'

$$P'(p,k) = ig_0\int (J(k+q,p+q) + \tag{18}$$
$$S_R(p+q)\gamma_5 P'(p+q,k+q)S_R(k+q)\gamma_5)V(q)dq.$$

This last equation can be solved by iteration. The solution can be represented with a Green's function T acting upon the source J

$$P'(p,k) = \int T(p,k,p',k')J(p',k')dp'dk'. \tag{19}$$

The function T is given explicitly by the following series

$$T^{\alpha\alpha'\beta\beta'}(p,k,p',k') = -(-ig_0)V(k-k',p-p')\delta^{\alpha\alpha'}\delta^{\beta\beta'} - \tag{20}$$
$$(-ig_0)^2\int V(k-k_1,p-p_1)V(k_1-k',p_1-p')$$
$$(S_R(p_1)\gamma_5)^{\alpha\alpha'}(S_R(k_1))^{\beta\beta'}dk_1dp_1 -$$
$$(-ig_0)^3\int V(k-k_1,p-p_1)V(k_1-k_2,p_1-p_2)$$
$$V(k_2-k',p_2-p') \; (S_R(p_1)\gamma_5 S_R(p_2)\gamma_5)^{\alpha\alpha'}$$
$$(S_R(k_2)S_R(k_1))^{\beta'\beta}dk_i dp_i + \ldots$$

where $V(k-k',p-p') = V(k-k')\delta(k-k'-p+p')$.

The generating functional of the fermion Green's functions is given by

$$Z(\bar{\eta},\eta,J) = \exp(-i(2\pi)^{-4}\int\bar{\eta}_a(k)R_{\delta/\delta J}(k,p)\eta_a(p)\,dkdp + S_I(\delta/\delta J))\exp((1/2)\int J(p,k)T(p,k,p',k')J(p',k')\,dp'dk'). \tag{21}$$

As an important example we can apply this generating functional to the calculation of the four-fermion function. The expression we get for this function is

$$\langle \bar{f}_{a_1}^{\alpha_1}(p_1)f_{a_2}^{\alpha_2}(p_2)\bar{f}_{a_3}^{\alpha_3}(p_3)f_{a_4}^{\alpha_4}(p_4)\rangle$$
$$= (-i(2\pi)^{-4})^2(R^{\alpha_4\alpha_3}_{\delta/\delta J}(p_4,p_3)R^{\alpha_2\alpha_1}_{\delta/\delta J}(p_2,p_1)$$
$$\delta_{a_4a_3}\delta_{a_2a_1} + (1\leftrightarrow 3))\exp S_I \exp(1/2)\int JTJ. \tag{22}$$

The first contribution given by the last formula is the disconnected piece proportional to $S_R S_R$. The second contribution, which is N^{-1} smaller than the first, contains one insertion of T function, and S_I does not contribute. These two contributions are the whole ladder approximation with connected and disconnected graphs, to the four-point function.

The function T represents the ladder kernel. Thus the homogeneous part (J=0) of eq. (18) is the bound state equation for the fermion-antifermion pairs. Introducing the following change of variables

$$S_R(p)\gamma_5 P'(p,k) S_R(k)\gamma_5 = w(p,k-p) \qquad (23)$$

in eq.(18) and taking J=0 we obtain

$$w(p,k-p) = -ig_0 S_R(p)\gamma_5 \int dq\, w(q,k-q) V(q-p) S_R(k)\gamma_5 \qquad (24)$$

which is the usual form of the bound state equation. The eigenfunctions are represented by w.

Expression (24) is the Bethe-Salpeter equation in the large-N limit, which is given by the ladder approximation with corrected fermion propagators.

This last equation can also be derived from the action S_0 given by (8). Since equation (18) comes from the quadratic term in P', the second functional derivative of S_0 with respect to P also produces the operator that yields the bound state equation. The second functional derivative is

$$\delta^2 S_0(P)/\delta P^{\alpha\beta}(q_1,q_2)\delta P^{\varepsilon\sigma}(q_3,q_4) = ig_0^{-1}\delta^{\beta\varepsilon}\delta^{\sigma\alpha} \cdot Q(q_1-q_4,q_2-q_3) - (S_P(q_2,q_3)\gamma_5)^{\beta\varepsilon}(S_P(q_4,q_1)\gamma_5)^{\sigma\alpha}. \quad (25)$$

The homogeneous part of equation (18) is reproduced by the following equation

$$\int(\delta^2 S_0(P)/\delta P^{\alpha\beta}(p,q)\delta P^{\varepsilon\sigma}(r,s))_{P=R_\delta} P'^{\varepsilon\sigma}(r,s)drds = 0. \quad (26)$$

V. REDUCED MODEL FOR THE LARGE-N LIMIT

The quenched reduced[33] model associated to the theory we have been working with, is given by the following action

$$S = L^4 (2\pi)^4 i\left((1/2)((p_a-p_b)^2-m_1^2)A_{ab}A_{ba} - \bar{f}_a(\gamma p_a+m_2)f_a + ig\bar{f}_a\gamma_5 A_{ab}f_b\right) \quad (27)$$

where L has dimensions of coordinates and a,b = 1,2,...N.

This type of reduced model was introduced because its large-N limit reproduces the large-N limit of the whole model, represented by action (1). This property was originally proven by means of conventional perturbation theory in the coupling constant g. A correspondence was established for the planar graphs of the complete theory with the dominant graphs in the large N limit of the reduced model. Reduced models were also established for gauge theories in the lattice and in the continuum framework[33].

The formalism of stochastic quantization provides a more global view of the large-N limit of the reduced model[33].

The simplicity of the reduced models has turned out to be only superficial. So far, to find the equations for the large-N limit of reduced models seems to be a task comparable to the same problem formulated in the complete model.

The self-energy expansion is applied to a new version of the reduced model and we derive the equations for the large-N limit which are, for the leading large-N contribution, only the same as the corresponding equations of the complete model. We find that the large-N limit of the reduced model yields a model slightly over-simplified with respect to the complete model.

We introduce a new reduced model taking the large-N limit at the level of the reduced action. We assign a continuous momentum variable to each color index variable, according the following rule: $A_{ab} \rightarrow L^{-4}A(p_a,p_b)$, $f_b \rightarrow L^{-4}f(p_b)$ and $(L^{-4}/N)\sum_{a=1}^{N} \rightarrow d^4p$.

This new reduced model we shall call "continuous color reduced model" and its action is

$$S=i(2\pi)^4((L^4N^2/2)\int((p-k)^2-m_1^2)A(p,k)A(k,p)\,dpdk$$
$$-N\int\bar{f}(p)(\gamma p+m_2)f(p)\,dp + N^2ig\int\bar{f}(p)\gamma_5A(p,q)f(q)\,dpdq$$
$$+ N\int(\bar{f}(p)\eta(p) + \bar{\eta}(p)f(p))\,dp \tag{28}$$

where we have added sources $\bar{\eta}$, η for the fermion variables. Note the similarities of the fermion pieces of the action to the corresponding one component field theory; however the matrix field becomes bilocal. We shall show that the continuous color reduced model reproduces correctly only the leading large-N contribution for the fermion Green's functions.

The starting point of the self-energy expansion method is the integration of the integration of the matrix field, which yields the following action

$$S^{(1)} = i(2\pi)^4 (-N\int \bar{f}(p)(\gamma p+m_2)f(p)dp + (N^2g^2/2L^4)\int V(p-q)\bar{f}(p)\gamma_5 f(q)\bar{f}(q)\gamma_5 f(p)dpdq + N\int(\bar{\eta}f + \bar{f}\eta)dp.$$

We introduce the self-energy field P to substitute the four fermion interaction

$$S^{(2)} = i(2\pi)^4(-N\int \bar{f}(p)(\gamma p+m_2-\gamma_5 P(p))f(p)dp + (L^4/2g^2)\int P^{\alpha\beta}(p)Q_0(p-k)P^{\beta\alpha}(k)dpdk) + N\int(\bar{\eta}f + \bar{f}\eta)dp$$

where Q_0 has been defined in the paragraph below eq. (5). The integration of P reproduces the action with the four-fermion interaction $S^{(1)}$. After the integration of the fermion fields we get the following action

$$S^{(3)} = -i(2\pi)^{-4}\int\bar{\eta}(p)S_P(p)\eta(p)\,dp -$$

$$\delta^4(0)\,\mathrm{tr}\left(\int S(p)\gamma_5 P(p)\,dp + (1/2)\int(S(p)\gamma_5 P(p))^2 dp +\right.$$

$$\left.(1/3)\int(\quad)^3 + \ldots\right) - (iL^4(2\pi)^4/2g^2)\,\mathrm{tr}\int P(p)P(k)Q_0(p-k)\,dpdk$$

$$= -i(2-)^{-4}\int\bar{\eta}S_P\eta + NS_0$$

where $S_P(p) = (\gamma p + m_2 - \gamma_5 P(p))^{-1}$.

In this last form of the action we have to interpret the momentum delta as $\delta^4(0) = NL^4$. Up to here the value of L has been irrelevant. We require that NL^4 approaches infinity as N increases. Since NL^4 becomes infinity the saddle point of the action becomes important in the large-N limit.

The saddle point equation emerges from the condition $\delta S_0/\delta P(q) = 0$ and it is explicitly given by $P(p) = -ig_0\int V(p-q)S_P(q)\gamma_5\,dq$, which is exactly equation (14) derived previously for the large-N limit of the self-energy contribution to the fermion propagator.

To investigate the contributions beyond the leading order, we calculate the second functional derivative

$$\delta^2 S_0/\delta P^{\alpha\beta}(q)\delta P^{\varepsilon\sigma}(k) = -(S_P(q)\gamma_5)^{\beta\varepsilon}(S_P(q))^{\sigma\alpha}\delta(q-k) + (i/g_0)\delta^{\beta\varepsilon}\delta^{\sigma\alpha}Q_0(q-k).$$

Comparing this last expression with eq. (25) we find that this second derivative does not have the correct form in order to obtain the bound state equation. For further discussion on this problem see last section of chapter 3.

3. THE LARGE-N EXPANSION OF THE TWO-DIMENSIONAL U(N) QCD

I. INTRODUCTION

The masses of mesons, which are bound states of a quark with an antiquark, are given in the large-N expansion of the two-dimensional U(N) QCD by an equation first derived[4] by G. 't Hooft.

Since then, different aspects of 't Hooft's results have been extensively discussed from different points of view. In this chapter, we present the derivation of the large-N expansion based on the singlet field method. We follow closely the presentation given in our first derivation[23]. Now all the steps are justified from the general presentation given in the previous chapter. In Section II the quark and gluon fields are replaced by the singlet field in the generating functional. In Section III we make the derivation of the bound states equation for the quark-antiquark pairs. In Section IV we show that cubic and higher terms in the large-N expansion of the action generate the multimeson amplitudes which were first obtained by C.G. Callan, Jr., N. Coote, and D.J. Gross with the diagrammatic method[5]. Section V exibits the application of the singlet field method to the large-N reduced model of two-dimensional QCD .

The original reduced model is considered first. The correct leading large-N contribution to the fermion propagator is derived. The first part suggests a new form of the reduced model which avoids the quenched momentum integration in the calculation of the fermion Green's functions. A general rule is given to obtain this new reduced model[35]. Finally, we show that the reduced model does not predict the correct meson bound state equation. We identify the origin of this problem, and some necessary modifications are pointed out.

The QCD Lagrangian is given by

$$L = (1/4)\,\mathrm{tr}(G_{\mu\nu}G_{\mu\nu}) - \bar{q}(\gamma_\mu D_\mu + m)q \tag{1}$$

with its various terms defined as

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \tag{2a}$$

$$D_\mu q_a = \partial_\mu q_a + gA_\mu^{ab} q_b \tag{2b}$$

$$A_\mu = A_\mu^k T^k \qquad , k= 1, 2, \ldots, N^2 . \tag{2c}$$

$$A_\mu^+ = -A_\mu \qquad , T^{k+} = T^k \tag{2d}$$

$$tr(T^j T^k) = \delta^{jk} \quad , T^a_{rs} T^a_{en} = \delta_{rn}\delta_{se} \tag{2e}$$

where T^b are the generators of U(N) group.

To deal with the QCD theory defined by this Lagrangian in one space and one time dimensions, it is advantageous to introduce the light cone coordinates defined through

$$x^{\pm} = 2^{-1/2}(x^1 \pm x^0) = x_{\mp} \tag{3a}$$

$$p_{\pm} = 2^{-1/2}(p_1 \pm p_0) \tag{3b}$$

$$A_{\pm} = 2^{-1/2}(A_1 \pm A_0) \tag{3c}$$

The scalar products of vectors and the properties of the γ matrices in these coordinates are

$$\begin{aligned} x^+p^- + x^-p^+ &= (x^+x^-)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}p^+\\ p^-\end{bmatrix} \\ &= (x^0x^1)\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}p^0\\ p^1\end{bmatrix} \end{aligned} \tag{4}$$

$$\gamma_{\pm} = 2^{-1/2}(\gamma_1 \pm \gamma_0) \tag{5a}$$

$$\gamma_-^2 = \gamma_+^2 = 0 \quad , \quad \gamma_+\gamma_- + \gamma_-\gamma_+ = 2\, I \tag{5b}$$

$$\mathrm{tr}(\gamma_+) = \mathrm{tr}(\gamma_-) = 0 \quad , \quad \mathrm{tr}(\gamma_+\gamma_-) = 2\ . \tag{5c}$$

The above definitions give us an appropriate set of coordinates to work in the light cone gauge which is defined by $A_- = 0$.

II. THE GENERATING FUNCTIONAL.

In this section we integrate the gluon field variables of the generating functional which then depends on the quark field variables only. Later, a bilocal singlet field is introduced, and through the integration of the quark field variables we find an action only in terms of the singlet field.

In the light cone gauge, the pure gauge field part of the Lagrangian becomes

$$G_{\mu\nu}G_{\mu\nu} = -2(\partial_- A_+)^2 \tag{7}$$

and using equations (3.2e) and (3.7) we see that (3.1) is now given as

$$L = (1/2)(\partial_- A_+^b)^2 - \bar{q}(\gamma\partial + m + ig\gamma_- A_+^a T^a)q. \tag{8}$$

This form of the Lagrangian is particularly simple because it allows us to directly eliminate the gauge field variables from the generating functional as we show in the next calculation.

The generating functional is defined by

$$Z = \int D\bar{q}Dq \prod_{b=1}^{N} DA_{+}^{b} \exp\left(i\, S(\bar{q},\, q,\, A_{+}) \right) \tag{9}$$

where

$$S' = \int L \; dx^{+}dx^{-} \; . \tag{10}$$

In view of the quadratic dependence of the Lagrangian (8) on the gauge fields, to perform their integration in the generating functional it is convenient to use a functional expansion of the action (10) around the saddle point solution $(A=A^{(s.p.)})$. Therefore we have

$$\int DA_{+} \exp(i\; S(A_{+})\,) = \text{constant} \exp\left(i\; S(A^{(s.p.)})\right), \tag{11}$$

After this integration, the generating functional can be defined as

$$Z = \int D\bar{q}Dq \; \exp\left(i\; S(q,q)\right) \tag{12}$$

where $S(q,q)$ is given by

$$S(\bar{q},q) = S_0 - (g^2/4)\int V(y,\, x)\; J^b(y) J^b(x)\, d^2x d^2y$$

and

$$S_0 = - \int \bar{q}_b(\gamma\partial + m) q_b d^2x, \qquad b = 1, \ldots, N$$

$$J^k(x) = \bar{q}(x) \gamma_- T^k q(x) \qquad k = 1, \ldots, N^2$$

$$V(y,x) = \delta(y^+ - x^+) \left| y^- - x^- \right| . \tag{13}$$

Finally, using (2e), one finds that

$$S(\bar{q},q) = S_0 + S_1 = - \int q_r(\gamma\partial + m) q_r +$$

$$- (g^2/4) \int \bar{q}_r(y) \gamma_- q_s(y) \bar{q}_s(x) \gamma_- q_r(x) V(y,x) dxdy. \tag{14}$$

In order to study the possible bound states of quarks, it is useful to define a new bilocal singlet field. The generating functional (12) in terms of our singlet field is

$$Z = \int D\bar{q} Dq D\lambda \, \exp(\, i(S_0 + \int \bar{q}_r(y) \gamma_- \lambda(y,x) q_r(x) d^2x d^2y$$

$$- (1/g^2) \int d^2x d^2y \lambda_{\alpha\beta}(y,x) \lambda_{\beta\alpha}(x,y) / V(x,y)) \,) \tag{15}$$

The integration of the λ-field shows immediately that the above generating functional is the same as shown in (12). The exact solution of the λ-field is

$$\lambda_{\alpha\beta}(y,x) = (g^2/2)V(x,y)\bar{q}_b^{\rho}(x)\gamma_-^{\rho\beta}q_b^{\alpha}(y). \qquad (16)$$

At this point, we recall the anticommuting nature of the quark field variables on the generating functional. It is now possible to integrate the quark field variables in expression (15). After this integration, the generating functional will contain only the λ-field variable, and the bound state equation will then be derived from the second functional derivative with respect to the λ-field evaluated at the classical solution.

In order to carry out the integration over the quark field variables of the generating functional (3.15),the following integral is relevant:

$$F(\lambda) = \int D\bar{q}Dq \exp i(S_0 + \int q_b(y)\gamma_-\lambda(y,x)q_b(x)d^2xd^2y), \qquad (17)$$

The functional (17) contains the quark field variables as a bilinear form. Thus the result of the integration is

$$F(\lambda) = (\det(\delta(x-y)I - \int \underline{S}_F(z-x)\lambda(y,z)d^2z))^N \qquad (18)$$

where

$$\underline{S}_F = S_F \gamma_-$$

$$(\gamma\partial + Im)^{\alpha\beta}(x)\ S_F^{\beta\varepsilon}(x-z) = \delta^{\alpha\varepsilon}\delta(x-z). \tag{19}$$

The determinant given in (18) can be expanded in terms of an exponential function according to the formula

$$\det(\delta(x-y)I - M(x,y)) = \exp(\ \mathrm{tr}\ln(\delta(x-y)I - M(x,y))\ )$$

$$= \exp(\ -\mathrm{tr}(\int M(x,x)dx + (1/2)\int M(x,y)M(y,x)dxdy + \ldots)\ ) \tag{20}$$

From (17), (18), and (20) the generating functional acquires a form such that only the λ-field remains as an integration variable

$$Z = \int D\lambda \exp(\ -N\mathrm{tr}\,(i/g^2N)\int d^2x d^2y\ \lambda(y,x)\lambda(x,y)/V(x,y)\ +$$

$$+ \int \underline{S}_F(z-x)\lambda(x,z)d^2z d^2x + 1/2\int \underline{S}_F(z-x)\lambda(y,z)\underline{S}_F(v-y)\lambda(x,v)dzdvdxdy + \ldots$$

$$= \int D\lambda \exp(\ -NS(\lambda)\ ). \tag{21}$$

III. THE EQUATION FOR THE BOUND STATES.

Following Witten's idea, the next task is to expand the action $S(\lambda)$ defined in (21) around the classical solution λ_0 of the λ-field. As (21) shows, N appears as an overall factor multiplying the entire action. As N becomes large, and thinking that N corresponds to 1/h, we can establish a formal analogy between the standard semi-classical approximation in field theory, and the following large-N expansion

$$NS(\lambda) = NS(\lambda_0 + \delta\lambda/N^{-1/2}) = NS(\lambda_0) + (1/2!)\int dxdy \left.\frac{\delta^2\lambda}{\delta\lambda(x)\delta\lambda(y)}\right|_{\lambda=\lambda_0} \delta\lambda(x)\delta\lambda(y)$$

$$+ (1/3!N^{1/2})\int dxdydz \left.\frac{\delta^3\lambda}{\delta\lambda(x)\delta\lambda(y)\delta\lambda(z)}\right|_{\lambda=\lambda_0} \delta\lambda(x)\delta\lambda(y)\delta\lambda(z) + \ldots. \qquad (22a)$$

where

$$(\delta S(\lambda)/\delta\lambda(x))_{\lambda=\lambda_0} = 0 \qquad (22b)$$

For the purpose of this expansion, it is necessary to solve equation (22b). The first functional derivative is given by

$$\delta S(\lambda)/\delta\lambda_{\alpha\beta}(u,v) = (2i/g^2N)\lambda_{\beta\alpha}(v,u)/V(v,u) + \underline{S}_F^{\beta\alpha}(v-u) +$$

$$+ \int (\underline{S}_F(z-u)\lambda(y,z)S_F(v-y))^{\beta\alpha} d^2z d^2y + \ldots \qquad (23)$$

It will be shown that the solution λ_0 of (22b) is proportional to the sum of the irreducible self-energy parts of the quark propagator in the limit when $N \to \infty$. We recall here that in this limit g^2N should remain finite. Indeed, the trace of the Fourier transform of equation (22b) is the equation for the self-energy[3]. To show this explicitly, we introduce the following notation:

$$\lambda(v-u) = 1/(2\pi)^2 \int e^{ik(v-u)} \lambda(k) d^2k \qquad (24a)$$

$$V(v,u) = \int e^{-ik(v-u)} V(k) d^2k \qquad (24b)$$

$$\underline{S}_F(v-u) = 1/(2\pi)^2 \int e^{ik(v-u)} \underline{S}_F(k) d^2k. \qquad (24c)$$

then, using (23) and (24), the Fourier transform of equation (22b) is

$$\begin{aligned}\lambda(p) &= ig^2N/2\int V(k)(\underline{S}_F(k+p) + \underline{S}_F(k+p)\lambda(k+p)\underline{S}_F(k+p) + \ldots)d^2k \\ &= ig^2N/2\int V(k)\underline{S}_F(k+p)(I - \lambda(k+p)\underline{S}_F(k+p))^{-1}d^2k\end{aligned} \qquad (25)$$

where the propagator S_F is given by

$$S_F(k) = (m - i(k_+\gamma_- + k_-\gamma_+)/(m^2 + 2k_+k_- - i\epsilon) \qquad (26)$$

and

$$\underline{S}_F = S_F(k)\gamma_- = \underline{S}(k)M(k) \qquad (27)$$

where

$$\underline{S}(k) = -ik_-/(m^2 +k_+k_- - i\epsilon) \qquad (28a)$$

$$M(k) = ((im/k_-)\gamma_- + \gamma_+\gamma_-). \qquad (28b)$$

In order to solve equation (25), we notice that it is possible to simplify its matrix form to a non-matrix equation as follows; note that a basis of 2 x 2 matrices is the set

$$\gamma_-,\ \gamma_+,\ \gamma_-\gamma_+,\ \gamma_+\gamma_- . \qquad (29)$$

We can therefore establish in general a linear combination of the following type

$$\lambda = (i/4)\,(\,2\lambda^{(1)}\gamma_- + 2\lambda^{(2)}\gamma_+ + \lambda^{(3)}\gamma_-\gamma_+ + \lambda^{(4)}\gamma_+\gamma_-\,). \tag{30}$$

Now, substituting (30) into (25), multiplying (25) by some of the various matrices given in (29), and taking trace of the resulting equation, we see that $\lambda_0^{(2)} = \lambda_0^{(3)} = 0$. Inserting (30) in equation (25) and taking its trace gives the following equation for $\lambda_0^{(4)}$

$$\lambda_0^{(4)}(p) = (4g^2N/2)\int d^2k V(k - p)\underline{S}(k)/(1 - \underline{S}(k)\lambda_0^{(4)}(k)\,). \tag{31}$$

A similar procedure is useful to derive the equation for $\lambda_0^{(1)}$. This function need not be known to calculate the dressed propagator which is given by

$$R_F(k) = \underline{S}_F(k)\,(\,1 + \lambda_0(k)\underline{S}_F(k) + \lambda_0(k)\underline{S}_F(k)\lambda_0(k)\underline{S}_F(k) + \ldots)$$

$$= \underline{S}(k)M(k)/(\,1 - i\underline{S}(k)\lambda_0^{(4)}(k)\,) = R(k)M(k). \tag{32}$$

Hence, to calculate R_F, the relevant part of λ is $\lambda^{(4)}$, and the relevant part of $\underline{S}_F$ is $\underline{S}$. (See equation (28)).

If we use the following potential (see (24) and (13))

$$V(x,y) = |x^- - y^-|\,\delta(x^+ - y^+) = \int e^{-ik(x-y)} V(k)\, d^2k_-$$

(33a)

where

$$V(k)/2 = (-1/(2\pi)^2) P(1/k_-^2)$$

$$P(1/k_-^2) = (1/2)\ (\ (k_- + i\varepsilon)^{-2} + (k_- - i\varepsilon)^{-2})\ .$$

(33b)

The solution of equation (31) is

$$\lambda_0^{(4)}(p_+, p_-) = g^2 N/\pi p_- .$$

(34)

Note that the prescription $P(1/k^2)$ as defined above eliminates the infrared difficulties treated in references 4, 5.

The final field theory after the expansion of the bilocal singlet field around its classical solution is given by expansion (3.22a). For the large-N limit the quadratic term in the singlet field $\delta\lambda$ remains. This quadratic term of the action gives us the corresponding free field theory for the bilocal singlet field $\delta\lambda$. By solving the wave equation for the free field theory we can obtain the mass spectrum of the mesons (see chapter 2).

More precisely, the equation for the mass spectrum of the possible bound states of $\bar{q}$ and q is given by

$$\int (\delta^2 S(\lambda)/\delta\lambda^{(4)}(u,v)\delta\lambda^{(4)}(a,b))_{\lambda=\lambda_0} V(u,v)\Psi(u,v)\,d^2u\,d^2v = 0 \tag{35}$$

with

$$(\delta S(\lambda)/\delta\lambda^{(j)}(u,v))_{\lambda=\lambda_0} = 0\,, \qquad j = 1,2,3,4.$$

Equation (35) is the coordinate representation for the bound state equation given in formula (15) of reference 4. The unknown function $\Psi(u,v)$ is the eigenfunction. To see clearly the form of equation (35), it is first necessary to compute the second functional derivative of the action $S(\lambda)$ with respect to $\lambda^{(4)}$

$$(\delta^2 S(\lambda)/\delta\lambda^{(4)}(u,v)\delta\lambda^{(4)}(a,b))_{\lambda=\lambda_0} = -(i/2g^2N)\,\delta(v-a)\,\delta(b-u)/V(v,u) - R(v-a)R(b-u) \tag{36}$$

and therefore equation (35) becomes

$$\Psi(b,a) = i2g^2N\int R(b-u)R(v-a)V(u,v)\Psi(u,v)\,d^2u\,d^2v. \tag{37}$$

By introducing the Fourier transforms(see (24), (32), (33))

$$\psi(b,a) = 1/(2\pi)^4 \int e^{ikb-ipa} (k,p)\, d^2k\, d^2p$$

(38a)

$$R(b-u) = 1/(2\pi)^2 \int e^{ik(b-u)} R(k)\, d^2k.$$

(38b)

The momentum representation of (37) is

$$\psi(k_1,k_2) = i2g^2 N R(k_1) R(k_2) \int \psi(k_1+k, k_2+k) V(k)\, d^2k$$

(39)

and changing to new variables according to

$$k_1 = p - r, \quad k_2 = p$$

$$\psi(k_1,k_2) = s(k_2, k_2 - k_1)$$

the equation for $s(p,r)$ is finally given by

$$s(p,r) = i2g^2 N\, R(p-r)\, R(p) \int s(p+k,r) V(k)\, d^2k.$$

(40)

which is the equation for the quark-antiquark bound states written in (15) of reference 4. The solutions of equation (40) are discussed in references 4 and 5.

IV. HIGHER DERIVATIVES AND SCATTERING AMPLITUDES OF MESONS.

We now turn to the investigation of the higher order terms in the large-N expansion given in equation (22a). Since the relevant part of λ_0 is $\lambda_0^{(4)}$ (see (30)), we compute the third functional derivative with respect to $\lambda^{(4)}$

$$i\left(\delta^3 S(\lambda)/\delta\lambda^{(4)}(u_1,v_1)\,\delta\lambda^{(4)}(u_2,v_2)\,\delta\lambda^{(4)}(u_3,v_3) \right)_{\lambda=\lambda_0}$$

$$= R(v_1-u_3)R(v_3-u_2)R(v_2-u_1) + R(v_1-u_2)R(v_2-u_3)R(v_3-u_1) \tag{41}$$

where R was defined in (32). The cubic term of the action of the large-N expansion is therefore given by

$$-(i/3N^{-1/2})\int R(v_1-u_2)\,\delta\lambda^{(4)}(u_2,v_2)R(v_2-u_3)\,\delta\lambda^{(4)}(u_3,v_3) \\ R(v_3-u_1)\,\delta\lambda^{(4)}(u_1,v_1)\prod_k du_k dv_k. \tag{42}$$

We may regard $\delta\lambda^{(4)}$ as the effective field of the mesons of the theory. We see that equation (42) describes a disintegration process of a meson into two mesons. The amplitude for this process in momentum space is given by

$$CN^{-1/2}\int dk R(k) R(k-r_2) R(k-r_1) \Omega_1(k,r_1-k)\Omega_2(k,r_2-k)\Omega_3(r_2,r_1-k)$$

where C is a numerical constant and Ω_j are the bound state wave functions. This agrees with the expression derived by the diagrammatic method.

All the higher derivatives with respect to $\lambda^{(4)}$ can be calculated, and the expressions are analogous to the third derivative shown in equation (41). The term with a derivative of order n generates a process with n external mesons, and is of order $(N^{-1/2})^{n-2}$.

V. REDUCED MODEL

To keep uniformity with previous notations we use the prescription $-i\partial f \rightarrow p_a f_a$ to build the associated reduced model. The lagrangian to start with is given at the begining of section II. The reduced action[33-35] in this case is

$$i(2\pi)^{-2} S_R = -\tfrac{1}{2}(p_{-a}-p_{-c})^2 A_{ac}A_{ca} - \bar{f}_a\left((i\gamma p_a+m)\delta_{ac}+ig\gamma_- A_{ac}\right)f_c + \bar{\eta}_a f_a + \bar{f}_a \eta_a \tag{2}$$

where A stands for A_+. The quarks variables $\bar{f}, f$ are anticonmuting variables as well as the sources $\eta, \bar{\eta}$.

The generating functional of the reduced model would be, $Z(P,\bar{\eta},\eta) = \int \prod dA\, d\bar{f}\, df \exp S_R$, where $P_{ab} = \delta_{ab} p_b$.

The goal is to modified the action in such a way that only the field representing the self-energy of the fermion propagator remains as integration variable. For this purpose we integrate the gluon variables. Thus the reduced action is given by

$$i(2\pi)^{-2} S_R^{(1)} = \tfrac{1}{2} g^2 V_{ab} \bar{f}_a \gamma_- f_b \bar{f}_b \gamma_- f_a - \bar{f}(i\gamma P+m)f + \bar{\eta} f + \bar{f}\eta\,, \tag{3}$$

where $V_{ab} = -1/(p_{-a}-p_{-b})^2$.

The infrared divergence of the gluon propagator will be troublesome at the level of the integral equations governing the self-energy in the large-N limit. However is already known that a proper definition of the propagator using a small momentum cutoff or a special principal value prescription defined by

$$P(1/k^2)=\tfrac{1}{2}((k+ie)^{-2}+(k-ie)^{-2}), \qquad e \rightarrow 0$$

removes completely the infrared singularity in the two-dimensional QCD.

The fermion self-energy variable is introduced as follows,

$$i(2\pi)^{-2}S_R^{(2)} = -\bar{f}(i\gamma P+m)f + \tfrac{1}{4}i\bar{f}_a\gamma_- E_a f_a - (1/32g^2)E_a^{\alpha\beta}V_{ab}^{-1}E_b^{\beta\alpha} + \bar{\eta}f + \bar{f}\eta \ . \quad (4)$$

The integration over E would yield exactly the action given by (3). Before doing the fermion integration we can make some simplification of the E-variable due to the lightcone coordinates. We use the following general decomposition,

$$E^{\alpha\beta}=\left(E^{(1)}\gamma_- + E^{(2)}\gamma_+ + E^{(3)}\gamma_-\gamma_+ + E\gamma_+\gamma_-\right)^{\alpha\beta}$$

and replace it in the generating functional (the following properties of the γ matrices are important: $\gamma_-^2=\gamma_+^2=0$; $\gamma_+\gamma_-+\gamma_-\gamma_+=2I$; $\mathrm{tr}\gamma_+=\mathrm{tr}\gamma_-=0$).

We find that $E^{(1)}$ can be integrated, which imply that $E^{(2)}=0$ and $E^{(3)}$ decouples from the action, so just E is left as integration variable,

$$i(2\pi)^{-2}S_R^{(3)} = -\bar{f}(i\gamma P+m)f + i\tfrac{1}{2}\bar{f}_a\gamma_- f_a E_a \\ -(i/8g^2)E_a V_{ab}^{-1}E_b + \bar{\eta}f + \bar{f}\eta . \qquad (5)$$

The integration over the fermion yields a determinant which after being exponentiated induces the following action,

$$S_R^{(4)} = \sum_{a=1}^{N} \ln(1-i\underline{S}(p_a)E_a) + (2\pi)^2 i(\ (1/8g^2) \\ E_a V_{ab}^{-1}E_b - \bar{\eta}_a S_E(p_a)\eta_a\ ) , \qquad (6)$$

where

$$\underline{S}(p) = -ip_-/(m^2+2p_+p_--ie),$$

$$S_E(p_a) = S(p_a)(1+ (i/2)E_a\gamma_- S(p_a) + ((i/2)E_a\gamma_- S(p_a))^2+\ldots)$$

and

$$S(p) = (m - i(p_+\gamma_- + p_-\gamma_+))/(m^2 + 2p_+p_- - ie).$$

In the source term of action (6) we can identify the full fermion propagator $S_{\bar{E}}$, E being the self-energy correction. The saddle point equation for the E variable is

$$E_b = (g/\pi)^2 V_{bc}\underline{S}(p_c)/(1 - i\underline{S}(p_c)E_c) \ . \tag{7}$$

Now we take the large-N limit of this last equation. We interpret the sum over p_c as a random evaluation of an integral with a volume element equal to 1/N. We may choose the volume of integration as large as we want to approximate the infinite volume limit. Thus N should be as large as the number of momentum space points we choose to approximate the continuous space. Therefore in the large-N limit equation (7) is given by,

$$E(p) = -(g^2N/\pi^2)\int d^2k \ \underline{S}(k)/(p_- - k_-)^2(1 - i\underline{S}(k)E(k)) \tag{8}$$

which is the equation for the leading large-N contribution to the quark propagator. Nevertheless, there is a remaining question, why the saddle point of action (6) is the good approximation for the large-N limit?. The answer lies in the process that changed the algebraic equation (7) into the integral equation (8). We apply the same procedure to the original reduced action (2) and a general rule could be stated to build the new reduced model: identify, $A_{ab} \rightarrow A(p_a, p_b)$, $f_b \rightarrow f(p_b)$ and $\sum_{a=1}^{N} \rightarrow N\, dp$. Applying this rule to eq. (2) we obtain

$$S_0 = -i4\pi^2 N \int (-\tfrac{1}{2} N \int (p_- - k_-)^2 A(p,k) A(k,p) d^2k - \bar{f}(p)(i\gamma p + m) f(p) - igN \int \bar{f}(p) \gamma_- A(p,k) f(k) d^2k + \bar{\eta}(p) f(p) + \bar{f}(p) \eta(p)) d^2p. \tag{9}$$

The generating functional is

$$Z(\bar{\eta}, \eta) = \int D\bar{f}(p) Df(k) DA(k,k') \exp S_0 .$$

Note that the fields do not have color index and the generating functional does not depend on P. Action (9) generates the same large-N limit as the original reduced action (2) for the fermion Green's functions globally invariant under U(N). These Green's functions are given directly in momentum space without additional "quenching" procedures.

The large-N limit of $\langle \bar{f}_a(p) f_a(q) \rangle_{\text{whole } S'}$ is given by the large-N limit of $\langle f(p) f(q) \rangle_{S_0}$. In analogy to the derivation of (6) the integration over A in (9) produces an action with quartic fermion interaction, which is equivalent to,

$$S_0^{(2)} = -i4\pi^2 N \int (-\bar{f}(p)(i\gamma p+m-i\tfrac{1}{4}\gamma\_E(p)f(p) + \qquad (10)$$
$$- (1/32g^2 N)\int E^{\alpha\beta}(p)\ Q(p-k)\ E^{\beta\alpha}(k)dk + \bar{\eta}f + \bar{f}\eta\ )d^2p\ ,$$

where Q(p-p') is a function such that if, $g(p) = -\int d^2q h(q)/(p\_-q\_)^2$ then $h(p) = \int Q(p-q)g(q)d^2q$.
The matrix $E^{\alpha\beta}(p)$ can be simplify with the same decomposition used previously, from where only E variable remains. Therefore, integrating over the fermions we get the following action,

$$S_0^{(3)} = N\int (\ln(1-i\underline{S}(p)E(p)) + (i\pi^2/2g^2N)\int E(p)Q(p-q)E(q)dq$$
$$- i4\pi^2\bar{\eta}(p)S_{E(p)}\ n(p)\ )dp\ , \qquad (11)$$

where it has been taken $\delta^2(0)=N$. The saddle point equation for E(p) is exactly the equation given in (8).

Expression (11) is the analog of (6) and we can see that actually there is an overall factor $\bar{N}$ which insures that the saddle point equation with respect to E is the correct approximation for the leading large-N contribution. Formally, in (11) the overall factor N of the action is already infinite, therefore only the saddle point of the action is meaningfull within the reduced model context.

Taking functional derivatives respect to $\bar{\eta}$ and η we get that the four fermion Green's function is proportional to $S_{E(p)} S_{E(k)}$ where E is evaluated at the saddle point solution E_0. This is just the first term of the ladder approximation which is the second contribution in the large-N expansion of the four fermion function. The whole ladder contribution is necessary to obtain the correct large-N amplitude for the meson bound states. If we want to extend the meaning of the reduced model beyond the leading large-N contribution we have to keep N large but finite in action (11), though we do not expect the reduced model would yield the correct contribution beyond the leading large N term. It has been shown (see chapter 2) that the second functional derivative evaluated at saddle point solution produces the operator to obtain the meson bound state equation of the model in the large-N expansion. This second derivative is

$$S_F(p)S_F(k)\,\delta(p-k) + i(\pi^2/g^2N)Q(p-k)$$

where, $S_F(p) = \underline{S}(p)/(1 - \underline{S}(p)\ E_0(p))$. This is not the correct amplitude to obtain the meson bound state equation. From the complete theory given by eq. (1) we realize that this amplitud B should be,

$$B(p_1,k_1,p_2,k_2) = S_F(p_1)S_F(k_1)\delta(p_1-k_2)\delta(k_1-p_2) + (i\pi^2/g^2N)G(p_1-k_2,k_1-p_2).$$

The meson bound state equation is,

$$\int B(p_1,k_1,p_2,k_2)s(k',p_2-k_2)V(k'-k_2)dk'dk_2dp_2 = 0\ ,$$

here V is the gluon propagator $(V=-1/q_-^2)$, s the eigenfunctions and G is a function such that if $\int dqX(k+q,p+q)V(q^2) = Y(p,k)$ then $X(k,p) = \int G(k-k',p-p')Y(p',k')\ dk'dp'$.

A more standard form of the meson bound states equation is

$$s(p,k_1-p) = (g^2N/\pi^2)S_F(p)S_F(k_1)\int s(k',k_1-p)V(k'-p)dk'.$$

Indeed, the reduced model could not predict the amplitud B, since the self-energy variable depend only on one coordinate E(p), whereas in the whole model (1) it depends on two coordinates, E(p,k). However this is not a serious drawback of the reduced model since the crucial point is to find the correct leading large-N contribution to the fermion propagator, which once is known it can be used directly in the standard form of the meson bound state equation in the ladder approximation.

4. THE SINGLET FIELDS METHOD AND THE LARGE-N EXPANSION OF THE MATRIX MODEL

I. INTRODUCTION

This chapter considers a four-dimensional matrix model coupled to fermions and proposes an action from which a systematic large-N expansion can be obtained.

To deal with matrix self-interactions, additional fields are introduced[36]. The quartic self-interaction is reduced to Yukawa couplings with auxiliary anticonmuting fields.

The whole model can be reformulated in terms of singlet combinations of the fermions and the auxiliary fields. The resulting action has not any dependence on the group indeces. There remains an overall factor N, once the coupling constants are rescaled by the appropriate N factors.

The overall N factor and the explicit dependence on the fermion sources allow a systematic expansion of the fermion Green's functions in powers of $N^{-1/2}$.

This chapter is organized in the following way. In Section II, the preliminary calculations for the full four-dimensional case are illustrated by making use of the one-dimensional case. Many of the functional manipulations in the continuum can be more easily understood in the formulation given by a finite number of variables. In the one-dimensional version the discrete representation of path integrals can be readily derived at each step of the calculations .

The four-dimensional case is developed in Section III. Here, many formulas are directly analogous to their one-dimensional counterparts. The goal of this section is the derivation of the singlet fields action. The last section describes in general terms how the large-N expansion would proceed from the singlet fields action.

II. THE ONE-DIMENSIONAL MODEL

The matrix model in one dimension is a quantum mechanical analogy of the field theory model. The interaction between the matrix field and the fermions is not unique. Here we make a choice that has been already considered[11,12] and has given good results.

The model is defined by the following lagrangian

$$L = (1/2)\,\mathrm{tr}\,\left((dA/dt)^2 - m_1^2 A^2 - (g_1/2) A^4 \right) + \bar{f}_a d_t f_a + g_2 \bar{f}_a^{\alpha} A^{ab} E^{\alpha\beta} f_b^{\beta} \qquad (1)$$

where $d_t = i(d/dt) - m\sigma_3$ and $E = -\sigma_1$. The NxN matrix A is hermitian and $\bar{f}, f$ are the fermion variables. The lagrangian is invariant under global U(N) group transformations.

The generating functional is given by

$$Z(\bar{\eta},\eta) = \int DAD\bar{f}Df \exp\int(iL + \bar{\eta}f + \bar{f}\eta)\,dt$$

where $\bar{\eta}$, η are sources for the fermion variables. The fermions and their sources are anticonmutinq variables in the path integral formalism.

The discrete definition of the path integral is given by the following integration measures

$$DA = \lim_{n\to\infty} \prod_{k=1}^{n} \left(\prod_{a\leq b}^{N} dR_k^{ab} \prod_{a<b}^{N} dG_k^{ab} \right) \qquad (2)$$

where R = Real(A) and G = Imaginary(A). For the fermions

$$Df = \lim_{n\to\infty} \prod_{k=1}^{n} \prod_{a=1}^{N} df_k^a \,. \qquad (3)$$

The integrals of the action are defined by

$$\int q(t)\,dt = \lim_{n\to\infty} \sum_{k=1}^{n} q_k/n \,. \qquad (4)$$

Other related formulas can be found in the appendix.

The first step is to reduce the quartic interaction to a cubic interaction with an auxiliary matrix field

$$\exp(-(ig_1/4)\int trA^4 dt) = C(a_0)\int DP \, \exp(\int (i/2)\,tr\,((a_0^2/g_1 2)P^2 - a_0 A^2 P)\,dt\,). \qquad (5)$$

The constant a_0 has been introduced to rescale the coupling constant. The constant factor $C(a_0)$ dissapears once P is integrated.

The cubic interaction can be reduced[36] to Yukawa couplings with auxiliary anticonmuting variables. The key formulas to reduce higher power interactions are given by

$$\exp(\ \mathrm{tr}\int M(t)dt)) = \lim_{n\to\infty} c_0(n)\prod_{k=1}^{n}\prod_{a=1}^{N} dw'^a_k dw^a_k \exp(\sum_{k=1}^{n}(w'^a_k w^a_k + \sum_{j=1}^{n} w'^a_j T_{jk} M^{ab}_k w^b_k/n\ )/n)$$

$$= c_0\int Dw'Dw\exp(\int(w'(t)w(t) + \int ds w'(s)T(s,t)M(t)w(t)\ )dt\ ), \tag{6}$$

and

$$\exp(\int w'^a A^{ab} B^{bc} w^c dt = \int DuDv\ \exp(\int(w'Au + v(-u + Bw)dt\ )$$

where w', w, u, v are anticonmuting variables and T is a function such that the above equality is satisfied. Formulas similar to (6), using conmuting variables, have already been introduced[36] in a preliminary application of the quadratic singlet field method to QCD without fermions. The choice of T is not unique. Two simple elections are

$$T(s,t) = \delta_{t,s} = \delta(t-s)/\delta(0)$$
$$T(s,t) = \theta(t-s) \ .$$

The function θ is defined by $x \geqslant 0$, $\theta(x) = 1$; $x < 0$, $\theta(x) = 0$. The formula can be directly derived by integration of w'and w. Applying this formula to the cubic interaction we obtain

$$\exp(-(a_0 i/2)\, tr\int A^2 P dt) = \int Dw'Dw \exp(\int(w'w - (a_0 i/2) w_T A^2 P w) dt)$$

where we have defined $w_T(t) = \int w'(s) T(s,t) ds$.

In this last expression the matrix interactions can be splitted by the introduction of delta functions

$$\exp(-(a_0 i/2)\, tr\int A^2 P dt) = \int Dw'DwDuDv \exp(S') \tag{7}$$

where

$$S' = \int(w'w - v_1^a u_1^a - v_2^a u_2^a + b_A w_T^a A^{ab} u_1^b + b_P v_1^a P^{ab} u_2^b + b_A v_2^a A^{ab} w^b) dt \ .$$

In this last formula the constants satisfy the condition $2b_A^2 b_P = -ia_0$.

The complete action can now be given by

$$S^{(1)} = \int((i/2) tr((dA/dt)^2 - m_1^2 A^2 - (a_0^2/2g_1) P^2) + ig_2 \bar{f} A E f) dt + S' + S_F \tag{8}$$

where $S_F = \int(i\bar{f}d_t f + \bar{\eta}f + \bar{f}\eta)dt.$

In this last form of the action the matrix variables A, P can be integrated. The result of such integration is the following action

$$S^{(2)} = \int(w'w - vu + S_F + b_1(v_1^a u_2^a)^2)dt + (i/2)\int G(t-s)\,tr(K(t)K(s))dtds \tag{9}$$

where we have defined

$$(d^2/dt^2 + m_1^2)G(t-s) = \delta(t-s), \tag{10}$$
$$K^{ab}(t) = b_A(w_T^a u_1^b + v_2^a w^b) + g_2 f^a E f^b,$$
$$b_1 = ib_P^2 g_1/a_0^2 .$$

Expanding the product tr(KK) we find only singlet combinations of the auxiliary and fermion variables. The next step is to substitute these singlet products by invariant fields. From here on the cases in one and four dimensions are treated exactly in the same way. To avoid unnecesary repetitions we leave the present case and continue in the next section with the matrix model in four dimensions.

III. THE SINGLET FIELDS ACTION IN FOUR DIMENSIONS.

The model to be considered here is given by the action $S = S_{AF} + S_{IA} + S_{FN}$, where the various pieces of the action are defined as

$$S_{AF} = i\int((1/2)\,\mathrm{tr}(\partial_\mu A\partial^\mu A - m_1^2A^2) + ig_2\bar{f}_a\Gamma A_{ab}f_b)dx \quad (11)$$

$$S_{IA} = i\int(-g_1/4)\,\mathrm{tr}A^4(x)\;dx$$

$$S_{FN} = i\int(\bar{f}(i\gamma\partial - m_2)f + \bar{\eta}f + \bar{f}\eta)\;dx$$

The field A is a scalar hermitian matrix of NxN size, and $\bar{\eta},\eta$ are the sources of the fermion fields $f,\bar{f}$. The action is invariant under global U(N) transformations.

The generating functional iś given by $Z(\bar{\eta},\eta) = \int DAD\bar{f}Df e^S$.

The following convention is adopted: given an action $S^{(j)}(F)$ where F are the fields, then $S^{(k)}(F,F')$ is an action equivalent to $S^{(i)}$ The functional integration over the extra fields F' yields back the initial action, $\exp(S^{(j)}(F)) = \int DF'\exp(S^{(k)}(F,F'))$.

The term S_{IA} is given in terms of an auxiliary matrix field P

$$S_{IA}^{(1)} = i\int(1/2)\,tr(-a_0APA + (a_0^2/2g_1)P^2)\,dx \qquad (12)$$

The constant a_0 has been introduced for rescaling purposes and disappears completely once P is integrated. By introducing the auxiliary anticonmuting fields u_k^a, v_k^a, w_k^a ,a=1,..N , k=1,2; the interaction of P and A can be reduced to Yukawa couplings as follows

$$S_{IA}^{(2)} = \int(w_1^a(x)w_2^a(x) - v_k^a u_k^a + b_A(w_T A u_1 + b_A v_2 A w_2 + b_P v_1 P u_2 + (ia_0^2/4g_1)\,trP^2)\,dx.$$

In this last formula we have defined $w_T(x) = \int w_1(z)T(z,x)dz$. Also it is necesary that $-ia_0 = 2b_A^2 b_P$.

The field P can be integrated from the action $S_{IA}^{(2)}$,

$$S_{IA}^{(3)} = \int(w_1 w_2 - v_k u_k + g_2(w_T A u_1 + v_2 A w_2) + b_1(v_1 u_2)^2)\,dx \qquad (13)$$

where we have defined $b_1 = ig_1 b_P^2/a_0^2$. A choice we can make is $b_A = g_2$.

Now we consider the terms $S_{AF} + S_{IA}^{(3)} = S_{XF}$, which can be arranged as follows

$$S_{XF} = i\int tr((1/2)(\partial_\mu A \partial^\mu A - m_1^2 A^2) + AK)dx + S_X \qquad (14)$$

where $K_{ba} = g_2(\bar{f}_a \Gamma f_b + w_T^a u_1^b + v_2^a w_2^b)$

and $S_X = \int(w_1 w_2 + v_k u_k + b_1 (v_1 u_2)^2)dx$

We are in position of integrating A from S_{XF},

$$S_{XF}^{(1)} = (i/2)\int G(x-y) tr(K(x)K(y))dxdy + S_X \qquad (15)$$

where $(\Box + m_1^2)G(x-y) = \delta(x-y)$.

Expanding all the terms of $S_{XF}^{(1)}$,

$$S_{XF}{}^{(1)} = (i/2)g_2{}^2\int G(x-y)((\bar{f}_a(x)\Gamma)^\alpha f_a^\beta(y)(\bar{f}_b(y)\Gamma)^\beta f_b^\alpha(x) +$$
$$2(\bar{f}_a(x)\Gamma)^\alpha u_1^a(y) w_1^b(y) f_b^\alpha(x) + 2(\bar{f}_a(x)\Gamma)^\alpha w_2^a(y) v_2^b(y) f_b^\alpha(x)$$
$$w_1^a(x) u_1^a(y) w_1^b(y) u_1^b(x) + 2w_1^a(x) w_2^a(y) v_2^b(y) u_1^b(x)$$
$$+ v_2^a(x) w_2^a(y) v_2^b(y) w_2^b(x))dxdy + S_X.$$

The total action is given by $S^{(1)} = S_{FN} + S_{XF}^{(1)}$.

The term $S_{XF}^{(1)}$ contains only singlet combinations of the fermion fields and the auxiliary fields. To substitute this combinations, the following set of singlet fields are introduced through delta functions.

$$
\begin{aligned}
S_D = \int & V_1(x)(V'_1(x) - v_1(x)u_2(x))dx + \\
& dxdy(V_2(x,y)(V'_2(x,y) - v_2(x)u_1(y)) + \\
& W_1(x,y)(W'_1(x,y) - w_1(x)u_1(y)) + \\
& W_2(x,y)(W'_2(x,y) - v_2(x)w_2(y)) + \\
& W_0(x,y)(W'_0(x,y) - w_1(x)w_2(y)) + \\
& F_2^{\alpha}(x,y)(F'^{\alpha}_2(x,y) - (\bar{f}(x)\Gamma)^{\alpha}w_2(y)) + \\
& G_1^{\alpha}(x,y)(G'^{\alpha}_1(x,y) - (\bar{f}(x)\Gamma)^{\alpha}u_1(y)) + \\
& F_1^{\alpha}(x,y)(F'^{\alpha}_1(x,y) - w_1(x)f^{\alpha}(y)) + \\
& G_2^{\alpha}(x,y)(G'^{\alpha}_2(x,y) - v_2(x)f^{\alpha}(y)) + \\
& Q_0^{\alpha\beta}(x,y)(Q'^{\alpha\beta}_0(x,y) - (\bar{f}(x)\Gamma)^{\alpha}f^{\beta}(y))).
\end{aligned}
$$

In this last formula the sumation over the group indeces has been ommitted in all the singlet products, for example $v_2u_1 = v_2^a u_1^a$.

Let us introduce S_D in the total action. The pieces where the singlet fields would be replaced are contained in $S_X + S_{XF}^{(1)}$, therefore

$$S_D + S_{XF}^{(1)} = S_G + S_{XG}$$

where

$$\begin{aligned} S_G = (g_2^2/2)\int dxdy G(x-y)\,(& Q'^{\alpha\beta}_0(x,y)Q'^{\beta\alpha}_0(y,x) + 2G'^{\alpha}_1 F'^{\alpha}_1 + \\ & 2F'^{\alpha}_2 G'^{\alpha}_2 + W'_1 W'_1 + W'_2 W'_2 + 2W'_0 V'_2) \\ & + b_1 \int V'^2_1(x)dx + \\ & \int dxdy\, (Q^{\alpha\beta}_0(x,y)Q'^{\alpha\beta}_0(x,y) + V_2(x,y)V'_2(x,y) + \\ & F^{\alpha}_1 F'^{\alpha}_1 + F^{\alpha}_2 F'^{\alpha}_2 + G^{\alpha}_1 G'^{\alpha}_1 + G^{\alpha}_2 G'^{\alpha}_2 + \\ & W_1 W'_1 + W_2 W'_2 + W_0 W'_0) + \int V_1 V'_1 dx \end{aligned}$$

and

$$\begin{aligned} S_{XG} = & \int (w_1 w_2 - v_k u_k - V_1 v_1 u_2)dx \\ & - dxdy(V_2(x,y)v_2(x)u_1(y) + W_1(x,y)w_1(x)u_1(y) + \\ & W_2(x,y)v_2(x)w_2(y) + W_0(x,y)w_1(x)w_2(y) + \\ & F^{\alpha}_2 (x,y)(\bar{f}(x)\Gamma)^{\alpha} w_2(y) + G^{\alpha}_1 (x,y)(\bar{f}(x)\Gamma)^{\alpha} u_1(y) + \\ & F^{\alpha}_1(x,y)w_1(x)f(y) + G^{\alpha}_2(x,y)v_2(x)f^{\alpha}(y) + Q^{\alpha\beta}_0(x,y)(\bar{f}_a(x)\Gamma)^{\alpha} f^{\beta}_a(y)). \end{aligned}$$

The whole action is now given by

$$S^{(2)} = S_{FN} + S_G + S_{XG}$$

The fields u, v and w can be completely integrated from S_{XG}. The result of integrating u and v is given by

$$S_{XG}^{(1)} = \int w_1 w_2 dx - \int w_1(x)T(x,y)W(x,y)w_2(y)\, dxdy - \int(\, w_1(x)H_1(x) + H_2(x)w_2(x)\,)dx - \int \bar{f}(x)\Gamma(Q_0 + Q_1)(x,y)f(y)dxdy.$$

In the last formula we have defined the following terms

$$W(x,y) = W_0+W_{12} = W_0(x,y) + \int W_1(x,x')Z(x',y')W_2(y',y)dx'dy'$$

$$H_1^a(x) = \int F_1^{\alpha}(x,x')f_a^{\alpha}(x')dx' + \int W_1(x,r)Z(r,s)G_2^{\alpha}(s,t)f_a^{\alpha}(t)drdsdt$$

$$H_2^a(x) = \int \bar{f}_a(z)\Gamma F_2(z,x)dz + \int(\bar{f}_a(t)\Gamma)\; G_1(t,r)Z(r,s)W_2(s,x)drdsdt$$

$$Q_1^{\alpha\beta}(x,y) = \int G_1^{\alpha}(x,r)Z(r,s)G_2^{\beta}(s,y)drds$$

where $Z(r,s) = V_1(r)V^{-1}(r,s)$ and

$$\int V^{-1}(x,z)(\quad (z-y) - V_2(z,y)V_1(y)\;)\, dz = \delta(x-y)$$

The result of the integration over $\underline{w}$ field is given by the action

$$S_{XG}^{(2)} = N\mathrm{trln}(\,\delta(x-y) - W(x,y)\,) - \int \bar{f}(x)Q(x,y)f(y)\,dxdy.$$

where $Q = Q_0 + Q_1 + Q_2$ and

$$Q_2(x,y) = \int F_2(x,x')W_T(x',y')F_1(y',y)\,dx'dy' +$$
$$(\,F_2(x,x')W_T(x',y')W_1(y',u')Z(u',z')G_2(z',y) +$$
$$G_1(x,z')Z(z',u')W_2(u',x')W_T(x',y')F_1(y',y)\,)\,dx'dy'du'dz' +$$
$$G_1(x,z')Z(z',u')W_2(u',x')W_T(x',y')W_1(y',v')Z(v',w')G_2(w',y)\;d()'.$$

The term W_T is defined by,

$$W_T(x,y) = \delta(x-y) + \int T(x,z)W(z,y)\,dz + \int TWTW + \;\ldots\; .$$

The whole action is now given by

$$S^{(3)} = S_{FN} + S_G + S_{XG}^{(2)}.$$

The fermions fields are contained in the the term $S_{FN} + S_{XG}^{(2)} = S_N$. the integration over the fermion fields yields

$$S_N^{(1)} = NS_L - i\int \bar{\eta}^a(x)S_Q(x,y)\eta^a(y)\;dxdy$$

where

$$S_L = \mathrm{trln}(\,\delta(x-y) - \int T(x,z)W(z,y)\,dz\,) + \mathrm{trln}(\,\delta(x-y)I - \int S_F(x-z)(-i\Gamma)Q(z,y)\,dz\,)$$

$$(i\gamma\partial - m_2)S_F(x-y) = I\delta(x-y)$$

and

$$S_Q(x,y) = S_F(x-y) + \int S_F(x,u)(-i\Gamma)Q(u,v)S_F(v,y)\,dudv + \ldots$$

The whole action is given by $S^{(4)} = S_G + S_N^{(1)}$.

By examining S_G we see that all fields with prime superscript can be integrated and the result is given by

$$S_G^{(1)} = ig_2^{-2}\int(1/2G(x-y))(\,Q_0^{\alpha\beta}(x,y)Q_0^{\beta\alpha}(y,x) + W_1(x,y)W_1(y,x) + W_2W_2 + 2W_0V_2 + 2F_1G_1 + 2F_2G_2\,)dxdy - (1/4b_1)\,V_1^2(x)dx$$
$$= NS_C$$

Finally the action is given by $S_G^{(1)} + S_N^{(1)} = S^{(5)}$,

$$S^{(5)} = N(S_L + S_C) - i\int\bar{\eta}^a(x)S_Q(x,y)\eta^a(y)\,dxdy$$

From action $S^{(5)}$ the whole large N expansion of the fermion Green's functions can be derived. From $S^{(5)}$ we can find the Green's function which are invariant under the symmetry group as well as the non-invariant Green's functions. As we can see the non-invariant Green's functions are always combinations of Kroneker deltas and invariant Green's functions.

Before qoing into additional details of the large N expansion let us consider the important particular case given by $g_{1N} = 0$. So far it has been convenient to keep g_{1N} only as a factor of V_1, however as it is there we can not set $g_{1N} = 0$. To consider this case we have to rescale some singlet fields as follows,

$$g_{1N}^{-1} V_1 \rightarrow V_1, \quad g_{1N} V_2 \rightarrow V_2, \quad g_{1N}^{-1} W_0 \rightarrow W_0.$$

This rescaling changes Z into $g_{1N} Z$. Now we can set $g_{1N} = 0$. As a result we obtain $W = Q_1 = Q_2 = 0$, and the only surviving singlet field is Q_0.

IV. LARGE-N EXPANSION.

The basis for the large N expansion is provided by the large N action derived in the previuos section , which is $S = N(S_L + S_C) - i\int \bar{\eta}(x) S_Q(x,y)\eta(y)\,dxdy$.

The explicit dependence on the fermion sources provides the mechanism to derive any fermion (quark) Green's function in a power series expansion of $N^{(-1/2)}$.

For examnle the whole two point fermion Green's function (the propagator) is represented by

$$\langle \bar{f}^{a}(x) f^{b}(y) \rangle = \delta^{ab} \int DF \; S_Q(x,y) e^{N(S_L(F) + S_C(F))} \; ,$$

where all the singlet fields has been represented by F.

In general terms, the procedure for the large N expansion is as follows. The first step is to evaluate the saddle point equation of the singlet fields. Then the fields are shifted by a solution of the saddle point equation and the quadratic piece of the action has to be calculated. Additional large N corrections are calculated as perturbations around the quadratic piece.

To illustrate more explicitly how the large-N expansion would proceed let us take the case $\bar{g}_{1N} = 0$. In this case the only singlet field is given by $Q = Q_0$. The action is now given by $S_L + S_C = S(Q_0)$. The power expansion in $N^{(-1/2)}$ is obtained by the introduction of the shift of variables $Q_0 = R + N^{(-1/2)}R'$ in the generating functional. The action is expanded as follows

$$NS(Q_0) = NS(R + N^{(-1/2)}R') = NS(R) + N^{1/2}\int(\delta S/\delta Q_0(x,y))_{Q_0=R}R'(x,y)\,dxdy + (1/2!)\int(\delta S^2/\delta Q_0\delta Q_0)_{Q_0=R}R'R' + (1/3!N^{1/2})\int(\delta^3 S/\delta Q \quad) + \ldots$$

The function R is chosen such that is a solution of the saddle point equation

$$(\delta S/\delta Q_0)_{Q_0=R} = 0$$

This last equation in momentun space is given by

$$R(p) = -i(2\pi)^{-4}g^2_{2N}\int V(q)S_R(p+q)\,dq.$$

where the following propagator has been defined

$$S_R(k) = S(k)(1 + \Gamma R(k)S(k) + (\Gamma R(k)S(k))^2 + \ldots).$$

The free propagators are given by $S(k)(\gamma k + m_2) = I$ and $V(k) = 1/(k^2 - m_1^2)$. The equation for R is the leading large N contribiution to the fermion propagator.

The next step is to obtain the quadratic contribiution of the field R' and treat the higher powers as perturbations around the quadratic piece.

By looking at the four point fermion Green's function it can be shown that the quadratic contribution of R' is equivalent to the Bethe-Salpeter kernel in the first large N connected contribution (the leading large N contribution is disconnected). The large N approximation of the Bethe-Salpeter kernel is given by the ladder approximation taking the fermion propagator as S_R.

The equation for the posible bound states is given by the homogeneous kernel, which can be derived directly fron the second functional derivative respect to R'.

As a final remark, it is important to notice that the procedure used in this paper is more general and our main motivation has been to prepare the groundwork to be able to apply it to QCD. The details of the extension of this method to QCD are presented in Chapter 5.

5. THE SINGLET FIELDS FORMULATION AND THE LARGE-N EXPANSION OF THE QUARK SECTOR OF QCD

I. INTRODUCTION

One of the major analytical problems in formulating the large-N expansion has been the treatment of matrix fields interactions. To deal with matrix interactions, additional scalar fields were introduced[36]. This initial effort, done to derive a suitable action for the large-N expansion of the gluon sector,[36] led to difficulties. The resulting action could not be completely given in terms of singlet fields combinations. However, if we focus on the fermion sector, then the QSF method can be successfully used.

The main goal of the present chapter is to show the application of the QSF method to the quark sector of QCD. In essence we want to verify that the calculations performed in chapter 4 for the matrix model are also extensive to QCD. The argument is presented at the formal level of the path integral representation of field theory. From there, the framework to calculate the large-N expansion of the quark Green's functions is developed in general terms.

The starting point of our derivation is the reduction of the gluon self-interaction to Yukawa couplings with auxiliary anticonmuting fields.

The whole QCD action can be reformulated in terms of quadratic singlet combinations of the quarks and auxiliary fields. The resulting action has not any dependence on the group indeces and there remains an overall factor N, once the coupling constant is rescaled from g^2 to Ng^2. We call this resulting action the QSF action.

The explicit dependence on the quark sources of the QSF action makes it possible to apply the conventional procedure based on functional derivatives to derive a systematic expansion of the quark Green's functions in powers of $N^{-1/2}$.

To actually calculate the large-N limit, the saddle point equation of the QSF action should be solved. The saddle point equation is a finite system of integral equations. These equations determine the leading large-N contribution to the quark propagator.

To proceed with the large-N expansion to all orders we should solve a second system of integral equations, derived from the second functional derivative of the QSF action.

These are essentially equations for the connected leading large-N form of the four-point quark Green's function. We are faced with the difficult problem of solving these large systems of integral equations. It is out of the scope of this presentation to elaborate on the solution of these equations. However, we comment on some possible simplifications of this problem.

The next section of this chapter is devoted to the derivation of the QSF action from the conventional local field theory formulation of QCD. In the last section we show in general terms the procedure to derive the large-N expansion from the QSF action.

II. THE QSF ACTION.

The action of QCD is considered as follows: $S = S_{FA} + S_{AE} + S_{EN}$, where the various parts of the action are given by

$$S_{FA} = i\int (c_0/2)\,\mathrm{tr}((c_0/2)F_{\mu\nu}^2 - F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]))dx \qquad (1)$$

$$S_{AE} = i\int (-(1/2\alpha)\,\mathrm{tr}(\partial^\mu A_\mu)^2 - ig\partial^\mu h_a A_\mu^{ab} h_b + g\bar{f}_a \gamma^\mu A_\mu^{ab} f_b)\,dx$$

$$S_{EN} = i\int (\bar{f}(i\gamma\partial - m)f + \partial^\mu \bar{h}\partial_\mu h + \bar{\eta} f + \bar{f}\eta)\,dx.$$

The fields $\bar{h}, h$ are the anticonmuting variables associated with the relativistic gauge condition and α is the gauge parameter. The variables $\bar{\eta}$, η are the sources of the quark fields $\bar{f}$, f. For simplicity, we consider the U(N) QCD action without flavor indeces. The parameter c_0 has been introduced to perform a field rescaling and it disappears completely from the action once $F_{\mu\nu}$ is integrated over.

The generating functional of the quarks Green's functions is given by

$$Z(n,n) = \int DFDAD\bar{f}DfD\bar{h}Dh\; e^S. \qquad (2)$$

The notational conventions in this chapter are the same adopted in chapter 4.

The first step is to reduce the cubic term of S_{FA} to Yukawa interactions with auxiliary anticonmuting fields according to the following formula

$$\exp(-(c_0 g/2)\int tr(F^{\mu\nu}[A_\mu, A_\nu])dx) = \int DvDuDw \exp(S_{XQ} +$$

$$g\int(w_T A_\mu u_1^\mu + v_2^\mu A_\mu w_2 + F_{\nu\mu}(v_1^\mu u_2^\nu - v_1^\nu u_2^\mu)dx) \qquad (3)$$

where

$$S_{XQ} = \int(w_1(x)w_2(X) - v_k u_k)dx.$$

To derive equation (3) we have to set $c_0 = -2g^2$. The auxiliary anticonmuting fields are $v_k^{a\mu}$, $u_k^{a\mu}$ and w_k^a (k=1,2; a=1,2,...N).

Substituting equation (3) in the generating functional, we find that $F_{\mu\nu}$ can be integrated over. The new form of S_{FA} is given by

$$S_{FA}^{(1)} = i\int(-(1/4)\,tr\,(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - (2ig/c_0)v_{1\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu)u_{2\mu}$$

$$+ g\int(w_T A_\mu u_1^\mu + v_2^\mu A_\mu w_2)dx + S_{XI} + S_{XQ} \qquad (4)$$

where

$$S_{XI} = -(2ig^2/c_0^2)\int((v_{1\nu}^a u_{2\mu}^a)^2 - (v_{1\mu}^a u_{2\mu}^a)^2)dx.$$

The terms $S_{FA}^{(1)} + S_{AE}$ are more conveniently arranged in the following way: $S_{FA}^{(1)} + S_{AE} = S_K + S_{XI} + S_{XQ}$, where

$$S_K = -i\int tr((1/2)(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + (1/\alpha)(\partial^\mu A_\mu)^2) + K^\mu A_\mu)dx.$$

In this last equation we have defined

$$K_\mu^{ab} = gi((2/c_0)\partial^\nu(v_{1\nu}^b u_{2\mu}^a - v_{1\mu}^b u_{2\nu}^a) +$$

$$w_T^b u_{1\mu}^a + v_{2\mu}^b w_2^a + (\partial_\mu \bar{h}_b)h_a + i\bar{f}_b \gamma_\mu f_a).$$

The field A can now be integrated over. The result of this integration is given by

$$S_K^{(1)} = -(i/2)\int G_{\mu\nu}(x-y)\,\mathrm{tr}\,(K^{\mu}(x)\,K^{\nu}(y))\,dxdy. \qquad (5)$$

In this last equation we have defined the following free propagator

$$G_{\mu\nu}(x) = -(2\pi)^{-4}\int d^4p\ e^{-ipx}(g_{\mu\nu} + (\alpha-1)(p_{\mu}p_{\nu}/p^2+i\varepsilon))/(p^2+i\varepsilon).$$

The whole action is now given by

$$S^{(1)} = S_K^{(1)} + S_{XI} + S_{XQ} + S_{EN}. \qquad (6)$$

Let us consider the part of the action given by $S_K^{(1)} + S_{XI}$. Here we find the interacting terms of all anticonmuting fields. The interactions are quartic polynomials that are only given by singlet combinations of the anticonmuting fields. Since these singlet combinations are always quadratic, we can introduce a set of bilocal fields to replace them. After this substitution is performed, it is possible to integrate out all the anticonmuting fields. The result of this integration is the QSF action.

The quadratic singlet products are substituted by the bilocal singlet fields, using delta functions with the following action

$$S_D = \int dxdy\, (W_{mk\mu\nu}(x,y) (g^{-2} W'^{\mu\nu}_{mk}(x,y) - v^{a\mu}_{m}(x) u^{a\nu}_{k}(y)) + \tag{7}$$
$$V_{k\mu}(x,y) (V'^{\mu}_{k}(x,y) - v^{a}_{k}(x) w^{a}_{2}(y)) +$$
$$V_{k1\mu}(x,y) (V'^{\mu}_{k1}(x,y) - v^{a\mu}_{k}(x) h_a(y)) +$$
$$V_{k2\mu}{}^{\alpha}(x,y) (V'^{\alpha\mu}_{k2}(x,y) - v^{a\mu}_{k}(x) f^{\alpha}_{a}(y)) +$$
$$U_{k\mu}(x,y) (U'^{\mu}_{k}(x,y) - w^{a}_{1}(x) u^{a\mu}_{k}(y)) +$$
$$U_{1k\mu}(x,y) (U'^{\mu}_{1k}(x,y) - h_a(x) u^{a\mu}_{k}(y)) +$$
$$U^{\alpha}_{2k\mu}(x,y) (U'^{\mu\alpha}_{2k}(x,y) - \bar{f}^{\alpha}_{a}(x) u^{a\mu}_{k}(y)) +$$
$$W_0(x,y) (W'_0(x,y) - w^{a}_{1}(x) w^{a}_{2}(y)) +$$
$$W_{11}(x,y) (W'_{11}(x,y) - w^{a}_{1}(x) h_a(y)) +$$
$$W_{12}(x,y) (W'_{12}(x,y) - \bar{h}_a(x) w^{a}_{2}(y)) +$$
$$W^{\alpha}_{12}(x,y) (W'^{\alpha}_{12}(x,y) - w^{a}_{1}(x) f^{\alpha}_{a}(y)) +$$
$$W^{\alpha}_{22}(x,y) (W'^{\alpha}_{22}(x,y) - \bar{f}^{\alpha}_{a}(x) w^{a}_{2}(y)) +$$
$$H_0(x,y) (H'_0(x,y) - \bar{h}_a(x) h_a(y)) +$$
$$H^{\alpha}_{1}(x,y) (H'^{\alpha}_{1}(x,y) - \bar{h}_a(x) f^{\alpha}_{a}(y)) +$$
$$H^{\alpha}_{2}(x,y) (H'^{\alpha}_{2}(x,y) - \bar{f}^{\alpha}_{a}(x) h_a(y)) +$$
$$Q^{\beta\alpha}_{0}(x,y) (Q'^{\alpha\beta}_{0}(x,y) - \bar{f}^{\beta}_{a}(x) f^{\alpha}_{a}(y))).$$

The term S_D is added to the complete action given by equation (6). The purpose of S_D is to introduce all the singlet fields in $S^{(1)}_{K}$ and in S_{XI}. After this substitution is performed, the resulting action is arranged as follows

$$S^{(1)}_{K} + S_{XI} + S_{XQ} + S_D = g^{-2} S_T + S_X. \tag{8}$$

The terms on the left side of equation (8) are given by: $S_X = S_{XQ} + S_{DX}$, where S_{DX} are all the terms of S_D that contain the quadratic products of the anticonmuting fields. Defining $S_D = S_{DX} + S_{DL}$, then $S_T = S_{KD} + S_{DL}$, where S_{KD} is the result of substituting the bilocal fields into $S_K^{(1)} + S_{XI}$.

Let us consider the part of the action given by S_X. We notice here that the anticonmuting fields v,u, and w can be integrated out. The result of that integration is given by

$$S_X^{(1)} = -\int dxdy(\bar{f}_a^{\alpha}(x)(Q_0+Q_1)^{\alpha\beta}(x,y)f_a^{\beta}(y) + \bar{h}_a(x)G(x,y)h_a(y) + \bar{h}_a(x)G_1^{\alpha}(x,y)f_a^{\alpha}(y) + \bar{f}_a^{\alpha}(x)G_2^{\alpha}(x,y)h_a(y)) + NS_{L1} \quad (9)$$

where

$$S_{L1} = \mathrm{tr}\ \ln (\delta(x-y) - \int T(x-z)W(z,y)dz).$$

To reduce the large form of $S_X^{(1)}$ we have defined the following expressions

$$G(x,y) = H_0(x,y) + \int dzdu\ U_{1j\mu}(x,z)W_{jk}^{-1\mu\nu}(z,u)V_{k1\nu}(u,y) + \int dsdt\ (W_{12}(x,s) + \int dzduU_{1j\mu}(x,z)W_{jk}^{-1\mu\nu}(z,u)V_{k\nu}(u,s)) W_T(s,t)(W_{11}(t,y) + \int dzduU_{j\mu}(t,z)W_{jk}^{-1\mu\nu}(z,u)V_{k1\nu}(u,y)), \quad (10)$$

$$G_1^{\alpha}(x,y) = H_1^{\alpha}(x,y) + \int dzduU_{1j\mu}(x,z)\, W_{jk}^{-1\mu\nu}(z,u)V_{k2\nu}^{\alpha}(u,y) +$$
$$\int dsdt\,(\, W_{12}(x,s) + \int dzduU_{1j\mu}(x,z)W_{jk}^{-1\mu\nu}(z,u)V_{k\nu}(u,s)\,)W_T(s,t)$$
$$(\, W_{12}^{\alpha}(t,y) + \int dzduU_{j\mu}(t,z)W_{jk}^{-1\mu\nu}(z,u)V_{k2\nu}^{\alpha}(u,y)\,)$$

$$G_2^{\alpha}(x,y) = H_2^{\alpha}(x,y) + \int dzduU_{2j}^{\alpha}\ (x,z)\, W_{jk}^{-1\mu\nu}(z,u)V_{k1\nu}(u,y) +$$
$$\int dsdt\,(\, W_{22}^{\alpha}(x,s) + \int dzduU_{2j\mu}^{\alpha}(x,z)W_{jk}^{-1\mu\nu}(z,u)V_{k\nu}(u,s)\,)W_T(s,t)$$
$$(\, W_{11}(t,y) + \int dzduU_{j\mu}(t,z)W_{jk}^{-1\mu\nu}(z,u)V_{k1\nu}(u,y)\,)$$

and

$$Q_1^{\alpha\beta}(x,y) = \int dzduU_{2j\mu}^{\alpha}(x,z)\, W_{jk}^{-1\mu\nu}(z,u)V_{k2\nu}^{\beta}(u,y) +$$
$$\int dsdt\,(\, W_{22}^{\alpha}(x,s) + \int dzduU_{2j\mu}^{\alpha}(x,z)W_{jk}^{-1\mu\nu}(z,u)V_{k\nu}(u,s)\,)W_T(s,t)$$
$$(\, W_{12}^{\beta}(t,y) + \int dzduU_{j\mu}(t,z)W_{jk}^{-1\mu\nu}(z,u)V_{k2\nu}(u,y)\,).$$

Let us consider the complete action that is now given by $S^{(2)} = g^{-2}S_T + S_X^{(1)} + S_{EN}$. Here we notice that the fields $\bar{h}$, h and $\bar{f}$, f can also be integrated out. The result of this integration is what we call the QSF action, that is

$$S^{(3)} = S_G = N(\ (1/g^2N)S_T + S_L\ ) - i\int dxdy\bar{\eta}^a(x)S_Q(x,y)\eta^a(y). \tag{11}$$

In this last expression we have defined the following quantities

$$S_L = S_{L1} + S_{L2} + S_{L3}$$

$$Q = Q_0 + Q_1 + Q_2$$

where

$$S_{L2} = \mathrm{tr}\ \ln(\delta(x-y) - \int dz\, \Delta(x-z)G(z,y))$$

$$S_{L3} = \mathrm{tr}\ \ln(\delta(x-y)I - \int dz S_F(x-z)Q(z,x))$$

and

$$Q_2^{\alpha\beta}(x,y) = -\int dudz G_2^{\alpha}(x,u)G(u,z)G_1^{\beta}(z,y).$$

The singlet fields action given by equation (11) is dependent on a rather large set of singlet fields. The singlet fields which are under functional integration are all the singlet bilocal fields defined in equation (7). Half of these bilocal fields, which are denoted by a prime, can be integrated out. The part of the action that contains them is given by S_T.

To simplify the discussion of the next section we introduce Q as a field in the generating functional. We shall refer to Q as the self-energy field. The introduction of Q as a field can be accomplished by a delta function.

Once Q has been introduced as a field, then the field Q_0 can be integrated out from the generating functional. This last step is possible because the action becomes gaussian in the field Q_0. We do not display the result of this change of field variables because the specific details are not relevant for the general presentation of the section that follows.

III. LARGE-N EXPANSION OF THE SINGLET FIELDS ACTION.

The basis for the large-N expansion is provided by the QSF action derived in the previous section and given by equation (11). The same general type of the singlet fields action has also been derived for the case of the matrix model in chapter 4.

The explicit dependence of S_G on the quark sources makes possible the application of the standard mechanism based on the functional derivatives to obtain any quark Green's function in a power series expansion of $N^{-1/2}$.

For example, the whole two-point fermion Green's function (i.e. the propagator) is represented by

$$\langle \bar{f}^a(x) f^b(y) \rangle = \delta^{ab} \int DG \; S_Q(x,y) e^{S_G} \Big|_{\bar{\eta},\eta=0} , \tag{12}$$

where all the singlet fields and the self-energy field have been represented by G.

In general terms, the procedure for the large-N expansion is as follows. The first step is to evaluate from the action S_G the saddle point equation of the singlet fields.

The next step is to shift the singlet fields by a solution of the saddle point equation and then the quadratic piece of the action has to be calculated. Finally, additional large-N corrections should be calculated as perturbations around the quadratic piece.

We now proceed to show how the large-N expansion of the QSF action is implemented. To emphasize the general structure we gather here the expression of the various parts of S_G and introduce a new notation to show explicitly the functional dependence of each term

$$S_{L3} = L(S_F,Q) = \mathrm{tr}\ \ln\ (\ \delta(x-y)I - \int du S_F(x-u)Q(u,y)\ ) \qquad (13)$$

$$S_Q(x,y) = P(S_F,Q)(x,y) = S_F(x-y) + \int dudv S_F(x-u)Q(u,v)S_F(v-y) + \ldots$$

$$(1/g^2N)S_T + S_{L2} + S_{L3} = I(F_j,Q).$$

From these last definitions the QSF action is given by

$$S_G = N(\ L(S_F,Q) + I(F_j,Q)\ ) - i\int dxdy\bar{\eta}^a(x)P(S_F,Q)(x,y)\eta^a(y). \qquad (14)$$

In these last expressions F_j represents the singlet fields. The generating functional of the quark Green's functions is now given by

$$Z(\bar{\eta},\eta) = \int DFDQ\ e^{S_G} . \tag{15}$$

In order to expedite the shift of field variables in the generating functional, it is useful to apply the following two properties. If we take $Q = A + B$, then the functionals L and P satisfy the following relationship

$$L(S,A + B) = L(S,A) + L(P(S,A),B) \tag{16}$$

$$P(S,A + B) = P(P(S,A),B).$$

These properties can be proven by rearranging the series that define P and L. The following functional derivatives are also useful

$$\delta L(S,Q)/\delta Q \quad (y,x) = - P^{\beta\alpha}(S,Q)(x,y) \tag{17}$$

$$\delta P^{\alpha\beta}(x,y)/\delta Q^{\gamma\rho}(u,v) = P^{\alpha\gamma}(x,u)P^{\rho\beta}(v,y).$$

We introduce the following shift of variables

$$Q = R + N^{-1/2}M , \quad F_j = F_{0j} + N^{-1/2}F_{1j}$$

in the QSF action

$$S_G(F_0 + N^{-1/2}F_1, R + N^{-1/2}M) = NS_G(F_0,R) + NL(P(S_F,R), N^{-1/2}M)$$
$$+ N^{1/2}\int((\ I(F,Q)/\ F)F_1 + (\delta I(F,Q)/\delta Q)M)\Big|_{F=F_0, Q=R} + I_2 + I_3$$
$$-i\int dxdy\bar{\eta}(x)P(P(S_F,R), N^{-1/2}M)(x,y)\eta(y). \qquad (18)$$

In this last expression, I_2 represents the quadratic piece, and I_3 all the higher powers of the functional series of I. The initial field configuration (F_0,R) is determined by the saddle point equation of S_G. This equation arises from the condition that the factor proportional to F_1 and M vanishes

$$\delta I(F_0,Q)/\delta Q(y,x)\Big|_{Q=R} - P(S_F,R)(x,y) = 0 \qquad (19)$$

$$\delta I(F,R)/\delta F\Big|_{F=F_0} = 0 .$$

From this last system of equations we see that the important quantity to be determined is the self-energy function R. The self-energy function R, of the corrected propagator $P(S_F,R)$, is the term needed to evaluate the large-N limit of the quark Green's functions in terms of the quark sources $\bar{\eta}$, η. We shall denote the corrected propagator by P_R ($= P(S_F,R)$). The leading large-N part of the four-point Green's functions is disconnected and therefore the next contribution is necessary.

Once the self-energy function R is determined, we treat $\bar{M}$ as a new field. With the fields F_1 and M we can devise a perturbation expansion to calculate all the contributions beyond the large-N limit. To obtain this perturbation expansion, source variables J and J_1 have to be introduced for the M and F_1 fields as follows

$$\int dxdy(J(x,y)M(x,y) + J_1(x,y)F_1(x,y)). \qquad (20)$$

This source term is added to S_G and has to be considered together with the quadratic piece of the action given by equation (18).

The generating fuctional for the large-N expansion has the following form

$$Z(\bar{\eta},\eta,J,J_1) = e^{-i\int dxdy\bar{\eta}(x)P(P_R,N^{-1/2}\delta/\delta J)\eta(y)}$$

$$e^{NL_3(P_R,N^{-1/2}\delta/\delta J) + I_3(\delta/\delta J,\delta/\delta J_1)}$$

$$e^{\int(JT_1J + J_1T_2J + JT_3J_1 + J_1T_4J_1)}.$$

In the last expression we have denoted by I_3 all the terms of I involving powers of M equal or higher than the cubic. The terms T_k are the Green's functions associated with the inversion of the quadratic form of the fields M and F_1.

The problem of inverting the quadratic form of the fields M and F_1 induces a system of integral equations. These equations determine the connected leading large-N contribution to the four-point quark Green's function. The homogeneous part of this equation is the bound states equation for the meson states.

The explicit form of the bound states equation is very large and cumbersome. The analysis of this equation as well as of equation (19) is out of the scope of this report. We consider premature to tackle this problem at this stage of development of the QSF method.

In order to investigate the integral equations, additional preliminary research has to be done. We have to gain experience by looking at similar systems of equations in simpler models, as for example in the matrix model. Special attention has to be paid to the problem of regularization and renormalization of these equations.

It is possible that a substantial reduction in the number of integral equations may occur if the QSF method is applied to QCD in a non-relativistic and free of ghost gauge fixing. The elimination of the anticonmuting fields associated with a gauge fixing condition decreases the number of singlet fields and the complexity of their interaction.

A few additional simplifications are attained in the formal functional manipulations by transforming the action to the momentum representation. This procedure would give the integral equations directly in momentum space, where they are more easily analyzed. Also in this way, a relationship with the reduced model[33] could be more easily obtained.

APPENDIX

Here we collect the most common and usefull formulas and definitions used in path integration.

Gaussian Integrals

Vector Variables

$$\int_{-\infty}^{\infty} dx\ e^{-(a/2)x^2 + bx} = (2\pi/a)^{1/2} e^{b^2/2a};\quad a > 0$$

$$\int_{-\infty}^{\infty} dx\ e^{-(ia/2)x^2 + bx} = (2\pi/ia) e^{b^2/2ia}$$

$$\int_{-\infty}^{\infty} d^n x \exp(\ -(1/2) x_c A_{cd} x_d + b_c x_c\ ) = (2\pi)^{n/2} \exp(\ (1/2) b_c A^{-1}_{cd} b_d\ )\ (\det A)^{-1/2}$$

$$\int \prod_{k=1}^{n} (dz^+_k dz_k / 2\pi i) \exp(\ -z^+ A z + u^+ z + u z^+\ ) = \exp(\ u^+ A^{-1} u\ )\ (\det A)^{-1};\quad z = x+iy,\ z^+ = x-iy\ .$$

Exponentiation of a Determinant

$$\det(\ A\ ) = \exp \operatorname{tr} \ln (\ A\ )$$

$$\det(\ I + B\ ) = \exp \operatorname{tr}(\ B - (1/2)B^2 + (1/3)B^3 - \ldots)$$

Matrix variables

$$\int \prod_{j \leq k}^{N} dX_{jk} \exp(\operatorname{tr}(-X^2/2 + RX)) = 2^{N/2} \pi^{N(N+1)/4} \exp(\frac{1}{8} \operatorname{tr}(R + R^T)^2) \; ; \quad X = X^T .$$

$$\int \prod_{j < k}^{N} dY_{jk} \exp(\operatorname{tr}(Y^2/2 + SY)) = \pi^{N(N-1)/4} \exp(\frac{1}{8} \operatorname{tr}(S - S^T)^2) \; ; \quad Y = -Y^T .$$

$$\int \prod_{j \leq k}^{N} dX_{jk} \prod_{j < k}^{N} dY_{jk} \exp(\operatorname{tr}(-M^2/2 + KM)) = 2^{N/2} \pi^{N^2/2} \exp(\operatorname{tr} K^2/2) ,$$

where X = Real(M) , Y = Imaginary(M) and M is hermitian.

$$\int \prod_{a,k}^{N,n} dx_k^a / 2\pi \exp(x_j^a M_{jk}^{ab} x_k^b) = (\det M)^{-1/2} ,$$

where $\det M = \det \begin{bmatrix} M_{jk}^{11} \cdots M_{jk}^{1N} \\ \\ M_{jk}^{N1} \cdots M_{jk}^{NN} \end{bmatrix}$.

Path Integral

Given a hamiltonian $H = p^2/2m + V(q)$, the path integral is defined in terms of the following matrix element

$$F(q',t',q,t) = \langle q'| \exp(-i(t'-t)H)|q\rangle$$
$$= \sum_k f_k(q')f^+(q)\exp(-iE_k(t'-t))$$

where $Hf_k = E_k f_k$. The path integral is given by

$$F(q',t',q,t) = \lim_{n \to \infty} \int \prod_{j=1}^{n} \prod_{k=1}^{n+1} dq_j dp_k/2\pi \exp(i\sum_{j=1}^{n+1} (p_j(q_j-q_{j-1})-H(p_j,q_j)(t_j-t_{j-1}))$$

where $q_0 = q$; $q_{n+1} = q'$; $t_k = k\varepsilon+t$; $\varepsilon = (t'-t)/(n+1)$.
In the continuous notation

$$F(q',t',q,t) = \int DqDp\exp(i\int_t^{t'}(pdq/dt - H)dt).$$

After the momentum integration

$$F(q',t',q,t) = \lim_{n \to \infty}\int \prod_{k=1}^{n} dq_k/(2\pi i\varepsilon)^{1/2} \exp(i\sum_{j=1}^{n+1} \varepsilon(((q_j-q_{j-1})/\varepsilon)^2/2 - V(q_j)))$$

$$= \int DQ\exp(i\int_t^{t'} L(q,dq/dt)dt)$$

Anticonmuting Variables

A set of symbols $w_1, w_2, \ldots w_n$ is a set of anticonmuting variables if the following algebraic rules are satisfied

$$w_j w_k + w_k w_j = 0$$

and the following formal integration and derivation operations

$$\partial w_a/\partial w_b = \delta_{ab}$$
$$(\partial/\partial w_a)(\partial f(w)/\partial w_b) = -(\partial/\partial w_b)(\partial f(w)/\partial w_a)$$
$$(dw_a)w_b + w_b(dw_a) = 0$$
$$w_a(\partial f/\partial w_b) + (\partial f/\partial w_a)w_b = 0$$
$$\int dw_a = 0$$
$$\int w_a dw_{(a)} = 1$$
$$\int(\partial f/\partial w_a)dw_{(a)} = 0.$$

For example, a general function of two anticonmuting variables is

$$f(w_1,w_2) = f_0 + f_1w_1 + f_2w_2 + f_3w_1w_2$$

where f_k are real constants.

Gaussian Integrals

Anticonmuting variables

$$\int \prod_{a=1}^{n} dw_a \exp(-(1/2)wAw + uw) = \frac{\exp((1/2)uA^{-1}u)}{(\det A)^{1/2}}$$

where n is even and $A = -A^T$.

$$\int \prod_{k=1}^{n} du_k dv_k \exp(-uAv + rv + us) = \exp(rA^{-1}s)\det A$$

where u, v, w, r, s are a set of anticonmuting variables.

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