74-20,652

SIMENAUER, Ronald Felix, 1946-
A SPACE OF MODULAR FORMS.

The City University of New York, Ph.D., 1974
Mathematics

A SPACE OF MODULAR FORMS

by

RONALD SIMENAUER

A dissertation submitted to the Graduate Faculty
in Mathematics in partial fulfillment of the
requirements for the degree of Doctor of Philosophy,
The City University of New York.

June 1974

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

May 10, 1974
DATE

Joseph Lewittes
CHAIRMAN OF EXAMINING COMMITTEE
Professor Joseph Lewittes

May 10, 1974
DATE

Richard Sacksteder
EXECUTIVE OFFICER
Professor Richard Sacksteder

Professor Harry E. Rauch

Professor Herve Jacquet

Professor Burton Randol

SUPERVISORY COMMITTEE

The City University of New York

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my advisor Professor Joseph Lewittes for his many hours of advice and encouragement. Without his help, this paper could not have been written.

My thanks also go to my parents Oskar and Ruth Simenauer and my wife Suzanne for their support which was constant and unfailing.

Also I would like to thank the faculty of the Mathematics Department who spurred my interest in Riemann surfaces, particularly Professors H. E. Rauch and B. Randol.

Finally, I would like to thank The City University of New York for their generous cooperation.

TABLE OF CONTENTS

Introduction

This paper is divided into three main parts. In Part I we take special linear combinations of generalized Eisenstein series of level N, and obtain a finite dimensional vector space of abelian differentials of third kind on the compactified Riemann surface $S_N = \overline{\mathcal{H}/\Gamma(N)}$, where $\mathcal{H}$ is the upper half plane and $\Gamma(N)$ is the principal congruence subgroup of level N in the inhomogeneous modular group Γ . We exhibit a group representation of $\Gamma/\Gamma(N)$ acting on this space of differentials, and for N a prime, we find the multiplicities of the irreducible components of the representation.

In Part II we take a particular set of these differentials from Part I and show a new derivation for obtaining the periods of these differentials on S_N .

In Part III we consider the classical Poincaré series $\varphi_g(z,k)$ of integer weight $k \geq 2$. We generalize to a series $P(z,s,g)$ of complex dimension $-s$, and show that although the series does not converge absolutely for $\mathrm{Re}(s) \leq 2$, it has an analytic continuation to $\mathrm{Re}(s) > 7/4$. This analytic continuation is given by a convergent Fourier series in powers of $\exp(2\pi iz)$.

PART I

§0. Introduction

Let $\mathcal{H}$ be the upper half plane. Let Γ be the inhomogeneous modular group, and $\Gamma(N)$ its principal congruence subgroup. In this part of the paper, using results of Hecke [6] and Lewittes [10] on generalized Eisenstein series, we construct a space of differentials of third kind on the Riemann surface $S_N = \overline{(\mathcal{H}/\Gamma(N))}$, where the bar denotes compactification by addition of cusps. Our differentials form a finite dimensional complex vector space, H_N, and we define a faithful group representation, L, of $\Gamma/\Gamma(N)$ acting on H_N. From Frobenius [4], we know the irreducible representations of $\Gamma/\Gamma(q)$, for q prime ≥ 3, and their characters.

The main result of this part of my paper is that the decomposition of L into irreducible components has the form, for q prime ≥ 3:

$$L = \begin{cases} L_q + 2 \sum_{i=1}^{\frac{q-3}{4}} L_{q+1}^{(i)} & \text{if } q \equiv 3(4) \\ L_q + L_{\frac{q+1}{2}}^{(1)} + L_{\frac{q+1}{2}}^{(2)} + 2 \sum_{i=1}^{\frac{q-5}{4}} L_{q+1}^{(i)} & \text{if } q \equiv 1(4) \end{cases}$$

where L_n denotes an irreducible representation of dimension n.

§1. Preliminaries

$\mathbb{C}$,R,Q,Z will stand for the sets of complex numbers, reals, rationals, and integers respectively.

$$\mathcal{H} = \{z \in C \mid \mathrm{Im}(z) > 0\} = \text{upper half plane}.$$

Γ = inhomogeneous modular group, namely, the group of fractional linear transformations $z \to \frac{az+b}{cz+d}$, where $a,b,c,d \in Z$, $ad - bc = 1$.

SL(2,Z) = homogeneous modular group, namely, the group of 2×2 integer matrices of determinant one.

Note: $\Gamma \cong SL(2,Z)/\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\}$.

We will use $\{c,d\}$ to denote an ordered pair, so as not to confuse this with the greatest common divisor of two integers, denoted by (c,d).

If $V \in \Gamma$, we can write $V(z) = (az+b)/(cz+d)$ where $a,b,c,d \in Z$, $ad - bc = 1$, and either $c > 0$, or $\{c,d\} = \{0,1\}$. Then we define:

(0) $\underline{V} \underset{\text{def}}{=} \begin{pmatrix}a&b\\c&d\end{pmatrix}$ where a,b,c,d satisfy the above conditions.

Let $\Gamma(N)$ = the principal congruence subgroup of level N in Γ, namely the group of elements $V \in \Gamma$ such that, componentwise, we have $\underline{V} \equiv \pm \begin{pmatrix}1&0\\0&1\end{pmatrix} \pmod N$. $\Gamma(N)$ acts as a discontinuous group of conformal homeomorphisms of $\mathcal{H}$ onto itself. Thus, we can form the quotient space $\mathcal{H}/\Gamma(N)$, compactify it by adding ∞ and other rational cusps, and give it a Riemann surface structure (see, for example, Ahlfors and Sario, Riemann Surfaces, Ch. II. 4 p. 119-124).

Let:

$S_N = \overline{\mathcal{H}/\Gamma(N)}$ denote the compact Riemann surface thus obtained. By an abelian differential on a Riemann surface we will mean a differential form which is expressible locally as $f(t)dt$, where f is a meromorphic function of the local coordinate t. We will say the abelian differential is of third kind if at some point on the Riemann surface the local expression $f(t)$ has a pole with non-zero residue. Note, the Riemann surface S_N is compact, and thus by general theory we know that for an abelian differential on S_N , the sum of the residues is zero.

In the next section we construct a finite family of analytic functions $h_k(z)$ on $\mathcal{H}$ which satisfy:

$$h_k(V(z))\,\frac{dV}{dz} = h_j(z) \quad \text{for} \quad V \in \Gamma\ , \tag{1}$$

where j depends on k and V, and

$$j = k \Leftrightarrow V \in \Gamma(N)\ .$$

Thus we get:

$$h_k(V(z))dV = h_k(z)dz \quad \text{for} \quad V \in \Gamma(N)\ . \tag{2}$$

The functions $h_k(z)$ are analytic at $z = i\infty$, and therefore at all rational cusps by formula (1). Thus we use formula (2) to define $h_k(z)dz$ as abelian differentials on S_N. These differentials will be of third kind.

§2. Construction of a Space of Abelian Differentials of Third Kind on S_N.

In [10], Lewittes defines the generalized Eisenstein series $G(z,s,r_1,r_2)$ for $z \in \mathcal{H}$, $s \in \mathbb{C}$, r_1 and $r_2 \in R$ as follows: For $\mathrm{Re}(s) > 2$:

$$G(z,s,r_1,r_2) = \sum_{m,n}{}' ((m+r_1)z + n + r_2)^{-s} \tag{3}$$

where $\{m,n\}$ runs through all integer pairs except $\{-r_1,-r_2\}$ if $\{r_1,r_2\} \in Z^2$. The complex power is made unambiguous by agreeing that for any $w \in C$, $w \neq 0$, we define $\mathrm{Arg}(w)$ as that value of the argument such that $-\pi \leq \mathrm{Arg}(w) < \pi$, and then letting

$$w^s \underset{\mathrm{def}}{=} e^{s(\ell n|w| + i\,\mathrm{Arg}(w))}$$

As is pointed out in [10], the series in (3) defines a function which is analytic in z and s for $\mathrm{Im}(z) > 0$, $\mathrm{Re}(s) > 2$.

It is further shown that $G(z,s,r_1,r_2)$ has an everywhere finite analytic continuation to $\{z,s\} \in \mathcal{H} \times \mathbb{C}$, given by:

$$\begin{aligned} G(z,s,r_1,r_2) &= \chi(r_1)\{\zeta(s,r_2) + e^{\pi is}\zeta(s,-r_2)\} \\ &+ \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} \exp(2\pi ikr_2) \exp(2\pi ik(m+r_1)z) \\ &+ e^{\pi is} \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{m>r_1} \sum_{k=1}^{\infty} k^{s-1} \exp(-2\pi ikr_2)\exp(2\pi ik(m-r_1)z) \end{aligned} \tag{4}$$

where $\Gamma(s)$ is the usual gamma function,

$$\chi(t) = \left\{\begin{matrix} 1 & \text{if} & t \in Z, \\ 0 & \text{if} & t \notin Z \end{matrix}\right\},$$

and for $\mathrm{Re}(s) > 1$, $\zeta(s,r_2) = \Sigma_{n>-r_2}(n+r_2)^{-s}$.

For our purposes we are only interested in the case $s = 2$,

$r_1 = a_1/N$, $r_2 = a_2/N$, as this will lead us to abelian differentials on S_N. For this case, (4) becomes:

$$G(z,2,a_1/N,a_2/N) = \chi(a_1/N) \sum_n{}' (n+a_2/N)^{-2}$$

$$-4\pi^2 \sum_{m>-a_1/N} \sum_{k=1}^{\infty} k \exp(2\pi ika_2/N) \exp(2\pi ik(m+a_1/N)z)$$

$$-4\pi^2 \sum_{m>a_1/N} \sum_{k=1}^{\infty} k \exp(-2\pi ika_2/N) \exp(2\pi ik(m-a_1/N)z)$$

$$= \chi(a_1/N) \cdot N^2 \sum_n{}' (Nn+a_2)^{-2}$$

$$-4\pi^2 \sum_{Nm+a_1>0} \sum_{k=1}^{\infty} k \exp(2\pi ika_2/N) \exp(2\pi ik(Nm+a_1)z/N)$$

$$-4\pi^2 \sum_{Nm-a_1>0} \sum_{k=1}^{\infty} k \exp(-2\pi ika_2/N) \exp(2\pi ik(Nm-a_1)z/N) \quad .$$

Thus

$$G(z,2,a_1/N,a_2/N) = \chi(a_1/N) \cdot N^2 \sum_{d\equiv a_2 (N)} 1/d^2$$

$$-4\pi^2 \sum_{\substack{j>0 \\ j\equiv a_1 (N)}} \sum_{k=1}^{\infty} k \exp(2\pi ika_2/N) \exp(2\pi ikja/N)$$

$$-4\pi^2 \sum_{\substack{j>0 \\ j\equiv -a_1 (N)}} \sum_{k=1}^{\infty} k \exp(-2\pi ika_2/N) \exp(2\pi ikjz/N) \quad .$$

Finally, writing n for kj in the double sums, and collecting terms we have the Fourier expansion:

$$(5) \qquad G(z,2,a_1/N,a_2/N) = \sum_{n=0}^{\infty} \alpha_n(a_1,a_2,N) \exp(2\pi inz/N)$$

where

$$\alpha_0(a_1,a_2,N) = N^2\chi(a_1/N) \sum_{d\equiv a_2(N)} 1/d^2$$

and for $n \geq 1$

$$\alpha_n(a_1,a_2,N) = -4\pi^2\Big\{ \sum_{\substack{k|n \\ \frac{n}{k}\equiv a_1(N) \\ k>0}} k\exp(2\pi i a_2 k/N) + \sum_{\substack{k|n \\ \frac{n}{k}\equiv -a_1(N) \\ k>0}} k\exp(-2\pi i a_2 k/N)\Big\} .$$

To construct our differentials we will also make use of the transformation formula in [10] (p. 486,(49)) for $G(z,2,r_1,r_2)$, namely: if $V \in \Gamma$, with $\underline{V} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ as in (0), then

$$\text{(6)} \qquad \frac{G(V(z),2,r_1,r_2)}{(cz+d)^2} = G(z,2,R_1,R_2) - \frac{2\pi i c}{cz+d}$$

where $\{R_1,R_2\} = \{r_1,r_2\}\cdot\underline{V} = \{ar_1 + cr_2, br_1 + dr_2\}$.

<u>Note 1</u>: In this case, where $s = 2$, there is no need to specify $c > 0$ or $\{c = 0,\ d = 1\}$, as is needed in the general case for $s \in \mathbb{C}$. For suppose $c < 0$ or $\{c = 0,\ d = -1\}$, then by (1) we have

$$\underline{V} = \begin{pmatrix} -a & -b \\ -c & -d\end{pmatrix}$$

and (6) becomes:

$$\text{(7)} \qquad \frac{G(V(z),2,r_1,r_2)}{(-cz-d)^2} = G(z,2,-R_1,-R_2) - \frac{2\pi i(-c)}{-cz-d}$$

where $\{R_1,R_2\} = \{ar_1 + cr_2, br_1 + dr_2\}$. But using $s = 2$ in (4) it is easy to see that

$$\text{(8)} \qquad G(z,2,-R_1,-R_2) = G(z,2,R_1,R_2) .$$

Thus (7) can be written as

$$\frac{G(V(z),2,r_1,r_2)}{(cz+d)^2} = G(z,2,R_1,R_2) - \frac{2\pi ic}{cz+d}$$

where $\{R_1,R_2\} = \{r_1,r_2\}\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \{ar_1 + cr_2, br_1 + dr_2\}$ with $c < 0$ or $\{c = 0, d = -1\}$. But this is the same formula as (6), so we see that we can use (6) with no restriction on c or d, except $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,Z)$.

<u>Note 2:</u> As is pointed our in [10], the function $G(z,2,a_1/N,a_2/N)$ is closely related to the function

$$(9) \qquad G_2(z;a_1,a_2,N) \underset{\text{def}}{=} \lim_{s\to 0^+} \sum_{m_i\equiv a_i(N)} \frac{1}{(m_1z+m_2)^2|m_1z+m_2|^s}$$

defined by Hecke [6]. In fact by comparing the Fourier series of the two functions it is shown that

$$(10) \qquad N^2G_2(z;a_1,a_2,N) = \frac{-2\pi i}{z-\bar{z}} + G(z,2,a_1/N,a_2/N) \quad .$$

Thus, G_2 has the non-analytic term $\dfrac{-2\pi i}{N^2(z-\bar{z})}$. However,

$$\frac{-2\pi i}{V(z)-\overline{V(z)}} \cdot \frac{1}{(cz+d)^2} = \frac{-2\pi i}{z-\bar{z}} + \frac{2\pi ic}{cz+d} \quad ,$$

and so by (10) and (6) we get the formula:

$$(11) \qquad \frac{G_2(V(z);a_1,a_2,N)}{(cz+d)^2} = G_2(a;A_1,A_2,N)$$

where $\{A_1,A_2\} = \{a_1,a_2\}\underline{V}$.

It is easy to see either by analytic continuation from (3) or by using (4) that:

$$(12) \qquad G(z,2,r_1+h,r_2+k) = G(z,2,r_1,r_2) \quad \text{if} \quad \{h,k\} \in Z^2 \ .$$

Furthermore, from (4) we have:

(13) $$G(z,2,-r_1,r_2) = G(z,2,r_1,r_2) .$$

Thus we can write:

(14) $$G(z;2,a_1/N,a_2/N) = G(z;2,b_1/N,b_2/N) \text{ if } \{a_1,a_2\} \equiv \pm \{b_1,b_2\}(N),$$

or, by (10), in terms of G_2 :

(15) $$G_2(z;a_1,a_2,N) = G_2(z;b_1,b_2,N)$$

if $\{a_1,a_2\} \equiv \pm \{b_1,b_2\}(N)$.

Now consider the following functions defined by Hecke [6]:

(16) $$G_2^*(z;a_1,a_2,N) = \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t \cdot G_2(z;ta_1,ta_2,N)$$

where $(a_1,a_2,N) = 1$, and $c_t = \sum_{\substack{n>0 \\ tn\equiv 1(N)}} \frac{\mu(n)}{n^2}$, μ being the Moebius function of number theory. By (15) we have:

(17) $$G_2^*(z;a_1,a_2,N) = G_2^*(z;b_1,b_2,N)$$

if $\{a_1,a_2\} \equiv \pm \{b_1,b_2\}(N)$.

Furthermore by (16) and our previous results on the function G_2 we have the transformation formula:

(18) $$\frac{G_2^*(V(z;a_1,a_2,N)}{(cz+d)^2} = G_2^*(z;A_1,A_2,N)$$

for $V \in \Gamma$, $\underline{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\{A_1,A_2\} = \{a_1,a_2\} \cdot \underline{V}$. We also get the Fourier expansion:

(19) $$G_2^*(z;a_1,a_2,N) = \frac{-2\pi i}{N^2(z-\bar{z})} \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t$$

$$+ \frac{1}{N^2} \cdot \sum_{n=0}^{\infty} \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t \cdot \alpha_n(ta_1,ta_2,N)\exp(2\pi inz/N)$$

where α_n is given in formula (5).

(20) Definition A: We say that $G_2^*(z;a_1,a_2,N)$ vanishes (or doesn't vanish) at a cusp $-d/c$, $(c,d) = 1$, according as the constant term of the expansion of $(cz+d)^2\, G_2^*(z;a_1,a_2,N)$ in powers of

$$t = \exp\{2\pi i(az+b)/(cz+d)N\} \text{ vanishes (or doesn't vanish).}$$

But $(cz+d)^2\, G_2^*(z;a_1,a_2,N) = G_2^*(V(z);B_1,B_2,N)$ where $V \in \Gamma$, $\underset{-}{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\{B_1,B_2\} = \{a_1,a_2\} \cdot \underset{-}{V}^{-1} = \{da_1 - ca_2, -ba_1 + aa_2\}$. By (19) the constant term of the expansion of $G_2^*(V(z),B_1,B_2,N)$, in powers of $t = \exp(2\pi iV(z)/N)$, is

$$\frac{1}{N^2} \cdot \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t\, \alpha_0(tB_1,tB_2,N)$$

where recall,

$$\alpha_0(a_1,a_2,N) = N^2 \chi(a_1/N) \sum_{d \equiv a_2 (N)} 1/d^2 .$$

It is shown in [6], (see also appendix 1 here) that

$$(21) \qquad \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t\, \alpha_0(tx,ty,N) = \begin{cases} 1 & \text{if } \{x,y\} \equiv \pm\{0,1\}(N), \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$(22) \qquad \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t\, \alpha_0(tB_1,tB_2,N) = \begin{cases} 1 & \text{if } \{c,d\} \equiv \pm\{a_1,a_2\}(N) \\ 0 & \text{otherwise.} \end{cases},$$

where $\{B_1,B_2\} = \{da_1 - ca_2, -ba_1 + aa_2\}$.

Thus in the sense of Definition A, the function $G_2^*(z;a_1,a_2,N)$ vanishes at all cusps $-d/c$, $(c,d) = 1$, except those for which $\{c,d\} \equiv \pm\{a_1,a_2\}(N)$. These latter cusps determine one cusp on S_N as follows:

Given an ordered pair $\{a_1,a_2\}$ with $(a_1,a_2,N) = 1$, there exists an integer k_0 such that $(a_1,a_2+k_0N) = 1$. Furthermore, if $(c_1,d_1) = (c_2,d_2) = 1$, then $-d_1/c_1$ is identified with $-d_2/c_2$ by

$$\Gamma(N) \Leftrightarrow \{c_1,d_1\} \equiv \pm \{c_2,d_2\}(N) .$$

Thus,

(23) given an ordered pair $\{a_1,a_2\}$ with $(a_1,a_2,N) = 1$, let
$\gamma_N(a_1,a_2)$ = the cusp on S_N represented by
$z = -(a_2+k_0N)/a_1$ where $(a_1,a_2+k_0N) = 1$.

We obviously get all the cusps of S_N from (23) since any cusp is represented by $z = -d/c$, where $(c,d) = 1$, and $\pm 1/0 \underset{\text{def}}{=} \infty$. Furthermore, it is easy to verify that:

$$\gamma_N(a_1,a_2) = \gamma_N(b_1,b_2) \Leftrightarrow \{a_1,a_2\} \equiv \pm \{b_1,b_2\}(N) . \tag{24}$$

Let

$\mathcal{P}_N$ = the set of ordered pairs of integers $\{a_1,a_2\}$ mod N such that $(a_1,a_2,N) = 1$.

Let $\{a_1,a_2\} \sim \{b_1,b_2\}$ mean $\{a_1,a_2\} \equiv \pm \{b_1,b_2\}(N)$. Then there is a one-to-one correspondence between the set $\mathcal{P}_N/\!\sim$ and the set of cusps of S_N .

If $\lambda(N)$ = cardinality of $\mathcal{P}_N$ then it can be shown (see Gunning [5], p. 9), that

$$\lambda(N) = \begin{cases} 1 & \text{for } N = 1 , \\ N^2 \prod\limits_{\substack{p \mid N \\ p \text{ prime}}} (1 - \frac{1}{p^2}) & \text{for } N \geq 2 \end{cases}$$

If $\sigma(N)$ = cardinality of $\mathcal{P}_N/\!\sim$, then it follows that

$$\sigma(N) = \begin{cases} 1 & \text{for } N = 1 , \\ 3 & \text{for } N = 2 , \\ \frac{N^2}{2} \prod\limits_{\substack{p \mid N \\ p \text{ prime}}} (1 - \frac{1}{p^2} & \text{for } N \geq 3 \end{cases} \tag{25}$$

By (17) and (24) we see that each cusp $\gamma_N(a_1,a_2)$ can be associated uniquely to the function $G_2^*(z;a_1,a_2,N)$. Furthermore, if

$$F(z) = \sum_{\{a_1,a_2\}\in\mathcal{P}_N/\sim} C(a_1,a_2)\ G_2^*(z;a_1,a_2,N)$$

where some $C(x,y) \neq 0$, then we know that in the sense of (20), the function $F(z)$ does not vanish at the cusp $\gamma_N(x,y)$ and so $F(z)$ is not identically zero. Thus the set $\{G_2^*(z;a_1,a_2,N) | \{a_1,a_2\} \in \mathcal{P}_N/\sim\}$ is linearly independent over $\mathbb{C}$. Therefore,if

(26) W_N = the complex vector space spanned by the functions $G_2^*(z;a_1,a_2,N)$,

then

(27) $\dim W_N = \sigma(N)$, $\sigma(N)$ given by (25).

As we shall see in the proof of Theorem 1, we cannot define $G_2^*(z;a_1,a_2,N)dz$ as an abelian differential on S_N, since then in the local coordinate at a cusp of S_N we would have an abelian differential with exactly one pole of non-zero residue. This would contradict the fact that on a compact Riemann surface, the sum of the residues of an abelian differential must be zero. [This also shows that $\sum_{\substack{t \bmod N \\ (t,N)=1}} c_t \neq 0$, for otherwise we could use (18) and (19) to define $G_2^*(z;a_1,a_2,N)dz$ as an abelian differential on S_N.] Thus we define for $(a_1,a_2,N) = 1$:

(28) $$h(z;a_1,a_2,N) = G_2^*(z;a_1,a_2,N) - G_2^*(z;0,1,N) .$$

Note that the non-analytic term $\frac{-2\pi i}{N^2(z-\bar{z})} \sum c_t$ in (19) is cancelled out.

We have the following facts about the functions h :

(29) $$h(z;0,1,N) = 0 , \quad \forall\ z \in \mathcal{H}.$$

(30) If $\{a_1,a_2\} \not\equiv \pm \{0,1\}(N)$ then, in the sense of (20), $h(z;a_1,a_2,N)$ vanishes at all rational cusps except ∞ and $-d/c$, $(c,d) = 1$, $\{c,d\} \equiv \pm \{a_1,a_2\}(N)$.

[We know $\exists$ such an ordered pair $\{c,d\}$ since $(a_1,a_2,N) = 1$.] Furthermore, $h(z;a_1,a_2,N)$ is analytic for $z \in \mathcal{H}$, and satisfies:

$$\frac{h(V(z);a_1,a_2,N)}{(cz+d)^2} = h(z;A_1,A_2,N) - h(z;c,d,N) \tag{31}$$

for $V \in \Gamma$, $\underline{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\{A_1,A_2\} = \{a_1,a_2\} \cdot \underline{V}$. In particular if $V \in \Gamma(N)$, we have:

$$\begin{aligned} \{A_1,A_2\} &\equiv \pm \{a_1,a_2\}(N) \\ \{c,d\} &\equiv \pm \{0,1\}(N) \end{aligned}$$

and thus

$$\frac{h(V(z;a_1,a_2,N)}{(cz+d)^2} = h(z;a_1,a_2,N) \quad \text{for} \quad V \in \Gamma(N) \ . \tag{32}$$

<u>Theorem 1</u>: On S_N , the differential $h(z;a_1,a_2,N)dz$ has a simple pole of residue $\frac{-N}{2\pi i}$ at ∞ , a simple pole of residue $\frac{N}{2\pi i}$ at $-d/c$, $(c,d) = 1$, $\{c,d\} \equiv \pm \{a_1,a_2\}(N)$, $(\{a_1,a_2\} \not\equiv \pm \{0,1\}(N))$, and is analytic at all other points of S_N .

<u>Proof</u>: At a regular point of S_N , that is, a point which has a representative $z_0 \in \mathcal{H}$, we can use $t_0 = z - z_0$ as a local parameter in a disc about z_0. Since h is analytic there, it has a convergent Taylor series in t_0 .

At ∞ we use the local coordinate $t = \exp(2\pi iz/N)$ in a vertical strip of width N in $\mathcal{H}$.

By (25) and (22) we have:

$$(33) \qquad h(z;a_1,a_2,N) = \sum_{n=0}^{\infty} \beta_n(a_1,a_2,N)\ \exp(2\pi inz/N)$$

where

$$\beta_n(a_1,a_2,N) = \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t \cdot \{\alpha_n(ta_1,ta_2,N) - \alpha_n(0,t,N)\} \ ,$$

α_n being defined in formula (5).

Since $\{a_1,a_2\} \not\equiv \pm \{0,1\}(N)$ we know by (21) that:

$$\sum_{\substack{t \bmod N \\ (t,N)=1}} c_t\ \alpha_0(ta_1,ta_2,N) = 0$$

and

$$\sum_{\substack{t \bmod N \\ (t,N)=1}} c_n\ \alpha_0(0,t,N) = 1 \ .$$

Thus we have:

$$(34) \qquad \beta_0(a_1,a_2,N) = -1 \ .$$

In a neighborhood of ∞ (a vertical strip of width N) we have two local expressions $\varphi(t)dt$ and $h(z;a_1,a_2,N)dz$ where $t = \exp(2\pi iz/N)$ is a local coordinate at ∞. Since the two expressions must agree at a regular point in this neighborhood, we set:

$$(35) \qquad \varphi(t) = \frac{h(z;a_1,a_2,N)}{\frac{dt}{dz}} = \frac{\sum_{n=0}^{\infty} \beta_n(a_1,a_2,N)t^n}{2\pi it/N} \ .$$

Thus $\varphi(t)$ has a simple pole with residue $-N/2\pi i$ at the cusp ∞. At a rational cusp $-d/c$, $(c,d) = 1$, $\{c,d\} \not\equiv \{0,1\}(\bmod N)$ we have:

$$(36) \qquad (cz+d)^2\ h(z;a_1,a_2,N) = h(\tfrac{az+b}{cz+d}, da_1 - ca_2, -ba_1+aa_2, N) - h(\tfrac{az+b}{cz+d}, -c, a, N) \ .$$

$$= \sum_{n=0}^{\infty} \delta_n(a_1,a_2,N;a,b,c,d)\, \exp(2\pi i n(az+b)/N(cz+d))$$

where

$$\delta_n(a_1,a_2,N;a,b,c,d) = \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t\{\alpha_n(da_1-ca_2,-ba_1+aa_2,N)-\alpha_n(-c,a,N)\}.$$

Again by (21) and (22), we have since $\{-c,a\} \not\equiv \{0,1\}(N)$:

$$(37) \quad \delta_0(a_1,a_2,N;a,b,c,d) = \begin{cases} 1 & \text{if } \{c,d\} \equiv \pm \{a_1,a_2\}(N) \\ 0 & \text{otherwise} \end{cases} .$$

In a cusp neighborhood of $z = -d/c$ we have the two local expressions $\psi(t)dt$ and $h(z;a_1,a_2,N)dz$, where $t = \exp\{2\pi i(az+b)/N(cz+d)\}$, a local coordinate at $z = -d/c$. Since the two expressions must agree at a regular point in this neighborhood, we get:

$$\psi(t) = \frac{h(z;a_1,a_2,N)}{\frac{dt}{dz}}$$

$$= \frac{h(z;a_1,a_2,N)}{t \cdot \frac{2\pi i}{N} \cdot \frac{1}{(cz+d)^2}}$$

which by (31) gives:

$$\psi(t) = \frac{\sum_{n=0}^{\infty} \delta_n(a_1,a_2,N;a,b,c,d)t^n}{t \cdot \frac{2\pi i}{N}} .$$

Thus by (32), $\psi(t)$ has a simple pole at $t = 0$, $(z = -d/c)$ if $\{c,d\} \equiv \pm \{a_1,a_2\}(N)$ and is analytic at $t = 0$ otherwise. This concludes the proof of Theorem 1.

To summarize, we have shown that for $\{a_1,a_2\}$ such that $(a_1,a_2,N) = 1$ and $\{a_1,a_2\} \not\equiv \pm \{0,1\}(N)$, the functions $h(z;a_1,a_2,N)$ can be used to define abelian differentials of third kind on $S_N = \overline{(\mathcal{H}/\Gamma(N))}$.

The differential $h(z;a_1,a_2,N)dz$ is analytic at all points of S_N except at the cusp ∞ where it has a simple pole of residue $-N/2\pi i$, and at the cusp $-d/c$, $(c,d) = 1$, $\{c,d\} \equiv \pm \{a_1,a_2\}(N)$, where it has a simple pole of residue $N/2\pi i$. Let:

(38) H_N = the vector space of abelian differentials on S_N generated by the differentials $h(z;a_1,a_2,N)dz$.

By (26- 28) we see that

(39) $\dim_{\mathbb{C}} H_N = \sigma(N) - 1$ where $\sigma(N)$ is given in (25).

§3. A Faithful Finite Dimensional Group Representation of $\Gamma/\Gamma(N)$.

In this section we define the group representation $L\colon \Gamma/\Gamma(N) \to GL(H_N)$ where $GL(H_N)$ is the group of 1-1 linear transformations of H_N onto itself. For $N = q$, a prime ≥ 3, we find the multiplicities of the irreducible components of L .

From the previous section we have the vector space H_N of abelian differential on S_N, generated by $\{h(z;a_1,a_2,N)dz\}$. We saw that:

$$\dim_{\mathbb{C}} H_N = \sigma(N) - 1$$

where

$$\sigma(N) = \begin{cases} 1 & \text{for } N = 1 \\ 3 & \text{for } N = 2 \\ \dfrac{N^2}{2} \prod\limits_{\substack{p|N \\ p \text{ prime}}} (1 - \dfrac{1}{p^2}) & \text{for } N \geq 3 \end{cases} .$$

For ease of notation we will write for fixed N :

(40) h_{a_1,a_2} = the differential on S_N defined by $h(z;a_1,a_2,N)dz$.

For an analytic function $f\colon \mathcal{H} \to \mathcal{H}$ we define:

$$f(z)|_V \underset{\text{def}}{=} f(V(z)) \frac{dV}{dz} \quad \text{for} \quad V \in \Gamma \quad . \tag{41}$$

For the corresponding differential $f(z)dz$ we define:

$$(f(z)dz)|_V = f(z)|_V\, dz = f(V(z))dV(z) \quad . \tag{42}$$

By (29) we have for $f(z) = h(z;a_1,a_2,N)$:

(43) $h(z;a_1,a_2,N)|_V = h(z;A_1,A_2,N) - h(z;c,d,N)$ and

thus for our differentials we have:

$$h_{a_1,a_2}\Big|_V = h_{A_1,A_2} - h_{c,d} \tag{44}$$

where $V \in \Gamma$, $\underline{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\{A_1,A_2\} = \{a_1,a_2\} \cdot \underline{V}$.

In particular, if $V \in \Gamma(N)$, then by (30) we have:

(45) $$h_{a_1,a_2}\Big|_V = h_{a_1,a_2} \quad \text{for } V \in \Gamma(N) .$$

By the chain rule for derivatives it follows that:

(46) $$h|_{V_1V_2} = (h|_{V_1})|_{V_2} .$$

Furthermore by our definitions, we have for α and $\beta \in \mathbb{C}$, f and $g \in H_N$:

(47) $$(\alpha f + \beta g)|_V = \alpha(f|_V) + \beta(g|_V) .$$

Thus any $V \in \Gamma$ defines a linear map $L(V)$ of H_N into itself via:

(48) $h \xrightarrow{L(V)} h|_{V^{-1}}$. [We use V^{-1} so that we will have

$L(V_1 \circ V_2) = L(V_1) \circ L(V_2)$].

(49) <u>Claim (1)</u>: $L(V)$ is <u>onto</u> H_N .

<u>Verification</u>: By (46) we see that given $h \in H_N$ we have:

$$(h|_V)|_{V^{-1}} = h|_{VV^{-1}} = h|_{id} = h .$$

<u>Q.E.D. claim (1)</u>.

Since H_N is finite dimensional, $L(V)$ is also 1-1 and thus $L(V) \in GL(H_N)$, the 1-1 linear maps of H_N <u>onto</u> itself.

(50) <u>Claim (2)</u>: $L(V)$ depends only on the coset of V (left or right) mod $\Gamma(N)$.

<u>Verification</u>: Suppose $V,W \in \Gamma$, $K \in \Gamma(N)$, $V = KW$.

Then for $h \in H_N$ we have:

$$h|_{V^{-1}} = h|_{W^{-1}K^{-1}} = (h|_{W^{-1}})|_{K^{-1}} = h|_{W^{-1}} .$$

If $V = WK$ for $K \in \Gamma(N)$ a similar argument works.

Note: The latter case also follows from the first case since $\Gamma(N)$ is normal in Γ .

Q.E.D. Claim (2).

We now define the mapping $L\colon \Gamma/\Gamma(N) \to GL(W_N)$ by

(51) $$L(\Gamma(N)\cdot V) = L(V) \ .$$

(52) Lemma 1: L is a group representation.

Proof: L is a homomorphism since

$$h|_{(V_1 \circ V_2)^{-1}} = h|_{V_2^{-1} \circ V_1^{-1}} = (h|_{V_2^{-1}})|_{V_1^{-1}}$$

and thus

$$L(V_1 \circ V_2) = L(V_1) \circ L(V_2) \ .$$

Q.E.D. Lemma 1.

(53) Lemma 2: L is a faithful representation, that is, 1-1 .

Proof: Let $V \in \Gamma$, with $\underline{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let I = identity map on H_N .

Suppose $L(V) = I$. Then $h_{a_1,a_2}|_V = h_{a_1,a_2}$ for all basis vectors h_{a_1,a_2} . But $h_{a_1,a_2}|_V = h_{A_1,A_2} - h_{c,d} = h_{\{aa_1+ca_2,ba_1+da_2\}} - h_{c,d}$. Thus

$$h_{a_1,a_2}|_V = h_{a_1,a_2} \underset{(1)}{\Rightarrow} \begin{cases} \{aa_1 + ca_2, ba_1 + da_2\} \equiv \pm \{a_1,a_2\}(N) \\ \text{and} \\ \{c,d\} \equiv \pm \{0,1\}(N) \ . \end{cases}$$

[(1) since the h's are linearly independent]

$$\Rightarrow \{aa_1, ba_1 + a_2\} \equiv \pm \{a_1,a_2\}(N) \ ,$$
$$\Rightarrow \{a,b\} \equiv \pm \{1,0\}(N) \ .$$

Thus $\underline{V} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(N)$, i.e., $V \in \Gamma(N)$.

Note: $\underline{V} \not\equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(N)$ since then $V \notin \Gamma$ if $N \geq 3$.

Q.E.D. Lemma 2.

To summarize, we have shown that $L\colon \Gamma/\Gamma(N) \to GL(H_N)$ is a faithful group representation. The number of different possible irreducible representations of $\Gamma/\Gamma(q)$ for q a prime ≥ 3, was calculated by Frobenius [4]. If we let

$$\epsilon = (-1)^{\frac{q-1}{2}} \quad ,$$

and L_n stand for an irreducible representation of dimension n, then we have the following list:

Besides the identity representation we have:

$$(54) \qquad \begin{array}{rll} 1 & \text{representation} & L_q \\ 2 & \text{representations} & L^{(1)}_{\frac{q+\epsilon}{2}} \text{ and } L^{(2)}_{\frac{q+\epsilon}{2}} \\ \frac{q-\epsilon}{4} - 1 & \text{representations} & L^{(i)}_{q+1} \quad (i = 1,\ldots,\frac{q-\epsilon}{4} - 1) \\ \frac{q+\epsilon}{4} - \frac{1}{2} & \text{representations} & L^{(j)}_{q-1} \quad (j = 1,\ldots,\frac{q+\epsilon}{4} - \frac{1}{2}) \end{array}$$

In our case we will show that the identity representation does not occur as a component, so our representation L has the following decomposition:

$$(55) \qquad L = xL_q + y_1 L^{(1)}_{\frac{q+\epsilon}{2}} + y_2 L^{(2)}_{\frac{q+\epsilon}{2}} + \sum_i u_i \cdot L^{(i)}_{q+1} + \sum_i v_i \cdot L^{(i)}_{q-1}$$

where x, y_i, u_i, v_i are the multiplicities of the irreducible components, and thus non-negative integers.

In order to show that the identity representation does not occur as a component of L, we need a function theoretic result of Lewittes [8], and some preliminary notation.

Since $\Gamma(N)$ is normal in Γ, Γ acts in a natural way on $S_N = \overline{H/\Gamma(N)}$; namely, for $V \in \Gamma$ and $[z_0] \in S_N$ we define

$$(56) \qquad V([z_0]) = [V(z_0)] \ .$$

This is well defined, since if $[z_0] = [z_1]$ then $z_1 = M(z_0)$ for some $M \in \Gamma(N)$ and so $V(z_1) = V \circ M(z_0)$. But $V \circ M = K \circ V$ for some $K \in \Gamma(N)$ since $\Gamma(N)$ is normal in Γ . Thus $V(z_1) = K \circ V(z_1)$ and so

$$[V(z_1)] = [V(z_0)]$$

as needed. By a similar argument we can define $\Gamma/\Gamma(N)$ acting on S_N by

$$(57) \qquad (\Gamma(N) \cdot V)([z_0]) = [V(z_0)] \in S_N \ .$$

In this way $\Gamma(N) \cdot V$ is a conformal homeomorphism of S_N onto itself, also called an <u>automorphism</u> of S_N , Hence $\Gamma/\Gamma(N)$ is a group of automorphisms of S_N . Thus if we let

$$(58) \qquad \mathfrak{m}(N) = \Gamma/\Gamma(N) \ ,$$

we can define the surface

$$(59) \qquad \widetilde{S}_N = S_N/\mathfrak{m}(N) \ .$$

Since in what follows, the results depend only on the coset of an element in Γ mod $\Gamma(N)$, we will let V denote both the element of Γ and its coset in $\Gamma/\Gamma(N)$.

(60) <u>Lemma 3</u>: There is no non-zero vector in H_N which is left fixed by $L(V)$ for all $V \in \Gamma/\Gamma(N)$.

<u>Proof</u>: By [8], a differential on S_N which is left fixed under the action of $L(V)$ for every $V \in \Gamma/\Gamma(N)$, determines a differential on $\widetilde{S}_N$ and conversely. In fact on page 738-9 of [8], there are formulas for going from a Taylor series of one to a Taylor series of the other.

Now suppose we have a non-zero differential $\theta \in H_N$ which is left fixed by $L(V)$ for all $V \in \Gamma/\Gamma(N)$. Then θ is a linear combination of the differentials h_{a_1,a_2} and so must have simple poles at two or more cusps and be analytic elsewhere. Let P_o be any point on S_N. An invariant differential θ has the expansion in local coordinate t at P_o given by:

$$(61)\quad \theta = (a_{n_o h(P_o)-1}\, t^{n_o h(P_o)-1} + \ldots + a_{(n_o+k)h(P_o)-1}\, t^{(n_o+k)h(P_o)-1} + \ldots)dt$$

where n_o is an integer to be determined and $h(P_o)$ is the order of the stability subgroup of P_o, that is, the subgroup of $\mathfrak{m}(N)$ leaving P_o fixed. Then by [8], the corresponding $\tilde{\theta}$ on $\tilde{S}$ has the expansion:

$$(62)\quad \tilde{\theta} = \frac{1}{h(P_o)}\,(a_{n_o h(P_o)-1}\, \tilde{t}^{(n_o-1)} + \ldots +)d\tilde{t}\ ,$$

where $\tilde{t} = t^{h\,(P_o)}$ is a local coordinate on $\tilde{S}_N$ at $\tilde{P}_o$. But in our case,

$$\begin{aligned} \tilde{S}_N &= S_N/\mathfrak{m}(N) \\ &= \overline{(\mathcal{H}/\Gamma(N))}/(\Gamma/\Gamma(N)) \\ \overset{\approx}{\text{conformal}}&\ \ \overline{(\mathcal{H}/\Gamma)} \\ \overset{\approx}{\text{conformal}}&\ \ S_0 = \text{Riemann sphere}\ . \end{aligned}$$

At a cusp where θ has a simple pole, the order of θ in local coordinate t must be -1 <u>and</u> since $h(P_o) > 0$, it follows that $n_o = 0$. On $\tilde{S}$ all the cusps are identified to one point, say $[\tilde{\infty}]$. But $\tilde{\theta}$ at $[\tilde{\infty}]$, in local coordinate $\tilde{t}$, then has order -1 since $n_o = 0$. On the other hand, at a point P_o where θ has no simple pole, θ is analytic and so in (61) we have $n_o \geq 1$. Then by (62), $\tilde{\theta}$ is analytic at the corresponding $\tilde{P}_o$.

Thus $\tilde{\theta}$ is a differential on the Riemann sphere with one simple pole and otherwise analytic. This is <u>impossible</u>, since for a one-differential on a compact Riemann surface we know that the sum of the residues must be zero. Thus $\tilde{\theta}$ would have to have at least one other pole on S_0^{σ}. Thus there cannot exist such a $\tilde{\theta}$ as we arrived at, and therefore no such invariant θ .

<u>Q.E.D. Lemma 3.</u>

Thus we have shown that in the representation $L: \Gamma/\Gamma(N) \to GL(H_N)$, the multiplicity of the identity representation is zero, which justifies its omission in the decomposition (55).

Frobenius [5] calculated the characters of the elements of $\Gamma/\Gamma(q)$ with respect to the irreducible representations. In particular, if

$$T(z) = z + 1$$

and R is any element of $\Gamma/\Gamma(q)$ of order $\frac{q-1}{2}$ then we have the following table:

	L_q	$L^{(i)}_{\frac{q+\epsilon}{2}}$	$L^{(i)}_{q+1}$	$L^{(i)}_{q-1}$
dim	q	$\frac{q+\epsilon}{2}$	$q+1$	$q-1$
$\chi(T)$	0	$\frac{\epsilon \pm \sqrt{\epsilon q}}{2}$	1	-1
$\chi(R^a)$	1	$\left(\frac{1+\epsilon}{2}\right)(-1)^a$	$\rho_i^a + \rho_i^{-a}$	0

where ρ_i is such that $\rho_i^{\frac{q-1}{2}} = 1$, $\rho_i \neq \pm 1$, and $\underline{a}$ runs through the residue classes mod $\frac{q-1}{2}$ except 0 .

<u>Note</u>: Reciprocal roots ρ_i and ρ_i^{-1} obviously define the same character in the last line.

Now, writing H_q as the direct sum of its invariant subspaces we get the following equation from (55):

$$(64) \qquad \chi(V) = x\cdot\chi_q(V) + y_1\,\chi^{(1)}_{\frac{q+\varepsilon}{2}}(V) + y_2\,\chi^{(2)}_{\frac{q+\varepsilon}{2}}(V) + \sum_{i=1}^{\frac{q-\varepsilon}{4}-1} u_i\,\chi^{(i)}_{q+1}(V) + \sum_{i=1}^{\frac{q+\varepsilon}{4}-\frac{1}{2}} v_i\,\chi^{(i)}_{q-1}(V)$$

where $\chi(V)$ is the character of V with respect to the representation L and $\chi_n(V)$ is the character of V with respect to the irreducible representation L_n. Furthermore,

$$(65) \qquad \dim_{\mathbb{C}} H_q = \sigma(q) - 1 = \frac{q^2-3}{2} .$$

Thus by adding the dimensions of the invariant subspaces with their appropriate multiplicities we have:

$$(66) \qquad \frac{q^2-3}{2} = x\cdot q + (y_1+y_2)\cdot\frac{q+\varepsilon}{2} + (q+1)\sum_{i=1}^{\frac{q-\varepsilon}{4}-1} u_i + (q-1)\sum_{i=1}^{\frac{q+\varepsilon}{4}-\frac{1}{2}} v_i .$$

<u>Note</u>: (66) can be obtained from (64) by using $V = \text{identity} \in \Gamma/\Gamma(q)$.

As a basis for H_q we have the differentials h_{a_1,a_2} subject to the restrictions given earlier, namely:

(67) $(a_1,a_2,q) = 1$ [which for q prime means $\{a_1,a_2\} \not\equiv \{0,0\}(q)$] ,

and by (15) and definitions (16),(23), and (35) we have:

$$(68) \qquad h_{a_1,a_2} = h_{b_1,b_2} \Leftrightarrow \{a_1,a_2\} \equiv \pm\{b_1,b_2\}(q) .$$

Thus, if we define:

$$(69) \qquad A_q = \{\{0,2\},\{0,3\},\ldots,\{0,\tfrac{q-1}{2}\},\{1,0\},\{1,1\},\ldots,\{1,q-1\},\ldots,$$
$$\{\tfrac{q-1}{2},0\},\{\tfrac{q-1}{2},1\},\ldots,\{\tfrac{q-1}{2},q-1\}\}$$

then the ordered set $\mathcal{B}_q = \{h_{a_1,a_2}\}_{\{a_1,a_2\}\in A_q}$ is a basis for H_q, where the ordering in $\mathcal{B}_q$ corresponds to the ordering in A_q.

Using this basis for H_q, let us calculate $\chi(T)$, where $T(z) = z+1$.

Proposition 1:

$$\chi(T) = \frac{q-3}{2}, \quad T(z) = z + 1. \tag{70}$$

Proof: $\underline{T}^{-1} = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$.

By (44) we have: $L(T)(h_{a_1,a_2}) =$

$$h_{a_1,a_2}\Big|_{T^{-1}} = h_{a_1,a_2-a_1} - h_{0,1} = h_{a_1,a_2-a_1} \text{ since } h_{0,1} = \vec{0}.$$

If we write the linear map $L(T)$ as a matrix with respect to the basis $\{h_{a_1,a_2}\}_{\{a_1,a_2\}\in A_q}$ we will get a contribution to the diagonal $\Leftrightarrow h_{a_1,a_2-a_1} = \pm h_{a_1,a_2}$. But h_{a_1,a_2-a_1} is a basis vector itself and so it could not equal $-h_{a_1,a_2}$. Now, $h_{a_1,a_2-a_1} = h_{a_1,a_2} \Leftrightarrow \{a_1,a_2-a_1\} \equiv \pm\{a_1,a_2\}(q)$. But $\{a_1,a_2-a_1\} \equiv \{a_1,a_2\}(q) \Leftrightarrow a_1 \equiv 0(q)$. Similarly $\{a_1,a_2-a_1\} \equiv -\{a_1,a_2\}(q) \Rightarrow a_1 \equiv -a_1(q)$ which $\Rightarrow a_1 \equiv 0(q)$ since q is odd. But $a_1 \equiv 0(q) \Rightarrow a_2 \equiv -a_2(q)$ which $\Rightarrow a_2 \equiv 0(q)$. Thus $\{a_1,a_2-a_1\} \equiv -\{a_1,a_2\}(q) \Rightarrow \{a_1,a_2\} \equiv \{0,0\}(q)$ which is <u>impossible</u> since $(a_1,a_2,q) = 1$. Thus we have:

$$h_{a_1,a_2-a_1} = h_{a_1,a_2} \Leftrightarrow a_1 \equiv 0\ (q). \tag{71}$$

Thus for the matrix of $L(T)$ we get $+1$ on the diagonal from each of the following $(q-3)/2$ equations:

$$h_{\{0,2\}}\Big|_{T^{-1}} = h_{\{0,2\}}, \ldots, h_{\{0,(q-1)/2\}}\Big|_{T^{-1}} = h_{\{0,(q-1)/2\}};$$

and we get a <u>zero</u> contribution to the diagonal from the transformation of the other basis vectors. Thus we have:

$$\chi(T) = \frac{q-3}{2} \quad \text{where} \quad T(z) = z + 1 \ .$$

This result is of course independent of the basis chosen for H_q since any basis would give the same trace in the matrix of $L(V)$, $V \in \Gamma$.

Q.E.D. Proposition 1.

Proposition 2: For $q \equiv 3(4)$ we have in equation (55):

$$(72) \qquad L = L_q + \sum_{i=1}^{\frac{q-3}{4}} u_i L_{q+1}^{(i)} \quad \text{with} \quad \sum_{i=1}^{\frac{q-3}{4}} u_i = \frac{q-3}{2} \ .$$

Proof: We look at the characters of T relative to the irreducible representations L_n, using table (63),

$$\epsilon = (-1)^{\frac{q-1}{2}} = -1 \ ,$$

and we get from our table:

$$(73) \qquad \chi_q(T) = 0 \ .$$

$$(74) \qquad \chi^{(1)}_{\frac{q-1}{2}}(T) = \frac{-1+\sqrt{-q}}{2} \ , \quad \chi^{(2)}_{\frac{q-1}{2}}(T) = \frac{-1-\sqrt{-q}}{2} \ .$$

$$(75) \qquad \chi^{(i)}_{q+1}(T) = 1 \ , \ i = 1,\dots,\frac{q-3}{4} \ .$$

$$(76) \qquad \chi^{(i)}_{q-1}(T) = -1 \ , \ i = 1,\dots,\frac{q-3}{4} \ .$$

Thus using (70), and (73-76), equation (64) becomes for $V = T$:

$$(77) \qquad \frac{q-3}{2} = y_1\left(\frac{-1+\sqrt{-q}}{2}\right) + y_2\left(\frac{-1-\sqrt{-q}}{2}\right) + \sum_{i=1}^{\frac{q-3}{4}} u_i - \sum_{i=1}^{\frac{q-3}{4}} v_i \ .$$

From (77) we immediately deduce:

(78) $y_1 = y_2$ (call their common value y), for otherwise there would be an imaginary part on the right hand side of (77) and none on the left hand side. Thus (77) can be written:

(79) $$\frac{q-3}{2} = -y + \sum_i u_i - \sum_i v_i \ ,$$

and so we get with $U = \sum_i u_i$,

(80) $$\frac{q-3}{2} \le U \ .$$

On the other hand (66) gives:

(81) $$\frac{q^2-3}{2} \ge (q+1)U \ .$$

If $U \ge \frac{q-1}{2}$ then (81) would give $\frac{q^2-3}{2} \ge \frac{q^2-1}{2}$, a contradiction. Thus we have

(82) $$U < \frac{q-1}{2} \ , \text{ or equivalently,}$$
$$U \le \frac{q-3}{2} \quad \text{since } U \text{ is an integer.}$$

Thus (80) and (82) give us:

(83) $$\sum_{i=1}^{\frac{q-3}{4}} u_i = U = \frac{q-3}{2} \ .$$

If we now let $V = \sum_{i=1}^{\frac{q-3}{4}} v_i$, then (66) becomes:

(84) $$\frac{q^2-3}{2} = x\cdot q + y(q-1) + (q+1)\left(\frac{q-3}{2}\right) + (q-1)\cdot V$$

which upon simplifying gives the Diophantine equation:

(85) $$q = x\cdot q + y(q-1) + (q-1)\cdot V$$

where x,y, and V are required to be integers ≥ 0 . One solution is

(86) $$x = 1, \ y = 0, \ V = 0 \ .$$

Certainly $x \le 1$, so if $x \ne 1$, then $x = 0$, and then (85) becomes:

(87) $$\frac{q}{q-1} = y + V \quad \text{an integer}$$

which for $q \ge 3$ is impossible. Thus (86) gives the only solution and so our decomposition of L, (see (55)) becomes:

$$L = L_q + \sum_{i=1}^{\frac{q-3}{4}} u_i L_{q+1}^{(i)} ,$$

with

$$\sum_{i=1}^{\frac{q-3}{4}} u_i = \frac{q-3}{2} .$$

Q.E.D. Proposition 2.

If $q = 3$, then $u_i = 0 \ \forall \ i$ and we have that our representation is irreducible:

(88) $$L = L_3 , \qquad (q = 3) .$$

[It will be shown later that for $q \equiv 3(4)$, $q \geq 7$, (q prime), we have $u_i = 2$ for all i .]

Proposition 3: If $q \equiv 1(4)$, then our decomposition in (55) is:

(89) $$L = L_q + y\left(L_{\frac{q+1}{2}}^{(1)} + L_{\frac{q+1}{2}}^{(2)}\right) + \sum_{i=1}^{\frac{q-5}{4}} u_i L_{q+1}^{(i)}$$

with

$$y + \sum_{i=1}^{\frac{q-5}{4}} u_i = \frac{q-3}{2} .$$

Proof: Since $\frac{q-1}{2}$ is even, $\epsilon = (-1)^{\frac{q-1}{2}} = +1$, and thus from table (58) we have:

(90) $$\chi_q(T) = 0 .$$

(91) $$\chi_{\frac{q+1}{2}}^{(1)}(T) = \frac{1+\sqrt{q}}{2} , \quad \chi_{\frac{q+1}{2}}^{(2)}(T) = \frac{1-\sqrt{q}}{2} .$$

(92) $$\chi_{q+1}^{(i)}(T) = 1 , \quad i = 1,\ldots,\frac{q-5}{4} .$$

(93) $$\chi_{q-1}^{(i)}(T) = -1 , \quad i = 1,\ldots,\frac{q-1}{4} .$$

Thus equation (64) becomes:

(94) $$\frac{q-3}{2} = y_1\left(\frac{1+\sqrt{q}}{2}\right) + y_2\left(\frac{1-\sqrt{q}}{2}\right) + \sum_{i=1}^{\frac{q-5}{4}} u_i - \sum_{i=1}^{\frac{q-1}{4}} v_i \quad .$$

From (94) we deduce:

(95) $y_1 = y_2$, for otherwise the right hand side would have an irrational part while the left hand side would not. Secondly:

(96) $$\frac{q-3}{2} \le y + U \quad \text{where } y = y_1 = y_2 \text{ and } U = \sum_i u_i \quad .$$

But equation (66) gives:

(97) $$\frac{q^2-3}{2} \ge (q+1)(y+U) \quad .$$

Thus if in (96) we had $\frac{q-3}{2} < y + U$, which means $\frac{q-1}{2} \le y + U$, then (97) would give:

$$\frac{q^2-3}{2} \ge \frac{q^2-1}{2} \text{ , a contradiction.}$$

Thus from (96) we must have:

(98) $$\frac{q-3}{2} = y + U \ .$$

Substituting into equation (66) with $\epsilon = 1$, $V = \sum_i v_i$, we have

$$\frac{q^2-3}{2} = q\cdot x + (q+1)\left(\frac{q-3}{2}\right) + (q-1)\cdot V \ ,$$

which simplifies to:

(99) $$q = q\cdot x + (q-1)\, V \ .$$

Clearly $x = 1, V = 0$ is the only non-negative integer solution and thus our decomposition (55) becomes:

(100) $$L = L_q + y\left(L^{(1)}_{\frac{q+1}{2}} + L^{(2)}_{\frac{q+1}{2}}\right) + \sum_{i=1}^{\frac{q-5}{4}} u_i\, L^{(i)}_{q+1}$$

with

$$y + \sum_{i=1}^{\frac{q-5}{4}} u_i = \frac{q-3}{2} .$$

<u>Q.E.D. Proposition 3</u>.

For $q = 5$, the unique solution is $y = 1$, $u_i = 0$. Thus for $q = 5$ our decomposition (55) becomes:

(101) $$L = L_5 + L_3^{(1)} + L_3^{(2)} , \quad (q = 5) .$$

<u>Note</u>: Since (88) and (101) take care of the cases $q = 3$, $q = 5$, let us assume from now on that $q \geq 7$.

In table (63) R stands for a map in $\Gamma/\Gamma(q)$ of order $\frac{q-1}{2} \geq 3$. Taking a result from [3] we know that for such a map we can take:

(102) $$R = ST^{\alpha} \, ST^{\beta} \, ST^{\alpha}$$

where $S(z) = \frac{-1}{z}$, $T(z) = z + 1$, α is a primitive root mod q and $\alpha\beta \equiv 1 (\text{mod } q)$. This yields

$$R \equiv \pm \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} (q) ,$$

so

$$\underline{R}^{\frac{q-1}{2}} \equiv \pm \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} (q) .$$

If $V \underset{\text{def}}{=} R^{-1}$ then,

(103) $$\underline{V} = (\underline{R}^{-1}) = \pm (\underline{R})^{-1} \equiv \pm \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} (q) ,$$

and

(104) $$(\underline{R})^{-\lambda} \equiv \pm \underline{V}^{\lambda} \equiv \pm \begin{pmatrix} \alpha^{\lambda} & 0 \\ 0 & \beta^{\lambda} \end{pmatrix} (q).$$

Thus we have:

$$h_{x,y}\Big|_{R^{-\lambda}} = h_{\{x,y\}\underline{V}^{\lambda}} - h_{0,\beta^{\lambda}} = h_{x\alpha^{\lambda}, y\beta^{\lambda}} - h_{0,\beta^{\lambda}} .$$

(105) <u>Proposition 4</u>: For $\lambda \not\equiv 0 \left(\frac{q-1}{2}\right)$ we have

$$\chi(R^{\lambda}) = -1 .$$

[Note: If $\lambda \equiv 0\ (\frac{q-1}{2})$, then $R^{\lambda} = I \in \Gamma/\Gamma(N)$ and $\chi(I)$ = dim of representation].

Proof: To get any contributions to the diagonal of the matrix of $L(R^{\lambda})$ with respect to basis $\{h_{x,y}\}_{\{x,y\}\in A_q}$ we must get them separately from $h_{x\alpha^{\lambda},y\beta^{\lambda}}$ and $h_{0,\beta^{\lambda}}$ since the differentials $h_{x,y}$ are linearly independent. To get a contribution from $h_{x\alpha^{\lambda},y\beta^{\lambda}}$ we must have:

$$\{x\alpha^{\lambda},y\beta^{\lambda}\} \equiv \pm \{x,y\}\ (q) \quad . \tag{106}$$

But since $\{x,y\} \not\equiv \{0,0\}\ (q)$ then we must have either

$$\alpha^{\lambda} \equiv \pm 1\ (q) \quad \text{or} \quad \beta^{\lambda} \equiv \pm 1\ (q) \quad .$$

Assume $\alpha^{\lambda} \equiv \pm 1\ (q)$. Then $\alpha^{2\lambda} \equiv 1\ (q)$ and so $2\lambda \equiv 0\ (q-1)$ since α is a primitive root mod q. This gives $\lambda \equiv 0\ (\frac{q-1}{2})$ which contradicts the hypothesis of our lemma. Thus $\alpha^{\lambda} \not\equiv \pm 1\ (q)$ and similarly $\beta^{\lambda} \not\equiv \pm 1 (q)$. Thus there can be no contribution to the diagonal of $L(R^{\lambda})$ from the term $h_{x\alpha^{\lambda},y\beta^{\lambda}}$.

To get a contribution from the term $-h_{0,\beta^{\lambda}}$ we must have:

$$\{x,y\} \equiv \pm \{0,\beta^{\lambda}\}\ (q) \ . \tag{107}$$

Note: $\beta^{\lambda} \not\equiv \pm 1\ (q)$ since $\lambda \not\equiv 0\ (\frac{q-1}{2})$ and therefore $h_{0,\beta^{\lambda}} \neq \vec{0}$.

Now (107) does result in a contribution to the diagonal but only one contribution, since $h_{0,\beta^{\lambda}} = h_{0,-\beta^{\lambda}}$. We know

$$h_{0,\beta^{\lambda}}\Big|_{R^{-\lambda}} = h_{0,\beta^{2\lambda}} - h_{0,\beta^{\lambda}} \ .$$

Thus we get a contribution of -1 to the diagonal of $L(R^{\lambda})$ from

$h_{0,\beta^\lambda}\big|_{R^{-\lambda}}$, and all other contributions are zero. Therefore if $\lambda \not\equiv 0 \ (\frac{q-1}{2})$ we have

$$\chi(R^\lambda) = -1 . \tag{108}$$

<u>Q.E.D. Proposition 4</u>.

From our table (63), we have for $\lambda \not\equiv 0 \ (\frac{q-1}{2})$:

$$\chi_q(R^\lambda) = 1 . \tag{109}$$

$$\chi^{(k)}_{\frac{q+\varepsilon}{2}}(R^\lambda) = \left(\frac{1+\varepsilon}{2}\right)(-1)^\lambda , \quad k = 1,2 . \tag{110}$$

$$\chi^{(k)}_{q+1}(R^\lambda) = \rho_k^\lambda + \rho_k^{-\lambda} , \qquad k = 1,\ldots,\frac{q-\varepsilon}{4} - 1 \tag{111}$$

where $\rho_k^{(q-1)/2} = 1$, $\rho_k \neq \pm 1$.

For definiteness, let us define:

$$\rho \underset{\text{def}}{=} \exp\{4\pi i/(q-1)\} , \quad i = \sqrt{-1} , \tag{112}$$

and

$$\rho_k \underset{\text{def}}{=} \rho^k = \exp\{4\pi k i/(q-1)\} . \tag{113}$$

We will use (108)-(111) to find the multiplicities u_i in (72), and y and u_i in (89). First, we need the following two lemmas:

<u>Lemma 1</u>: Suppose m is odd, $m \geq 3$, ρ is a primitive m^{th} root of 1, and λ is an integer, $\lambda \not\equiv 0$ (m) . Then,

$$\sum_{k=1}^{\frac{m-1}{2}} (\rho^{k\lambda} + \rho^{-k\lambda}) = -1 . \tag{114}$$

<u>Proof</u>: By hypothesis $\rho^\lambda \neq 1$ and $\rho^{\lambda m} = 1$. Thus ρ^λ is a root of the polynomial $x^m - 1 = (x-1)(x^{m-1} + x^{m-2} + \ldots + x + 1)$ but since $\rho^\lambda - 1 \neq 0$, it follows that ρ^λ is a root of $x^{m-1} + x^{m-2} + \ldots + x + 1$. Thus we have:

$$(115) \qquad \sum_{k=1}^{m-1} \rho^{\lambda k} = -1 .$$

Furthermore, since $\rho^{\lambda m} = 1$ we can write

$$\rho^{-\lambda k} = \rho^{-\lambda k} \cdot \rho^{\lambda m} = \rho^{\lambda(m-k)} .$$

Thus we have:

$$\sum_{k=1}^{\frac{m-1}{2}} (\rho^{\lambda k}+\rho^{-\ k}) = \sum_{k=1}^{\frac{m-1}{2}} (\rho^{\lambda k}+\rho^{\lambda(m-k)}) = \sum_{k=1}^{m-1} \rho^{\lambda k} = -1$$

by (115).

Q.E.D. Lemma 1.

Lemma 2: Suppose m is even, $m \geq 4$, ρ is a primitive m^{th} root of 1, and λ is an integer, $\lambda \not\equiv 0 \ (m)$, Then,

$$(116) \qquad \sum_{k=1}^{\frac{m}{2}-1} (\rho^{\lambda k}+\rho^{-\lambda k}) = \begin{matrix} 0 & \text{if} & \lambda \text{ is odd,} \\ -2 & \text{if} & \lambda \text{ is even.} \end{matrix}$$

Proof: As in the proof of (114) we have:

$$(117) \qquad \sum_{k=1}^{m-1} \rho^{\lambda k} = -1 .$$

Furthermore,

$$\sum_{k=1}^{\frac{m}{2}-1} (\rho^{\lambda k}+\rho^{-\lambda k}) = \left(\sum_{k=1}^{m-1} \rho^{\lambda k}\right) - (\rho^{m/2})^{\lambda} = -1 - (\rho^{m/2})^{\lambda} .$$

Since ρ is a primitive m^{th} root of 1, m even, $m \geq 4$, we have:

$$\rho^{m/2} = -1 .$$

Thus

$$\sum_{k=1}^{\frac{m}{2}-1} (\rho^{\lambda k}+\rho^{-\lambda k}) = -1 - (-1)^{\lambda} .$$

Q.E.D. Lemma 2.

We are now ready to prove the following propositions about the multiplicities in (72) and (89), and thus the decomposition of our representation L into irreducible components L_n .

<u>Proposition 5</u>: In (72), we have $u_k = 2 \ \forall \ k$. That is, for $q \equiv 3(4)$ we have:

(118) $$L = L_q + \sum_{k=1}^{\frac{q-3}{4}} 2 \cdot L_{q+1}^{(k)} \quad \text{where } L_q \text{ and } L_{q+1}^{(k)} \text{ are irreducible.}$$

<u>Proof</u>: Using (72), (108), (109), and (111) we can write a system of equations from the characters of R^λ , $\lambda \not\equiv 0(\frac{q-1}{2})$; namely:

(119) $$-1 = 1 + \sum_{k=1}^{\frac{q-3}{4}} u_k(\rho_k^{\lambda} + \rho_k^{-\lambda})$$

or equivalently:

(120) $$-2 = \sum_{k=1}^{\frac{q-3}{4}} u_k(\rho^{k\lambda} + \rho^{-k\lambda}) \quad \text{where, recall } \rho = \exp\{4\pi i/(q-1)\}.$$

For $\lambda_1 \equiv \pm \lambda_2 \ (\frac{q-1}{2})$ we have:

$$\rho^{k\lambda_1} + \rho^{-k\lambda_1} = \rho^{k\lambda_2} + \rho^{-k\lambda_2} \text{ , since } \rho^{\frac{q-1}{2}} = 1 \ .$$

Thus from (120) for $\lambda = 1,2,\ldots, \frac{q-3}{4}$, we get $\frac{q-3}{4}$ equations in the $\frac{q-3}{4}$ unknowns u_i .

By Lemma 1 (114), with $m = (q-1)/2$ we see that one solution to (120) is

(121) $$u_k = 2 \ \forall \ k \ .$$

The matrix of coefficients (a_{ij}) where $a_{ij} = \rho^{ij} + \rho^{-ij}$ has a non-zero determinant (see Proposition 7). Thus our solution (121) is unique.

<u>Q.E.D. Proposition 5</u>.

<u>Proposition 6</u>: In (89) we have $y = 1$, $u_k = 2 \ \forall \ k$. Thus for $q \equiv 1(4)$ we have:

$$(122) \qquad L = L_q + L_{\frac{q+1}{2}}^{(1)} + L_{\frac{q+1}{2}}^{(2)} + \sum_{k=1}^{\frac{q-5}{4}} 2\, L_{q+1}^{(k)} \quad .$$

Proof: Using (89) and (108)-(111) we can write a system of equations from the characters of R^{λ} , $\lambda \not\equiv 0(\frac{q-1}{2})$; namely:

$$-1 = 1 + y\{(-1)^{\lambda} + (-1)^{\lambda}\} + \sum_{k=1}^{\frac{q-5}{4}} u_k(\rho^{k\lambda} + \rho^{-k\lambda})$$

where $\rho = \exp\{4\pi i/(q-1)\}$. Furthermore from (89) we know that

$$\frac{q-3}{2} = y + \sum_{k=1}^{\frac{q-5}{4}} u_k \quad .$$

Thus we have the system of equations:

$$(123) \qquad \begin{aligned} -2 &= 2y(-1)^{\lambda} + \sum_{k=1}^{\frac{q-5}{4}} u_k(\rho^{k\lambda} + \rho^{-k\lambda}) \ , \ \lambda = 1,2,\ldots,\frac{q-5}{4} \\ \frac{q-3}{2} &= y + \sum_{k=1}^{\frac{q-5}{4}} u_k \quad . \end{aligned}$$

This is a system of $\frac{q-1}{4}$ equations in the $\frac{q-1}{4}$ unknowns y , $u_1,\ldots,u_{\frac{q-5}{4}}$.

By Lemma 2, (116), with $m = \frac{q-1}{2}$, we see that one solution to (123) is:

$$(124) \qquad y = 1, \quad u_k = 2 \ \forall \ k \ .$$

But the matrix of coefficients has non-zero determinant (see Proposition 8). Thus our solution (124) is unique.

Q.E.D. Proposition 6.

We combine Propositions 5 and 6 into the following theorem.

Theorem 2: For q a prime ≥ 3, the representation $L: \Gamma/\Gamma(q) \to GL(H_q)$ has the decomposition into irreducibles given by:

$$L = L_q + \sum_{k=1}^{\frac{q-3}{4}} 2 \cdot L_{q+1}^{(k)} \quad \text{for} \quad q \equiv 3(4)$$

(125)

$$L = L_q + L_{\frac{q+1}{2}}^{(1)} + L_{\frac{q+1}{2}}^{(2)} + \sum_{k=1}^{\frac{q-5}{4}} 2\, L_{q+1}^{(k)} \quad \text{for} \quad q \equiv 1(4) \quad .$$

<u>Proof</u>: We need only prove Propositions 7 and 8 which were referred to in Propositions 5 and 6.

<u>Proposition 7</u>: Suppose m is an odd integer, and ρ is a <u>primitive</u> m^{th} root of 1 . Then, $\det(\rho^{ij} + \rho^{-ij}) \neq 0$ where $1 \leq i, j \leq \frac{m-1}{2}$. In fact, we will find:

$$(\det(\rho^{ij} + \rho^{-ij}))^2 = m^{\frac{m-3}{2}} \quad .$$

<u>Proof</u>: Let the matrix:

$$(\alpha_{ij}) = ((\rho^{ij} + \rho^{-ij}))^2 \quad . \tag{126}$$

Then

$$\alpha_{ij} = \sum_{k=1}^{\frac{m-1}{2}} \{\rho^{k(i+j)} + \rho^{-k(i+j)} + \rho^{k(i-j)} + \rho^{-k(i-j)}\} \ .$$

Now ρ^{i+j} is an m^{th} root of 1, and $\rho^{i+j} \neq 1$, since $2 \leq i + j \leq m-1$. Furthermore, ρ^{i-j} is an m^{th} root of 1, and

$$\rho^{i-j} = 1 \Leftrightarrow i = j \ .$$

Thus, by Lemma 1, (114), we have:

$$\sum_{k=1}^{\frac{m-1}{2}} (\rho^{k(i+j)} + \rho^{-k(i+j)}) = -1 \ ,$$

and

$$\sum_{k=1}^{\frac{m-1}{2}} (\rho^{k(i-j)} + \rho^{-k(i-j)}) = \begin{cases} -1 & \text{if } i \neq j \\ m-1 & \text{if } i = j \end{cases} \ .$$

Therefore:

$$(127) \qquad \alpha_{ij} = \begin{cases} -2 & \text{if} \quad i \neq j \\ m-2 & \text{if} \quad i = j \end{cases} .$$

m is fixed throughout (recall m is odd, $m \geq 3$).

For $k \geq 2$ let:

$$(128) \qquad A_k = \det \begin{bmatrix} (m-2) & -2 & \cdots & -2 \\ -2 & (m-2) & -2 \cdots & -2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -2 \\ -2 & \cdots -2 & \cdots & (m-2) \end{bmatrix} \qquad (k \times k)$$

Thus,

$$\det(\alpha_{ij}) = A_{\frac{m-1}{2}} .$$

In A_k we can subtract the second column from the first and we get for $k \geq 3$:

$$(129) \qquad A_k = \det \begin{bmatrix} m & -2 & -2 & \cdots & -2 \\ -m & (m-2) & -2 & \cdots & -2 \\ 0 & -2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2 \\ 0 & -2 & \cdots & -2 & (m-2) \end{bmatrix}$$

Thus for $k \geq 3$:

$$(130) \qquad A_k = m \cdot A_{k-1} + mB_{k-1} ,$$

where for $k \geq 2$:

(131) $$B_k = \det \begin{bmatrix} -2 & -2 & -2 & \cdots & -2 \\ -2 & (m-2) & -2 & \cdots & -2 \\ \vdots & -2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2 \\ -2 & \cdots & \cdots & -2 & (m-2) \end{bmatrix} \qquad (k \times k) \ .$$

In B_k we can subtract the second column from the first and get for $k \geq 3$:

(132) $$B_k = mB_{k-1} \ .$$

Using the recursive formulas (130)and (132) we get:

(133) $$A_k = m^{k-2}\{A_2 + (k-2)B_2\} = m^{k-2}\{m^2 - 4m + (k-2)(-2m)\} \ .$$

Now if we take $k = \frac{m-1}{2}$ we get

$$A_{\frac{m-1}{2}} = m^{\frac{m-3}{2}} \ .$$

<u>Q.E.D. Proposition 7</u>.

<u>Proposition 8</u>: Suppose m is an even integer, $m \geq 4$, and ρ is a <u>primitive</u> m^{th} root of 1.

Let (a_{ij}) be the $\frac{m}{2} \times \frac{m}{2}$ matrix defined by:

(134) $$a_{ij} = \begin{cases} \rho^{ij} + \rho^{-ij} & \text{for } 1 \leq i \leq \frac{m}{2} - 1 \text{ and } \\ & \quad 1 \leq j \leq \frac{m}{2} \, , \\ 1 & \text{for } i = \frac{m}{2} \, , \ 1 \leq j \leq \frac{m}{2} \ . \end{cases}$$

Then $(\det(a_{ij}))^2 = m^{\frac{m}{2} - 1}$.

<u>Proof</u>: If we have the matrix equation:

(135) $$(\beta_{ij}) = (a_{ij})^2 \quad \text{then}$$

(136) $$\beta_{ij} = \begin{cases} \{\sum_{k=1}^{\frac{m}{2}-1} (\rho^{k(i+j)} + \rho^{-k(i+j)} + \rho^{k(i-j)} + \rho^{-k(i-j)})\} + 2(-1)^i & \text{if } 1 \le i \le \frac{m}{2}-1 \\ & \phantom{\text{if }} 1 \le j \le \frac{m}{2}, \\ \{\sum_{k=1}^{\frac{m}{2}-1} (\rho^{kj} + \rho^{-kj})\} + 1 & \text{if } i = \frac{m}{2} \\ & \phantom{\text{if }} 1 \le j \le \frac{m}{2}. \end{cases}$$

For $1 \le i \le \frac{m}{2} - 1$ we see that

$$\rho^{i+j} \text{ is an } m^{\text{th}} \text{ root of } 1, \text{ and } \rho^{i+j} \ne 1$$
$$\rho^{i-j} \text{ is an } m^{\text{th}} \text{ root of } 1, \text{ and}$$
$$\rho^{i-j} = 1 \Leftrightarrow i = j .$$

Thus by Lemma 2, (116), we have for $i \ne \frac{m}{2}$,

$$\beta_{ij} = \begin{cases} 2(-1)^i & \text{if } i + j \text{ is odd,} \\ -4 + 2(-1)^i & \text{if } i + j \text{ even, } i \ne j, \\ m - 4 + 2(-1)^i & \text{if } i + j \text{ even, } i = j . \end{cases}$$

On the other hand, if $i = \frac{m}{2}$, then

$$\beta_{\frac{m}{2},j} = \begin{cases} 1 & \text{if } j \text{ is odd,} \\ -1 & \text{if } j \text{ is even.} \end{cases}$$

We will now consider two cases:

Case 1: $\frac{m}{2}$ odd. Then

(137)

$$(\beta_{ij}) = \begin{bmatrix} (m-6 & -2 & -6 & -2 & & -6 & \cdots & -2 & -6 \\ 2 & (m-2) & 2 & -2 & & 2 & \cdots & -2 & 2 \\ -6 & -2 & (m-6) & -2 & & -6 & \cdots & -2 & -6 \\ 2 & -2 & 2 & (m-2) & & 2 & \cdots & -2 & 2 \\ -6 & -2 & -6 & -2 & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & -2 & -6 \\ 2 & -2 & 2 & -2 & \cdots & 2 & & (m-2) & 2 \\ 1 & -1 & 1 & -1 & \cdots & 1 & & -1 & 1 \end{bmatrix}$$

This is an $\frac{m}{2} \times \frac{m}{2}$ matrix.

<u>Note</u>: m is fixed throughout as above.

For k odd, $k \geq 3$, let:

(138) $(\beta_{ij})_k$ = the $k \times k$ matrix having the form given in (137).

The last row of $(\beta_{ij})_k$ is the k-tuple $(1,-1,1,-1,\ldots,1,-1,1)$, and

$$(\beta_{ij})_{\frac{m}{2}} = (\beta_{ij}) .$$

Let

(139) $$D_k = \det(\beta_{ij})_k \quad .$$

Subtracting the last column from the first in (137), we get for

$k \geq 5$:

(140)

$$D_k = m \cdot \begin{vmatrix} (m-2) & 2 & -2 & \cdots & 2 & -2 & 2 \\ -2 & (m-6) & -2 & \cdots & -6 & -2 & -6 \\ -2 & 2 & (m-2) & \cdots & 2 & -2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ -2 & -6 & -2 & \cdots & (m-6) & -2 & -6 \\ -2 & 2 & -2 & \cdots & 2 & (m-2) & 2 \\ -1 & 1 & -1 & \cdots & 1 & -1 & 1 \end{vmatrix}$$

Now subtracting twice the last <u>row</u> from the first we get for $k \geq 5$:

(141) $$D_k = m^2 \cdot D_{k-2} \quad .$$

Using (141) recursively we get:

(142) $$D_k = m^{k-3} \cdot D_3 \quad .$$

Using $k = \frac{m}{2}$ in (142) we have:

$$D_{\frac{m}{2}} = m^{\frac{m}{2}-3} \cdot m^2 = m^{\frac{m}{2}-1} \quad .$$

This shows Proposition (8) for the case $\frac{m}{2}$ odd.

<u>Case 2</u>: $\frac{m}{2}$ even. In this case we have:

(143)

$$(\beta_{ij}) = \begin{bmatrix} (m-6) & -2 & -6 & \cdots & -2 & -6 & -2 \\ 2 & (m-2) & 2 & \cdots & -2 & 2 & -2 \\ -6 & -2 & (m-6) & \cdots & -2 & -6 & -2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 2 & -2 & 2 & \cdots & (m-2) & 2 & -2 \\ -6 & -2 & -6 & \cdots & -2 & (m-6) & -2 \\ 1 & -1 & 1 & \cdots & -1 & 1 & -1 \end{bmatrix}$$

This is an $\frac{m}{2} \times \frac{m}{2}$ matrix.

For k even, $k \geq 2$, let:

(144) $(\beta_{ij})_k = k \times k$ matrix having the form of (143).

Thus $(\beta_{ij})_{\frac{m}{2}} = (\beta_{ij})$. Let

(145) $E_k = \det(\beta_{ij})_k$.

Subtracting the last column from the second, fourth, sixth,...,$(k-2)^{th}$ columns, and expanding, we get:

(146)
$$E_k = m^{\frac{1}{2}(k-2)} \cdot \begin{vmatrix} (m-6) & -6 & -6 & \cdots & -6 & -2 \\ -6 & (m-6) & -6 & \cdots & -6 & -2 \\ -6 & -6 & (m-6) & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & -6 & -2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -6 & -6 & -6 & \cdots & (m-6) & -2 \\ +1 & +1 & +1 & \cdots & +1 & +1 \end{vmatrix}$$

$(\frac{k}{2} + 1) \times (\frac{k}{2} + 1)$ determinant.

Subtracting 3 times the last column from all other columns we get:

(147) $E_k = m^{\frac{k}{2} - 1} \, G_{\frac{k}{2} + 1}$,

where for $\mu \geq 2$:

(148)
$$G_\mu = \begin{vmatrix} m & 0 & 0 & \cdots & 0 & 0 & -2 \\ 0 & m & 0 & \cdots & 0 & 0 & -2 \\ 0 & 0 & m & \cdots & 0 & 0 & -2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m & 0 & -2 \\ 0 & 0 & 0 & \cdots & 0 & m & -2 \\ 4 & 4 & 4 & \cdots & 4 & 4 & -1 \end{vmatrix} \quad (\mu \times \mu)$$

Expanding by the first column, we get for $\mu \geq 3$:

(149) $$G_{\mu} = mG_{\mu-1} + 4(-1)^{\mu+1} \cdot \{(-2)(-1)^{\mu}\, m^{\mu-2}\} = mG_{\mu-1} - 8(-1)^{2\mu+1} \cdot m^{\mu-2} .$$

Thus (147) becomes for $k \geq 4$: $E_k = m^{\frac{k}{2}-1}\{mG_{\frac{k}{2}} - 8(-1)^{k+3} \cdot m^{\frac{k}{2}-1}\}$.

That is,

(150) $$E_k = m^{\frac{k}{2}}\, G_{\frac{k}{2}} + 8m^{k-2} \quad \text{since } k \text{ is even,}$$

But by (147)

$$E_{k-2} = m^{\frac{k}{2}-2}\, G_{\frac{k}{2}} .$$

Thus (150) can be written as:

(151) $$E_k = m^2\, E_{k-2} + 8m^{k-2} .$$

Using this formula recursively we get: $E_k = m^{k-2}\, E_2 + 8(\frac{k-2}{2})m^{k-2}$.

That is,

(152) $$E_k = m^{k-2}\, \{E_2 + 4(k-2)\} .$$

Using $k = \frac{m}{2}$ we get, since $E_2 = 8 - m$:

(153) $$E_{\frac{m}{2}} = m^{\frac{m}{2}-1} .$$

This completes the proof of case (2) and thus of Proposition 8.

Q.E.D. Proposition 8.

This also completes the proof of Theorem 2.

APPENDIX TO PART I

In Part I, formula (21), we claimed the following (actually just for $N \geq 3$, there):

Proposition: For a_1, a_2, N integers, $N \geq 1$, such that $(a_1, a_2, N) = 1$, we define:

$$v_N(a_1,a_2) = \frac{1}{N^2} \sum_{\substack{t \bmod N \\ (t,N)=1}} c_t\, \alpha_0(ta_1, ta_2, N) \tag{1}$$

where

$$\alpha_0(a_1,a_2,N) = N^2 \chi\left(\frac{a_1}{N}\right) \sum_{n \equiv a_2 (N)} \frac{1}{n^2} ,$$

and

$$c_t = \sum_{\substack{k>0 \\ tk \equiv 1 (N)}} \frac{\mu(k)}{k^2} .$$

Then,

$$v_N(a_1,a_2) = \begin{cases} 1 & \text{if } \{a_1,a_2\} \equiv \pm \{0,1\} (N) , \; N \geq 3 , \\ 2 & \text{if } \{a_1,a_2\} \equiv \pm \{0,1\} (N) , \; N = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Proof: (Note: this proof, without many details, is given in [6], p. 466-7).

For $(t,N) = 1$, we have

$$\chi\left(\frac{ta_1}{N}\right) = \chi\left(\frac{a_1}{N}\right) .$$

Thus,

$$\alpha_0(ta_1,ta_2,N) = N^2 \chi\left(\frac{a_1}{N}\right) \sum_{n \equiv ta_2 (N)} \frac{1}{n^2} . \tag{3}$$

Thus we have

$$v_N(a_1,a_2) = \sum_{\substack{t \bmod N \\ (t,N)=1}} \left\{ \sum_{\substack{k>0 \\ tk \equiv 1 (N)}} \frac{\mu(k)}{k^2} \right\} \left\{ \chi\left(\frac{a_1}{N}\right) \sum_{n \equiv ta_2 (N)} \frac{1}{n^2} \right\} . \tag{4}$$

If we let $\bar{t}$ denote an inverse of $t \bmod N$, and agreeing that all congruences will be mod N, we can write (4) as:

$$(5) \qquad v_N(a_1,a_2) = \chi\left(\frac{a_1}{N}\right) \sum_{\substack{t \bmod N\\(t,N)=1}} \sum_{\substack{k\equiv\bar{t}\\k>0}} \sum_{n\equiv ta_2} \frac{\mu(k)}{(kn)^2} .$$

Thus

$$v_N(a_1,a_2) = \chi\left(\frac{a_1}{N}\right) \sum_{\substack{k=1\\(k,N)=1}}^{\infty} \sum_{n\equiv\bar{k}a_2} \frac{\mu(k)}{(kn)^2} = \chi\left(\frac{a_1}{N}\right) \sum_{\substack{k=1\\(k,N)=1}}^{\infty} \sum_{kn\equiv a_2} \frac{\mu(k)}{(kn)^2}$$

$$= \chi\left(\frac{a_1}{N}\right) \sum_{\substack{k=1\\(k,N)=1}}^{\infty} \sum_{\substack{j\equiv a_2\\k|j}} \frac{\mu(k)}{(j)^2} ,$$

and thus,

$$(6) \quad v_N(a_1,a_2) = \chi\left(\frac{a_1}{N}\right) \sum_{j\equiv a_2} \frac{1}{j^2} \sum_{\substack{k|j\\k>0\\(k,N)=1}} \mu(k)$$

where we have interchanged the order of summation by absolute convergence of the series.

If $N \nmid a_1$, then $\chi\left(\frac{a_1}{N}\right) = 0$ and so $v_N(a_1,a_2) = 0$.

If $N|a_1$, then since $(a_1,a_2,N) = 1$, we must have $(a_2,N) = 1$, and thus the condition $(k,N) = 1$ in (6) is redundant. Therefore (6) can be written:

$$(7) \qquad v_N(a_1,a_2) = \chi\left(\frac{a_1}{N}\right) \sum_{j\equiv a_2} \frac{1}{j^2} \sum_{\substack{k|j\\k>0}} \mu(k) .$$

But $\sum_{\substack{k|j\\k>0}} \mu(k) = \begin{cases} 1 & \text{if } j = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$

Thus if $a_2 \equiv 1 (\text{mod } N)$ then we get a factor of 1 from the $j = 1$ term, all other terms in the double sum giving zero. Similarly, if $a_2 \equiv -1(N)$, the $j = -1$ term gives the only contribution to the double sum, If $N = 1$ or 2 then $a_2 \equiv 1(N) \Leftrightarrow a_2 \equiv -1(N)$, and so we get contributions from both the $j = 1$ and $j = -1$ terms. Thus we have:

$$(8) \qquad v_N(a_1,a_2) = \begin{cases} \chi\left(\frac{a_1}{N}\right) & \text{if } a_2 \equiv \pm 1(N),\ N > 2, \\ 2\chi\left(\frac{a_1}{N}\right) & \text{if } a_2 \equiv \pm 1(N),\ N = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

But

$$\chi(t) = \begin{cases} 1 & \text{if } t \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$v_N(a_1,a_2) = \begin{cases} 1 & \text{if } \{a_1,a_2\} \equiv \pm \{0,1\}(N),\ N > 2, \\ 2 & \text{if } \{a_1,a_2\} \equiv \pm \{0,1\}(N),\ N = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

<u>Q.E.D.</u>

PART II

§0. Introduction

In Part I we discussed the functions $G(z,s,r_1,r_2)$ of Lewittes [10] and $G_2(z;a_1,a_2,N)$ of Hecke [6]. As was mentioned in Part I, $G(z,2,a_1/N,a_2/N) = 2\pi i/(z-\bar{z})+N^2G_2(z;a_1,a_2,N)$. In [6] Hecke defined the functions

$$(1)\quad \Phi(z;b_1,b_2,N) = \sum_{\substack{a_1,a_2\\ \text{mod } N}} \zeta_N^{a_1b_2-a_2b_1}\{G_2(z;a_1,a_2,N) - G_2(z;0,0,N)\}$$

where

$$\zeta_N = \exp(2\pi i/N) .$$

Since $G_2(z;a_1,a_2,N) - G_2(z;0,0,N)$ is an abelian differential for $\Gamma(N)$, so is $\Phi(z;b_1,b_2,N)$. Furthermore it is shown in [11] that

$$(2)\quad \Phi(V(z);b_1,b_2,N)\,\frac{dV}{dz} = \Phi(z;B_1,B_2,N) \quad \text{for} \quad V \in \Gamma , \quad \text{where}$$

$\{B_1,B_2\} = \{b_1,b_2\}\underline{V}$ as in Part I. Since for $\{b_1,b_2\} \equiv \{0,0\}(N)$ we have $\Phi(z,0,0,N) = N^2G_2(z,0,0,N) - \sum_{\substack{a_1,a_2\\ \text{mod } N}} G_2(z,a_1a_2,N) = 0$, we will assume unless otherwise stated that $\{b_1,b_2\} \not\equiv \{0,0\}(N)$. The Fourier series of $\Phi(z;b_1,b_2,N)$ will show that in local coordinates on S_N, the differential $\Phi(z;b_1,b_2,N)dz$ has simple poles at the cusps of S_N.

In §1 we note that:

$$(3)\quad \Phi(z;b_1,b_2,N) = 2\pi^2\, P_2\left(\frac{b_1}{N}\right) + 2\pi i H'\left(z,0,\frac{b_1}{N},\frac{b_2}{N}\right) ,$$

where $H(z,s,r_1,r_2)$ is defined in [10], the prime in (3) denoting $\frac{d}{dz}$. Since $P_2(r_1)$ and $H'(z,0,r_1,r_2)$ are defined for r_1,r_2 any real numbers, we define:

(4) $\varphi(z,r_1,r_2) = 2\pi^2 P_2(r_1) + 2\pi i H'(z,0,r_1,r_2)$, $\{r_1,r_2\} \notin Z^2$,

so that $\varphi(z,b_1/N,b_2/N) = \Phi(z;b_1,b_2,N)$, and (2) becomes by a continuity argument:

(5) $\varphi(V(z),r_1,r_2)\frac{dV}{dz} = \varphi(z,\{r_1,r_2\}\cdot\underline{V})$ for $V \in \Gamma$, r_i real.

<u>Note</u>: since $\Phi(z;b_1,b_2,N) = \Phi(z;-b_1,-b_2),N)$, and $\varphi(z,r_1,r_2) = \varphi(z,-r_1,-r_2)$ we can use $-\underline{V}$ rather than $\underline{V}$ in formulas (2) and (5). For ease of notation then, let V denote a map in Γ as well as either of its matrix representative in $SL(2,Z)$.

If we define:

(6) $\psi(z,r_1,r_2) = \pi i z P_2(r_1) - H(z,0,r_1,r_2)$, that is:

$\psi(z,r_1,r_2) = (i/2\pi)\int \varphi(z,r_1,r_2)dz$, then,

(7) $\psi(z,\frac{b_1}{N},\frac{b_2}{N}) = \frac{i}{2\pi}\Psi(z;b_1,b_2,N)$ where

$\Psi(z;b_1,b_2,N) = \int \Phi(z;b_1,b_2,N)dz$ as in [6]. By formula (5) we get that the following numbers are independent of z :

(8) $Q(V,r_1,r_2) = \psi(V(z),r_1,r_2) - \psi(z,R_1,R_2)$ for $V \in \Gamma$,

$\{R_1,R_2\} = \{r_1,r_2\}\cdot V$. For $r_i = a_i/N$, the numbers $Q(V,a_1/N,a_2/N)$ were calculated by Schoeneberg [11] using methods of Riemann, Dedekind, and Hecke, involving asymptotic expansions of theta functions. [Note: In [11], Schoeneberg uses $\Phi(z,0,0,N) = N^2 G_2(z,0,0,N)$, while here we use $\Phi(z;0,0,N) = 0$, thus our formulas differ for the case $\{r_1,r_2\} \in Z^2$].

In §2, using formula (6) and the transformation formula for the function H , we get a new derivation of the formulas for the numbers $Q(V,r_1,r_2)$, r_1,r_2 real, which agree with those of [11] for r_1,r_2 rational, $\{r_1,r_2\} \notin Z^2$.

§1. The Abelian Differential $\Phi(z;b_1,b_2,N)$ and its Generalization $\varphi(z,r_1,r_2)$.

Consider the functions defined by formula (1). Since for $\{b_1,b_2\} \not\equiv \{0,0\}(N)$ we have:

$$\sum_{\substack{a_1,a_2\\(N)}} \zeta_N^{a_1b_2-a_2b_1} = 0$$

we can write, for $\{b_1,b_2\} \not\equiv \{0,0\}(N)$:

$$(9) \qquad \Phi(z;b_1,b_2,N) = \sum_{\substack{a_1,a_2\\(N)}} \zeta_N^{a_1b_2-a_2b_1} G_2(z;a_1,a_2,N) .$$

While for $\{b_1,b_2\} \equiv \{0,0\}(N)$ we have:

$$(10) \qquad \Phi(z;b_1,b_2,N) = 0 \quad \forall \quad z,$$

since $N^2G_2(z;0,0,N) = \sum_{\substack{a_1,a_2\\(N)}} G_2(z,a_1,a_2,N)$ as can be seen from the definition of G_2. Since the non-analytic term of $G_2(z;a_1,a_2,N)$, namely, $2\pi i/N^2(z-\bar{z})$, is independent of $\{a_1,a_2\}$ we see that $\Phi(z,b_1,b_2,N)$ is an analytic function of $z \in \mathcal{H}$.

As is pointed out in [11], the functions Φ satisfy the relation:

$$(11) \qquad \Phi(V(z),b_1,b_2,N)\frac{dV}{dz} = \Phi(z;B_1,B_2,N) \quad \text{for} \quad V \in \Gamma , \{B_1,B_2\} = \{b_1,b_2\} \cdot V .$$

This follows since the functions G_2 satisfy (11), and since $A_1B_2 - A_2B_1 = a_1b_2 - a_2b_1$, and $\{A_1,A_2\}$ runs once through all pairs mod N as $\{a_1,a_2\}$ does. In particular, of course, for $V \in \Gamma(N)$ we have $\{B_1,B_2\} \equiv \pm \{b_1,b_2\}(N)$ and thus $\Phi(z,b_1,b_2,N)$ transforms like a differential for $\Gamma(N)$.

<u>Note</u>: Since $\Phi(z,0,0,N) = 0$, we will assume from now on that $\{b_1,b_2\} \neq \{0,0\}(N)$, and $\{r_1,r_2\} \not\subset Z^2$.

Now, using the Fourier expansions of $G_2(z;a_1,a_2,N)$ from [6], or of $G(z,2,a_1/N,a_2/N)$ from Part I, formula (5), we get as in [6]:

$$(12) \qquad \Phi(z;b_1,b_2,N) = 2\pi^2 P_2\left(\frac{b_1}{N}\right) - \frac{4\pi^2}{N} \sum_{\substack{m\cdot m_1>0 \\ m\equiv b_1(N)}} |m|\, \zeta_N^{m_1 b_2} \exp\left(\frac{2\pi i m m_1 z}{N}\right)$$

where $P_2(x) = \langle x\rangle^2 - \langle x\rangle + \frac{1}{6}$, $\langle x\rangle = x - [x]$ = fractional part of x . Note that the roots of $P_2(x)$ are irrational so that $\Phi(z;b_1,b_2,N)$ does not vanish at ∞ , and thus in local coordinates on S_N , the differential $\Phi(z;b_1,b_2,N)dz$ has a simple pole at ∞ , as well as the other cusps by formula (11).

If we now look at [10], formula (17), namely:

$$(13) \qquad A(z,s,r_1,r_2) = \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} \exp(2\pi i r_2) \exp\{2\pi i k(m+r_1)z\}$$

we see that:

$$(14) \qquad \frac{2\pi i}{N} \cdot \sum_{\substack{m\cdot m_1>0 \\ m\equiv b_1(N)}} |m|\, \zeta_N^{m_1 b_2} \exp\left(\frac{2\pi i m m_1 z}{N}\right) = A'\left(z,0,\frac{b_1}{N},\frac{b_2}{N}\right) + A'\left(z,0,\frac{-b_1}{N},\frac{-b_2}{N}\right)$$

where the prime on A denotes differentiation with respect to z .

Using, as in [10],

$$(15) \qquad H(z,s,r_1,r_2) = A(z,s,r_1,r_2) + \exp(\pi i s) A(z,s,-r_1,-r_2) ,$$

we get from (12) and (14):

$$(16) \qquad \Phi(z,b_1,b_2,N) = 2\pi^2 P_2\left(\frac{b_1}{N}\right) + 2\pi i H'\left(z,0,\frac{b_1}{N},\frac{b_2}{N}\right) .$$

The right hand side of (16) is still defined if we replace b_1/N, b_2/N by r_1, r_2 respectively, where r_i are any real numbers. Thus if we define:

$$(17) \qquad \varphi(z,r_1,r_2) = \begin{cases} 2\pi^2 P_2(r_1) + 2\pi i H'(z,0,r_1,r_2) & \text{if } \{r_1,r_2\} \notin Z^2, \\ 0 \quad \text{if } \{r_1,r_2\} \in Z^2 . \end{cases}$$

We have

$$(18) \qquad \varphi\left(z, \frac{b_1}{N}, \frac{b_2}{N}\right) = \Phi(z;b_1,b_2,N) .$$

$P_2(r_1)$ is continuous for $r_1 \notin Z$, and $H'(z,0,r_1,r_2)$ is continuous as a function of the two variables $\{r_1,r_2\}$ for $r_1 \notin Z$, as can be seen in the definitions (12), (13) and (15). Thus formula (11) gives by a continuity argument:

$$(19) \qquad \varphi(V(z),r_1,r_2) \frac{dV}{dz} = \varphi(z,R_1,R_2) , \ r_i \text{ real}, \ V \in \Gamma ,$$

$\{R_1,R_2\} = \{r_1,r_2\}\cdot V$.

Note that by continuity we have also:

$$(20) \qquad \varphi(z,-r_1,-r_2) = \varphi(z,r_1,r_2) , \ r_1, r_2 \text{ real.}$$

In [6] and [11], the Fourier series (12) is integrated term by term and there is obtained (after multiplying by $i/2\pi$ as in [11]) ;

$$(21) \quad \frac{i}{2\pi}\cdot\Psi(z;b_1,b_2,N) = \pi i z P_2\left(\frac{b_1}{N}\right) - \sum_{\substack{m\cdot m_1>0 \\ m\equiv b_1 (N)}} \frac{1}{|m_1|} \zeta_N^{m_1 a_2} \exp\left(\frac{2\pi i z m m_1}{N}\right) ,$$

In the notation of [10] this would be:

$$(22) \quad \frac{i}{2\pi}\cdot\Psi(z;b_1,b_2,N) = \pi i z P_2\left(\frac{b_1}{N}\right) - H\left(z,0, \frac{b_1}{N} , \frac{b_2}{N}\right) .$$

In general we take for r_1, r_2 real:

$$(23)\qquad \psi(z,r_1,r_2) = \pi i z P_2(r_1) - H(z,0,r_1,r_2)\ ,$$

so that,

$$(24)\qquad \psi'(z,r_1,r_2) = \frac{i}{2\pi}\,\varphi(z,r_1,r_2)\ .$$

Consider the numbers:

$$(25)\qquad Q(V,r_1,r_2) = \psi(V(z),r_1,r_2) - \psi(z,R_1,R_2)$$

where $V \in \Gamma$, $\{R_1,R_2\} = \{r_1,r_2\} \cdot V$. By differentiating the right hand side of (25) with respect to z and using (19), we see that $Q(V,r_1,r_2)$ is independent of z .

If $V \in \Gamma(N)$, and $r_i = a_i/N$, then $R_i \equiv a_i/N \pmod{Z}$, and the numbers $Q(V,a_1/N,a_2/N)$ have a geometric significance. Namely, for $V \in \Gamma(N)$ we have:

$$(26)\qquad Q(V, \frac{a_1}{N}, \frac{a_2}{N}) = \psi(V(z), \frac{a_1}{N}, \frac{a_2}{N}) - \psi(z, \frac{a_1}{N}, \frac{a_2}{N})\ .$$

But since $\psi'(z,r_1,r_2) = \frac{i}{2\pi}\,\varphi(z,r_1,r_2)$ we know that

$$(27)\qquad \psi(z,r_1,r_2) = \frac{i}{2\pi}\int_{z_0}^{z} \varphi(\tau,r_1,r_2)\,d\tau + \psi(z_0,r_1,r_2)$$

where $z_0 \in \mathcal{H}$ is arbitrary, as is the path of integration in $\mathcal{H}$. Then,

$$\frac{2\pi}{i}\,Q(V, \frac{a_1}{N}, \frac{a_2}{N}) = \int_{z_0}^{V(z)} \varphi(\tau, \frac{a_1}{N}, \frac{a_2}{N})\,d\tau - \int_{z_0}^{z} \varphi(\tau, \frac{a_1}{N}, \frac{a_2}{N})\,d\tau$$

and finally,

$$(28)\qquad \frac{2\pi}{i}\,Q(V, \frac{a_1}{N}, \frac{a_2}{N}) = \int_{z}^{V(z)} \varphi(\tau, \frac{a_1}{N}, \frac{a_2}{N})\,d\tau\ .$$

This is the integral around a closed loop on $\mathcal{H}/\Gamma(N)$ at z . Note that the integral depends on $V, a_1/N$, a_2/N , but not on z . $Q(V,a_1/N,a_2/N)$ is a period of the abelian integral $\psi(z,a_1/N,a_2/N)$ on $\mathcal{H}/\Gamma(N)$.

§2. A New Calculation of the Numbers $Q(V,a_1/N,a_2/N)$ and their Generalization $Q(V,r_1,r_2)$.

Recall, we assume $\{r_1,r_2\} \notin Z^2$. By (25) and (23) we have for $V \in \Gamma$:

$$(29) \qquad Q(V,r_1,r_2) = \pi i V(z) P_2(r_1) - H(V(z),0,r_1,r_2) - \{\pi i z P_2(R_1) - H(z,0,R_1,R_2)\} \ .$$

We now use the transformation formula for $H(z,s,r_1,r_2)$ given in [10], (51), or actually the simplified version for $H(z,0,r_1,r_2)$ given in [1], (22). Namely:

Case (1): $c = 0$.

Then $V(z) = z + b$, $\underset{-}{V} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and

$$(30) \qquad H(z+b,0,r_1,r_2) - H(z,0,r_1,br_1+r_2) = 0 \ .$$

Case (2): $c > 0$.

Then $V(z) = \frac{az+b}{cz+d}$, $\underset{-}{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$$(31) \qquad H(V(z),0,r_1,r_2) - H(z,0,R_1,R_2) =$$

$$= 2\pi i \left\{ \frac{-P_2(\rho)}{2c(cz+d)} - \frac{(cz+d)P_2(R_1)}{2c} + \sum_{j \bmod c} \left(\left(d\frac{j-R_1}{c} + R_2 \right)\right) \left(\left(\frac{j-R_1}{c} \right)\right) \right\}$$

$$+ \chi(r_1) \mathrm{Log}\{2\sin(\pi \langle r_2 \rangle)\} - \chi(R_1) \mathrm{Log}\{2\sin(\pi \langle R_2 \rangle)\} \ ,$$

where $P_2(x) = \langle x \rangle^2 - \langle x \rangle + \frac{1}{6}$, $\langle x \rangle = x - [x]$, $[x]$ = greatest integer $\leq x$, $\rho = \langle R_2 \rangle c - \langle R_1 \rangle d$, and

$$((x)) = \begin{cases} \langle x\rangle - \frac{1}{2} & \text{if } x \notin Z , \\ 0 & \text{if } x \in Z . \end{cases}$$

Log denotes the branch using $-\pi \le \arg < \pi$. Thus (29) gives:

<u>Case (1)</u>: For $V(z) = z + b$, and $\underline{V} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ we have:

$$Q(V,r_1,r_2) = \pi i b P_2(r_1) . \tag{32}$$

<u>Case (2)</u>: For $V(z) = \frac{az+b}{cz+d}$, $c > 0$ we have

$$\begin{aligned} Q(V,r_1,r_2) &= \pi i\{V(z)P_2(R_1) - zP_2(r_1)\} \\ &-2\pi i\left\{ \frac{P_2(\rho)}{2c(cz+d)} - \frac{(cz+d)P_2(R_1)}{2c} + \sum_{j \bmod c} \left(\left(d\,\frac{j-R_1}{c} + R_2\right)\right)\left(\left(\frac{j-R_1}{c}\right)\right)\right\} \\ &- \chi(r_1)\mathrm{Log}\{2\sin(\pi\langle r_2\rangle)\} + \chi(R_1)\mathrm{Log}\{2\sin(\pi\langle R_2\rangle)\} . \end{aligned} \tag{33}$$

Thus for $c > 0$:

$$\begin{aligned} Q(V,r_1,r_2) &= \pi i\left\{\left(\frac{az+b}{cz+d}\right)P_2(r_1) + \frac{P_2(\langle R_2\rangle c-\langle R_1\rangle d)}{c(cz+d)}\right\} + \frac{\pi i d}{c}P_2(R_1) \\ &- 2\pi i \sum_{j \bmod c} \left(\left(\frac{d(j-R_1)}{c} + R_2\right)\right)\left(\left(\frac{j-R_1}{c}\right)\right) \\ &+ \chi(r_1)\mathrm{Log}\{2\sin(\pi\langle r_2\rangle)\} + \chi(R_1)\mathrm{Log}\{2\sin(\pi\langle R_2\rangle)\} . \end{aligned}$$

But since $Q(V,r_1,r_2)$ is independent of z, it must be that

$$\frac{c(az+b)P_2(r_1)+P_2\langle R_2\rangle c - \langle R_1\rangle d)}{c(cz+d)}$$

is independent of z . In fact, since $P_2(x+k) = P_2(x)$ for $k \in \mathbb{Z}$ we have:

$$P_2(\langle R_2\rangle c - \langle R_1\rangle d) = P_2(R_2 c - R_1 d) .$$

But $R_2c - R_1d = -r_1$, using $ad - bc = 1$. Thus $P_2(\langle R_2\rangle c - \langle R_1\rangle d) = P_2(-r_1) = P_2(r_1)$. Also by $ad - bc = 1$, we have

$$\frac{c(az+b) + 1}{c(cz+d)} = \frac{a}{c} .$$

Thus

$$\frac{c(az+b)P_2(r_1) + P_2(\langle R_2\rangle c - \langle R_1\rangle d)}{c(cz+d)} = \frac{a}{c} P_2(r_1)$$

and so we get for $c > 0$,

$$(34) \qquad Q(V,r_1,r_2) = \pi i\left\{\frac{a}{c} P_2(r_1) + \frac{d}{c} P_2(R_1)\right\} - 2\pi i \sum_{j \bmod c} \left(\left(\frac{d(j-R_1)}{c} + R_2\right)\right)\left(\left(\frac{j-R_1}{c}\right)\right) - \beta(r_1,r_2) + \beta(R_1,R_2)$$

where,

$$(35) \qquad \beta(r_1,r_2) \underset{\text{def}}{=} \chi(r_1)\mathrm{Log}\{2\sin(\pi\langle r_2\rangle)\} .$$

The annoying part about the formula is that we insist that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the representative $\underline{V}$, where $c > 0$. Suppose we write $V(z) = \frac{az+b}{cz+d}$ with $c < 0$, the case $c = 0$ being omitted temporarily. Then $\underline{V} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ and by our convention $\{R_1,R_2\} = \{r_1,r_2\}\cdot\underline{V} = \{r_1,r_2\}\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. Let $\{R_1',R_2'\} = \{r_1,r_2\}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{-R_1,-R_2\}$. Formula (34) gives:

$$Q(V,r_1,r_2) = \pi i\left\{\frac{-aP_2(r_1)}{-c} + \frac{-dP_2(-R_1')}{-c}\right\} - 2\pi i \sum_{j \bmod (-c)} \left(\left(\frac{-d(j+R_1')}{-c} - R_2'\right)\right)\left(\left(\frac{j+R_1'}{-c}\right)\right) - \beta(r_1,r_2) + \beta(-R_1', -R_2') .$$

It is easily verified in the definition (35) that

$$\beta(x,y) = \beta(-x,-y) .$$

Also we know: $P_2(x) = P_2(-x)$ as can be verified in its definition in (31).

Thus

$$Q(V,r_1,r_2) = \pi i\left\{\frac{a}{c}\, P_2(r_1) + \frac{d}{c}\, P_2(R_1')\right\}$$

$$- 2\pi i \sum_{j \bmod c} \left(\left(\frac{d(j+R_1')}{c} - R_2'\right)\right)\left(\left(\frac{j+R_1'}{-c}\right)\right)$$

$$- \beta(r_1,r_2) + \beta(R_1',R_2') .$$

Now in the sum, use $k = -j$ and the fact that $((-x)) = -((x))$ as can be verified in its definition in (31). We get,

$$\sum_{j \bmod c} \left(\left(\frac{d(j+R_1')}{c} - R_2'\right)\right)\left(\left(\frac{j+R_1'}{-c}\right)\right)$$

$$= -\sum_{k \bmod c} \left(\left(\frac{d(k-R_1')}{c} + R_2'\right)\right)\left(\left(\frac{k-R_1'}{c}\right)\right) ,$$

and thus

$$Q(V,r_1,r_2) = \pi i\left\{\frac{a}{c}\, P_2(r_1) + \frac{d}{c}\, P_2(R_1')\right\}$$

$$+ 2\pi i \sum_{j \bmod c} \left(\left(\frac{d(j-R_1')}{c} + R_2'\right)\right)\left(\left(\frac{j-R_1'}{c}\right)\right)$$

$$- \beta(r_1,r_2) + \beta(R_1',R_2')$$

where $c < 0$ and $\{R_1',R_2'\} = \{r_1,r_2\} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Comparing this with (34) we see that if $c \neq 0$ we always have the formula:

$$(36) \quad Q(V,r_1,r_2) = \pi i\left\{\frac{a}{c}\, P_2(r_1) + \frac{d}{c}\, P_2(R_1)\right\}$$

$$-2\pi i \operatorname{sgn} c \sum_{j \bmod c} \left(\left(\frac{d(j-R_1)}{c} + R_2\right)\right)\left(\left(\frac{j-R_1}{c}\right)\right)$$

$$- \beta(r_1,r_2,) + \beta(R_1,R_2)$$

where $\{R_1,R_2\} = \{r_1,r_2\}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $V(z) = \frac{az+b}{cz+d}$ and $\beta(x,y)$ is given in (35).

Thus we need not specify which representative matrix we are using to get $\{R_1, R_2\}$. Now, if $c = 0$, then we previously took $\underset{\sim}{V} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ to get formula (32) namely,

$$Q(V, r_1, r_2) = \pi i b P_2(r_1) .$$

Had we used $\{R_1', R_2'\} = \{r_1, r_2\}\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$ to get $Q(V, r_1, r_2) = \psi(V(z), r_1, r_2) - \psi(z, R_1', R_2')$ we would still have obtained $Q(V, r_1, r_2) = \pi i b P_2(r_1)$ where b would be the upper right hand entry in $\underset{\sim}{V} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

Thus, if we want to avoid having to say from which representative b comes, we can write:

$$(37) \qquad Q(V, r_1, r_2) = \pi i \, \frac{b}{d} \, P_2(R_1) \quad \text{if} \quad c = 0 .$$

We now wish to compare our formulas (36) and (37) with Schoeneberg's [11], (39). First of all, comparing our definitions with [11], we see that for $\{a_1, a_2\} \not\equiv \{0, 0\}(N)$ we have:

$$Q(V, a_1/N,\ a_2/N) + \beta(a_1/N,\ a_2/N) - \beta(a_1'/N,\ a_2'/N) = \psi_{a_1,a_2}(\tau') - \psi_{a_1',a_2'}(\tau)$$

where ψ_{a_1,a_2} is defined in [11] with $\{a_1', a_2'\} = \{a_1, a_2\} \cdot V$, $V(z) = \frac{az+b}{cz+d}$.

For $c \neq 0$, we see that Schoeneberg has:

$$\frac{1}{\pi i} \{\psi_{a_1,a_2}(\tau') - \psi_{a_1',a_2'}(\tau)\} = \frac{a}{c} P_2\left(\frac{a_1}{N}\right) + \frac{d}{c} P_2\left(\frac{a_1'}{N}\right)$$

$$- 2 \operatorname{sgn} c \sum_{j \bmod c} \left(\left(\frac{a_1}{N} + \frac{j}{c}\right)\right)\left(\left(\frac{a_1'}{cN} + \frac{aj}{c}\right)\right) .$$

Comparing this with (36) we see that where we have:

$$(38) \qquad -2\pi i \operatorname{sgn} c \sum_{j \bmod c} \left(\left(\frac{d(j - a_1'/N)}{c} + \frac{a_2'}{N}\right)\right)\left(\left(\frac{j - a_1'/N}{c}\right)\right) ;$$

Schoeneberg has:

(39) $$-2\pi i \operatorname{sgn} c \sum_{j \bmod c} \left(\left(\frac{a_1}{cN} + \frac{j}{c}\right)\right)\left(\left(\frac{a_1'}{cN} + \frac{aj}{c}\right)\right) .$$

The two sums, (38) and (39), can be shown equal with the aid of a lemma of independent interest.

Lemma:

(40) $$Q(V_2 \circ V_1, r_1, r_2) = Q(V_2, r_1, r_2) + Q(V_1, \{r_1, r_2\} \cdot V_2) .$$

Proof: By (25) we have:

$$Q(V_2 \circ V_1, r_1, r_2) = \psi(V_2 \circ V_1(z), r_1, r_2) - \psi(z, \{r_1, r_2\} \cdot V_2 \cdot V_1)$$

for any $z \in \mathcal{H}$. Thus,

$$\begin{aligned} Q(V_2 \circ V_1, r_1, r_2) &= \psi(V_2 \circ V_1(z), r_1, r_2) - \psi(V_1(z), \{r_1, r_2\} \cdot V_2) \\ &\quad + \psi(V_1(z), \{r_1, r_2\} \cdot V_2) - \psi(z, \{r_1, r_2\} \cdot V_2 \cdot V_1) \\ &= Q(V_2, r_1, r_2) + Q(V_1, \{r_1, r_2\} \cdot V_2) \quad . \end{aligned}$$

Q.E.D. Lemma.

Recall, we intend to show that (38) and (39) are equal. If $I(z) = z$ for $z \in \mathcal{H}$, then:

$$\begin{aligned} 0 = Q(I, r_1, r_2) &= Q(V \circ V^{-1}, r_1, r_2) \\ &= Q(V, r_1, r_2) + Q(V^{-1}, \{r_1, r_2\} \cdot V) \quad \text{by (40).} \end{aligned}$$

Thus,

(41) $$Q(V, r_1, r_2) = -Q(V^{-1}, R_1, R_2) \quad \text{where} \quad \{R_1, R_2\} = \{r_1, r_2\} \cdot V \quad .$$

But for $c > 0$:

$$\begin{aligned} Q(V, r_1, r_2) = {} & \pi i \{\tfrac{a}{c} P_2(r_1) + \tfrac{d}{c} P_2(R_1)\} \\ & -2\pi i \sum_{j \bmod c} \left(\left(\frac{d(j-R_1)}{c} + R_2\right)\right)\left(\left(\frac{j-R_1}{c}\right)\right) \\ & -\beta(r_1, r_2) + \beta(R_1, R_2) \ ; \end{aligned}$$

$$Q(V^{-1},R_1,R_2) = \pi i\{\tfrac{-d}{c}\,P_2(R_1) + \tfrac{-a}{c}\,P_2(R_1'')\}$$

$$-2\pi i \sum_{j \bmod c} \left(\left(\frac{-a(j-R_1'')}{c} + R_2''\right)\right)\left(\left(\frac{j-R_1''}{c}\right)\right)$$

$$-\beta(R_1,R_2) + \beta(R_1'',R_2'')$$

where $\{R_1'',R_2''\} = \{R_1,R_2\}\begin{pmatrix}-d & b\\ c & -a\end{pmatrix} = \{r_1,r_2\}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}-d & b\\ c & -a\end{pmatrix} = \{r_1,r_2\}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} =$ $\{-r_1,-r_2\}$. Thus,

$$Q(V^{-1},R_1,R_2) = -\pi i\{\tfrac{a}{c}\,P_2(-r_1) + \tfrac{d}{c}\,P_2(R_1)\}$$

$$-2\pi i \sum_{j \bmod c} \left(\left(\frac{-a(j+r_1)}{c} - r_2\right)\right)\left(\left(\frac{j+r_1}{c}\right)\right)$$

$$-\beta(R_1,R_2) + \beta(-r_1,-r_2) \;.$$

But $P_2(x) = P_2(-x)$ and $\beta(x,y) = \beta(-x,-y)$. Taking the negative of both sides we get

$$-Q(V^{-1},R_1,R_2) = \pi i\{\tfrac{a}{c}\,P_2(r_1) + \tfrac{d}{c}\,P_2(R_1)\}$$

$$+2\pi i \sum_{j \bmod c} \left(\left(\frac{-a(j+r_1)}{c} - r_2\right)\right)\left(\left(\frac{j+r_1}{c}\right)\right)$$

$$+\beta(R_1,R_2) - \beta(r_1,r_2) \;.$$

Comparing this with $Q(V,r_1,r_2)$ we see that

$$+\sum_{j \bmod c} \left(\left(\frac{d(j-R_1)}{c} + R_2\right)\right)\left(\left(\frac{j-R_1}{c}\right)\right) = -\sum_{j \bmod c} \left(\left(\frac{-a(j+r_1)}{c} - r_2\right)\right)\left(\left(\frac{j+r_1}{c}\right)\right)$$

and using $((-x)) = -((x))$ we get:

$$\sum_{j \bmod c} \left(\left(\frac{d(j-R_1)}{c} + R_2\right)\right)\left(\left(\frac{j-R_1}{c}\right)\right) = \sum_{j \bmod c} \left(\left(\frac{a(j+r_1)}{c} + r_2\right)\right)\left(\left(\frac{j+r_1}{c}\right)\right) \;. \tag{42}$$

To get the right hand side of (42) to look like the sum in (39) we first use the fact that $R_1 = ar_1 + cr_2$ to get

$$\sum_{j \bmod c} \left(\left(\frac{aj+R_1}{c}\right)\right)\left(\left(\frac{j+r_1}{c}\right)\right) .$$

Now plugging in $r_1 = \frac{a_1}{N}$, $r_2 = \frac{a_2}{N}$, $R_1 = ar_1 + cr_2 = \frac{a_1'}{N}$, $R_2 = br_1 + cr_2 = \frac{a_2'}{N}$, into both sides of (42) we see that (38) and (39) are equal.

A similar argument works with $c < 0$.

PART III

§ 0. Introduction: The Classical Poincaré Series.

We will use C,R, and Z to stand for the complex numbers, real numbers, and integers respectively. Furthermore, we will use the following notations:

(1) $s = \sigma + it \;, \quad \sigma \in R \;, \; t \in R$

(2) $z = x + iy \;, \quad x \in R \;, \; y \in R$

(3) $\mathcal{H} = \{z \,|\, y > 0\}$

(4) $\mathcal{K}_a = \{s \,|\, \sigma > a\}$.

If $w \in C$, $w \neq 0$ we make the definitions:

(5) Arg(w) is that value of the argument such that $-\pi \leq \text{Arg}(w) < \pi$,

(6) $\text{Log}(w) = \ln|w| + i\,\text{Arg}(w)$,

and,

(7) for $s \in C$, $w^s = e^{s\,\text{Log}(w)} = \exp(s\,\text{Log}(w))$.

(8) Γ will denote the <u>inhomogeneous modular group</u>, namely the group of transformations $z \rightarrow \frac{az+b}{cz+d}$, $a,b,c,d \in Z$, $ad - bc = 1$.

(9) SL(2,Z) will denote the <u>homogeneous modular group</u>, namely the group of integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1 .

<u>Note</u>: $\Gamma \cong SL(2,Z)/(\pm I)$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

m and n will always denote integers, (m,n) will denote the greatest common divisor of m and n , and $\{x,y\}$ will denote an ordered pair.

For $(m,n) = 1$, we define:

(10) $V_{m,n}(z) = \frac{Mz+N}{mz+n}$, where $\begin{pmatrix} M & N \\ m & n \end{pmatrix} \in SL(2,Z)$.

$V_{m,n}(z)$ is well defined only up to an additive integer constant, as follows: if $\begin{pmatrix} M' & N' \\ m & n \end{pmatrix} \in SL(2,Z)$, then

$$\frac{Mz+N}{mz+n} = \frac{M'z+N'}{mz+n} + k \ , \quad k \in Z \ .$$

This ambiguity in the definition of $V_{m,n}$ does not concern us however, since we only use $V_{m,n}(z)$ in the expression $\exp(2\pi i g V_{m,n}(z))$ where $g \in Z$.

If $(m,n) = d$, then we define

$$(11) \qquad V_{m,n}(z) \underset{\text{def}}{=} V_{\frac{m}{d},\frac{n}{d}}(z) \ .$$

We then define for $m,n,g \in Z$, and $z,s \in C$:

$$(12) \qquad f_{m,n}(z,s,g) = (mz+n)^{-s} \exp(2\pi i g V_{m,n}(z)) \ .$$

Furthermore, when s and g are fixed, we suppress the dependence on s and g and write:

$$(13) \qquad f(m,n,z) = f_{m,n}(z,s,g) \ .$$

<u>Note</u>: When we choose the map $V_{m,n}$ as defined in (10), we are choosing a coset representative for Γ/Γ_0 where Γ_0 = group of translations in Γ.

Let $\mathfrak{R}$ be a set of coset representatives for Γ/Γ_0 , namely:

$$\mathfrak{R} = \{V_{m,n}(z) \,|\, (m,n) = 1,\ m > 0\} \cup \{V_{0,1}(z)\}$$

where each ordered pair $\{m,n\}$ is used <u>only</u> <u>once</u>.

The classical Poincaré series are defined as follows: for $z \in \mathcal{H}$, $g \in Z$, $g \geq 0$, $k \in Z$, $k \geq 2$ we define:

$$(14)\qquad \varphi_g(z,k) = \sum_{V_{m,n}\in\mathcal{R}} (mz+n)^{-2k} \exp(2\pi i g V_{m,n}(z))$$

$$= f_{0,1}(z,2k,g) + \sum_{\substack{(m,n)=1\\ m>0}} f_{m,n}(z,2k,g)$$

$$= \exp(2\pi i g z) + \sum_{\substack{(m,n)=1\\ m>0}} f_{m,n}(z,2k,g) \quad .$$

It is known that this series converges absolutely so that no order of summation need be given. (For proofs of this and the following facts, in a slightly more general setting, see [5], p. 28-34). The series converges uniformly absolutely in $\{z \in \mathcal{H} \,|\, \mathrm{Im}(z) \geq y_0\}$ for each $y_0 > 0$. Thus (14) defines an analytic function of z for $z \in \mathcal{H}$. It is also shown that:

$$(15)\qquad \varphi_g(V(z),k)\left(\frac{dV}{dz}\right)^k = \varphi_g(z,k) \quad \text{for} \quad V \in \Gamma ,$$

and that:

$$(16)\qquad \lim_{\mathrm{Im}(z)\to\infty} \varphi_g(z,k) = \begin{cases} 1 & \text{if} \quad g = 0 \\ 0 & \text{if} \quad g \geq 1 \end{cases}$$

Thus $\varphi_g(z,k)$ is a modular form of weight k for the modular group Γ, and for $g \geq 1$ in fact a cusp form, that is, vanishing at the cusp of $\mathcal{H}/\Gamma$.

Note: In the older literature the terminology "dimension -2k" is used rather than "weight k".

The case $g = 0$ gives the Eisenstein series $\varphi_0(z,k)$. In [9],& [10], Lewittes defined a generalization of $\varphi_0(z,k)$, namely $G(z,s) = \{1 + \exp(\pi i s)\}\zeta(s)\cdot\varphi_0(z, \frac{s}{2})$ where s is complex with $\mathrm{Re}(s) > 2$, $z \in \mathcal{H}$. He obtained a finite analytic continuation of $G(z,s)$ to $\{z,s\} \in \mathcal{H} \times \mathbb{C}$. In this paper we define a generalization of $\varphi_g(z,k)$

for $g \geq 1$, namely

$$P(z,s,g) = \varphi_g(z, \tfrac{s}{2}) \quad \text{for} \quad \mathrm{Re}(s) > 2 \ , \ \mathrm{Im}(z) > 0 \ , \text{ and obtain}$$

a finite analytic continuation of

$$P(z,s,g) \quad \text{to} \quad \{z,s\} \in \mathcal{H} \times \mathcal{K}_{\frac{7}{4}} \ , \quad \mathcal{K}_{\frac{7}{4}} = \{s \,|\, \mathrm{Re}(s) > \tfrac{7}{4}\} \ .$$

§1. Poincaré Series of Complex Weight (or Dimension).

For $\mathrm{Im}(z) > 0$, $\mathrm{Re}(s) > 2$, and $g \in Z$, $g \geq 0$, we define:

$$(17) \qquad P(z,s,g) = \exp(2\pi i g z) + \sum_{\substack{(m,n)=1 \\ m>0}} f_{m,n}(z,s,g) \ ,$$

where $f_{m,n}$ is as in (12). Thus for $s = 2k$, k an integer ≥ 2 , we have:

$$(18) \qquad P(z,2k,g) = \varphi_g(z,k) = \text{the classical Poincaré series of weight}$$

k for the full modular group, defined in §0, (14).

Instead of (17) we could have defined

$$(19) \qquad R(z,s,g) = \sum_{m,n}{}' f_{m,n}(z,s,g) \quad \text{where the prime denotes the term}$$

$\{m,n\} = \{0,0\}$ is omitted.

As will be shown later, in Proposition 1, we have the relation:

$$(20) \qquad R(z,s,g) = \{1 + \exp(\pi i s)\}\zeta(s) \ P(z,s,g) \quad .$$

We first prove the following theorem.

__Theorem 1__: $R(z,s,g)$, defined by (19), converges uniformly absolutely on compact subsets of $\mathcal{H} \times \mathcal{K}_2$, and thus defines an analytic function of $\{z,s\} \in \mathcal{H} \times \mathcal{K}_2$, $\mathcal{K}_2 = \{s \,|\, \mathrm{Re}(s) > 2\}$.

__Proof__: With $s = \sigma + it$, σ and t real we have:

$$(21) \qquad |R(z,s,g)| \leq \exp(\pi |t|) \sum_{m,n}{}' |mz+n|^{-\sigma} \exp(-2\pi g \cdot \mathrm{Im}(V_{m,n}(z))) \ .$$

If $\mathrm{Im}(z) > 0$, then $\mathrm{Im}(V_{m,n}(z)) > 0$ and thus from (20) we have:

$$|R(z,s,g)| \le \exp(\pi|t|) \sum_{m,n}{}' |mz+n|^{-\sigma} \quad . \tag{22}$$

It is well known that the series in (22) converges uniformly for $\{z,s\}$ in any compact subset of $\mathcal{H} \times \mathcal{H}_2$.

This proves our convergence claims for $R(z,s,g)$, and the same estimates work for $P(z,s,g)$ which is a subseries of $R(z,s,g)$. Since the terms of the series are analytic in $\mathcal{H} \times \mathcal{H}_2$, we have $R(z,s,g)$ and $P(z,s,g)$ analytic as functions of $\{z,s\} \in \mathcal{H} \times \mathcal{H}_2$.

Q.E.D. Theorem 1.

We now introduce some useful lemmas.

Lemma 1: If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)$, then up to an additive integer constant we have:

$$V_{m,n}\left(\frac{az+b}{cz+d}\right) = V_{\{m,n\}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) \quad , \tag{23}$$

where $V_{m,n}$ is given in (10) and (11).

Proof: Just use direct substitution into the definition (11) and thus into (10).

Q.E.D. Lemma 1.

As special cases of Lemma 1 we note:

$$V_{m,n}(z) = V_{m,n}\left(\frac{-z}{-1}\right) = V_{-m,-n}(z) \quad ; \tag{24}$$

$$V_{m,n}\left(\frac{-1}{z}\right) = V_{n,-m}(z) \quad ; \tag{25}$$

$$V_{m,n}(z+b) = V_{m,mb+n}(z) \quad . \tag{26}$$

Lemma 2: Let A,B,C,D be real numbers, A,B not both zero, and C positive. Then for $z \in \mathcal{H}$ we have:

$$\text{Arg}\left(\frac{Az+B}{Cz+D}\right) = \text{Arg}(Az+B) - \text{Arg}(Cz+D) + 2\pi k \tag{27}$$

where $k = \begin{cases} 1 & \text{if } A \le 0 \text{ , } AD - BC > 0 \text{ ,} \\ 0 & \text{otherwise.} \end{cases}$

Proof: Straightforward (see [10], p. 470).

Q.E.D. Lemma 2.

Special cases used frequently are:

(28) If $z \neq 0$, $u > 0$, then $\text{Arg}(uz) = \text{Arg}(z)$, and thus $(uz)^s = u^s z^s$ by (7).

(29) If $z \in \mathcal{H}$ (or $z > 0$), and $u > 0$, then $\text{Arg}(-uz) = \text{Arg}(z) - \pi$, and thus by (7) $(-uz)^s = e^{-\pi i s} u^s z^s$.

Proposition 1: For $\{z,s\} \in \mathcal{H} \times \mathcal{K}_2$ we have the equation in (20), namely: $R(z,s,g) = (1 + \exp(\pi i s)) \cdot \zeta(s) \cdot P(z,s,g)$.

Proof: By absolute convergence we can write (19) as:

$$R(z,s,g) = \sum_{d=1}^{\infty} \sum_{\substack{m,n \\ (m,n)=d}} f_{m,n}(z,s,g) \quad . \tag{30}$$

By (28) we have:

$$f_{m,n}(z,s,g) = d^{-s}\left(\frac{mz}{d} + \frac{n}{d}\right)^{-s} \exp(2\pi i g V_{m,n}(z)) \ .$$

Thus if $(m,n) = d$, we have:

$$f_{m,n}(z,s,g) = d^{-s}\left(\frac{mz}{d} + \frac{n}{d}\right)^{-s} \exp(2\pi i g V_{\frac{m}{d},\frac{n}{d}}(z)) = d^{-s} f_{\frac{m}{d},\frac{n}{d}}(z,s,g) \ . \tag{31}$$

For ease of notation, and for fixed s and g we can use (13), namely: $f(m,n,z) = f_{m,n}(z,s,g)$.

Thus (30) becomes:

$$(32)\qquad R(z,s,g) = \sum_{d=1}^{\infty} d^{-s} \sum_{(m,n)=d} f(\tfrac{m}{d},\tfrac{n}{d},z) \quad ,$$

or rewriting, we get:

$$(33)\qquad R(z,s,g) = \zeta(s) \sum_{(m_1,n_1)=1} f(m_1,n_1,z) \quad .$$

Furthermore,

$$(34)\qquad \sum_{(m,n)=1} f(m,n,z) = \left(\sum_{\substack{m=0\\ n=1,-1}} + \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty\\ (m,n)=1}}^{\infty} + \sum_{m=-\infty}^{-1} \sum_{\substack{n=-\infty\\ (m,n)=1}}^{\infty} \right) f(m,n,z) \quad .$$

By (24) and (29) we have:

$$V_{-m,-n}(z) = V_{m,n}(z) \quad \text{and} \quad (-mz-n)^{-s} = e^{\pi is}(mz+n)^{-s} \quad .$$

It thus follows that:

$$(35)\qquad f(-m,-n,z) = e^{\pi is} f(m,n,z) \quad .$$

Therefore, (34) can be written as:

$$(36)\qquad \sum_{(m,n)=1} f(m,n,z) = (1+e^{\pi is})\{f(0,1,z) + \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty\\ (m,n)=1}}^{\infty} f(m,n,z)\}$$

$$= (1+e^{\pi is})P(z,s,g) \quad .$$

<u>Q.E.D. Proposition 1</u>.

§ 2. Analytic Continuation of $P(z,s,g)$.

Theorem 2: $P(z,s,g)$, $g \geq 1$, has an analytic continuation for z and s to $\{z,s\} \in \mathcal{H} \times \mathcal{H}_{\frac{7}{4}}$, given explicitly by the Fourier series:

$$(37) \qquad P(z,s,g) = \exp(2\pi igz) + \sum_{p=1}^{\infty} \left\{ \sum_{m=1}^{\infty} m^{-s}\, S(p,g,m)\, I\left(2\pi p, \frac{2\pi}{m^2}, s\right) \right\} \exp(2\pi ipz)$$

where

$$(38) \qquad S(p,g,m) = \sum_{\substack{r \bmod m \\ (r,m)=1}} \exp\left\{\frac{2\pi i}{m}(pr + g\bar{r})\right\}, \quad \text{[a Kloosterman sum]},$$

and for $\mathrm{Re}(s) > 1$, A and B real:

$$(39) \qquad I(A,B,s) = \int_{y=y_0} z^{-s} \exp\left(-iAz - \frac{iB}{z}\right) dz\,, \quad \text{[a Bessell function]}.$$

for any $y_0 > 0$.

Proof of Theorem 2: Let

$$Q(z,s,g) = P(z,s,g) - \exp(2\pi igz) = \sum_{\substack{(m,n)=1 \\ m>0}} f_{m,n}(z,s,g)\,.$$

Thus,

$$(40) \qquad Q(z,s,g) = \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ (m,n)=1}}^{\infty} f_{m,n}(z,s,g)$$

by absolute convergence.

We can simplify the inner sum of (40) by letting n run through congruence classes mod m and then summing over a reduced residue system mod m. Namely,

$$(41) \qquad Q(z,s,g) = \sum_{m=1}^{\infty} \sum_{\substack{1 \leq r \leq m \\ (r,m)=1}} \sum_{\substack{n=-\infty \\ n \equiv r\,(m)}}^{\infty} f(m,n,z)$$

where we are suppressing the dependence of f on s and g for ease of notation. Equivalently we have

$$Q(z,s,g) = \sum_{m=1}^{\infty} \sum_{\substack{1\le r\le m \\ (r,m)=1}} \sum_{k=-\infty}^{\infty} f(m,r+km,z) \quad .$$

But $V_{m,r+km}(z) = V_{m,r}(z+k)$ by (26), and so $f(m,r+km,z) = f(m,r,z+k)$ by direct substitution. Thus we have:

$$(42) \qquad Q(z,s,g) = \sum_{m=1}^{\infty} \sum_{\substack{1\le r\le m \\ (r,m)=1}} F_{m,r}(z,s,g) \ ,$$

where,

$$(43) \qquad F_{m,r}(z,s,g) = \sum_{k=-\infty}^{\infty} f(m,r,z+k) \ .$$

Clearly,

$$(44) \qquad F_{m,r}(z+1,s,g) = F_{m,r}(z,s,g) \ .$$

Furthermore,

$$F_{m,r}(z,s,g) = \sum_{k=-\infty}^{\infty} f(m,r+km,z) \ ,$$

and so $F_{m,r}(z,s,g)$ is a subseries of $R(z,s,g)$ for $\mathrm{Re}(s) > 2$. Thus by Theorem 1, $F_{m,r}(z,s,g)$ defines an analytic function of $\{z,s\} \in \mathcal{H} \times \mathcal{K}_2$. Since $F_{m,r}$ satisfies (44), it has a Fourier series:

$$(45) \qquad F_{m,r}(z,s,g) = \sum_{p=-\infty}^{\infty} a_p^{m,r}(s,g) \exp(2\pi ipz)$$

where,

$$(46) \qquad a_p^{m,r}(s,g) = \int_{L_1} F_{m,r}(z,s,g) \exp(-2\pi ipz)dz \quad .$$

L_1 being any line segment in $\mathcal{H}$ of length one, parallel to the x-axis, and oriented from left to right. (43) can now be written as:

$$(47) \qquad Q(z,s,g) = \sum_{m=1}^{\infty} \sum_{\substack{1\le r\le m \\ (r,m)=1}} \sum_{p=-\infty}^{\infty} a_p^{m,r}(s,g) \exp(2\pi ipz) \ .$$

Note: For brevity we will suppress the dependence of $a_p^{m,r}$ on s and g, since s and g are fixed in the following discussion.

Choose a number $y_0 > 0$, and let $L(k,k+1)$ be the line segment from $k + iy_0$ to $k + 1 + iy_0$ oriented from left to right. Then by (46) and (43) we have:

$$a_p^{m,r} = \int_{L(0,1)} f(m,r,z+k) \exp(-2\pi iz)dz$$

or interchanging order of sum and integral, since the sum converges uniformly absolutely, we have:

$$a_p^{m,r} = \sum_{k=-\infty}^{\infty} \int_{L(0,1)} f(m,r,z+k) \exp(-2\pi iz)dz$$

$$= \sum_{k=-\infty}^{\infty} \int_{L(k,k+1)} f(m,r,z) \exp(-2\pi i(z-k))d(z-k)$$

$$= \sum_{k=-\infty}^{\infty} \int_{L(k,k+1)} f(m,r,z) \exp(-2\pi ipz)dz \quad .$$

Thus

$$a_p^{m,r} = \int_L f(m,r,z) \exp(-2\pi ipz)dz \tag{48}$$

where L is the line $\mathrm{Im}(z) = y_0$ oriented from left to right, for any $y_0 > 0$.

We will now show that $a_p^{m,r}(s,g)$ is analytic as a function of s. Putting in:

$$f(m,r,z) = f_{m,r}(z,s,g) = (mz+n)^{-s} \exp(2\pi ig)\ V_{m,r}(z))$$

where $V_{m,r}(z) = \frac{Mz+R}{mz+r}$, we get:

$$a_p^{m,r} = \int_L (mz+r)^{-s} \exp\{2\pi i\{g(\frac{Mz+R}{mz+r}) - pz\}\} dz \ .$$

Now, to simplify the integral, we let $z = z' - \frac{r}{m}$, and obtain:

$$a_p^{m,r} = \int_L (mz)^{-s} \exp\{2\pi i\{g(\frac{M}{m} - \frac{Mr-Rm}{m^2 z}) - pz + \frac{pr}{m}\}\}dz$$

(after calling z',z again). Thus, since $Mr - Rm = 1$, and using (28) we get:

$$a_p^{m,r} = m^{-s} \int_L z^{-s} \exp\{2\pi i(\frac{gM+pr}{m} - \frac{g}{m^2 z} - pz)\}dz$$

and finally:

$$(49) \qquad a_p^{m,r}(s,g) = m^{-s} \exp\{2\pi i(\frac{gM+pr}{m})\} \int_L z^{-s} \exp\{2\pi i(\frac{-g}{m^2 z} - pz)\}dz$$

where L is the line $\operatorname{Im}(z) = y_0$, $y_0 > 0$, oriented from left to right.

For $\operatorname{Re}(s) > 1$, A and B real, let

$$(50) \qquad I(A,B,s) = \int_L z^{-s} \exp(-iAz - \frac{iB}{z})dz ,$$

for L as above, so that (49) becomes:

$$(51) \qquad a_p^{m,r}(s,g) = m^{-s} \exp(2\pi i(\frac{gM+pr}{m}))\ I(2\pi p, \frac{2\pi g}{m^2}, s) \quad .$$

Using $s = \sigma + it$, we have by elementary estimates that:

$$(52) \qquad |I(A,B,S)| \le \exp(\pi|t|+A) \int_{x=-\infty}^{\infty} |x+iy_0|^{-\sigma} \exp(-Bx/|x+iy_0|^2)dx$$

which converges uniformly for $\{A,B,s\}$ is any compact subset of $R \times R \times \mathcal{K}_1$, $\mathcal{K}_1 = \{s|\sigma > 1\}$, and thus I is continuous as a function of $A \in R$, continuous as a function of $B \in R$, and analytic as a function of s in $\mathcal{K}_1$.

<u>Lemma 3</u>: <u>Case (1)</u>: If $A \le 0$, then

$$(53) \qquad I(A,B,s) = 0 .$$

<u>Case (2)</u>: If $A > 0$, then for any $\delta > 0$,

$$(54)\qquad I(A,B,s) = i\{\exp(\tfrac{-3\pi is}{2}) - \exp(\tfrac{\pi is}{2})\}\int_{\delta}^{\infty} y^{-s}\exp(-Ay+\tfrac{B}{y})dy$$

$$-\oint_{|z|=\delta} \langle z^{-s}\rangle \exp(-iAz - \tfrac{iB}{z})dz$$

where the closed contour integral on $|z| = \delta$ is taken counterclockwise, and

$$\langle z^{-s}\rangle \underset{\text{def}}{=} \begin{cases} z^{-s} & \text{if } \frac{-\pi}{2} \le \operatorname{Arg}(z) < \pi, \\ \exp(-2\pi is)z^{-s} & \text{if } -\pi \le \operatorname{Arg}(z) < \frac{-\pi}{2} \end{cases} .$$

Proof: For definiteness in (50) let us take $y_0 = 1$, that is, L is the line $\operatorname{Im}(z) = 1$ oriented from left to right.

Case 1: $A \le 0$.

Consider the closed contour shown below made up of the following two curves, oriented as shown:

$$C_R = \{z:\ |z| = R,\ \alpha_R \le \operatorname{Arg} z \le \pi - \alpha_R\} ,$$

where $\alpha_R = \operatorname{Arcsin}(\frac{1}{R}),\ 0 < \alpha_R < \frac{\pi}{2}$;

$$L_{R'} = \{z:\ \operatorname{Im}(z) = 1, \text{ and } |\operatorname{Re}(z)| \le R'\} ,$$

where $R' = \sqrt{R^2 - 1}$.

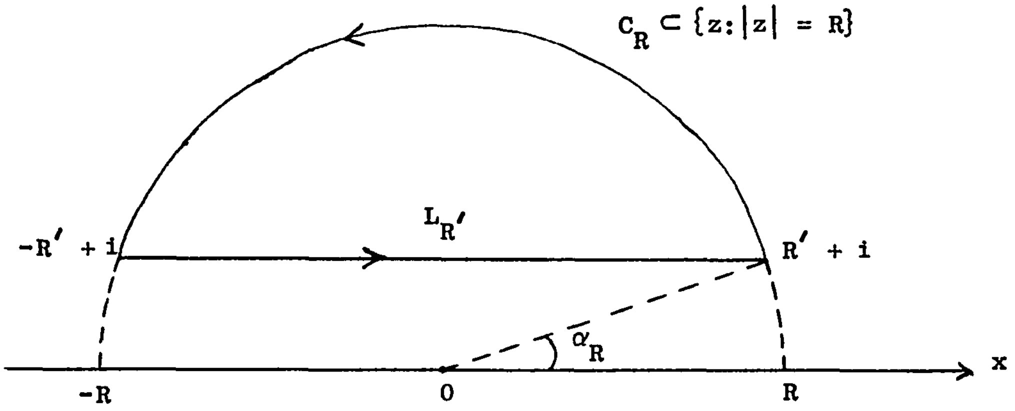

Figure 1

Then, since our integrand in (50), namely $h(z,s,A,B) = z^{-s}\exp(-iAz - \frac{iB}{z})$, is analytic in the upper half plane, we have by Cauchy's theorem that

$$\int_{L_{R'}} h(z,s,A,B)dz = - \int_{C_R} h(z,s,A,B)dz \quad .$$

But

$$\int_L h(z,s,A,B)dz = \lim_{R'\to\infty} \int_{L_{R'}} h(z,s,A,B)dz = - \lim_{R\to\infty} \int_{C_R} h(z,s,A,B)dz \ ,$$

where recall

$$R = \sqrt{(R')^2 + 1} \ .$$

On C_R we have $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$, $dz = i\,Re^{i\theta}d\theta$. Thus

$$\int_{C_R} h(z,s,A,B)dz = \int_{\theta=\alpha_R}^{\pi-\alpha_R} (Re^{i\theta})^{-\sigma-it}\exp\{-iAR(\cos\theta+i\sin\theta) - \frac{iB}{R}(\cos\theta-i\sin\theta)\}iRe^{i\theta}d\theta .$$

Therefore,

$$\left|\int_{C_R} h(z,s,A,B)dz\right| \le \int_{\theta=\alpha_R}^{\pi-\alpha_R} R^{1-\sigma} \cdot e^{\theta t}\exp\{(AR - \frac{B}{R})\sin\theta\}d\theta$$

$$\le R^{1-\sigma} \cdot e^{\pi|t|} \int_{\theta=\alpha_R}^{\pi-\alpha_R} \exp(\frac{|B|}{R})d\theta$$

since $A \le 0$ and $0 \le \sin\theta \le 1$ for $0 \le \theta \le \pi$. Thus

$$\left|\int_{C_R} h(z,s,A,B)dz\right| \le R^{1-\sigma} \cdot e^{\pi|t|} \cdot e^{\frac{|B|}{R}} \cdot (\pi - 2\alpha_R) \ .$$

But $\alpha_R \to 0$ as $R \to \infty$. Therefore, for $\sigma > 1$ and $A \le 0$ we see that

$$\lim_{R\to\infty} \int_{C_R} h(z,s,A,B)dz = 0 \ ,$$

that is, $I(A,B,s) = 0$. By (51) this gives $a_p^{m,r} = 0$ for $p \le 0$ in (47).

Q.E.D. Case 1.

Case 2: $A > 0$.

In this case, $\left|\int_{C_R}\right|$ would go to ∞ as $R \to \infty$. Thus we go below the line L_R, for our contour. Furthermore, if s is not an integer, then our integrand is discontinuous on the negative real axis $\{z \mid y = 0, x \le 0\}$, and so to use Cauchy's theorem we would have to avoid this line. It turns out to be useful for later estimates to use a branch of z^{-s} , called here $\langle z^{-s}\rangle$, which agrees with our original branch on the path of integration L, but has its discontinuity on the negative imaginary axis $\{z: x = 0, y \le 0\}$. Recall,

$$z^{-s} = \exp\{(-s)(\ln|z|+i\,\mathrm{Arg}(z))\} \quad \text{with} \quad -\pi \le \mathrm{Arg}(z) < \pi \, .$$

We define:

$$(55) \qquad \langle z^{-s}\rangle = \begin{cases} z^{-s} & \text{if } \frac{-\pi}{2} \le \mathrm{Arg}(z) < \pi \, ; \\ \exp(-2\pi i s) z^{-s} & \text{, if } -\pi \le \mathrm{Arg}(z) < \frac{-\pi}{2} \, . \end{cases}$$

Thus, if we define:

$$(56) \qquad g(z,s,A,B) = \langle z^{-s}\rangle \exp(-iAz - \frac{iB}{z}) \, ,$$

then (50) becomes (recall L is any line $\mathrm{Im}(z) = y_0 > 0$ oriented left to right):

$$(57) \qquad I(A,B,s) = \int_L g(z,s,A,B)dz = \int_L g \quad \text{for brevity.}$$

Our new integrand, g, is analytic in the plane slit along the negative imaginary axis, and thus we can apply Cauchy's theorem to the contour shown below in Figure 2.

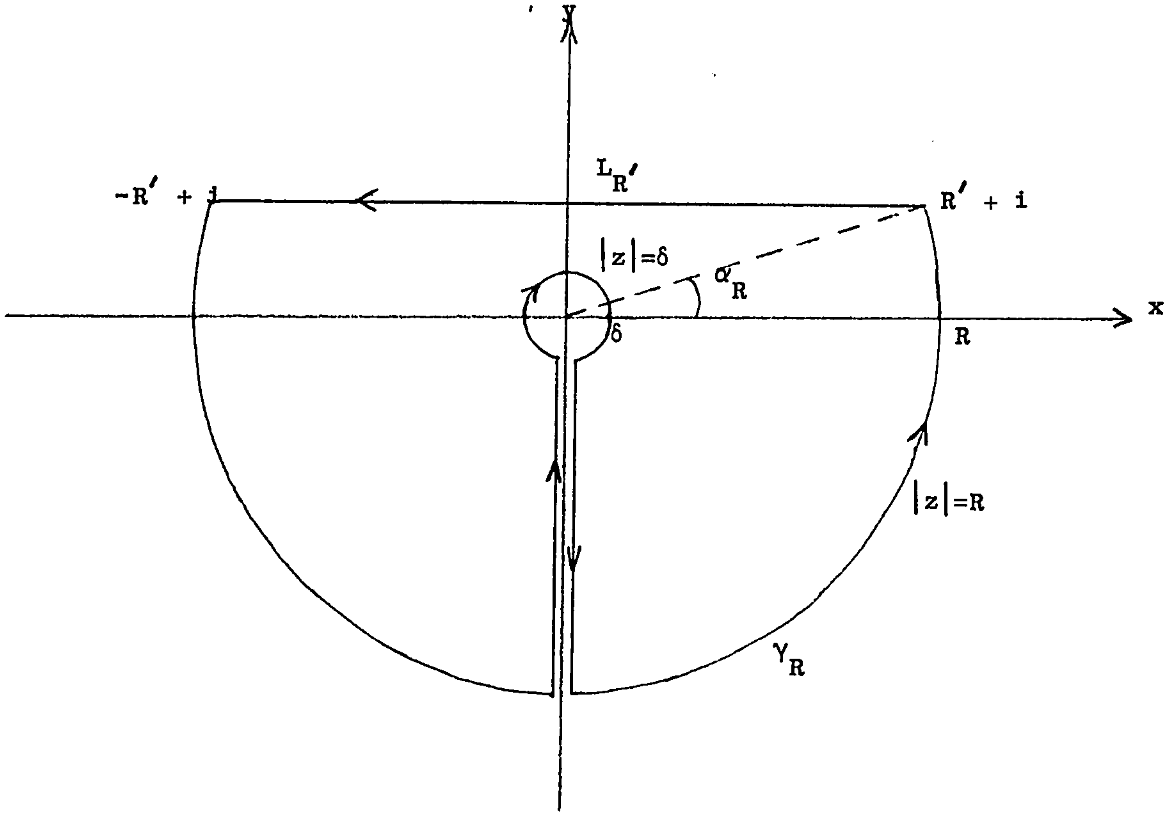

Figure 2.

Let $L_{R'}$ be the oriented line segment from $R' + i$ to $-R' + i$ and let γ_R be the rest of the oriented contour shown in Figure 2. Thus we have:

$$\int_L g = -\lim_{R'\to\infty} \int_{L_{R'}} g$$

and by Cauchy's theorem:

$$-\int_{L_{R'}} g = \int_{\gamma_R} g \; .$$

Therefore

$$\int_L g = \lim_{R\to\infty} \int_{\gamma_R} g \; . \qquad (58)$$

Let C_R denote the part of γ_R which belongs to $|z| = R$. On C_R we have $z = Re^{i\theta}$ for $\theta \in [\pi - \alpha_R, \frac{3\pi}{2}) \cup [\frac{-\pi}{2}, \alpha_R]$. Thus

$$\int_{C_R} g = \int_{\theta=\pi-\alpha_R}^{\frac{3\pi}{2}} R^{-(s-1)} \; e^{-i\theta(s-1)} k(\theta)d\theta + \int_{\theta=\frac{-\pi}{2}}^{\alpha_R} R^{-(s-1)} \; e^{-i\theta(s-1)} k(\theta)d\theta$$

where $k(\theta) = \exp\{-iAR(\cos\theta + i\sin\theta) - \frac{iB}{R}(\cos\theta - i\sin\theta)\}$.

Therefore

$$\left|\int_{C_R} g\right| \le \int_{\theta=\pi-\alpha_R}^{\frac{3\pi}{2}} R^{1-\sigma} \; e^{-\theta t} \; h(\theta)d\theta + \int_{\theta=\frac{-\pi}{2}}^{\alpha_R} R^{1-\sigma} \; e^{-\theta t} \; h(\theta)d\theta ,$$

where $h(\theta) = \exp(AR\sin\theta - \frac{B}{R}\sin\theta)$. But for $\theta \in [\pi, \frac{3\pi}{2}] \cup [-\frac{\pi}{2}, 0]$ we have $-1 \le \sin\theta \le 0$, and since $A > 0$, we have $|h(\theta)| \le \exp(\frac{|B|}{R})$. For $\theta \in [\pi - \alpha_R, \pi] \cup [0, \alpha_R]$, we have $0 \le \sin\theta \le \frac{1}{R}$ as can be seen from the diagram of the contour, Figure 2, so that $|h(\theta)| \le \exp(A + \frac{|B|}{R^2})$.

Thus for $R > 1$, and since $A > 0$:

$$\left|\int_{C_R} g\right| \le R^{1-\sigma}\exp\left(\frac{3\pi|t|}{2}\right)\left\{\exp\left(A + \frac{|B|}{R^2}\right)\left(\int_{\pi-\alpha_R}^{\pi} 1d\theta + \int_{0}^{\alpha_R} 1d\theta\right) + \exp\left(\frac{|B|}{R}\right)\left(\int_{\pi}^{\frac{3\pi}{2}} 1d\theta + \int_{\frac{-\pi}{2}}^{0} 1d\theta\right)\right\}$$

$$= R^{1-\sigma} \exp\left(\frac{3\pi|t|}{2}\right)\left\{2\alpha_R \exp\left(A + \frac{|B|}{R^2}\right) + \pi \exp\left(\frac{|B|}{R}\right)\right\} .$$

But $2\alpha_R \le \pi$, and so for $\sigma > 1$ we have

(59) $$\lim_{R\to\infty} \int_{C_R} g = 0 .$$

Thus if we let K_R^δ (shown below in Figure 3) denote the part of γ_R not including C_R, then we have

(60) $$I(A,B,s) = \lim_{R\to\infty} \int_{K_R^\delta} g(z,s,A,B)dz .$$

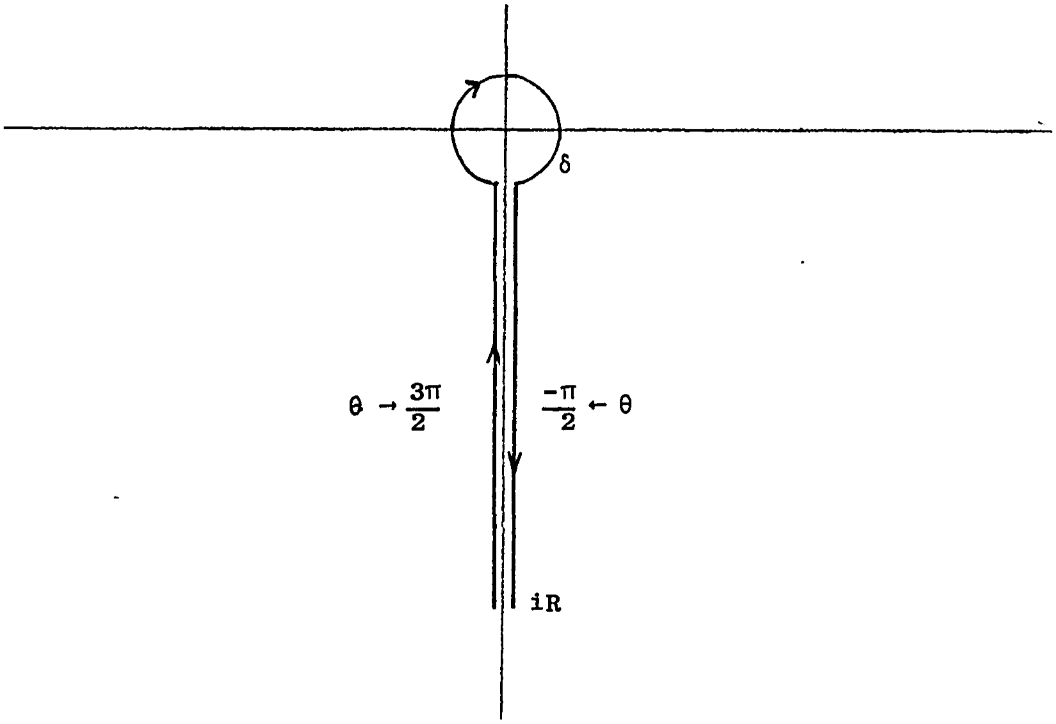

Figure 3.

On the left straight edge of K_R^δ we have $z = -iy$, $y > 0$, and $<z^{-s}> = y^{-s} \exp(\frac{-3\pi is}{2})$, while on the right straight edge we have $z = -iy$, $y > 0$ and $<z^{-s}> = y^{-s} \exp(\frac{\pi is}{2})$. Taking into account the orientation of K_R^δ and denoting by $\int_{|z|=\delta}$ the integral over the circle in the counterclockwise direction, we get:

$$\lim_{R\to\infty} \int_{K_R^\delta} <z^{-s}>\exp(-iAz - \frac{iB}{z})dz = -i\left\{\exp\left(\frac{-3\pi is}{2}\right) - \exp\left(\frac{\pi is}{2}\right)\right\}\int_{+\infty}^{+\delta} y^{-s}\exp(-Ay + \frac{B}{y})dy$$

$$- \int_{|z|=\delta} <z^{-s}> \exp(-iAz - \frac{iB}{z})dz$$

$$= I(A,B,s)$$

by (60).

This shows (54).

<u>Q.E.D. Lemma 3</u>.

The right hand side of (54) is the difference of two entire functions in s. Thus (53) and (54) give the analytic continuation of $I(A,B,s)$ to $s \in \mathbb{C}$.

Substituting (51) into (47) we now have:

(61)

$$Q(z,s,g) = \sum_{m=1}^{\infty} \sum_{\substack{1\le r\le m\\(r,m)=1}} \sum_{p=1}^{\infty} m^{-s} \exp\left\{\frac{2\pi i}{m}(gM+pr)\right\} I\left(2\pi p, \frac{2\pi g}{m^2}, s\right) \exp(2\pi ipz)$$

(recall $a_p^{m,r}(s,g) = 0$ if $p \le 0$)

$$= \sum_{m=1}^{\infty} m^{-s} \sum_{p=1}^{\infty} \sum_{\substack{1\le r\le m\\(r,m)=1}} \exp\left\{\frac{2\pi i}{m}(gM+pr)\right\} I\left(2\pi p, \frac{2\pi g}{m^2}, s\right) \exp(2\pi ipz) \quad .$$

Recall $Mr - mR = 1$, and thus $Mr \equiv 1 \pmod m$. In the integers mod m let $\bar{r}$ = the multiplicative inverse of r mod m and so $M \equiv \bar{r} \pmod m$.

Let,

(62) $$S(p,g;m) = \sum_{\substack{r \bmod m\\(r,m)=1}} \exp\left\{\frac{2\pi i}{m}(pr + g\bar{r})\right\} \quad \text{where} \quad r\bar{r} \equiv 1 \ (m) \ .$$

This function is known as a Kloosterman sum. Thus,

(63) $$Q(z,s,g) = \sum_{m=1}^{\infty} m^{-s} \sum_{p=1}^{\infty} S(p,g;m) I\left(2\pi p, \frac{2\pi g}{m^2}, s\right) \exp(2\pi ipz) \quad .$$

By Estermann [2], (p. 82), we have:

(64) $$|S(p,g;m)| \le d^{\frac{3}{4}}\left(\frac{m}{(g,m)}\right) m^{\frac{3}{4}} (g,m)^{\frac{1}{4}}$$

where $d(n)$ = the number of divisors of n. It is easy to see (see for example Landau, <u>Zahlentheorie</u>, Vol. I, Part VI, p. 250-251) that $\forall\ n \ge 1$ and $\forall\ \epsilon > 0$ we have:

(65) $$d(n) \le C_\epsilon\, n^\epsilon$$

where C_ϵ depends only on ϵ. Therefore taking $n = \frac{m}{(g,m)}$ in (65), (64) gives:

$$(66) \qquad |S(p,g;m)| \le \frac{(C_\epsilon m^\epsilon)^{\frac{3}{4}} \cdot m^{\frac{3}{4}} \cdot (g,m)^{\frac{1}{4}}}{(g,m)^{\frac{3}{4}\epsilon}}$$

$$\le C_\epsilon^{\frac{3}{4}} \cdot m^{\frac{3}{4}+\epsilon} \cdot (g,m)^{\frac{1}{4}-\frac{3}{4}\epsilon}$$

$$\le K_\epsilon(g) \cdot m^{\frac{3}{4}+\epsilon} ,$$

where

$$K_\epsilon(g) = C_\epsilon^{\frac{3}{4}} g^{\frac{1}{4}} \text{ with } g \ge 1 .$$

So $\forall\ \epsilon > 0$ we have from (63):

$$(67) \qquad |Q(z,s,g)| \le K_\epsilon(g) \sum_{m=1}^{\infty} m^{-\sigma+\frac{3}{4}+\epsilon} \sum_{p=1}^{\infty} \left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right| \exp(-2\pi y p) .$$

We can see the behavior of $\left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right|$ as $m \to \infty$, as follows:

$I(A,B,s)$ is continuous as a function of $B \in \mathbb{R}$, and so at $B = 0$ we have:

$$(68) \qquad \lim_{B \to 0} I(A,B,s) = I(A,0,s) = \int_L z^{-s} \exp(-iAz)dz$$

where L is any line $\operatorname{Im}(z) = y_0 > 0$ oriented left to right. Using $A = 2\pi p$, $B = \frac{2\pi g}{m^2}$ we have:

$$(69) \qquad \lim_{m \to \infty} I(2\pi p, \frac{2\pi g}{m^2}, s) = \int_L z^{-s} \exp(-2\pi i p z)dz .$$

This integral has been calculated in [10], p. 473 formulas (11) and (12), namely

$$\int_L z^{-s} \exp(-2\pi i p z)dz = (-2\pi i)^s\ p^{s-1}/\Gamma(s), \text{ for } p > 0, \sigma > 1 .$$

Thus on the right hand side of (67) we have:

$$(70) \qquad \lim_{m \to \infty} \left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right| = \left|(-2\pi i)^s\ p^{s-1}/\Gamma(s)\right|$$

which is some non-zero number depending on p. Thus on the right hand side of (67), our only hope for convergence is for $\sigma > \frac{7}{4}$.

In fact, the following lemma will show uniform convergence in (67) for $\{z,s\}$ in compact subsets of $\mathcal{H} \times \mathcal{K}_{\frac{7}{4}}$.

Lemma 4:

(71) $$\sum_{p=1}^{\infty} \left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right| e^{-2\pi y p}$$

converges uniformly for $y \geq y_0 > 0$ and s in a compact subset of $\{\mathrm{Re}(s) > 0\}$. Furthermore, the sum of this series, as a function of m, is bounded.

Proof of Lemma 4: If we try to use (52) we get:

(72) $$\left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right| \leq e^{2\pi p} K(s,g,m) \quad \text{where}$$

$$K(s,g,m) = e^{\pi|t|} \int_{-\infty}^{\infty} |x+i|^{-\sigma} \exp\left(\frac{-2\pi g}{m^2 |x+i|^2}\right) dx .$$

Thus,

$$K(s,g,m) \leq e^{\pi|t|} \int_{-\infty}^{\infty} |x+i|^{-\sigma}\, dx$$

and so we have

(73) $$\left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right| \leq e^{2\pi p} K(s) , \quad \text{where}$$

$$K(s) = e^{\pi|t|} \int_{-\infty}^{\infty} |x+i|^{-\sigma}\, dx .$$

This estimate would show convergence in (71) for $y > 1$, $\sigma > 1$, and uniform convergence for $\{y,s\}$ in $\{y \geq y_0 > 1\} \times \mathcal{K}$ where $\mathcal{K}$ is any compact subset of $\{s \mid \sigma > 1\}$. To lower the restriction on y to $y \geq y_0 > 0$, we need to look at (54), as follows.

For $A > 0$, and using $z = \delta e^{i\theta}$, $\frac{-\pi}{2} \leq \theta < \frac{3\pi}{2}$, in the second integral of (54), (54) gives $\forall\ \delta > 0$:

$$(74)\qquad |I(A,B,s)| \le \{\exp(3\pi|t|/2) + \exp(\pi|t|/2)\} \int_{\delta}^{\infty} y^{-\sigma} \exp(-Ay + \frac{B}{y})dy$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |\delta^{-s} \cdot e^{-i\theta s} \cdot h(\delta,\theta) \cdot \delta i e^{i\theta}| d\theta$$

where

$$h(\delta,\theta) = \exp\{-iA\delta(\cos\theta + i\sin\theta) - \frac{iB}{\delta}(\cos\theta - i\sin\theta)\} .$$

Simplifying we get:

$$(75)\qquad |I(A,B,s)| \le 2\exp(3\pi|t|/2) \cdot I_1 + \delta^{1-\sigma} \cdot I_2 \quad \text{where}$$

$$I_1 = \int_{\delta}^{\infty} y^{-\sigma} \exp(-Ay + \frac{B}{y})dy$$

and

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \exp\{\sin\theta(A\delta - \frac{B}{\delta})\}d\theta \quad .$$

Now for $\sigma > 0$ (recall $A > 0$) we have:

$$(76)\qquad I_1 \le \frac{\exp(\frac{|B|}{\delta})}{\delta^{\sigma}} \int_{\delta}^{\infty} \exp(-Ay)dy = \frac{\exp(\frac{|B|}{\delta})}{\delta^{\sigma}} \cdot \frac{\exp(-A\delta)}{A}$$

$$= \frac{\exp(\frac{|B|}{\delta} - A\delta)}{\delta^{\sigma}A} \quad .$$

Furthermore, we have:

$$(77)\qquad I_2 \le \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \exp(A\delta + \frac{|B|}{\delta})d\theta = 2\pi \exp(A\delta + \frac{|B|}{\delta}) .$$

Thus for $\sigma > 0$ we have in (75):

$$(78)\qquad \frac{|I(A,B,s)|}{2\pi\exp\left(\frac{3\pi|t|}{2}\right)} \le \frac{\exp(\frac{|B|}{\delta} - A\delta)}{\delta^{\sigma}A} + \frac{\exp(\frac{|B|}{\delta} + A\delta)}{\delta^{\sigma-1}} \quad .$$

Now putting in $A = 2\pi p > 0$ and $B = \frac{2\pi g}{m^2} > 0$, we get:

$$(79) \qquad \frac{\left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right|}{2\pi \exp\left(\frac{3\pi|t|}{2}\right)} \le \frac{\exp\left(\frac{2\pi g}{m^2\delta} - 2\pi p\delta\right)}{\delta^\sigma 2\pi p} + \frac{\exp\left(\frac{2\pi g}{m^2\delta} + 2\pi p\delta\right)}{\delta^{\sigma-1}} .$$

Since (79) is true for all $\delta > 0$, let us take for any given p, $\delta = \frac{1}{p^{\frac{1}{2}}}$, where $p \ge 1$. Thus (79) gives:

$$(80) \qquad \frac{\left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right|}{2\pi \exp\left(\frac{3\pi|t|}{2}\right)} \le \frac{\exp\left(\frac{2\pi g p^{\frac{1}{2}}}{m^2} - 2\pi p^{\frac{1}{2}}\right)}{p^{\frac{2-\sigma}{2}}} + \frac{\exp\left(\frac{2\pi g p^{\frac{1}{2}}}{m^2} + 2\pi p^{\frac{1}{2}}\right)}{p^{\frac{1-\sigma}{2}}} .$$

Since the second term on the right hand side of (80) is larger than the first we have:

$$(81) \qquad \frac{\left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right|}{2\pi \exp\left(\frac{3\pi|t|}{2}\right)} \le \frac{2\exp\left(\frac{2\pi g p^{\frac{1}{2}}}{m^2} + 2\pi p^{\frac{1}{2}}\right)}{p^{\frac{1-\sigma}{2}}} \le \frac{2\exp(2\pi g p^{\frac{1}{2}} + 2\pi p^{\frac{1}{2}})}{p^{\frac{1-\sigma}{2}}} ,$$

since $m \ge 1$. Thus finally, we have for $\sigma > 0$:

$$(82) \qquad \left|I(2\pi p, \frac{2\pi g}{m^2}, s)\right| \le \frac{4\pi \exp(3\pi|t|/2)\exp(4\pi g p^{\frac{1}{2}})}{p^{\frac{1-\sigma}{2}}} .$$

Therefore (71) certainly converges for $\sigma > 0$, and uniformly for $\{y,s\}$ in $\{y \ge y_0 > 0\} \times \mathcal{K}$ where $\mathcal{K}$ is any compact subset of $\{s \mid \sigma > 0\}$. Furthermore, since the estimate (82) is independent of m, the value of (71) as a function of m is bounded.

Q.E.D. Lemma 4.

Now, going back to (67) we see that the right hand side converges for

$y > 0$, $\sigma > \frac{7}{4}$, and converges uniformly for $\{z,s\} \in \{z\colon y \geq y_0 > 0\} \times \mathcal{K}$, where $\mathcal{K}$ is any compact subset of $\{s \mid \sigma > \frac{7}{4}\} = \mathcal{K}_{\frac{7}{4}}$. Thus (61) gives $Q(z,s,g)$ as an analytic function of $\{z,s\} \in \mathcal{H} \times \mathcal{K}_{\frac{7}{4}}$ and likewise for $P(z,s,g) = \exp(2\pi igz) + Q(z,s,g)$.

This completes the proof of Theorem 2 with which we began this section.

Note: Since $R(z,s,g) = \zeta(s)\{1 + \exp(\pi is)\}P(z,s,g)$, $R(z,s,g)$ also has an analytic continuation to $\{z,s\} \in \mathcal{H} \times \mathcal{K}_{\frac{7}{4}}$, where $R(z,s,g)$ given in (19).

BIBLIOGRAPHY

[1] B. C. Berndt: Generalized Dedekind eta-functions and generalized Dedekind Sums, Transactions of Amer. Math. Soc., vol. 178(April 1973), p. 495-508.

[2] T. Estermann: Vereinfachter Beweis eines Satzes von Kloosterman, Hamburg U. Math. Sem. Abh., vol. 5(1927), p. 199-224.

[3] H. Frasch: Die Erzeugenden der Hauptkongruezgruppen für Primzahlstufen, Math. Amm., 108(1933), p. 229-252.

[4] G. F. Frobenius: Über Gruppencharaktere, Stizungsberichte der Berliner Akademie (1896), p. 985-1021.

[5] R. C. Gunning: Lectures on Modular Forms, Princeton University Press, 1962.

[6] E. Hecke: Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, Hamburg U. Math. Sem. Abh. 5(1927), p. 199-224, Note corrections in Werke p. 461-486.

[7] ________: Über ein Fundamental problem aus der Theorie der Elliptischen Modulfunktionen, Hamburg U. Math. Sem. Abh. 6(1928), p. 235-257.

[8] J. Lewittes: Automorphisms of a compact Riemann surface, Amer. J. of Math., vol. 85, no. 4, (Oct. 1963), p. 734-752.

[9] ___________: Analytic continuation of the series $\Sigma(mz+n)^{-s}$, Transactions of Amer. Math. Soc., 159(1971), p. 594-598.

[10] ___________: Analytic continuation of Eisenstein series, Transactions of Amer. Math. Soc., 171(1972), p. 469-490.

[11] B. Schoeneberg: Verhalten von speziellen Integrallen 3. Gattung bei Modultransformationen und verallgemeinerte Dedekindsche Summen, Hamburg U. Math. Sem. Abh., 30(1967), p. 1-10.

[12] C. L. Siegel: Lectures on Advanced Analytic Number Theory, Tata Institute of Fundamental Research, Bombay, 1961.