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# ROUGH ISOMETRIES BETWEEN NON-COMPACT RIEMANNIAN MANIFOLDS

by
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# Abstract


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Rough Isometries, in the sense of M. Kanai [16], preserve geometric properties of Riemannian manifolds such as volume growth rate, isoperimetric dimensions and Sobolev constants, Liouville property, transience of Brownian motion and Harnack inequalities. In this manuscript we first look at *Asymptotic Ends*. We show that in a Riemannian manifold asymptotic ends are preserved under rough isometries. Secondly, we study *Mappings with Maximal Rank*. Here we use the jargon of fiber bundles. These projection-types define a decomposition of each tangent space to the domain, into the orthogonal sum of *Horizontal* plus *Vertical* subspaces, where vertical and horizontal mean tangent to the fibers and orthogonal to the fibers, respectively. Motivated by [25] we investigated the question: when is a Riemannian manifold roughly isometric to a Riemannian product manifold? Firstly, for

Riemannian submersions, a concept defined in [25], we show that, if the base manifold is compact then the fibers can be roughly isometrically immersed into the domain, and thus the domain is roughly isometric to the product of any fiber and the base space. Under assumptions on the fibers the Riemannian submersion is a rough isometry, and if a fixed fiber is compact then the domain is roughly isometric to the product of that fiber and the base space. For onto maximal rank maps that are not necessarily submersions, by adding assumptions on the Horizontal Vectors Space we have the same consequences. We provide *Counterexamples* to show that if any of the assumptions are removed those results cease to follow. Finally, as an answer to the question above we provide the main result. We show that for maximal rank onto mappings between Riemannian manifolds with bounded geometry, under assumptions on the fibers and assumptions on the subspaces of horizontal vectors, the domain of such mapping is roughly isometric to the product of the base manifold and a fixed fiber of the domain.

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# Chapter 1

# Introduction

As essential tools in this work we are interested in equivalence relations between non-compact Riemannian manifolds given by *Rough Isometries*, a concept first introduced in [16], which we define in **Definition 2.1**.

When given such an equivalence relation, one can investigate geometric properties of Riemannian manifolds which are preserved under the equivalence relation. For instance, properties such as upper bounds on volume growth, isoperimetric dimensions and Sobolev constants, Liouville property, transience of Brownian motion and Harnack inequalities.

To study geometric properties of Riemannian manifolds, which are invariant under rough isometries, we introduce in **Definition 2.2** what is called in [16], a *Net.* A net is a discrete or combinatorial structure that provides approximations of Riemannian manifolds.

In a Riemannian manifold, one of the entities that a rough isometry preserves is an *Asymptotic End.* In **Definition 2.6** we give the definition of an asymptotic end, according to [1], and we show in **Theorem 3.5** that asymptotic ends are preserved under rough isometries.

We begin **Chapter 4** by introducing key tools along with their properties that were used in this research. We use the jargon of fiber bundles throughout. Properties of *Fibers* of $M$ are described in **Chapter 2**.

Let $M$ and $B$ be Riemannian manifolds and let $\pi : M \to B$ denote a mapping of $M$ onto $B$.

To establish some known results in **Chapter 4** we utilize a description of each tangent space to $M$ as a direct sum of two orthogonal spaces, where the restriction of the derivative map $\pi_*$ of $\pi$, to one of those spaces, is an isomorphism. Maximal rank onto mappings give such decompositions, namely the subspaces of *Horizontal* and *Vertical* vectors (see comments preceding **Definition 2.7** for meaning).

In the remaining part of this work, we initially explore the motivation behind the main **Theorem 4.2.10**, before we state and prove this theorem.

*B. O'Neill*'s article [25] furnished the inspiration for this work. In this article O'Neill defines *Riemannian submersions* (see **Definition 2.7**) and gives necessary and sufficient conditions for a Riemannian submersion to differ by an isometry from the simplest type of Riemannian submersions, the projection of a Riemannian product manifold on one of its factors. In our research, **Theorem 4.2.10** provides an answer to the same kind of question as above, with maximal rank mapping replacing Riemannian submersion, and with rough isometry instead of isometry.

In sections **4.1 Riemannian Submersions** and **4.2 Non-Submersions Surjective Maximal Rank Maps** we investigate this question: when is a Riemannian manifold roughly isometric to a Riemannian product manifold?

Our first response is given by **Theorems 4.1.1, 4.1.2** from section **4.1**, which motivated the following two results in section **4.2**.

We show in **Theorem 4.2.1** that, under certain assumptions, a maximal rank onto mapping between Riemannian manifolds allows us to roughly isometrically immerse into $M$, its fibers. In that case, since the base space is compact the domain is roughly isometric to the product of a fixed fiber and the base space.

Under some other assumptions we prove in **Theorem 4.2.3** that a maximal rank onto mapping between Riemannian manifolds provides us with an equivalence relation between them, given by a rough isometry. In that case, if a fixed fiber is compact the domain is roughly isometric to the product of that fiber and the base space.

In **Counterexamples 4.2.2, 4.2.4, 4.2.5, 4.2.6** we provide 4 counterexamples showing that if we drop any of the hypothesis in either **Theorem 4.2.1** or **Theorem 4.2.3**, those results cease to follow.

Finally, we state and prove the main result **Theorem 4.2.10**, which says that for maximal rank onto mappings $\pi : M \to B$ between Riemannian manifolds with bounded geometry (see **Section 4.2** for a definition), under assumptions on the fibers and assumptions on the subspaces of horizontal vectors, $M$ is roughly isometric to $F \times B$, where $F$ is a fixed fiber of $M$.

# Chapter 2

# Preliminaries

In this section we define some notation, provide some definitions according to [16], [1] and [25] and state some results without proofs, providing references whenever necessary.

We will be interested in equivalence relations given by *rough isometries*, a concept first introduced in [16].

**Definition 2.1** *A map $\varphi : M \to N$, between two metric spaces $(M, \delta)$ and $(N, d)$, not necessarily continuous, is called a* <u>*rough isometry,*</u> *if it satisfies the following two axioms:*

**(RI.1)** *There are constants $A \geq 1, C \geq 0$, such that,*

$$\frac{1}{A}\delta(p_1, p_2) - C \leq d(\varphi(p_1), \varphi(p_2)) \leq A\delta(p_1, p_2) + C, \quad \forall p_1, p_2 \in M$$

**(RI.2)** *The set* $Im\varphi := \{q = \varphi(p), \forall p \in M\}$ *is* *<u>full</u>* *in* $N$, *i.e.*

$$\varepsilon > 0 : N = B_\varepsilon(Im\varphi) = \{q \in N : d(q, Im\varphi) < \varepsilon\}$$

*In this case we say that* $Im\varphi$ *is* *<u>$\varepsilon$-full</u>* *in* $N$.

We will denote by $\varphi^- : N \to M$ a *rough inverse* of $\varphi$, defined as follows: for each $q \in N$, choose $p \in M$ so that $d(\varphi(p), q) < \varepsilon$, we point out here that such a $p$ exists because of the condition **(RI.2)**. $\varphi^-$ is a rough isometry such that both $\delta(\varphi^- \circ \varphi(p), p)$ and $d(\varphi \circ \varphi^-(q), q)$ are bounded in $p \in M$ and in $q \in N$, respectively.

To study geometric properties of manifolds, which are invariant under rough isometries, we next introduce what is called in [16], a *net.* A net is a discrete or combinatorial structure that provides approximations of Riemannian manifolds.

**Definition 2.2** *Let* $P$ *be a countable set. A family* $N = \{N(p) : p \in P\}$ *is called a* *<u>net structure</u>* *of* $P$ *if the following conditions hold for all* $p, q \in P$*:*

**(N.1)** $N(p)$ *is a finite subset of* $P$

**(N.2)** $q \in N(p)$ *iff* $p \in N(q)$

Let $M$ be a complete Riemannian manifold, and let $d$ be the induced metric. A subset $P$ of $M$ is said to be *$\varepsilon$-separated* for $\varepsilon > 0$, if $d(p,q) \geq \varepsilon$ whenever $p$ and $q$ are distinct points of $P$, and an $\varepsilon$-separated set is called *maximal* if it is maximal with respect to the order relation of inclusion.

We have the following,

**Proposition 2.3** *If $P$ is a countable maximal $\varepsilon-$separated set in a Riemannian manifold $(M,d)$, then $P$ is $\varepsilon-$full in $M$, where $\varepsilon > 0$.*

*Proof.* We want to show that,

$$d(x,P) < \varepsilon, \quad \forall x \in M$$

If $x \in P$, then $d(x,P) = 0 < \varepsilon$.

If $x \in M \setminus P$, by the maximality of $P$, there exists $\bar{p} \in P$ such that $d(x,\bar{p}) < \varepsilon$, and finally the definition of infimum implies that $d(x,P) := \inf_{p \in P} d(x,p) \leq d(x,\bar{p}) < \varepsilon$.

□

Let $P$ be a maximal $\varepsilon$-separated subset of $M$. We define a *net structure* $N = \{N(p) : p \in P\}$ of $P$ by $N(p) = \{q \in P : 0 < d(p,q) \leq 2\varepsilon\}$. A maximal $\varepsilon$-separated subset of a complete Riemannian manifold with the net structure described above will be called an *$\varepsilon$-net in $M$*.

For a point $p \in P$, each element of $N(p)$ is called a *neighbor* of $p$. A sequence $p = (p_0, \cdots, p_l)$ of points in $P$ is called a *path* from $p_0$ to $p_l$ of length $l$ if each $p_k$ is a neighbor of $p_{k-1}$. A net $P$ is said to be *connected* if any two points in $P$ are joined by a path. For points $p$ and $q$ of a connected net $P$, $\delta(p,q)$ denotes the minimum of the lengths of paths from $p$ to $q$. This $\delta$ satisfies the axioms of metric and it is called, according to [16], the *combinatorial metric* of $P$.

We observe that an $\epsilon-$net in a complete Riemannian manifold is connected if the manifold is connected (see [16]).

The following Lemma makes it possible to interpret the geometry of a Riemannian manifold into the combinatorial geometry of an $\epsilon$-net in the manifold.

**Lemma 2.4** *Let $(M,d)$ be a complete Riemannian $n$-manifold whose Ricci curvature is bounded from below by $-(n-1)K^2$, $K$ a positive constant, and $P$ an $\epsilon-$net in $M$. Then $P$ with the combinatorial metric $\delta$ is roughly isometric to $M$. In fact we have*

$$\frac{1}{2\epsilon} d(p_1,p_2) \leq \delta(p_1,p_2) \leq a d(p_1,p_2) + c, \quad \forall p_1, p_2 \in P$$

*where $a \geq 1$ and $c \geq 0$ are constants depending only on $n, K$ and $\epsilon$, and* *consequently the inclusion of $P$ into $M$ is a rough isometry.*

*Proof.* For a proof see Lemma (2.5) in [16].

□

In a Riemannian manifold, one of the entities that a rough isometry preserves is an *asymptotic end.* In what follows we introduce the definition of an asymptotic end, according to [1].

Let $M$ be a connected Riemannian $n$-dimensional manifold, complete, noncompact, without boundary.

We will denote by $X = \{x_0, x_1, x_2, x_3, \ldots, x_m, \ldots\}$ any infinite, discrete, countable set of points in $M$.

If $X$ satisfies the condition:

For any given compact $K \subseteq M$, there exists $\eta^K = \eta^K(K) > 0$ such that, $\forall m \geq \eta^K$, the points $x_m$ lie in a single connected component of $M \setminus K$

we will refer to such a set of points $X$, as an *admissible sequence* in $M$.

**Definition 2.5** *Two admissible sequences $X$ and $Y$ are said to be cofinal if $\forall K \subseteq M$, compact, $\exists \eta^K = \eta^K(K, X, Y) > 0$ such that, for all $m_1, m_2 \geq \eta^K$, the points $x_{m_1}$ and $y_{m_2}$ lie in the same connected component of $M \setminus K$. In that case we write $\underline{X \sim Y}$.*

The relation $\sim$ is an equivalence relation. We will denote by $[X]$ the set $\{Y : X \sim Y\}$ of all admissible sequences, equivalent to $X$.

**Definition 2.6** *An end $E$ of $M$ is defined by $E := [X]$, where $X$ is an admissible sequence in $M$.*

Let $M$ and $B$ be Riemannian manifolds and let $\pi : M \to B$ denote a mapping of $M$ onto $B$.

To establish some known results in **Chapter 4** we will utilize a description of each tangent space to $M$ as a direct sum of two orthogonal spaces, where the restriction of the derivative map $\pi_*$ of $\pi$, to one of those spaces, is an isomorphism. Maximal rank onto mappings give such decompositions.

As in [25] we will use the language of fiber bundles throughout.

If $\pi : M \to B$ is a maximal rank mapping of $M$ onto $B$, then for all $b \in B$, $\pi^{-1}(b)$ is a submanifold of $M$ of dimension dim $M$- dim $B$, as it is proved in **Lemma 4.1**. For each $b \in B, \pi^{-1}(b)$ is called a *fiber*.

According to [25], a tangent vector on $M$ is said to be *vertical* if it is tangent to a fiber, *horizontal* if it is orthogonal to a fiber. A vector field on $M$ is *vertical* if it is always tangent to fibers , *horizontal* if it is always orthogonal to fibers.

We show in **Theorem 4.2.3** that, under certain assumptions, a maximal rank onto mapping between Riemannian manifolds provides us with an equivalence relation between them, given by a rough isometry, which we defined in **Definition 2.1**. In that case, if a fixed fiber is compact the domain is roughly isometric to the product of that fiber and the base space.

Under some other assumptions we prove in **Theorem 4.2.1** that a maximal rank onto mapping between Riemannian manifolds allows us to roughly isometrically immerse into a manifold $M$, its fibers. In that case, since the base space is compact the domain is roughly isometric to the product of a fixed fiber and the base space.

We also have the same results as above (see **Theorems 4.1.2, 4.1.1**) for a more restricted class of maximal rank mappings, *Riemannian submersions* in the sense of [25], which we next introduce.

**Definition 2.7** *A Riemannian submersion* $\pi : M \to B$ *is an onto mapping satisfying the following two axioms:*

**(S.1)** $\pi$ *has maximal rank, i.e., the derivative map* $\pi_*$ *is surjective;*

**(S.2)** $\pi_*$ *preserves lengths of horizontal vectors*

# Chapter 3

# Asymptotic Ends

In this chapter we show that a rough isometry takes ends to ends, in other words, that it preserves the equivalence relation $\sim$ from **Definition 2.5**.

First we verify that a rough isometry $\varphi : M \to N$ induces a well-defined map $\tilde{\varphi}$ between the set of equivalence classes $M/_\sim = \{\text{ends of } M\}$ and $N/_\sim = \{\text{ends of } N\}$, in the following way:

If $E = [X]$ is an end of $M$, through $\varphi$ we obtain a new sequence we will denote by,

$$\varphi \circ X := \{\varphi(x_0), \varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m), \ldots\}$$

Since $\varphi$ is a rough isometry $\varphi \circ X$ is also an infinite, discrete, countable set of points in $N$.

Next, we show that $\varphi \circ X$ is still an admissible sequence of points in $N$.

For, we need the following Lemmas.

**Lemma 3.1** *Let $K \subseteq N$ be any compact set. Then, there exists a constant $\eta^{K,1} = \eta^{K,1}(K, x_0, \varphi(x_0), a, b) > 0$ such that, $\forall m_1 \geq \eta^{K,1} \Rightarrow \varphi(x_{m_1}) \in N \setminus K$.*

*Proof of Lemma.* The definition of an end guarantees that $\forall r > 0, \exists \tilde{\eta} = \tilde{\eta}(r, a, b, x_0) > 0$, such that,

$$x_{m_1} \in M \setminus \overline{B_{(r+b)a}(x_0)}, \forall m_1 \geq \tilde{\eta}$$

where, $\overline{B_{(r+b)a}(x_0)}$ denotes a closed ball centered at $x_0$ with radius $(r+b)a > 0$ with respect to the distance $\delta$.

By **RI.1**, for any $m_1 \geq \tilde{\eta}$ we have,

$$\begin{aligned} d(\varphi(x_{m_1}), \varphi(x_0)) &\geq a^{-1}\delta(x_{m_1}, x_0) - b \\ &> a^{-1}(r+b)a - b = r \end{aligned} \tag{3.1}$$

Let $r_0 := d(\varphi(x_0), K) + \text{diam}K + 1 > 0$.

We claim that,

$$K \subseteq B_{r_0}(\varphi(x_0)) \subseteq \overline{B_{r_0}(\varphi(x_0))}$$

Indeed, since $K$ is compact, $\exists q_K \in K$ such that,

$$d(\varphi(x_0), K) = d(\varphi(x_0), q_K)$$

Therefore, for any given $q \in K$,

$$d(\varphi(x_0), q) \leq d(\varphi(x_0), q_K) + d(q_K, q) < d(\varphi(x_0), K) + \text{diam}K + 1 = r_0$$

If we let $\eta^{K,1} := \tilde{\eta}(r_0, x_0, a, b) = \tilde{\eta}(K, x_0, \varphi(x_0), a, b) > 0$, (3.1) implies that $\forall m_1 \geq \eta^{K,1}$

$$\varphi(x_{m_1}) \in N \setminus \overline{B_r(\varphi(x_0))} \subseteq N \setminus K$$

which is the claim.

□

**Lemma 3.2** *Given an arbitrary compact set $K$, there exists a positive real number $\eta^{K,2} = \eta^{K,2}(K, x_0, \varphi(x_0), a, b)$ such that, whenever $m_1 \geq \eta^{K,2}$, it follows that, $\varphi(x_{m_1}) \in N \setminus \overline{B_{2(a+b+1)}(K)}$, where $\overline{B_{2(a+b+1)}(K)} = \{q \in N : d(q, K) \leq 2(a+b+1)\}$.*

*Proof of Lemma.* This is **Lemma 3.1**, where $K$ is replaced by $\overline{B_{2(a+b+1)}(K)}$.

□

**Lemma 3.3** *For any compact set $K$, there exists $J_{K,2} = J_{K,2}(\eta^{K,2}) \subset \mathbb{N}$, where $\eta^{K,2}$ is given by* **Lemma 3.2**, *such that $m \in J_{K,2}$ implies $\varphi(x_m) \in B_{\frac{d_{K,2}}{2}}(\varphi(x_{\eta^{K,2}})) \cap N \setminus K$, which is entirely contained in a single connected component of $N \setminus K$.*

*Proof of Lemma.* Since $\varphi(x_{\eta^{K,2}}) \in N \setminus \overline{B_{2(a+b+1)}(K)} \in N \setminus K$, we have

$$d_{K,2} := d(\varphi(x_{\eta^{K,2}}), K) \geq 2(a+b+1) > 0 \tag{3.2}$$

by **RI.1**,

$$d(\varphi(x_{m_1}), \varphi(x_{\eta^{K,2}})) \leq a\delta(x_{m_1}, x_{\eta^{K,2}}) + b \tag{3.3}$$

Let

$$J_{K,2} := \{m \in \mathbb{N} : \delta(x_m, x_{\eta^{K,2}}) < 1\}$$

Since $x_{\eta^{K,2}} \in J_{K,2}$, we have $J_{K,2} \neq \emptyset$.

Thus, by (3.3) and (3.2), for any $m_1 \in J_{K,2}$,

$$d(\varphi(x_{m_1}), \varphi(x_{\eta^{K,2}})) \leq a\delta(x_{m_1}, x_{\eta^{K,2}}) + b < a + b + 1 \leq \frac{d_{K,2}}{2},$$

Finally, we conclude that,

$$\varphi(x_m) \in \mathcal{B}_{K,2} := B_{\frac{d_{K,2}}{2}}(\varphi(x_{\eta^{K,2}})) \cap N \setminus K, \forall m \in J_{K,2}$$

Since, $B_{\frac{d_{K,2}}{2}}(\varphi(x_{\eta^{K,2}}))$ is connected and open, it follows that $\mathcal{B}_{K,2}$ is entirely contained in a single connected component of $N \setminus K$.

$\square$

We denote by $\varphi \circ X_\eta$ the sequence $\{(\varphi(x_m))_{m \in N, m \geq \eta}\}$ of points in $N$, where $X$ is an admissible sequence in $M$.

Next, we prove the theorem,

**Theorem 3.4** *If $X$ is an admissible sequence in $M$, $\varphi \circ X$ is still an admissible sequence of points in $N$.*

*Proof.* We need to prove that, for any given compact $K \subseteq N$, the sequence

$\varphi \circ X_{\eta^{K,2}}$ is entirely contained in a connected component of $N \setminus K$,

which we shall denote by $D_K^X$, where $\eta^{K,2}$ is as in **Lemma 3.2**.

First, notice that in the proof of **Lemma 3.3**, if we replace $\eta^{K,2}$ with any $\eta \geq \eta^{K,2}$, there will be a corresponding set

$$J_\eta := \{m \in I\!N : \delta(x_m, x_\eta) < 1\} \neq \emptyset$$

for which,

$$\varphi(x_m) \in B_{\frac{d_{K,2}}{2}}(\varphi(x_\eta)) \cap N \setminus K, \forall m \in J_\eta$$

And so,

$$\varphi \circ X_{\eta^{K,2}} \subseteq \left[ \bigcup_{\eta \geq \eta^{K,2}} B_{\frac{d_{K,2}}{2}}(\varphi(x_\eta)) \right] \cap N \setminus K,$$

Since, by construction, $\bigcup_{\eta \geq \eta^{K,2}} B_{\frac{d_{K,2}}{2}}(\varphi(x_\eta))$ is a union of open balls and pathwise connected, it is open and connected itself.

Thus, $\varphi \circ X_{\eta^{K,2}}$ is entirely contained in some connected component $D_K^X$ of $N \setminus K$.

□

In other words, the above **Theorem 3.4** says that it makes sense to consider the equivalence class $[\varphi \circ X]$ in $N/_\sim$, for any admissible sequence $X$ in $M$.

For any $E = [X]$ end of $M$, consider the following map,

$$\begin{array}{rccc} \tilde{\varphi}: & M/_{\sim} & \Longrightarrow & N/_{\sim} \\ & [X] & \longmapsto & [\varphi \circ X] \end{array}$$

Next **Theorem 3.5** shows that the definition above is consistent, and so a rough isometry takes ends to ends.

**Theorem 3.5** *$\tilde{\varphi}$ as above is well-defined, i.e., for any two given equivalent sequences $X \sim Y$ in $M$ it follows that $\varphi \circ X \sim \varphi \circ Y$ in $N$.*

We will give a proof of this theorem based on the following Lemma,

**Lemma 3.6** *If $K \subseteq N$ is an arbitrary compact set and $X$ and $Y$ are co-final sequences in $M$, it follows that the connected components of $N \setminus K$, corresponding to $X$ and $Y$, as defined in* **Theorem 3.4**, *coincide.*

*Proof of Lemma.* Given any compact $K \subseteq N$, by **Theorem 3.4**,

$$\varphi \circ Z_{\eta_Z^{K,2}} \subseteq D_K^Z, \text{ for } Z = X, Y \tag{3.4}$$

where, $D_K^X$ and $D_K^Y$ are the connected components of $N \setminus K$, corresponding respectively to $X$ and $Y$, as in the proof of **Theorem 3.4**, and $\eta_X^{K,2}, \eta_Y^{K,2}$ are as in **Lemma 3.2**.

We will show that $D_K^X = D_K^Y$.

Let

$$\bar{B} := \overline{B_{a+b+1}(K)} = \{q \in N : d(q,K) \leq a+b+1\}$$

which is a compact subset of $N$.

By **Theorem 3.4**, $\exists \eta_X^{\bar{B},2} = \eta_X^{\bar{B},2}(\bar{B}, x_0, \varphi(x_0), a, b) > 0$ such that,

$$\varphi \circ X_{\eta_X^{\bar{B},2}} \subseteq D_{\bar{B}}^X$$

and $\exists \eta_Y^{\bar{B},2} = \eta_Y^{\bar{B},2}(\bar{B}, y_0, \varphi(y_0), a, b) > 0$ such that,

$$\varphi \circ Y_{\eta_Y^{\bar{B},2}} \subseteq D_{\bar{B}}^Y$$

where $D_{\bar{B}}^X$ and $D_{\bar{B}}^Y$ are connected components of $N \setminus \bar{B} \subseteq N \setminus K$.

For each <u>fixed</u> $m_1 \geq \eta_X^{\bar{B},2}$, and for every $p$ satisfying $\delta(x_{m_1}, p) \leq 1$, one has by **RI.1**,

$$d(\varphi(x_{m_1}), \varphi(p)) \leq a\delta(x_{m_1}, p) + b \leq a + b < a + b + \frac{1}{2} < a + b + 1$$

which implies that

$$\varphi(p) \in \overline{B_{a+b}(\varphi(x_{m_1}))} \subseteq B_{a+b+\frac{1}{2}}(\varphi(x_{m_1})) \subseteq N \setminus K$$

Since, $B_{a+b+\frac{1}{2}}(\varphi(x_{m_1})) \cap D_{\bar{B}}^X \neq \emptyset$

and $B_{a+b+\frac{1}{2}}(\varphi(x_{m_1}))$ is open and connected, we conclude that

$$\varphi(p) \in D_{\bar{B}}^X \subseteq A^X \text{ a connected component of } N \setminus K \tag{3.5}$$

For the same reason, for each <u>fixed</u> $m_2 \geq \eta_Y^{\bar{B},2}$, if $p$ satisfies $\delta(x_{m_2}, p) \leq 1$, one has

$$\varphi(p) \in D_{\bar{B}}^Y \subseteq A^Y \text{ a connected component of } N \setminus K \tag{3.6}$$

Since $\overline{B_{a+b+1}(K)}$ is compact (it is closed and bounded, and $N$ is complete) and $\varphi^-$ is a rough isometry, the set $\tilde{K} := \overline{\varphi^-(\bar{B})}$ is compact.

Because $X$ and $Y$ are cofinal sequences, by **Definition 2.5**, there exists an integer $\eta^{\tilde{K}} > 0$, such that, $\forall m_1, m_2 \geq \eta^{\tilde{K}}$

$$x_{m_1} \text{ and } y_{m_2} \text{ lie in the same connected component } D_{\tilde{K}} \text{ of } N \setminus \tilde{K} \tag{3.7}$$

Let $\hat{\eta}_X^K := \max\{\eta_X^{K,2}, \eta_X^{\bar{B},2}, \eta^{\tilde{K}}\}$ and $\hat{\eta}_Y^K := \max\{\eta_Y^{K,2}, \eta_Y^{\bar{B},2}, \eta^{\tilde{K}}\}$. Notice that the constant $\hat{\eta}_X^K > 0$ depends only on $(K, x_0, \varphi(x_0), a, b, \bar{B}, \tilde{K})$ and the constant $\hat{\eta}_Y^K > 0$ depends only on $(K, y_0, \varphi(y_0), a, b, \bar{B}, \tilde{K})$.

By (3.4), (3.5), (3.6) and (3.7), for each pair $(Z, z) \in \{(X, x), (Y, y)\}$ we have,

$$\begin{aligned} \varphi \circ Z_{\hat{\eta}_Z^K} &\subseteq D_K^Z \\ p \in M : \delta(z_{\hat{\eta}_Z^K}, p) \leq 1 &\Rightarrow \varphi(p) \in A^Z \\ Z_{\hat{\eta}_Z^K} &\subseteq D_{\tilde{K}} \end{aligned} \tag{3.8}$$

Furthermore, for each pair $(Z, z) \in \{(X, x), (Y, y)\}$ the following condition holds,

$$\varphi(z_{\hat{\eta}_Z^K}) \in D_K^Z \cap A^Z \neq \emptyset$$

and thus,

$$A^Z = D_K^Z$$

Also,

$$\varphi(p) \in \bar{B} \Rightarrow p \in \varphi^{-}(\bar{B}) \subseteq \tilde{K} \Rightarrow p \in \tilde{K}$$

which implies that,

$$p \in M \setminus \tilde{K} \Rightarrow \varphi(p) \in N \setminus \bar{B} \subseteq N \setminus K \tag{3.9}$$

By the definition of $D_{\tilde{K}}$,* there exists a continuous curve

$$\gamma : [0,1] \longrightarrow D_{\tilde{K}} \subseteq M \setminus \tilde{K}$$

joining $x_{\hat{\eta}_X^K} = \gamma(0)$ to $y_{\hat{\eta}_Y^K} = \gamma(1)$, entirely contained in $D_{\tilde{K}}$.

Let $\tilde{d} := \delta(\gamma([0,1]), \tilde{K})(. : \tilde{d} > 0)$ and $\tilde{r} := \min(\frac{\tilde{d}}{2}, 1)(. : \tilde{r} > 0)$.

By using the fact that $\gamma([0,1])$ is compact, it is possible to construct a finite covering for it, by open balls with fixed radius $\tilde{r}$, with the following properties:

*Since any manifold $M$ is locally path connected–because balls are path connected–for each open set $A \subset M$, the connected components of $A$ are open subsets of $M$ and thus locally path connected, too. Because, a connected, locally path connected space is path connected, the connected components of $A$ are path connected.

There exists a finite sequence of numbers $\{0 = t_0 < t_1 < \cdots < t_{\ell-1} < t_\ell = 1\}$, for which, the union of the balls of radius $\tilde{r}$, centered at $\gamma(t_\imath)$ is a covering for $\gamma([0,1])$, and there exists another finite sequence of numbers $\{s_0 < s_1 < s_2 < \cdots < s_{\ell-2} < s_{\ell-1}\}$ satisfying $t_\imath < s_\imath < t_{\imath+1}, \imath = 0, 1, \ldots, \ell-1$, such that,

$$\gamma(s_\imath) \in B_{\tilde{r}}(\gamma(t_\imath)) \cap B_{\tilde{r}}(\gamma(t_{\imath+1})), \imath = 0, 1, \ldots, \ell-1 \tag{3.10}$$

By (3.8), we have for each pair $(Z, z) \in \{(X, x), (Y, y)\}$,

$$\varphi\left(B_{\tilde{r}}(z_{\hat{\eta}^K_Z})\right) \subseteq D^Z_K \tag{3.11}$$

Now, for each $\imath = 0, 1, \ldots, \ell$,

$$\begin{aligned} &\forall p \in B_{\tilde{r}}(\gamma(t_\imath)) \\ &\quad \Rightarrow \quad d(\varphi(p), \varphi \circ \gamma(t_\imath)) \le a\delta(p, \gamma(t_\imath)) + b \le a\tilde{r} + b \le a + b \\ &\quad \Rightarrow \quad \varphi(p) \in \overline{B_{a+b}(\varphi \circ \gamma(t_\imath))} \subseteq B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_\imath)) \end{aligned}$$

Moreover, by (3.9) it follows that

$$\gamma(t_\imath) \in M \setminus \tilde{K} \Rightarrow \varphi \circ \gamma(t_\imath) \in N \setminus \bar{B} \subseteq N \setminus K$$

and by the definition of $\bar{B}$, we have

$$B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_\imath)) \subseteq N \setminus K$$

Therefore,

$$\varphi\left(B_{\tilde{r}}(\gamma(t_\imath))\right) \subseteq B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_\imath)) \subseteq N\setminus K \tag{3.12}$$

By (3.10) and (3.12), we conclude that for each $\imath = 0, 1, \ldots, \ell-1$,

$$\begin{aligned}\varphi\circ\gamma(s_\imath) &\in \varphi\left\{B_{\tilde{r}}(\gamma(t_\imath))\cap B_{\tilde{r}}(\gamma(t_{\imath+1}))\right\} \subseteq \\ &\subseteq \varphi\left\{B_{\tilde{r}}(\gamma(t_\imath))\right\}\cap\varphi\left\{B_{\tilde{r}}(\gamma(t_{\imath+1}))\right\} \subseteq \\ &\subseteq B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_\imath))\cap B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_{\imath+1})) \subseteq N\setminus K \end{aligned} \tag{3.13}$$

In particular, for $\imath = 0$, it follows from (3.11) and (3.12), that

$$\varphi(x_{\hat{\eta}^K_X}) = \varphi\circ\gamma(0) \in D^X_K \cap B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_0)) \subseteq N\setminus K$$

and from (3.13),

$$\varphi\circ\gamma(s_0) \in B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_0))\cap B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_1)) \subseteq N\setminus K$$

Then, by the definition of a connected component, we must have,

$$B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_0))\cup B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_1)) \subseteq D^X_K \subseteq N\setminus K$$

To conclude the proof, we will proceed by induction and show that,

$$B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_{\imath+1}))\cup B_{a+b+\frac{1}{2}}(\varphi\circ\gamma(t_{\imath+2})) \subseteq D^X_K$$

holds, by assuming the induction hypothesis,

$$B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_\imath)) \cup B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_{\imath+1})) \subseteq D_K^X$$

where $\imath = 0, 1, \dots, \ell - 2$.

By (3.13), for every $\imath = 0, 1, \dots, \ell - 2$, we have,

$$\varphi \circ \gamma(s_{\imath+1}) \in B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_{\imath+1})) \cap B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_{\imath+2})) \subseteq N \setminus K$$

Thus, by the induction hypothesis and the definition of a connected component of $N \setminus K$,

$$B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_{\imath+2})) \subseteq D_K^X$$

for each $\imath = 0, 1, \dots, \ell - 2$.

In particular, when $\imath = \ell - 2$,

$$\varphi \circ \gamma(t_\ell) \in B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_{\ell-1})) \cup B_{a+b+\frac{1}{2}}(\varphi \circ \gamma(t_\ell)) \subseteq D_K^X$$

Since, by the definition of $y_{\hat{\eta}_Y^K}$,

$$\varphi \circ \gamma(t_\ell) = \gamma(1) = y_{\hat{\eta}_Y^K} \in D_K^Y$$

we can conclude, finally, that the two connected components coincide, i.e., $D_K^X = D_K^Y$.

$\square$

*Proof of Theorem 3.5.* Let $E = [X]$ be an end of $M$ and $Y \in E = [X]$, in other words, $X$ and $Y$ are cofinal sequences in $M$.

Given any compact $K \subseteq N$, by **Lemma 3.6**, there exists a connected component $D_K^{X,Y} := D_K^X = D_K^Y$, of $N \setminus K$, such that,

$$\varphi \circ Z_{\hat{\eta}_Z^{K,2}} \subseteq D_K^{X,Y}, \text{ for } Z = X, Y \tag{3.14}$$

where $\hat{\eta}_Z^{K,2}$ are defined in **Lemma 3.6**.

Let us define a new constant, by

$$\eta^K := \max\{\hat{\eta}_X^{K,2}, \hat{\eta}_Y^{K,2}\}$$

Notice that the dependence of the new constant $\eta^K$ is described by $\eta^K(K, x_0, y_0, \varphi(x_0), \varphi(y_0), a, b, \tilde{K}) > 0$.†

Therefore, by (3.14), we have,

$$\varphi \circ X_{\eta^K} \subseteq D_K^{X,Y} \text{ and } \varphi \circ Y_{\eta^K} \subseteq D_K^{X,Y}$$

and, the two admissible sequences $\varphi \circ X$ and $\varphi \circ Y$ in $N$, are cofinal.

In other words,

$$\tilde{\varphi}([X]) = [\varphi \circ X] = [\varphi \circ Y] = \tilde{\varphi}([Y])$$

and thus, $\tilde{\varphi}$ is well-defined.

□

†see definition of $\tilde{K}$ in the proof of **Lemma 3.6**

# Chapter 4

# Mappings with Maximal Rank

We begin this Chapter by introducing key tools along with their properties that were used in this research. We use the jargon of fiber bundles throughout. Properties of *Fibers* of $M$ are described in **Chapter 2**.

Let $M$ and $B$ be Riemannian manifolds with dimensions $m$ and $n$, respectively, where $m \geq n$. Let $\pi : M \to B$ denote a mapping with maximal rank $n$ from $M$ onto $B$.

We first show that the fibers at each $b \in B$ are submanifols of $M$ of dimension $m - n$.

**Lemma 4.1** *For each $b \in B$, the fiber of $b$ denoted by $\pi^{-1}(b)$, is a submanifold of $M$ of dimension $m - n$.*

*Proof.* Let $x \in \pi^{-1}(b)$.

Since $\pi$ has maximal rank $n$, we may choose a direct sum decomposition

$\mathbb{R}^{m-n} \oplus \mathbb{R}^n = \mathbb{R}^m$, such that, in coordinate systems the restriction of $\pi_*$ to $\mathbb{R}^n$ is an isomorphism.

In that case we may apply the Inverse Function Theorem to find an open neighborhood $Z \in \mathbb{R}^m$ of $x$ in coordinates, such that $Z$ is the graph of a smooth function $g_b$ defined in an open neighborhood of $proj_{\mathbb{R}^{m-n}}(x)$ , which is the projection of $x$ in coordinates onto $\mathbb{R}^{m-n}$.

Therefore, we can define a smooth coordinate map of an open neighborhood of $x$ in $\pi^{-1}(b)$ into $\mathbb{R}^{m-n}$, by using the smooth function $g_b$, and thus conclude, by the Maximal Rank Theorem, that $\pi^{-1}(b)$ is an $(m-n)-$dimensional submanifold of $M$.

$\square$

As a generalization of **Lemma 4.1** we have the following Proposition,

**Proposition 4.2** *Let $\gamma : [t_1, t_2] \longrightarrow B$ a smooth embedded closed curve. Then $\pi^{-1}(\gamma) := \{x \in M : \pi(x) \in \gamma([t_1, t_2])\}$ is a closed submanifold of $M$ of dimension $(m-n)+1 = \dim(\pi^{-1}(b)) + 1$, for any $b \in B$.*

*Proof.*

Let $x \in \pi^{-1}(\gamma)$ and $b = \pi(x) = \gamma(\bar{t})$, where $\bar{t} \in [t_1, t_2]$.

That $\pi^{-1}(\gamma) := \{x \in M : \pi(x) \in \gamma([t_1, t_2])\}$ is closed it follows from $\pi$

being continuous and $\gamma : [t_1, t_2]$ being closed.

To show that $\pi^{-1}(\gamma)$ is a submanifold of $M$ of dimension $(m-n)+1$, we proceed as in the proof of **Lemma 4.1**, with the additional remark that $\gamma$ is an embedding.

In that case, we can define by means of The Inverse Function Theorem, a smooth function $g_{\bar{t}}$ defined in an open neighborhood of $(proj_{\mathbb{R}^{m-n}}(x), \bar{t})$ in $\mathbb{R}^{m-n} \times \mathbb{R}$, which can be used to define a smooth coordinate map of an open neighborhood of $x$ in $\pi^{-1}(\gamma) \subseteq M$ onto $\mathbb{R}^{m-n} \times \mathbb{R} = \mathbb{R}^{(m-n)+1}$.

Therefore, we conclude by the Maximal Rank Theorem that $\pi^{-1}(\gamma)$ is an $((m-n)+1)$-dimensional submanifold of $M$.

□

Since each derivative map $\pi_*$ of $\pi$ is surjective, we can define the projections of the tangent space of $M$ onto the subspaces of horizontal and vertical vectors, and denote them respectively by $(VT)_x$ and $(HT)_x$ for each $x \in M$. In that case, we can decompose each tangent space to $M$ into a direct orthogonal sum $T_xM = (VT)_x \oplus (HT)_x$, which is justified by **Lemma 4.3**.

Next, we show that at each $x \in M$ the subspace of vertical vectors is the kernel of the derivative of $\pi$ at $x$.

**Lemma 4.3** $(VT)_x = \ker(\pi_*)_x, \forall x \in M$

*Proof.* First, we will show that

$$(VT)_x \subseteq \ker(\pi_*)_x, \quad \forall x \in M \tag{4.1}$$

Let $x \in M, b = \pi x, v \in (VT)_x = T_x\pi^{-1}(b)$, and let $c : [0,1] \to \pi^{-1}(b)$ be a differentiable curve satisfying $c(0) = x$ and $c'(0) = v$.

Since $\pi \circ c(t) = b, \ \forall t \in [0,1]$, it follows that $(\pi_*)_{c(t)}(c'(t)) = 0$. In particular, $v \in \ker(\pi_*)_x$.

Now, by **Lemma 4.1** $\dim\ker(\pi_*)_x = m-n = \dim T_x\pi^{-1}(b) = \dim(VT)_x$, which together with (4.1) give us $(VT)_x \subseteq \ker(\pi_*)_x$.

Notice that $(HT)_x := (VT)_x^\perp = (\ker(\pi_*)_x)^\perp$.

Thus, $T_xM = (\ker(\pi_*)_x) \oplus (\ker(\pi_*)_x)^\perp = (VT)_x \oplus (HT)_x$.

□

From the maximality of the rank of the onto mapping $\pi$ we have the unique horizontal vector property, as follows.

**Lemma 4.4** *Let $b \in B$. Given any $w \in T_bB$ and $x \in M$ satisfying $\pi(x) = b$, there exists a unique horizontal vector $v \in T_xM$ which is $\pi$-related to $w$, i.e. satisfying $v \in (HT)_x$ and $(\pi_*)_x(v) = w$.*

*Proof.* Let $b, w$ and $x$ be as above.

Since $(\pi_*)_x$ is onto, there exists $u \in (VT)_x$ and $v \in (HT)_x$ such that $(\pi_*)_x(u+v) = w$, where $u + v \in T_xM = (VT)_x \oplus (HT)_x$.

By **Lemma 4.3**, we have $w = (\pi_*)_x(u+v) = (\pi_*)_x(v)$, which proves the existence for the claim.

For the uniqueness, assume there exists $\bar{v} \in T_xM$ satisfying $\bar{v} \in (HT)_x$ and $(\pi_*)_x(\bar{v}) = w$.

So, $(\pi_*)_x(\bar{v}) = (\pi_*)_x(v) \Rightarrow (\pi_*)_x(\bar{v} - v) = 0 \Rightarrow \bar{v} - v \in \ker(\pi_*)_x = (VT)_x$, but $\bar{v} - v \in (HT)_x$ implies that $\bar{v} - v \in (VT)_x \cap (HT)_x = \{0\}$, and thus $\bar{v} = v$.

$\square$

In particular, under additional assumptions on the space of horizontal vectors, we can compare lengths of minimal geodesics in $M$ with those in $B$, according to the following Lemma, which will be used in several proofs ahead.

**Lemma 4.5** *Assume that $M$ and $B$ are both connected and geodesically complete. Let $x, x' \in M$, $\Gamma_{min} \subset M$ be a minimal geodesic joining $x$ to $x'$, and $\gamma_{min} \subset B$ be a minimal geodesic joining $\pi(x)$ to $\pi(x')$. Suppose that for all $b \in B$ and for all $x \in F_b$ there exist constants $\alpha \geq 1$ and $\beta > 0$, both* *independent of $b$ and $x$, such that*

$$\frac{1}{\alpha}||w||_B - \beta \leq ||v||_M$$

*for all $w \in T_bB$, where $v$ is the unique horizontal lift of $w$ through $x$ that we assume satisfies $||v||_M \leq 1$, and $||\ \ ||_M$, $||\ \ ||_B$ denote the inner product on $TM$ and $TB$, respectively.*

*Then,*

$$d_M(x,x') = \ell(\Gamma_{min}) \geq \frac{1}{\alpha}\ell(\gamma_{min}) - \beta = \frac{1}{\alpha}d_B(\pi(x),\pi(x')) - \beta$$

*Proof.* Firstly, we note that in the assumption, we assumed that the horizontal lift $v \in (HT)_x$ of $w$ satisfies $||v||_M \leq 1$. Otherwise, if $||v||_M > 1$ we define $\tilde{v} := \dfrac{v}{||v||_M}$, with the properties

- $\tilde{v} := \frac{v}{||v||_M} \in (HT)_x$
- $||\tilde{v}||_M = 1$
- $(\pi_*)_x(\tilde{v}) = \dfrac{w}{||v||_M}$

and if we use $\frac{w}{||v||_M}$ and $\tilde{v}$ in the assumption, we thus obtain the equivalent inequality,

$$\frac{1}{\alpha}\left|\left|\frac{w}{||v||_M}\right|\right|_B - \beta \leq ||\tilde{v}||_M \Rightarrow \frac{1}{\alpha}\frac{||w||_B}{||v||_M} - \beta \leq \frac{||v||_M}{||v||_M} \Rightarrow$$

$$\Rightarrow \frac{1}{\alpha}||w||_B - \beta||v||_M \leq ||v||_M \Rightarrow \frac{1}{\alpha}||w||_B \leq (\beta+1)\,||v||_M \Rightarrow$$
$$\Rightarrow \frac{1}{\alpha\,(\beta+1)}||w||_B \leq ||v||_M$$

for $w \in T_bB$, where $v$ is the unique horizontal lift of $w$ through $x$ with $||v||_M > 1$.

For the sake of simplicity, assume that both parametrizations of $\Gamma_{min}$ and $\gamma_{min}$ are defined in the interval $[0,1]$.

Since, $T_xM = (VT)_x \oplus (HT)_x$, we can write the tangent vectors to $\Gamma_{min}$ in the form $\Gamma'_{min}(t) = \Gamma'_V(t) \oplus \Gamma'_H(t), \forall t \in [0,1]$.

Notice that by **Lemma 4.4**, $\Gamma'_H(t) \in T_{\Gamma_{min}(t)}M$ is the unique horizontal vector which is $\pi-$related to $\frac{d}{dt}\left(\pi \circ \Gamma_{min}\right)(t)$, for each $t \in [0,1]$. Assume that $\Gamma$ is p.p.a.l. so that $||\Gamma'_H(t)||_M \leq 1$.

We have,

$$d_M(x,x') = \ell(\Gamma_{min}) = \int_0^1 ||\Gamma'_V(t) \oplus \Gamma'_H(t)||_M dt \geq \int_0^1 ||\Gamma'_H(t)||_M dt \overset{\text{hyp.}}{\geq}$$
$$\geq \frac{1}{\alpha}\int_0^1 \left|\left|\frac{d}{dt}\left(\pi \circ \Gamma_{min}\right)(t)\right|\right|_M dt - \beta = \frac{1}{\alpha}\ell(\pi \circ \Gamma_{min}) - \beta \overset{\text{min.geod.}}{\geq}$$
$$\geq \frac{1}{\alpha}\ell(\gamma_{min}) - \beta \overset{\text{dist.}}{=} \frac{1}{\alpha}d_B(\pi(x),\pi(x')) - \beta$$

which concludes the Lemma.

□

Let $\gamma$ be a smooth embedded curve in $B$ and $\Gamma$ be any curve in $M$

satisfying $\pi \circ \Gamma = \gamma$. $\Gamma$ is called a *lift* of $\gamma$.

Next, **Proposition 4.8** proves the uniqueness of a horizontal lift of a curve in $B$, through a point $x \in M$.

We will use the following **Propositions 4.6, 4.7** in the proof of **Proposition 4.8**.

**Proposition 4.6** *(Fundamental Theorem for Systems of Ordinary Differential Equations). Let $f(t,\zeta,\xi)$ be a family of $n$ functions defined in $\mid t \mid < \delta$ and $(\zeta,\xi) = (\zeta_1,\ldots,\zeta_n,\xi_1,\ldots,\xi_m) \in \mathcal{D}$, where $\mathcal{D}$ is an open set in $\mathbb{R}^{n+m}$. If $f(t,\zeta,\xi)$ is continuous in $t$ and differentiable of class $C^1$ in $\zeta$, then there exists a unique family $\varphi(t,\eta,\xi)$ of $n$ functions defined in $\mid t \mid < \delta'$ and $(\eta,\xi) \in \mathcal{D}'$, where $0 < \delta' < \delta$ and $\mathcal{D}'$ is an open subset of $D$, such that:*

1. *$\varphi(t,\eta,\xi)$ is differentiable of class $C^1$ in $t$ and $\eta$;*
2. $\dfrac{\partial\varphi(t,\eta,\xi)}{\partial t} = f(t,\varphi(t,\eta,\xi),\xi)$;
3. $\varphi(0,\eta,\xi) = \eta$.

*If $f(t,\zeta,\xi)$ is differentiable of class $C^p, 0 \le p \le \omega$, in $t$ and of class $C^q, 1 \le q \le \omega$, in $\zeta$ and $\xi$, then $\varphi(t,\eta,\xi)$ is differentiable of class $C^{p+1}$, in $t$ and of class $C^q$, in $\eta$ and $\xi$. If we consider the system of differential equations:*

$$\frac{d\zeta}{dt} = f(t,\zeta,\xi)$$

*which depend on the parameters $\xi$, then $\zeta = \varphi(t, \eta, \xi)$ is called the solution satisfying the initial condition $\zeta = \eta$ when $t = 0$.*

*Proof.* (see [20], Appendix 1).

□

**Proposition 4.7** *Let $X$ be a vector field on an $m$-manifold $M$. For any point $x_0$ of $M$, there is a unique integral curve $\Gamma(t)$ of $M$, defined for $\mid t \mid < \epsilon$ for some $\epsilon > 0$, such that $x_0 = \Gamma(0)$.*

*Proof.* (see [20], Chapter I.1) Let $(\xi_1, \dots, \xi_m)$ be a local coordinate system in a neighborhood $U$ of $x_0$ and let $X = \sum_{\iota=1}^{m} \theta_\iota \frac{\partial}{\partial \xi^\iota}$ in $U$. Then an integral curve is a solution of the following system of ordinary differential equations:

$$\frac{d\xi_\iota}{dt} = \theta_\iota \left(\xi_1(t), \dots, \xi_m(t)\right), \qquad \iota = 1, \dots, m.$$

Our assertion follows from **Proposition 4.6**.

□

We now state and prove **Proposition 4.8**.

**Proposition 4.8** *Given $x_0 \in M$ and $\gamma : [t_1, t_2] \to B$ a smooth embedded curve starting at $b := \pi(x_0)$, i.e. with $\gamma(t_1) = \pi(x_0) = b$, there exists a unique smooth curve $\Gamma : [t_1, t_2] \to M$ satisfying*

**(hl.1)** $\pi \circ \Gamma = \gamma$,

**(hl.2)** $\Gamma(t_1) = x_0$, *and*

**(hl.3)** $\Gamma$ *is horizontal, i.e.,* $\Gamma'(t) \in (HT)_{\Gamma(t)}$.

$\Gamma$ *is called the unique* <u>*horizontal lift*</u> *of* $\gamma$ *through* $x_0$.

*Proof.* For a given $x \in \pi^{-1}(\gamma) \subseteq M$, since $\gamma$ is an embedded curve, there exists a unique $t_x \in [t_1, t_2]$ such that $\pi(x) = \gamma(t_x)$.

By **Lemma 4.4** there exists a unique horizontal vector $\pi$-related to $\gamma'(t_x)$, i.e. $v_x \in (HT)_x \subseteq T_xM$ and $(\pi_*)_x(v_x) = \gamma'(t_x)$, which gives us a vector field $X$ on $\pi^{-1}(\gamma)$ defined by

$$\begin{aligned} \pi^{-1}(\gamma) \subseteq M \quad &\rightarrow \quad (HT)_x \subseteq T_xM \\ x \quad &\mapsto \quad X_x := v_x = \{(\pi_*)_x\}^{-1} \circ \{\gamma'(t_x)\} \end{aligned}$$

Since $\pi^{-1}(\gamma) \subset M$ is a closed submanifold, $(\pi_*)$ is smooth in $x \in \pi^{-1}(\gamma)$, $(\pi_*)_x \mid_{(HT)_x}$ is an isomorphism, and $\gamma'$ provides a smooth vector field on $B$, it follows that the vector field $X$ is smooth in $x \in \pi^{-1}(\gamma) \subset M$.

From **Proposition 4.7** there exists a unique smooth curve $\Gamma(t)$ entirely contained in $\pi^{-1}(\gamma) \subseteq M$ such that $x_0 = \Gamma(t_1)$, $\Gamma'(t) = X_{\Gamma(t)} = v_{\Gamma(t)} \in (HT)_{\Gamma(t)}, \forall t \in [t_1, t_2]$ and $\Gamma([t_1, t_2]) \subseteq \pi^{-1}(\gamma)$, which is the required result.

We finally notice that according to **Proposition 4.6**, $\Gamma$ depends smoothly on the initial condition $x_0$.

□

Let $\gamma : [t_1, t_2] \to B$ denote a smooth embedded curve and let its lift be denoted by $\Gamma : [t_1, t_2] \to M$.

In the next two Propositions, under certain assumptions on the derivative of the submersion $\pi$, we have control over the length of the lift of a curve. For instance, in **Proposition 4.9** for a long curve $\gamma$ in $B$ its lift $\Gamma$ in $M$ cannot be short, and **Proposition 4.11** the length of a lift $\Gamma$ of a long curve $\gamma$ is bounded above by the length of $\gamma$.

We denote by $|| \ ||_M$ and $|| \ ||_B$ the Riemannian norms in $TM$ and $TB$, respectively.

**Proposition 4.9** *Assume there are constants $\alpha \geq 1$ and $\beta > 0$ such that,*

$$||(\pi_*)_x v||_B \leq \alpha \, ||v||_M + \beta \tag{4.2}$$

*for all $x \in M$, for all $v \in T_xM$ satisfying $||v||_M \leq 1$.*

*If $\gamma$ is any smooth long curve in $B$, then,*

$$\ell(\Gamma) \geq \frac{1}{\alpha} \left[ \ell(\gamma) - \beta(t_2 - t_1) \right] > 0$$

*where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves $\Gamma$ and $\gamma$, respectively.*

*Proof.* First, we choose a parametrization proportional to arc length of $\Gamma : t \in [t_1, t_2] \to \Gamma(t) \in M$, an arbitrary lift of $\gamma \subset B$. We may assume w.l.o.g. that $||\Gamma'(t)||_M \leq 1$.

If we use $v = \Gamma'(t)$ in (4.2), we obtain

$$\left|\left|(\pi_*)_{\Gamma(t)}\Gamma'(t)\right|\right|_B \leq \alpha\,||\Gamma'(t)||_M + \beta, \qquad \forall t \in [t_1, t_2]$$

Since $\pi \circ \Gamma = \gamma$, we can rewrite the above inequality as

$$||\gamma'(t)||_B \leq \alpha\,||\Gamma'(t)||_M + \beta, \qquad \forall t \in [t_1, t_2] \tag{4.3}$$

Finally, if we integrate (4.3), we get

$$\begin{aligned} \ell(\gamma) &= \int_{t_1}^{t_2} ||\gamma'(t)||_B\, dt \leq \alpha \int_{t_1}^{t_2} ||\Gamma'(t)||_M\, dt + \beta \int_{t_1}^{t_2} dt = \\ &= \alpha \cdot \ell(\Gamma) + \beta(t_2 - t_1) \Rightarrow \\ &\Rightarrow \ell(\Gamma) \geq \tfrac{1}{\alpha}\left[\ell(\gamma) - \beta(t_2 - t_1)\right] > 0 \end{aligned}$$

which proves the proposition.

That the second hand side of the last inequality above is positive follows from the assumption that $\gamma$ is a long curve. Therefore, as it can be interpreted from the inequality shown, for a long curve $\gamma$ its lift $\Gamma$ cannot be short.

□

Now, in the proof of **Proposition 4.11** we will need what is stated and proved in the following Lemma: for a long curve in $B$, its lift in $M$ is non-vertical.

**Lemma 4.10** *Let* $\gamma : [t_1, t_2] \to B$ *be a smooth embedded curve in* $B$ *and let* $\Gamma : [t_1, t_2] \to M$ *be its lift. If* $\gamma$ *is a long curve, then,* $\Gamma$ *is non-vertical, i.e. there exists an interval* $[t_1, t_2]$*, such that,*

$$(\Gamma'(t))_H \neq 0, \qquad \forall t \in [t_1, t_2]$$

*where* $\Gamma'(t) = (\Gamma'(t))_V \oplus (\Gamma'(t))_H \in T_{\Gamma(t)}M = (VT)_{\Gamma(t)} \oplus (HT)_{\Gamma(t)}$.

*Proof.* Since $\gamma$ is a smooth, embedded long curve, there exists an interval let us say $[t_1, t_2]$, for which,

$$\gamma'(t) \neq 0, \ \ \forall t \in [t_1, t_2]$$

Moreover, since for all $t \in [t_1, t_2]$ the derivative map $(\pi_*)_{\Gamma(t)} \mid_{(HT)_{\Gamma(t)}}$ is an isomorphism, we thus obtain

$$(\pi_*)_{\Gamma(t)} \, (\Gamma'(t))_H = (\pi_*)_{\Gamma(t)} \, \{(\Gamma'(t))_V \oplus (\Gamma'(t))_H\} = (\pi_*)_{\Gamma(t)} \, \{\Gamma'(t)\} = \gamma'(t) \neq 0$$

$$\overset{isom.}{\Longrightarrow} (\Gamma'(t))_H \neq 0$$

for all $t \in [t_1, t_2]$, and thus $\Gamma$ is non-vertical.

□

**Proposition 4.11** *Let $\gamma : [t_1, t_2] \to B$ denote a smooth embedded curve and let $\Gamma : [t_1, t_2] \to M$ denote its lift. For horizontal*[(†)] *vectors $v \in TM$ only, assume that there is a universal constant $\alpha \geq 1$ such that,*

$$||(\pi_*)_x v||_B \geq \frac{1}{\alpha}||v||_M - \beta \tag{4.4}$$

*for all $x \in M$, for all $v \in T_xM \setminus (VT)_x = (HT)_x = [\ker(\pi_*)_x]^{\perp}$.*

*If $\gamma$ is a long curve then,*

$$\ell(\Gamma) \leq \alpha\,[\ell(\gamma) + \beta(t_2 - t_1)]$$

*where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves $\Gamma$ and $\gamma$, respectively.*

*Proof.* We first notice that because $\gamma$ is a long curve, by **Lemma 4.10**, $\Gamma$ is non-vertical.

If we use $v = \Gamma'(t)$ in (4.4), we may write

$$0 \overset{(\dagger)}{\neq} ||(\pi_*)_{\Gamma(t)}\Gamma'(t)||_B \geq \frac{1}{\alpha}||\Gamma'(t)||_M - \beta, \qquad \forall t \in [t_1, t_2]$$

and using $\pi \circ \Gamma = \gamma$ in the above inequality, we obtain

$$0 \neq ||\gamma'(t)||_B \geq \frac{1}{\alpha}||\Gamma'(t)||_M - \beta, \qquad \forall t \in [t_1, t_2]$$

which in turn implies that

$$||\Gamma'(t)||_M \leq \alpha\,(||\gamma'(t)||_B + \beta), \qquad \forall t \in [t_1, t_2]$$

Finally, integrating the above inequality gives us

$$\begin{aligned}
\ell(\Gamma) &= \int_{t_1}^{t_2} ||\Gamma'(t)||_M dt \leq \\
&\leq \alpha \int_{t_1}^{t_2} ||\gamma'(t)||_B dt + \beta\alpha \int_{t_1}^{t_2} dt = \alpha \cdot \ell(\gamma) + \beta\alpha(t_2 - t_1) = \\
&= \alpha \left[\ell(\gamma) + \beta(t_2 - t_1)\right] > 0
\end{aligned}$$

which proves the proposition.

Therefore, as it can be interpreted from the above inequality, the length of a lift $\Gamma$ of a long curve $\gamma$ is bounded above by the length of $\gamma$.

□

Next, for onto smooth mappings with maximal rank, we define special diffeomorphisms between any two fibers, a useful tool that will feature in many of our proofs. In **Theorem 4.12** we give its definition and state and prove several of its properties.

**Theorem 4.12** *Let $\pi : M \to B$ be an onto smooth map with maximal rank, where $B$ is connected. Let $b_1, b_2$ be distinct elements of $B$ and let $\gamma \subset B$ be a piecewise smooth embedded curve joining $b_1$ and $b_2$ and parametrized proportionally to arc-length (p.p.a.l.), given by $\gamma : [t_1, t_2] \longrightarrow B$, where $\gamma(t_1) = b_1, \gamma(t_2) = b_2$.*

*If we denote by $\pi^{-1}(b_1) = F_{b_1}$ and $\pi^{-1}(b_2) = F_{b_2}$ their corresponding* *fibers, we thus define the map*

$$\varphi_{(\gamma)} : F_{b_1} \longrightarrow F_{b_2}$$

*by the following rule:*

*Given* $x \in F_{b_1}$, *let* $\Gamma_x$ *be the unique horizontal lift of* $\gamma$ *through* $x$, *and set*

$$\varphi_{(\gamma)}(x) := \Gamma_x(t_2) \in F_{b_2}$$

*Then, the map* $\varphi_{(\gamma)}$ *has the following properties:*

1. $\varphi_{(\gamma)} : F_{b_1} \longrightarrow F_{b_2}$ *is well-defined;*

2. *Let* $\gamma_1 : [t_1, t_2] \longrightarrow B$ *and* $\gamma_2 : [t_2, t_3] \longrightarrow B$ *be smooth embedded curves parametrized proportional to arc-length and satisfying* $\gamma_1(t_2) = \gamma_2(t_2)$. *If* $\gamma_3$ *is the composition of* $\gamma_1$ *and* $\gamma_2$, *denoted by* $\gamma_3 = \gamma_2 \circ \gamma_1$, *and defined as follows:*

$$\begin{array}{rccl} \gamma_3 : & [t_1, t_3] & \longrightarrow & B \\ & t & \longmapsto & \gamma_3(t) := \left\{ \begin{array}{ll} \gamma_1(t), & \text{if } t_1 \leq t \leq t_2 \\ \gamma_2(t), & \text{if } t_2 \leq t \leq t_3 \end{array} \right. \end{array}$$

   *Then* $\varphi_{(\gamma_3)} : F_{\gamma_1(t_1)} \longrightarrow F_{\gamma_2(t_3)}$ *satisfies* $\varphi_{(\gamma_3)} = \varphi_{(\gamma_2)} \circ \varphi_{(\gamma_1)}$.

3. $\varphi_{(\gamma)} : F_{b_1} \longrightarrow F_{b_2}$ *is a diffeomorphism;*

4. $\varphi_{(\gamma)}$ *depends continuously on* $\gamma$, *i.e., for sufficiently small displacements of* $\gamma$ *within a tubular neighborhood, keeping the endpoints fixed,*

*their horizontal lifts through $x$ lie entirely within any given small tubular neighborhood of $\Gamma_x$. Furthermore, since horizontal lifts are, by definition, curves tangent to $(HT) \subset TM$, this fact forces the end-points of horizontal lifts through $x$ to belong to arbitrary balls around $\varphi_{(\gamma)}(x) = \Gamma_x(t_2)$, as long as those horizontal lifts through $x$ lie entirely within a sufficiently small tubular neighborhood of $\Gamma_x$.*

5. *A group of diffeomorphisms between fibers is defined as follows.*

   *Let $b \in B$ be a fixed base point and $\gamma : [t_1, t_2] \longrightarrow B$ be a piece-wise smooth embedded geodesic loop parametrized proportionally to arc-length, where $\gamma(t_1) = \gamma(t_2) =: b$. The properties of $\varphi_{(\gamma)} : F_b \longrightarrow F_b$, imply that the set of mappings $G_b$ given by,*

   $$G_b := \{\varphi_{(\gamma)} : F_b \to F_b,\ \ \forall\gamma \text{ piec. smooth embedded geodesic loop at } b\}$$

   *defines a group of diffeomorphisms of the fiber $F_b$, which is called the* ***holonomy*** *group of the assignment,*

   $$x \in M \mapsto (HT)_x \subset T_xM$$

   *with reference to the point $b \in B$.*

*Proof.* We shall proceed proving item by item.

*Proof of 1.* $\varphi_{(\gamma)}$ is well-defined.

Indeed, since $\gamma$ is fixed and for each $x \in F_{b_1}$, $\Gamma_x$ is unique, the map $\varphi_{(\gamma)}$ is well-defined (see Fig. 4.1).

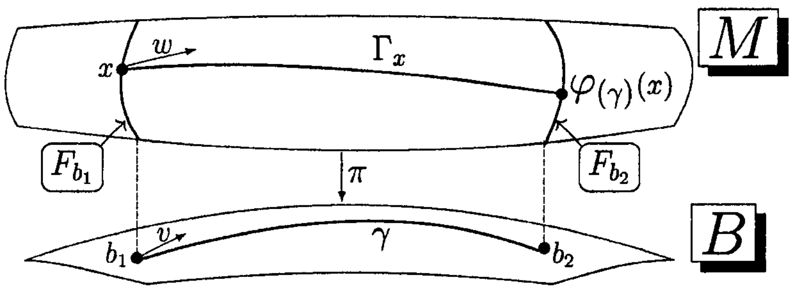


Figure 4.1: The map $\varphi_{(\gamma)}$.

*Proof of 2.* $\varphi_{(\gamma_3)} = \varphi_{(\gamma_2)} \circ \varphi_{(\gamma_1)}$.

Let $\gamma_\iota : [t_\iota, t_{\iota+1}] \longrightarrow B$, $\iota = 1, 2$, as it was introduced in *2.* For every $x_\iota \in F_{\gamma_\iota(t_\iota)}$, the following are well-defined:

$$\begin{array}{rccl} \varphi(\gamma_\iota): & F_{\gamma_\iota(t_\iota)} & \longrightarrow & F_{\gamma_\iota(t_{\iota+1})} \\ & x_\iota & \longmapsto & \Gamma^\iota_{x_\iota}(t_{\iota+1}) \end{array}$$

where $\Gamma^\iota_{x_\iota} : [t_\iota, t_{\iota+1}] \to M$ is the unique horizontal lift of $\gamma_\iota$ through $x_\iota$.

Let $x \in F_{\gamma_1(t_1)}$ and define $\hat{\Gamma}$ by

$$\begin{array}{rccl} \hat{\Gamma}: & [t_1, t_3] & \longrightarrow & M \\ & t & \longmapsto & \hat{\Gamma}(t) := \begin{cases} \Gamma^1_x(t), & \text{if } t_1 \le t \le t_2 \\ \Gamma^2_{\{\Gamma^1_x(t_2)\}}(t), & \text{if } t_2 \le t \le t_3 \end{cases} \end{array}$$

We want to show that $\hat{\Gamma}(t) = \Gamma^3_x(t), \forall t \in [t_1, t_3]$, where $\Gamma^3_x : [t_1, t_3] \to M$ is the unique horizontal lift of $\gamma_3 = \gamma_2 \circ \gamma_1$ through $x$.

Therefore, we must verify properties **(hl.1,2,3)** of $\Gamma^3_x$ (see **Proposition 4.8**) for $\hat{\Gamma}$.

**(hl.1)** $\hat{\Gamma}(t_1) = \Gamma^1_x(t_1) = x$

**(hl.2)** $\pi \circ \hat{\Gamma}(t) =$

$$\begin{aligned} &= \left\{ \begin{array}{lcll} \pi \circ \Gamma^1_x(t) & = & \gamma_1(t), & \text{if } t_1 \leq t \leq t_2 \\ \pi \circ \Gamma^1_x(t_2) & = & \gamma_1(t_2) = \gamma_2(t_2) & = \pi \circ \Gamma^2_{\Gamma^1_x(t_2)}(t_2) \\ \pi \circ \Gamma^2_{\Gamma^1_x(t_2)}(t) & = & \gamma_2(t), & \text{if } t_2 \leq t \leq t_3 \end{array} \right\} = \\ &= \gamma_3(t), \quad \forall t : t_1 \leq t \leq t_3 \end{aligned}$$

**(hl.3)** $\dfrac{d}{dt}\hat{\Gamma}(t) =$

$$= \left\{ \begin{array}{lcl} \dfrac{d}{dt}\Gamma^1_x(t) & \in & (HT)_{\Gamma^1_x(t)} \subseteq T_{\Gamma^1_x(t)}M, \quad \forall t : t_1 \leq t < t_2 \\ \dfrac{d}{dt}\Gamma^2_{\Gamma^1_x(t_2)}(t) & \in & (HT)_{\Gamma^2_{\Gamma^1_x(t_2)}(t)} \subseteq T_{\Gamma^2_{\Gamma^1_x(t_2)}}M, \quad \forall t : t_2 < t \leq t_3 \end{array} \right.$$

which are horizontal vectors, for all $t : t_1 \leq t < t_2 \vee t_2 < t \leq t_3$.

It remains to verify **(hl.3)** for the case $t = t_2$, for we will show that

$$\frac{d}{dt}\Gamma^1_x(t_2) = \frac{d}{dt}\Gamma^2_{\Gamma^1_x(t_2)}(t_2)$$

We have the following

$$\begin{aligned} \gamma_1(t_2) &= \gamma_2(t_2) \\ \frac{\gamma_1'(t_2)}{||\gamma_1'(t_2)||} &= \pm\frac{\gamma_2'(t_2)}{||\gamma_2'(t_2)||} \\ (\pi_*)_{\Gamma^1_x(t)}\left(\frac{d}{dt}\Gamma^1_x(t)\right) &= \frac{d}{dt}\gamma_1(t), \quad \forall t : t_1 \leq t \leq t_2 \end{aligned}$$

$$\begin{aligned}
(\pi_*)_{\Gamma^2_{\Gamma^1_x(t_2)}(t)}\left(\frac{d}{dt}\Gamma^2_{\Gamma^1_x(t_2)}(t)\right) &= \frac{d}{dt}\gamma_2(t), \quad \forall t : t_2 \le t \le t_3 \\
\Gamma^1_x(t_2) &= \Gamma^2_{\Gamma^1_x(t_2)}(t_2)
\end{aligned}$$

which implies by uniqueness of horizontal lifts that $\frac{d}{dt}\Gamma^1_x(t_2) = \frac{d}{dt}\Gamma^2_{\Gamma^1_x(t_2)}(t_2)$.

Thus, $\frac{d}{dt}\hat{\Gamma}(t)$ is well-defined and horizontal, for all $t : t_1 \le t \le t_3$.

After verifying **(hl.1,2,3)** for $\hat{\Gamma}$, we conclude, by uniqueness of horizontal lifts, that $\hat{\Gamma}(t) = \Gamma^3_x(t)$, for all $t : t_1 \le t \le t_3$.

Finally, we obtain

$$\begin{aligned}
\varphi(\gamma_3)(x) &= \Gamma^3_x(t_3) = \hat{\Gamma}(t_3) = \Gamma^2_{\Gamma^1_x(t_2)}(t_3) \\
\varphi(\gamma_2)\circ\varphi(\gamma_1)(x) &= \varphi(\gamma_2)(\Gamma^1_x(t_2)) = \Gamma^2_{\Gamma^1_x(t_2)}(t_3)
\end{aligned}$$

and since $x \in F_{\gamma_1(t_1)}$ is arbitrary, we conclude that $\varphi(\gamma_3) = \varphi(\gamma_2)\circ\varphi(\gamma_1)$ (see Fig. 4.2).

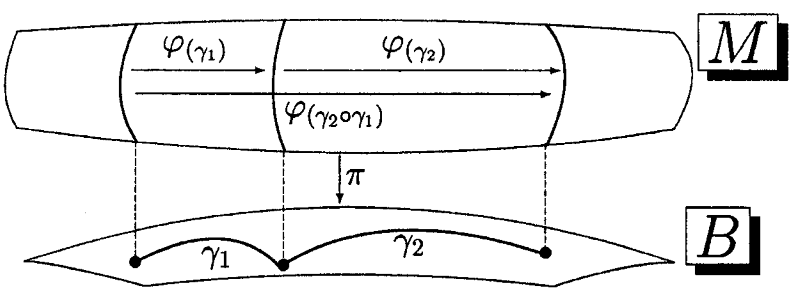


Figure 4.2: Property $\varphi(\gamma_2 \circ \gamma_1) = \varphi(\gamma_2)\circ\varphi(\gamma_1)$.

*Proof of 3.* $\varphi_{(\gamma)} : F_{b_1} \longrightarrow F_{b_2}$ is a diffeomorphism.

We will prove this in 3 steps. In (*Step 1.*) we exhibit the inverse of $\varphi_{(\gamma)}$, in (*Step 2.*) we prove that $\varphi_{(\gamma)}$ is smooth and in (*Step 3.*) we conclude the argument.

(*Step 1.*) $\varphi_{(\gamma)}$ has an inverse of the same type $\varphi_{(\bar{\gamma})}$, where $\bar{\gamma}$ is the opposite curve to $\gamma$.

If $x \in F_{b_1}$, then $\Gamma_x$, the unique horizontal lift of $\gamma$ through $x$, satisfies:

$$\begin{cases} \textbf{(hl.1)} & \Gamma_x(t_1) & = & x \\ \textbf{(hl.2)} & \pi \circ \Gamma_x(t) & = & \gamma(t), \quad \forall t : t_1 \le t \le t_2 \\ \textbf{(hl.3)} & \dfrac{d}{dt}\Gamma_x(t) & \in & (HT)_{\Gamma_x(t)} \subseteq T_{\Gamma_x(t)}M, \quad \forall t : t_1 \le t \le t_2 \end{cases}$$

Let $\bar{\gamma}$ denote the opposite curve to $\gamma$, defined over the real line interval $[t_2, t_2 + (t_2 - t_1)] = [t_2, 2t_2 - t_1]$, by

$$\begin{array}{rccl} \bar{\gamma}: & [t_2, 2t_2 - t_1] & \longrightarrow & B \\ & u & \longmapsto & \bar{\gamma}(u) = \gamma(2t_2 - u) \end{array}$$

So, $\bar{\gamma}$ is an embedded p.p.a.l. smooth curve joining $b_2$ and $b_1$.

We denote by $\bar{\Gamma}_{\Gamma_x(t_2)} : [t_2, 2t_2 - t_1] \longrightarrow M$ be the unique horizontal lift of $\bar{\gamma}$ through $\Gamma_x(t_2)$, i.e.,

**(hl.$\bar{1}$)** $\bar{\Gamma}_{\Gamma_x(t_2)}(t_2) = \Gamma_x(t_2)$

**(hl.$\bar{2}$)** $\pi \circ \bar{\Gamma}_{\Gamma_x(t_2)}(u) = \bar{\gamma}(u), \quad \forall u : t_2 \le u \le 2t_2 - t_1$

**(hl.$\bar{3}$)** $\dfrac{d}{du}\bar{\Gamma}_{\Gamma_x(t_2)}(u) \in (HT)_{\bar{\Gamma}_{\Gamma_x(t_2)}(u)} \subseteq T_{\bar{\Gamma}_{\Gamma_x(t_2)}(u)}M, \quad \forall u : t_2 \le u \le 2t_2 - t_1,$

a horizontal vector.

We claim that

$$\bar{\Gamma}_{\Gamma_x(t_2)}(u) = \Gamma_x(2t_2 - u), \quad \forall u : t_2 \leq u \leq 2t_2 - t_1$$

In order to prove the claim we must verify that the curve

$$u \in [t_2, 2t_2 - t_1] \mapsto \Gamma_x(2t_2 - u)$$

satisfies **(hl.$\bar{1}$)**, **(hl.$\bar{2}$)**, **(hl.$\bar{3}$)**, and then from the uniqueness of the horizontal lift the claim follows.

In verifying those 3 items, we use **(hl.1,2,3)** for $\Gamma_x$:

**(hl.$\bar{1}$)** $\Gamma_x(2t_2 - (-t_2)) = \Gamma_x(t_2)$ $\surd$

**(hl.$\bar{2}$)** $\pi \circ \Gamma_x(2t_2 - u) = \gamma(2t_2 - u) = \bar{\gamma}(u), \quad \forall u : t_2 \leq u \leq 2t_2 - t_1$ $\surd$

**(hl.$\bar{3}$)** $\dfrac{d}{du}\Gamma_x(2t_2 - u) =$
$-\dfrac{d}{dt}\Gamma_x(t)\mid_{t=2t_2-u} \in (HT)_{\Gamma_x(t)|_{t=2t_2-u}} \subseteq T_{\Gamma_x(t)|_{t=2t_2-u}}M$, $\forall u \in [t_2, 2t_2 - t_1]$,
a horizontal vector. $\surd$

In that case,

$$\begin{aligned} \varphi_{(\bar{\gamma}\circ\gamma)}(x) &= \varphi_{(\bar{\gamma})} \circ \varphi_{(\gamma)}(x) = \varphi_{(\bar{\gamma})}(\Gamma_x(t_2)) := \bar{\Gamma}_{\Gamma_x(t_2)}(2t_2 - t_1) = \\ &= \Gamma_x(2t_2 - (2t_2 - t_1)) = \Gamma_x(t_1) = x \end{aligned}$$

and since $x \in F_{b_1}$ is arbitrary, it follows that $\varphi_{(\bar{\gamma}\circ\gamma)} = id\mid_{F_{b_1}}$, i.e. $\varphi_{(\bar{\gamma})} = \varphi_{(\gamma)}^{-1}$

(*Step 2.*) $\varphi_{(\gamma)}$ is smooth.

From the last remark in the proof of **Proposition 4.8**, we have that $\Gamma$ depends smoothly on the initial condition $x \in F_{b_1}$, and since $\varphi_{(\gamma)}(x) := \Gamma_x(t_2) \in F_{b_2}$, we conclude that $\varphi_{(\gamma)}$ depends smoothly on $x \in F_{b_1}$.

(*Step 3.*) $\varphi_{(\gamma)}$ is a diffeomorphism.

Since $\gamma$ is arbitrary in (*Step 2.*) it follows that $\varphi_{(\bar{\gamma})}$ is also smooth, which implies together with (*Step 1.*) that $\varphi_{(\gamma)}$ is a diffeomorphism.

*Proof of 4.* $\varphi_{(\gamma)}$ depends continuously on $\gamma$.

We firstly remark that since $\gamma \subset B$ is an embedded curve, there exists a single coordinate neighborhood in $B$, which entirely contains the curve $\gamma$. Therefore, in order to prove the *Theorem* is enough to look at one single coordinate neighborhood.

Initially, let us introduce some notation.

Assume there exists $\lambda$, a piecewise smooth p.p.a.l. embedded curve, distinct from $\gamma$, joining $b_1$ and $b_2$, where

$$\begin{aligned} \lambda : [t_1, t_2] &\rightarrow B \\ t_1 &\mapsto \lambda(t_1) := b_1 = \gamma(t_1) \\ t_2 &\mapsto \lambda(t_2) := b_2 = \gamma(t_2) \end{aligned}$$

For each $x \in F_{b_1}$, let $(U, \phi)$ be a coordinate system around $x$, and $(V, \psi)$ a coordinate system around $b_1$, such that

- $\pi(U) = V$, and
- $\pi_{\phi\psi} = \psi \circ \pi \circ \phi^{-1} : \phi(U) \subseteq I\!\!R^m \to I\!\!R^n$ has maximal rank $n$

We may assume w.l.o.g. that the images of $\lambda$ and $\gamma$ are entirely contained in the coordinate neighborhood $V$.

Since, by The Inverse Function Theorem, maximal rank maps behave locally like projections, w.l.o.g., we may assume that $(U, \phi)$ and $(V, \psi)$ are such that

$$\phi(U) \subseteq Z \text{ and } \psi(V) \subseteq W$$

where $Z \subseteq I\!\!R^m(\approx I\!\!R^{m-n} \times I\!\!R^n)$ is a neighborhood of $\phi(x) = (\xi_x, \zeta_x)$, and $W \subseteq I\!\!R^n$ is a neighborhood of $\psi(b_1)$, with the following properties:

- $X \times W$ is an open neighborhood of $(\xi_x, \psi(b_1))$
- $Z$ is an open neighborhood of $\phi(x)$
- $h : X \times W \to Z$ is the inverse diffeomorphism of

$$\begin{aligned} f : Z \subseteq \phi(U) \subseteq I\!\!R^m &\to X \times W \\ (\xi, \zeta) &\mapsto (\xi, \pi_{\phi\psi}(\xi, \zeta)) \end{aligned}$$

satisfying $\pi_{\phi\psi} \circ h(\xi, w) = w, \forall(\xi, w) \in X \times W$.

In particular, $\pi \circ \phi^{-1} \circ h(\xi_x, \psi \circ \gamma(t)) = \gamma(t)$, for all $t \in [t_1, t_2]$ and $b_1 = \pi \circ \phi^{-1}(\xi_x, \zeta_x)$.

In local coordinates, the unique horizontal lift of $\gamma$ through $x$, is given by

$$\Gamma_x(t) = \phi^{-1} \circ h(\xi_x, \psi \circ \gamma(t)), \forall t \in [t_1, t_2]$$

and similarly, for the unique horizontal lift of $\lambda$ through $x$,

$$\Lambda_x(t) = \phi^{-1} \circ h(\xi_x, \psi \circ \lambda(t)), \forall t \in [t_1, t_2]$$

In particular,

$$\varphi_{(\gamma)}(x) = \Gamma_x(t_2) = \Lambda_x(t_2) = \varphi_{(\lambda)}(x)$$

Finally, by the local forms of $\Gamma_x$ and $\Lambda_x$ as above, and since all the maps considered in these expressions are continuous, we can conclude that:

Given any small tubular neighborhood $\mathcal{T}_{\Gamma_x}$ of $\Gamma_x$ in $M^m$,

$$\Gamma_x([t_1, t_2]) \subset \mathcal{T}_{\Gamma_x} \subseteq M^m$$

there exists a sufficiently small tubular neighborhood $\mathcal{T}_\gamma$ of $\gamma$ in $B^n$,

$$\gamma([t_1, t_2]) \subset \mathcal{T}_\gamma \subseteq B^n$$

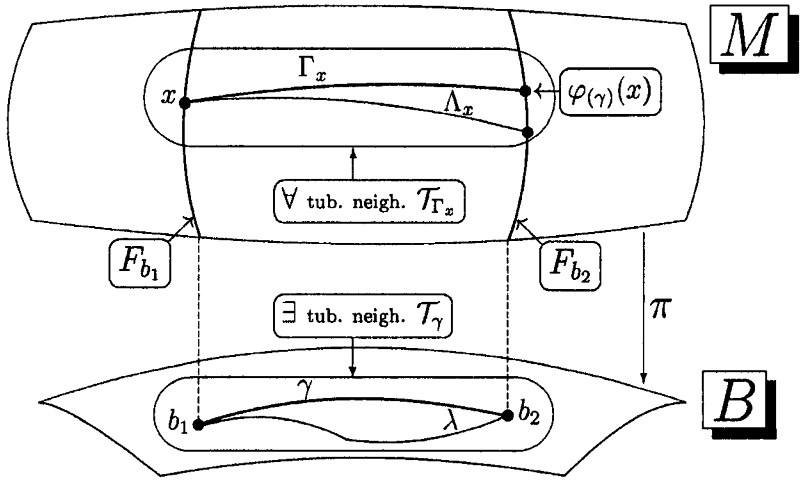


Figure 4.3: $\varphi_{(\gamma)}$ depends continuously on $\gamma$.

such that, for any curve $\lambda$ in $B^n$ which lies entirely inside $\mathcal{T}_\gamma$, its horizontal lift $\Lambda_x$ in $M^m$, lies entirely inside $\mathcal{T}_{\Gamma_x}$ (see Fig. 4.3).

Consequently, $\varphi(\gamma)$ depends continuously on $\gamma$.

□

In the remaining part of this work, we will initially explore the motivation behind the main **Theorem 4.2.10**, before we state and prove it.

*B. O'Neill*'s article [25] furnished the inspiration for this work. In this article O'Neill defines *Riemannian submersions* (see **Definition 2.7**) and gives necessary and sufficient conditions for a Riemannian submersion to differ by an isometry from the simplest type of Riemannian submersions, the projection of a Riemannian product manifold on one of its factors. In our research, **Theorem 4.2.10** provides an answer to the same kind of question as above, with maximal rank mapping replacing Riemannian submersion, and with rough isometry instead of isometry.

In the next two sections we will investigate this question: when is a Riemannian manifold roughly isometric to a Riemannian product manifold?

## 4.1 Riemannian Submersions

Let $M$ and $B$ be Riemannian manifolds, and $\pi : M \to B$ a Riemannian submersion.

In the next 2 Theorems, we obtain equivalence relations given by rough isometries. That will allow us to see $M$, through the eyes of a rough isometry, as a Riemannian product manifold.

**Theorem 4.1.1** *Let $\pi : M \to B$ be a Riemannian submersion. Suppose $B$ is compact and connected, and for each $b \in B$ the fiber $\pi^{-1}(b)$ has the induced metric from $(M, d)$. Then for each $b \in B$, the inclusion $\iota : \pi^{-1}(b) \hookrightarrow M$ is a rough isometry.*

*In particular, since $B$ is compact, $M$ is roughly isometric to the product $\pi^{-1}(b) \times B$.*

*Proof.* We must verify axioms **(RI.1)** and **(RI.2)** for $\iota$, on each fiber.

Let $b \in B$.

We claim that $\iota : \pi^{-1}(b) \hookrightarrow M$ satisfies **(RI.1)**.

In fact, first notice that the induced metric $d_b$ on the fiber is defined by

$$d_b(x, y) := d(\iota(x), \iota(y)) = \inf_{\gamma_{x,y}} \ell(\gamma_{x,y}), \quad \forall x, y \in \pi^{-1}(b)$$

where, $\gamma_{x,y} \subseteq M$ is any curve in $M$ joining $x$ and $y$, and $d$ is the Riemannian metric on $M$.

We can rewrite the definition of $d_b$ in the form,

$$1 \cdot d_b(x, y) - 1 \leq d(\iota(x), \iota(y)) \leq 1 \cdot d_b(x, y) + 1$$

which is axiom **(RI.1)** with $A = C = 1$.

Now, we verify axiom **(RI.2)** for $\iota : \pi^{-1}(b) \hookrightarrow M$.

We want to show that for some $\epsilon > 0$, $M$ is an $\epsilon$-neighborhood of $\iota\,(\pi^{-1}(b))$ viewed as a subset of $M$, i.e. there exists an $\epsilon > 0$, such that for all $y \in M$ we have $d(y, \mathrm{Im}(\iota)) < \epsilon$.

If $M = \iota\,(\pi^{-1}(b))$, then **(RI.2)** follows for any $\epsilon > 0$.

Otherwise, if $y \in \{M \setminus \pi^{-1}(b)\} \neq \emptyset$, since $B$ is compact and connected, and thus complete, we can consider the minimal geodesic $\lambda$ in $B$, joining $b$ and $\pi(y)$.

Notice that in this case $\pi(y) \neq b$ implies that

$$\text{diam } B := \sup\{d(x,y) : \forall x, y \in B\} > 0$$

and the fact that $B$ is compact gives us,

$$0 < \text{diam } B < \infty \tag{4.5}$$

Let $\Lambda$ be the unique horizontal lift, in $M$, of $\lambda$ through $y$, in other words $\Lambda$ satisfies,

**(hl.1)** $\pi \circ \Lambda = \lambda$,

**(hl.2)** $\Lambda$ passes through $y$, connecting $y$ to the fiber $\pi^{-1}(b)$. In particular, $\Lambda$ intersects $\pi^{-1}(b)$.

**(hl.3)** $\Lambda$ is horizontal, i.e., its velocity $\Lambda' \in (HT)_\Lambda$ defines a horizontal vector field on $M$.

Thus,

$$\ell(\Lambda) \;=\; \int |\, \Lambda' \,|_M \overset{\textbf{(S.2)}}{=} \int |\, \pi_* \circ \Lambda' \,|_B = \int |\, \lambda' \,|_B = \ell(\lambda) \tag{4.6}$$

It follows from ( 4.5) and the fact that $\lambda$ is a minimal geodesic,

$$\ell(\lambda) \leq \text{diam } B := \sup\{d(x,y) : \forall x, y \in B\} \tag{4.7}$$

Finally,

$$d(y, \mathrm{Im}(\iota)) := \inf_{x \in \pi^{-1}(b)} d(y, \iota(x)) \overset{\mathbf{(hl.2)}}{\leq} \ell(\Lambda) \overset{(4.6)}{=} \ell(\lambda) \overset{(4.7)}{\leq} \mathrm{diam}\, B$$

and this is axiom **(RI.2)**, with $\epsilon := \mathrm{diam}\, B > 0$.

□

**Theorem 4.1.2** *Let $\pi : M \to B$ be a Riemannian submersion, where $B$ is connected and complete. Suppose that, for some constant $m > 0$, all fibers satisfy the universal property:*

**(UP)** *diam* $(\pi^{-1}(b)) \leq m < \infty, \forall b \in B$.

*Then, $\pi : M \to B$ is a rough isometry.*

*In particular, if for some $b_0$ the fiber $\pi^{-1}(b_0)$ is compact, then $M$ is roughly isometric to the product $\pi^{-1}(b_0) \times B$.*

*Proof.* Let us denote the Riemannian metric spaces by $(M, d_M)$ and $(B, d_B)$.

We must verify axioms **(RI.1)** and **(RI.2)** for $\pi$.

By definition $\pi$ is a surjective map, i.e. $\mathrm{Im}\pi = B$, which implies that $B = B_\epsilon(\mathrm{Im}\pi) = \{y \in B : d(y, \mathrm{Im}\pi) = 0 < \epsilon\}$ for any choice of $\epsilon > 0$, and this gives us axiom **(RI.2)**.

To verify axiom **(RI.1)** we need to find constants $A \geq 1$ and $C > 0$, satisfying

$$\frac{1}{A} \cdot d_M(x,y) - C \leq d_B(\pi(x), \pi(y)) \leq A \cdot d_M(x,y) + C, \ \ \forall x, y \in M$$

Let $x, y \in M$.

Since $B$ is connected and complete, there is a minimal geodesic $\lambda$ in $B$, connecting $\pi(x)$ and $\pi(y)$, with the property,

$$d_B(\pi(x), \pi(y)) = \ell(\lambda) \tag{4.8}$$

In $M$, there exists a unique $\Lambda_x$, horizontal lift of $\lambda$, passing through $x$,

**(hl.1)** $\pi \circ \Lambda_x = \lambda$;

**(hl.2)** $\Lambda_x$ intersects the fiber passing through $y$, namely $\pi^{-1}(\pi(y))$, at some $z \in M : \pi(z) = \pi(y)$;

**(hl.3)** $\Lambda_x$ is horizontal, i.e., its velocity $\Lambda_x{}' \in (HT)_{\Lambda_x}$ defines a horizontal vector field on $M$.

Thus,

$$\ell(\Lambda_x) \ = \ \int |\, \Lambda_x' \,|_M \overset{\textbf{(S.2)}}{=} \int |\, \pi_* \circ \Lambda_x' \,|_B = \int |\, \lambda' \,|_B = \ell(\lambda) \tag{4.9}$$

Now, we observe that **(UP)** for $\pi^{-1}(\pi(y))$, means,

$$\text{diam}\ \{\pi^{-1}(\pi(y))\} \leq m$$

By using ( 4.8), ( 4.9) and the triangle inequality, we can write,

$$\begin{aligned} d_M(x,y) &\leq d_M(x,z) + d_M(y,z) = \inf_{\gamma_{x,z}} \ell(\gamma_{x,z}) + d_M(y,z) \leq \\ &\leq \ell(\Lambda_x) + d_M(y,z) \leq \ell(\Lambda_x) + \text{diam}\ \{\pi^{-1}(\pi(y))\} \overset{\textbf{(UP)}}{\leq} \\ &\leq \ell(\Lambda_x) + m \overset{(\ 4.9)}{=} \ell(\lambda) + m \overset{(\ 4.8)}{=} d_B(\pi(x),\pi(y)) + m \end{aligned}$$

which can be rewritten as

$$d_B(\pi(x),\pi(y)) \geq \frac{1}{1} \cdot d_M(x,y) - m \tag{4.10}$$

On the other hand, by ( 4.8) and ( 4.9), we have

$$d_B(\pi(x),\pi(y)) \overset{(\ 4.8)}{=} \ell(\lambda) \overset{(\ 4.9)}{=} \ell(\Lambda_x) \tag{4.11}$$

We claim that

$$\ell(\Lambda_x) \leq \ell(\alpha), \quad \forall \alpha \tag{4.12}$$

where $\alpha : [t_1, t_2] \to M$ is any smooth connected curve joining $x$ and $y$ in $M$.

In fact, if we write

$$\alpha'(t) = \alpha'_H(t) \oplus \alpha'_V(t) \in T_{\alpha(t)}M = (HT)_{\alpha(t)} \oplus (VT)_{\alpha(t)}, \quad \forall t \in [t_1, t_2]$$

and observe that for any orthogonal vectors $U$ and $W$,

$$||U \oplus W||^2 = ||U||^2 + ||W||^2 \geq \max\{||U||^2, ||W||^2\}$$

then,

$$\ell(\alpha) = \int_{t_1}^{t_2} ||\alpha'(t)||_M dt \geq \int_{t_1}^{t_2} ||\alpha'_H(t)||_M dt$$

Moreover,

$$\begin{aligned} \ell(\pi \circ \alpha) &= \int_{t_1}^{t_2} ||(\pi_*)_{\alpha(t)}\alpha'(t)||_B dt \overset{\text{Lem 4.3}}{=} \int_{t_1}^{t_2} ||(\pi_*)_{\alpha(t)}\alpha'_H(t)||_B dt \overset{\text{(S.2)}}{=} \\ &= \int_{t_1}^{t_2} ||\alpha'_H(t)||_M dt \end{aligned}$$

and consequently,

$$\ell(\alpha) \geq \int_{t_1}^{t_2} ||\alpha'_H(t)||_M dt = \ell(\pi \circ \alpha) \geq d_B(\pi(x), \pi(y)) \overset{(\ 4.11)}{\geq} \ell(\Lambda_x)$$

for all $\alpha : [t_1, t_2] \to M$, a smooth connected curve joining $x$ and $y$ in $M$, which is the claim.

Now, by ( 4.12) and the definition of infimum, we can write,

$$\inf_{\alpha} \ell(\alpha) =: d_M(x, y) \geq \ell(\Lambda_x) \overset{(\ 4.11)}{=} d_B(\pi(x), \pi(y))$$

which can be rewritten in the form,

$$d_B(\pi(x), \pi(y)) \leq d_M(x, y) \leq 1 \cdot d_M(x, y) + m \tag{4.13}$$

Finally, if we combine ( 4.10) and ( 4.13), we obtain,

$$\frac{1}{1} \cdot d_M(x,y) - m \leq d_B(\pi(x), \pi(y)) \leq 1 \cdot d_M(x,y) + m$$

which is axiom **(RI.2)** for $A = 1$ and $C = m > 0$

□

## 4.2 Non-Submersions Surjective Maximal Rank Maps

Let $M$ and $B$ be Riemannian manifolds, and $\pi : M \to B$ an onto smooth map with maximal rank.

In this section, we show that for onto smooth maximal rank maps the same results as in **Theorems 4.1.1, 4.1.2** hold, as long as we make extra assumptions on the subspaces of horizontal vectors.

We also provide in **Counterexamples 4.2.2, 4.2.4, 4.2.5, 4.2.6** evidence showing that if we drop any of the hypothesis in either **Theorem 4.2.1** or **Theorem 4.2.3**, those results cease to follow.

**Theorem 4.2.1** *Let $\pi : M \to B$ be an onto smooth map with maximal rank, where $B$ is compact. Assume that, for each $b \in B$, the fiber $\pi^{-1}(b) = F_b$ is endowed with the induced metric from $(M, d)$. Suppose there are constants*

*$\alpha \geq 1$ and $\beta > 0$, such that, for all $b \in B$ the following inequality holds:*

$$||v||_M \leq \alpha||w||_B + \beta \tag{4.14}$$

*for all $x \in F_b$ and $w \in T_bB$, where $v \in (HT)_x \subset T_xM$ is the horizontal lift of $w$ through $x$.*

*Then, for each $b \in B$, the inclusion map $\iota : F_b \hookrightarrow M$ is a rough isometry.*

*In particular, since $B$ is compact, $M$ is roughly isometric to the product $\pi^{-1}(b) \times B$.*

*Proof.* We must verify axioms **(RI.1)** and **(RI.2)** for $\iota$, given any $b \in B$.

Let us denote by $(M, d_M)$ and $(B, d_B)$ the Riemannian metric spaces.

As in the proof of **Theorem 4.1.1**, if we rewrite the definition of the induced metric $d_b$ on the fiber $F_b$ in the form,

$$1 \cdot d_b(x, y) - 1 \leq d_M(\iota(x), \iota(y)) \leq 1 \cdot d_b(x, y) + 1$$

this is axiom **(RI.1)** with $A = C = 1$.

To verify axiom **(RI.2)**, we need to prove that $M$ is an $\epsilon$-neighborhood of $\iota(F_b) \subseteq M$, for some $\epsilon > 0$, i.e. we must find a constant $\epsilon > 0$ for which $d_M(y, \iota(F_b)) < \epsilon$, for all $y \in M$.

W.l.o.g. we may assume that $B$ is connected, since if that is not the case, we can repeat, on each connected component of $B$, the argument that will follow.

Now, since $B$ is compact and connected it is also complete.

Thus, for any $y \in M$ there exists a minimal geodesic $\gamma$ joining $\pi(y)$ to $b$, with $\gamma : [0,1] \to B, \gamma(0) = \pi(y), \gamma(1) = b$.

By **Proposition 4.8**, $\gamma$ has a unique horizontal lift $\Gamma_y : [0,1] \to M$, through $y$, i.e.,

**(hl.1)** $\pi \circ \Gamma_y = \gamma$;

**(hl.2)** $\Gamma_y(0) = y$ and $\Gamma_y(1) \in \pi^{-1}\{\gamma(1)\} = F_b$. In particular, $\Gamma_y$ connects $y$ to the fiber $F_b$;

**(hl.3)** $\Gamma_y$ is horizontal, i.e., $\Gamma_y{}'(t) \in (HT)_{\Gamma_y(t)}$.

In that case, we can write,

$$\begin{aligned} d_M(y, \iota(F_b)) &\leq \ell(\Gamma_y) := \int_0^1 ||\Gamma'_y||_M dt \overset{(\,4.14)}{\leq} \\ &\leq \alpha \int_0^1 ||(\pi_*)_{\Gamma_y(t)}\Gamma'_y(t)||_B dt + \beta = \\ &= \alpha \int_0^1 ||\gamma'(t)||_B dt + \beta = \alpha \cdot \ell(\gamma) + \beta \end{aligned} \tag{4.15}$$

Now, by the compactness of $B$,

$$\text{diam } B := \sup_{b_1, b_2 \in B} \{d_B(b_1, b_2)\} < \infty$$

Moreover, since $\gamma$ is a minimal geodesic joining $\pi(y)$ to $b$,

$$\ell(\gamma) = d_B(\pi(y), b) \leq \text{diam } B < \infty \tag{4.16}$$

By substituting ( 4.16) in ( 4.15), we obtain,

$$d_M(y, \iota(F_b)) \overset{(\ 4.15)}{\leq} \alpha \cdot \ell(\gamma) + \beta \overset{(\ 4.16)}{\leq} \alpha \cdot (\text{diam } B) + \beta \tag{4.17}$$

Define $\epsilon := \alpha \cdot (\text{diam } B) + \beta$, which is a positive constant independent of $y$, and in particular, independent of $b \in B$.

For that choice of $\epsilon$, since $y \in M$ is arbitrary, we see that ( 4.17) is exactly axiom **(RI.2)**.

□

In what follows, we provide a Counterexample to illustrate how assumption ( 4.14) is essential in **Theorem 4.2.1.** We show that if ( 4.14) doesn't hold for some $b \in B$, then the inclusion map $\iota : F_b \hookrightarrow M$ ceases to be a rough isometry.

**Counterexample 4.2.2** *We will exhibit $M, B, \pi$ satisfying all the conditions in Theorem 4.2.1 with the exception of ( 4.14), which does* not *hold in this Counterexample, i.e.,*

*For any given constants $\alpha \geq 1$ and $\beta > 0$ there exist $b \in B$, $w \in T_bB$ and*

*$x \in F_b$ satisfying:*

$$||v||_M > \alpha ||w||_B + \beta \qquad \textbf{[CE.1]}$$

*where $v$ is the horizontal lift of $w$ in $(HT)_x \subset T_x M$.*

*In this case, the inclusion map $\iota : F_b \hookrightarrow M$ is not a rough isometry.*

Let $M$ and $B$ be the following Riemannian manifolds,

$$M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2 + 1\}$$

and the compact unit circle,

$$B = \mathcal{S}^1 = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$$

where the metrics on $M$ and $B$ are induced by the Euclidean metric on $\mathbb{R}^3$.

Let $\pi : M \to B$ be defined by,

$$\pi(x_1, x_2, x_3) = \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, 0 \right)$$

The map $\pi$ defined as above (see Fig. 4.4) is the composition of the two maps,

$$\begin{aligned} (x_1, x_2, x_3) &\mapsto \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, x_3 \right) \\ (\tilde{a}, \tilde{b}, \tilde{c}) &\mapsto (\tilde{a}, \tilde{b}, 0) \end{aligned}$$

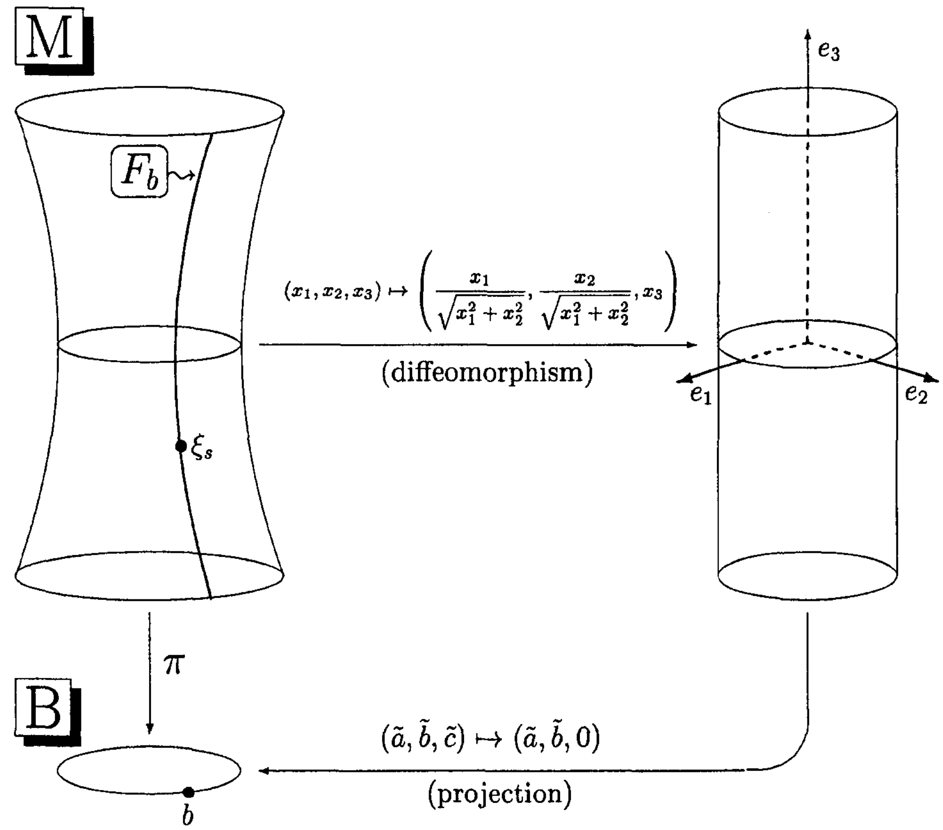


Figure 4.4: The map $\pi : M \to B$ in **Counterexample 4.2.2.**

where the first is a diffeomorphism between $M$ and the cylinder described by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1; x_3 \in \mathbb{R}\}$, and the second is the projection onto the first two coordinates, which is an onto smooth maximal rank map.

Consequently, $\pi : M \to B$ is an onto smooth maximal rank map.

Notice that for any compact connected smooth curve $\gamma$ in $B = \mathcal{S}^1$, which minimizes distances between two points in $B$, its length satisfy $\ell(\gamma) \leq 2\pi$.

On the other hand, if we denote by $\Gamma_r$ the horizontal lift of $\gamma$ through $(x_1, x_2, r)$, where

$$(x_1, x_2, r) \in \mathbb{R}^3 : x_1^2 + x_2^2 = r^2 + 1, \quad r > 0 \text{ constant}$$

the lengths of the horizontal lifts $\ell(\Gamma_r)$ grow without any bounds as $r \to +\infty$.

Next, we find suitable parametrizations for $\gamma$ and $\Gamma_r$ in order to show that **[CE.1]** holds.

Let $b = (b_1, b_2, 0) \in B$.

Assume that $\gamma$ passes through $b$, and let us parametrize $\gamma$ by,

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), 0) \in B, \quad \forall t \in [0,1]$$

with $b = \gamma(0)$ and $\gamma_1^2(t) + \gamma_2^2(t) = 1$ where, $0 \leq t \leq 1$.

A generic element in the fiber $F_b \subseteq M$ can be described as,

$$\xi_s := \left(b_1 \cdot \sqrt{s^2+1}, b_2 \cdot \sqrt{s^2+1}, s\right) \in M$$

where $s \in \mathbb{R}$ is constant.

We claim that the unique horizontal lift $\Gamma_r$ (see Fig. 4.5) of $\gamma$ through $\xi_r$, where $r > 0$, can be parametrized by,

$$\Gamma_r(t) = \left(\gamma_1(t) \cdot \sqrt{r^2+1}, \gamma_2(t) \cdot \sqrt{r^2+1}, r\right) \quad \forall t \in [0,1] \qquad \textbf{[CE.2]}$$

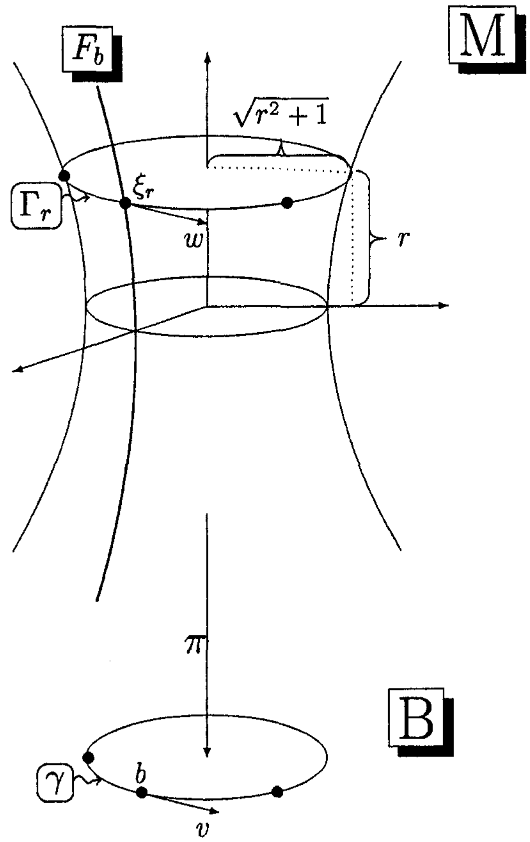


Figure 4.5: Curve $\gamma$ and its horizontal lift $\Gamma_r$.

First, we show that the Subspace of Vertical Vectors is a 2-dimensional vector space of the form,

$$(VT)_{(x_1,x_2,x_3)} \overset{\textbf{Lem 4.3}}{=} \ker(\pi_*)_{(x_1,x_2,x_3)} = \{(\xi,\zeta,\eta) \in I\!\!R^3 : x_2 \cdot \xi - x_1 \cdot \zeta = 0\}$$

where $(x_1, x_2, x_3) \in M$.

The Jacobian of $\pi$ at $(x_1, x_2, x_3) \in M$ is given by,

$$(\pi_*)_{(x_1,x_2,x_3)}(\xi,\zeta,\eta) =$$
$$= \begin{bmatrix} \frac{\partial \pi_1}{\partial x_1}(x_1,x_2,x_3) & \frac{\partial \pi_1}{\partial x_2}(x_1,x_2,x_3) & \frac{\partial \pi_1}{\partial x_3}(x_1,x_2,x_3) \\ \frac{\partial \pi_2}{\partial x_1}(x_1,x_2,x_3) & \frac{\partial \pi_2}{\partial x_2}(x_1,x_2,x_3) & \frac{\partial \pi_2}{\partial x_3}(x_1,x_2,x_3) \\ 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} \xi \\ \zeta \\ \eta \end{bmatrix} =$$
$$= \begin{bmatrix} \frac{x_2^2}{(x_1^2+x_2^2)^{3/2}} & \frac{-x_1x_2}{(x_1^2+x_2^2)^{3/2}} & 0 \\ \frac{-x_1x_2}{(x_1^2+x_2^2)^{3/2}} & \frac{x_1^2}{(x_1^2+x_2^2)^{3/2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} \xi \\ \zeta \\ \eta \end{bmatrix} =$$
$$= \frac{1}{(x_1^2+x_2^2)^{3/2}} \bullet \begin{bmatrix} x_2^2\xi - x_1x_2\zeta \\ -x_1x_2\xi + x_1^2\zeta \\ 0 \end{bmatrix}$$

for any $(\xi, \zeta, \eta) \in I\!R^3$.

Thus, for each $(x_1, x_2, x_3) \in M$,

$$(\xi,\zeta,\eta) \in \ker(\pi_*)_{(x_1,x_2,x_3)} \iff$$
$$\iff \begin{cases} x_2^2\xi - x_1x_2\zeta = 0 \\ -x_1x_2\xi + x_1^2\zeta = 0 \end{cases} \iff \begin{cases} x_2 \cdot (x_2\xi - x_1\zeta) = 0 \\ -x_1 \cdot (x_2\xi - x_1\zeta) = 0 \end{cases} \overset{x_1^2+x_2^2 \neq 0}{\iff}$$
$$\iff x_2 \cdot \xi - x_1 \cdot \zeta = 0 \qquad \textbf{[CE.3]}$$

Notice that, the map $(\pi_*)_{(x_1,x_2,x_3)}$ has rank 1, $\forall (x_1, x_2, x_3) \in M$, since,

$$\det \begin{bmatrix} \frac{x_2^2}{(x_1^2+x_2^2)^{3/2}} & \frac{-x_1x_2}{(x_1^2+x_2^2)^{3/2}} \\ \frac{-x_1x_2}{(x_1^2+x_2^2)^{3/2}} & \frac{x_1^2}{(x_1^2+x_2^2)^{3/2}} \end{bmatrix} = 0$$

and

$$x_1^2 + x_2^2 \neq 0 \Rightarrow \text{ either } \det \left[ \frac{x_2^2}{(x_1^2 + x_2^2)^{3/2}} \right] \neq 0 \text{ or } \det \left[ \frac{x_1^2}{(x_1^2 + x_2^2)^{3/2}} \right] \neq 0$$

which implies that,

$$\dim \left( (VT)_{(x_1,x_2,x_3)} \overset{\text{Lem 4.3}}{=} \ker(\pi_*)_{(x_1,x_2,x_3)} \right) = 2$$

and

$$\dim \left( (HT)_{(x_1,x_2,x_3)} \overset{\text{Lem 4.3}}{=} \left( \ker(\pi_*)_{(x_1,x_2,x_3)} \right)^{\perp} \right) = 1$$

In order to prove claim [CE.2], we must verify properties **(hl.1,2,3)** according to **Proposition 4.8**, for $\Gamma_r$.

[**(hl.1)**] $\pi \circ \Gamma_r = \gamma$

$$\begin{aligned} \pi \circ \Gamma_r(t) &= \pi \left( \gamma_1(t) \cdot \sqrt{r^2+1}, \gamma_2(t) \cdot \sqrt{r^2+1}, r \right) = \\ &= \left( \frac{\gamma_1(t) \cdot \sqrt{r^2+1}}{\sqrt{\gamma_1^2(t)(r^2+1) + \gamma_2^2(t)(r^2+1)}}, \frac{\gamma_2(t) \cdot \sqrt{r^2+1}}{\sqrt{\gamma_1^2(t)(r^2+1) + \gamma_2^2(t)(r^2+1)}}, 0 \right) \\ &= (\gamma_1(t), \gamma_2(t), 0) = \gamma(t), \quad \forall t \in [0,1] \end{aligned}$$

where, $\gamma_1^2(t) + \gamma_2^2(t) = 1, \ \ \forall t \in [0,1]$.

[**(hl.2)**] $\Gamma_r(0) = \xi_r$

$$\begin{aligned} \Gamma_r(0) = &= \left( \gamma_1(0) \cdot \sqrt{r^2+1}, \gamma_2(0) \cdot \sqrt{r^2+1}, r \right) = \\ &= (b_1 \cdot \sqrt{r^2+1}, b_2 \cdot \sqrt{r^2+1}, r) = \xi_r \end{aligned}$$

**[(hl.3)]** $\Gamma_r$ is horizontal: $\Gamma_r'(t) \in (HT)_{\Gamma_r(t)} = \left(\ker(\pi_*)_{\Gamma_r(t)}\right)^\perp, \forall t \in [0,1]$

We have, for all $t \in [0,1]$,

$$\begin{aligned}\gamma_1^2(t) + \gamma_2^2(t) = 1 \ &\Rightarrow \ \gamma_1(t)\cdot\gamma_1'(t) + \gamma_2(t)\cdot\gamma_2'(t) = 0 \Rightarrow \\ &\Rightarrow \ \gamma(t) \bullet \gamma'(t) = 0\end{aligned}$$

and,

$$\begin{aligned}(\gamma_2(t), -\gamma_1(t), 0) \bullet \gamma(t) &= (\gamma_2(t), -\gamma_1(t), 0) \bullet (\gamma_1(t), \gamma_2(t), 0) = \\ &= \gamma_2(t)\cdot\gamma_1(t) - \gamma_1(t)\cdot\gamma_2(t) = 0\end{aligned}$$

implying in $\mathbb{R}^2 \times \{0\}$ that,

$$(\gamma_2(t), -\gamma_1(t), 0) \bullet (\gamma_1'(t), \gamma_2'(t), 0) \neq 0$$

Moreover,

$$\Gamma_r'(t) = \left(\gamma_1'(t)\cdot\sqrt{r^2+1}, \gamma_2'(t)\cdot\sqrt{r^2+1}, 0\right), \quad \forall t \in [0,1] \qquad \textbf{[CE.4]}$$

Then, condition **[CE.3]** determining whether a tangent vector belongs to the kernel of $\pi_*$, writes,

$$\begin{aligned}&\left(\gamma_2(t)\sqrt{r^2+1}\right)\cdot\gamma_1'(t)\sqrt{r^2+1} - \left(\gamma_1(t)\sqrt{r^2+1}\right)\cdot\gamma_2'(t)\sqrt{r^2+1} = \\ = \ &\sqrt{r^2+1}\,(\gamma_2(t)\cdot\gamma_1'(t) - \gamma_1(t)\cdot\gamma_2'(t)) = \\ = \ &\sqrt{r^2+1}(\gamma_2(t), -\gamma_1(t), 0) \bullet (\gamma_1'(t), \gamma_2'(t), 0) \neq 0\end{aligned}$$

and consequently, $\forall t \in [0,1], \Gamma_r'(t)$ is not a vertical vector.

Now, we show that the tangent vector to $\Gamma_r(t)$ is perpendicular to the fiber $F_{\gamma(t)}$, for each $t \in [0,1]$.

Recall that a generic element in the fiber $F_{\gamma(t)}$, can be written as,

$$\xi_s := \left(\gamma_1(t)\cdot\sqrt{s^2+1}, \gamma_2(t)\cdot\sqrt{s^2+1}, s\right), s \in \mathbb{R} \qquad \textbf{[CE.5]}$$

Therefore, a tangent vector to the fiber is given by,

$$\frac{d}{ds}\xi_s = \left(\gamma_1(t)\cdot\frac{s}{\sqrt{s^2+1}}, \gamma_2(t)\cdot\frac{s}{\sqrt{s^2+1}}, 1\right), s \in \mathbb{R}$$

and thus,

$$\begin{aligned}
&\frac{d}{dr}\xi_r \bullet \Gamma_r'(t) = \\
&= \gamma_1(t)\cdot\frac{r}{\sqrt{r^2+1}}\cdot\gamma_1'(t)\cdot\sqrt{r^2+1} + \gamma_2(t)\cdot\frac{r}{\sqrt{r^2+1}}\cdot\gamma_2'(t)\cdot\sqrt{r^2+1} = \\
&= r(\gamma_1(t)\cdot\gamma_1'(t) + \gamma_2(t)\cdot\gamma_2'(t)) = 0, \quad \forall t \in [0,1]
\end{aligned}$$

So, $\Gamma_r'(t) \perp F\gamma(t)$ for all $t \in [0,1]$ and $r \in \mathbb{R}$, and we conclude that $\Gamma_r'(t) \in (HT)_{\Gamma_r(t)}$.

Next, we show that **[CE.1]** holds for $\pi : M \to B$.

Let $b \in B$ be arbitrary, and any given real numbers $\alpha \geq 1$ and $\beta > 0$.

For the smooth curve $\gamma$ in $B$, parametrized by,

$$\begin{array}{rcl}\gamma : [0,2\pi] & \to & B \\ t & \mapsto & \gamma(t) = (\cos t, \sin t, 0)\end{array}$$

where,

$$\begin{array}{rcl}\gamma' : [0, 2\pi] & \to & B \\ t & \mapsto & \gamma'(t) = (-\sin t, \cos t, 0)\end{array}$$

and,

$$||\gamma'(t)||_M = 1, \quad 0 \le t \le 2\pi \qquad \textbf{[CE.6]}$$

there exists a unique $t_b \in [0, 2\pi)$, such that $b = \gamma(t_b)$.

If we choose $r = \sqrt{\{(\alpha+\beta)+1\}^2 - 1}$, which is a number greater than 1, we obtain for $v = \Gamma'_r(t_b), w = \gamma'(t_b), x = \Gamma_r(t_b)$ the following,

$$\begin{aligned} ||v||_M &= ||\Gamma'_r(t_b)||_M \overset{\textbf{[CE.4]}}{=} ||\gamma'(t_b)||_M \sqrt{r^2+1} \overset{\textbf{[CE.6]}}{=} \sqrt{r^2+1} = \\ &= (\alpha+\beta)+1 > \alpha+\beta = \alpha \cdot 1 + \beta \overset{\textbf{[CE.6]}}{=} \alpha||\gamma'(t_b)||_M + \beta = \\ &= \alpha||w||_B + \beta \end{aligned}$$

which is **[CE.1]**.

Finally, we show that the inclusion $\iota : F_b \to M$ is <u>not</u> a rough isometry.

In that direction, we claim that **(RI.2)** fails, i.e.

$$\forall \epsilon > 0, \exists y_\epsilon = y_\epsilon(\epsilon, b) \in M, \text{ satisfying } d_M(y_\epsilon, \iota(F_b)) \ge \epsilon$$

From **[CE.5]** the fiber $F_b$, where $b = \gamma(t_b) = (\cos t_b, \sin t_b, 0)$, can be described as,

$$F_b = \{\xi_r := (\sqrt{r^2+1} \cdot \cos t_b, \sqrt{r^2+1} \cdot \sin t_b, r), r \in I\!R\}$$

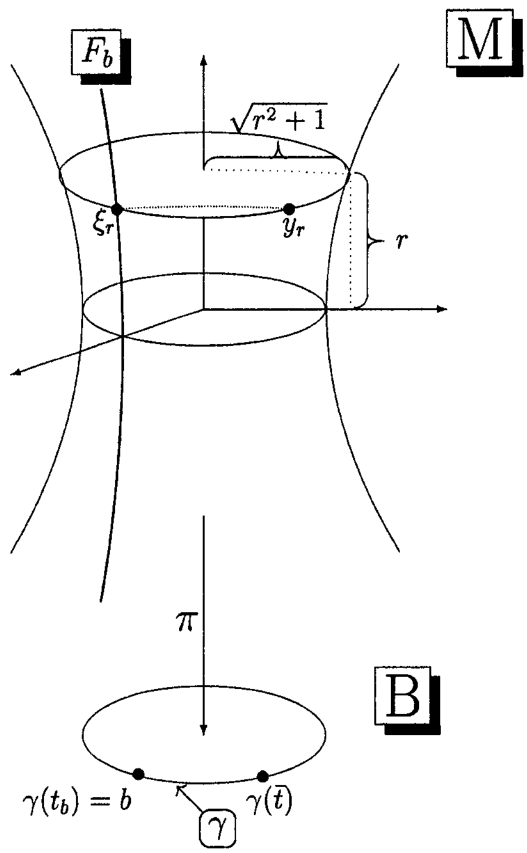


Figure 4.6: Generic elements $\xi_r$ in the fiber $F_b$, and $y_r$ in $M \setminus F_b$.

So, $M \setminus F_b \neq \emptyset$.

Now, any element $y_r$ of $M \setminus F_b$ is of the form,

$$y_r = \left(\sqrt{r^2+1} \cdot \cos\bar{t}, \sqrt{r^2+1} \cdot \sin\bar{t}, r\right)$$

for some $r \in \mathbb{R}$ and $\bar{t} \in [0, 2\pi), \bar{t} \neq t_b$, where $\gamma(t_b) = (\cos t_b, \sin t_b, 0) = b \neq \pi(y_r) = (\cos\bar{t}, \sin\bar{t}, 0) = \gamma(\bar{t})$ (see Fig. 4.6).

Define $\bar{t} \in [0, 2\pi)$, by,

$$\bar{t} := \begin{cases} t_b + \pi, & \text{if} \quad 0 \leq t_b < \pi \quad (:. \pi \leq \bar{t} < 2\pi) \\ t_b - \pi, & \text{if} \quad \pi \leq t_b < 2\pi \quad (:. 0 \leq \bar{t} < \pi) \end{cases}$$

In particular, $\bar{t} \neq t_b$ and $|\bar{t} - t_b| = \pi$.

Moreover,

$$d_M(\xi_r, y_r) = \sqrt{(r^2+1)(\cos t_b - \cos \bar{t})^2 + (r^2+1)(\sin t_b - \sin \bar{t})^2} =$$

$$= \sqrt{r^2+1}\sqrt{1 - 2\cos t_b \cos \bar{t} - 2\sin t_b \sin \bar{t} + 1} =$$

$$= \sqrt{r^2+1}\sqrt{2}\sqrt{1-\cos(t_b - \bar{t})} = \sqrt{r^2+1}\sqrt{2}\sqrt{2} = 2\sqrt{r^2+1} \quad \textbf{[CE.7]}$$

Let $\epsilon > 0$ be arbitrary.

If $\epsilon \leq 2$, by **[CE.7]** we have (see Fig. 4.7),

$$d_M(\iota(F_b), y_0) = d_M(\xi_0, y_0) \overset{[\mathrm{CE.7}]}{=} 2 \geq \epsilon$$

which shows that **[RI.2]** fails for $y_\epsilon := y_0 = (\cos \bar{t}, \sin \bar{t}, 0) \in M \setminus F_b$.

If $\epsilon > 2$, consider any $r \in I\!R$ satisfying

$$r > \frac{\sqrt{\epsilon^2 - 4}}{2} = \frac{\sqrt{\epsilon - 2}\sqrt{\epsilon + 2}}{2} > 0$$

this choice of $r$ being possible, because property **[CE.1]** holds for this Counterexample.

In that case, we have,

$$2r > \sqrt{\epsilon^2 - 4} > 0 \Longrightarrow 4r^2 > \epsilon^2 - 4 > 0 \Longrightarrow 4(r^2+1) > \epsilon^2 \Longrightarrow$$

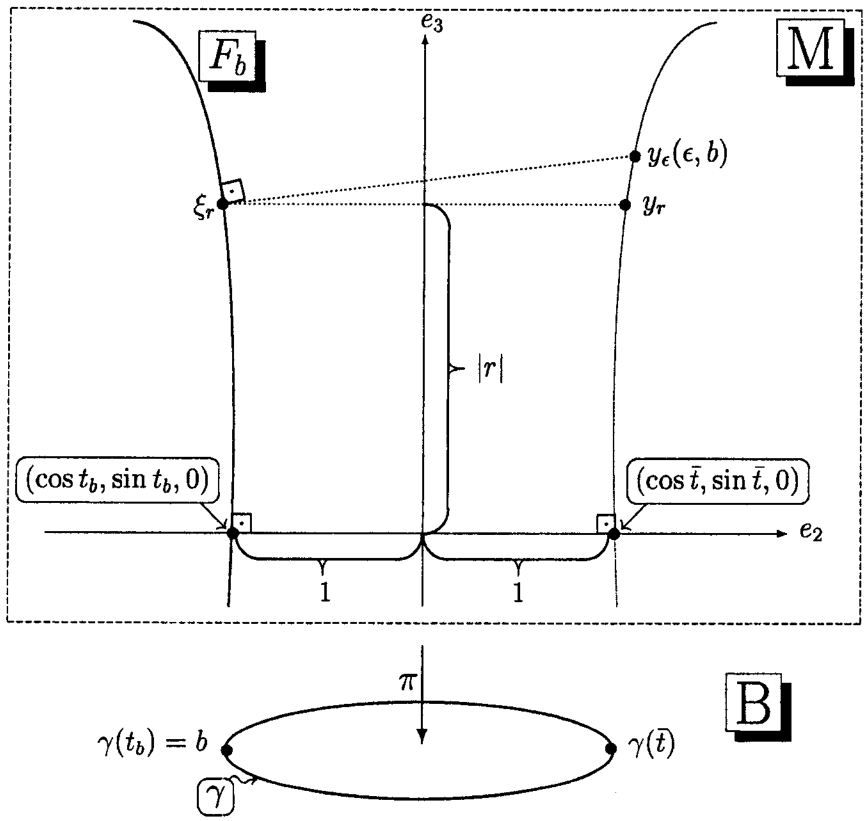


Figure 4.7: Realization of the distance between $y_\epsilon$ and the fiber $F_b$.

$$\Longrightarrow 2\sqrt{r^2+1} > \epsilon \qquad \textbf{[CE.8]}$$

In what follows we will define (see Fig. 4.7),

$$y_\epsilon = \left(\sqrt{r_\epsilon^2+1}\cdot\cos\bar{t}, \sqrt{r_\epsilon^2+1}\cdot\sin\bar{t}, r_\epsilon\right) \in M \setminus F_b$$

satisfying the 2 conditions,

- $r_\epsilon > r$; and

- the unique straight line passing through $y_\epsilon$ and $\xi_r$ is perpendicular to $F_b$ at $\xi_r$, thus giving us the realization of the distance $d_M(y_\epsilon, \iota(F_b)) = d_M(y_{\epsilon,\xi_r})$.

We may assume w.l.o.g., for the sake of a much simplified calculation, that, $t_b = \dfrac{3\pi}{2}$ and $\bar{t} = \dfrac{\pi}{2}$, since $M$ is symmetric with respect to both axis $e_3$ and $e_2$ (see Fig. 4.8).

Thus we have, $\gamma\left(\frac{3\pi}{2}\right) = (0,-1,0) = b, \gamma\left(\frac{\pi}{2}\right) = (0,1,0)$, a generic element $\xi_r = (0, -\sqrt{r^2+1}, r)$ in the fiber $F_b$, and a generic element $y_r = (0, \sqrt{r^2+1}, r)$ in $M$, but not in the fiber $F_b$.

In this case, $F_b$ is given by $x_2 = -\sqrt{x_3^2+1}$, and the perpendicular line to $F_b$ at $\xi_r$ has equation,

$$\begin{aligned} x_2 + \sqrt{r^2+1} &= \frac{-1}{\mu}(x_3 - r) = \\ &= \frac{\sqrt{r^2+1}}{r}(x_3 - r), \quad \forall x_3 \in I\!R \end{aligned} \qquad \textbf{[CE.9]}$$

where $\mu = \dfrac{d}{du} - \sqrt{u^2+1}\,|_{u=r} = \dfrac{-2r}{2\sqrt{r^2+1}}$.

Since $y_\epsilon = \left(0, \sqrt{r_\epsilon^2+1}, r_\epsilon\right)$ is on the line **[CE.9]**, we obtain the following equation,

$$\sqrt{r_\epsilon^2+1} + \sqrt{r^2+1} = \frac{\sqrt{r^2+1}}{r}(r_\epsilon - r) > 0 \qquad \textbf{[CE.10]}$$

which defines $r_\epsilon$.

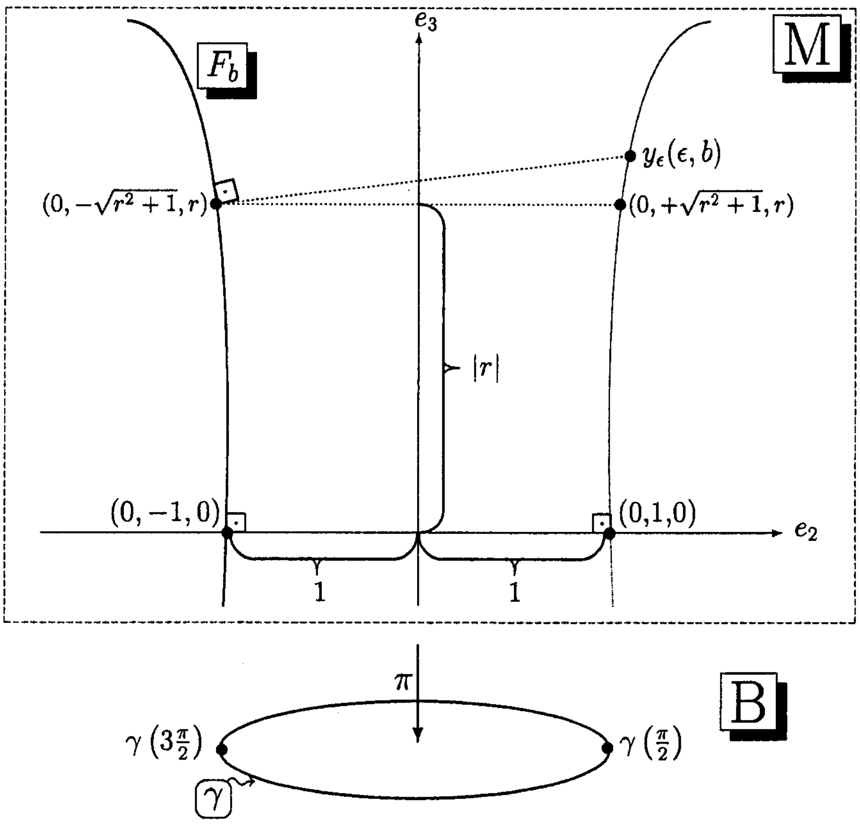


Figure 4.8: Construction of $y_\epsilon = y_\epsilon(\epsilon, b)$ when $t_b = \frac{3\pi}{2}$ and $\bar{t} = \frac{\pi}{2}$.

Indeed, equation **[CE.10]** has only one solution,

$$\sqrt{r_\epsilon^2+1} = \frac{\sqrt{r^2+1}}{r} r_\epsilon - 2\sqrt{r^2+1} = \sqrt{r^2+1}\left(\frac{r_\epsilon}{r} - 2\right) \Rightarrow$$

$$\Rightarrow r_\epsilon^2 + 1 = (r^2+1)\left(\frac{r_\epsilon^2}{r^2} + 4 - 4\frac{r_\epsilon}{r}\right) \Rightarrow$$

$$\Rightarrow r_\epsilon^2 + 1 = r_\epsilon^2 + \frac{r_\epsilon^2}{r^2} + 4(r^2+1) - 4\left(r + \frac{1}{r}\right) r_\epsilon \Rightarrow$$

$$\Rightarrow \frac{1}{r^2}\mathbf{r}_\epsilon^2 - 4\left(r+\frac{1}{r}\right)\mathbf{r}_\epsilon + 4r^2 + 3 = 0 \Rightarrow$$

$$\Rightarrow \mathbf{r}_\epsilon = \frac{4\left(r+\frac{1}{r}\right) \pm \sqrt{\Delta}}{\frac{2}{r^2}}$$

where

$$\begin{aligned}
\Delta &= 16\left(r+\frac{1}{r}\right)^2 - 4\frac{1}{r^2}\left(4r^2+3\right) = 16\left(r^2+2+\frac{1}{r^2}\right) - 16 - \frac{12}{r^2} = \\
&= 16r^2 + 16 + \frac{4}{r^2} = 4\left(4r^2+4+\frac{1}{r^2}\right) = 4\left(2r+\frac{1}{r}\right)^2 \Rightarrow \\
\Rightarrow \sqrt{\Delta} &= 2\left(2r+\frac{1}{r}\right) = \left(4r+\frac{2}{r}\right)
\end{aligned}$$

which implies that,

$$\mathbf{r}_\epsilon = \frac{4\left(r+\frac{1}{r}\right) \pm \left(4r+\frac{2}{r}\right)}{\frac{2}{r^2}} = 2r^3 + 2r \pm \left(2r^3+r\right) = \begin{cases} 4^3 + 3r \text{ (solution)} \\ r \text{ (\underline{not} a solution)} \end{cases}$$

Then, the only solution of equation **[CE.10]** is,

$$\mathbf{r}_\epsilon = 4^3 + 3r = r\left(4^2+3\right) > r > 0 \qquad \textbf{[CE.11]}$$

Finally, by employing the expressions,

$$y_\epsilon = \left(0, \sqrt{r_\epsilon^2+1}, r_\epsilon\right) \quad \text{and} \quad \xi_r = (0, -\sqrt{r^2+1}, r)$$

and by using **[CE.10]** and **[CE.11]** , we can estimate the distance from $y_\epsilon$ to the fiber $F_b$, only in terms of $r$,

$$d_M(y_\epsilon, \xi_r) = \sqrt{0^2 + \left(\sqrt{r_\epsilon^2+1} + \sqrt{r^2+1}\right)^2 + (r_\epsilon - r)^2} \overset{\textbf{[CE.10]}}{=}$$

$$
\begin{aligned}
&= \sqrt{\left(\frac{r^2+1}{r^2}\right)(r_\epsilon - r)^2 + (r_\epsilon - r)^2} = \sqrt{\left(\frac{r^2+1}{r^2}+1\right)(r_\epsilon - r)^2} = \\
&= \sqrt{\frac{(2r^2+1)}{r^2}(r_\epsilon - r)^2} \overset{[\mathbf{CE.11}]}{=} \sqrt{\frac{(2r^2+1)}{r^2}(4r^3+3r-r)^2} = \\
&= \sqrt{(2r^2+1)(4r^2+2)} = \sqrt{2r^2+1}\left(4r^2+2\right) = 2\sqrt{2r^2+1}\left(2r^2+1\right) = \\
&= 2\left(2r^2+1\right)^{\frac{3}{2}}
\end{aligned}
$$

which reduces to,

$$d_M(y_\epsilon, \xi_r) = 2\left(2r^2+1\right)^{\frac{3}{2}} \qquad \textbf{[CE.12]}$$

Next, we claim that,

$$\left(2r^2+1\right)^{\frac{3}{2}} > \sqrt{r^2+1} \qquad \textbf{[CE.13]}$$

In order to prove the claim we define the auxiliary $C^\infty$ function,

$$f(r) := \left(2r^2+1\right)^3 - \left(r^2+1\right), \quad r \in \mathbb{R}$$

and we prove that $f\mid_{[0,\infty]}$ is strictly increasing.

The first and second derivatives of $f$,

$$
\begin{aligned}
\frac{d}{dr}f(r) &= 3\left(2r^2+1\right)^2 4r - 2r = 3\left(4r^4+4r^2+1\right)4r - 2r = \\
&= 2r\underbrace{\left(24r^4+24r^2+5\right)}_{>0} \\
\frac{d^2}{dr^2}f(r) &= 240r^4 + 144r^2 + 10 > 0
\end{aligned}
$$

imply that $f\ |_{[-\infty,0]}$ is strictly decreasing, $f$ has an absolute minimum at $r = 0$, and $f\ |_{[0,\infty]}$ is strictly increasing.

These facts give us the following,

$$f(r) > f(0), \quad \forall r > 0 \Rightarrow$$

$$\Rightarrow \left(2r^2+1\right)^3 - \left(r^2+1\right) > 0 = \left(2(0)^2+1\right)^3 - \left(0^2+1\right), \quad \forall r > 0 \Rightarrow$$

$$\Rightarrow \left(2r^2+1\right)^3 > \left(r^2+1\right) > 0, \quad \forall r > 0 \Rightarrow \left(2r^2+1\right)^{\frac{3}{2}} > \left(r^2+1\right)^{\frac{1}{2}}, \quad \forall r > 0$$

which is claim **[CE.13]**.

Now, if we combine **[CE.12]**, **[CE.13]** and **[CE.8]**, we get,

$$d_M(y_\epsilon, \iota(F_b)) = d_M(y_\epsilon, \xi_r) > 2\sqrt{r^2+1} > \epsilon$$

which shows that **(RI.2)** fails for $\iota : F_b \to M$, and consequently, the inclusion map $\iota$ is NOT a rough isometry.

This describes the Counterexample.

Next, including a lower bound in assumption ( 4.14), with an extra universal property of the fibers, we can show that $\pi : M \to B$ is a rough isometry, as follows.

**Theorem 4.2.3** *Let $\pi : M \to B$ be an onto smooth map with maximal rank, where $B$ is complete. Assume the following,*

**(UP)** *$\exists m > 0$, a universal constant, such that diam $\{\pi^{-1}(b)\} \le m < \infty$, for all $b \in B$; and*

**(HLC)** *$\exists \alpha \ge 1$ and $\beta > 0$ such that, for all $b \in B$ the inequality holds:*

$$\frac{1}{\alpha}||w||_B - \beta \le ||v||_M \le \alpha||w||_B + \beta$$

*for all $x \in F_b$ and $w \in T_bB$, where $v \in (HT)_x \subset T_xM$ is the horizontal lift of $w$ through $x$ and we assume that $v$ satisfies $||v||_M \le 1$.*

*Then, $\pi : M \to B$ is a rough isometry.*

*In particular, if for some $b_0$ the fiber $\pi^{-1}(b_0)$ is compact, then $M$ is roughly isometric to the product $\pi^{-1}(b_0) \times B$.*

*Proof.* Firstly, note that in **(HLC)** the horizontal lift $v \in (HT)_x$ of $w$ is assumed to satisfy $||v||_M \le 1$. Otherwise, if $||v||_M > 1$ we define $\tilde{v} := \frac{v}{||v||_M}$, with the properties

- $\tilde{v} := \frac{v}{||v||_M} \in (HT)_x$
- $||\tilde{v}||_M = 1$
- $(\pi_*)_x(\tilde{v}) = \frac{w}{||v||_M}$

and if we use $\dfrac{w}{||v||_M}$ and $\tilde{v}$ in **(HLC)**, we thus obtain the equivalent inequality,

$$
\begin{aligned}
& \frac{1}{\alpha}\left|\left|\frac{w}{||v||_M}\right|\right|_B - \beta \leq ||\tilde{v}||_M \leq \alpha\left|\left|\frac{w}{||v||_M}\right|\right|_B + \beta \Rightarrow \\
\Rightarrow\ & \frac{1}{\alpha}\frac{||w||_B}{||v||_M} - \beta \leq \frac{||v||_M}{||v||_M} \leq \alpha\frac{||w||_B}{||v||_M} + \beta \Rightarrow \\
\Rightarrow\ & \frac{1}{\alpha}||w||_B - \beta||v||_M \leq ||v||_M \leq \alpha||w||_B + \beta||v||_M \Rightarrow \\
\Rightarrow\ & \frac{1}{\alpha}||w||_B \leq (\beta+1)\,||v||_M \wedge (1-\beta)\,||v||_M \leq \alpha||w||_B \Rightarrow \\
\Rightarrow\ & \begin{cases} \dfrac{1}{\alpha\,(\beta+1)}||w||_B \leq ||v||_M \leq \dfrac{\alpha}{(1-\beta)}||w||_B, & \text{if } \beta \neq 1 \\ \dfrac{1}{\alpha\,(\beta+1)}||w||_B \leq ||v||_M, & \text{if } \beta = 1 \end{cases}
\end{aligned}
$$

for $w \in T_bB$, where $v$ is the unique horizontal lift of $w$ through $x$ with $||v||_M > 1$.

We must verify the validity of **(RI.1)** and **(RI.2)**.

Since $\pi$ is onto,

$$B = \mathrm{Im}\pi = \{y \in B : d(y, \mathrm{Im}\pi) < \epsilon\} = B_\epsilon(\mathrm{Im}\pi), \quad \forall\epsilon > 0$$

which is **(RI.2)** for any $\epsilon > 0$.

To verify **(RI.1)**, let $x, y \in M$.

W.l.o.g. we may assume that $B$ is connected. Otherwise, we repeat the argument which will be utilized in this proof, on each connected component and the result will follow.

$B$ being complete, implies that there exists a minimal geodesic $\gamma$ joining

$\pi(x)$ to $\pi(y)$, where,

$$\begin{cases} \gamma : [0,1] & \to & B, \\ \gamma(0) & := & \pi(x), \quad \text{and} \quad \ell(\gamma) = d_B(\pi(x), \pi(y)) \\ \gamma(1) & := & \pi(y), \end{cases}$$

Now, **Proposition 4.8** guarantees that $\gamma$ has a unique horizontal lift $\Gamma_x : [0,1] \to M$, through $x$, satisfying,

**(hl.1)** $\pi \circ \Gamma_x = \gamma$;

**(hl.2)** $\Gamma_x(0) = x$ and $\Gamma_x(1) \in \pi^{-1}\{\gamma(1)\} = \pi^{-1}\{\pi(y)\} = F_{\pi(y)}$. In particular, $\Gamma_x$ intersects the fiber $F_{\pi(y)}$ containing $y$;

**(hl.3)** $\Gamma_x$ is horizontal, i.e., ${\Gamma_x}'(t) \in (HT)_{\Gamma_x(t)}$.

We may assume w.l.o.g. that $\Gamma_x$ is p.p.a.l. and $||{\Gamma_x}'(t)||_M \leq 1$ for all $t \in [0,1]$.

Thus we can write,

$$\begin{aligned} \ell(\Gamma_x) &= \int_0^1 ||\Gamma_x'||_M dt \overset{(HLC)}{\leq} \alpha \int_0^1 ||(\pi_*)_{\Gamma_x(t)}\Gamma_x'(t)||_B dt + \beta = \\ &= \alpha \int_0^1 ||\gamma'(t)||_B dt + \beta = \alpha \cdot \ell(\gamma) + \beta \overset{def \gamma}{=} \\ &= \alpha \cdot d_B(\pi(x), \pi(y)) + \beta \end{aligned} \tag{4.18}$$

By the triangle inequality, by hypothesis and the above, we have,

$$d_M(x,y) \;\leq\; d_M(x, \Gamma_x(1)) + d_M(\Gamma_x(1), y) \leq$$

$$\begin{aligned} &\leq \ell(\Gamma_x) + d_M(\Gamma_x(1), y) \overset{(\mathbf{UP})}{\leq} \ell(\Gamma_x) + m \overset{(4.18)}{\leq} \\ &\leq \alpha \cdot d_B(\pi(x), \pi(y)) + \beta + m \Rightarrow \\ &\Rightarrow d_B(\pi(x), \pi(y)) \geq \frac{1}{\alpha} d_M(x, y) - \frac{(\beta + m)}{\alpha} \end{aligned} \tag{4.19}$$

Now, we claim that for $\gamma$, the minimal geodesic joining $\pi(x)$ to $\pi(y)$, its length $\ell(\gamma)$ satisfies,

$$d_B(\pi(x), \pi(y)) = \ell(\gamma) \leq \alpha \cdot \ell(\varsigma) + \alpha \cdot \beta \tag{4.20}$$

for any smooth curve $\varsigma : [0, 1] \to M$, joining $x$ to $y$.

First, observe that for any orthogonal vectors $U$ and $W$,

$$||U \oplus W||^2 = ||U||^2 + ||W||^2 \geq \max\{||U||^2, ||W||^2\}$$

Now, since each tangent vector is the direct sum of a horizontal and a vertical vector, we can write,

$$\ell(\varsigma) = \int_0^1 ||\varsigma'(t)||_M dt = \int_0^1 ||\varsigma'_H(t) \oplus \varsigma'_V(t)||_M dt \geq \int_0^1 ||\varsigma'_H(t)||_M dt \tag{4.21}$$

where we are assuming here that $\varsigma_H$ is a p.p.a.l. and $||\varsigma'_H(t)||_M \leq 1, \forall t \in [0, 1]$.

By **Lemma 4.3** we have,

$$\ell(\pi \circ \varsigma) = \int_0^1 ||(\pi_*)_{\varsigma(t)} \varsigma'(t)||_B dt = \int_0^1 ||(\pi_*)_{\varsigma(t)} \varsigma'_H(t)||_B dt \tag{4.22}$$

From the left-hand side of **(HLC)**,

$$\frac{1}{\alpha} \cdot ||(\pi_*)_{\varsigma(t)}\varsigma'_H(t)||_B - \beta \leq ||\varsigma'_H(t)||_M \Rightarrow$$
$$\Rightarrow \quad ||(\pi_*)_{\varsigma(t)}\varsigma'_H(t)||_B \leq \alpha \cdot ||\varsigma'_H(t)||_M + \alpha \cdot \beta \tag{4.23}$$

for all $t \in [0,1]$.

If we combine ( 4.21), ( 4.22) and ( 4.23), we get,

$$\ell(\pi \circ \varsigma) \overset{(\,4.22)}{=} \int_0^1 ||(\pi_*)_{\varsigma(t)}\varsigma'_H(t)||_B dt \overset{(\,4.23)}{\leq} \alpha \int_0^1 ||\varsigma'_H(t)||_M + \alpha\beta \leq$$
$$\overset{(\,4.21)}{\leq} \alpha\ell(\varsigma) + \alpha\beta$$

The above together with the fact that $\gamma$ is the minimal geodesic joining $\pi(x)$ to $\pi(y)$ implies that we can finally write,

$$d_B(\pi(x), \pi(y)) = \ell(\gamma) \leq \ell(\pi \circ \varsigma) \leq \alpha\ell(\varsigma) + \alpha\beta$$

for any smooth curve $\varsigma : [0,1] \to M$, joining $x$ to $y$, which is claim ( 4.20).

We recall that by definition of infimum, $d_M(x,y)$ is the greatest lower bound for $\{\ell(\varsigma)$, where $\varsigma : [0,1] \to M$ is any smooth curve joining $x$ to $y\}$, and since $\varsigma$ is arbitrary in ( 4.20), we obtain,

$$d_B(\pi(x), \pi(y)) \leq \alpha \cdot d_M(x,y) + \alpha \cdot \beta \tag{4.24}$$

Let $A := \alpha \geq 1$ and $C := \max\left\{\dfrac{\beta + m}{\alpha}, \alpha \cdot \beta\right\} > 0$.

If we now, rewrite ( 4.19) and ( 4.24) in terms of $A$ and $C$, as follows,

$$\begin{aligned}\frac{1}{A}d_M(x,y) - C &\leq \frac{1}{\alpha}d_M(x,y) - \frac{(\beta+m)}{\alpha} \overset{(4.19)}{\leq} \\ &\leq d_B(\pi(x),\pi(y)) \overset{(4.24)}{\leq} \alpha d_M(x,y) + \alpha\beta \leq A d_M(x,y) + C\end{aligned}$$

we obtain **(RI.1)** for $\pi$.

□

In the following 3 Counterexamples, we show that if in **Theorem 4.2.3** we drop either the universal property of the fibers or the control over the length of horizontal lifts of tangent vectors, $\pi : M \to B$ ceases to be a rough isometry.

**Counterexample 4.2.4** *We will exhibit $M, B, \pi$, where $B$ is connected and $\forall b \in B$ the fiber $F_b$ is compact, satisfying all conditions in Theorem 4.2.3 with the exception of* **(HLC)**, *which does not hold in this Counterexample, i.e.,*

*For each given constants $\alpha \geq 1$ and $\beta > 0$, there exist $\bar{b} \in B, \bar{x} \in F_{\bar{b}}, \bar{w} \in T_{\bar{b}}B$ such that either one of the following holds:*

$$||\bar{v}||_M > \alpha||\bar{w}||_B + \beta \text{ or } ||\bar{v}||_M < \frac{1}{\alpha}||\bar{w}||_B - \beta \qquad \textbf{[CE.14]}$$

*where $\bar{v}$ is the unique horizontal lift of $\bar{w}$ through $\bar{x}$.*

*In this case, the map $\pi : M \to B$ is not a rough isometry.*

Let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 1, y \in \mathbb{R}\}$ and $B = \mathbb{R}$, a complete and connected Riemannian manifold (see Fig. 4.9).

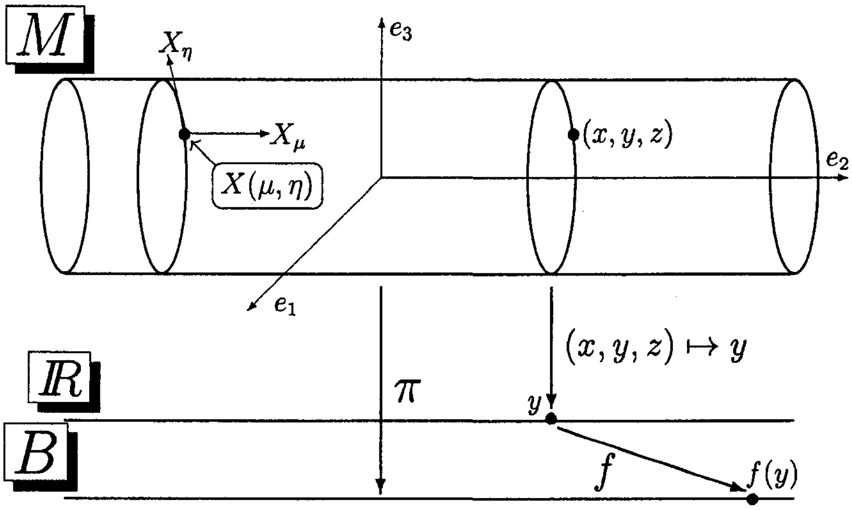


Figure 4.9: Manifolds $M, B$ and the map $\pi$ in **Counterexample 4.2.4**.

We first define an auxiliary $C^1$-diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ by,

$$f(y) = \begin{cases} e^y - 1 & \in [0, \infty) \quad \text{if } y \geq 0 \\ 1 - e^{-y} & \in (-\infty, 0] \quad \text{if } y \leq 0 \end{cases} \qquad \textbf{[CE.15]}$$

with inverse,

$$f^{-1}(z) = \begin{cases} \ln(z+1) & \in [0, \infty) \quad \text{if } z \geq 0 \\ -\ln(1-z) & \in (-\infty, 0] \quad \text{if } z \leq 0 \end{cases} \qquad \textbf{[CE.16]}$$

and derivative,

$$f'(y) = \begin{cases} e^y & \text{if } y \geq 0 \\ e^{-y} & \text{if } y \leq 0 \end{cases} \qquad \textbf{[CE.17]}$$

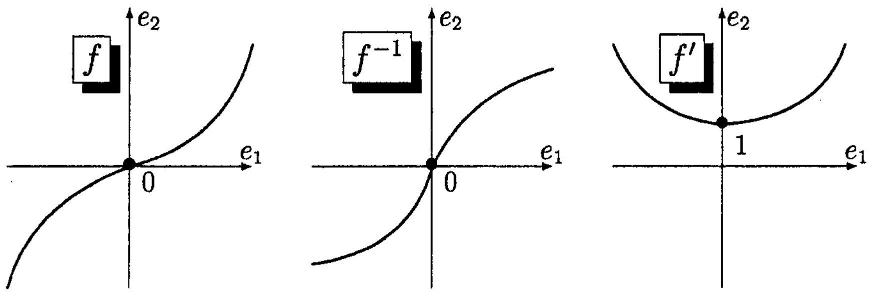


Figure 4.10: Graphs of $f, f^{-1}, f'$.

which is never zero.

Let $\pi : M \to B$ be given by $\pi(x, y, z) := f(y)$ (see Fig. 4.9). The map $\pi$ is onto and $\mathcal{C}^\infty$, since both the projection $(x, y, z) \mapsto y$ and $f$ have those properties.

*(Claim 1)* The rank of $\pi$ is maximal.

The Jacobian of $\pi_*$ is given by,

$$(\pi_*)_{(x,y,z)} = \begin{bmatrix} 0 & f'(y) & 0 \end{bmatrix} \quad \forall (x, y, z) \in M$$

which has maximal rank 1, since $f'$ never vanishes (see **[CE.17]**).

Next, we observe that the space of vertical vectors at $(x, y, z) \in M$ is,

$$\begin{aligned} (VT)_{(x,y,z)} &= \ker(\pi_*)_{(x,y,z)} = \\ &= \{\text{vectors tangent to the parallels } y{=}\text{constant}\} \cong T\mathcal{S}^1 \end{aligned}$$

and the space of horizontal vectors at $(x, y, z) \in M$ is,

$$(HT)_{(x,y,z)} = \left(\ker(\pi_*)_{(x,y,z)}\right)^{\perp} =$$

$$= \{\text{vectors tangent to the meridians } x{=}\text{constant}\wedge z{=}\text{constant}\} \cong \mathbb{R}$$

In order to describe explicitly the horizontal and vertical vectors, we will use the following global parametrization of $M$,

$$X(\mu, \eta) = (\cos\eta, \mu, \sin\eta), \qquad \mu \in \mathbb{R}, \eta \in [0, 2\pi]$$

Note that $\pi \circ X(\mu, \eta) = \mu, \quad \forall \mu \in \mathbb{R}, \forall \eta \in [0, 2\pi]$.

The parallels are the curves,

$$\eta \in [0, 2\pi] \mapsto X(\mu_{\text{const.}}, \eta) = (\cos\eta, \mu_{\text{const.}}, \sin\eta) \in M$$

so that the space of vertical vectors is generated by,

$$\left\langle X_\eta(\mu_{\text{const.}}, \eta) := \frac{\partial X}{\partial \eta}(\mu_{\text{const.}}, \eta) = (-\sin\eta, 0, \cos\eta)\right\rangle$$

The meridians are the curves,

$$\mu \in \mathbb{R} \mapsto X(\mu, \eta_{\text{const.}}) = (\cos\eta_{\text{const.}}, \mu, \sin\eta_{\text{const.}}) \in M$$

so that the space of horizontal vectors is generated by,

$$\left\langle X_\mu(\mu, \eta_{\text{const.}}) := \frac{\partial X}{\partial \mu}(\mu, \eta_{\text{const.}}) = (0, 1, 0)\right\rangle$$

*(Claim 2)* The fiber $F_b$ is <u>compact</u> and diam $F_b \leq m, \forall b \in B$, where $m = 3 > 0$ is the <u>universal</u> upper bound for the fibers' diameters.

Each fiber is given by,

$$F_b = \pi^{-1}(b) = \{(x, y, z) \in M : f(y) = b\}, \quad b \in B = I\!R$$

and the fact that,

$$f(y) = b \Leftrightarrow y = f^{-1}(b) \overset{\textbf{[CE.16]}}{\Leftrightarrow}$$

$$\Leftrightarrow \quad \{y = \ln(b+1), \text{ if } b \geq 0\} \vee \{y = -\ln(1-b), \text{ if } b \leq 0\}$$

together with the definition of $M$, imply that the fibers have either form,

$$\begin{aligned} F_b &= \{(x, \ln(b+1), z) \in I\!R^3 : x^2 + z^2 = 1\}, \text{ if } b \geq 0 \;\; \vee \\ F_b &= \{(x, -\ln(1-b), z) \in I\!R^3 : x^2 + z^2 = 1\}, \text{ if } b \leq 0 \end{aligned}$$

which is a unit circle in the plane $y = f^{-1}(b)$, and therefore a compact subset of $I\!R^3$ with diameter $2 < m = 3$.

*(Claim 3)* $\forall \alpha \geq 1, \forall \beta > 0, \exists \bar{b} \in B, \exists \bar{x} \in F_{\bar{b}}, \exists \bar{w} \in T_{\bar{b}}B, \exists \bar{v} \in T_{\bar{x}}M$ : $(\pi_*)_{\bar{x}}\bar{v} = \bar{w}$, such that **[CE.14]** verifies.

Let $\alpha \geq 1$ and $\beta > 0$ be any given constants.

Define,

$$\bar{b} \;:=\; \overbrace{2\alpha - 1}^{>0} \in B$$

$$\begin{aligned}
\bar{x} &:= (1, \underbrace{\ln(2\alpha)}_{>0}, 0) \in F_{\bar{b}}, \text{ since } \pi\bar{x} = f(\ln 2\alpha) = e^{\ln 2\alpha} - 1 = 2\alpha - 1 = \bar{b} \\
\bar{w} &:= 2\alpha(\beta+1) \in T_{\bar{b}}B \\
\bar{v} &:= (0, \beta+1, 0) \in T_{\bar{x}}M, \text{ the unique horizontal lift of } \bar{w} \text{ through } \bar{x}, \\
& \qquad \text{since } (\pi_*)_{\bar{x}}\bar{v} = f'(\ln 2\alpha)(\beta+1) = e^{\ln 2\alpha}(\beta+1) = 2\alpha(\beta+1) = \bar{w}
\end{aligned}$$

Moreover,

$$||\bar{w}||_B = 2\alpha(\beta+1), ||\bar{v}||_M = \beta + 1$$

implies that,

$$\begin{aligned}
||\bar{v}||_M - \frac{1}{\alpha}||\bar{w}||_B + \beta &= \beta + 1 - \frac{1}{\alpha}2\alpha(\beta+1) + \beta = -\beta - 1 + \beta = -1 < 0 \\
&\Rightarrow ||\bar{v}||_M < \frac{1}{\alpha}||\bar{w}||_B - \beta
\end{aligned}$$

which is **[CE.14]**.

We remark here that there is also a choice for $\tilde{b} < 0$ an element of $B$, as well as for $\tilde{x} \in F_{\tilde{b}}, \tilde{w} \in T_{\tilde{b}}B, \tilde{v} \in T_{\tilde{x}}M : (\pi_*)_{\tilde{x}}\tilde{v} = \tilde{w}$, providing us with $||\tilde{v}||_M > \alpha||\tilde{w}||_B + \beta$.

*(Claim 4)* $\pi$ does not satisfy **(RI.1)**.

It suffices to verify that **(RI.1)** fails for $\pi$, for particular pairs of elements in $M$. We will show that $\forall A \geq 1, \forall C > 0, \exists y_{AC} \in \mathbb{R}$, a positive number such that,

$$d_B(\pi(x,0,z), \pi(x,y,z)) > A \cdot d_M((x,0,z),(x,y,z)) + C \Leftrightarrow$$

$$\Leftrightarrow \quad |f(0) - f(y)| > A \cdot y + C \Leftrightarrow e^y - 1 > A \cdot y + C \qquad \text{[CE.18]}$$

for all $y > y_{AC}$, where $x, z \in \mathbb{R} : x^2 + z^2 = 1$ are arbitrary.

Let $A \geq 1$ and $C > 0$ be any given constants.

We begin by introducing an auxiliary $C^\infty$-function $g$,

$$\begin{aligned} g : \mathbb{R} &\rightarrow \mathbb{R} \\ y &\mapsto g(y) := e^y - 1 - Ay - C \end{aligned}$$

We will guarantee **[CE.18]**, by showing that,

$$\exists y_{AC} > 0 : \forall y > y_{AC} \Rightarrow g(y) > 0 \qquad \text{[CE.19]}$$

In what follows, we study the functional behavior of $g$ through its derivatives, which will enable us to conclude **[CE.19]**.

| $y$ | $g(y) = e^y - 1 - Ay - C$ | $g'(y) = e^y - A$ | $g^{(\kappa)}(y) = e^y, \kappa \geq 2$ |
|---|---|---|---|
| $0$ | $-C$ | $1 - A$ | $1$ |
| $\ln A$ | $A \cdot \ln \frac{e}{A} - 1 - C$ | $0$ | $A$ |
| $+\infty$ | $+\infty$ | $+\infty$ | $+\infty$ |
| $-\infty$ | $+\infty$ | $(-A)^+$ | $0^+$ |

| *(case $A > 1$)* | $(-\infty, 0)$ | $(0, \ln A)$ | $(\ln A, +\infty)$ |
|---|---|---|---|
| $g'$ | $-$ | $-$ | $+$ |
| $g$ (monotonicity) | $\searrow$ | $\searrow$ | $\nearrow$ |
| $g''$ | $+$ | $+$ | $+$ |
| $g$ (concavity) | $\cup$ | $\cup$ | $\cup$ |

| *(case $A = 1$)* | $(-\infty, 0)$ | $(0, +\infty)$ |
|---|---|---|
| $g'$ | $-$ | $+$ |
| $g$ (monotonicity) | $\searrow$ | $\nearrow$ |
| $g''$ | $+$ | $+$ |
| $g$ (concavity) | $\cup$ | $\cup$ |

The tables above, the facts that $g$ is continuous, $g(0) = -C < 0$, $\lim_{y\to+\infty} g(y) = +\infty$ and $g\,|_{(\ln A,+\infty)}$ is strictly increasing, allow us to conclude that, there exists a real number $y_{AC} > 0$ for which $g(y_{AC}) = 0$ and $g\,|_{(y_{AC},+\infty)}> 0$, which is exactly **[CE.19]**.

Therefore, **[CE.18]** holds and (Claim 4) follows, which shows that $\pi$ is NOT a rough isometry.

This describes the Counterexample.

**Counterexample 4.2.5** *We will exhibit $M, B, \pi$, where $B$ is connected and $\forall b \in B$ the fiber $F_b$ is compact, satisfying all the conditions in Theorem 4.2.3, with the exception of* **(UP)**, *which does not hold in this Counterexample, i.e.,*

*The fibers' diameters are not uniformly bounded, in other words:*

$$\forall m > 0, \exists b_m \in B : diam\ F_{b_m} > m$$

*In this case, the map $\pi : M \to B$ is not a rough isometry.*

*In particular, we note that in this Counterexample* **(HLC)** *does not hold either, i.e.,*

*For given constants $\alpha \geq 1$ and $\beta > 0$, there exist $\bar{b} \in B, \bar{x} \in F_{\bar{b}}, \bar{w} \in T_{\bar{b}}B$ such that either one of the following holds:*

$$||\bar{v}||_M > \alpha||\bar{w}||_B + \beta \ or\ ||\bar{v}||_M < \frac{1}{\alpha}||\bar{w}||_B - \beta \qquad \textbf{[CE.20]}$$

*where $\bar{v}$ is the unique horizontal lift of $\bar{w}$ through $\bar{x}$.*

Let $\vartheta = \{(x,y,z) \in I\!R^3 : x = 0 \wedge z = y^2 + 1\}$ be a curve in $I\!R^3$, and let $M$ be the rotation hypersurface of $\vartheta$ , around the $y$-axis, described by $M = \{(x,y,z) \in I\!R^3 : x^2 + z^2 = (y^2+1)^2\}$. Let $B = I\!R$, a complete and connected Riemannian manifold (see Fig. 4.11).

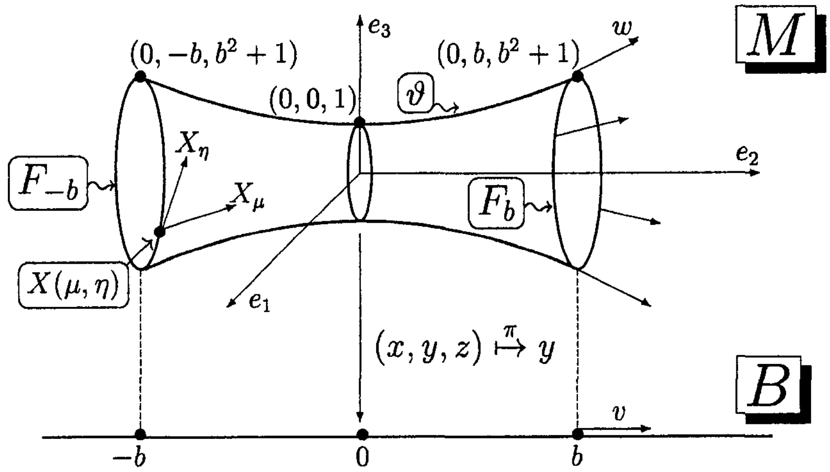


Figure 4.11: Manifolds $M, B$ and the map $\pi$ in **Counterexample 4.2.5**.

Let $\pi : M \to B$ be the projection $\pi(x,y,z) := y$, which is onto and $\mathcal{C}^\infty$. Note that $\pi$ has maximal rank=1, since the Jacobian of $\pi_*$ is given by,

$$(\pi_*)_{(x,y,z)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \forall (x,y,z) \in M$$

Each fiber is given by,

$$F_b \;=\; \pi^{-1}(b) = \{(x,y,z) \in M : y = b\} =$$

$$= \{(x,b,z) \in \mathbb{R}^3 : x^2 + z^2 = (b^2+1)^2\}, \quad b \in B = \mathbb{R}$$

which is a circle centered at $(0,b,0)$, with radius $(b^2+1)$, contained in the plane $y = b$. Hence each fiber is a compact subset of $\mathbb{R}^3$ with diameter diam $F_b = 2(b^2+1)$, which grows without any bounds as $b \to \pm\infty$ and therefore not uniformly bounded.

Next, we want to justify **[CE.20]**.

The space of vertical vectors at $(x,y,z) \in M$ is,

$$(VT)_{(x,y,z)} = \ker(\pi_*)_{(x,y,z)} =$$

$$= \{\text{vectors tangent to the parallels } y{=}\text{constant}\} \cong T\mathcal{S}^1$$

and the space of horizontal vectors at $(x,y,z) \in M$ is,

$$(HT)_{(x,y,z)} = \left(\ker(\pi_*)_{(x,y,z)}\right)^{\perp} =$$

$$= \{\text{vectors tangent to the meridians } \{(x,y,z) \in M, x = az : a \in \mathbb{R}\}\}$$

$$= \left\{\text{vectors tangent to the curves } y \in \mathbb{R} \mapsto \left(\frac{a\,(y^2+1)}{\sqrt{a^2+1}}, y, \frac{y^2+1}{\sqrt{a^2+1}}\right)\right\}$$

In order to describe explicitly the horizontal and vertical vectors, we will use the following global parametrization of $M$,

$$X(\mu,\eta) = \left((\mu^2+1)\cos\eta, \mu, (\mu^2+1)\sin\eta\right), \quad \mu \in \mathbb{R}, \eta \in [0,2\pi]$$

Note that $\pi \circ X(\mu,\eta) = \mu, \quad \forall \mu \in \mathbb{R}, \forall \eta \in [0,2\pi]$.

The parallels are the curves,

$$\eta \in [0, 2\pi] \mapsto X(\mu_{\text{const.}}, \eta) =$$
$$= \left(\left(\mu_{\text{const.}}^2 + 1\right)\cos\eta, \mu_{\text{const.}}, \left(\mu_{\text{const.}}^2 + 1\right)\sin\eta\right) \in M$$

so that the space of vertical vectors is generated by,

$$\left\langle X_\eta(\mu_{\text{const.}}, \eta) := \frac{\partial X}{\partial \eta}(\mu_{\text{const.}}, \eta) = \left(\mu_{\text{const.}}^2 + 1\right)(-\sin\eta, 0, \cos\eta)\right\rangle$$

The meridians are the curves,

$$\mu \in \mathbb{R} \mapsto X(\mu, \eta_{\text{const.}}) = \left(\left(\mu^2 + 1\right)\cos\eta_{\text{const.}}, \mu, \left(\mu^2 + 1\right)\sin\eta_{\text{const.}}\right) \in M$$

so that the space of horizontal vectors is generated by,

$$\left\langle X_\mu(\mu, \eta_{\text{const.}}) := \frac{\partial X}{\partial \mu}(\mu, \eta_{\text{const.}}) = (2\mu \cdot \cos\eta_{\text{const.}}, 1, 2\mu \cdot \sin\eta_{\text{const.}})\right\rangle$$

The above determines the general expression of a horizontal lift: for each $b \in B = \mathbb{R}$ and $w \in T_bB \cong \mathbb{R}$, the unique horizontal lift $v$ of $w$ through a point $X(b, \bar{\eta}) \in F_b \subset M$ is represented by the vector,

$$v = w \cdot X_\mu(b, \bar{\eta}) = w \cdot (2b \cdot \cos\bar{\eta}, 1, 2b \cdot \sin\bar{\eta}) \in (HT)_{X(b,\bar{\eta})} \subset T_{X(b,\bar{\eta})}M$$

since $(\pi_*)_{X(b,\bar{\eta})}(v) = w \cdot (\pi_*)_{X(b,\bar{\eta})}(2b \cdot \cos\bar{\eta}, 1, 2b \cdot \sin\bar{\eta}) = w \cdot 1 = w$.

In particular, $||v||_M = |w| \cdot \sqrt{4b^2 + 1} = ||w||_B \cdot \sqrt{4b^2 + 1}$, which implies that,

$$\forall \alpha \geq 1, \forall \beta > 0, \exists b_{\alpha,\beta} > 0, \exists w_{\alpha,\beta} \in T_{b_{\alpha,\beta}}B \cong \mathbb{R} : ||v_{\alpha,\beta}||_M > \alpha ||w_{\alpha,\beta}||_B + \beta$$

where $w_{\alpha,\beta}$ is the unique horizontal lift of $v_{\alpha,\beta}$ through $X(b_{\alpha,\beta},\bar{\eta}) \in F_b \subset M$, for $\bar{\eta} \in [0, 2\pi]$. This guarantees **[CE.20]**.

Our goal here is to show that $\pi$ is not a rough isometry. One can accomplish that by verifying the failure of **(RI.1)** for $\pi$. It suffices to verify that **(RI.1)** fails for $\pi$, for particular pairs of elements in $M$.

Hence, we will accomplish our goal if we show that $\forall A \geq 1$, $\forall C > 0$, $\exists y_{AC} \in \mathbb{R}$, a positive number such that,

$$d_B\left(\pi(0,0,1), \pi(0,y,y^2+1)\right) < \frac{1}{A} \cdot d_M\left((0,0,1),(0,y,y^2+1)\right) - C \Leftrightarrow$$
$$\Leftrightarrow |0-y| < \frac{1}{A} \cdot d_M\left((0,0,1),(0,y,y^2+1)\right) - C, \quad \forall y > y_{AC} \qquad \textbf{[CE.21]}$$

where, the curve $\{y \in \mathbb{R} \mapsto \vartheta(y) := (0, y, y^2+1) = X(y, \frac{\pi}{2}) \in M\}$ is a meridian of $M$, and therefore a geodesic.

Observe that one may express distance in terms of $\vartheta$, since it is a geodesic, so we may write for all $y \geq 0$,

$$d_M\left((0,0,1),(0,y,y^2+1)\right) = \int_0^y ||\vartheta'(y)||_M dy = \int_0^y \sqrt{1+4y^2}dy \qquad \textbf{[CE.22]}$$

If we incorporate **[CE.22]**reformulation of claim **[CE.21]**

$$\forall A \geq 1, \forall C > 0, \exists y_{AC} \in \mathbb{R}, y_{AC} > 0 :$$
$$d_B\left(\pi(0,0,1), \pi(0,y,y^2+1)\right) < \frac{1}{A} \cdot d_M\left((0,0,1),(0,y,y^2+1)\right) - C \Leftrightarrow$$
$$\Leftrightarrow y < \frac{1}{A} \cdot \int_0^y \sqrt{1+4y^2}\ dy - C, \quad \forall y > y_{AC} > 0 \qquad \textbf{[CE.23]}$$

Let $A \geq 1$ and $C > 0$ be any given constants and define an auxiliary $C^\infty$-function $h$ by,

$$\begin{aligned} h : I\!R &\rightarrow I\!R \\ y &\mapsto h(y) := y - \frac{1}{A} \cdot \int_0^y \sqrt{1+4y^2}\ dy + C \end{aligned}$$

Note that claim **[CE.23]**

$$\exists y_{AC} > 0 : \forall y > y_{AC} \Rightarrow h(y) < 0 \qquad \textbf{[CE.24]}$$

In what follows, we will study the functional behavior of $h$ through its derivatives, which will enable us to conclude **[CE.24]**

The algebraic expressions of $h, h', h''$ are, respectively,

$$\begin{aligned} h(y) &= y - \frac{1}{A} \cdot \int_0^y \sqrt{1+4y^2}\ dy + C \\ h'(y) &= 1 - \frac{1}{A} \cdot \sqrt{1+4y^2} \\ h''(y) &= \frac{-4y}{A\sqrt{1+4y^2}} \end{aligned}$$

| $y$ | $h(y)$ | $h'(y)$ | $h''(y)$ |
|---|---|---|---|
| $0$ | $C > 0$ | $1 - \frac{1}{A} \geq 0$ | $0$ |
| $\frac{\sqrt{A^2-1}}{2} \geq 0$ | $h\left(\frac{\sqrt{A^2-1}}{2}\right)$ | $0$ | $\frac{-2\sqrt{A^2-1}}{A^2} \leq 0$ |
| $+\infty$ | $-\infty$ | $-\infty$ | $\frac{-2}{A} \leq 0$ |

| *(case $A > 1$)* | $\left(0, \dfrac{\sqrt{A^2-1}}{2}\right)$ | $\left(\dfrac{\sqrt{A^2-1}}{2}, +\infty\right)$ |
|---|---|---|
| $h'$ | $+$ | $-$ |
| $h$ (monotonicity) | $\nearrow$ | $\searrow$ |
| $h''$ | $-$ | $-$ |
| $h$ (concavity) | $\cap$ | $\cap$ |

| *(case $A = 1$)* | $(0, +\infty)$ |
|---|---|
| $h'$ | $-$ |
| $h$ (monotonicity) | $\searrow$ |
| $h''$ | $-$ |
| $h$ (concavity) | $\cap$ |

The tables above, plus the facts that $h$ is continuous, $h(0) = C > 0$, $\lim_{y\to+\infty} h(y) = -\infty$, $h$ has an absolute maximum at $\dfrac{\sqrt{A^2-1}}{2}$ and $h$ restricted to the open interval $\left(\dfrac{\sqrt{A^2-1}}{2}, +\infty\right)$ is strictly decreasing, allow us to conclude that, there exists a real number $y_{AC} > 0$ for which $h(y_{AC}) = 0$ and $h\,|_{(y_{AC},+\infty)} < 0$, which is exactly **[CE.24]**

Therefore, **[CE.21]** holds and **(RI.1)** fails for $\pi$, which shows that $\pi$ is NOT a rough isometry.

This describes the Counterexample.

**Counterexample 4.2.6** *We will exhibit $M, B, \pi$, where $B$ is connected, satisfying all the conditions from* Theorem 4.2.3, *with the exception of* **(UP)**, *which does* <u>*not*</u> *hold in this Counterexample, i.e.,* *The fibers' diameters are not uniformly bounded, in other words:*

$$\forall m > 0, \exists b_m \in B : diam\ F_{b_m} > m$$

*In this case, the map $\pi : M \to B$ is not a rough isometry.*

*In particular, we note that for each $b \in B$, the fiber $F_b$ is not compact and condition* **(HLC)** *holds.*

*Proof.* Let $M = \{(0, y, z) \in I\!R^3\} \cong \{0\} \times I\!R^2$ and $B = I\!R$, a complete and connected Riemannian manifold (see Fig. 4.12).

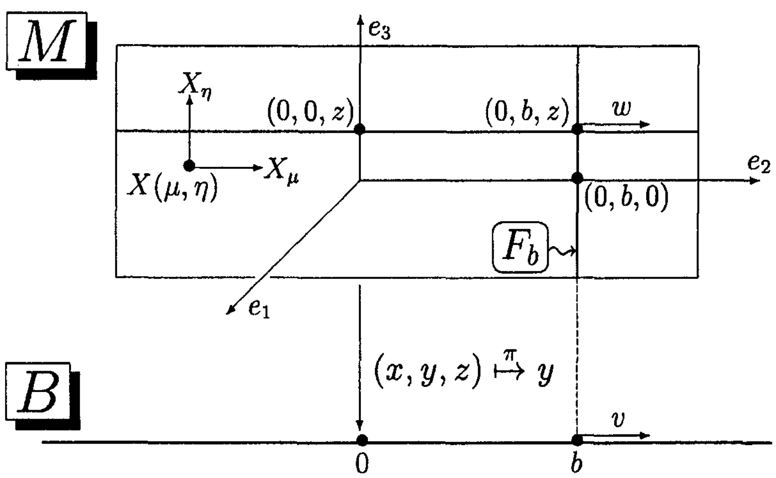


Figure 4.12: Manifolds $M, B$ and the map $\pi$ in **Counterexample 4.2.6**.

Let $\pi : M \to B$ be the projection $\pi(x, y, z) := y$, which is onto and $C^\infty$. Note that $\pi$ has maximal rank=1, since the Jacobian of $\pi_*$ is given by,

$$(\pi_*)_{(x,y,z)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \forall (x, y, z) \in M$$

Each fiber is given by,

$$F_b = \pi^{-1}(b) = \{(0,y,z) \in M : y = b\} = \{(0,b,z) : z \in \mathbb{R}\}, \quad b \in B = \mathbb{R}$$

which is a line passing through $(0, b, 0)$, determined by the intersection of $M$ with the plane $y = b$. Hence each fiber is <u>not compact</u> as a subset of $\mathbb{R}^3$, they all have infinite diameter, and therefore the fibers' diameters are <u>not uniformly bounded</u>.

Next, we want to justify **(HLC)**

The space of <u>vertical</u> vectors at $(x, y, z) \in M$ is,

$$\begin{aligned}(VT)_{(x,y,z)} &= \ker(\pi_*)_{(x,y,z)} = \\ &= \{\text{vectors tangent to the vertical lines } y{=}\text{constant}\} \cong \mathbb{R}\end{aligned}$$

and the space of <u>horizontal</u> vectors at $(x, y, z) \in M$ is,

$$\begin{aligned}(HT)_{(x,y,z)} &= \left(\ker(\pi_*)_{(x,y,z)}\right)^{\perp} = \\ &= \{\text{vectors tangent to the horizontal lines } z{=}\text{constant}\} \cong \mathbb{R}\end{aligned}$$

In order to describe explicitly the horizontal and vertical vectors, we will use the following global parametrization of $M$,

$$X(\mu, \eta) = (0, \mu, \eta), \quad \mu \in \mathbb{R}, \eta \in \mathbb{R}$$

Note that $\pi \circ X(\mu, \eta) = \mu, \quad \forall \mu \in \mathbb{R}, \forall \eta \in \mathbb{R}$.

The parallels are the curves,

$$\eta \in \mathbb{R} \quad \mapsto \quad X(\mu_{\text{const.}}, \eta) = (0, \mu_{\text{const.}}, \eta) \in M$$

so that the space of vertical vectors is generated by,

$$\left\langle X_\eta(\mu_{\text{const.}}, \eta) := \frac{\partial X}{\partial \eta}(\mu_{\text{const.}}, \eta) = (0,0,1) \right\rangle$$

The meridians are the curves,

$$\mu \in \mathbb{R} \quad \mapsto \quad X(\mu, \eta_{\text{const.}}) = (0, \mu, \eta_{\text{const.}}) \in M$$

so that the space of horizontal vectors is generated by,

$$\left\langle X_\mu(\mu, \eta_{\text{const.}}) := \frac{\partial X}{\partial \mu}(\mu, \eta_{\text{const.}}) = (0,1,0) \right\rangle$$

The above determines the general expression of a horizontal lift: for each $b \in B = \mathbb{R}$ and $w \in T_bB \cong \mathbb{R}$, the unique horizontal lift $v$ of $w$ through a point $X(b, \bar{\eta}) \in F_b \subset M$ is represented by the vector,

$$v = w \cdot X_\mu(b, \bar{\eta}) = w \cdot (0,1,0) \in (HT)_{X(b,\bar{\eta})} \subset T_{X(b,\bar{\eta})}M$$

since $(\pi_*)_{X(b,\bar{\eta})}\,(v) = w \cdot (\pi_*)_{X(b,\bar{\eta})}\,(0,1,0) = w \cdot 1 = w$.

In particular, $||v||_M = |w| \cdot 1 = ||w||_B$, which implies that, if $\alpha = 2 > 1$ and $\beta = 1 > 0$,

$$\frac{1}{\alpha}||w||_B - \beta \leq ||w||_B = ||v||_M \leq \alpha||w||_B + \beta$$

for all $b \in B, w \in T_bB, x \in F_b$, which is **HLC**.

Our goal here is to show that $\pi$ is not a rough isometry. One can accomplish that by verifying the failure of **(RI.1)** for $\pi$. It suffices to verify that **(RI.1)** fails for $\pi$, for particular pairs of elements in $M$.

Hence, we will accomplish our goal if we show that $\forall A \geq 1$, $\forall C > 0$, $\exists \eta_{AC} \in I\!R$, a positive number such that,

$$d_B(\pi \circ X(\mu,\eta), \pi \circ X(\mu,0)) < \frac{1}{A} \cdot d_M(X(\mu,\eta), X(\mu,0)) - C \Leftrightarrow$$
$$\Leftrightarrow \quad |\mu - \mu| < \frac{1}{A} \cdot d_M((0,\mu,\eta),(0,\mu,0)) - C \Leftrightarrow$$
$$\Leftrightarrow \quad 0 < \frac{1}{A} \cdot |\eta| - C, \quad \forall \eta \geq \eta_{AC} \qquad \textbf{[CE.25]}$$

Let $A \geq 1, C > 0$ be arbitrary.

Finally, if we define the real positive number $\eta_{AC} := AC + 1 > 0$, we see that,

$$\eta_{AC} = AC + 1 > AC \Leftrightarrow \frac{1}{A}\eta_{AC} > C \Leftrightarrow \frac{1}{A}\eta_{AC} - C > 0 \qquad \textbf{[CE.26]}$$

and since, for all $\eta \geq \eta_{AC}$,

$$\frac{1}{A}\eta - C \geq \frac{1}{A}\eta_{AC} - C \overset{\textbf{[CE.26]}}{>} 0$$

inequality **[CE.25]** is verified.

Therefore, **[CE.25]** holds and **(RI.1)** fails for $\pi$, which shows that $\pi$ is NOT a rough isometry. This describes the Counterexample.

In what follows, we introduce some notation (see [27], [9], [16] and [8]) and we define a *bounded geometry* condition for manifolds.

Let $(M, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$ and curvature tensor $R$.

The *Ricci curvature tensor* of $(M, g)$, at each $x \in M$ is a symmetric bilinear form $Ric$ defined by

$$\begin{array}{rccl} Ric: & T_xM \times T_xM & \longrightarrow & \mathbb{R} \\ & (\xi, \mu) & \longmapsto & Ric(\xi, \mu) := trace(\zeta \mapsto R(\xi, \zeta)\mu) \end{array}$$

If $M$ is complete, the *injectivity radius at $x \in M$* is given by

$$\imath_x(M) := \sup\{r > 0 : \exp_x|_{B(x,r)} \text{ is a diffeomorphism}\}$$

and $\imath(M) := \inf\{\imath_x(M) : x \in M\}$ is called the *injectivity radius* of $M$.

**Definition 4.2.7** *Let $M$ be a complete $m$-dimensional Riemannian manifold. We say that $M$ has* **bounded geometry** *if it satisfies:*

**(BG.R)** *the Ricci curvature is bounded from below by $-(m-1)k_M^2$, where $k_M$ is a positive constant;*

**(BG.I)** *the injectivity radius $\imath(M)$ is positive.*

We recall that a complete Riemannian manifold satisfying a bounded geometry condition has its geometry reflected by that of any net that approximates the manifold (see **Lemma 2.4**).

Moreover, if we denote by $\gamma_\xi : [0, +\infty) \to M$ the geodesic emanating from $x$ with the initial direction $\xi \in T_xM, ||\xi||_M = 1$, the *distance to the cut point of $x$ along $\gamma_\xi$* is defined by

$$c(\xi) := \sup\{t > 0 : \gamma_\xi \mid_{[0,t]} \text{ is minimal, namely, } d_M(x, \gamma_\xi(t)) = t\}$$

and when $\underline{c(\xi) < +\infty}$, we refer to $\gamma_\xi(c(\xi))$ as the *cut point of $x$ along $\gamma_\xi$*, and we refer to $C(x) := \exp \mathcal{C}(x)$ as the *cut locus of $x$ in $M$*, where $\mathcal{C}(x)$ is the *tangential cut locus of $x$ in $T_xM$*, given by

$$\mathcal{C}(x) := \{c(\xi)\xi : c(\xi) < +\infty, \xi \in T_xM, ||\xi||_M = 1\}$$

furthermore, one knows that $\mathcal{C}(x)$ has zero $n$-dimensional Lebesgue measure in $T_xM$ and its image in $M$, $C(x)$, also has zero $n$-dimensional Riemannian measure in $M$.

Next, in **Theorem 4.2.10** we show that for an onto smooth map with maximal rank $\pi : M \to B$ between manifolds with bounded geometry, under assumptions on the fibers and assumptions on the subspaces of horizontal vectors, $M$ is roughly isometric to $F \times B$, where $F$ is a fixed fiber of $M$.

In order to prove **Theorem 4.2.10** we will need the following technical

Lemma and Proposition.

**Lemma 4.2.8** *Let $A > 1$, $C > 0$, $\alpha \geq 1$, $\beta > 0$, be given constants, and let $\epsilon_0 > 0$, $\epsilon_B > 0$ be such that,*

$$\epsilon_0 > C \cdot (1 + A^2) \tag{4.25}$$

$$\epsilon_B > \left(\frac{\epsilon_0 - C}{A} + \beta\right) \cdot \alpha \tag{4.26}$$

*The following hold:*

**(1.)** $\left(\frac{\epsilon_0 - C}{A}\right) > 0$

**(2.)** $\frac{\epsilon_0 - C}{A} < \epsilon_0$

**(3.)** $\frac{\epsilon_0 - C}{A^2} - C > 0$

**(4.)** $\frac{\epsilon_0 - C}{A} > \frac{\epsilon_0 - C}{A^2} - C$

**(5.)** $\frac{1}{\alpha}\epsilon_B - \beta > 0$

**(6.)** $\frac{\epsilon_0 - C}{A} < (\epsilon_0 + C) \cdot A$

**(7.)** $2\left(\frac{\epsilon_0 - C}{A}\right) < (2\epsilon_0 + C) \cdot A$

**(8.)** $2\epsilon_B > 2\left(\frac{\epsilon_0 - C}{A} + \beta\right)\alpha$

*Proof.*

**(1.)** By (4.25),

$$\epsilon_0 > C\underbrace{(1+A^2)}_{>1} > C \Rightarrow \left(\frac{\epsilon_0 - C}{A}\right) > 0$$

**(2.)** Since $A > 1$,

$$\begin{aligned}
\epsilon_0\underbrace{(1-A)}_{<0} < 0 < C &\Rightarrow \frac{\epsilon_0}{A}(1-A) < \frac{C}{A} \Rightarrow \\
&\Rightarrow \frac{\epsilon_0}{A} - \epsilon_0 - \frac{C}{A} < 0 \Rightarrow \frac{\epsilon_0}{A} - \frac{C}{A} < \epsilon_0 \Rightarrow \\
&\Rightarrow \frac{\epsilon_0 - C}{A} < \epsilon_0
\end{aligned}$$

**(3.)** By (4.25),

$$\epsilon_0 > C(1+A^2) \Rightarrow \epsilon_0 - C - CA^2 > 0 \Rightarrow \frac{\epsilon_0 - C}{A^2} - C > 0$$

**(4.)** Since $A > 1$,

$$\begin{aligned}
&\epsilon_0\underbrace{(1-A)}_{<0} < 0 < C + CA\underbrace{(A-1)}_{>0} \Rightarrow \\
\Rightarrow\ &\epsilon_0(1-A) < C + CA^2 - CA \Rightarrow \epsilon_0 - C - CA^2 < \epsilon_0 A - CA \Rightarrow \\
\Rightarrow\ &\frac{\epsilon_0 - C}{A^2} - C < \frac{\epsilon_0 - C}{A}
\end{aligned}$$

**(5.)** By (4.26),

$$\epsilon_B > \left(\frac{\epsilon_0 - C}{A} + \beta\right)\alpha = \underbrace{\left(\frac{\epsilon_0 - C}{A}\right)\alpha}_{>0} + \beta\alpha > \beta\alpha \Rightarrow$$

$$\overset{\div\alpha}{\Rightarrow} \frac{1}{\alpha}\epsilon_B - \beta > 0$$

**(6.)** $A > 1, C > 0$ and $\epsilon_0 > 0$ imply that,

$$\epsilon_0(1-A)(1+A) < 0 \Rightarrow$$

$$\Rightarrow \epsilon_0(1-A)(1+A) < 0 < C(1+A^2) \Rightarrow \epsilon_0(1-A^2) < C(1+A^2) \Rightarrow$$

$$\Rightarrow \epsilon_0 - \epsilon_0 A^2 < C + CA^2 \Rightarrow \epsilon_0 - C < \epsilon_0 A^2 + CA^2 \Rightarrow$$

$$\Rightarrow \frac{\epsilon_0 - C}{A} < (\epsilon_0 + C)A$$

**(7.)** From $A > 1, C > 0$ and $\epsilon_0 > 0$ we have,

$$\begin{cases} 2\epsilon_0(1-A)(1+A) < 0 \\ 0 < C(2+A^2) \end{cases} \Rightarrow 2\epsilon_0(1-A)(1+A) < 0 < 2C + CA^2 \Rightarrow$$

$$\Rightarrow 2\epsilon_0(1-A^2) < 2C + CA^2 \Rightarrow 2\epsilon_0 - 2\epsilon_0 A^2 < 2C + CA^2 \Rightarrow$$

$$\Rightarrow 2\epsilon_0 - 2C < 2\epsilon_0 A^2 + CA^2 \Rightarrow$$

$$\Rightarrow 2\left(\frac{\epsilon_0 - C}{A}\right) < (2\epsilon_0 + C)A$$

**(8.)** By (4.26),

$$\epsilon_B > \left(\frac{\epsilon_0 - C}{A}\right)\alpha + \beta\alpha > \left(\frac{\epsilon_0 - C}{A}\right)\alpha + \frac{\beta\alpha}{2} \Rightarrow$$

$$\overset{\times 2}{\Rightarrow} 2\epsilon_B > 2\left(\frac{\epsilon_0 - C}{A}\right)\alpha + \beta\alpha \Rightarrow$$

$$\Rightarrow 2\epsilon_B > \left[2\left(\frac{\epsilon_0 - C}{A}\right) + \beta\right]\alpha$$

□

**Proposition 4.2.9** *Suppose that $M^m$ and $B^n$ are complete $m-$dimensional and $n-$dimensional Riemannian manifolds, respectively, both having bounded* *geometry (see Definition 4.2.7). Let $\pi : M \to B$ be an onto smooth map with maximal rank.*

*Let $b_0 \in B$ be fixed. Assume that $\{\phi_b : F_b \to F_{b_0}\}_{b\in B}$ is a family of bijective rough isometries satisfying,*

$$\forall b \in B, \exists A > 1, \exists C > 0 : \forall x, x' \in F_b$$

$$\frac{1}{A} d_M(x, x') - C \leq d_M(\phi_b(x), \phi_b(x')) \leq A d_M(x, x') + C, \qquad \textbf{[URI]}$$

*where $A$ and $C$ are universal constants independent of $b$.*

*Then, the following hold:*

1. *If $P_0$ is an $\epsilon_0$−separated set and $\epsilon_0$−full in $F_{b_0}$, where we assume that $\epsilon_0 > C$. Then, for all $b \in B$, the set*

   $$P_b := \phi_b^{-1}(P_0)$$

   *is an $\hat{\epsilon}$−separated set and $\tilde{\epsilon}$−full in $F_b$, where $\hat{\epsilon} := \left(\frac{\epsilon_0 - C}{A}\right) > 0$ and $\tilde{\epsilon} := (A\epsilon_0 + C) > 0$.*

2. *For all $b \in B$ the corresponding nets $P_b$ are uniformly roughly isometric to $P_0$ with respect to the combinatorial metric $\delta$.*

*Proof.*

*1.* Let $b \in B$ be arbitrary.

We first show that, $P_b := \phi_b^{-1}(P_0)$ is a countable $\hat{\epsilon}$-separated subset of $F_b$.

The set $P_b$ is countable, since $P_0$ is countable and $\phi_b$ is bijective.

Notice that by **Lemma 4.2.8** (*1.*), $\hat{\epsilon} = \left(\frac{\epsilon_0 - C}{A}\right) > 0$.

Consider now the options for $b \in P_B$.

If $b = b_0$, since $P_0$ is $\epsilon_0-$separated, by **Lemma 4.2.8** (*2.*), we conclude that $P_b$ is $\hat{\epsilon}-$separated.

If $b \neq b_0$, because $\phi_b$ is a bijection, we have,

$$\begin{aligned}
&\forall p, q \in P_b := \phi_b^{-1}(P_0) : p \neq q \Rightarrow \\
&\Rightarrow \quad \phi_b(p) \neq \phi_b(q) \text{ in } P_0 \Rightarrow \epsilon_0 \leq d_M(\phi_b(p), \phi_b(q)) \leq \\
&\overset{[\mathbf{URI}]}{\leq} \quad A d_M(p,q) + C \Rightarrow d_M(p,q) \geq \hat{\epsilon}
\end{aligned}$$

and $P_b$ is $\hat{\epsilon}-$separated.

Next, we prove that $P_b := \phi_b^{-1}(P_0)$ is $\tilde{\epsilon}-$full in $F_b$, i.e.

$$F_b = B_{\tilde{\epsilon}} P_b := \{x \in F_b : d_M(x, P_b) < \tilde{\epsilon}\}$$

We want to show that,

$$d_M(x, P_b) := \inf_{p \in P_0} d_M\left(x, \phi_b^{-1}(p)\right) < \tilde{\epsilon}, \quad \forall x \in F_b$$

Let $x \in F_b$ be arbitrary. Either $x \in P_b$ or $x \in F_b \setminus P_b$.

If $x \in P_b$, then $d_M(x, P_b) = 0 < \tilde{\epsilon}$.

If $x \in F_b \setminus P_b$, let us look at $\phi_b^{-1}(x) \in F_{b_0}$.

Since $P_0$ is $\epsilon_0$−full in $F_{b_0}$, by the definition of infimum $\exists \bar{p}_0 \in P_0$ satisfying

$$d_M\left(\phi_b^{-1}(x), \bar{p}_0\right) < \epsilon_0$$

which, in turn together with **[URI]** implies that

$$d_M(x, P_b) \overset{\text{inf.}}{\leq} d_M\left(x, \phi_b(\bar{p}_0)\right) \overset{\text{[URI]}}{\leq} A d_M\left(\phi_b^{-1}(x), \bar{p}_0\right) + C < A\epsilon_0 + C =: \tilde{\epsilon}$$

and since $x$ is arbitrary in $F_b$, claim *1.* is proved.

*2.* From **Lemma 2.4** a complete Riemannian manifold with bounded geometry is roughly isometric to each of its nets.

This implies that for each $b \in B$,

$$(P_b, \delta) \xrightarrow{\text{R.I.}} (F_b, d_M) \overset{\text{Lem 2.4}}{\Longrightarrow}$$
$$\frac{1}{2\tilde{\epsilon}} d_M(p_1, p_2) \leq \delta(p_1, p_2) \leq \tilde{a} d_M(p_1, p_2) + \tilde{c}, \quad \forall p_1, p_2 \in P_b \quad (4.27)$$

where $\tilde{a} := \tilde{a}(m, K_M, \tilde{\epsilon}) > 1, \tilde{c} := \tilde{c}(m, K_M, \tilde{\epsilon}) > 0$, and $(F_b, d_M)$ indicates that on each fiber $F_b$ we will use the induced Riemannian metric from $M$.

Also, by **Lemma 2.4** we have,

$$(P_0, \delta_0) \xrightarrow{\text{R.I.}} (F_0, d_M) \overset{\text{Lem 2.4}}{\Longrightarrow}$$
$$\Rightarrow \frac{1}{2\epsilon_0} d_M(p_3, p_4) \leq \delta_0(p_3, p_4) \leq \tilde{a}_0 d_M(p_3, p_4) + \tilde{c}_0 \Rightarrow$$
$$\frac{1}{\tilde{a}_0}\delta_0(p_3, p_4) - \frac{\tilde{c}_0}{\tilde{a}_0} \leq d_M(p_3, p_4) \leq 2\epsilon_0 \delta_0(p_3, p_4), \quad \forall p_3, p_4 \in P_0 \quad (4.28)$$

where $\tilde{a}_0 := \tilde{a}_0(m, K_M, \epsilon_0) > 1, \tilde{c}_0 := \tilde{c}_0(m, K_M, \epsilon_0) > 0$, and $(F_0, d_M)$ indicates that on the fiber $F_0$ the induced Riemannian metric from $M$ is used.

From **[URI]**, we have,

$$(F_b, d_M) \overset{\text{R.I.}}{\longrightarrow} (F_0, d_M) \overset{\text{[URI]}}{\Longrightarrow}$$

$$\frac{1}{A} d_M(\phi_b(p_1), \phi_b(p_2)) - \frac{C}{A} \leq d_M(p_1, p_2) \leq A d_M(\phi_b(p_1), \phi_b(p_2)) + AC \quad (4.29)$$

for all $p_1, p_2 \in F_b$.

Next, we observe the following diagram for $\iota = 1, 2$,

$$\begin{array}{ccccccc} P_b & & F_b & & F_0 & & P_0 \\ p_\iota & \hookrightarrow & p_\iota & \overset{\phi_b}{\to} & \phi_b(p_\iota) & \hookrightarrow & \phi_b(p_\iota) \end{array}$$

where, $p_\iota \in P_b := \phi_b^{-1} P_0 \Rightarrow \phi_b(p_\iota) \in P_0$.

We claim that,

$$(P_b, \delta) \overset{\text{unif.R.I.}}{\longrightarrow} (P_0, \delta_0)$$

Indeed, let $p_1, p_2 \in P_b$.

By (4.27), (4.29) and (4.28), we may write,

$$\overset{(4.27)}{\Longrightarrow} \frac{1}{2\hat{\epsilon}} d_M(p_1, p_2) \leq \delta(p_1, p_2) \leq \tilde{a} d_M(p_1, p_2) + \tilde{c} \Rightarrow$$

$$\overset{(4.29)}{\Longrightarrow} \frac{1}{2\hat{\epsilon}A} d_M(\phi_b(p_1), \phi_b(p_2)) - \frac{C}{2\hat{\epsilon}A} \leq \delta(p_1, p_2) =$$

$$= \delta(p_1, p_2) \leq \tilde{a} A d_M(\phi_b(p_1), \phi_b(p_2)) + \tilde{a}AC + \tilde{c} \Rightarrow$$

$$\overset{(4.28)}{\Longrightarrow} \frac{1}{2\hat{\epsilon}A\tilde{a}_0} \delta_0(\phi_b(p_1), \phi_b(p_2)) - \frac{\tilde{c}_0}{2\hat{\epsilon}A\tilde{a}_0} - \frac{C}{2\hat{\epsilon}A} \leq \delta(p_1, p_2) =$$

$$= \delta(p_1, p_2) \leq \tilde{a}_0 A 2\epsilon_0 \delta_0(\phi_b(p_1), \phi_b(p_2))\tilde{a}AC + \tilde{c}$$

which can be rewritten in the ,

$$\frac{1}{A_{net}}\delta_0(\phi_b(p_1), \phi_b(p_2)) - C_{net} \leq \delta(p_1, p_2) \leq A_{net}\delta_0(\phi_b(p_1), \phi_b(p_2)) + C_{net}$$

where,

$$\begin{aligned} A_{net} &:= A_{net}(m, K_M, \epsilon_0, C, A) := 2A\max\{\hat{\epsilon}\tilde{a}_0, \tilde{a}\epsilon_0\} \geq 1 \\ C_{net} &:= C_{net}(m, K_M, \epsilon_0, C, A) := \max\left\{\tilde{a}AC + \tilde{c}, \frac{1}{2\hat{\epsilon}A\tilde{a}_0}\left(\frac{\tilde{c}_0}{\tilde{a}_0} + C\right)\right\} > 0 \end{aligned}$$

and the Proposition is proved.

□

Finally, we state and prove the main result.

**Theorem 4.2.10** *Let $M$ and $B$ be complete Riemannian manifolds, where $\dim M = m$ and $\dim B = n$. Assume that both $M$ and $B$ have bounded geometry and let $\pi : M \to B$ be an onto smooth map with maximal rank.*

*Let $b_0 \in B$ be a fixed base point.*

*Choose $P_B \subset B$ a maximal $\epsilon_B-$separated set with $b_0 \in P_B$, and $P_0 \subset F_{b_0}$ a maximal $\epsilon_0-$separated set, where $\epsilon_B$ and $\epsilon_0$ are as in* Lemma 4.2.8.

*For every $b_1, b_2 \in B$, we will denote by $\gamma_{[b_1 b_2]}$ a broken geodesic in $B$ joining $b_1$ to $b_2$. In particular, $\gamma_{b_1} := \gamma_{[b_1 b_1]}$ will represent a broken geodesic loop in $B$ at $b_1$.*

*Let $\varphi_{(\gamma_{[b_1 b_2]})} : F_{b_1} \longrightarrow F_{b_2}$ be the corresponding diffeomorphism to $\gamma_{[b_1 b_2]}$, as in Theorem 4.12.*

*Assume the following 3 conditions hold:*

**(LLL)** *For any $b \in P_B$ and any broken geodesic loop $\gamma_b$, the corresponding diffeomorphism $\varphi_{(\gamma_b)} : F_b \longrightarrow F_b$ is the identity map on $F_b$.*

*(see Fig. 4.13)*

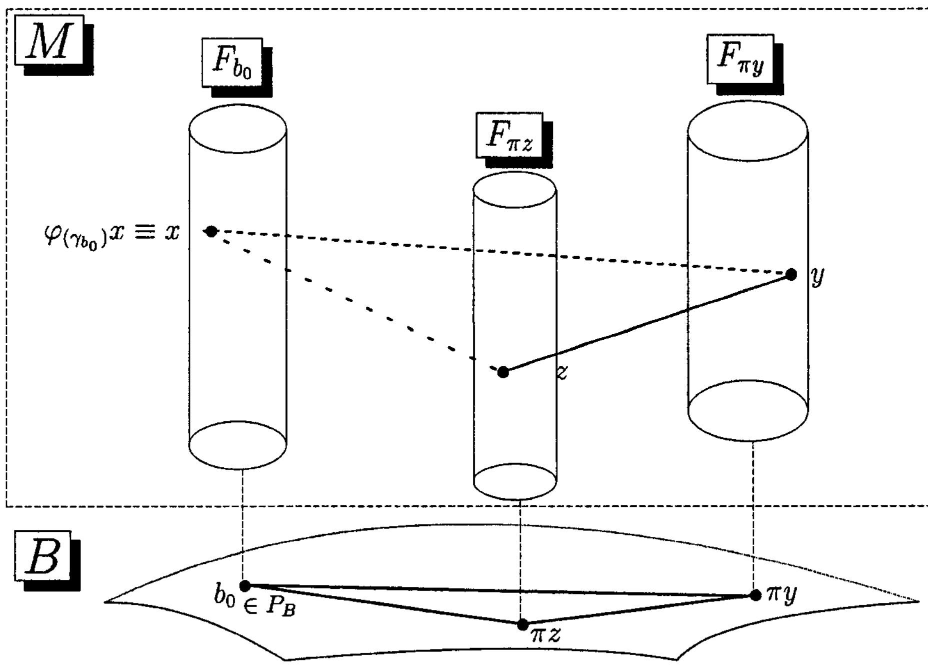


Figure 4.13: **(LLL)**.

**(RIF)** $\forall b \in P_B$, $\exists A > 1, \exists C > 0$, *constants independent of* $b$, *such that,*

$$\frac{1}{A} d_M(x, x') - C \leq d_M(\varphi_{(\gamma_{[b,b_0]})}(x), \varphi_{(\gamma_{[b,b_0]})}(x')) \leq A d_M(x, x') + C$$

$\forall x, x' \in F_b$, *where* $d_M$ *denotes the Riemannian metric on* $M$.

*In this case, since* $\varphi_{(\gamma_{[b,b_0]})} : F_b \longrightarrow F_{b_0}$ *is onto, it is also a rough isometry for each* $b \in B$, *and therefore the fibers are uniformly roughly isometric. (see Fig. 4.14)*

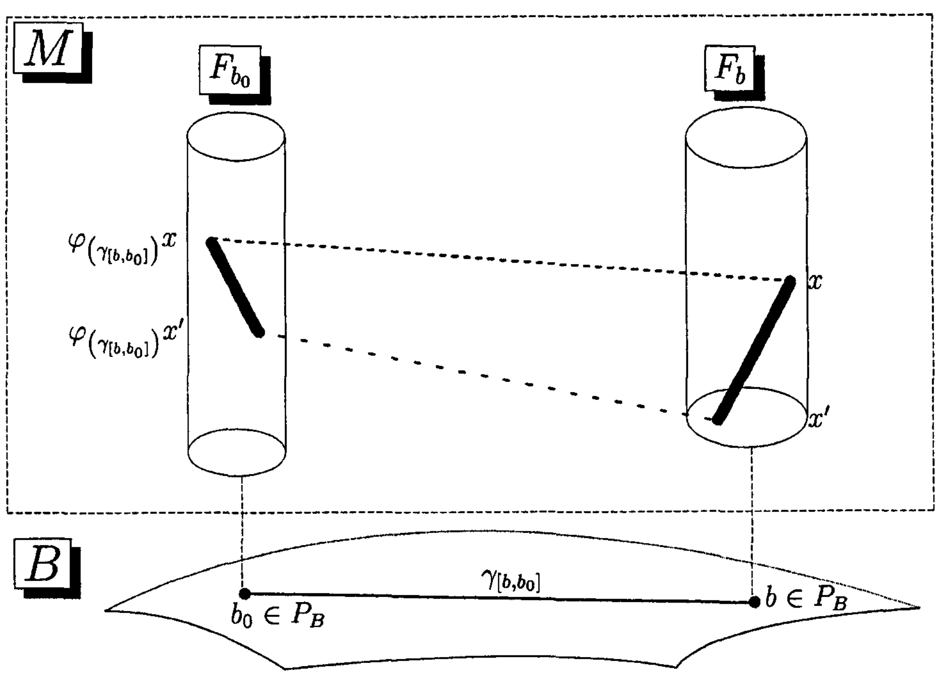


Figure 4.14: **(RIF)**.

**(HLC)** $\forall b \in B, \forall x \in F_b, \exists \alpha \geq 1, \exists \beta > 0,$ *constants independent of* $b$ *and* $x$*,*

*such that,*

$$\frac{1}{\alpha}||w||_B - \beta \leq ||v||_M \leq \alpha||w||_B + \beta$$

$\forall w \in T_bB$*, where* $v$ *is the unique horizontal lift of* $w$ *through* $x$ *satisfying* $||v||_M \leq 1$*, and* $||\ \ ||_M$*,* $||\ \ ||_B$ *denote the inner product on* $TM$ *and* $TB$*, respectively. (see Fig. 4.15)*

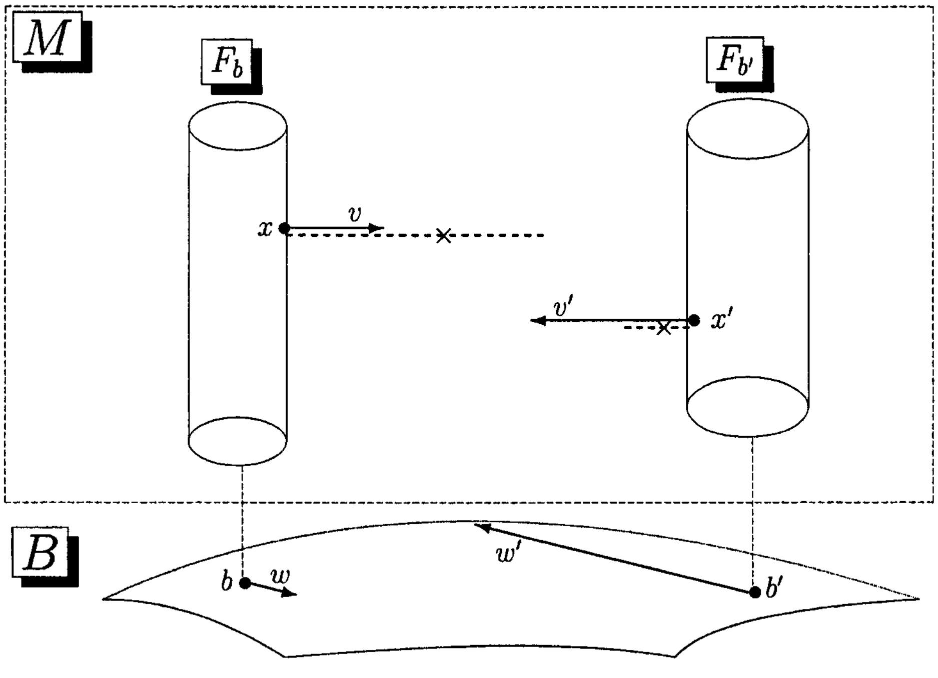


Figure 4.15: **(HLC)**, where $w$ is short and $w'$ is long.

*Then,* $M$ *is roughly isometric to the product* $F_{b_0} \times B$*.*

*Proof.* In order to prove the Theorem, by **Lemma 2.4**, it suffices to show that an $\epsilon-$net in $M$ is roughly isometric to an $\epsilon'-$net in the product $F_{b_0} \times B$. We remark here that in the proof of **Lemma 2.4** the maximal property of an $\epsilon-$net is not required, it sufficient that the "net" be a countable, $\epsilon-$separated and $\epsilon-$full set.

We will proceed with the proof by constructing in 2 steps a rough isometry $\phi$ between countable, separated full sets in $M$ and in $F_{b_0} \times B$.

In *Step 1.* we combine the diffeomorphisms $\varphi_{(\gamma_{[b,b_0]})}$ with two countable maximal separated sets, $P_0$ in the fiber $F_{b_0} \subset M$ and $P_B$ in $B$, in a fashion that will produce a suitable countable separated full set $P$ in $M$. We also show that the product $P_0 \times P_B$ is a countable separated full set in $F_{b_0} \times B$.

Then, in *Step 2.* we introduce a bijection $\phi$ from $P$ to $P_0 \times P_B$, which will turn out to be the rough isometry between discrete approximations of $M$ and $F_{b_0} \times B$, as mentioned above.

*Step 1.*

Let $\epsilon_0 > 0$ and $\epsilon_B > 0$ be, as in **Lemma 4.2.8**, where,

$$\epsilon_0 > C \cdot (1 + A^2) \qquad \text{and} \qquad \epsilon_B > \left(\frac{\epsilon_0 - C}{A} + \beta\right) \cdot \alpha$$

We first define two countable sets $P_0 \subsetneq F_{b_0} \subset M$ and $P_B \subsetneq B$, with

$b_0 \in P_B$, where $P_0$ is a maximal $\epsilon_0$−separated set,

$$\forall p, q \in P_0, p \neq q \Rightarrow d_M(p, q) \geq \epsilon_0$$

and $P_B$ is a maximal $\epsilon_B$−separated set,

$$\forall b_1, b_2 \in P_B, b_1 \neq b_2 \Rightarrow d_B(b_1, b_2) \geq \epsilon_B$$

and then we introduce the net structure $N_0 = \{N_0(p) : p \in P_0\}$ of $P_0$ given by,

$$N_0(p) = \{q \in P_0 : 0 < d_M(p, q) \leq 2\epsilon_0\}$$

and $N_B = \{N_B(b) : b \in P_B\}$ the net structure of $P_B$ defined by,

$$N_B(b) = \{\hat{b} \in P_B : 0 < d_B(b, \hat{b}) \leq 2\epsilon_B\}$$

Observe that **Proposition 2.3** implies $P_0$ is $\epsilon_0$−full in $F_{b_0}$ and $P_B$ is $\epsilon_B$−full in $B$.

We remark that since the net $P_B$ is a countable subset of $B$ with empty interior, and the cut-locus, $C(b)$, of any $b \in B$, has $n$−dimensional Riemannian measure zero in $B$, we may assume that in the net $P_B \subset B$, which by the way contains $b_0$, no two points $b_1, b_2$ belong to the cut locus of each other, i.e. $b_1 \in P_B \setminus C(b_2)$ and $b_2 \in P_B \setminus C(b_1)$. Otherwise, if there exists $\tilde{b} \in P_B \cap C(b)$, since $C(b)$ has $n$−dimensional Riemannian measure zero in

$B$, we can replace $\tilde{b} \in P_B$ with another element in $B$ with the same required properties that $\tilde{b}$ has as an element of the net $P_B$, but with the advantage that the new element is not in the cut locus, $C(b)$, of $b$.

In that case, for every $b_1, b_2 \in P_B$ there is a unique minimal geodesic, $\gamma_{[b_1 b_2]}$, in $B$ joining $b_1$ to $b_2$.

We now, construct $P$ a countable $\left(\frac{\epsilon_0 - C}{A}\right)$ –separated full set in $M$.

For each $b \in B$, let us look first at $\varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0) \subseteq F_b$.

By **Proposition 4.2.9,** *1.*, it follows that,

$$\varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0) \quad \text{is a countable} \quad \left(\frac{\epsilon_0 - C}{A}\right) - \text{separated subset of } F_b \qquad (4.30)$$

Let (see Fig. 4.16),

$$P := \bigcup_{b \in P_B} \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0)$$

Since $P_0, P_B$ are countable sets and $\varphi_{(\gamma_{[b,b_0]})}$ is a bijection for all $b \in B$, the set $P$ is also countable.

To show that $P$ is $\left(\frac{\epsilon_0 - C}{A}\right)$ –separated, we proceed as follows.

For any $p, q \in P$ such that $p \neq q$, we have only two possibilities,

*(POSSIBILITY 1:)* $\exists b \in P_B : p, q \in \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0)$.

In that case (4.30) gives us, $d_M(p, q) \geq \dfrac{\epsilon_0 - C}{A}$.

*(POSSIBILITY 2:)* $\exists b \in P_B : p \in \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0)$ and $\exists \tilde{b} \in P_B$ :

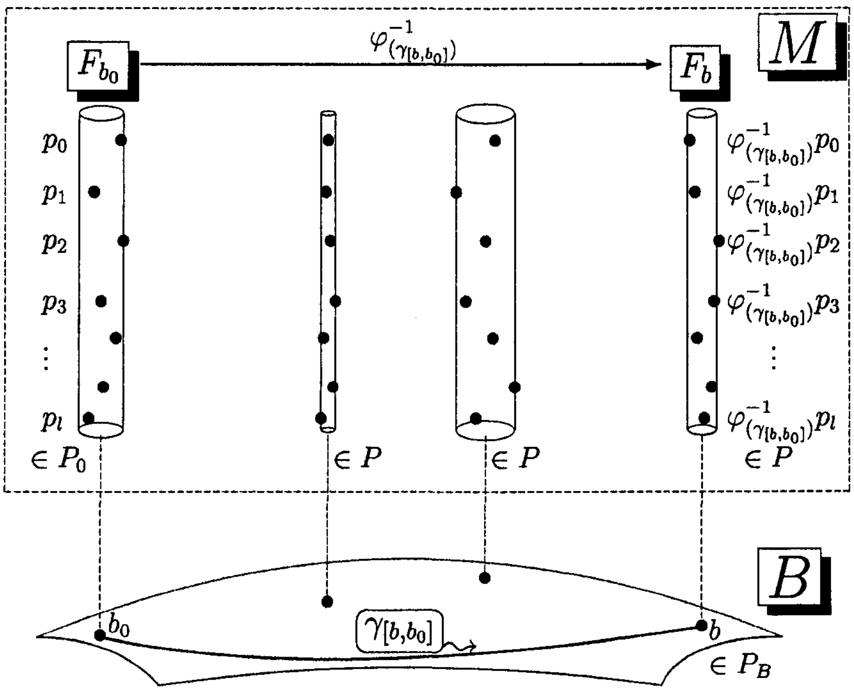


Figure 4.16: The net $P = \bigcup_{b \in P_B} \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0)$.

$q \in \varphi^{-1}_{(\gamma_{[\tilde{b},b_0]})}(P_0)$, where $\pi p = b \neq \tilde{b} = \pi \tilde{b}$.

We claim,

$$d_M(p,q) \geq \frac{1}{\alpha}\epsilon_B - \beta \overset{(4.26)}{>} \frac{\epsilon_0 - C}{A}$$

We will only verify the first inequality, since the second one is the requirement (4.26) on $\epsilon_B$.

Let $\varsigma$ be a general p.p.a.l. curve joining $p$ and $q$ in $M$, with length $\ell(\varsigma)$. In this case $\pi \circ \varsigma$ is a curve joining $b$ and $\tilde{b}$. In addition, we will denote by

$\gamma_{b\tilde{b}}$ the minimal geodesic joining $b$ and $\tilde{b}$.

For all $x \in M$, we can write $v = v_V \oplus v_H \in T_xM = (VT)_x \oplus (HT)_x$ in a unique way, where by **Lemma 4.3** $v_V \in (\ker(\pi_*)_x)$ and $v_H \in (\ker(\pi_*)_x)^{\perp}$. Furthermore, $||v||_M = ||v_V \oplus v_H||_M = \sqrt{||v_V||_M^2 + ||v_H||_M^2}$.

By assumption **(HLC)**, the facts that $\gamma_{b\tilde{b}}$ is a minimal geodesic and $B$ is complete,

$$\begin{aligned}
\ell(\varsigma) &= \int_0^1 ||\varsigma'(t)||_M dt = \int_0^1 ||\varsigma_H'(t) \oplus \varsigma_V'(t)||_M dt \geq \int_0^1 ||\varsigma_H'(t)||_M dt \overset{\textbf{(HLC)}}{\geq} \\
&\geq \frac{1}{\alpha}\int_0^1 ||(\pi \circ \varsigma)'(t)||_B dt - \beta = \frac{1}{\alpha}\ell(\pi \circ \varsigma) - \beta \overset{min.geod.}{\geq} \\
&\geq \frac{1}{\alpha}\ell(\gamma_{b\tilde{b}}) - \beta \overset{Bcomplete}{=} \frac{1}{\alpha} d_B(b, \tilde{b}) - \beta, \quad \forall \varsigma
\end{aligned}$$

which is a lower bound on the length of any curve $\varsigma$ joining $p$ and $q$ in $M$, independent of the curve $\varsigma$.

Finally, by definition of infimum and from $b, \tilde{b} \in P_B$,

$$d_M(p,q) := \inf_{\varsigma \subset M} \ell(\varsigma) \geq \frac{1}{\alpha} d_B(b, \tilde{b}) - \beta \overset{b,\tilde{b}\in P_B}{\geq} \frac{1}{\alpha}\epsilon_B - \beta \overset{Lem\ 4.2.8(5.)}{>} 0$$

and the claim is proved.

So, $P$ is $\left(\frac{\epsilon_0 - C}{A}\right)$ − <u>separated.</u>

We introduce the net structure $N_P = \{N_P(p) : p \in P\}$ of $P$ given by,

$$N_P(p) = \left\{ q \in P : 0 < d_M(p,q) \leq 2\left(\frac{\epsilon_0 - C}{A}\right)\right\}$$

Next, we prove that $P$ is $[(\epsilon_0 + C)A + \alpha\epsilon_B + \beta]$−full in $M$, i.e.

$$M = B_{[(\epsilon_0+C)A+\alpha\epsilon_B+\beta]}P = \{x \in M : d_M(x,P) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta\}$$

We want to show that,

$$d_M(x,P) := \inf_{p\in P} d_M(x,p) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta, \quad \forall x \in M$$

Let $x \in M$. Either $x \in P$ or $x \in M \setminus P$.

If $x \in P$, then $d_M(x,P) = 0 < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta$.

If $x \in M \setminus P$, let $b := \pi x \in B$.

There are two possibilities for $b$,

*(POSSIBILITY 1:)* If $b \in P_B$, since $x$ is not in $P$, $P_0$ is maximal, assumption (**RIF**) holds, and $b \in P_B$ implies $\varphi^{-1}_{(\gamma_{[b,b_0]})}(p_0) \in P, \forall p_0 \in P_0$, then,

$$\begin{aligned}
b \in P_B \overset{x\in M\setminus P}{\Longrightarrow} \; & \varphi_{(\gamma_{[b,b_0]})}x \in F_{b_0} \setminus P_0 \overset{P_0 max.}{\Longrightarrow} \exists p_0 \in P_0 : d_M(\varphi_{(\gamma_{[b,b_0]})}x, p_0) < \epsilon_0 \\
\overset{\textbf{(RIF)}}{\Longrightarrow} \; & \frac{1}{A} d_M(x, \varphi^{-1}_{(\gamma_{[b,b_0]})}(p_0)) - C \le d_M(\varphi_{(\gamma_{[b,b_0]})}x, p_0) < \epsilon_0 \Rightarrow \\
\Rightarrow \; & d_M(x, \underbrace{\varphi^{-1}_{(\gamma_{[b,b_0]})}(p_0)}_{\in P}) < (\epsilon_0 + C)A < (\epsilon_0 + C)A + \underbrace{\alpha\epsilon_B + \beta}_{>0} \Rightarrow \\
\Rightarrow \; & \inf_{p\in P} d_M(x,p) \le d_M(x, \varphi^{-1}_{(\gamma_{[b,b_0]})}(p_0)) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta \\
\Rightarrow \; & d_M(x,P) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta
\end{aligned}$$

*(POSSIBILITY 2:)* If $b \in B \setminus P_B$, by the maximality of $P_B$ there exists $\bar{b} \in P_B : d_B(b, \bar{b}) < \epsilon_B$.

We wish to obtain $\bar{p} \in P$ satisfying $d_M(x, \bar{p}) \leq (\epsilon_0 + C)A + \alpha\epsilon_B + \beta$, which will be accomplished as follows.

Let,

$$\begin{aligned} \gamma_{b\bar{b}} : [t_1, t_2] &\longrightarrow B \\ t_1 &\longmapsto \gamma_{b\bar{b}}(t_1) := b \\ t_2 &\longmapsto \gamma_{b\bar{b}}(t_2) := \bar{b} \end{aligned}$$

be a p.p.a.l. of the unique minimal geodesic joining $b$ and $\bar{b}$ in $B$, and let $\Gamma_{b\bar{b}}$ be its unique horizontal lift through $x$, i.e.

**(hl.1)** $\pi \circ \Gamma_{b\bar{b}}(t) = \gamma_{b\bar{b}}(t), t \in [t_1, t_2]$

**(hl.2)** $\Gamma_{b\bar{b}}(t_1) = x$, and in particular $\Gamma_{b\bar{b}}(t_2) \in F_{\bar{b}}$.

**(hl.3)** $\Gamma_{b\bar{b}}$ is horizontal, i.e. $\Gamma'_{b\bar{b}}(t) \in (HT)_{\Gamma_{b\bar{b}}(t)}, t \in [t_1, t_2]$.

We have, by **(hl.2)**, the fact that $P_0$ is $\epsilon_0-$full in $F_{b_0}$ and the definition of infimum,

$$\begin{aligned} &\Gamma_{b\bar{b}}(t_2) \in F_{\bar{b}} \Rightarrow \varphi_{(\gamma_{[\bar{b}, b_0]})}\left(\Gamma_{b\bar{b}}(t_2)\right) \in F_{b_0} \Longrightarrow \\ \overset{P_0-full}{\Longrightarrow} \quad & d_M\left(\varphi_{(\gamma_{[\bar{b}, b_0]})}\left(\Gamma_{b\bar{b}}(t_2)\right), P_0\right) < \epsilon_0 \Rightarrow \end{aligned}$$

$$\overset{infimum}{\Longrightarrow} \quad \exists p_0 \in P_0 : d_M\left(\varphi_{(\gamma_{[\bar{b},b_0]})}\left(\Gamma_{b\bar{b}}(t_2)\right), p_0\right) < \epsilon_0 \tag{4.31}$$

We claim that the desired $\bar{p} \in P$ is exactly $\varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0)$.

Indeed, $\bar{b} \in P_B$ and $\varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0) \in \varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(P_0)$ imply that $\varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0) \in P$.

Furthermore, by the triangle inequality,

$$d_M\left(\varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0), x\right) \leq d_M\left(\varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0), \Gamma_{b\bar{b}}(t_2)\right) + d_M\left(\Gamma_{b\bar{b}}(t_2), x\right) \tag{4.32}$$

In addition we have,

$$\begin{aligned} &\frac{1}{A} d_M\left(\Gamma_{b\bar{b}}(t_2), \varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0)\right) - C \overset{\textbf{(RIF)}}{\leq} d_M\left(\varphi_{(\gamma_{[\bar{b},b_0]})}\left(\Gamma_{b\bar{b}}(t_2)\right), p_0\right) \Rightarrow \\ \Rightarrow \quad & d_M\left(\Gamma_{b\bar{b}}(t_2), \varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0)\right) \leq \left[d_M\left(\varphi_{(\gamma_{[\bar{b},b_0]})}\left(\Gamma_{b\bar{b}}(t_2)\right), p_0\right) + C\right] A < \\ &\overset{(4.31)}{<} (\epsilon_0 + C)A \end{aligned} \tag{4.33}$$

Also,

$$\begin{aligned} d_M\left(\Gamma_{b\bar{b}}(t_2), x\right) &= d_M\left(\Gamma_{b\bar{b}}(t_2), \Gamma_{b\bar{b}}(t_1)\right) \overset{infimum}{\leq} \ell(\Gamma_{b\bar{b}}) = \int_0^1 ||\Gamma'_{b\bar{b}}(t)||_M dt \\ &\overset{\textbf{(HLC)}}{\leq} \alpha \int_0^1 ||\left(\pi \circ \Gamma_{b\bar{b}}\right)'(t)||_B dt + \beta \overset{\textbf{(hl.1)}}{=} \alpha \int_0^1 ||\gamma'_{b\bar{b}}(t)||_B dt + \beta = \\ &= \alpha\ell(\gamma_{b\bar{b}}) + \beta \overset{min.geod.}{=} \alpha d_B(b, \bar{b}) + \beta \overset{def.\bar{b}}{<} \alpha\epsilon_B + \beta \end{aligned} \tag{4.34}$$

Thus, by combining (4.32), (4.33) and (4.34),

$$d_M\left(\varphi^{-1}_{(\gamma_{[\bar{b},b_0]})}(p_0), x\right) \leq (\epsilon_0 + C)A + \alpha\epsilon_B + \beta$$

which in turn implies,

$$d_M(x,P) = \inf_{p\in P} d_M(x,p) \leq d_M(x, \underbrace{\varphi^{-1}_{(\gamma_{[\tilde{b},b_0]})}(p_0)}_{\in P}) \leq (\epsilon_0 + C)A + \alpha\epsilon_B + \beta$$

and we conclude that $P$ is $[(\epsilon_0 + C)A + \alpha\epsilon_B + \beta]-$ full in $M$.

In what follows, we show that $P_0 \times P_B$ is countable, $(\epsilon_0 + \epsilon_B)$-separated, and $(\epsilon_0 + \epsilon_B)$-full in $F_{b_0} \times B$.

In $F_{b_0} \times B$ we have the induced product metric from $M \times B$,

$$d_\times\Big((x,b),(\tilde{x},\tilde{b})\Big) := d_M(x,\tilde{x}) + d_B(b,\tilde{b})$$

for all $x, \tilde{x} \in F_{b_0}$ and $b, \tilde{b} \in B$.

$P_0 \times P_B$ being countable comes from the fact that both $P_0$ and $P_B$ have that property.

Let $(x,b) \neq (\tilde{x},\tilde{b}) \in P_0 \times P_B$. Since $P_0$ is $\epsilon_0-$separated and $P_B$ is $\epsilon_B-$separated,

$$d_\times\Big((x,b),(\tilde{x},\tilde{b})\Big) = d_M(x,\tilde{x}) + d_B(b,\tilde{b}) \geq \epsilon_0 + \epsilon_B$$

and $P_0 \times P_B$ is $(\epsilon_0 + \epsilon_B)$-separated.

To prove that $P_0 \times P_B$ is $(\epsilon_0 + \epsilon_B)$-full in $F_{b_0} \times B$, i.e.,

$$F_{b_0} \times B = \{(x,b) \in F_{b_0} \times B : d_\times\left((x,b), P_0 \times P_B\right) < \epsilon_0 + \epsilon_B\}$$

we need to show that,

$$d_\times\left((x,b), P_0 \times P_B\right) < \epsilon_0 + \epsilon_B, \quad \forall (x,b) \in F_{b_0} \times B$$

Let $(x,b) \in F_{b_0} \times B$.

Since, $P_0$ is $\epsilon_0$-full in $F_{b_0}$, there exists $p_0 \in P_0 : d_M(p_0, x) < \epsilon_0$. Similarly, $P_B$ being $\epsilon_B$-full in $B$, implies that there exists $\bar{b} \in B : d_B(\bar{b}, b) < \epsilon_B$.

Therefore,

$$\begin{aligned} d_\times\left((x,b), P_0 \times P_B\right) &\overset{inf.}{\leq} d_\times\left((x,b),(p_0,\bar{b})\right) = d_M(x,p_0) + d_B(b,\bar{b}) < \\ &< \epsilon_0 + \epsilon_B \end{aligned}$$

and since $(x,b)$ is arbitrary, we conclude that $P_0 \times P_B$ is $(\epsilon_0 + \epsilon_B)$-full in $F_{b_0} \times B$.

*Step 2.*

Let us initially define some notation as well as provide a geometric interpretation of a "net". We will assume that all nets are connected, since we can repeat the argument on each connected component of the underlying manifold.

Graphically, we will describe an element of an $\epsilon-$net as the center of a ball of radius $\frac{\epsilon}{2}$, which can be visualized as a coin with diameter $\epsilon$. So, the control of distances between elements in an $\epsilon-$net allows us to describe it as a countable set of coins, which can touch boundaries but can never overlap. We will call such element an $\epsilon$**−coin** (see Fig. 4.17).

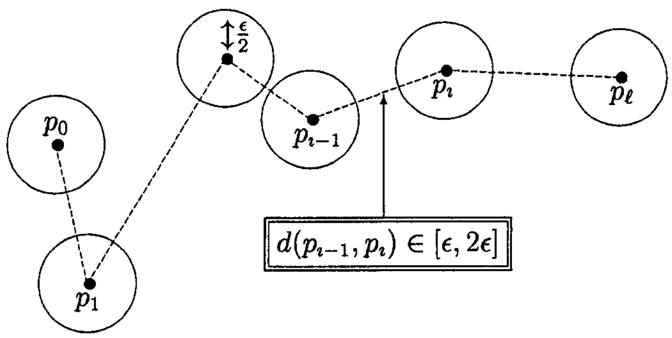


Figure 4.17: A discrete path $(p_0, p_1, \ldots, p_\ell)$ in an $\epsilon$−net, and its elements regarded as centers of coins with diameter $\epsilon$.

We define a map $\phi$ from $P \subset M$ into $P_0 \times P_B \subset F_{b_0} \times B$ as follows,

$$\begin{array}{rrcl} \phi : & P = \bigcup_{b \in P_B} \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0) & \longrightarrow & P_0 \times P_B \\ & p & \longmapsto & \phi p := \left(\varphi_{(\gamma_{[\pi p, b_0]})} p, \pi p\right) \end{array}$$

*(Claim 1)* $\phi$ is well-defined.

If $p \in \varphi^{-1}_{(\gamma_{b_0})}(P_0) = P_0$, then $\pi p = b_0$ and $\phi p = \left(\varphi_{(\gamma_{b_0})} p, b_0\right) = (p, b_0) \in P_0 \times P_B$.

If $p \in \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0)$, where $b \in P_B, b \neq b_0$, then $\pi p = b$ and $\phi p = \left(\varphi_{(\gamma_{[b,b_0]})} p, b\right) \in P_0 \times P_B$.

*(Claim 2)* $\phi$ is injective.

Let $p, q \in P$ and assume that $\phi p = \phi q$.

Thus,

$$\phi p = \phi q \Rightarrow \begin{cases} \varphi_{(\gamma_{[\pi p, b_0]})} p = \varphi_{(\gamma_{[\pi q, b_0]})} q \\ \pi p = \pi q =: b \end{cases} \Rightarrow \varphi_{(\gamma_{[b,b_0]})} p = \varphi_{(\gamma_{[b,b_0]})} q \overset{diffeo.}{\Rightarrow} p = q$$

*(Claim 3)* $\phi$ is surjective.

Let $(p_0, b) \in P_0 \times P_B$ and define $p \in F_b$ by,

$$p := \varphi^{-1}_{(\gamma_{[b,b_0]})} p_0 \in \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0) \subseteq F_b$$

where, $\pi p = b$ and $p = p_0$ if $b = b_0$.

Thus,

$$\phi p = \left(\varphi_{(\gamma_{[\pi p, b_0]})} p, \pi p\right) = \left(\varphi_{(\gamma_{[b,b_0]})} p, b\right) = (p_0, b)$$

*(Claim 4)* $\phi$ satisfies **(RI.1)**.

By (Claim 3), for any given $\epsilon > 0$, we have,

$$P_0 \times P_B = \phi(P) = \mathcal{B}_\epsilon \phi(P) = \{(p_0, b) \in P_0 \times P_B : d_\times((p_0, b), \phi(P)) < \epsilon\}$$

so, $\phi$ is $\epsilon-$full in $P_0 \times P_B$ for any $\epsilon > 0$, which is exactly **(RI.1)** for $\phi$.

*(Claim 5)* $\phi$ satisfies **(RI.2)**.

We want to show that there exist constants $a \geq 1$ and $c > 0$ satisfying,

$$\frac{1}{a}\delta_P(p, q) - c \leq \delta_\times(\phi p, \phi q) \leq a\delta_P(p, q) + c, \quad \forall p, q \in P$$

where $\delta_P$ is the combinatorial metric of $P$, and $\delta_\times$ is the discrete product metric of $P_0 \times P_B$ given by,

$$\delta_\times((p,b),(\tilde{p},\tilde{b})) := \delta_{P_0}(p,\tilde{p}) + \delta_{P_B}(b,\tilde{b})$$

for all $(p,b),(\tilde{p},\tilde{b}) \in P_0 \times P_B$.

In terms of $\delta_{P_0}$ and $\delta_{P_B}$, the condition we want to verify for $\phi$, translates into, $\exists a \geq 1, \exists c > 0 : \forall p,q \in P$,

$$\frac{1}{a}\delta_P(p,q) - c \leq \delta_{P_0}\left(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi q,b_0]})}q\right) + \delta_{P_B}(\pi p, \pi q) \leq a\delta_P(p,q) + c \tag{4.35}$$

Indeed, let $p,q \in P = \bigcup_{b\in P_B} \varphi^{-1}_{(\gamma_{[b,b_0]})}(P_0)$.

Let $\gamma_{\min}$ be a minimal geodesic joining $\pi_p$ to $\pi_q$ in $B$, and let its unique horizontal lift through $q$ be parametrized by $\Gamma_q : [t_1,t_2] \longrightarrow M$.

By the triangle inequality, the definition of distance, **Proposition 4.11**, assumptions **(LLL)**and **(RIF)**, we have (see Fig. 4.18)

$$\begin{aligned}
d_M(p,q) &\overset{\triangle}{\leq} d_M(p,\Gamma_q(t_2)) + d_M(\Gamma_q(t_2),q) \leq \\
&\overset{dist.}{\leq} d_M(p,\Gamma_q(t_2)) + \ell(\Gamma_q) \leq \\
&\overset{(RIF)}{\leq} Ad_M(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi p,b_0]})}\Gamma_q(t_2)) + AC + \ell(\Gamma_q) \leq \\
&\overset{Prop\ 4.11}{\leq} Ad_M(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi p,b_0]})}\Gamma_q(t_2)) + AC + \alpha\left[\ell(\gamma_{\min}) + \beta(t_2 - t_1)\right]
\end{aligned}$$

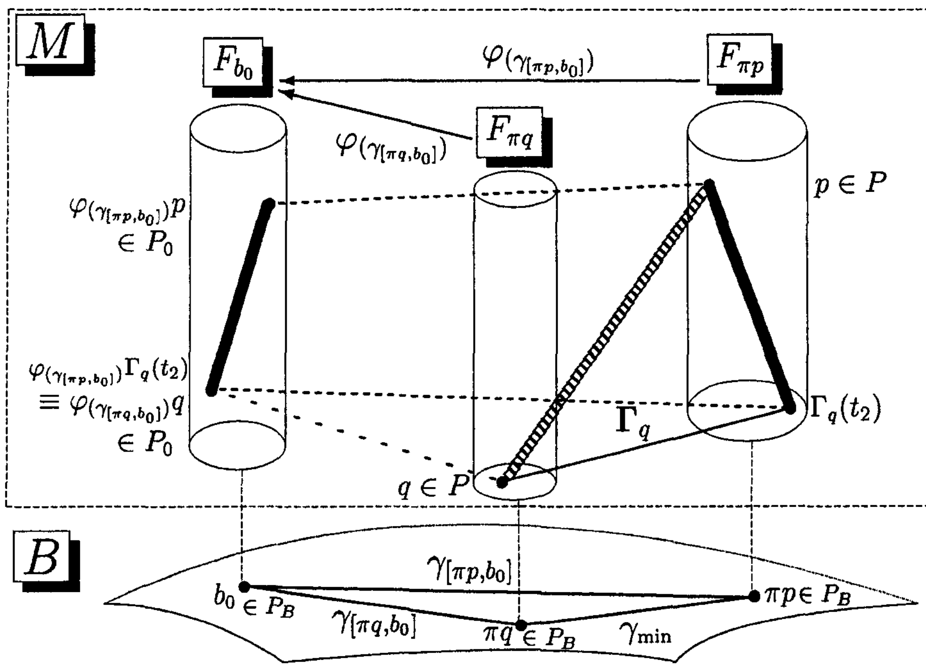


Figure 4.18: An upper bound for the distance $d_M(p,q)$, via the triangle inequality.

$$\overset{(LLL)}{\leq} \quad Ad_M(\varphi_{(\gamma_{[\pi p, b_0]})}p, \varphi_{(\gamma_{[\pi p, b_0]})}q) + AC + \alpha\left[\ell(\gamma_{\min}) + \beta(t_2 - t_1)\right]$$

$$\overset{dist.}{=} \quad \alpha d_B(\pi p, \pi q) + Ad_M(\varphi_{(\gamma_{[\pi p, b_0]})}p, \varphi_{(\gamma_{[\pi q, b_0]})}q) + \alpha\beta(t_2 - t_1) + AC$$

which implies that,

$$d_M(p,q) \leq \alpha d_B(\pi p, \pi q) + Ad_M(\varphi_{(\gamma_{[\pi p, b_0]})}p, \varphi_{(\gamma_{[\pi q, b_0]})}q) + AC + \alpha\beta(t_2 - t_1) \tag{4.36}$$

Now, by **Lemma 2.4** we have for the nets $P$, $P_0$ and $P_B$, respectively,

$$\exists \hat{a}(m, k_M, \hat{\epsilon}) \geq 1, \exists \hat{c}(m, k_M, \hat{\epsilon}) > 0 : \frac{1}{\hat{a}}\delta_P(p,q) - \frac{\hat{c}}{\hat{a}} \leq d_M(p,q) \tag{4.37}$$

$$d_M(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi q,b_0]})}q) \leq 2\epsilon_0 \delta_{P_0}(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi q,b_0]})}q) \tag{4.38}$$

$$d_B(\pi p, \pi q) \leq 2\epsilon_B \delta_{P_B}(\pi p, \pi q) \tag{4.39}$$

If we combine (4.37), (4.38), and (4.39) into (4.36), we obtain,

$$\begin{aligned}
&\frac{1}{\hat{a}}\delta_P(p,q) - \frac{\hat{c}}{\hat{a}} \overset{(4.37)}{\leq} d_M(p,q) \leq \\
&\overset{(4.36)}{\leq} \alpha d_B(\pi p, \pi q) + A d_M(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi q,b_0]})}q) + AC + \alpha\beta(t_2 - t_1) \\
&\overset{(4.38)}{\leq} \alpha d_B(\pi p, \pi q) + A\left(2\epsilon_0 \delta_{P_0}(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi q,b_0]})}q)\right) + AC + \\
&\qquad +\alpha\beta(t_2 - t_1) \leq \\
&\overset{(4.39)}{\leq} \alpha 2\epsilon_B \delta_{P_B}(\pi p, \pi q) + A\left(2\epsilon_0 \delta_{P_0}(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi q,b_0]})}q)\right) + AC + \\
&\qquad +\alpha\beta(t_2 - t_1) \Rightarrow \\
\Rightarrow \; &\frac{1}{\hat{a}}\delta_P(p,q) - \frac{\hat{c}}{\hat{a}} \leq 2A\epsilon_0 \delta_{P_0}(\varphi_{(\gamma_{[\pi p,b_0]})}p, \varphi_{(\gamma_{[\pi q,b_0]})}q) + 2\alpha\epsilon_B \delta_{P_B}(\pi p, \pi q) + \\
&+AC + \alpha\beta(t_2 - t_1) \Rightarrow \\
\Rightarrow \; &\frac{1}{\hat{a}}\delta_P(p,q) - \frac{\hat{c}}{\hat{a}} \leq \max\{2A\epsilon_0, 2\alpha\epsilon_B\} \cdot \delta_\times(\phi p, \phi q) + AC + \alpha\beta(t_2 - t_1) \\
\Rightarrow \; &\frac{1}{\hat{a} \cdot \max\{2A\epsilon_0, 2\alpha\epsilon_B\}}\delta_P(p,q) - \\
&-\left[\frac{\hat{c}}{\hat{a}} + AC + \alpha\beta(t_2 - t_1)\right]\frac{1}{\max\{2A\epsilon_0, 2\alpha\epsilon_B\}} \leq \delta_\times(\phi p, \phi q)
\end{aligned} \tag{4.40}$$

In what follows, we will produce the inequality that completes (4.40) into the searched condition (4.35).

Let us denote $l := \delta_P(p,q)$.

Define $(y_0, y_1, \ldots, y_l)$ a discrete path in $P$ of minimum length $l$ joining $p$ to $q$. Hence, $(y_0, y_1, \ldots, y_l)$ has the following properties,

$$
\begin{aligned}
y_0 := p \in P, &\quad y_l := q \in P, \\
d_M(y_\imath, y_\jmath) \geq \hat{\epsilon}, &\quad \imath, \jmath = 0, 1, \ldots, l \ \ (\imath \neq \jmath) \\
d_M(y_{\iota-1}, y_\iota) \leq 2\hat{\epsilon}, &\quad \iota = 1, \ldots, l \\
\Rightarrow \delta_P(y_{\iota-1}, y_\iota) = 1, &\quad \iota = 1, \ldots, l \\
\overset{\text{def } P}{\Rightarrow} \pi y_\iota \in P_B, &\quad \iota = 0, 1, \ldots, l \\
\overset{\text{def } P}{\Rightarrow} \varphi_{(\gamma_{[\pi y_\iota, b_0]})} y_\iota \in P_0, &\quad \iota = 0, 1, \ldots, l \\
\overset{\text{def } P_B}{\Rightarrow} d_B(\pi y_\imath, \pi y_\jmath) \geq \epsilon_B, &\quad \imath, \jmath = 0, 1, \ldots, l \ \ (\imath \neq \jmath)
\end{aligned}
\tag{4.41}
$$

Next, we will compare $l$ with $\delta_{P_B}(\pi p, \pi q)$.

Notice that because we are assuming (**HLC**), by **Lemma 4.5** we obtain for any $x, y \in M$,

$$
d_M(x, y) \geq \frac{1}{\alpha} d_B(\pi x, \pi y) - \beta \tag{4.42}
$$

Therefore, from (4.41), (4.42) and **Lemma 4.2.8** (*8.*), we get,

$$
\begin{aligned}
&\Rightarrow \quad 2\hat{\epsilon} \overset{(4.41)}{\geq} d_M(y_{\iota-1}, y_\iota) \overset{(4.42)}{\geq} \frac{1}{\alpha} d_B(\pi y_{\iota-1}, \pi y_\iota) - \beta \Rightarrow \\
&\Rightarrow \quad \alpha(2\hat{\epsilon} + \beta) \geq d_B(\pi y_\iota, \pi y_{\iota-1}) \Rightarrow \\
&\Rightarrow \quad 2\epsilon_B \overset{\text{Lem 4.2.8}(8.)}{>} \alpha(2\hat{\epsilon} + \beta) \geq d_B(\pi y_\iota, \pi y_{\iota-1}) \overset{(4.41)}{\geq} \epsilon_B \Rightarrow
\end{aligned}
$$

$$\Rightarrow \quad 2\epsilon_B \geq d_B(\pi y_\iota, \pi y_{\iota-1}) \geq \epsilon_B, \qquad \forall \iota = 1, \ldots, l \tag{4.43}$$

Since (4.43) holds, we obtain a discrete path $(\pi y_0, \pi y_1, \ldots, \pi y_{l-1}, \pi y_l)$ in $P_B$, connecting $\pi p$ to $\pi q$.

Therefore, by the definition of $\delta_{P_B}$, we conclude that,

$$\delta_{P_B}(\pi p, \pi q) \leq l = \delta_P(p, q) \tag{4.44}$$

Now, we will compare $l$ with $\delta_{P_0}\left(\varphi_{(\gamma_{[\pi p, b_0]})}p, \varphi_{(\gamma_{[\pi q, b_0]})}q\right)$.

By **Lemma 2.4**, we have for the nets $P_0$, $P$ and $P_B$, respectively,

$$\exists a_0(m, k_M, \epsilon_0) \geq 1, \exists c_0(m, k_M, \epsilon_0) > 0 :$$

$$\delta_{P_0}(p_1, p_2) \leq a_0 \cdot d_M(p_1, p_2) + c_0, \quad \forall p_1, p_2 \in P_0 \tag{4.45}$$

$$d_M(p_3, p_4) \leq 2\hat{\epsilon}\delta_P(p_3, p_4), \quad \forall p_3, p_4 \in P \tag{4.46}$$

$$d_B(\pi y_{\imath-1}, \pi y_\imath) \leq 2\epsilon_B \delta_{P_B}(\pi y_{\imath-1}, \pi y_\imath), \quad \forall \imath = 1, \ldots, l \tag{4.47}$$

Initially, for each $\imath = 1, \ldots, l$, let us look at

$$\delta_{P_0}\left(\varphi_{(\gamma_{[\pi y_{\imath-1}, b_0]})}y_{\imath-1}, \varphi_{(\gamma_{[\pi y_\imath, b_0]})}y_\imath\right)$$

which is well defined because of properties (4.41).

For each unique (see remark in the beginning of *Step 1.*) minimal geodesic in $B$ joining $\pi y_{\imath-1}$ to $\pi y_\imath$, let its unique horizontal lift through $y_{\imath-1}$ be denoted by $\Gamma_{y_{\imath-1}} : [t_1, t_2] \longrightarrow M$.

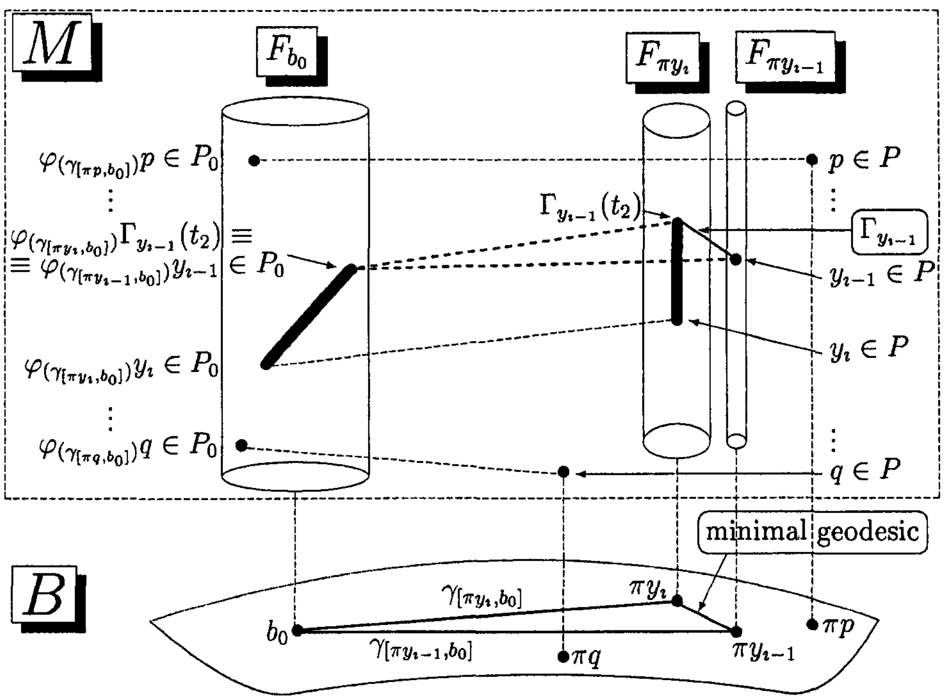


Figure 4.19: $y_{\imath-1}$, the lift $\Gamma_{y_{\imath-1}}(t_2)$, and their counterparts in the fiber $F_{b_0}$.

By **(LLL)**, (4.45) and **(RIF)**, we may write (see Fig. 4.19)

$$
\begin{aligned}
&\delta_{P_0}\left(\varphi(\gamma_{[\pi y_{\imath-1}, b_0]})y_{\imath-1}, \varphi(\gamma_{[\pi y_\imath, b_0]})y_\imath\right) \leq \\
\overset{\textbf{(LLL)}}{\leq}\quad & \delta_{P_0}\left(\varphi(\gamma_{[\pi y_\imath, b_0]})\Gamma_{y_{\imath-1}}(t_2), \varphi(\gamma_{[\pi y_\imath, b_0]})y_\imath\right) \leq \\
\overset{(4.45)}{\leq}\quad & a_0 d_M\left(\varphi(\gamma_{[\pi y_\imath, b_0]})\Gamma_{y_{\imath-1}}(t_2), \varphi(\gamma_{[\pi y_\imath, b_0]})y_\imath\right) + c_0 \leq \\
\overset{\textbf{(RIF)}}{\leq}\quad & a_0\left[A d_M\left(\Gamma_{y_{\imath-1}}(t_2), y_\imath\right) + C\right] + c_0
\end{aligned}
\tag{4.48}
$$

Now, that (4.46), **Proposition 4.11** and (4.47) hold, implies

$$
d_M\left(\Gamma_{y_{\imath-1}}(t_2), y_\imath\right) \overset{\triangle}{\leq} d_M\left(\Gamma_{y_{\imath-1}}(t_2), y_{\imath-1}\right) + d_M\left(y_{\imath-1}, y_\imath\right) \leq
$$

$$
\begin{aligned}
&\overset{(4.46)}{\leq} && d_M\left(\Gamma_{y_{\imath-1}}(t_2), y_{\imath-1}\right) + 2\hat{\epsilon}\,\overbrace{\delta_P(y_{\imath-1}, y_\imath)}^{\overset{(4.43)_1}{=}} \overset{dist.}{\leq} \ell(\Gamma_{y_{\imath-1}}) + 2\hat{\epsilon} \leq \\
&\overset{Prop4.11}{\leq} && \alpha\left[\ell\left(\gamma_{\pi y_{\imath-1}\pi y_\imath}\right) + \beta(t_2 - t_1)\right] + 2\hat{\epsilon} = \\
&\overset{dist.}{=} && \alpha d_B(\pi y_{\imath-1}, \pi y_\imath) + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon} \leq \\
&\overset{(4.47)}{\leq} && \alpha[2\epsilon_B\,\underbrace{\delta_{P_B}(\pi y_{\imath-1}, \pi y_\imath)]}_{\overset{(4.43)_1}{=}} + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon} = \\
&\overset{(4.43)}{=} && \alpha 2\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon} \qquad (4.49)
\end{aligned}
$$

By combining and (4.48) and (4.49), we get

$$
\begin{aligned}
&\delta_{P_0}\left(\varphi_{(\gamma_{[\pi y_{\imath-1}, b_0]})} y_{\imath-1}, \varphi_{(\gamma_{[\pi y_\imath, b_0]})} y_\imath\right) \leq \\
&\quad \overset{(4.48)}{\leq} \quad a_0\left[A d_M\left(\Gamma_{y_{\imath-1}}(t_2), y_\imath\right) + C\right] + c_0 \leq \\
&\quad \overset{(4.48)}{\leq} \quad a_0\left[A d_M\left(\Gamma_{y_{\imath-1}}(t_2), y_\imath\right) + C\right] + c_0 \leq \\
&\quad \overset{(4.49)}{\leq} \quad a_0\left\{A\left[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}\right] + C\right\} + c_0 = \\
&\quad = \quad a_0 A\left[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}\right] + a_0 C + c_0 \qquad (4.50)
\end{aligned}
$$

If we sum (4.50) over $\imath = 1, \ldots, l$, we may write

$$
\begin{aligned}
&\delta_{P_0}\left(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi q, b_0]})} q\right) \leq \\
&\quad \overset{\triangle}{\leq} \quad \sum_{\imath=1}^{l} \delta_{P_0}\left(\varphi_{(\gamma_{[\pi y_{\imath-1}, b_0]})} y_{\imath-1}, \varphi_{(\gamma_{[\pi y_\imath, b_0]})} y_\imath\right) \leq \\
&\quad \overset{(4.50)}{\leq} \quad l\left\{a_0 A\left[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}\right] + a_0 C + c_0\right\} = \\
&\quad = \quad \delta_P(p, q)\left\{a_0 A\left[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}\right] + a_0 C + c_0\right\} \qquad (4.51)
\end{aligned}
$$

Next, by using inequalities (4.44) and (4.51),

$$\delta_\times(\phi p, \phi q) = \delta_{P_0}\left(\varphi_{(\gamma_{[\pi p, b_0]})}p, \varphi_{(\gamma_{[\pi q, b_0]})}q\right) + \delta_{P_B}(\pi p, \pi q) \overset{(4.44)}{\leq}$$

$$\overset{(4.51)}{\leq} \quad \{a_0 A\,[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0\}\,\delta_P(p,q) + \delta_P(p,q) =$$

$$= \quad \{a_0 A\,[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0 + 1\}\,\delta_P(p,q) \qquad (4.52)$$

Finally, we combine inequalities (4.40) and (4.52), and obtain

$$\begin{aligned}
&\frac{1}{\hat{a}\cdot\max\{2A\epsilon_0, 2\alpha\epsilon_B\}}\delta_P(p,q) - \\
&-\left[\frac{\hat{c}}{\hat{a}} + \alpha\beta(t_2 - t_1) + AC\right]\frac{1}{\max\{2A\epsilon_0, 2\alpha\epsilon_B\}} \overset{(4.40)}{\leq} \delta_\times(\phi p, \phi q) \leq \\
&\overset{(4.52)}{\leq} \{a_0 A\,[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0 + 1\}\,\delta_P(p,q)
\end{aligned}$$

which is the required property (4.35) as follows

$$\frac{1}{a}\delta_P(p,q) - c \leq \delta_\times(\phi p, \phi q) \leq a\delta_P(p,q) + c$$

with the universal constants given by

$$\begin{aligned}
a :=&\ \max\{a_0 A\,[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0 + 1, \\
&\qquad \hat{a}\cdot\max\{2A\epsilon_0, 2\alpha\epsilon_B\}\} \geq 1 \\
c :=&\ \left[\frac{\hat{c}}{\hat{a}} + \alpha\beta(t_2 - t_1) + AC\right]\frac{1}{\max\{2A\epsilon_0, 2\alpha\epsilon_B\}} > 0
\end{aligned}$$

and so $\phi$ satisfies **(RI.2)**.

This concludes the proof of the Theorem.

□

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# Autobiographical Statement

Cristina Abreu Suzuki was born and raised in São Paulo, SP, Brazil. She received a Bachelor of Science and Licentiate in Mathematics from Pontifical Catholic University of São Paulo, SP, Brazil in 1980. She went on to work for the State of São Paulo, Brazil first as full-time mathematics high-school teacher at "Conde José Vicente de Azevedo" State High School–Senior Level and next as a full-time tenured mathematics high-school teacher at "Jardim Maria Tereza" State High School–Junior Level. From 1986 until 1988 she worked as a tenured instructor in the Department of Mathematics at The College of Economical Sciences of São Paulo of The Foundation School of Commerce "Álvares Penteado", SP, Brazil. In 1989 she received a Master of Science in Mathematics from University of São Paulo, Brazil. In 1988 she obtained a full-time tenure-track position as Assistant Professor in Mathematics at University of São Paulo, Brazil, and she retained this position until the year 2000. In 1991 she was awarded a governmental fellowship and a paid leave of absence to pursue a Ph.D. degree in Mathematics at Graduate Center of City University of New York. She received a Ph.D. in Mathematics from Graduate Center of City University of New York in June 1$^{St}$, 2004. Her main research interests are differential geometry, geometric inequalities, partial differential equations, combinatorial geometry, graph theory, Riemannian geometry and metric spaces.

She is a long distance runner, who has won many races, a fervent animal rights activist and a connoisseur of vegan cooked and gourmet raw foods. She has been an ethical vegetarian since 1977 and a frugivore since 1996. Her hobbies include book collecting; playing musical instruments and singing; acting; creating art using GIMP–The GNU Image Manipulation Program and practicing read, written and spoken: Portuguese (native language), Japanese, French, Russian, Spanish and Italian, besides learning new languages. Her favorite pastimes include swimming, cycling, hiking, doing yoga and meditating, practicing oha-shiatsu and Swedish massages both on human and non-human friends, and volunteering her time to charitable organizations.