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A DECOMPOSITION THEOREM FOR A CLASS OF 2-COMPLEXES

by
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PREFACE

"Let three points be specified on a line. Then one of the points is in the interior of the line segment joining the other two, and one of the points is exterior to the line segment joining the other two." This elementary statement concerning the ordering of three points on a line is capable of various extensions. Thus, e.g., let n+2 points which are not all on an n-sphere be specified in E_n, with some n+1 of them linearly independent. Then one of the points must be in the interior of the n-sphere passing through the other n+1 points, and one of the points must be exterior to the n-sphere passing through the other n+1 points. Analogous questions may be posed for non-spherical situations, but this area remains essentially unexplored.

F. Supnick opened this field of research with his Theorem Concerning Six Points (Proc. Am. Math. Soc. 1960, pp. 498-504). Cf. H. Simpson's On F. Supnick's Six Conic Theorem in Proc. Am. Math. Soc. 1961, p.931. Further, Prof. Seidel of Holland has informed F. Supnick of a paper he is writing which makes use of the above-mentioned work.

Another direction of generalizing the three-point statement above was undertaken by F. Supnick. Denoting the points A, B, C in linear order, he considered this as a decomposition of the line segment AC into three components, namely, the stars of the end-points and the point B. F. Supnick generalized this to the case of the linear graph. By a substar (sst(v)) of a star (st(v)) of a vertex of a linear graph, he means the set consisting of v and open segments of some edges of st(v) which have v as common end-point. If it is possible to decompose a linear graph G into mutually exclusive sets T_a, T_b, Q, such that $G = T_a \cup T_b \cup Q$ where

T_a: $\{sst(v_{a_1}),\ldots,sst(v_{a_k})\}$, T_b: $\{sst(v_{b_1}),\ldots,sst(v_{b_k})\}$,

$sst(v_{a_i})$ and $sst(v_{b_i})$ are of the same degree $(i = 1,2,\ldots,k)$, Q is a finite set of points in G (considered as a point set), and v_{a_i}, v_{b_i} $(i = 1,2,\ldots,k)$ are all distinct vertices of G, then he calls $G = T_a \cup T_b \cup Q$ a <u>bisection</u> of G. He proved that <u>any nonsingular linear graph G which contains no circuit formed by precisely one or two closed edges of G can be bisected</u>.

The question now remains of generalizing this work to complexes of higher dimension. The present paper generalizes to the case of finite, rectilinear, geometric 2-complexes. The problem of generalizing for arbitrary 2-complexes and for higher dimensions remains to be resolved.

FLOW CHART OF PROOF

K_2 is reducible if some 2 edges of the same degree either do not lie on the boundary of the same face or lie on the boundary of a common face having at least 4 bounding edges.

A 3-edged face is called a primitive irreducible if all its edges are of degree 1.

Thm. If K_2 is not a primitive irreducible, it is reducible.

Pf. Assume K_2 is irreducible but not primitive, and show that K_2 has at least 1 more edge of degree less than n+1 than possible, leading to a contradiction.

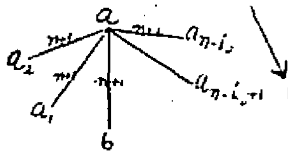
Suppose K_2 has exactly: $n-(j-2)-i_j$ edges of degree $n-(j-1)$ $(j = 0,1,\ldots,n-1)$

Case (A): Suppose all edges of degree n+1 cluster at a vertex a.

→ Case (A_1): $i_0=0$

→ Case (A_2): $i_0 = 1$ or 2

→ Case (A_3): $3 \leq i_0 \leq n+1$



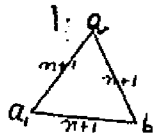
The sequence of numbers $S = (i_0,i_1,\ldots,i_{i_0-2})$ has 0 generation if $i_0 \leq i_k+k+1$ $(k=1,2,\ldots,i_0-2)$; 1 generation if $i_0 \leq i_k+k+1$ $(k=1,2,\ldots,j_0-1)$ and $i_0 > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq i_0-2$; s generations if $i_0 \leq i_k+k+1$ $(k=1,2,\ldots,j_0-1)$, $i_0 > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq i_0-2$, $i_{j_p} \leq i_k+(k-j_p)$ for all k, $j_p \leq k < j_{p+1}$, and $i_{j_p} > i_{j_{p+1}}+j_{p+1}-j_p$ for some j_{p+1} $(j_p < j_{p+1} \leq i_0-2)$ $(p=1,2,\ldots,s-2)$

→ Case (A_3^1): S has 0 generation.

→ Case (A_3^2): S has only 1 generation

→ Case (A_3^3): S has only r generations

Case (B): Suppose K_2 has only 3 edges of degree n+1 forming a triangular configuration.



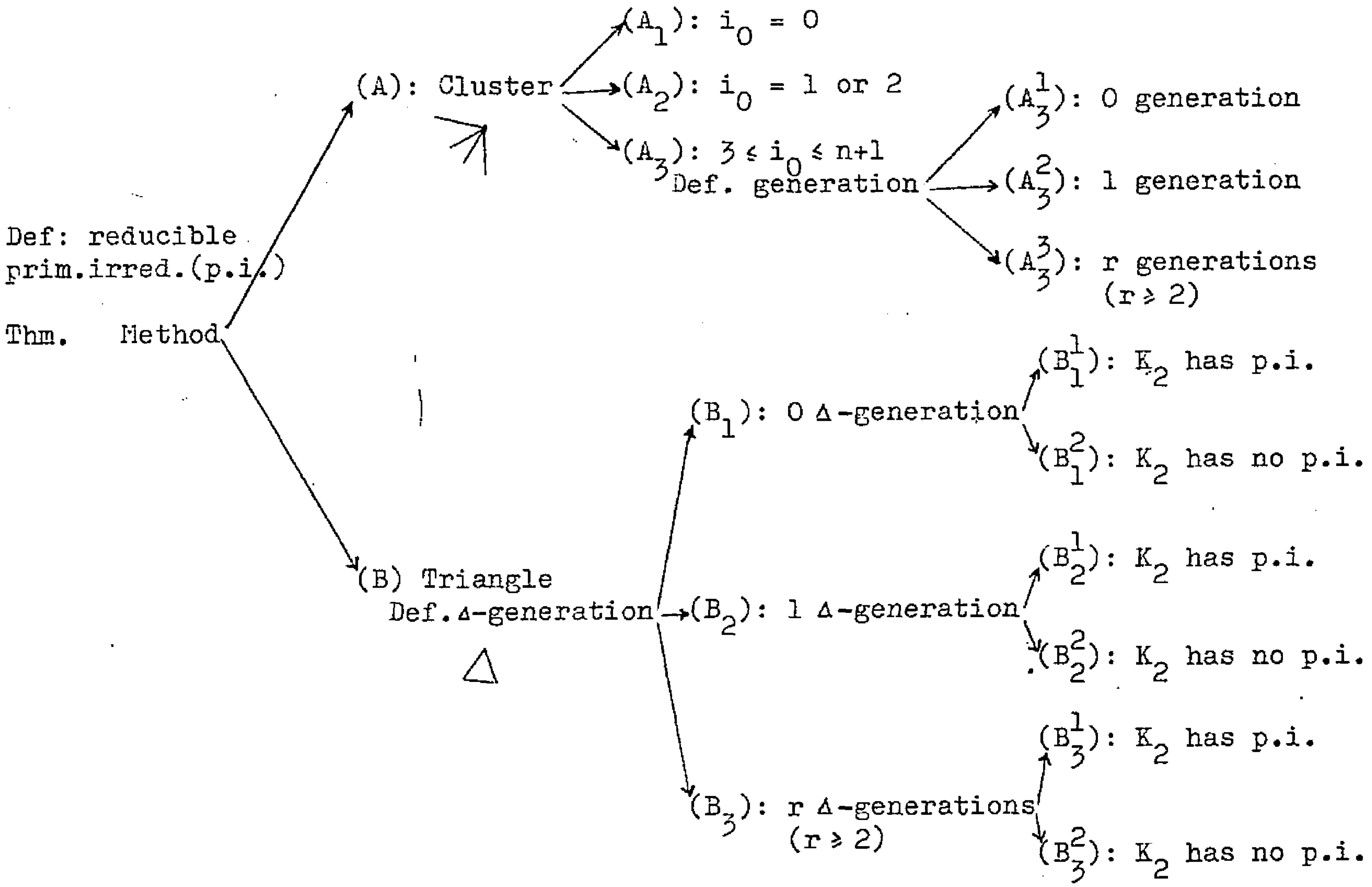
The sequence $(i_1,i_2,\ldots,i_{n-2})$ is said to have 0 Δ-generation if $n \leq i_k+k+1$ $(k=1,\ldots,n-2)$; 1 Δ-gen. if $n \leq i_k+k+1$ $(k=1,2,\ldots,j_0-1)$ and $n > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq n-2$; s Δ-gens.: same as generations with n substituted for i_0

→ Case (B_1): S has 0 Δ-gens → (B_1^1): p.i. → (B_1^2): no p.i.

→ Case (B_2): S has 1 Δ gen. → (B_2^1): p.i. → (B_2^2): no p.i.

→ Case (B_3): S has r Δ-gens. → (B_3^1): p.i. → (B_1^2): no p.i.

SKELETON OF FLOW CHART



INTRODUCTION: BISECTING A LINEAR GRAPH

This section consists of an (unpublished) note by F. Supnick which we now include for introductory reasons: —

"Consider a nonsingular geometric linear graph G consisting of a finite set of points (0-cells, vertices) and a finite set of open simple arcs (1-cells, edges) so that each 0-cell is an end-point of (is incident with) some 1-cell and each 1-cell has precisely two 0-cells as end-points. By the star of a vertex v (abbr. st(v)) of a linear graph we mean the set consisting of v and all the 1-cells $vp_1, \ldots, vp_j$ of G which are incident with v, i.e.,

$$\text{st}(v) = \{v, (vp_1), \ldots, (vp_j)\} \tag{1.1}$$

where ( ) and [ ] will denote that edges are open or closed respectively; j is called the degree of the star.

Let $q_1, \ldots, q_j$ be points on $[vp_1], \ldots, [vp_j]$; then the set

$$\{v, (vq_1), \ldots, (vq_j)\} \tag{1.2}$$

will be denoted sst(v) and called a sub-star of st(v).

Now, consider a line segment [AB] . In bisecting [AB] , we may view the bisection process as follows: we mark a point C on (AB), thus decomposing [AB] into sets T_a, T_b, and Q , where T_a consists of sst(A) = $\{A,(AC)\}$, T_b consists of sst(B) = $\{B,(BC)\}$ and Q = $\{C\}$. We note (with the purpose of motivating the generalization below) that sst(A) and sst(B) are of the same degree. We pose the question: can this bisection decomposition be extended to a linear graph (or to an n-complex)?

Definition: If it is possible to decompose a linear graph G into mutually exclusive sets T_a, T_b, Q, such that

$$G = T_a \cup T_b \cup Q \tag{1.3}$$

T_a: $\{sst(v_{a_1}),..,sst(v_{a_k})\}$, T_b: $\{sst(v_{b_1}),..,sst(v_{b_k})\}$

where $sst(v_{a_i})$ and $sst(v_{b_i})$ are of the same degree

(i = 1,2,..,k), Q is a finite set of points in G (considered as a point set), and v_{a_i}, v_{b_i} (i = 1,2,..,k) are all distinct vertices of G, then we shall say that the decomposition (1.3) is a bisection of G.

Theorem: Any nonsingular linear graph G which contains no circuit formed by precisely one or two closed edges of G can be bisected.

Proof: Let v_0 denote a vertex in G of maximum degree d, (i.e. $st(v_0)$ has maximum degree). Then G has at least d+1 vertices (since G has no one or two-edged circuits). Thus, two vertices of G, say v_{a_1} and v_{b_1} must have the same degree. If G contains the 1-cell $(v_{a_1}v_{b_1})$, let P be a point on $(v_{a_1}v_{b_1})$. Let us remove $(v_{a_1}v_{b_1})$ from G and replace it by (Pv_{a_1}), P, (Pv_{b_1}), denoting the new graph by G_0. If G does not contain the 1-cell $(v_{a_1}v_{b_1})$, let $G_0 = G$. Let Q_0 denote the set of isolated points of

$$G_0 - st(v_{a_1}) - st(v_{b_1}). \tag{1.4}$$

Remove Q_0 from (1.4) and denote the resulting linear graph by G_1.

This describes a process which may be iterated, whereby G_1 is treated as was G in the above argument, etc., eventually obtaining T_a, T_b, and $Q = \bigcup Q_i$. It must be remembered, however, that the star of a vertex of G_i $(i \geqslant 1)$ is a sub-star of G. Thus the theorem is established.

It might be of interest to ask how far the idea of bisection might be extended to an n-complex and to study classes of n-complexes for which bisection is possible (defining bisection of an n-complex as an extension of that above, but now permitting Q to contain 0,1, . . , (n-1)-cells)."

The present paper constitutes an extension of this work to finite, rectilinear, nonsingular, geometric 2-complexes.

A DECOMPOSITION THEOREM FOR A CLASS OF 2-COMPLEXES

The finite geometric complexes K_2 considered in this section will be composed of vertices, (open) rectilinear edges, and (open) planar faces which are the interiors of finite, convex planar polygons in R^m $(m > 1)$, where K_2 is nonsingular and connected, so that each vertex is an end-point of some edge, each edge has precisely two vertices as end-points, each edge bounds at least one face, each face has at least three bounding edges, and any pair of faces have at most one closed edge or one vertex in common.

By the star of an edge (abbr. st(E)), we mean the set $\{E; F_1, F_2, \ldots, F_j\}$ consisting of E and all open faces F_i $(i = 1,2,\ldots,j)$ having E as an edge of its bounding polygon. We shall refer to E as the hub of st(E).

By a sub-star of a st(E): $\{E; F_1, F_2, \ldots, F_j\}$ (abbr. sst(E)), we mean the set $\{E; G_1, G_2, \ldots, G_j\}$ where $G_i \in \{F_i, \emptyset, F_i'\}$, F_i' being that one of the two (open) components of F_i formed by a partitioning rectilinear diagonal (i.e. joining two non-adjacent vertices) which contains E on its boundary.

Consider, e.g., the star (fig.1) st(E):$\{AB; F_1, F_2, F_3\}$.

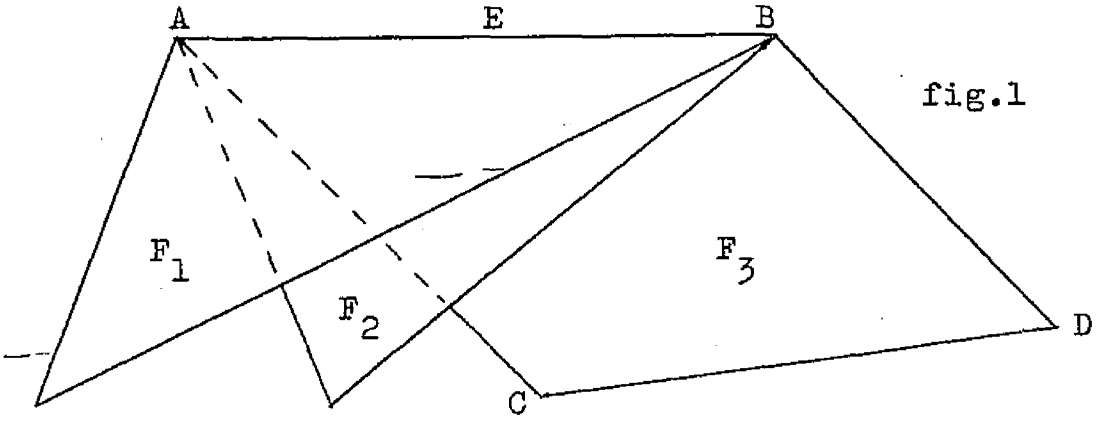


fig.1

Then sst($\underline{E}$): $\{AB; G_1, G_2, G_3\}$, where $G_1=F_1$, $G_2=\emptyset$, $G_3=F_3'=\Delta ABC$,

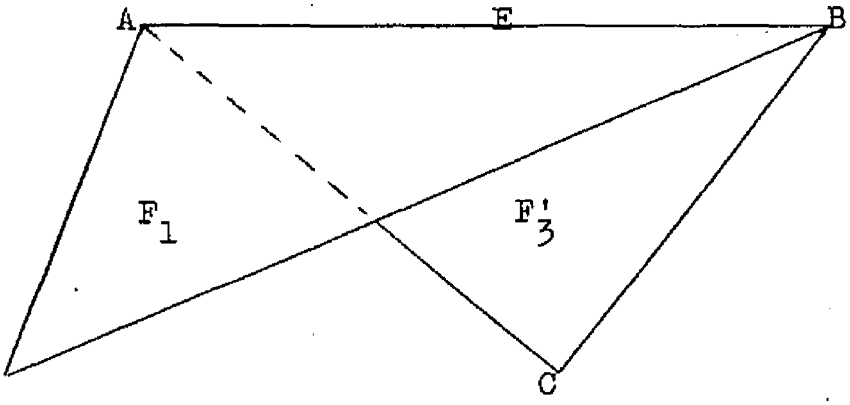


fig.2

would be a sub-star of st(E).

By the <u>degree of a sub-star of a st(E)</u>:

$$(2.1) \qquad \{E;\ G_1,\ G_2,\ .\ .\ ,\ G_j\}$$

(abbr. <u>deg(sst(E))</u>), we mean the number of non-null G_i's in (2.1).

Definition: If it is possible to decompose a 2-complex K_2 into mutually exclusive sets T_a, T_b, Q,

where $T_a: \{sst(E_{a_1}), sst(E_{a_2}), \ldots, sst(E_{a_k})\}$

$T_b: \{sst(E_{b_1}), sst(E_{b_2}), \ldots, sst(E_{b_k})\}$

$deg(sst(E_{a_i})) = deg(sst(E_{b_i})) \quad (i = 1,2,..,k)$

Q: a finite set of vertices, edges, and partitioning diagonals of K_2

E_{a_i}, E_{b_i} $(i = 1,2,..,k)$: distinct edges of K_2

such that

$$K_2 = T_a \cup T_b \cup Q \quad \text{(as a point set)} \tag{2.2}$$

then we say that the decomposition (2.2) is pairwise homologous. It is to be understood that all decompositions in this paper are pairwise homologous.

By the degree of an edge E of a complex K_2 (abbr. $deg(E)$), we shall mean the number of faces of K_2 having E in their boundaries.

We shall call a complex K_2 reducible if some two of its edges of the same degree either do not lie on the boundary of the same face, or lie on the boundary of a common face having at least four bounding edges.

A three-edged face is called a primitive irreducible if all its edges are of degree 1.

Theorem: If K_2 is not a single triangular face, it is reducible.

If a complex is reducible, we may begin the process of decomposing, by extracting the stars of two edges of the same degree if they do not lie on the boundary of the same face, or by introducing a partitioning diagonal in the common face having at least four bounding edges and extracting the appropriate sub-stars, and then removing any isolated edges and possibly a diagonal into the Q set. If the remaining sub-complex is not a primitive irreducible, we may iterate the above procedure until we have completely decomposed K_2 or are left with a single triangular face. Note that if the last sub-complex is a single triangular face, this does not necessarily mean that K_2 cannot be decomposed. Another sequence of choices of sub-stars may achieve the desired decomposition.

By the degree of a complex K_2 (abbr. $\deg(K_2)$) we shall mean the maximum degree of all the edges of K_2.

Proof: Let $\deg(K_2) = n+1$, $n \geq 0$. Suppose K_2 has

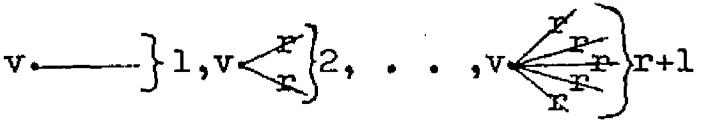
$$k_j = n-(j-2)-i_j \tag{2.3}$$

edges of degree $n-(j-1)$ $(j=0,1,..,n-1)$. Note that:

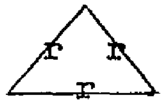
$$\deg(K_2) = n+1 \Rightarrow k_0 \geq 1 \Rightarrow i_0 \leq n+1 \ . \tag{2.4}$$

All edges of degree r can appear in at most two configurations in an irreducible K_2:

(A) Cluster at v (no more than r+1)



(B) Triangle



This is true since K_2 cannot have more than r+1 edges clustering at v, each of degree r. For, if K_2 had at least r+2 edges of degree r clustering at v, any two such edges would have to have a face in common (else reducible), and so each edge would be of degree at least r+1 - contradiction. Hence in (2.3), i_j represents the <u>localized</u> <u>defect</u>, i.e., there exist at most $n-(j-2)$ edges of degree $n-(j-1)$ in an irreducible K_2. Thus the actual number of edges of degree $n-(j-1)$ in an irreducible

K_2 may be represented as in (2.3), where i_j is the difference between the maximum number of edges of degree $n-(j-1)$ and the actual number.

If $\deg(K_2) = 1$, and K_2 is not a primitive irreducible, then K_2 has at least four edges of degree 1, proving reducibility by the discussion in the preceding paragraph. Hence, we may presume that $n \geq 1$, $i_j \geq 0$ ($j=0,1,\ldots,n-1$). Otherwise, reducibility follows immediately by the above remarks.

We shall assume that K_2 is irreducible, but not a primitive irreducible, and show that this leads to a contradiction.

Case (A): <u>Suppose all edges of degree n+1 cluster at a vertex a</u>. Denote these distinct edges by:

$$(2.5) \qquad ab,\ aa_1,\ aa_2,\ \ldots,\ aa_{n+1-i_0}\ .$$

(If $n=i_0-1$, ab is the only edge of degree $n+1$.) Since $\deg(ab) = n+1$, at least $n+2$ distinct edges must emanate from a, denoted by: ab, aa_i ($i=1,2,\ldots,n+1$), and at least $n+2$ distinct edges must emanate from b, denoted by: ab, bb_i ($i=1,2,\ldots,n+1$). Since K_2 is irreducible, ab and

aa_j $(j=1,2,..,n+1-i_0)$ must share a three-edged face, denoted by F_j, with edges: aa_j, ab, ba_j. Let x be a vertex in K_2, and let <u>$E(st(x))$</u> denote the set of edges of $st(x)$. Since $\deg(aa_s) = n+1$ $(s=1,2,..,n+1-i_0)$, at least $n+2$ distinct edges must emanate from a_s. At most one edge from each a_i $(i=1,2,..,s-1)$ can terminate at a_s, and only aa_s emanating from a and ba_s emanating from b terminate at a_s. Hence, at least $n+1-s$ distinct edges, each of which does not belong to

$$(2.6)\left\{E(st(a))\cup E(st(b))\cup E(st(a_1))\cup E(st(a_2))\cup..\cup E(st(a_{s-1})\right\},$$

emanate from a_s. Call these edges $a_s a_s^{j_s}$ $(j_s=1,2,..,n+1-s)$.

Let $c(S)$ denote the number of elements of the set S. K_2 has at least the following distinct edges of degree less than $n+1$, each of which belongs to (2.6) for $s = n+2-i_0$:

$$(2.7)\quad \begin{aligned} W_0 &= \{ba_1, ba_2, .., ba_{n+1-i_0}, bb_{n+2-i_0}, .., bb_{n+1}\} \\ W_1 &= \{a_1a_1^1, a_1a_1^2, \ . \ . \ . \ . \ , a_1a_1^n\} \\ W_2 &= \{a_2a_2^1, a_2a_2^2, \ . \ . \ . \ , a_2a_2^{n-1}\} \end{aligned}$$

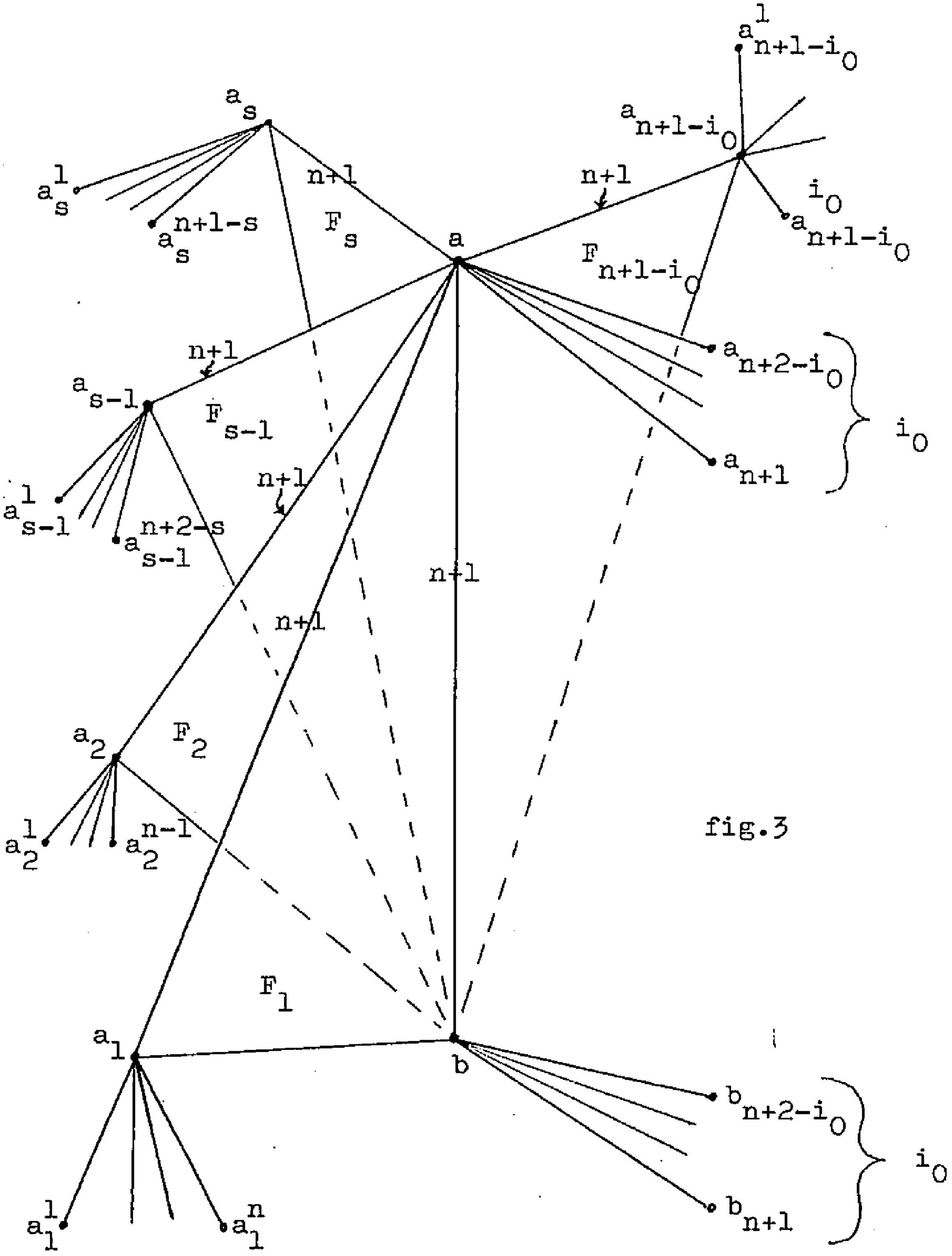

a_{n+1-i_0}^1
a_s
a_{n+1-i_0}
a_s^1
n+1
n+1
i_0
a_s^{n+1-s}
F_s
a
a_{n+1-i_0}
F_{n+1-i_0}
n+1
a_{s-1}
F_{s-1}
a_{n+2-i_0}
i_0
a_{n+1}
n+1
a_{s-1}^1
a_{s-1}^{n+2-s}
n+1
n+1
a_2
F_2
a_2^1
a_2^{n-1}
F_1
a_1
b
b_{n+2-i_0}
i_0
b_{n+1}
a_1^1
a_1^n

fig.3

$$\begin{aligned}
&\cdots\cdots\\
&W_s = \left\{a_s a_s^1, a_s a_s^2, \ . \ . \ . \ , a_s a_s^{n+1-s}\right\}\\
(2.7)\qquad &\cdots\cdots\\
&W_{n-i_0} = \left\{a_{n-i_0} a_{n-i_0}^1, a_{n-i_0} a_{n-i_0}^2, \ldots, a_{n-i_0} a_{n-i_0}^{i_0+1}\right\}\\
&W_{n+1-i_0} = \left\{a_{n+1-i_0} a_{n+1-i_0}^1, \ldots, a_{n+1-i_0} a_{n+1-i_0}^{i_0}\right\}\\
&W_{n+2-i_0} = \left\{a a_{n+2-i_0}, a a_{n+3-i_0}, \ldots, a a_{n+1}\right\}\\
&(s=0,1,\ldots,n+1-i_0) \quad (\text{If } i_0=0, \text{ then } W_{n+1-i_0}=\emptyset\\
&\qquad\qquad \text{and } W_{n+2-i_0}=\emptyset).
\end{aligned}$$

Note that:

$$\begin{aligned}
(2.8)\qquad & c(W_s) = n+1-s \quad (s=0,1,\ldots,n+1-i_0)\\
& c(W_{n+2-i_0}) = i_0 \ .
\end{aligned}$$

Adding up (2.8), we find that K_2 has at least:

$$(2.9)\qquad \frac{(n+1)(n+2)}{2} + \frac{-i_0^2+3i_0}{2}$$

distinct edges of degree less than n+1.

K_2 has at most:

$$(2.10)\qquad \begin{array}{l} n-(j-2) \text{ distinct edges of degree } n-(j-1) \\ \text{and } 3 \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad 1 \\ (j=1,2,..,n-1) \end{array}$$

by the discussion on p.9, since K_2 is irreducible. Hence, adding up the values in (2.10), we find that K_2 has at most:

$$(2.11)\qquad \frac{(n+1)(n+2)}{2}$$

distinct edges of degree less than n+1.

Case (A_1): $\underline{i_0 = 0}$. By (2.9), K_2 has at least $\frac{(n+1)(n+2)}{2}$ distinct edges of degree less than n+1. If K_2 has no primitive irreducible, then K_2 has at most 2 edges of degree 1, and hence at most $\frac{(n+1)(n+2)}{2} - 1$ distinct edges of degree less than n+1, contradicting the irreducibility of K_2. If K_2 has a primitive irreducible, say F, then none of the edges of F are in (2.7). For, all of the 1-cells in (2.7) are edges of faces which have another edge of degree n+1, and hence these faces cannot be primitive irreducibles. If one of the edges in (2.7) were an edge of F, it would also be an edge of another face, not a primitive irreducible, and so would be of degree at least 2, contradicting its being an edge of F. Hence, K_2 must have 3 additional edges, distinct from those in (2.5) and (2.7) and therefore K_2 must have at least $\frac{(n+1)(n+2)}{2} + 3$ distinct edges of degree less than n+1. This contradicts (2.11), and proves that K_2 cannot be irreducible in this case.

Case (A_2): $\underline{i_0 = 1 \text{ or } 2}$. By (2.9), K_2 has at least $\frac{(n+1)(n+2)}{2} + 1$ distinct edges of degree less than n+1. This contradicts (2.11), and establishes reducibility.

Case (A_3): $\underline{3 \leq i_0 \leq n+1}$. By (2.3), K_2 has exactly:

(2.12) $n-(j-2)-i_j$ distinct edges of degree $n-(j-1)$
$(j=1,2,..,i_0-2)$

and by (2.10), K_2 has at most:

(2.13) $n-(j-2)$ distinct edges of degree $n-(j-1)$

(2.13) and 3 distinct edges of degree 1 $(j=i_0-1, i_0, \ldots, n-1)$.

Hence, adding the values in (2.12) and (2.13), we find that K_2 has at most:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_0-2} i_k \tag{2.14}$$

distinct edges of degree less than n+1.

Generation Descendants

The remainder of the proof will make use of the fact that at least x+1 distinct edges must emanate from a point which is the vertex of an edge of degree x. Each of the edges in (2.5) and (2.7) emanates from one of the $n-i_0+3$ distinct vertices:

$$a,\ b,\ a_i (i=1,2,\ldots,n-i_0+1) \tag{2.15}$$

Since, by (2.3), K_2 has exactly $n-(j-2)-i_j$ distinct edges of degree $n-(j-1)$ $(j=1,2,\ldots,i_0-2)$, K_2 must have at least $n-(j-2)-i_j$ distinct vertices of edges of degree $n-(j-1)$. If $n-(j-2)-i_j \leq n-i_0+3$, i.e., if $i_0 \leq i_j+j+1$ $(j=1,2,..,i_0-2)$, then we cannot be sure that the vertices of edges of any particular degree lie outside the set of vertices (2.15). In this case, no edges of degree less than n+1 which are not already listed in (2.5) and (2.7) can be shown to exist, and hence we speak of the sequence of localized defects $S = (i_0, i_1, .., i_{i_0-2})$ as having <u>no descendants</u>.

If, however, $n-(j-2)-i_j \leq n-i_0+3$, i.e., if $i_0 \leq i_j+j+1$ $(j=1,2,..,j_0-1)$, but $n-(j_0-2)-i_{j_0} > n-i_0+3$, i.e., $i_0 > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq i_0-2$, this means that at least $i_0-i_{j_0}-j_0-1$ distinct vertices of edges of degree $n-(j_0-1)$ lie outside the set of vertices (2.15). Denote these vertices by:

$$v^k_{n-(j_0-1)} \qquad (k=1,2,..,i_0-i_{j_0}-j_0-1) \tag{2.16}$$

K_2 must have at least $n-(j_0-2)$ distinct edges emanating from each of the vertices in (2.16). Even allowing for all possibilities of edges already listed in (2.5) and

(2.7) terminating at the vertices in (2.16), and all possibilities of these vertices sharing edges among themselves, we find (see Case (A^2_3)) that K_2 must have additional edges, distinct from those already listed, which emanate from these vertices. We call these edges *first generation*, and speak of the sequence of localized defects $S = (i_0, i_1, .., i_{i_0-2})$ as having *first generation descendants*.

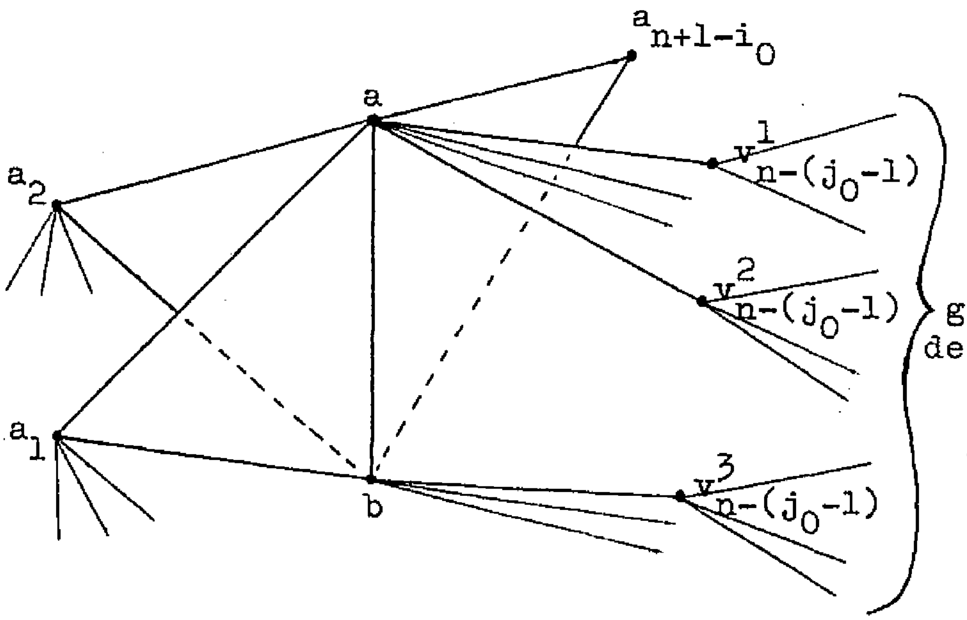


fig.4

Now all the edges of the first generation are of degree less than n+1 since all edges of degree n+1 have already been listed in (2.5), and the first generation edges were chosen as distinct from these. Each of the edges in (2.5),(2.7), and the first generation emanates from one of the $(n-i_0+3) + (i_0-i_{j_0}-j_0-1) = n-i_{j_0}-j_0+2$

distinct vertices:

$$(2.17)\qquad \begin{array}{l} a,\ b,\ a_i (i=1,2,..,n-i_0+1) \\ v^k_{n-(j_0-1)} (k=1,2,..,i_0-i_{j_0}-j_0-1)\ . \end{array}$$

If $n-(j-2)-i_j \leqslant n-i_{j_0}-j_0+2$, i.e., $i_{j_0} \leqslant i_j+(j-j_0)$ $(j=j_0+1, j_0+2,..,i_0-2)$, then we cannot be sure that the vertices of edges of any particular degree less than n+1 lie outside the vertices in (2.17). In this case, no edges of degree less than n+1 which are not already listed in (2.7) and the first generation can be shown to exist, and hence we speak of the sequence of localized defects $S = (i_0,i_1,..,i_{i_0-2})$ as having <u>only one generation</u>.

If, however, $n-(j-2)-i_j \leqslant n-i_{j_0}-j_0+2$, i.e., $i_{j_0} \leqslant i_j+(j-j_0)$ $(j=j_0,j_0+1,..,j_1-1)$, but $n-(j_1-2)-i_{j_1} > n-i_{j_0}-j_0+2$, i.e., $i_{j_0} > i_{j_1}+(j_1-j_0)$ for some j_1, $j_0 < j_1 \leqslant i_0-2$, this means that at least $i_{j_0}-i_{j_1}+j_0-j_1$ distinct vertices of edges of degree $n-(j_1-1)$ lie outside the set of vertices (2.17). Denote these vertices by:

$$(2.18)\qquad v^k_{n-(j_1-1)} (k=1,2,..,i_{j_0}-i_{j_1}+j_0-j_1).$$

K_2 must have at least $n-(j_1-2)$ distinct edges emanating from each of the vertices in (2.18). Even allowing for all possibilities of edges already listed terminating at these vertices and all possibilities of these vertices sharing edges among themselves, we find (see Case (A_3^3)) that K_2 must have additional edges, distinct from those already listed, which emanate from these vertices. We call these edges <u>second generation</u>, and speak of the sequence of localized defects $S = (i_0, i_1, .., i_{i_0-2})$ as having <u>second generation descendants</u>.

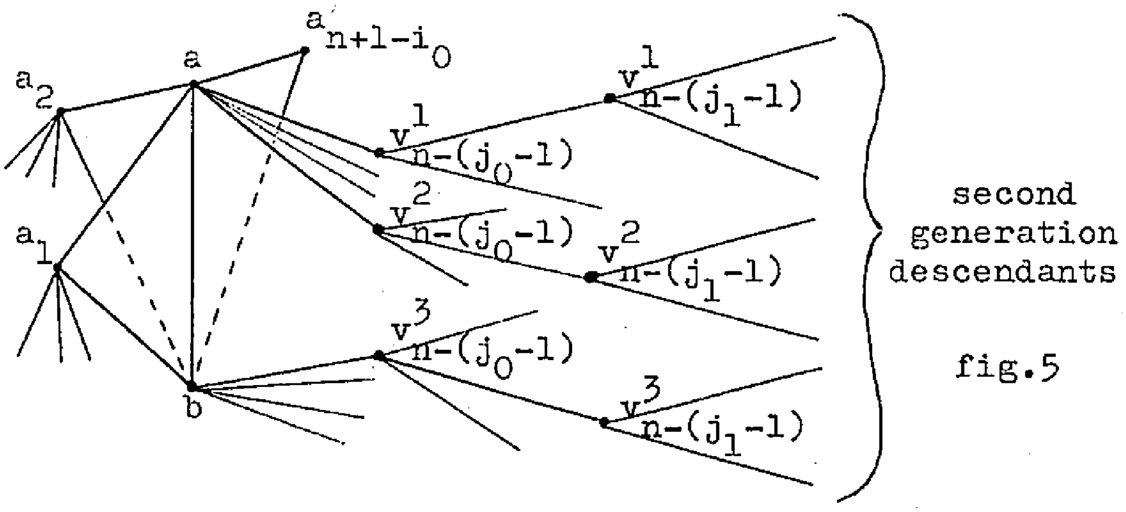


fig.5

Preserving the consistency of the emerging pattern, we are led to define the <u>s^{th} generation</u> by the inequalities: $n-(j-2)-i_j \leq n-i_{j_{s-2}}-j_{s-2}+2$, i.e., $i_{j_{s-2}} \leq i_j+(j-j_{s-2})$ $(j=j_{s-2}, j_{s-2}+1, .., j_{s-1}-1)$, but $n-(j_{s-1}-2)-i_{j_{s-1}} > n-i_{j_{s-2}}$ $-j_{s-2}+2$, i.e., $i_{j_{s-2}} > i_{j_{s-1}}+(j_{s-1}-j_{s-2})$ for some j_{s-1},

$j_{s-2} < j_{s-1} \leq i_0-2$. Summarizing and formalizing the above discussion, we are led to make the following definitions.

The sequence of localized defects $S = (i_0, i_1, .., i_{i_0-2})$ is said to have <u>no descendants</u> if $i_0 \leq i_k+k+1$ $(k=1,2,..,i_0-2)$. S is said to have <u>first generation descendants</u> if $i_0 > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq i_0-2$, and if $i_0 \leq i_k+k+1$ $(k=0,1,..,j_0-1)$. S is said to have <u>$s^{\underline{th}}$ generation descendants</u> if $i_0 > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq i_0-2$, if $i_0 \leq i_k+k+1$ $(k=0,1,..,j_0-1)$, and if $i_{j_p} > i_{j_{p+1}}+(j_{p+1}-j_p)$ for some j_{p+1}, $j_p < j_{p+1} \leq i_0-2$, and $i_{j_p} \leq i_k+(k-j_p)$ for all k, $j_p \leq k < j_{p+1}$ $(p=0,1,..,s-2)$.

The proof of the theorem (Case (A_3)) will proceed by a consideration of the number of generations of S.

Case (A_3^1): <u>S has no descendants</u>.

Hence, $i_0(i_0-2) \leq \sum_{k=1}^{i_0-2}(i_k+k+1) = \sum_{k=1}^{i_0-2}(k+1) + \sum_{k=1}^{i_0-2} i_k \Rightarrow$

$$(2.19) \qquad \frac{-i_0^2+3i_0}{2} - 1 \geqslant -\sum_{k=1}^{i_0-2} i_k$$

So, by (2.9), the number of distinct edges in K_2 of degree less than n+1 is at least:

$$\frac{(n+1)(n+2)}{2} + \frac{-i_0^2+3i_0}{2} > \frac{(n+1)(n+2)}{2} + \frac{-i_0^2+3i_0}{2} - 1 ,$$

which, by (2.19), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_0-2} i_k ,$$

which, by (2.14), is a maximum possible number of distinct edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (A_3^2): S has only 1 generation. Hence, we are given that:

$$(2.20)\qquad \begin{aligned} &i_0 > i_{j_0}+j_0+1 \text{ for some } j_0,\ 1 \leq j_0 \leq i_0-2 \\ &i_0 \leq i_k+k+1 \quad (k=1,2,..,j_0-1) \quad (\text{if } j_0 \neq 1) \\ &i_{j_0} \leq i_k+(k-j_0) \quad (k=j_0,j_0+1,..,i_0-2) \end{aligned}$$

K_2 must have $n-(j_0-2)$ distinct edges emanating from each vertex in (2.16). Any pair of vertices $v^k_{n-(j_0-1)}$ and $v^m_{n-(j_0-1)}$ $(k\neq m)$ can share at most 1 edge. While K_2 must have $n-(j_0-2)$ distinct edges emanating from $v^1_{n-(j_0-1)}$,

K_2 must have $n-(j_0-2)-1$ distinct edges emanating from $v^2_{n-(j_0-1)}$ since an edge emanating from $v^1_{n-(j_0-1)}$ may terminate at $v^2_{n-(j_0-1)}$. K_2 must have $n-(j_0-2)-2$ distinct edges emanating from $v^3_{n-(j_0-1)}$ since an edge from $v^1_{n-(j_0-1)}$ and an edge from $v^2_{n-(j_0-1)}$ may terminate at $v^3_{n-(j_0-1)}$. In general, K_2 must have $n-(j_0-2)-t$ distinct edges emanating from $v^{t+1}_{n-(j_0-1)}$ $(0 \leq t \leq i_0-i_{j_0}-j_0-2)$ since an edge from $v^k_{n-(j_0-1)}$ $(k=1,2,..,t)$ may terminate at $v^{t+1}_{n-(j_0-1)}$. In all, then, K_2 must have:

$$(n-(j_0-2)) + (n-(j_0-2)-1) + \;.\;.\; +(n-(j_0-2)-(i_0-i_{j_0}-j_0-2)) =$$

$$(2.21) \quad (i_0-i_{j_0}-j_0-1)n + \frac{j_0^2+7i_0-7i_{j_0}-5j_0-i_0^2-i_{j_0}^2+2i_0i_{j_0}-6}{2}$$

distinct edges emanating from some vertex in (2.16).

At most one edge from a, at most one edge from b, and at most one edge from each vertex a_j $(j=1,2,..,n+1-i_0)$ can terminate at each $v^k_{n-(j_0-1)}$ $(k=1,2,..,i_0-i_{j_0}-j_0-1)$. Even if all these possibilities occur, this accounts for only $(n-i_0+3)(i_0-i_{j_0}-j_0-1)$ =

$$(2.22) \quad (i_0-i_{j_0}-j_0-1)n - \frac{-8i_0+6i_{j_0}+6j_0+2i_0^2-2i_0i_{j_0}-2i_0j_0+6}{2}$$

distinct edges emanating from some vertex in (2.16). Subtracting (2.22) from (2.21), we find that K_2 has at least:

$$(2.23) \qquad \frac{i_0^2-i_{j_0}^2-i_0-i_{j_0}+j_0^2+j_0-2i_0j_0}{2}$$

distinct edges, each of which does not belong to the set (2.5) or (2.7). The value in (2.23) is positive, since by (2.20), $i_0 > i_{j_0}+j_0+1 \Rightarrow i_0-j_0-1 > i_{j_0} \geq 0 \Rightarrow i_0^2+j_0^2+j_0 > i_{j_0}^2+i_0+2i_0j_0+(i_0-j_0-1) = i_{j_0}^2+i_0+i_{j_0}+2i_0j_0+(i_0-(i_{j_0}+j_0+1))$, which, again by (2.20), is greater than $i_{j_0}^2+i_0+i_{j_0}+2i_0j_0$. These edges are all of degree less than n+1 since all edges of degree n+1 are listed in (2.5). Denote these distinct edges by:

$$(2.24) \qquad E_1^{j_1} \quad \left(j_1 = 1,2,..,\frac{i_0^2-i_{j_0}^2-i_0-i_{j_0}+j_0^2+j_0-2i_0j_0}{2} \right).$$

Therefore K_2 has at least:

$$\frac{(n+1)(n+2)}{2} + \frac{-i_0^2+3i_0}{2} + \frac{i_0^2-i_{j_0}^2-i_0-i_{j_0}+j_0^2+j_0-2i_0j_0}{2} =$$

$$(2.25) \qquad \frac{(n+1)(n+2)}{2} + i_0(1-j_0) + \frac{-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2}$$

distinct edges not of degree n+1. We note that:

$i_0 > i_{j_0}+j_0+1 \Rightarrow i_0-2 \geqslant i_{j_0}+j_0 \Rightarrow$

$$(2.26) \qquad \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{j_0+i_{j_0}} i_k \geqslant \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_0-2} i_k \ .$$

Now, from (2.20), $i_0 \leqslant i_k+k+1$ $(k=1,2,..,j_0-1) \Rightarrow$

$$i_0(j_0-1) \leqslant \sum_{k=1}^{j_0-1} (i_k+k+1) = \sum_{k=1}^{j_0-1} (k+1) + \sum_{k=1}^{j_0-1} i_k \Rightarrow$$

$$(2.27) \qquad i_0(1-j_0) + \frac{j_0^2+j_0}{2} - 1 \geqslant -\sum_{k=1}^{j_0-1} i_k .$$

Also, from (2.20), $i_{j_0} \leqslant i_k+(k-j_0) \quad (k=j_0,j_0+1,\ldots,i_{j_0}+j_0)$

$$i_{j_0}(i_{j_0}+1) \leqslant \sum_{k=j_0}^{j_0+i_{j_0}} (i_k+k-j_0) = \sum_{k=j_0}^{j_0+i_{j_0}} (k-j_0) + \sum_{k=j_0}^{j_0+i_{j_0}} i_k$$

$$(2.28) \qquad \frac{-i_{j_0}^2-i_{j_0}}{2} \geqslant -\sum_{k=j_0}^{j_0+i_{j_0}} i_k .$$

So, by (2.25), the number of distinct edges in K_2 of degree less than n+1 is at least:

$$\frac{(n+1)(n+2)}{2} + i_0(1-j_0) + \frac{-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2} >$$

$$\frac{(n+1)(n+2)}{2} + i_0(1-j_0) + \frac{j_0^2+j_0}{2} - 1 + \frac{-i_{j_0}^2-i_{j_0}}{2} ,$$

which, by (2.27) and (2.28), is greater than or equal to

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{j_0+i_{j_0}} i_k ,$$

which, by (2.26), is greater than or equal to

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_0-2} i_k ,$$

which, by (2.14), is a maximum possible number of distinct edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (A_3^3): S has only r generations $(r \geq 2)$.

We must first establish three lemmas.

Remark 1:

$$\frac{j_0(j_0-1)}{2} + \frac{(j_1-j_0-1)(j_1-j_0)}{2} - i_{j_0}(j_1-j_0) + \frac{(j_2-j_1-1)(j_2-j_1)}{2} - i_{j_1}(j_2-j_1) + . + \frac{(j_t-j_{t-1}-1)(j_t-j_{t-1})}{2} - i_{j_{t-1}}(j_t-j_{t-1}) + . + \frac{(j_s-j_{s-1}-1)(j_s-j_{s-1})}{2} - i_{j_{s-1}}(j_s-j_{s-1})$$

$$= (j_0-j_1)(i_{j_0}+j_0) + . + (j_{t-1}-j_t)(i_{j_{t-1}}+j_{t-1}) + . + (j_{s-1}-j_s)(i_{j_{s-1}}+j_{s-1}) + \frac{j_s^2-j_s}{2} \qquad (t = 1,2,..,s).$$

Proof of Remark 1: The proof proceeds by induction.

Let s = 1. Then $\frac{j_0(j_0-1)}{2} + \frac{(j_1-j_0-1)(j_1-j_0)}{2} - i_{j_0}(j_1-j_0) =$

$$\frac{j_0^2}{2} + \frac{(j_0-j_1)j_0}{2} - \frac{j_0j_1}{2} + i_{j_0}(j_0-j_1) + \frac{j_1^2-j_1}{2} =$$

$(j_0-j_1)(i_{j_0}+j_0) + \frac{j_1^2-j_1}{2}$. We now assume the remark true for all $m \leq s-1$, and show it true for $m = s$. Hence,

$$\frac{j_0(j_0-1)}{2} + \frac{(j_1-j_0-1)(j_1-j_0)}{2} - i_{j_0}(j_1-j_0) + \frac{(j_2-j_1-1)(j_2-j_1)}{2}$$

$$-i_{j_1}(j_2-j_1) +.+ \frac{(j_{s-1}-j_{s-2}-1)(j_{s-1}-j_{s-2})}{2} - i_{j_{s-2}}(j_{s-1}-j_{s-2})$$

$$+ \frac{(j_s-j_{s-1}-1)(j_s-j_{s-1})}{2} - i_{j_{s-1}}(j_s-j_{s-1})$$ equals, by the induction hypothesis, $(j_0-j_1)(i_{j_0}+j_0) + (j_1-j_2)(i_{j_1}+j_1) +$

$$. . + (j_{s-2}-j_{s-1})(i_{j_{s-2}}+j_{s-2}) + \frac{j_{s-1}^2-j_{s-1}}{2} +$$

$$\frac{(j_s-j_{s-1}-1)(j_s-j_{s-1})}{2} - i_{j_{s-1}}(j_s-j_{s-1}) =$$

$$(j_0-j_1)(i_{j_0}+j_0) + (j_1-j_2)(i_{j_1}+j_1) + . . +$$

$$(j_{s-2}-j_{s-1})(i_{j_{s-2}}+j_{s-2}) + \frac{j_{s-1}(j_{s-1}-j_s)}{2} + \frac{j_{s-1}^2}{2} - \frac{j_{s-1}j_s}{2} +$$

$$i_{j_{s-1}}(j_{s-1}-j_s) + \frac{j_s^2}{2} - \frac{j_s}{2} = (j_0-j_1)(i_{j_0}+j_0) +$$

$$(j_1-j_2)(i_{j_1}+j_1) + .. + (j_{s-2}-j_{s-1})(i_{j_{s-2}}+j_{s-2}) +$$

$(j_{s-1}-j_s)(i_{j_{s-1}}+j_{s-1}) + \frac{j_s^2-j_s}{2}$. This concludes the proof of Remark 1.

Remark 2: $(n-i_0+3) + (i_0-i_{j_0}-j_0-1) + (i_{j_0}-i_{j_1}+j_0-j_1) + \ldots + (i_{j_t}-i_{j_{t+1}}+j_t-j_{t+1}) + \ldots + (i_{j_{s-1}}-i_{j_s}+j_{s-1}-j_s) = n - i_{j_s} - j_s + 2 \quad (t = 0,1,..,s-1)$.

Proof of Remark 2: The proof proceeds by induction. Let $s = 1$. Then, $(n-i_0+3) + (i_0-i_{j_0}-j_0-1) + (i_{j_0}-i_{j_1}+ j_0-j_1) = n - i_{j_1} - j_1 + 2$. We now assume the remark true for all $m \leqslant s-1$, and show true for $m = s$. Hence, by the induction assumption, $(n-i_0+3) + (i_0-i_{j_0}-j_0-1) + (i_{j_0}-i_{j_1}+j_0-j_1) + \ldots + (i_{j_{s-2}}-i_{j_{s-1}}+j_{s-2}-j_{s-1}) + (i_{j_{s-1}}-i_{j_s}+j_{s-1}-j_s) = (n-i_{j_{s-1}}-j_{s-1}+2) + (i_{j_{s-1}}-i_{j_s}+ j_{s-1}-j_s) = n - i_{j_s} - j_s + 2$. This concludes the proof of Remark 2.

Lemma 1: If S has r generations ($r \geqslant 2$), then K_2 has the following distinct vertices:

$$(2.29) \qquad \begin{array}{l} a,\ b,\ a_i \ (i = 1,2,..,n+1-i_0) \\ v^k_{n-(j_0-1)} \ (k = 1,2,..,i_0-i_{j_0}-j_0-1) \end{array}$$

$$
(2.29)\quad
\begin{array}{l}
v^k_{n-(j_1-1)}\ (k = 1,2,..,i_{j_0}-i_{j_1}+j_0-j_1) \\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
v^k_{n-(j_t-1)}\ (k = 1,2,..,i_{j_{t-1}}-i_{j_t}+j_{t-1}-j_t) \\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
v^k_{n-(j_{r-1}-1)}\ (k = 1,2,..,i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1})
\end{array}
$$

$(t = 1,2,..,r-1)$ where $v^k_{n-(j_t-1)}$ is a vertex of an edge of degree $n-(j_t-1)$.

<u>Proof of Lemma 1</u>: The proof proceeds by induction on the number r of generations of S.

Suppose S has only 2 generations $(r = 2)$, i.e.,

$$
(2.30)\quad
\begin{array}{l}
i_0 > i_{j_0}+j_0+1 \text{ for some } j_0,\ 1 \leq j_0 \leq i_0-2 \\
i_0 \leq i_k+k+1\ (k = 0,1,..,j_0-1) \\
i_{j_0} > i_{j_1}+(j_1-j_0) \text{ for some } j_1,\ j_0 < j_1 \leq i_0-2 \\
i_{j_0} \leq i_k+(k-j_0) \text{ for all } k,\ j_0 \leq k < j_1 \\
i_{j_1} \leq i_k+(k-j_1) \text{ for all } k,\ j_1 \leq k \leq i_0-2.
\end{array}
$$

Because of the first generation in S, K_2 has the distinct vertices in (2.15) and (2.16) by the argument on p.17. Since, by (2.3), K_2 has exactly $n-(j_1-2)-i_{j_1}$ edges of degree $n-(j_1-1)$, K_2 must have at least $n-(j_1-2)-i_{j_1}$ distinct vertices of edges of degree $n-(j_1-1)$. Now, by (2.30), $n-(j_1-2)-i_{j_1} - ((n-i_0+3)+(i_0-i_{j_0}-j_0-1)) = i_{j_0}-i_{j_1}+j_0-j_1 > 0$. Hence, at least $i_{j_0}-i_{j_1}+j_0-j_1$ distinct vertices of edges of degree $n-(j_1-1)$ lie outside the set of vertices (2.15) and (2.16). Denote these vertices by:

$$v^k_{n-(j_1-1)} \ (k = 1,2,..,i_{j_0}-i_{j_1}+j_0-j_1), \tag{2.31}$$

none of which are listed in (2.15) or (2.16).

We now assume the lemma true for all $m \leq r-1$, and show true for $m = r$. Hence, we presume that K_2 has the following distinct vertices:

$$a,\ b,\ a_i \ (i = 1,2,..,n+1-i_0) \tag{2.32}$$

$$v^k_{n-(j_0-1)} \quad (k = 1,2,..,i_0-i_{j_0}-j_0-1)$$

$$v^k_{n-(j_1-1)} \quad (k = 1,2,..,i_{j_0}-i_{j_1}+j_0-j_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$v^k_{n-(j_t-1)} \quad (k = 1,2,..,i_{j_{t-1}}-i_{j_t}+j_{t-1}-j_t)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \tag{2.32}$$

$$v^k_{n-(j_{r-2}-1)} \quad (k = 1,2,..,i_{j_{r-3}}-i_{j_{r-2}}+j_{r-3}-j_{r-2})$$

$(t = 1,2,..,r-2)$ where $v^k_{n-(j_t-1)}$ is a vertex of an edge of degree $n-(j_t-1)$.

We are supposing that S has only r generations, i.e.,

$$i_0 > i_{j_0}+j_0+1 \text{ for some } j_0,\ 1 \leqslant j_0 \leqslant i_0-2$$

$$i_0 \leqslant i_k+k+1 \quad (k = 1,2,..,j_0-1) \text{ (if } j_0 \neq 1)$$

$$i_{j_p} > i_{j_{p+1}}+(j_{p+1}-j_p) \text{ for some } j_{p+1},\ j_p < j_{p+1} \leqslant i_0-2 \tag{2.33}$$

$$i_{j_p} \leqslant i_k+(k-j_p) \text{ for all } k,\ j_p \leqslant k < j_{p+1}$$

$$i_{j_{r-1}} \leqslant i_k+(k-j_{r-1}) \text{ for all } k,\ j_{r-1} \leqslant k \leqslant i_0-2$$

$$(p = 0,1,..,r-2).$$

Since, by (2.3), K_2 has exactly $n-(j_{r-1}-2)-i_{j_{r-1}}$ edges of degree $n-(j_{r-1}-1)$, K_2 must have at least $n-(j_{r-1}-2)-i_{j_{r-1}}$ distinct vertices of edges of degree $n-(j_{r-1}-1)$.

Now, by Remark 2, $(t = 1,2,..,r-2)$,

$n-(j_{r-1}-2)-i_{j_{r-1}} - ((n-i_0+3)+(i_0-i_{j_0}-j_0-1)+ . . + (i_{j_{t-1}}-i_{j_t}+j_{t-1}-j_t)+ . . +(i_{j_{r-3}}-i_{j_{r-2}}+j_{r-3}-j_{r-2})) =$

$n-(j_{r-1}-2)-i_{j_{r-1}} - (n-i_{j_{r-2}}-j_{r-2}+2) =$

$i_{j_{r-2}}-i_{j_{r-1}}+(j_{r-2}-j_{r-1})$, which is greater than 0 by (2.33).

Hence, at least $i_{j_{r-2}}-i_{j_{r-1}}+(j_{r-2}-j_{r-1})$ distinct vertices lie outside the set (2.32). Denote these vertices by:

$$v^k_{n-(j_{r-1}-1)} \quad (k = 1,2,..,i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1}), \tag{2.34}$$

none of which are listed in (2.32). This concludes the proof of Lemma 1.

Lemma 2: If S has r generations $(r \geqslant 2)$, then K_2 has the following set of distinct edges:

$$\{ab,\ aa_i\ (i = 1,2,..,n+1-i_0)\} \cup$$

$$W_s\ (s = 0,1,2,..,n+2-i_0) \cup$$

$$\{E_1^{j_1}\ (j_1 = 1,2,..,\frac{i_0^2-i_{j_0}^2-i_0-i_{j_0}+j_0^2+j_0-2i_0j_0}{2})\} \cup$$

$$\{E_2^{j_2}\ (j_2 = 1,2,..,\frac{i_{j_0}^2+2i_{j_0}j_0-2i_{j_0}j_1+j_0^2+j_1^2+i_{j_0}}{2} +$$

$$\frac{-i_{j_1}+j_0-j_1-i_{j_1}^2-2j_0j_1}{2})\} \cup$$

(2.35) $$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\{E_q^{j_q}\ (j_q = 1,2,..,\frac{i_{j_{q-2}}^2+2i_{j_{q-2}}j_{q-2}-2i_{j_{q-2}}j_{q-1}}{2} +$$

$$\frac{+j_{q-2}^2+j_{q-1}^2+i_{j_{q-2}}-i_{j_{q-1}}+j_{q-2}-j_{q-1}-i_{j_{q-1}}^2}{2} +$$

$$\frac{-2j_{q-2}j_{q-1}}{2})\} \cup$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\{E_r^{j_r}\ (j_r = 1,2,..,\frac{i_{j_{r-2}}^2+2i_{j_{r-2}}j_{r-2}-2i_{j_{r-2}}j_{r-1}}{2} +$$

$$\frac{+j_{r-2}^2+j_{r-1}^2+i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1}-i_{j_{r-1}}^2}{2} +$$

$$\left.\left.\frac{-2j_{r-2}j_{r-1}}{2}\right)\right\} \quad (q = 2,3,..r) \quad \text{where } E_q^{j_q}$$

emanates from a vertex in $\left\{ v^k_{n-(j_{q-1}-1)} \; (k = 1, 2,..,i_{j_{q-2}}-i_{j_{q-1}}+j_{q-2}-j_{q-1}) \right\}$.

Proof of Lemma 2: The proof proceeds by induction on the number of generations of S.

Suppose S has only 2 generations. By Lemma 1, K_2 has the distinct vertices in (2.17) and (2.18). Because of the first generation of S, K_2 has the distinct edges in (2.5), (2.7), and (2.24) emanating from the vertices in (2.15) and (2.16). K_2 must have $n-(j_1-2)$ distinct edges emanating from each vertex in (2.31). By definition, $v^k_{n-(j_1-1)}$ and $v^m_{n-(j_1-1)}$ $(k \neq m)$ can share at most one edge. While K_2 must have $n-(j_1-2)$ distinct edges emanating from $v^1_{n-(j_1-1)}$, K_2 must have $n-(j_1-2)-1$ distinct edges emanating from $v^2_{n-(j_1-1)}$ since an edge from $v^{1}_{n-(j_1-1)}$ may terminate at $v^{2}_{n-(j_1-1)}$. K_2 must have $n-(j_1-2)-2$ distinct edges emanating from $v^{3}_{n-(j_1-1)}$ since an edge from $v^{1}_{n-(j_1-1)}$ and an edge from $v^{2}_{n-(j_1-1)}$ may terminate at $v^{3}_{n-(j_1-1)}$. In general, K_2 must have $n-(j_1-2)-t$ distinct edges emanating from $v^{t+1}_{n-(j_1-1)}$ $(0 \leqslant t \leqslant i_{j_0}-i_{j_1}+j_0-j_1-1)$ since an edge from $v^{k}_{n-(j_1-1)}$ $(k = 1,2,..,t)$ may terminate at $v^{t+1}_{n-(j_1-1)}$. In all, then, K_2 must have: $(n-(j_1-2)) + (n-(j_1-2)-1) + . . . + (n-(j_1-2)-(i_{j_0}-i_{j_1}+j_0-j_1-1))$ =

$$(2.36) \quad (i_{j_0}-i_{j_1}+j_0-j_1)n + \frac{j_1^2+5i_{j_0}-5i_{j_1}+5j_0-5j_1-i_{j_0}^2}{2} + \frac{+2i_{j_0}i_{j_1}-2i_{j_0}j_0-i_{j_1}^2+2i_{j_1}j_0-j_0^2}{2}$$

distinct edges emanating from some vertex in (2.31).

At most one edge from a, at most one edge from b, at most one edge from each vertex a_j $(j = 1,2,..,n+1-i_0)$, and at most one edge from each vertex $v^k_{n-(j_0-1)}$ $(k = 1, 2,..,i_0-i_{j_0}-j_0-1)$ can terminate at each vertex in (2.31). Even if all these possibilities occur, this accounts for only: $((n-i_0+3) + (i_0-i_{j_0}-j_0-1))(i_{j_0}-i_{j_1}+j_0-j_1) =$

$$(2.37) \qquad (i_{j_0}-i_{j_1}+j_0-j_1)n - \frac{+2i_{j_0}^2-2i_{j_0}i_{j_1}+4i_{j_0}j_0-2i_{j_0}j_1}{2} - \frac{-2i_{j_1}j_0+2j_0^2-2j_0j_1-4i_{j_0}+4i_{j_1}-4j_0+4j_1}{2}$$

distinct edges emanating from some vertex in (2.31). Subtracting (2.37) from (2.36), we find that K_2 has at least:

$$(2.38) \qquad \frac{i_{j_0}^2+2i_{j_0}j_0-2i_{j_0}j_1+j_0^2+j_1^2+i_{j_0}-i_{j_1}+j_0-j_1-i_{j_1}^2-2j_0j_1}{2}$$

distinct edges, each of which does not belong in the set of edges (2.5), (2.7), or (2.24). The value in (2.38) is positive, since by (2.30), $i_{j_0} > i_{j_1} + (j_1 - j_0) \Rightarrow$

$i_{j_0} - j_1 + j_0 > i_{j_1} \geqslant 0 \Rightarrow i^2_{j_0} + 2i_{j_0}j_0 + j_0^2 + j_1^2 > i^2_{j_1} + 2j_0j_1 + 2i_{j_0}j_1$.

Denote these distinct edges by:

$$(2.39) \quad E_2^{j_2} \quad (j_2 = 1,2,..,\frac{i^2_{j_0} + 2i_{j_0}j_0 - 2i_{j_0}j_1 + j_0^2 + j_1^2 + i_{j_0}}{2} + \frac{-i_{j_1} + j_0 - j_1 - i^2_{j_1} - 2j_0j_1}{2}),$$

where each $E_2^{j_2}$ emanates from a vertex in (2.31).

We now assume the lemma true for all $m \leqslant r-1$ and show it true for $m = r$. By Lemma 1, K_2 has the distinct vertices in (2.29). Because of the first r-1 generations of S and the induction assumption, K_2 has the following set of distinct edges:

$$\{ab,\ aa_i \ (i = 1,2,..,n+1-i_0)\} \cup$$

$$(2.40) \quad W_s \ (s = 0,1,2,..,n+2-i_0) \cup$$

$$\{E_1^{j_1} \ (j_1 = 1,2,..,\frac{i_0^2 - i^2_{j_0} - i_0 - i_{j_0} + j_0^2 + j_0 - 2i_0j_0}{2})\} \cup$$

$$\{E_2^{j_2}\ (j_2 = 1,2,..,\frac{i_{j_0}^2+2i_{j_0}j_0-2i_{j_0}j_1+j_0^2+j_1^2+i_{j_0}}{2}$$

$$\frac{-i_{j_1}+j_0-j_1-i_{j_1}^2-2j_0j_1}{2})\} \cup$$

$$\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot$$

$$\{E_q^{j_q}\ (j_q = 1,2,..,\frac{i_{j_{q-2}}^2+2i_{j_{q-2}}j_{q-2}-2i_{j_{q-2}}j_{q-1}}{2} + \tag{2.40}$$

$$\frac{+j_{q-2}^2+j_{q-1}^2+i_{j_{q-2}}-i_{j_{q-1}}+j_{q-2}-j_{q-1}-i_{j_{q-1}}^2}{2} +$$

$$\frac{-2j_{q-2}j_{q-1}}{2}\} \cup$$

$$\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot$$

$$\{E_{r-1}^{j_{r-1}}\ (j_{r-1} = 1,2,..,\frac{i_{j_{r-3}}^2+2i_{j_{r-3}}j_{r-3}-2i_{j_{r-3}}j_{r-2}}{2} +$$

$$\frac{+j_{r-3}^2+j_{r-2}^2+i_{j_{r-3}}-i_{j_{r-2}}+j_{r-3}-j_{r-2}-i_{j_{r-2}}^2}{2} +$$

$$\frac{-2j_{r-3}j_{r-2}}{2})\}$$

$(q = 2,3,..,r-1)$ where $E_q^{j_q}$ emanates from a vertex in $v_{n-(j_{q-1}-1)}^k$ $(k = 1,2,..,i_{j_{q-2}}-i_{j_{q-1}}+j_{q-2}-j_{q-1})$.

K_2 must have $n-(j_{r-1}-2)$ distinct edges emanating from

(2.41) $v^k_{n-(j_{r-1}-1)}$ $(k = 1,2,..,i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1})$

By definition, $v^k_{n-(j_{r-1}-1)}$ and $v^m_{n-(j_{r-1}-1)}$ $(k \neq m)$ can share at most one edge. While K_2 must have $n-(j_{r-1}-2)$ distinct edges emanating from $v^1_{n-(j_{r-1}-1)}$, K_2 must have $n-(j_{r-1}-2)-1$ distinct edges emanating from $v^2_{n-(j_{r-1}-1)}$, since an edge from $v^1_{n-(j_{r-1}-1)}$ may terminate at $v^2_{n-(j_{r-1}-1)}$. K_2 must have $n-(j_{r-1}-2)-2$ distinct edges emanating from $v^3_{n-(j_{r-1}-1)}$, since an edge from $v^1_{n-(j_{r-1}-1)}$ and an edge from $v^2_{n-(j_{r-1}-1)}$ may terminate at $v^3_{n-(j_{r-1}-1)}$. In general, K_2 must have $n-(j_{r-1}-2)-t$ distinct edges emanating from $v^{t+1}_{n-(j_{r-1}-1)}$ $(0 \leq t \leq i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1}-1)$ since an edge from $v^k_{n-(j_{r-1}-1)}$ $(k = 1,2,..,t)$ may terminate at $v^{t+1}_{n-(j_{r-1}-1)}$. In all, then, K_2 must have:

$(n-(j_{r-1}-2)) + (n-(j_{r-1}-2)-1) + (n-(j_{r-1}-2)-2) + . .$

$$. \; . + (n-(j_{r-1}-2)-(i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1}-1) \quad =$$

$$(i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1})n + \frac{-j_{r-1}^{2}+5i_{j_{r-2}}-5i_{j_{r-1}}}{2} +$$

$$\frac{+5j_{r-2}-5j_{r-1}-i_{j_{r-2}}^{2}+2i_{j_{r-2}}i_{j_{r-1}}-2i_{j_{r-2}}j_{r-2}-i_{j_{r-1}}^{2}}{2} + \tag{2.42}$$

$$\frac{+2i_{j_{r-1}}j_{r-2}-j_{r-2}^{2}}{2}$$

distinct edges emanating from some vertex in the set of vertices (2.41).

At most one edge from a, at most one edge from b, at most one edge from each a_j ($j = 1,2,..,n+1-i_0$), at most one edge from each vertex $v_{n-(j_0-1)}^{k}$ ($k = 1,2,.. ..,i_0-i_{j_0}-j_0-1$), and at most one edge from each vertex $v_{n-(j_{q-1}-1)}^{k}$ ($k = 1,2,..,i_{j_{q-2}}-i_{j_{q-1}}+j_{q-2}-j_{q-1}; q = 2,3,.. ..,r-1$) can terminate at each vertex in the set (2.41). Even if all these possibilities occur, this accounts for only the following number of distinct edges emanating from some vertex in (2.41): $((n-i_0+3)+(i_0-i_{j_0}-j_0-1)+(i_{j_0}-i_{j_1}+j_0-j_1)+\ .\ .\ .\ +(i_{j_t}-i_{j_{t+1}}+j_t-j_{t+1})+\ .\ .\ .\ +(i_{j_{r-3}}-i_{j_{r-2}}+j_{r-3}-j_{r-2}))(i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1})$,

(where t = 0,1,2,..,r-3), which, by Remark 2, equals:

$(n-i_{j_{r-2}}-j_{r-2}+2)(i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1}) =$

$$(2.43)\quad (i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1})n-\frac{2i^2_{j_{r-2}}-2i_{j_{r-2}}i_{j_{r-1}}}{2}-\frac{+4i_{j_{r-2}}j_{r-2}-2i_{j_{r-2}}j_{r-1}-2i_{j_{r-1}}j_{r-2}+2j^2_{r-2}}{2}-\frac{-2j_{r-2}j_{r-1}-4i_{j_{r-2}}+4i_{j_{r-1}}-4j_{r-2}+4j_{r-1}}{2}$$

Subtracting (2.43 from (2.42), we find that K_2 has at least:

$$(2.44)\quad \frac{i^2_{j_{r-2}}+2i_{j_{r-2}}j_{r-2}-2i_{j_{r-2}}j_{r-1}+j^2_{r-2}+j^2_{r-1}+i_{j_{r-2}}}{2}+$$

$$(2.44)\qquad \frac{-i_{j_{r-1}}+j_{r-2}-j_{r-1}-i^2_{j_{r-1}}-2j_{r-2}j_{r-1}}{2}$$

distinct edges, each of which is not listed in (2.40). The value in (2.44) is positive, since, by (2.33),

$$i_{j_{r-2}} > i_{j_{r-1}}+(j_{r-1}-j_{r-2}) \Rightarrow i_{j_{r-2}}-j_{r-1}+j_{r-2} > i_{j_{r-1}} \geq 0 \Rightarrow$$

$$i^2_{j_{r-2}}+2i_{j_{r-2}}j_{r-2}+j^2_{r-2}+j^2_{r-1} > i^2_{j_{r-1}}+2j_{r-2}j_{r-1}+2i_{j_{r-2}}j_{r-1}.$$

Denote these edges by:

$$(2.45)\qquad E_r^{j_r}\ \Bigl(j_r = 1,2,..,\frac{i^2_{j_{r-2}}+2i_{j_{r-2}}j_{r-2}-2i_{j_{r-2}}j_{r-1}}{2} + \frac{+j^2_{r-2}+j^2_{r-1}+i_{j_{r-2}}-i_{j_{r-1}}+j_{r-2}-j_{r-1}-i^2_{j_{r-1}}}{2} + \frac{-2j_{r-2}j_{r-1}}{2}\Bigr),$$

where each $E_r^{j_r}$ emanates from a vertex in (2.41). This concludes the proof of Lemma 2.

<u>Lemma 3</u>: If K_2 has r generations ($r \geq 2$), then K_2 has, in the set of edges (2.35),

$$\frac{(n+1)(n+2)}{2} + i_0(1-j_0) + j_0 + (j_0-j_1)(i_{j_0}+j_0) + (j_1-j_2)(i_{j_1}+j_1) + \cdot\cdot\cdot + (j_t-j_{t+1})(i_{j_t}+j_t) + \cdot\cdot \cdot\cdot + (j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) + \frac{-i_{j_{r-1}}^2 - i_{j_{r-1}}}{2} + \frac{+j_{r-1}^2 - j_{r-1}}{2} \,, \quad (t = 0,1,2,..,r-2) \,, \tag{2.46}$$

distinct edges of degree less than n+1.

Proof of Lemma 3: The proof proceeds by induction on the number of generations of S.

Suppose S has only 2 generations. By Lemma 2, K_2 has, in the set (2.35), the distinct edges listed in (2.7), (2.24), and (2.39), all of degree less than n+1. Adding up these edges, we find that the total number is:

$$\frac{(n+1)(n+2)}{2} + i_0(1-j_0) + j_0 + (j_0-j_1)(i_{j_0}+j_0) + \frac{-i_{j_1}^2 - i_{j_1} + j_1^2 - j_1}{2} \,,$$

which agrees with (2.46) for r = 2.

We now assume the lemma true for all $m \leq r-1$, and show true for m = r. By Lemma 2, K_2 has the set of distinct edges (2.35), where each $E_r^{j_r}$ emanates from a vertex in (2.34). Therefore, $\deg(E_r^{j_r}) < n+1$ since all edges of degree n+1 have both their vertices in (2.15). Because of the first r-1 generations of S and the induction assumption, K_2 has in the set of edges (2.40):

$$\begin{aligned} &\frac{(n+1)(n+2)}{2} + i_0(1-j_0) + j_0 + (j_0-j_1)(i_{j_0}+j_0) + \\ (2.47)\quad &(j_1-j_2)(i_{j_1}+j_1) + \ldots + (j_t-j_{t+1})(i_{j_t}+j_t) + \ldots + \\ &(j_{r-3}-j_{r-2})(i_{j_{r-3}}+j_{r-3}) + \frac{-i^2_{j_{r-2}}-i_{j_{r-2}}+j^2_{r-2}-j_{r-2}}{2} \end{aligned}$$

distinct edges of degree less than n+1, (t = 0,1,..,r-3). Adding the number of edges in (2.45) to the value in (2.47), we find that the number of distinct edges in (2.35) of degree less than n+1 is exactly as stated in (2.46). This concludes the proof of Lemma 3.

<u>Remark 3</u>: $i_0 - 2 \geq i_{j_{r-1}}+j_{r-1}$ for all $r \geq 2$.

<u>Proof of Remark 3</u>: The proof proceeds by induction. Let r = 2. Then, by (2.30), $i_0 > i_{j_0}+j_0+1 > i_{j_1}+j_1+1$,

which establishes the result in this case. We now assume the remark true for all $m \leq r-1$, and show true for $m = r$. By the induction assumption, $i_0 - 2 \geq i_{j_{r-2}} + j_{r-2}$, which, by (2.33), is greater than $i_{j_{r-1}} + (j_{r-1} - j_{r-2}) + j_{r-2} = i_{j_{r-1}} + j_{r-1}$. This concludes the proof of Remark 3.

<u>Corollary</u>: $$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_{j_{r-1}}+j_{r-1}} i_k \geq$$

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_0-2} i_k$$

Because of the first generation of S, (2.27) holds true. By (2.33), $i_{j_p} \leq i_k + (k - j_p)$ $(k = j_p, j_p+1, .., j_{p+1}-1)$

$$\Rightarrow \quad i_{j_p}(j_{p+1} - j_p) \leq \sum_{k=j_p}^{j_{p+1}-1} (i_k + k - j_p) \quad \Rightarrow$$

$$(2.48) \quad \frac{(j_{p+1} - j_p - 1)(j_{p+1} - j_p)}{2} - i_{j_p}(j_{p+1} - j_p) \geq - \sum_{k=j_p}^{j_{p+1}-1} i_k$$

$(p = 0, 1, .., r-2)$

Again by (2.33), we have that:

$$i_{j_{r-1}} \leqslant i_k+(k-j_{r-1}) \quad (k = j_{r-1},j_{r-1}+1,\ldots,j_{r-1}+i_{j_{r-1}}) \Rightarrow$$

$$i_{j_{r-1}}(i_{j_{r-1}}+1) \leqslant \sum_{k=j_{r-1}}^{j_{r-1}+i_{j_{r-1}}} (i_k+k-j_{r-1}) \Rightarrow$$

$$(2.49) \qquad \frac{-i^2_{j_{r-1}}-i_{j_{r-1}}}{2} \geqslant -\sum_{k=j_{r-1}}^{j_{r-1}+i_{j_{r-1}}} i_k$$

By Lemma 3, the number of distinct edges in K_2 of degree less than n+1 is at least:

$$\frac{(n+1)(n+2)}{2} + i_0(1-j_0) + j_0 + (j_0-j_1)(i_{j_0}+j_0) +$$
$$(j_1-j_2)(i_{j_1}+j_1) + \ldots + (j_t-j_{t+1})(i_{j_t}+j_t) + \ldots +$$
$$(j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) + \frac{-i^2_{j_{r-1}}-i_{j_{r-1}}+j^2_{r-1}-j_{r-1}}{2} \geqslant$$
$$\frac{(n+1)(n+2)}{2} +i_0(1-j_0) - 1 + j_0 + \Big((j_0-j_1)(i_{j_0}+j_0) +$$
$$(j_1-j_2)(i_{j_1}+j_1) + \ldots + (j_t-j_{t+1})(i_{j_t}+j_t) + \ldots +$$
$$(j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) + \frac{j^2_{r-1}-j_{r-1}}{2} \Big) + \frac{-i^2_{j_{r-1}}-i_{j_{r-1}}}{2},$$

which, by Remark 1, is equal to:

$$\frac{(n+1)(n+2)}{2} + i_0(1-j_0) - 1 + j_0 + \Big(\frac{j_0^2-j_0}{2} + \frac{(j_1-j_0-1)(j_1-j_0)}{2}$$

$$-i_{j_0}(j_1-j_0) + \frac{(j_2-j_1-1)(j_2-j_1)}{2} - i_{j_1}(j_2-j_1) + \ldots +$$

$$\frac{(j_t-j_{t-1}-1)(j_t-j_{t-1})}{2} - i_{j_{t-1}}(j_t-j_{t-1}) + \ldots +$$

$$\frac{(j_{r-1}-j_{r-2}-1)(j_{r-1}-j_{r-2})}{2} - i_{j_{r-2}}(j_{r-1}-j_{r-2}) \Big) + \frac{-i_{j_{r-1}}^2 - i_{j_{r-1}}}{2},$$

which, by (2.27), (2.48), and (2.49), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{j_{r-1}+i_{j_{r-1}}} i_k ,$$

which, by the Corollary to Remark 3, is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_0-2} i_k ,$$

which, by (2.14), is a maximum possible number of distinct edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (B): <u>Suppose K_2 has only 3 edges of degree n+1 forming a triangular configuration</u>. Denote these distinct edges by:

$$ab, \ aa_1, \ a_1b \ . \tag{2.50}$$

Since $\deg(ab) = \deg(aa_1) = \deg(a_1b) = n+1$, the edges in (2.50) must share a face F_1 because K_2 is irreducible. At least n+2 distinct edges must emanate from a, from b, and from a_1. Hence, K_2 must have n distinct edges emanating from a, from b, and from a_1, none of which are listed in (2.50). Call these edges:

$$\begin{gathered} aa_i(i = 2,3,\ldots,n+1), \ bb_i(i = 2,3,\ldots,n+1), \\ a_1a_1^{j_1}(j_1 = 1,2,\ldots,n) \end{gathered} \tag{2.51}$$

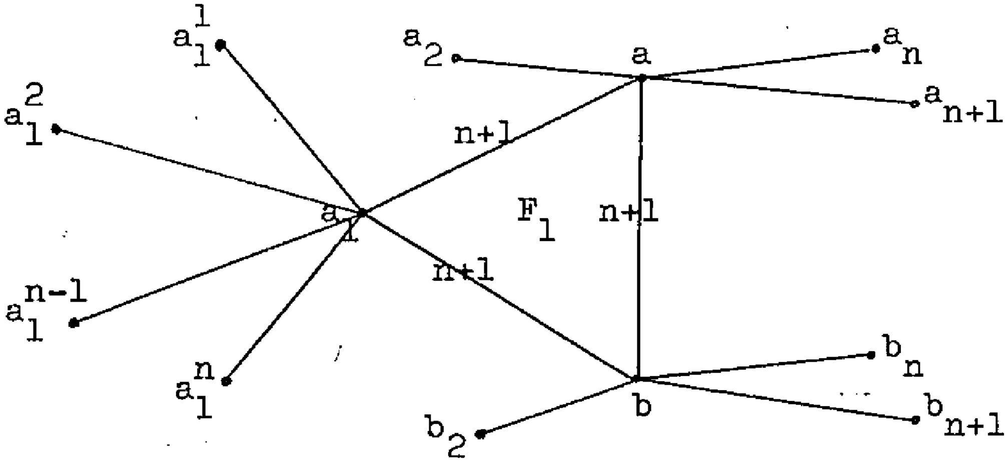


fig.6

All edges in (2.51) are of degree less than n+1 since they are all distinct from those in (2.50), the only edges of degree n+1. Hence, K_2 has at least:

(2.52) $$3n$$

distinct edges of degree less than n+1.

By (2.3), K_2 has exactly:

(2.53) $n-(j-2)-i_j$ distinct edges of degree $n-(j-1)$
$(j = 1,2,..,n-2)$,

and, by (2.10), K_2 has at most:

(2.54)
3 distinct edges of degree 2, and
3 " " " " 1,

by the discussion on p.9, since K_2 is irreducible.

Hence, by adding the values in (2.53) and (2.54), we find that K_2 has at most:

(2.55) $$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k$$

distinct edges of degree less than n+1.

Δ-Generation Descendants

Each of the edges in (2.50) and (2.51) emanates from one of the distinct vertices:

$$a,\ b,\ a_1\ . \tag{2.56}$$

Since K_2 has exactly $n-(j-2)-i_j$ distinct edges of degree $n-(j-1)$ $(j = 1,2,..,n-2)$, K_2 must have at least $n-(j-2)-i_j$ distinct vertices of edges of degree $n-(j-1)$. If $n-(j-2)-i_j \leq 3$, i.e., if $n \leq i_j+j+1$ for every j, $1 \leq j \leq n-2$, then we cannot be sure that the vertices of edges of any particular degree lie outside the set (2.56). In this case, no edges of degree less than n+1 which are not already listed in (2.51) can be shown to exist, and hence we speak of the sequence of localized defects $S = (i_0,i_1,\ldots,i_{n-2})$ as having <u>no Δ-descendants</u>. If, however, $n-(j-2)-i_j \leq 3$, i.e., $n \leq i_j+j+1$ $(j = 1,2,..,j_0-1)$, but $n-(j_0-2)-i_{j_0} > 3$, i.e., $n > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq n-2$, this means that at least $n-i_{j_0}-j_0-1$ distinct vertices of edges of degree $n-(j_0-1)$ lie outside the (2.56). Denote these vertices by:

$$(2.57) \qquad v^k_{n-(j_0-1)} \ (k = 1,2,..,n-i_{j_0}-j_0-1) \ .$$

K_2 must have at least $n-(j_0-2)$ distinct edges emanating from each of the vertices in (2.57). Even allowing for all possibilities of edges already listed in (2.51) terminating at the vertices in (2.57), and all possibilities of these vertices sharing edges among themselves, we find (see Case (B_2)) that K_2 must have additional edges, distinct from those already listed, which emanate from these vertices. We call these edges <u>first</u> <u>Δ-generation</u>, and speak of the sequence of localized defects $S = (i_0, i_1,..,i_{n-2})$ as having <u>first</u> <u>Δ-generation</u> <u>descendants</u>.

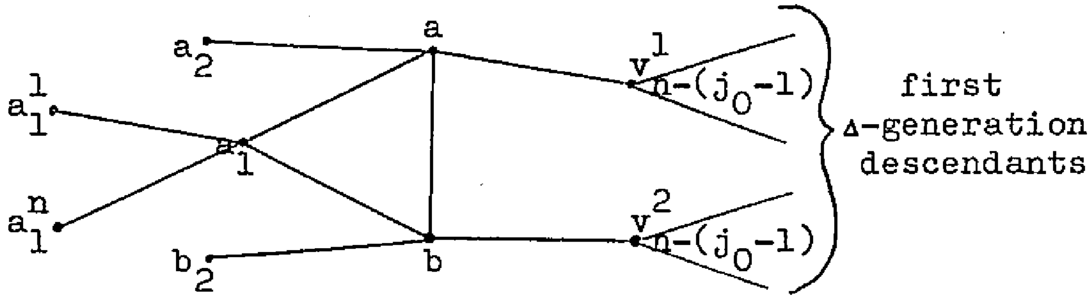


fig.7

Now, each of the edges in (2.50), (2.51), and the first Δ-generation emanates from one of the $3 + n-i_{j_0}-j_0-1 = n-i_{j_0}-j_0+2$ distinct vertices (2.56) and (2.57). If $n-(j-2)-i_j \leq n-i_{j_0}-j_0+2$, i.e., $i_{j_0} \leq i_j+(j-j_0)$, $(j = j_0+1, \ldots, n-2)$, then we cannot be sure that the vertices of edges of any particular degree less than n+1 lie outside the set of vertices (2.56) and (2.57). In this case, no edges of degree less than n+1 which are not already listed in (2.51) and the first Δ-generation can be shown to exist, and hence we speak of the sequence of localized defects $S = (i_0, i_1, \ldots, i_{n-2})$ as having <u>only one Δ-generation</u>.

The remainder of the discussion of Δ-generation follows very closely that of generation on pp.19-20. Summarizing and formalizing the above, we are led to make the following definitions.

The sequence of localized defects $S = (i_0, i_1, \ldots, i_{n-2})$ is said to have <u>no Δ-generation descendants</u> if $n \leq i_k+k+1$ for every k, $1 \leq k \leq n-2$. S is said to have <u>first Δ-generation descendants</u> if $n \leq i_k+k+1$ $(k = 1, 2, \ldots, j_0-1)$ but $n > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq n-2$. S is said to have

s^{th} Δ-generation descendants if $n \leq i_k+k+1$ $(k = 1,2,\ldots,j_0-1)$, if $n > i_{j_0}+j_0+1$ for some j_0, $1 \leq j_0 \leq n-2$, and if $i_{j_p} \leq i_k+(k-j_p)$ $(k = j_p, j_p+1, \ldots, j_{p+1}-1)$ and $i_{j_p} > i_{j_{p+1}}+(j_{p+1}-j_p)$ for some j_{p+1}, $j_p < j_{p+1} \leq n-2$ $(p = 0,1,2,\ldots,s-2)$. The proof of the theorem (Case (B)) will proceed by a consideration of the number of Δ-generations of S.

Case (B_1): Suppose S has no Δ-generation descendants.

Hence, $n(n-2) \leq \sum_{k=1}^{n-2} (i_k+k+1) \Rightarrow -\frac{n^2-3n+2}{2} \geq -\sum_{k=1}^{n-2} i_k \Rightarrow$

$$(2.58) \quad \frac{(n+1)(n+2)}{2} - \frac{n^2-3n+2}{2} = 3n \geq \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k \,.$$

Case (B_1^1): Suppose K_2 has a primitive irreducible F.

None of the edges in (2.51) can be edges of F since they are edges of faces, all of which have another edge in the set (2.50), and hence are not primitive irreducibles.

If one of the edges in (2.51) were an edge of F, it would also be an edge of another face, not a primitive irreducible, and so would be of degree at least 2, contradicting its being an edge of F. Hence, K_2 must have at least 3 additional edges of degree less than n+1, distinct from the 3n in (2.51). So, the total number of distinct edges in K_2 of degree less than n+1 is at least: $3n+3 > 3n$,

which, by (2.58), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k ,$$

which, by (2.55), is a maximum possible number of distinct edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (B_1^2): <u>Suppose K_2 has no primitive irreducible F.</u>

Then K_2 can have at most 2 distinct edges of degree 1, and hence, by (2.55), at most a total of:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k - 1$$ distinct edges of degree less than n+1. By (2.52), the number of distinct edges in K_2 of degree less than n+1 is at least: $3n$,

which, by (2.58), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k ,$$

which is greater than:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k - 1 ,$$

which is a maximum possible number of distinct edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (B_2): Suppose S has only 1 Δ-generation, i.e.,

$$\begin{aligned} & n \leq i_k+k+1 \quad (k=1,2,..,j_0-1) \text{ (if } j_0 \neq 1) \\ (2.59) \qquad & n > i_{j_0}+j_0+1 \text{ for some } j_0,\ 1 \leq j_0 \leq n-2 \\ & i_{j_0} \leq i_k+(k-j_0) \quad (k=j_0,j_0+1,..,n-2) \end{aligned}$$

K_2 must have $n-(j_0-2)$ distinct edges emanating from each vertex in (2.57). By the same argument as in Case (A_3^2), K_2 must have:

$$(2.60) \qquad n(n-i_{j_0}-j_0-1) + \frac{-n^2+2ni_{j_0}+7n-i_{j_0}^2-7i_{j_0}+j_0^2-5j_0-6}{2}$$

distinct edges emanating from some vertex in (2.57). At most one edge from a, at most one edge from b, and at most one edge from a_1 can terminate at each vertex in (2.57). Even if all these possibilities occur, this accounts for only:

$$(2.61) \qquad 3(n-i_{j_0}-j_0-1)$$

distinct edges emanating from some vertex in (2.57). Subtracting (2.61) from (2.60), we find that K_2 has at least:

$$(2.62) \qquad \frac{n^2-n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2}$$

distinct edges, each of which does not belong to the set of edges (2.50) or (2.51). The value in (2.62) is positive since by (2.59), $n > i_{j_0}+j_0+1 \Rightarrow n-j_0-1 > i_{j_0} \geq 0 \Rightarrow n^2+j_0^2+j_0 > i_{j_0}^2+n+2nj_0+(n-j_0-1) = i_{j_0}^2+n+2nj_0+i_{j_0}+(n-(i_{j_0}+j_0+1))$, which, again by (2.59), is greater than $i_{j_0}^2+n+2nj_0+i_{j_0}$. These edges are all of degree less than n+1 since (2.50) is a complete list of edges of degree n+1. Denote these distinct edges by:

$$(2.63) \qquad E_1^{j_1} \; (j_1 = 1,2,\ldots,\frac{n^2-n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2}).$$

Therefore, K_2 has at least:

$$3n + \frac{n^2-n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2} =$$

(2.64) $$\frac{n^2+5n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2}$$

distinct edges of degree less than n+1. Note that by (2.59), $n > i_{j_0}+j_0+1 \Rightarrow n-2 \geq i_{j_0}+j_0 \Rightarrow$

(2.65) $$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{i_{j_0}+j_0} i_k \geq \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k .$$

Now, from (2.59), $n \leq i_k+k+1$ $(k = 1,2,..,j_0-1) \Rightarrow$

$$n(j_0-1) \leq \sum_{k=1}^{j_0-1} (i_k+k+1) \Rightarrow$$

(2.66) $$\frac{-2nj_0+2n-2+j_0^2+j_0}{2} \geq -\sum_{k=1}^{j_0-1} i_k .$$

Also, from (2.59), $i_{j_0} \leq i_k+(k-j_0)$ $(k = j_0,j_0+1,..,j_0+i_{j_0})$

$$\Rightarrow\ i_{j_0}(i_{j_0}+1) \leq \sum_{k=j_0}^{j_0+i_{j_0}} (i_k+(k-j_0)) \quad \Rightarrow$$

$$(2.67) \qquad \frac{-i_{j_0}^2-i_{j_0}}{2} \geq -\sum_{k=j_0}^{j_0+i_{j_0}} i_k \ .$$

Hence, $\dfrac{n^2+5n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2} = \dfrac{n^2+3n+2}{2} +$

$\dfrac{-2nj_0+2n-2+j_0^2+j_0}{2} + \dfrac{-i_{j_0}^2-i_{j_0}}{2}$ by (2.66) and (2.67)

is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{j_0+i_{j_0}} i_k \ ,$$

which, by (2.65), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k \ .$$

So,

$$(2.68) \quad \frac{n^2+5n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2} \geq \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k \ .$$

Case (B_2^1): Suppose K_2 has a primitive irreducible F.

None of the edges in (2.51) or (2.63) can be edges of F since they are edges of faces all of which have another edge of degree greater than 1 ($n-(j_0-1)$ is at least 3) and hence are not primitive irreducibles. If one of the edges in (2.51) or (2.63) were an edge of F, it would also be an edge of another face, not a primitive irreducible, and so would be of degree at least 2, contradicting ~~its~~ being an edge of F. Hence, K_2 must have at least 3 additional edges of degree less than n+1, distinct from those in (2.51) and (2.63). So, the total number of distinct edges in K_2 of degree less than n+1 is at least:

$$\frac{n^2+5n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2}+3>\frac{n^2+5n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2},$$

which, by (2.68), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2}-\sum_{k=1}^{n-2} i_k ,$$

which, by (2.55), is a maximum possible number of distinct edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (B_2^2): Suppose K_2 has no primitive irreducible F.

Then K_2 can have at most 2 distinct edges of degree 1, and hence, by (2.55), at most a total of:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k - 1$$ distinct edges of degree less than n+1. By (2.64), the number of distinct edges in K_2 of degree less than n+1 is at least:

$$\frac{n^2+5n-2nj_0-i_{j_0}^2-i_{j_0}+j_0^2+j_0}{2},$$

which, by (2.68), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k > \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k - 1,$$

a maximum possible number of distinct edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (B_3): <u>Suppose S has only r Δ-generations ($r \geq 2$).</u> We must first establish three lemmas. By adding j_0 to both sides of the equality in Remark 1, we get

<u>Remark 4</u>: $$\frac{j_0(j_0+1)}{2} + \frac{(j_1-j_0-1)(j_1-j_0)}{2} - i_{j_0}(j_1-j_0) +$$

$$\frac{(j_2-j_1-1)(j_2-j_1)}{2} - i_{j_1}(j_2-j_1) + \cdot\cdot\cdot +$$

$$\frac{(j_t-j_{t-1}-1)(j_t-j_{t-1})}{2} - i_{j_{t-1}}(j_t-j_{t-1}) + \cdot\cdot\cdot +$$

$$\frac{(j_s-j_{s-1}-1)(j_s-j_{s-1})}{2} - i_{j_{s-1}}(j_s-j_{s-1}) = j_0 + (j_0-j_1)(i_{j_0}+j_0) + \cdots + (j_{t-1}-j_t)(i_{j_{t-1}}+j_{t-1}) + \cdots + (j_{s-1}-j_s)(i_{j_{s-1}}+j_{s-1}) + \frac{j_s^2-j_s}{2}, \quad (t = 1,2,..,s).$$

By adding $3 + (n-i_{j_0}-j_0-1) - ((n-i_0+3) + (i_0-i_{j_0}-j_0-1))$ to both sides of the equality in Remark 2, we get:

<u>Remark 5</u>: $3 + (n-i_{j_0}-j_0-1) + (i_{j_0}-i_{j_1}+j_0-j_1) + \cdots + (i_{j_t}-i_{j_{t+1}}+j_t-j_{t+1}) + \cdots + (i_{j_{s-1}}-i_{j_s}+j_{s-1}-j_s) = n - i_{j_s} - j_s + 2$, $(t = 0,1,..,s-1)$.

If the word "Δ-generation" is substituted for "generation", "n" for "i_0", and "Remark 5" for "Remark 2", the statements and proofs of Lemmas 1 and 2 hold true for Case (B) exactly as in Case (A).

<u>Lemma 4</u>: If S has r Δ-generations $(r \geq 2)$, then K_2 has in the set of edges (2.35):

$$(2.69) \quad \frac{n^2+5n}{2} + j_0(1-n) + (j_0-j_1)(i_{j_0}+j_0) + \cdots + (j_t-j_{t+1})(i_{j_t}+j_t) + \cdots + (j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2})$$

$$(2.69) \qquad + \frac{-i_{j_{r-1}}^2 - i_{j_{r-1}} + j_{r-1}^2 - j_{r-1}}{2} \quad , \quad (t = 0,1,..,r-2),$$

distinct edges of degree less than n+1.

<u>Proof of Lemma 4</u>: In Case (A) by Lemma 3, if $n = i_0$, K_2 has in the set of edges (2.35)

$$(2.70) \qquad \begin{aligned} &\frac{(n+1)(n+2)}{2} + n(1-j_0) + j_0 + (j_0-j_1)(i_{j_0}+j_0) + \\ &(j_1-j_2)(i_{j_1}+j_1) + \ldots + (j_t-j_{t+1})(i_{j_t}+j_t) + \ldots \\ &\ldots + (j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) + \frac{-i_{j_{r-1}}^2 - i_{j_{r-1}}}{2} + \\ &\frac{j_{r-1}^2 - j_{r-1}}{2} \quad , \quad (t = 0,1,..,r-2) \end{aligned}$$

distinct edges of degree less than n+1. Since $\deg(ba_1) = n+1$ in Case (B), there is one more edge of degree less than n+1 in (2.35) Case (A) than in (2.35) Case (B). Hence, (2.35) Case (B) has exactly one less edge of degrees 1 through n than the value in (2.70), which is precisely the value in (2.69).

<u>Remark 6</u>: $n-2 \geqslant i_{j_{r-1}} + j_{r-1}$ for all $r \geqslant 2$.

Proof of Remark 6: By substituting "n" for "i_0" in Remark 3, we have the proof of Remark 6.

Corollary to Remark 6:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{j_{r-1}+i_{j_{r-1}}} i_k \geqslant \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k .$$

Substituting "n" for "i_0", "Δ-generation" for "generation", "Lemma 4" for "Lemma 3", "Remark 4" for "Remark 1", and "Remark 6" for "Remark 3", we use the same argument as on pp.46-48 to conclude that:

$$\begin{aligned} &\frac{n^2+5n}{2} + j_0(1-n) + (j_0-j_1)(i_{j_0}+j_0) + \cdot \cdot \cdot + \\ (2.71) \qquad &(j_t-j_{t+1})(i_{j_t}+j_t) + \cdot \cdot \cdot + (j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) \\ &+ \frac{-i^2_{j_{r-1}}-i_{j_{r-1}}+j^2_{r-1}-j_{r-1}}{2} \geqslant \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k . \\ &(t = 0,1,..,r-2) \end{aligned}$$

Case (B^1_3): Suppose K_2 has a primitive irreducible F.

None of the edges in (2.35) can be edges of F since they are edges of faces all of which have another edge of degree greater than 1 (since $n-(j_k-1) > 1$ for all k, $0 \leq k \leq r-1$), and hence are not primitive irreducibles. If one of the edges in (2.35) were an edge of F, it would also be an edge of another face, not a primitive irreducible, and hence would be of degree at least 2, contradicting its being an edge of F. Hence, K_2 must have at least 3 additional edges of degree less than n+1, distinct from those in (2.35). So, by (2.69), the number of distinct edges in K_2 of degree less than n+1 is at least:

$$\begin{aligned}
&\frac{n^2+5n}{2} + j_0(1-n) + (j_0-j_1)(i_{j_0}+j_0) + (j_1-j_2)(i_{j_1}+j_1) + \cdot \\
&\cdot + (j_t-j_{t+1})(i_{j_t}+j_t) + \cdot\cdot\cdot + (j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) + \\
&\frac{-i^2_{j_{r-1}}-i_{j_{r-1}}+j^2_{r-1}-j_{r-1}}{2} + 3 > \frac{n^2+5n}{2} + j_0(1-n) + \\
&(j_0-j_1)(i_{j_0}+j_0) + (j_1-j_2)(i_{j_1}+j_1) + \cdot\cdot\cdot + \\
&(j_t-j_{t+1})(i_{j_t}+j_t) + \cdot\cdot\cdot + (j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) + \\
&\frac{-i^2_{j_{r-1}}-i_{j_{r-1}}+j^2_{r-1}-j_{r-1}}{2} \quad ,
\end{aligned}$$

which, by (2.71), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k \ ,$$

which, by (2.55), is a maximum possible number of edges in K_2 of degree less than n+1. This contradiction establishes the theorem in this case.

Case (B_3^2): Suppose K_2 has no primitive irreducible F. Then K_2 can have at most 2 distinct edges of degree 1, and hence, by (2.55), at most a total of:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k - 1$$ distinct edges of degree less than n+1. By (2.69), the number of distinct edges in K_2 of degree less than n+1 is at least:

$$\frac{n^2+5n}{2} + j_0(1-n) + (j_0-j_1)(i_{j_0}+j_0) + (j_1-j_2)(i_{j_1}+j_1) + \cdots + (j_t-j_{t+1})(i_{j_t}+j_t) + \cdots + (j_{r-2}-j_{r-1})(i_{j_{r-2}}+j_{r-2}) + \frac{-i_{j_{r-1}}^2 - i_{j_{r-1}} + j_{r-1}^2 - j_{r-1}}{2},$$

which, by (2.71), is greater than or equal to:

$$\frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k > \frac{(n+1)(n+2)}{2} - \sum_{k=1}^{n-2} i_k - 1,$$

which is a maximum number of distinct edges that K_2 can have of degree less than n+1. This contradiction establishes the theorem in this case and concludes the proof.

AUTOBIOGRAPHICAL STATEMENT

Shaun O'Connell was born on April 20, 1942, the only child of a loving, musical mother, and a devoted, studious father. Despite a strong early interest in the Latin and Greek classics, he decided to pursue mathematics after becoming fascinated by trigonometric identities ten years ago.

Shaun did his undergraduate work at Fordham College (B.A. 1965) and continued to study mathematics at Fordham University (M.A. 1966). He recently left the Jesuit Order after nine years to achieve a freer and fuller personal life and hopefully to make a more relevant contribution to society.

When not engaged in study, Shaun often plays the guitar. He composes his own material, has appeared on radio and television, and holds a professional recording contract.