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ON WEYL'S IDENTITY

by

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ABSTRACT

In 1916 Herman Weyl published the paper [5], "Uber die Bestimmung einer geschlossen konvexen Flache durch ihr Linienelement" in which the Weyl problem was introduced and partially solved. This is the problem of realizing an arbitrarily given two-dimensional metric of positive curvature as the boundary of a convex body. A survey of Weyl's problem and work done on it by Lewy, Nirenberg, Porgorelov and A.D. Alexanderov is given in the book [2], "Convex Surfaces" by Buseman. One of the crucial steps in solving the Weyl problem is to obtain an a priori bound for the mean curvature. Weyl deduces such a bound from his formula (33), which following Wintner we will refer to as Weyl's identity. Later writers derive essentially equivalent a priori bounds without using this identity. In 1956 Aurel Wintner published the paper [6], "On Weyl's identity in the differential geometry of surfaces," in which he observed that in addition to Weyl's purpose, this identity is of inherent interest. In addition to showing some interesting applications of Weyl's identity, Wintner pointed out some difficulties in Weyl's work due to the possible presence of umbilic points, and showed how to overcome them.

Actually Weyl does not include, in his paper, the details of the computations which lead to the Weyl identity. He describes these computations as a "langweilige Rechnung" and states "Es ist wahrscheinlich, dass ein geschickterer Rechner die Formel (33) auf viel leichterem Wege wird ermitteln konnen, als hier angeduetet wurde." As far as we know except for some related results derived by Chern

in [3], Weyl's request has thus far been unfulfilled.

In attempting to verify Weyl's identity and find a concise derivation of it, we discovered that the formula was not correct. A corrected identity (*) was found which leaves some applications unaffected but does show that some of Wintner's formulas are not correct. In particular, the a priori bound for the mean curvature is correct. The related results derived by Chern in [3] turn out to imply the identity (*) and not Weyl's identity as was claimed. In the course of this work, we shall correct other results in Wintner's paper, which are not consequences of his working with the wrong identity.

As Weyl predicted, a concise derivation of the identity (*) has been found, and is forthcoming in the doctoral thesis of Eleanor Zeitlin at The City University of New York.

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ABSTRACT

ACKNOWLEDGMENTS

1. Formulas of the theory of surfaces.

For the purposes of our work we need the fundamental formulas of the theory of surfaces. In 3-dimensional Euclidean space let S be a (sufficiently small) piece of surface which is of class C^k. By this is meant that S possesses a (local) one-to-one parametrization of the form $S : X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ where (u,v) ranges over a (sufficiently small, open) domain D in a (u,v) plane, the vector $X(u,v)$ is a function of class C^k on D, and the vectors, X_u and X_v are linearly independent at each point of D. In particular we will express the relationships of interest using a "lines of curvature coordinates" parametrization. We can do this because of the theorem:

> Let P be a point on a C^k surface S which is neither an umbilic nor a flat point. Then there is a neighborhood of P on the surface, and a C^{k-2} parametrization of that neighborhood such that the parametric curves are lines of curvature.

With respect to this parametrization, the coefficients of the first and second fundamental forms,

$$E(u,v)(du)^2 + 2F(u,v)\ dudv + G(u,v)(dv)^2 \tag{1}$$

$$L(u,v)(du)^2 + 2M(u,v)\ dudv + N(u,v)(dv)^2 \tag{2}$$

are functions of class C^{k-3} and C^{k-4} respectively. Furthermore we have:

(3) $F \equiv 0,\ M \equiv 0,\ L = k_1 E,\ N = k_2 G$

where k_1 and k_2 are the principal curvatures of the surface at the point $X(u,v)$. Hence the Gaussian curvature

(4) $K = LN/EG$

and the mean curvature,

(5) $H = \frac{1}{2}(L/E + N/G)$.

are functions of class C^{k-4} on D .

We now put theorem egregium and the Mainardi-Codazzi equations in these coordinates. First we observe that the general formula $\Gamma^r_{ik} = \frac{1}{2}\Sigma_{r,\ell} g^{r\ell}(\partial g_{\ell i}/\partial u^k - \partial g_{ik}/\partial g u^\ell + \partial g_{\ell k}/\partial u^i)$ for the Christoffel symbols of the second type where $u^1 = u,\ u^2 = v,\ i,j,k = 1,2$ (and the g_{ij} are the coefficients of the first fundamental form and the g^{ij} are the entries of the inverse matrix to that whose ij^{th} entry is g_{ij}) yields in our specific case, where,

(6) $g_{11} = E,\ g_{12} = g_{21} = 0,\ g_{22} = G$,

(7) $\Gamma^2_{11} = -E_v/2G,\ \Gamma^2_{12} = G_u/2G,\ \Gamma^1_{11} = E_u/2E,\ \Gamma^1_{12} = E_v/2E,\ \Gamma^2_{22} = G_v/2G$.

Theorem egregium, which in general is $K = \Sigma_j(g_{2j}/(g_{11}g_{22} - g^2_{12}))R^j_{112}$ where $R^j_{112} = \partial\Gamma^j_{11}/\partial v - \partial\Gamma^j_{12}/\partial u + \Sigma_r(\Gamma^r_{11}\Gamma^j_{r2} - \Gamma^r_{12}\Gamma^j_{r1})$ for "lines of curvature coordinates" (or any orthogonal coordinates) becomes on substituting (6) and (7):

(8) $K = -E_{vv}/2EG - G_{uu}/2EG + G^2_u/4G^2E + E^2_r/4E^2G + E_uG_u/4E^2G + E_vG_v/4G^2E$.

With the coefficients of the second fundamental form denoted by L_{ij} (where for our specific case of "lines of curvature coordinates," $L_{11} = L,\ L_{12} = L_{21} = 0,\ L_{22} = N$) . The Mainardi-Codazzi equations are $0 = \partial L_{ik}/\partial u^j - \partial L_{ij}/\partial u^k + \Sigma_r(\Gamma^r_{ik}L_{rj} - \Gamma^r_{ij}L_{rk})$. For "lines of curvature coordinates" upon substituting for the Christoffel symbols from (7), the Mainardi-Codazzi equations become:

$$(9) \qquad L_v = (E_v/2)(L/E + N/G) = E_vH$$

$$N_u = (G_u/2)(L/E + N/G) = G_uH \ .$$

2. The Beltrami parameters

Referring to our coordinates (u,v) of the surface S as (u^1,u^2), let $\theta(u^1,u^2)$ be any scalar function of class C^2. We denote the gradient of $\theta(u^1,u^2)$ by grad $\theta(u^1,u^2)$, which may be characterized as that vector in the tangent plane to S at the point $P = X(u^1,u^2)$, whose inner product with each vector α in this tangent plane is $d\theta \cdot \alpha$. From this characterization it follows that grad $\theta = (\Sigma_a \partial\theta/\partial u^a g^{a1})X_1 + (\Sigma_a \partial\theta/\partial u^a g^{a2})X_2$. Then the square of the length of grad θ equals $\Sigma_{\alpha,\beta}(\partial\theta/\partial u^\alpha g^{\alpha\beta}\partial\theta/\partial u^\beta)$, and is denoted by $\nabla'(\theta,\theta)$ or more briefly by $\nabla'\theta$. More generally if Φ is another scalar function on S, of class C^2, then grad $\theta \cdot$grad Φ, which equals $\Sigma_{\alpha\beta} g^{\alpha\beta}\partial\theta/\partial u^\alpha \, \partial\Phi/\partial u^\beta$ is denoted by $\nabla'(\theta,\Phi)$. Clearly this expression is an invariant with respect to allowable transformations of coordinates. ∇' is called the first differential parameter of Beltrami with respect to the first fundamental form. Formal substitution in the expression for $\nabla'(\theta,\Phi)$, of the coefficients of the second fundamental form for those of the first fundamental form yields the expression $\Sigma_{\alpha,\beta}L^{\alpha\beta}\partial\Phi/\partial u^\alpha \, \partial\theta/\partial u^\beta$, which we denote by $\nabla''(\theta,\Phi)$. ∇'' is called the first Beltrami parameter with respect to the second fundamental form. $L^{\alpha\beta}$ is the α,β entry in the inverse matrix of the one whose entries are the coefficients of the second fundamental form. That we may orient our surface so that the second fundamental form is positive definite and therefore may be used as a metric, follows from our assumption that $K > 0$.

If we take the "divergence" of grad Φ , the contraction of the covariant derivative of the contravariant vector field grad Φ , we arrive at the expression $(1/\sqrt{g})\ \partial/\partial u^{\alpha}(\sqrt{g}\ g^{\alpha\beta}\ \partial\Phi/\partial u^{\beta})$ where g denotes the determinant of the first fundamental form. We call this expression $\Delta'\Phi$ and refer to Δ' as the second differential parameter of Beltrami with respect to the first fundamental form. As before we arrive at Δ'' , the second differential parameter of Beltrami with respect to the second fundamental form. Specifically, $\Delta''\Phi = (\text{Det II})^{-\frac{1}{2}}\ \partial/\partial u^{\alpha}\ ((\text{Det II})^{\frac{1}{2}}\ L^{\alpha\beta}\ \partial\Phi/\partial u^{\beta})$ where Det II denotes the determinant of the second fundamental form.

In our "lines of curvature coordinates" some expressions we will be concerned with are:

(10) $\Delta''H = H_{uu}/L + H_{vv}/N + H_uN_u/2NL + H_vL_v/2NL - H_uL_u/2L^2 - H_vN_v/2N^2$

(11) $\Delta'K = K_{uu}/E + K_{vv}/G + K_uG_u/2EG - K_uE_u/2E^2 + K_vE_v/2EG - K_vG_v/2G^2$

(12) $\nabla'(K,J) = K_uJ_u/E + K_vJ_v/G$

(13) $\nabla''(H,J) = H_uJ_u/L + H_vJ_v/N$

(14) $\nabla''(K,H) = K_uH_u/L + K_vH_v/N$.

3. A counterexample to Weyl's identity

Since $K \leq H^2$, $J = (H^2 - K)^{\frac{1}{2}}$ defines a real valued function $J(u,v) = ((k_1/2 + k_2/2)^2 - k_1k_2)^{\frac{1}{2}} = | k_1/2 - k_2/2|$, so if we assume $k_1 > k_2$ we have by (3):

$$J = \tfrac{1}{2}(L/E - N/G) \ . \tag{15}$$

Thus at a non-umbilic point (these are the only points thus far considered as we are using "lines of curvature coordinates") J is C^{k-4}.

Weyl's identity referred to in Section 1 is the following connection between K,H and their differential parameters:

$$\Delta'' H - \tfrac{1}{2}\Delta' K - 2KJ^2 = (2/J)\nabla''(H,J) - (1/J)\nabla'(K,J)$$

The following brief argument shows that this is not a generally valid identity. Consider the one-parameter family of surfaces $X_\lambda(u,v) = \lambda X(u,v)$ for $\lambda \in R^1$. Then the first and second fundamental forms of the surface corresponding to λ are given by $(g_{ij})_\lambda = \lambda^2 g_{ij}$ and $(L_{ij})_\lambda = \lambda L_{ij}$ which implies by (4) and (5) that $H_\lambda = (1/\lambda)H$ and $K_\lambda = (1/\lambda)^2 K$. Therefore using (10)-(14) we see that:

$$\begin{aligned}
\Delta''_\lambda H_\lambda &= (1/\lambda)^2 \Delta'' H \\
\Delta'_\lambda K_\lambda &= (1/\lambda)^4 \Delta' K \\
\nabla'_\lambda(K_\lambda, J_\lambda) &= (1/\lambda)^5 \nabla'(K.J) \\
\nabla''_\lambda(H_\lambda, J_\lambda) &= (1/\lambda)^3 \nabla''(H,J) \\
\nabla''_\lambda(K_\lambda, H_\lambda) &= (1/\lambda)^4 \nabla''(K,H) \ .
\end{aligned}$$

On substituting these expressions into Weyl's identity applied to the surface $X_\lambda(u,v)$, we see that the terms are not homogeneous in λ, and arranging terms with respect to powers of λ, we get the equation,

$$(1/\lambda)^2(\Delta''H - (2/J)\nabla''(H,J)) = (1/\lambda)^4(\tfrac{1}{2}\Delta'K + 2KJ^2 - (1/J)\nabla'(K,J)) .$$

Thus if Weyl's identity were valid, the above equation would hold for $\lambda \neq 0$. Accordingly the coefficients of $1/\lambda^2$ and $1/\lambda^4$ must be zero. However, there exist surfaces on which the expression, $\Delta''H - (2/J)\nabla''(H,J)$ is not equal to zero. For example, one can find surfaces containing points at which the first partial derivatives of H vanish but $\Delta''H \neq 0$. At such points $\Delta''H - (2/J)\nabla''(H,J) \neq 0$, disproving Weyl's identity. In particular the surface defined by $X(x,y) = (x,y,x^2 + 2y^2)$ has these properties at the origin. In addition, the origin is neither an umbilic nor a point at which the Gaussian curvature vanishes, making this a counterexample to Weyl's identity.

4. Correction of Weyl's identity

In this section we assume that the point P of S is not an umbilic and that P is elliptic, that is $K > 0$ at P . We shall investigate the cases of umbilics and points of negative Gaussian curvature in Section 6. For the present S will be a C^k surface with $k \geq 6$. In this case we observe from (10)-(14) that the Beltrami parameters $\Delta''H, \Delta'K, \nabla'(K,J), \nabla''(H,J)$ and $\nabla''(K,H)$ exist and are continuous.

We now show that at the point P , the following connection between K,H and their differential parameters holds:

$$(*)\quad \Delta''H - \tfrac{1}{2}K^{-1}\Delta'K - 2J^2 = (2/J)\nabla''(H,J) - (1/KJ)\nabla'(K,J) - \tfrac{1}{2}K^{-1}\nabla''(K,H)$$

In order to look at the second order terms of $\Delta''H$, we differentiate the expression, $2HEG = LG + NE$, (which comes from (5)) twice, giving:

$$(16)\qquad 2(HEG)_{vv} = L_{vv}G + 2L_vG_v + LG_{vv} + N_{vv}E + 2N_vE_v + NE_{vv}$$

$$(17)\qquad 2(HEG)_{uu} = L_{uu}G + 2L_uG_u + LG_{uu} + N_{uu}E + 2N_uE_u + NE_{uu}$$

Following Weyl's advice we now seek to replace the second order derivatives of L and N by terms involving K,H,E,G ; their second order derivatives and terms involving L,N and their first order derivatives. To this end we differentiate the Mainardi-Codazzi equations (9), and differentiate this expression $KEG = LN$ from (4) twice, to obtain

$$(18)\qquad L_{vv} = E_{vv}H + E_vH_v$$

(19) $N_{uu} = G_{uu}H + G_u H_u$

(20) $(KEG)_{uu} = L_{uu}N + 2L_u N_u + LN_{uu}$

(21) $(KEG)_{vv} = L_{vv}N + 2L_v N_v + LN_{vv}$

In (16) we substitute the right side of (18) for L_{vv}, and for N_{vv} we use, $N_{vv} = (1/L)((KEG)_{vv} - 2L_v N_v - N(E_{vv}H + E_v H_v))$ which comes from (21) and (18). Upon expanding the left hand side of the resulting expression, solving for H_{vv} and dividing by N, we obtain:

(16′) $$H_{vv}/N = E_{vv}H/2NE + E_v H_v/2NE + LG_{vv}/2NEG + (KEG)_{vv}/2NGL$$
$$-E_{vv}H/2GL - E_v H_v/2GL - L_v N_v/NGL + N_v E_v/NEG$$
$$+L_v G_v/NEG + E_{vv}/2EG - 2H_v(EG)_v/NEG - H(EG)_{vv}/NEG$$

Similarly from (17), (19), and (20) we obtain:

(17′) $$H_{uu}/L = -G_{uu}H/2NE - G_u H_u/2NE + NE_{uu}/2LEG + (KEG)_{uu}/2NEL$$
$$+G_{uu}H/2GL + G_u H_u/2GL - L_u N_u/NEL + L_u G_u/LEG$$
$$+N_u E_u/LEG + G_{uu}/2EG - 2H_u(EG)_u/LEG - H(EG)_{uu}/LEG$$

If we substitute (16′) and (17′) into the expression (10), upon expanding and rearranging terms we obtain:

(22) $\Delta''H = (a) + (b) + (c) + (d)$, where

$(a) = K_{uu}/2KE + K_{vv}/2KG$

$(b) = G_{uu}/EG + E_{vv}/EG - G_{uu}H/2LG - E_{vv}H/2NE - E_{vv}H/2GL - G_{uu}H/2EN$

$(c) = E_{uu}/2E^2 + G_{vv}/2G^2 + LG_{vv}/2NEG - HG_{vv}/NG + NE_{uu}/2LEG - HE_{uu}/LE$

$$
\begin{aligned}
(d) = {} & K_u(EG)_u/NEL + K_v(EG)_v/NGL - G_uH_u/2EN + G_uH_u/2LG \\
& -2E_uH_u/LE - 2G_uH_u/LG + H_uN_u/2LN - H_uL_u/2L^2 \\
& -3E_vH_v/4GL - E_vH_v/2GL - 2G_vH_v/NG + H_vL_v/2LN - H_vN_v/2N^2 \\
& -L_uN_u/NEL + L_uG_u/LEG - 2E_uG_uH/LEG + L_vG_v/NEG \\
& -L_vN_v/NGL + N_vE_v/NEG - 2E_vG_vH/NEG + E_uG_u/E^2G + E_vG_v/EG^2 \ .
\end{aligned}
$$

Concerning the terms of (22): For line (a) we make use of

$$(a') \ K_{uu}/2KE + K_{vv}/2KG = \tfrac{1}{2}K^{-1}\Delta' K - \tfrac{1}{2}K^{-1}K_uG_u/2EG - \tfrac{1}{2}K^{-1}K_vG_v/2G^2 + \tfrac{1}{2}K^{-1}K_uE_u/2E^2$$

which follows from (11). Thus the second order derivatives of K are replaced by a Beltrami parameter of the second type and terms involving the products of pairs of first order derivatives which will ultimately fit into Beltrami parameters of the first type.

Adding the coefficients of the terms in (b) with respect to G_{uu} and E_{vv}, upon making use of the definitions of H and J, (5) and (15) we observe that:

$$(b) = 2J^2K^{-1}(-G_{uu}/2EG - E_{vv}/2EG) \ .$$

Since these terms are the second order terms in the right side of theorem egregium (8) multiplied by $2J^2/K$, we employ theorem egregium to conclude:

$$(b) = 2J^2 - (2J^2/K)(G_u^2/4G^2E - E_v^2/4E^2G + E_uG_u/4E^2G - E_vG_v/4G^2E) \ .$$

Adding the coefficients of the terms in line (c) with respect to E_{uu} and G_{vv}, and again using the definition of H, (5); yields that each coefficient is zero. So we have:

$$(c) = 0 \ .$$

Using these facts (22) becomes:

$$\Delta'' H = \tfrac{1}{2}K^{-1}\Delta' K + 2J^2 + R$$

where R denotes a sum of terms involving products of first order derivatives.

Specifically, we have:

$$(23) \qquad \Delta'' H - \tfrac{1}{2}K^{-1}\Delta' K - 2J^2 = (e) + (f) + (g) + (h) + (i) + (j) + (k),$$

where

$$(e) = (-2J^2/K)G_u^2/4G^2E - (2J^2/K)E_v^2/4E^2G - (2J^2/K)E_uG_u/4E^2G - (2J^2/K)E_vG_v/4G^2E$$

$$(f) = H_uG_u(-1/2EN + 1/2LG - 2/LG) + H_uE_u(-2/LE) + H_uN_u(1/2LN)$$

$$(g) = H_uL_u(-1/2L^2) + E_vH_v(1/2NE - 1/2LG - 2/NE) + H_vL_v(1/2LN)$$

$$(h) = G_vH_v(-2/NG) + H_vN_v(-1/2N^2) + K_vE_v(-1/4KEG)$$

$$(i) = K_vG_v(1/4KG^2) + K_uG_u(3/4NL) + K_uE_u(5G/4NEL)$$

$$(k) = L_vG_v/NEG - L_vN_v/NGL + N_vE_v/NEG - 2E_vG_vH/NEG + E_vG_v/EG^2 + K_v(EG)_v/NGL$$

In the right side of (23), we substitute for H_u and H_v by differentiating the definition (5) of H, and then we use the Mainardi-Codazzi equations to replace each L_v by E_vH and each N_u by G_uH. Adding the resulting terms with respect to like products of derivatives and again utilizing the definitions of H and J yields:

$$(24) \qquad \Delta'' H - \tfrac{1}{2}K^{-1}\Delta' K - 2J^2 = (\ell) + (m),$$

where

$$(\ell) = L_uG_u(-J/LEN - JH/2L^2N) + E_uG_u(J/E^2N + JH/2LEN) + G_u^2(3J^2/4LNG) +$$
$$+L_uE_u(1/2E^2L) + L_u^2(-1/4L^2E) + E_u^2(-1/4E^3),$$

and (m) is a sum of terms involving derivatives with respect to v, which may be obtained from (ℓ) by applying the transformation which replaces u,G,L,J,H by $v,N,E,-J,H$ respectively. In order to exploit the sort of symmetry in computing Beltrami parameters of the first kind, define:

$$\nabla'_u(\psi,\varphi) = \psi_u\varphi_u/E \text{ , and}$$

$$\nabla''_u(\psi,\varphi) = \psi_u\varphi_u/L \text{ .}$$

Using (13) for the Beltrami parameter, $\nabla''(H,J)$ we get

$$\nabla''_u(H,J) = H_uJ_u/L \text{ .}$$

Upon differentiating (5) and (15) for H_u and J_u, substituting therein for L_v and N_u via the Mainardi-Codazzi equations we see:

$$(25) \qquad 2\nabla''_u(H,J)/J = L_u^2(1/2E^2LJ) + L_uE_u(-1/JE^3) + E_u^2(L/2JE^4) + G_u^2(-J/2LG^2) \text{ .}$$

Similarly $\nabla'_u(K,J) = K_uJ_u/E$ yields:

$$(26) \qquad -\nabla'_u(K,J)/JK = G_uL_u(-J/LEN) + G_uE_u(J/E^2N) + G_u^2(J/2NEG) + \\ + L_u^2(-1/2JLE^2) + E_u^2(-L/2JE^4) + E_uL_u(1/JE^3)$$

and $\nabla''_u(K,H) = K_uH_u/L$ yields:

$$(27) \qquad -\nabla''_u(K,H)/2K = G_u^2(-J^2/4LNG) + G_uE_u(J/4NE^2 + J/4LEG) + L_u^2(-1/4L^2E) \\ + G_uL_u(-J/4LEN - J/4L^2G) + E_u^2(-\tfrac{1}{4}E^{-3}) + L_uE_u(1/2E^2) \text{ .}$$

Adding the respective sides of (25), (26), and (27) we get, using the definitions of J and H:

$$2\nabla''_u(H,J)/J - \nabla'_u(K,J)/KJ - \nabla''_u(K,H)/2K = (\ell) \text{ .}$$

The identity $(*)$ now follows from (24) since

$$\nabla(f,g) = \nabla_u(f,g) + \nabla_v(f,g) .$$

The proof of the identity $(*)$ given in this section is for surfaces of class C^k with $k \geq 6$. In applications of results of this type it is desirable to reduce the smoothness requirements as much as possible. Since the various terms in the identity are all meaningful for surfaces of class C^k with $k \geq 4$, we now investigate $(*)$ for C^4 surfaces. Previously, we increased k to 6 in order to use "lines of curvature coordinates." Actually the identity holds for $k \geq 4$. One way to see this is to note that given any C^4 surface, corresponding to each point there exists a C^∞ surface having the same values for all of the quantities appearing in $(*)$ at the specified point. To obtain a C^∞ surface we use the map $X(u,v)$ each of whose components is the first 5 terms of the two dimensional Taylor series expansion of the original C^4 surface with respect to the point of interest. Furthermore we can obtain a C^3 identity by integrating $(*)$ as follows:

First arrange the terms of $(*)$ so that all the Beltrami operators with respect to the first fundamental form are on the right side of the identity, and those with respect to the second fundamental form appear on the left side. Since $\nabla(x,y)/y = \nabla(x, \log y)$ we have:

$$\Delta''H + \nabla''(H,-2 \log J) + \nabla''(H,(\tfrac{1}{2})\log K) = \tfrac{1}{2}K^{-1}(\Delta'K + \nabla'(K,\log(1/J^2)) + 2J^2 ,$$

or

$$\Delta''H + \nabla''(H, \log(\sqrt{K}/J^2)) = \tfrac{1}{2}K^{-1}(\Delta'K + \nabla'(K, \log(1/J^2)) + 2J^2 . \tag{28}$$

Because $x\,\mathrm{grad}(\log x) = \mathrm{grad}\, x$, $(\sqrt{K}/J^2$ and $1/J^2$ are integrating factors for the left and right hand side of (28) respectively. Then multiplying both sides of (28) by $\sqrt{K}/J^2$ gives:

$$(29) \qquad \mathrm{div}_{II}((\sqrt{K}/J^2)(\mathrm{grad}_{II}H)) = (1/2\sqrt{k})\mathrm{div}_I((1/J^2)\mathrm{grad}_I K) + 2\sqrt{K}\ .$$

Now multiply (29) by $\sqrt{K}$ and integrating with respect to surface area gives:

$$\int \mathrm{div}_{II}((\sqrt{K}/J^2)\mathrm{grad}_{II}H)\ d\sigma_2 = (\tfrac{1}{2})\int \mathrm{div}_I((1/J^2)\mathrm{grad}_I K)\ d\sigma_1 + \int 2K d\sigma_1$$

where $d\sigma_2 = \sqrt{K} d\sigma_1$. Applying Green's theorem we get:

$$(30) \qquad \oint (\sqrt{K}/J^2)\mathrm{grad}_{II}H \cdot_I m_2 ds_2 = \tfrac{1}{2}\int (1/J^2)\mathrm{grad}_I K \cdot_{II} m_1 ds_1 + 2\int K d\sigma_1$$

where m_1 and m_2 are the outward unit normals with respect to I and II , respectively. To see that this identity is valid on C^3 surfaces, as well as on C^4 surfaces, which we have just proven, we note that for any C^3 surface $X(u,v)$ on a compact neighborhood of the origin, corresponding to each ϵ there exists a C^∞ function $X^*_\epsilon(u,v)$ such that $\| X^*_\epsilon - X \| < \epsilon$ and the analagous relationships between corresponding derivatives up to the third order hold. Then X^*_ϵ and its derivatives up to third order uniformly approximate X and its derivatives up to third order, respectively. Since (30) holds for each X^*_ϵ taking limits as ϵ approaches zero shows us that the identity (30) holds on $X(u,v)$.

5. The situation for umbilics and points at which $K < 0$

Thus far we have assumed for (*) that $K > 0$ on S. Wintner points out from the nature of the computations which Weyl described that "Weyl's Identity" holds for $K < 0$ as well. This remark is applicable to the computations in the derivation of (*). Although $(\det II)^{\frac{1}{2}} = (LN)^{\frac{1}{2}}$ is imaginary in this case, $\Delta'' H = (LN)^{-\frac{1}{2}}(\partial/\partial u^\alpha((LN)^{\frac{1}{2}} L^{\alpha\beta} \partial H/\partial u^\beta))$ is real and given by the right hand side of (10). Thus at the very beginning of our computations we have precisely the algebraic situation which exists if $\det II > 0$, and the same computation can be made.

We now try to find a relationship on surfaces with either $K > 0$ or $K < 0$ which reduces to (*) at non-umbilic points and is meaningful at umbilics as well. If we multiply both sides of (*) by J and try to take limits, the C^1 character of J becomes involved, and contrary to Wintner's claim, J is not in general C^1. (This will be shown by a counterexample in Section 10). In referring to "Weyl's identity" however, Wintner points the way to avoiding this matter by rewriting Beltrami parameters involving J by ones involving J^2, which of course is C^1 as $J^2 = H^2 - K$. Below we apply a similar argument to (*).

Replacing $\partial J/\partial u^i$ by $\frac{1}{2}J^{-1}\partial(J^2)/\partial u^i$ in the expressions for $\nabla'(J,K)$ and $\nabla''(J,H)$, we get

$$(31) \qquad \nabla'(J,K) = \tfrac{1}{2}J^{-1}\nabla'(J^2,K)$$

$$\nabla''(J,H) = \tfrac{1}{2}J^{-1}\nabla''(J^2,H) .$$

Substituting these expressions into the right hand side of (*) yields:

$$\Delta'' H - \tfrac{1}{2}K^{-1}\Delta' K - 2J^2 = \nabla''(J^2,H)/J^2 - \nabla'(J^2,K)/2KJ^2 - \nabla''(K,H)/2K \ .$$

On replacing J^2 by $H^2 - K$ and using the identities:

$$\nabla''(H,H^2 - K) = 2H\nabla'' H - \nabla''(H,K)$$

$$\nabla'(K,H^2 - K) = 2H\nabla'(K,H) - \nabla'(K)$$

we obtain:

$$(**) \quad (H^2 - K)(\Delta'' H - \tfrac{1}{2}K^{-1}\Delta' K) = 2(H^2 - K)^2 + 2H\nabla'' H - \nabla''(H,K) + \\ -H\nabla'(K,H)/K + \nabla'(K)/2K + \\ -(H^2 - K)\nabla''(K,H)/2K \ .$$

We will now show that (**) , which is equivalent to (*) when $J \neq 0$, is valid at umbilic points also. Consider the two possible cases at an umbilic point, P :

(a) P has a neighborhood in which each point is an umbilic point;

(b) there exists a sequence of non-umbilic points converging to P .

In case (a) it is well-known that the neighborhood is part of a sphere. $H^2 - K = 0$, and H and K are constant so the Beltrami operators involving them vanish. Accordingly in this case (**) is trivially meaningful. In case (b) since J^2, H and K are all C^1, we take the limit of both sides of (**) along a sequence of non-umbilics converging to P . The equality is then carried over to the limit. Thus (**) is valid at umbilics.

Although (**) has an integrating factor, this factor contains a power of J in the denominator, thus integrating (**) will not generalize (30).

6. Some applications of (*) and (**)

At the end of his paper, Weyl concludes from the invalid "Weyl identity," the inequality:

$$(32) \qquad H^2(u,v) \le \max_S \,(K - \Delta'(K)/4K)$$

for surfaces S, which are compact and on which the Gaussian curvature is positive. This inequality is valid and will be derived from (*) below.

As $K > 0$, the principal curvatures, k_1 and k_2 are of the same sign. We now choose our unit normals so that both k_1 and k_2 are positive. Then $H > 0$ on S. Let P_0 be the point on S at which H attains its maximum.

Case (a) P_0 is not an umbilic point. Let the coordinates of P_0 in lines of curvature coordinates be (u_0,v_0). As $H(u,v)$ is a maximum at (u_0,v_0), $H_{u^i} = 0$ and $H_{u^i u^i} \le 0$ at P_0. Then from (13) and (14), $\nabla''(H,J)$ and $\nabla''(K,H)$ vanish at P_0. From (10), $\Delta''H = (H_{uu}/L + H_{vv}/N) \le 0$ as L and N are positive due to the orientation mentioned above. Then at (u_0,v_0), (**) becomes:

$$\Delta''H - \Delta'(K)/2K = 2(H^2 - K) + \nabla'(K)/2K(H^2 - K)$$

and since $\nabla'K \ge 0$, $H^2 - K > 0$, $\Delta''H \le 0$ at P_0, this implies (30).

Case (b) P_0 is an umbilic. If Weyl's inequality does not hold we have:

$$H^2(u_0,v_0) > \max_S(K - \Delta'K/4K) \ge K(u_0,v_0) - (\Delta'K)(u_0,v_0)/4K(u_0,v_0)\ .$$

Since $X(u_0,v_0) = P_0$ is an umbilic, $H^2(u_0,v_0) = K(u_0,v_0)$ and we have $0 < \Delta'(K)/4K$ at P_0. But this contradicts the generally true fact that if $H(u,v)$ assumes its maximal value at an umbilic, K also assumes its maximal value there, for:

$$K(u,v) \leq H^2(u,v) \leq H^2(u_0,v_0) = K(u_0,v_0) .$$

Another application of (*) is Theorem 6 of Chern's paper [3]: "At a non-umbilical point of a convex surface it never occurs that the mean curvature has a relative maximum and the Gaussian curvature has at the same time a relative minimum."

To see this note that under the hypothesis first order derivatives of H and K vanish, causing (*) to become:

$$\Delta''H - \frac{1}{2}K^{-1}\Delta'K - 2J^2 = 0 .$$

Furthermore, in this case we must have $\Delta''H \leq 0$, $\Delta'K \geq 0$ and $K \geq 0$, so the above equation implies $J^2 \leq 0$. Thus $J = 0$, which contradicts our hypothesis.

7. A similar identity and Miranda's inequality

In Section 6 of his paper Wintner lets the third fundamental form play the role of the first fundamental form in Weyl's suggested proof of the "Weyl identity" to deduce a second invalid identity. If, however, we let the third fundamental form play the role of the first in the proof of the identity (*) , we do get another identity. Following Wintner we shall then use this identity to derive Miranda's inequality, which plays a role in the solution of Minkowski's problem, analagous to that of the Weyl inequality in the Weyl problem.

We still use "lines of curvature coordinates" on S , so $g_{11} = E$, $g_{22} = G$, $g_{12} = g_{21} = 0$; $\ell_{11} = L$, $\ell_{22} = N$, $\ell_{12} = \ell_{21} = 0$ where $L = k_1 E$ and $N = k_2 G$. Then the third fundamental form (whose ij^{th} entry is $\gamma_{ij} = N_i \cdot N_j$) is seen to have entries, $\gamma_{11} = P$, $\gamma_{22} = R$, $\gamma_{12} = \gamma_{21} = 0$. Furthermore, $P = (k_1)^2 E$ and $R = (k_2)^2 G$. This follows immediately from the Rodrigues formulas, $N_1 = -k_1 X_1$ and $N_2 = -k_2 X_2$ when the parametric curves are the lines of curvature.

Notice that III relates to II similarly to the way in which I relates to II , that is:

$$L = (k_1)E = (1/k_1)P$$

$$N = (k_2)G = (1/k_2)G \, .$$

With the aim of adapting our proof of the identity * , we define quantities K^* and H^* in terms of III and II analagously to the way in which K and H may be defined from I and II .

(33) $K^* = \text{Det II}/\text{Det I} = 1/K = LN/PR$

(34) $H^* = (\frac{1}{2})\,\text{trace}\,(\text{II}\cdot\text{III}^{-1}) = (\frac{1}{2})(1/k_1 + 1/k_2) = H/K = \frac{1}{2}(L/P + N/G)$

Thus if in the definition of K and H we replace E by P and G by R, we arrive at the formulas defining K^* and H^*. Weingarten shows in [6, pp. 252-261] that the Mainardi-Codazzi equations remain unchanged (with the understanding that the Christoffel symbols which originally refer to I now refer to III) if I is replaced by III. This fact may be seen directly, using "lines of curvature coordinates" as follows:

Differentiating the relation $P = (k_1)^2E$ with respect to v gives (a) : $P_v = 2k_1(k_1)_vE + (k_1)^2E_v$. But if in the Mainardi-Codazzi equation (b) : $L_v = E_vH$, L is replaced by k_1E, we get the equivalent relation (b') : $E_v = -(E/J)(k_1)_v$. Substituting this expression for E_v into (a) gives

$$(c) : P_v = ((k_1)_vEH/J)((2k_1J - (k_1)^2)/H)$$

From (b) and (b') we see that $L_v = -(E/J)(k_1)_vH$ so (c) may be rewritten:

$$P_v = L_v((k_1)^2 - 2k_1J)/H = L_vK/H = L_v/H^* .$$

(35) $L_v = P_vH^*$

and similarly we obtain the analogue of $N_u = G_uH$.

(36) $N_u = R_uH^*$.

Notice that the equations (33)-(36) are gotten from equations (3)-(5) upon replacing:

(A) : E by P
G by R
K by K^*
H by H^*

In our proof of (*) in Section 5, the equations (3), (4), and (5) are formally manipulated to give the expression (22) for $\Delta'' H$. So upon making the above substitution (A) , we get a similar expression for $\Delta'' H^*$. Concerning line (a') which follows (22), $\Delta' K$ becomes $\Delta^* K^*$, where Δ^* is the second Beltrami parameter with respect to the third fundamental form. Note that under our transformation, J is replaced by $(\frac{1}{2})(L/P - N/R)$ which we call $-J^*$. For later use observe $J^* = J/K$. Then under the transformation (A) line (c) in the right hand side of (22) becomes:

$$(C^*) = (2(J^*)^2/K)(-R_{uu}/2PR - P_{vv}/2PR + R_u^2/4R^2P + P_v^2/4P^2R + R_uP_u/4P^2R + R_vP_v/4R$$

where the second bracketed quantity is the right hand side of theorem egregium applied to a surface whose first fundamental form is III . Since this surface is isometric to part of a unit sphere, the referred to bracketed quantity must be equal to one. Thus unlike the proof of (*) , line $(c^*) = 2J^*/K^*$. The rest of the expression for $\Delta'' H^*$ is analagous to (22), and, as no further relations are introduced into the proof of (*) , it is clear that

$$(37) \qquad \Delta'' H^* - (1/2K^*)\Delta^* K^* - 2J^{*2}/K^* = 2\nabla''(H^*,J^*)/J^* - \nabla^*(K^*,J^*)/K^*J^* - \nabla''(K^*,H^*)$$

Noting that $(J^*)^2 = (H^*)^2 - K^*$, the same reasoning that permitted us to extend (*) to (**) yields:

$$((H^*)^2 - K^*)(\Delta''H^* - \Delta^*K^*/2K^* - 2((H^*)^2 - K^*)/K^*) = \tag{38}$$

$$= \nabla''(H^*, H^{*2} - K^*) - \nabla^*(K^*, (H^*)^2 - K^*)/2K^* - ((H^*)^2 - K^*)\nabla''(K^*, H^*)/2K^*$$

which is valid for surfaces on which $K > 0$ or $K < 0$ and at umbilic points as well as non-umbilics.

Miranda's inequality plays a role in the solution of the Minkowski problem, analagous to that of Weyl's inequality in the solution of the Weyl problem. For more information on this matter, the reader is again referred to the book [2] by Buseman. As an application of (38) we can derive Miranda's inequality without excluding umbilics. This will very closely resemble the derivation of Weyl's inequality from (**) .

Let (u_0^*, v_0^*) be the point at which H^* takes on its maximal value. As before only those coordinate patches on which $H > 0$ are admitted. Then H^* is a maximum at (u_0^*, v_0^*) implies $\Delta''H^* \le 0$ and $\nabla''(H^*, -) = 0$ there. So at (u_0^*, v_0^*) (38) becomes:

$$((H^*)^2 - K^*)(-\Delta^*K^*/2K^* - 2((H^*)^2 - K^*)/K^*) = \nabla^*K^*/2K^* .$$

As $K^* = 1/K > 0$, the right hand side of the preceding equation is non-negative.

(a) (u_0, v_0) is not an umbilic. Then $(H^*)^2 - K^* = (1/K^2)(H^2 - K) > 0$, so $-\Delta^*K^*/2K^* - 2((H^*)^2 - K^*)/K^* \ge 0$ and thus

$$(H^*)^2 \le \max_S(-\Delta^*(K^*)/4 + K^*) \tag{39}$$

which gives Miranda's inequality, $H^2 \le \max_S(K - (K/2)^2\Delta^*(K^{-1}))$ on replacing H^* by H/K and K^* by $1/K$.

(b) If (u_0^*, v_0^*) is an umbilic, we assume (39) does not hold and arrive at a contradiction precisely as in the corresponding part of the proof of Weyl's inequality.

The method used at the end of Section 5 to get an integrated form of the identity (*) also yields an integrated form of (37). First integrating factors are found which permit us to rewrite (37) as $\sqrt{K^*}\,\mathrm{div}_{II}((\sqrt{K^*}/2J^{*2})\,\mathrm{grad}_{II}H^*) = 1 + \mathrm{div}_{III}((1/4J^{*2})\,\mathrm{grad}_{III}K^*)$ which upon integration yields:

$$\int \mathrm{div}_{II}((\sqrt{K^*}/2J^{*2})\,\mathrm{grad}_{II}H^*)\,d\sigma_2 = \int d\sigma_3 + \int \mathrm{div}_{III}((1/4J^{*2})\,\mathrm{grad}_{III}K^*)\,d\sigma_3$$

since $d\sigma_2 = \sqrt{K^*}\,d\sigma_3$ = area element with respect to II . Then on applying Green's theorem, we obtain

$$\oint (\sqrt{K^*}/2J^{*2})\,\mathrm{grad}_{II}H^* \underset{II}{\cdot} m_2\,ds_2 = \int d\sigma_3 + \oint (1/4J^{*2})\,\mathrm{grad}_{III}K^* \underset{III}{\cdot} m_3\,ds_3 .$$

8. Additional applications of (*)

We now use the identity, (*) to derive necessary conditions for the existence of surfaces with constant mean curvature, and for surfaces of constant Gaussian curvature.

First let $H(u,v) = c$ on S. Then (**) becomes

$$(40) \qquad \Delta'(K)/(c^2 - K) + \nabla'(K)/(c^2 - K)^2 = -4K \, .$$

Wintner remarks that this is equivalent to $\Delta'(\log(c^2 - K)) = 4K$ which is easily seen upon writing the left hand side as $\operatorname{div} \operatorname{grad} \log(c^2 - K)$. This formula was found by Raffy, Rici, and Sannsone to be sufficient as well as necessary for S to be of constant mean curvature.

Now we let $K = c$ on S. Then (**) reduces to:

$$(41) \qquad (\tfrac{1}{2})(H^2 - c)\Delta''H - H\nabla''H - (H^2 - c)^2 = 0 \, , \text{ or}$$

$$\operatorname{div}_{II}(\operatorname{grad}_{II}(H)/(H^2 - c)) = 2 \, .$$

This expression is not equivalent to Wintner's formula 25, which is not correct due to the fact that Wintner employed Weyl's identity, rather than its corrected form (*) .

9. Smoothness of J

In Section 2 of his paper, Wintner claims that $J(u,v)$ is of class C^{n-2} whenever $S : X = X(u,v)$ is C^n for $n > 1$. In Section 9, Wintner attempts to prove this statement, but the proof is not valid because the functions he refers to as "a" and "b" cannot be defined in general to be continuous. In this section we present a counterexample to this statement, showing in fact that there exist C^∞ surfaces with $K > 0$ on which J is not everywhere C^1. Before doing so, however, it should be mentioned that Wintner shows how to avoid using (J)'s C^1 character by devising the method we employed in Section 6 to get (**). The counterexample is the surface

$$X(x,y) = (1/(x^2 + (y+1)^2 + x^2y^4))(x, y+1, xy^2) .$$

This surface is the inversion in the unit sphere of the surface $X(x,y) = (x, y+1, xy^2)$. First we shall show that J is not C^1 at $X(0,0)$ on $X(x,y) = (x, y+1, xy^2)$. The Gaussian curvature at $X(0,0)$ of this surface is 0. However, inversion in the unit sphere will give a surface X' such that J' is not C^1 at $X'(0,0)$ and $K' > 0$ there.

The first fundamental form for $X(x,y) = (x, y+1, xy^2)$ has entries: $g_{11} = 1 + y^4$, $g_{22} = 1 + 4x^2y^2$, $g_{12} = g_{21} = 2xy^3$ and its determinant, $g = 1 + y^4 + 4x^2y^2$. The second fundamental form has entries:

$$\ell_{11} = 0,\ \ell_{12} = \ell_{21} = 2y/\sqrt{g},\ \ell_{22} = 2x/\sqrt{g} .$$

The mean curvature $H = \frac{1}{2}$ trace $I^{-1} \cdot II = (x - 3xy^4)/g^{3/2}$. The Gaussian curvature $K = \mathrm{Det}(II)/\mathrm{Det}(I) = (-4y^2 - 4y^6 - 16x^2y^4)/g^3$. So $J = (H^2 - K)^{\frac{1}{2}} = (1/g^{3/2})(x^2 + 4y^2 + (9x^2y^8 + 4y^6 + 10x^2y^4))^{\frac{1}{2}}$. As $1/g^{3/2}$ is C^1 , that J is not C^1 follows from that fact that $(x^2 + 4y^2 + (9x^2y^8 + 4y^6 + 10x^2y^4))^{\frac{1}{2}}$ is not C^1 at the origin. This may quickly be checked by noting that

$$(\partial/\partial x)(x^2 + 4y^2 + (9x^2y^8 + 4y^6 + 10x^2y^4))^{\frac{1}{2}} =$$

$$= x(2 + 18y^8 + 20y^4)/2(x^2 + 4y^2 + (9x^2y^8 + 4y^6 + 10x^2y^2))^{\frac{1}{2}}$$

has a limit of zero if the origin is approached along the y-axis and a limit of ± 1 if it is approached along the x-axis. In Volume 1 of his "Differential Geometry of Three Dimensions," [4] C. E. Weatherburn computes the relation of J' to J and K' to K , for an inversion. These results show that for an inversion in the unit sphere:

(a) $J' = -|X|^2 J$

(b) $K' = |X|^4 K + 2|X|^2 \rho J + 4\rho^2$ where $\rho = N(u,v) \cdot X(u,v)$.

From (a) it is clear that if J is not C^1 neither is J' , as the inverted surface inherits (J_x)'s discontinuity. From (b) we see that $K'(0,0) = 4\rho^2(0,0) = 4 > 0$.

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Autobiographical Statement

Martin Katzen was born in New York City on January 10, 1939. He was graduated from Evander Childs High School in 1956 and received a Bachelor of Science degree from The College of the City of New York in 1960. At that time he married the former Helene Greenberger, and accepted a teaching position in the mathematics department of Christopher Columbus High School in New York City. After one and one-half years he became a full-time student in mathematics at New York University. In the spring of 1963 he taught mathematics at Bronx Vocational High School while completing the requirements for a Master of Science degree in mathematics. In the fall of 1963 he became an Instructor of Mathematics at the Newark College of Engineering, where he has remained except for a one-year leave of absence from 1967 to 1968. In September, 1966, he resumed graduate studies at The City University of New York where the following year he completed a dissertation while he was a National Science Foundation Science Faculty Fellow.

In November, 1965, his daughter, Jacqueline, was born.