ON CRITICAL POINTS FOR GAUSSIAN VECTORS WITH INFINITELY DIVISIBLE SQUARES

by

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Abstract

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This paper is concerned with the necessary conditions for infinite divisibility of the squares of Gaussian vectors with non-zero means. A Gaussian vector G with zero mean is said to have a critical point α_0;

$$0 < \alpha_0 < \infty$$

if the vector $((G_1+\alpha)^2, (G_2+\alpha)^2, ..)$ is infinitely divisible for all $|\alpha| \leq \alpha_0$ and is not infinitely divisible for all $|\alpha| > \alpha_0$. We derive an upper bound for the critical point of a Gaussian n-dimensional vector via the asymptotic analysis of its Laplace transform.

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Hana Kogan

Contents

1 Introduction

Let $G = (\eta_1, \eta_2, ...\eta_n)$ be an n-dimensional Gaussian vector. We say that G has infinitely divisible squares (or G^2 is infinitely divisible) if for any $m \in N$

$$G^2 := (\eta_1^2, \eta_2^2, ...\eta_n^2) \stackrel{law}{=} \sum_{i=1}^{m} Z_i \tag{1.1}$$

where Z_i are independent identically distributed n-dimensional random vectors.

It is very easy to see that the square of any one-dimensional mean zero Gaussian vector is infinitely divisible: if $G = \eta(0, \sigma^2)$ then the the Laplace transform of η^2 is $(1 + 2\lambda\sigma^2)^{-\frac{1}{2}}$ and it is easily established that $(1 + 2\lambda\sigma^2)^{-\frac{1}{2n}}$ is completely monotone. So $\eta(0, \sigma^2)$ is infinitely divisible. It is still possible to show infinite divisibility of a square of a Gaussian variable with non-zero mean by analyzing its Laplace transform, however already in dimension 2 the question of infinite divisibility becomes far from trivial.

The following definitions are necessary to state the main results:

> Let N be an $n \times n$ matrix such that $N_{i,j} = 0$ for all $i \neq j$; $N_{i,i} = \pm 1$ for all i. N is called a signature matrix.
>
> A non-singular matrix A is called an M-matrix if $A^{-1}_{i,j} \geq 0$ for all $1 < i, j < n$ and $A_{i,j} \leq 0$ for all $i \neq j$. Note that $A_{i,i}$ must be positive.
>
> A symmetric matrix A is irreducible if it can not be written as a direct sum of square matrices.

In what follows all covariance matrices considered are irreducible. The general case results will follow by considering any Gaussian vector as a direct sum of vectors with irreducible covariances. The following Theorem due to Griffiths and Bapat, see also [3], Theorem 13.2.1 and [4]. Theorem 1.1 completely characterizes zero-mean Gaussian processes with infinitely divisible squares.

Theorem 1.1. *Let $G = (G_1, G_2, ...G_n)$ be a mean zero Gaussian random variable with a strictly positive-definite covariance matrix $\Gamma = \{\Gamma_{i,j}\} = \{E(G_i G_j)\}$. G^2 is infinitely divisible if and only if $N\Gamma^{-1}N$ is an M-matrix for some signature matrix N.*

Since $G \overset{law}{=} -G$ it is clear that G^2 is infinitely divisible if and only if $(GN)^2$ is infinitely divisible for some signature matrix N. Hence one can restrict one's attention to the Gaussian vectors with positive covariance matrix . This observation suggests considering a relationship between the class of Gaussian processes with infinitely divisible squares and the class of associated Gaussian processes [Refer to sec. 3 below], since both are characterized by the positive covariance matrices. Once the mean zero Gaussian vectors with infinitely divisible squares are described the next question of interest is to consider the vector of the form

$$(G + \mathbf{1}c) := (G_1 + c, G_2 + c, \ldots) \qquad \text{for some } c \in R. \qquad (1.2)$$

where $\mathbf{1} = (1, 1, ...1)$ is an n-dimensional vector with all entries 1. The conditions for the infinite divisibility of the squares for the vectors of this form were first addressed by Eisenbaum and Kaspi. Complete characterization is provided by the following theorem, taken from [4], Theorem 1.2, see also [3], Theorem 13.3.1:

Theorem 1.2. *Let G be a mean zero Gaussian vector with strictly positive definite covariance matrix Γ. The following are equivalent:*

1. $(G+c)$ *has infinitely divisible squares for all* $c \in \mathbf{R}$*.*

2. There exists some $b \in \mathbf{R}, b \neq 0$ *such that for* $\eta = N(0,b)$*, independent of* G*,* $(G_1+\eta, G_2+\eta, ...G_n+\eta, \eta)$ *has infinitely divisible squares. In this case* $(G_1+\eta, G_2+\eta, ...G_n+\eta, \eta)$ *has infinitely divisible squares for all* $b \in \mathbf{R}$*.*

3. Γ^{-1} *is an* M*-matrix with non-negative row sums.*

Finally, let $\mathbf{c} = (c_1, c_2...)$ be any real n-dimensional vector. The question of infinite divisibility of $(G+\mathbf{c})^2 := (G_1+c_1, G_2+c_2, ...)$ is answered by Marcus and Rosen, [4], Theorem 1.3:

Theorem 1.3. *Let* G *be a mean zero Gaussian vector with strictly positive definite covariance matrix* Γ*. Let* $\mathbf{c} = (c_1, c_2...)$ *and* C *be an* $n \times n$ *matrix with* $C_{i,i} = \mathbf{c}_i, C_{i,j} = 0$ *for* $i \neq j$*. The following are equivalent:*

1. $(G+c\alpha)^2$ *is infinitely divisible for all* $\alpha \in \mathbf{R}$*.*

2. There exists some $b \in \mathbf{R}, b \neq 0$ *such that for* $\eta = N(0,b)$*, independent of* G*,* $(G_1+c_1\eta, G_2+c_2\eta, ...G_n+c_n\eta, \eta)$ *has infinitely divisible squares. In this case* $(G_1+c_1\eta, G_2+c_2\eta, ...G_n+c_n\eta, \eta)$ *has infinitely divisible squares for all* $b \in \mathbf{R}$*.*

3. $C\Gamma^{-1}C$ *is an* M*-matrix with non-negative row sums.*

Heuristically Theorem 1.3 is a logical consequence of Theorem 1.2 with Γ replaced by $C^{-1}\Gamma C^{-1}$. One corollary of Theorem 1.3 ([4], Corollary 1.1, # 4 and 5) is of interest here:

Corollary 1.1. *Let* G*,* Γ*,* C *be as above. When 1, 2 or 3 of Theorem 1.3 hold,* *1.* $C_{i,i}$ *have the same sign for all* i.

2. $C_{i,i} \neq 0$, *for all* i.

3. $\Gamma_{i,j} \neq 0, \quad \forall(i,j)$.

Here 1 follows from 3 in Theorem 1.3: since $C\Gamma^{-1}C$ is an M-matrix, it is invertible, and so $\det(C\Gamma^{-1}C) \neq 0$. Hence $\det C \neq 0$, which implies 1. Then 2 follows by representing $G = ([G_1 - G_k\frac{\Gamma_{1,k}}{\Gamma_{k,k}}] + \frac{G_1}{\Gamma_{k,k}}, ...G_k, ...)$ and noting that for each k this representation is of the same form as the vector in 2, Theorem 1.3 with $c_{i,i} = \frac{\Gamma_{i,k}}{\Gamma_{k,k}}$. Hence by 1 of this Corollary $\Gamma_{i,k} \neq 0$ for all (i,k).

The Theorems stated above provide complete characterization of:

1. Mean zero Gaussian vectors G with infinitely divisible squares.

2. The pairs $(G, \mathbf{c})$ of mean zero Gaussian vectors G and real vectors $\mathbf{c}$ with non zero entries for which $(G + \alpha\mathbf{c})^2$ is infinitely divisible for all α.

The natural question that could be asked at this point (and which was posed by Marcus and Rosen, see [4] and [5]) about existence and characteristics of the vectors G for which 2. is valid for some but NOT ALL α. To put this question formally we need

Definition 1.1. *Let* G *be a Gaussian vector. Let* α *be an extended real number* $0 \leq \alpha_0 \leq \infty$. *We say that* G *has a critical point* $\alpha_\circ$ *if*

$$(G+\alpha)^2 \text{ is infinitely divisible for all } \quad |\alpha| \leq \alpha_0$$

and

when $\alpha_0 \neq \infty$

$$(G+\alpha)^2 \text{ is not infinitely divisible for any } \quad |\alpha| > \alpha_0$$

Marcus and Rosen have shown in [5] the existence of a critical point $0 < \alpha_0 < \infty$, for all two-dimensional Gaussian vectors with infinitely divisible squares. The question about existence of a critical points for the vectors of higher dimension remains open.

We will attempt to explore the connection between the properties of the inverse covariance matrix Γ^{-1} and the infinite divisibility of $(G+\alpha)^2$ for different α. We will obtain some upper bounds on a critical point for Gaussian n-dimensional vectors G with $n > 2$ and also the proof of Theorem 1.2 which does not invoke the Markov process theory (see Corollary 4.1). In particular, the original proof of 1.2 relies on the identification of the process in Theorem 1.3 as an associated with a strongly symmetric transient Borel right process and then uses the Second Ray-Knight Theorem to establish an isomorphism between $(G+c)^2$ and an infinitely divisible random vector. We obtain an elementary proof of the same result.

Let G be a Gaussian vector with positive definite covariance Γ. Let

$$D_i = \sum_{k=1}^{n} \Gamma^{-1}_{i,k}; \tag{1.3}$$

i.e D_i denotes the i-th row sum of Γ^{-1}.

For any vector $\tilde{\Lambda} \in \mathbf{R}^n$ denote by $\Psi_G(\tilde{\Lambda})$ the Laplace transform of $(G+\alpha)^2$ and let S be a diagonal $n \times n$ matrix with entries $s_{i,i} \in (0,1]$. Let $t > 0$. Then for any $\tilde{\Lambda} \in \mathbf{R^n}$ we can write

$$\Lambda = \tilde{\Lambda} I = t(I - S) \tag{1.4}$$

for any t sufficiently large and some S , where I is an identity $n \times n$ matrix. Let

$$\Phi(t,S) = \log \Psi_G(\tilde{\Lambda}).$$

Theorem 1.4. *Let* $\Phi(t,S) = \Phi_1(t,S) + \Phi_2(t,S)$*;* $\quad \Phi_1(t,S) = \Phi(t,S)|_{\alpha=0}$.

Let

$$Q(t) = [I + (t\Gamma)^{-1}]^{-1} = I - (I + t\Gamma)^{-1}. \tag{1.5}$$

then

$$\Phi_1(S) = \frac{1}{2}\Big(\log|I - Q| + \sum_{m=1}^{\infty} \frac{\text{trace}\,(QS)^m}{m}\Big) \tag{1.6}$$

$$\Phi_2(S) = \frac{\alpha^2 t}{2} 1 \cdot [(Q - I) + (I - Q^{-1}) \sum_{m=1}^{\infty} (QS)^m (Q - I)] \cdot 1^T \tag{1.7}$$

Remark 1.1. *The result in (1.6) is a known result and is given in [1], see also [3, Lemma 13.2.1, p.566]. For the outline of the proof see Remark 2.2 at the end of section 2.*

When Γ^{-1}, the inverse covariance matrix of G, is an M-matrix, we have:

Theorem 1.5. *For all t sufficiently large,*

$$Q > 0$$

2 Proofs of Theorems 1.4 and 1.5

Proof of 1.4

By [3], Lemma 5.2.1,

$$\Psi_G(\tilde{\Lambda}) = \det(I + \Gamma\Lambda)^{-1/2} \cdot (\exp(\frac{\alpha^2}{2} 1[\Lambda\tilde{\Gamma}\Lambda - \Lambda]1^T)). \tag{2.1}$$

where $\tilde{\Gamma} = (\Gamma^{-1} + \Lambda)^{-1}$.

Let

$$\Lambda = t(I - S)$$

for sufficiently large t.

We now consider power series expansion for the logarithm of the Laplace transform of $(G + \alpha)^2$.

Let $\Phi(t, S) = \Phi_1(t, S) + \Phi_2(t, S)$, where

$$\Phi_1(t, S) = \frac{1}{2} \log |I + t\Gamma(I - S)|^{-1} \tag{2.2}$$

and

$$\Phi_2(t, S) = \frac{\alpha^2}{2} 1[\Lambda\tilde{\Gamma}\Lambda - \Lambda]1^T = \frac{\alpha^2}{2} P(t, S). \tag{2.3}$$

Let

$$Q(t) = [I + (t\Gamma)^{-1}]^{-1} = I - (I + t\Gamma)^{-1}. \tag{2.4}$$

For all sufficiently large t the expression in (2.4) renders itself to representation as an absolutely convergent geometric series:

$$Q(t) = \sum_{v=0}^{\infty} (-1)^v \Big(\frac{\Gamma^{-1}}{t}\Big)^v \tag{2.5}$$

Hence for all (i, j),

$$q_{i,j} := Q_{i,j} = \delta_{i,j} - \frac{(\Gamma^{-1})_{i,j}}{t} + \frac{(\Gamma^{-2})_{i,j}}{t^2} \ldots. \tag{2.6}$$

We proceed to find the series expansion for $\Phi_2(t, S)$ in terms of $Q(t)$. Henceforth the parameter t will be suppressed in the expressions for $Q(t)$ and $\Phi_i(t, S)$ for $i = 1, 2$.

Note that

$$\begin{aligned}
\Phi_1(t, S) &= \frac{1}{2} \log |(I + t\Gamma - t\Gamma S)^{-1}| \\
&= \frac{1}{2} \log |(I + t\Gamma)^{-1}(I - (I + (t\Gamma)^{-1})^{-1} S)^{-1}| \\
&= \frac{1}{2} (\log |I - Q| + \log |I - QS|^{-1})
\end{aligned} \tag{2.7}$$

The fact that

$$\log|I - QS|^{-1} = \left(\sum_{k=1}^{\infty} \frac{\operatorname{trace}(QS)^k}{k}\right) \tag{2.8}$$

is given in [1], see also [3, Lemma 13.2.1, p.566]. For the outline of the proof see Remark 2.2 at the end of this section.

To obtain (1.7) consider (2.3) and substituting for Λ using (3.3) write:

$$\Lambda\tilde{\Gamma}\Lambda - \Lambda = t(I-S)[(t\Gamma)^{-1} + I - S]^{-1}(I-S) - t(I-S) \tag{2.9}$$

Hence

$$\begin{aligned} P(S) &= t1 \cdot \{(I-S)[(t\Gamma)^{-1} + I - S]^{-1}(I-S) + (S-I)\} \cdot 1^T \\ &= t1 \cdot \{(I-S)[Q^{-1} - S]^{-1}(I-S) + (S-I)\} \cdot 1^T \\ &= t1 \cdot \{(I-S)[I - QS]^{-1}Q(I-S) + (S-I)\} \cdot 1^T \end{aligned} \tag{2.10}$$

Since $\det(QS) < 1$ for all t sufficiently large,

$$[I - QS]^{-1} = \sum_{m=0}^{\infty} (QS)^m.$$

Therefore

$$\begin{aligned} &(I-S)[I-QS]^{-1}Q(I-S) + (S-I) \\ &= (I-S)\left(\sum_{m=0}^{\infty}(QS)^m\right)Q(I-S) + (S-I) \\ &= \left(\sum_{m=0}^{\infty}(QS)^m\right)Q - S\left(\sum_{m=0}^{\infty}(QS)^m\right)Q - \\ &\qquad \left(\sum_{m=1}^{\infty}(QS)^m\right) + S\left(\sum_{m=1}^{\infty}(QS)^m\right) + (S-I). \end{aligned} \tag{2.11}$$

For $i = 0, 1$ we write

$$S\left(\sum_{m=i}^{\infty}(QS)^m\right) = Q^{-1}\left(\sum_{m=i+1}^{\infty}(QS)^m\right),$$

to see that the last line of (2.11)

$$= (Q - I) + (I - Q^{-1}) \left(\sum_{m=1}^{\infty} (QS)^m \right) (Q - I).$$

This gives us (1.7).

□

Corollary 2.1. *Let* $D = (D_1, \ldots, D_n)$*. Then*

$$\Phi_2(S) = \frac{\alpha^2}{2} \left(-1 \cdot \Gamma^{-1} \cdot 1^T + D \cdot \left(\frac{1}{t} \sum_{m=1}^{\infty} (QS)^m \right) \cdot D^T \right) (1 + O(1/t)) \tag{2.12}$$

Proof By (2.4):

$$I - Q^{-1} = I - (I + (t\Gamma)^{-1}) = \frac{-\Gamma^{-1}}{t}, \tag{2.13}$$

and

$$Q - I = \frac{-\Gamma^{-1}}{t}(1 + O(1/t)). \tag{2.14}$$

Let

$$\tilde{D} = DI$$

Now substituting this in (1.7) we see that

$$1 \cdot [t(I - Q^{-1}) \sum_{m=1}^{\infty} (QS)^m (Q - I)] \cdot 1^T \tag{2.15}$$

$$= \frac{1}{t} \sum_{m=1}^{\infty} 1 \cdot \Gamma^{-1} (QS)^m \Gamma^{-1} (1 + O(1/t)) \cdot 1^T$$

$$= \frac{1}{t} \sum_{m=1}^{\infty} \sum_{i,j=1}^{n} \{\Gamma^{-1} (QS)^m \Gamma^{-1}\}_{i,j} (1 + O(1/t))$$

$$= \frac{1}{t} \sum_{m=1}^{\infty} \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \Gamma^{-1}_{i,k} (QS)^m_{k,l} \Gamma^{-1}_{l,j} (1 + O(1/t))$$

$$= \frac{1}{t} \sum_{m=1}^{\infty} \sum_{k,l=1}^{n} (\sum_{i=1}^{n} \Gamma^{-1}_{i,k}) (QS)^m_{k,l} (\sum_{l=1}^{n} \Gamma^{-1}_{l,j}) (1 + O(1/t))$$

$$= \frac{1}{t} \sum_{m=1}^{\infty} \sum_{k,l=1}^{n} (QS)^m_{k,l} D_k D_l (1 + O(1/t))$$

$$= \frac{1}{t} \sum_{m=1}^{\infty} \sum_{k,l=1}^{n} \left\{ \tilde{D} (QS)^m \tilde{D} \right\}_{k,l} (1 + O(1/t))$$

$$= \frac{1}{t} \sum_{k,l=1}^{n} \left\{ \tilde{D} \left(\sum_{m=1}^{\infty} (QS)^m \right) \tilde{D} \right\}_{k,l} (1 + O(1/t))$$

Now substitute this into (1.7) to get (2.12). □

Proof of 1.5

This is the direct consequence of the following

Lemma 2.1. *Let* Γ^{-1} *be an* M *matrix and assume that for some* $i \neq j$, $\Gamma^{-1}_{i,j} = 0$. *Let*

$$k = \min\{l > 1 : \Gamma^{-l}_{i,j} \neq 0\}$$

Then

$$k \leq n - 1 \tag{2.16}$$

$$(\Gamma^{-k})_{i,j} = (-1)^k |(\Gamma^{-k})_{i,j}|. \tag{2.17}$$

Proof To show (2.17) make the following claim: if for all $1 \leq u \leq k$, $\Gamma^{-1}_{i,j} = 0$; then for any $h < k$, any term of the form

$$\sum_{r_1} ... \sum_{r_h} \Gamma^{-1}_{i,r_1} \Gamma^{-1}_{r_1,r_2} .... \Gamma^{-1}_{r_h,j}$$

equals 0.

For $k = 2$;

$$(\Gamma^{-2})_{i,j} = \sum_{r=1}^{n} \Gamma^{-1}_{i,r} \Gamma^{-1}_{r,j} \geq 0 \tag{2.18}$$

since all summands are the products of two negative factors, hence positive – the only potential negative summands are of the form $\Gamma^{-1}_{i,i}\Gamma^{-1}_{i,j}$, which is zero here, hence, if $(\Gamma^{-2})_{i,j} = 0$, then for all $1 \leq r \leq n$, $\Gamma^{-1}_{i,r}\Gamma^{-1}_{j,r} = 0$ – so the above claim holds.

In general, for arbitrary k, suppose the claim holds for all $l < k$; $\Gamma^{-k}_{i,j} = 0$. Then

$$(\Gamma^{-k})_{i,j} = \sum_{r_1} \sum_{r_2} .... \sum_{r_{k-1}} \Gamma^{-1}_{i,r_1} \Gamma^{-1}_{r_2,r_3} .... \Gamma^{-1}_{r_{k-1},j}. \tag{2.19}$$

Suppose a particular summand contains one or more factors of the form $\Gamma^{-1}_{i,i}$.

Rearranging the summation order we get:

$$\sum_{r_1} \sum_{r_2} ... \sum_{r_u} \Gamma^{-1}_{r_1,r_1} \Gamma^{-1}_{r_2,r_2} ... \Gamma^{-1}_{r_u,r_u} \Big[\sum_{r_{u+1}} ... \sum_{r_{k-1}} \Gamma^{-l}_{i,r_{u+1}} \Gamma^{-l}_{r_{u+1},r_{u+2}} ... \Gamma^{-l}_{r_{k-1},j} \Big]. \tag{2.20}$$

However, each of the summands in square brackets is 0 by assumption of our claim as it applies to $k - u < k$. So any potentially non-zero terms contain factors of the form $\Gamma^{-1}_{a,b}$ with $a \neq b$ only. But all such terms have the same sign as $(-1)^k$– so that no cancellation is possible. Hence $\Gamma^{-k}_{i,j}$ is either 0 or has the sign of $(-1)^k$. This is (2.17).

To obtain (2.16) note that $(\Gamma^{-k})_{i,j}$ must be non-zero for some $k \leq n$ for otherwise the eigenvectors of Γ will have to satisfy a homogeneous system of $n+1$ linear equations:

$$\begin{aligned}
\sum_{k=1}^{n} E_{i,k}E_{k,j} &= 0 \\
\sum_{k=1}^{n} E_{i,k}E_{k,j}u_k^{-1} &= 0 \\
\sum_{k=1}^{n} E_{i,k}E_{k,j}u_k^{-2} &= 0 \\
&. \\
&. \\
\sum_{k=1}^{n} E_{i,k}E_{k,j}u_k^{-(n-1)} &= 0. \qquad (2.21)
\end{aligned}$$

We bring an algebraical proof of (2.16). (For a sketch of a more intuitive geometric proof see Remark 2.3.) Let $\mathbf{E} = (E_{i,1}E_{1,j}, E_{i,2}E_{2,j}, ...E_{i,n}E_{n,j})$ and write (2.21) as:

$$U\mathbf{E}^T = \mathbf{0}$$

where U is a square matrix with $U_{i,j} = u_j^{-i-1}$ for $1 \leq i, j \leq n$.

We know that $\mathbf{E}$ is a non-zero vector – otherwise $\Gamma_{i,j}$ would be 0. Hence $\det U$ must be 0 – that is it must have linearly dependent rows. Let

$$\mathbf{u} = (u_1^{-1}, u_2^{-1}, \dots u_n^{-1})$$

and

$$\mathbf{u}^k = (u_1^{-k}, u_2^{-k}, \dots u_n^{-k})$$

Hence $\mathbf{u}^{k-1}$ is the k-th row of U. The linear dependence of these vectors then implies that for some real numbers a_k; for $0 \le k \le n-1$,

$$\sum_{k=0}^{n-1} a_k \mathbf{u}^k = \mathbf{0}.$$

Which is, writing the vector form explicitly: For all i,

$$\sum_{k=0}^{n-1} a_k u_i^k = 0.$$

So each of eigenvectors of Γ^{-1} is the solution to the same polynomial of degree $n-1$.

By the Fundamental Theorem of Algebra that would mean that at least two of our eigenvectors are identical. WLOG, let those be u_1 and u_2. Let $\tilde{\mathbf{E}} = (E_{i,1}E_{1,j} + E_{i,2}E_{2,j}, ...E_{i,n}E_{n,j})$. Applying now the reasoning exactly identical to the above to the $n-1$ dimensional vector $\tilde{\mathbf{E}}$ and matrix $\tilde{U}$, such that $\tilde{U}_{i,j} = u_{j+1}^{-i-1}$ for $1 \le i, j \le n-1$ conclude that $u_j = u_l$ for some $j \ne l$. Proceeding by induction conclude that all eigenvalues of Γ are equal. This contradicts the irreducibility of Γ.

Hence we know that (2.17) is indeed non-zero for some $k < n$.

□

Proof of Theorem1.5 concluded.

Using this result to substitute into (2.20) we conclude that whenever $\Gamma_{i,j}^{-l} = 0\ \forall l < k$ and $\Gamma_{i,j}^{-k} \ne 0$,

$$q_{i,j} = \frac{(-1)^k}{t^k}\Gamma_{i,j}^{-k} + O(t^{-(k+1)}) = \frac{1}{t^k}|\Gamma_{i,j}^{-k}| + O(t^{-(k+1)}), \tag{2.22}$$

and so is positive.

□

We will say that Γ^{-1} has a zero of order h at (i,j) if $\forall r \leq h$ $\Gamma^{-r}_{i,j} = 0$ and $\Gamma^{-(h+1)}_{i,j} \neq 0$.

Remark 2.1. *Note that the results of Lemmas 3.1, 2.1, Theorem 1.4 and Corollary 2.1 also hold for matrices that are not strictly positive definite.*

Remark 2.2. *Proof of (2.8) outline. This proof is due to Eisenbaum and Kaspi, [6], adapted from [3, Lemma 13.2.1, p.566].*

Let $u_i, i = 1, \ldots, n$ denote the eigenvalues of Γ then

$$v_i = \frac{tu_i}{1 + tu_i}, \qquad i = 1, \ldots, n \tag{2.23}$$

are the corresponding eigenvalues of Q. Hence, since $S_{i,i} < 1$ for all (i,i), all eigenvalues of QS lay in $[0,1)$ interval. Let $\mu_i, 1 \leq i \leq n$ be the eigenvalues of QS.

Then

$$\log |I - QS|^{-1} = -\log \prod_{k=1}^{n} (1 - \mu_i) = \sum_{h=1}^{n} \sum_{k=1}^{\infty} \frac{\mu_i^k}{k}, \tag{2.24}$$

which converges absolutely. Using the fact that the trace of the product of matrices is invariant with respect to the order of matrices in the product and Fubini's Theorem for sums get:

$$\log |I - QS|^{-1} = \sum_{k=1}^{\infty} (\sum_{h=1}^{n} \frac{\mu_i^k}{k}) = \sum_{k=1}^{\infty} \frac{\text{trace}\,(QS)^k}{k} \tag{2.25}$$

□

Remark 2.3. *Geometric proof of Lemma 2.1*

Let $\mathbf{E_i} = (E_{i,1}, E_{i,2}, ...E_{i,n})$, *i.e. is the i-th eigenvector of* Γ*; and* $\mathbf{E_j} = (E_{j,1}, E_{j,2}, ...E_{j,n})$, *i.e. is the j-th eigenvector of* Γ. *Let* W *be a* *diagonal matrix with* $W_{h,h} = u_h^{-1}$*. The system in (2.31) can then be written as:*

$$\mathbf{E_i} W^{k_1} \cdot \mathbf{E_j} W^{k_2} = 0 \qquad \textit{for all} \quad k_1 + k_2 \leq n \tag{2.26}$$

Since $\mathbf{E_i}$ *and* $\mathbf{E_j}$ *are orthogonal vectors in* n*-dimensional space, (2.26) means that all their images under repeated application of the linear transformation* W *remain orthogonal. Hence the images of these vectors will span two orthogonal linear spaces with sum of dimensions not greater than* n*. This, however, implies that* $\Gamma_{i,j} = \mathbf{E_i} W^{-1} \cdot \mathbf{E_j} = 0$*, which contradicts Corollary 1.1, 2. Hence (2.16) must hold for some* $k < n$*.*

3 Upper bound on the critical point for Gaussian vectors.

Theorem 3.1. *Let* G *be a mean zero Gaussian process with covariance matrix* Γ *and assume that* Γ^{-1} *is an* M *matrix. Let*

$$\mathcal{D} = \{(i,j) : D_i D_j < 0\}. \tag{3.1}$$

Then

if $\mathcal{D} \neq \emptyset$ *and* α_0 *is a critical point of* G*,*

$$\alpha_0 \leq \inf_{\mathcal{D}} \left\{ \left(\frac{\Gamma^{-1}_{i,j}}{2 D_i D_j} \right)^{\frac{1}{2}} \right\}. \tag{3.2}$$

Note that it is possible that $\alpha_0 = 0$*.*

if $\mathcal{D} = \emptyset$ *then* $\alpha_0 = \infty$

Note that for the $\mathcal{D}$ to be non-empty Γ^{-1} matrix has to have at least one negative row sum.

Proof

We will employ the necessary and sufficient conditions for infinite divisibility of a random vector (see [2], Ch. XIII.4), [1], [3, 13.2.2], and the application therein to Gaussian vectors. We bring the result as developed in [3, Lemmas 13.3.1 and 5.2.1], and [5]:

Lemma 3.1. *[3, Lemma 13.2.2] For any vector $\tilde{\Lambda} \in \mathbf{R}^n$ denote by $\Psi_G(\tilde{\Lambda})$ the Laplace transform of $(G+\alpha)^2$ and let S be a diagonal $n \times n$ matrix with entries $s_{i,i} \in (0,1]$. Let $t > 0$. Then for any $\tilde{\Lambda} \in \mathbf{R^n}$ we can write*

$$\Lambda = \tilde{\Lambda} I = t(I - S) \tag{3.3}$$

for any t sufficiently large and some S , where I is an identity $n \times n$ matrix. Let

$$\Phi(t,S) = \log \Psi_G(\tilde{\Lambda})$$

and suppose that $\Phi(t,S)$ has a power series expansion about $S = \mathbf{0}$. Then $(G+\alpha)^2$ is infinitely divisible if and only if for all t sufficiently large all coefficients of this expansion are non-negative, except the constant term.

Lemma 3.2. *Let A_{m_i,m_j} and B_{m_i,m_j} be the coefficients of the term $s_i^{m_i} s_j^{m_j}$ $(i \neq j)$ in $\Phi_1(S)$ and $\Phi_2(S)$ respectively. Then for all $m_i, m_j \geq 1$*

$$A_{m_i,m_j} = \frac{1}{2}\Big(\frac{\Gamma^{-1}_{i,j}}{t}\Big)^2 (1 + O(1/t)) \tag{3.4}$$

and

$$B_{m_i,m_j} = \frac{\alpha^2}{t^2} D_i D_j (-\Gamma^{-1}_{i,j})(1 + O(1/t)) \tag{3.5}$$

whenever $\Gamma^{-1}_{i,j} \neq 0$.

Proof Let $m = m_i + m_j$. It is clear from (1.6) that the only contribution to A_{m_i,m_j} comes from

$$\frac{\{(QS)^m\}_{i,i} + \{(QS)^m\}_{j,j}}{m}, \tag{3.6}$$

and, furthermore, we must have

$$\frac{\{(QS)^m\}_{i,i}}{s_i^{m_i} s_j^{m_j}} = \prod_{l=1}^{m} q_{p_0,p_1} q_{p_1,p_2} \cdots q_{p_{l-1},p_l} \cdots q_{p_{m-1},p_m} \tag{3.7}$$

with $p_0 = p_m = i$; and all the other p_l, $1 \le l \le m-1$, must be either i or j. Since

$$q_{i,i} = 1 + O(1/t), \quad q_{j,j} = 1 + O(1/t) \quad \text{and} \quad q_{i,j} = q_{j,i} = -\frac{\Gamma_{i,j}^{-1}}{t} + O(1/t^2), \tag{3.8}$$

the terms on the right-hand side of (3.7) that are not $O(1/t^3)$ are those terms in which $q_{i,j}$ and $q_{j,i}$ each occur only once. This can happen in the following m_i ways

$$q_{i,i}^r q_{i,j} q_{j,j}^s q_{j,i} q_{i,i}^u, \qquad r = 0, \ldots, m_i \tag{3.9}$$

(and, obviously, $s = m_j - 1$ and $u = m_i - r$). Clearly

$$q_{i,i}^r q_{i,j} q_{j,j}^s q_{j,i} q_{i,i}^u = \frac{(\Gamma_{i,j}^{-1})^2}{t^2} + O(1/t^3). \tag{3.10}$$

Repeating this argument with i and j interchanged we see that there are m_j ways that

$$\frac{\{(QS)^m\}_{j,j}}{s_i^{m_i} s_j^{m_j}} = \frac{(\Gamma_{i,j}^{-1})^2}{t^2} + O(1/t^3). \tag{3.11}$$

Using (3.7)–(3.11) we get (3.4) It is clear from (1.7) that the only contribution to the leading term of B_{m_i,m_j} comes from

$$\frac{\alpha^2}{2t} \left(\sum_{u=1}^{n} D_u \{q_{u,i} s_i (QS)^{m-1}\}_{i,j} D_j + \sum_{u=1}^{n} D_u \{q_{u,j} s_j \{(QS)^{m-1}\}_{j,i} D_i \right); \tag{3.12}$$

By (3.8) we see that (3.12) equals to

$$\frac{\alpha^2}{2t} \left(D_i \{(QS)^m\}_{i,j} D_j + D_j \{\{(QS)^m\}_{j,i} D_i \right) (1 + O(1/t)); \tag{3.13}$$

where the contribution to the first summand of B_{m_i,m_j} will be from a coefficient arising from m_i of s_i factors followed by m_j of s_j factors; and to the the second – from m_j of s_j factors followed by m_i of s_i factors. Furthermore, by (3.8) the part of B_{m_i,m_j} arising from $\{(QS)^m\}_{i,j}$ equals to

$$q_{i,i}^{m_i-1} q_{i,j} q_{j,j}^{m_j} + O(1/t^2) \tag{3.14}$$

and similarly with i and j interchanged. Since

$$q_{i,i}^{m_i-1} q_{i,j} q_{j,j}^{m_j} = \frac{-\Gamma_{i,j}^{-1}}{t} + O(1/t^2) \tag{3.15}$$

we get (3.5). □

Proof of Theorem 3.1 continued.

To begin, assume that $\Gamma_{i,j}^{-1} \neq 0$ for all (i,j).

Let

$$C_{m_i,m_j} = A_{m_i,m_j} + B_{m_i,m_j}$$

From (22) and (23):

$$C_{i,j}(\alpha) := C_{m_i,m_j} = \frac{-\Gamma_{i,j}^{-1}}{t^2} \left(\frac{-\Gamma_{i,j}^{-1}}{2} + \alpha^2 D_i D_j \right) (1 + O(1/t)) \tag{3.16}$$

Note that since the length of the term is fixed, for any N such that $m_i, m_j < N$ (3.16) holds for all t sufficiently large (i.e. $t >> N$) thus making $C_{i,j}(\alpha)$ independent of m_i, m_j.

For $C_{i,j}(\alpha)$ to remain non-negative when $D_i D_j < 0$ as $t \to \infty$ we need to have

$$\frac{|\Gamma_{i,j}^{-1}|}{2} + \alpha^2 D_i D_j > 0. \tag{3.17}$$

Therefore, using Lemma 3.1, we get (3.2), which is Theorem 3.1 for inverse covariance matrices with non-zero entries.

Removing restriction $\Gamma^{-1} > 0$

Now returning to (3.4) and (3.5) we see that if Γ^{-1} has a zero of order k at (i,j) for some $0 < k < n-1$ then using Theorem 1.4 again we obtain:

$$A_{i,j} = \frac{1}{2t^{2(k+1)}}((\Gamma^{-k})_{i,j})^2 + O(t^{-(2k+3)}) \tag{3.18}$$

And:

$$B_{i,j} = \alpha^2 D_i D_j \frac{1}{t^{k+2}} |(\Gamma^{-k})_{i,j}| + O(t^{-(k+3)}) \tag{3.19}$$

So that

$$C_{i,j}(\alpha) = \frac{1}{2t^{2(k+1)}}((\Gamma^{-k})_{i,j})^2 + \alpha^2 D_i D_j \frac{1}{t^{k+2}} |(\Gamma^{-k})_{i,j}|(1 + O(1/t)) \tag{3.20}$$

will be positive iff $D_i D_j > 0$, i.e. whenever i-th and j-th rows have the same sign. This concludes the proof of Theorem 3.1. □

Hence, to find an infinitely divisible Gaussian vector with characteristic point $\alpha_0 = 0$ it is enough to find a Γ^{-1} matrix with $\Gamma^{-1}_{i,j} = 0$; and D_i and D_j of different signs. (See section 5 for an example). Formally stating this last observation we have

Corollary 3.1. *If G has covariance matrix Γ such that for some pair (i,j),*

$$\Gamma^{-1}_{i,j} = 0 \qquad \textit{and} \qquad D_i D_j < 0$$

then $(G+\alpha)^2$ is not infinitely divisible for any $\alpha \in \mathbf{R}$

□

4 Some other applications.

We are now ready to give an elementary proof of the part $(3 \Rightarrow 1)$ of Theorem 1.2 in the following

Corollary 4.1. *Let G be a Gaussian vector with positive definite covariance matrix Γ. If $N\Gamma^{-1}N^t$ is an M-matrix with positive row sums for some signature matrix N then $(G+cN)^2$ is infinitely divisible for all real c.*

Proof WLOG assume that Γ^{-1} is an M-matrix, i.e. N is an identity matrix.

By Lemma 3.1 it is enough to show that the logarithm of the Laplace transform of $(G+c)^2$ has only positive coefficients in the expansion about $S = 0$ for non-constant terms. By Lemma 3.2 and Corollary 3.1 it is enough to show that $q_{i,j} \geq 0$ for all i, j for all t sufficiently large. By Lemma 2.5, $q_{i,j} > 0$ for such t. This gives the result. □

It is also possible to obtain some results about Gaussian vectors with non-infinitely divisible squares.

Lemma 4.1. *Let G be a Gaussian vector with positive, strictly positive definite covariance Γ, such that Γ^{-1} is not an M-matrix but is weakly diagonally dominant. If there exist a triple i, j, k such that*

$$q_{k,j}, q_{i,k} < 0 \quad \textit{and} \quad q_{j,i} > 0, \tag{4.1}$$

then

$(G+\alpha)^2$ is not infinitely divisible for all $\alpha \in \mathbf{R}$.

Proof

Consider the coefficient of the term $s_i s_j$:

$$\begin{aligned} & \frac{1}{2}\big(q_{i,j}^2 + \alpha_i^2 \frac{2D_iD_j}{t} q_{i,j}\big)(1+O(1/t)) \\ = \quad & \frac{1}{2t^2}\big((\Gamma^{-1}_{i,j})^2 - \alpha|D_iD_j|\Gamma^{-1}_{i,j}\big)(1+O(1/t)) \end{aligned} \tag{4.2}$$

This is negative when

$$\alpha^2 > \frac{\Gamma^{-1}_{i,j}}{2D_iD_j} \tag{4.3}$$

Now consider the term $s_i s_j s_k$. Its coefficient is:

$$\begin{aligned} & \big\{q_{i,j}q_{j,k}q_{k,i} + \frac{\alpha^2}{t}[|D_iD_j|q_{j,k}q_{k,i} + |D_jD_k|q_{i,j}q_{k,i} + |D_kD_i|q_{i,j}q_{j,k}]\big\} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot(1+O(1/t)) \\ = \quad & \frac{1}{t^3}\big\{-\Gamma^{-1}_{i,j}\Gamma^{-1}_{j,k}\Gamma^{-1}_{k,i} + \\ & \alpha^2[D_iD_j|\Gamma^{-1}_{j,k}\Gamma^{-1}_{k,i}| - D_jD_k|\Gamma^{-1}_{i,j}\Gamma^{-1}_{k,i}| - D_kD_i|\Gamma^{-1}_{i,j}\Gamma^{-1}_{j,k}|]\big\}(1+O(1/t)) \end{aligned} \tag{4.4}$$

Note that only the first term in square brackets is positive, while the other three terms are negative. Hence for this expression to be negative it is enough to have:

$$\alpha^2 < \frac{\Gamma^{-1}_{i,j}}{|D_iD_j|}. \tag{4.5}$$

□

Corollary 4.2. *Let* $n = 3, 4$ *or* 5. *Assume that* $\Gamma > 0$ *, is strictly positive definite; and* Γ^{-1} *is weakly diagonally dominant. If* Γ^{-1} *has at least one positive off-diagonal entry, then* $(G+\alpha)^2$ *is not infinitely divisible for all* $\alpha \in \mathbf{R}$.

Proof Note that to obtain this result it is enough to show that in dimension 3, 4 and 5 the inverse covariance matrix with positive off-diagonal elements has a triple of indices satisfying (4.1). Also note that Γ^{-1} can not have a row with only positive entries.

This is obvious for $n = 3$, since in this case one off diagonal element is positive, while two others are negative.

To see the result for $n = 4 \quad \text{and} \quad 5$ note that by row-column permutations it is possible to bring the matrix Γ^{-1} into the following form:

$$\text{if } \Gamma^{-1}_{i,j} > 0, \text{ then } \Gamma^{-1}_{i,k} > 0, \text{ for all } k > j$$

When written in this form, it is easy to see that the required triple indeed exists, unless $\Gamma^{-1} = A + B$, where A is the direct sum of M-matrices; $B > 0$ whenever $A = 0$, and $B = 0$ whenever $A \neq 0$. If $n = 4$ or 5 this is only possible if A is a direct sum of two such matrices, each of the size 2×2 for $n = 4$, or of size 2×2 and 3×3 for $n = 5$. However, if this is the case, then we can find a signature matrix N such that $N\Gamma^{-1}N$ is a weakly diagonally dominant matrix with negative off diagonal and positive diagonal elements. But the by [3], Remark 13.1.3, $(N\Gamma^{-1}N)^{-1} = N\Gamma N$ has positive entries. Contradiction. Hence the required triple exists for $n = 4 \quad \text{and} \quad 5$. □

Another application of Corollary 2.1 is the following result due to M.B. Marcus:

Lemma 4.2. *Let G be Gaussian vector with infinitely divisible squares and a finite critical point. Let α be such that $(G+\alpha)^2$ is infinitely divisible and $\Psi(\Lambda)$ be its Laplace transform and Φ_1, Φ_2 as above.*

Then $\exp(\Phi_2(t, \alpha, S))$ is not the Laplace transform of a random variable.

Proof Since

$$\Psi(\Lambda) = \exp(\Phi_1(t, S)) \exp(\Phi_2(t, S)),$$

and $\exp(\Phi_1(t, S))$ is a Laplace transform of a non-negative random variable, it follows that if $\exp(\Phi_2(t, S))$ is also a Laplace Transform of a random variable, then we can write G^2 as a sum of two independent random vectors, $V_1 + V_2$, where $P(V_1 < \epsilon)$ is positive for any $\epsilon > 0$. Since G^2, V_1 are continuous, so is V_2. Hence V_2 must be non-negative random variable (otherwise G^2 is negative with positive probability). It is therefore sufficient to show that $\exp(\Phi_2(t, S))$ is not the Laplace Transform of a non-negative random variable.

By [3], (13.62) $\exp(\Phi_2(t, S))$ is the Laplace Transform of a non-negative random variable if and only if all coefficients in its series expansion about S are positive.

By Corollary 2.1

$$\begin{aligned}
&\exp\left(\Phi_2(S)\right) \\
&\quad = \exp\left(\frac{\alpha^2}{2} 1 \cdot \Gamma^{-1} \cdot 1^T\right) \exp\left(\frac{\alpha^2}{2t} D \cdot \left(\sum_{m=1}^{\infty} (QS)^m\right) \cdot D\right) \\
&\qquad\qquad \cdot (1 + O(1/t)) \\
&\quad = K \cdot \exp\left(\frac{\alpha^2}{2t} D \cdot \left(\sum_{m=1}^{\infty} (QS)^m\right) \cdot D\right) (1 + O(1/t))
\end{aligned} \tag{4.6}$$

where K is some positive constant.

Let i, j be such that $D_i D_j < 0$. Consider the coefficient of the term $s_i s_j$ in the series expansion of $\exp \Phi_2(S)$. By the Taylor expansion of exponential function,

$$\exp \Phi_2(S) = \sum_{u=0}^{\infty} \frac{\{\Phi_2(S)\}^u}{u!}.$$

Hence the part of this coefficient coming from $\Phi_2(S)$ is

$$\frac{\alpha^2}{2t^2} 2 D_1 D_2 |\Gamma^{-1}_{1,2}| = \frac{\alpha^2}{t^2} D_1 D_2 |\Gamma^{-1}_{1,2}|;$$

and the part coming from $\frac{\Phi_2(S)^2}{2!}$ is

$$\frac{1}{2!}\left(\frac{\alpha^2}{2t}\right)^2 2D_1^2D_2^2 = \frac{\alpha^4}{4t^2}D_1^2D_2^2.$$

Hence this coefficient is positive when

$$\frac{\alpha^4}{4t^2}D_1^2D_2^2 > \frac{\alpha^2}{t^2}D_1D_2|\Gamma^{-1}_{1,2}|,$$

or

$$|\alpha| > 2\sqrt{\frac{|\Gamma^{-1}_{i,j}|}{|D_iD_j|}}.$$

By Theorem 3.1 this is outside of the range of α for which infinite divisibility of $(G+\alpha)^2$ is possible. $\square$

5 Analysis of terms with three or more indices.

Consider the term of the form $s_i^{m_i}s_j^{m_j}s_k^{m_k}$. We have seen that that each of the factors of the form $q_{i,i}$ approaches 1 as $t \to \infty$. Hence WLOG we can consider the case $m_i = m_j = m_k = 1$. By Lemma 3.2

$$A_{i,j,k} = q_{i,j}q_{j,k}q_{k,i}$$

and

$$B_{i,j,k} = \frac{\alpha^2}{2}[2D_iq_{i,j}q_{j,k}D_k + 2D_jq_{j,k}q_{k,i}D_i + 2D_jq_{j,i}q_{i,k}D_k]$$

Suppose now that row sums involved in the above equation are of different signs. WLOG we can assume $D_i < 0$ and $D_j, D_k > 0$. Then $B_{i,j,k}$ is negative in the case

$$|D_iq_{i,j}q_{j,k}D_k + D_jq_{j,k}q_{k,i}D_i| > D_jq_{j,i}q_{i,k}D_k. \tag{5.1}$$

When (5.1) holds, the restriction imposed by Lemma 3.1 on α looks as follows:

$$\alpha^2 \leq \Big[\frac{|D_i D_k|}{|\Gamma^{-1}_{i,k}|} + \frac{|D_i D_j|}{|\Gamma^{-1}_{i,j}|} - \frac{|D_j D_k|}{|\Gamma^{-1}_{j,k}|}\Big]^{-1} \tag{5.2}$$

So if α_3 is the upper bound obtained from three index analysis;

$$\alpha_3 = \inf_H \Big[\frac{|D_i D_k|}{|\Gamma^{-1}_{i,k}|} + \frac{|D_i D_j|}{|\Gamma^{-1}_{i,j}|} - \frac{|D_j D_k|}{|\Gamma^{-1}_{j,k}|}\Big]^{-1} \tag{5.3}$$

where H is a set of all (i, j, k) triples for which (5.1) holds.

Let r, v be the indices for which minimum is achieved in (5.3), i.e. a maximum is achieved at $\frac{2|D_r D_v|}{|\Gamma^{-1}_{r,v}|}$. To get a better upper bound result we need i, j, k such that:

$$\frac{|D_i D_k|}{|\Gamma^{-1}_{i,k}|} + \frac{|D_i D_j|}{|\Gamma^{-1}_{i,j}|} - \frac{D_j D_k}{|\Gamma^{-1}_{j,k}|} > \frac{2|D_r D_v|}{|\Gamma^{-1}_{r,v}|} \tag{5.4}$$

This is impossible, since the right side is greater than the sum of two first (positive) summands. Hence we have shown the following

Lemma 5.1. *For any non-associated Gaussian vector G with absolutely divisible squares, the upper bound on the critical point obtained from two term analysis is smaller than that obtained by the three term analysis.*

This result gives a gives a good idea about what happens to the structure of the leading term as the number of indices in the coefficient increases. The number of different sign row-column intersections is $\lceil \frac{n^2}{4} \rceil$ for $n \times n$ matrix. Total number of such intersections is $\frac{n(n-1)}{2}$. Hence with the increase of n the percentage of negative-positive intersections falls to a quarter. It is intuitively clear that the large override as described in (5.4) is either impossible or extremely unlikely for the terms with large number of indices.

This suggests that the tightest – or close to it – upper bound on the value of the critical point for many Gaussian vectors is obtained via the analysis of the term of two indices – initially chosen by the author for the reason of technical simplicity.

Remark 5.1. *If* $\beta = \inf\{\alpha_k\}$ *where* α_k *is the upper bound obtained by asymptotic analysis of the coefficient of the term with* k *indices, then*

$$\beta > 0 \text{ if whenever } \Gamma^{-1}_{i,j} = 0, \quad D_i D_j \geq 0.$$

$$\beta = 0 \text{ if for some } i, j \quad \Gamma^{-1}_{i,j} = 0, \text{ and } D_i D_j < 0.$$

The discussion of this section and the result for the critical point of the two-dimensional vectors (see [5]) suggest one may hope that all Gaussian vectors possessing covariance matrices with a certain degree of regularity should indeed have a positive critical point.

An example of a vector for which no improvement over the bound obtained from two index analysis is possible is the perturbed exchangeable Gaussian vector of dimension n for $n \geq 3$. The example in Lemma 5.2 uses the work of M.B. Marcus (unpublished).

Let G be an exchangeable n-dimensional Gaussian random vector with positive covariance matrix Γ such that $\Gamma_{i,i} = 1; \Gamma_{i,j} = a, i \neq j$. Solving a system of linear equations, find $\Gamma^{-1}_{i,i} = \beta_n$; $\Gamma^{-1}_{i,j} = \gamma_n$ for $i \neq j$.

$$\beta_n = \frac{1 + a(n-2)}{(1-a)(1 + a(n-1)}$$

$$\gamma_n = \frac{-a}{(1-a)(1 + a(n-1)}$$

Let

$$d_1 = \frac{a(n-1)}{1 + a(n-2)}$$

If $\mathbf{c}_0 = (d_1, 1, ..1)$, then $\frac{G}{\mathbf{c}_0}$ is an associated vector with the first row sum equal to zero. Hence letting $\mathbf{c} = (d_1 - \epsilon, 1, ..1)$, get $\frac{G}{\mathbf{c}}$ to have infinitely divisible squares and the inverse of its covariance matrix $\tilde{\Gamma}$ has the first row sum negative for all sufficiently small ϵ.

Lemma 5.2. *Let G, $\mathbf{c}$ be as above.*

Then the smallest upper bound for the critical point of $\frac{G}{\mathbf{c}}$ is obtained from the two-index term analysis.

$$D_- := D_1 = -\epsilon d_1 \beta_n + o(\epsilon);$$

$$D_+ := D_i = \beta_n + (n - 2 + d_1 - \epsilon)\gamma_n; \text{for all} \quad i \neq 1$$

Let

$$\tilde{q}_{-,-} := \tilde{q}_{1,1} = \tilde{Q}^{-1}_{1,1} = 1 + O(1/t);$$

$$\tilde{q}_{+,-} := \tilde{q}_{1,i} \text{ for } \{i \neq 1\} = \tilde{Q}^{-1}_{1,i} = -\frac{(d_1 - \epsilon)\gamma_n}{t};$$

$$\tilde{q}_c := \tilde{q}_{i,j} \text{for} \{i \neq j, i, j \neq 1\} = \tilde{Q}^{-1}_{i,j} = -\frac{\gamma_n}{t};$$

$$\tilde{q}_{+,+} := \tilde{q}_{i,i} \text{ for } \{i \neq j; i, j \neq 1\} = \tilde{Q}^{-1}_{i,j} = 1 + O(1/t);$$

We will attempt to apply the asymptotic analysis used earlier for the terms of 2 indices to the terms containing k indices for $2 \leq k \leq n$. In this case the $\tilde{\Gamma}^{-1}$ matrix is very regular and the results will be tractable. We are interested to see whether a significant improvement of the upper bound for α can be obtained thereby.

Consider the term $s_{i_1}^{m_{i_1}} s_{i_2}^{m_{i_2}} ... s_{i_k}^{m_{i_k}}$ for $m_{i_v} > 0$ for all v and assume $i_v = 1$ for some v (if index 1 is not present, all summands of the coefficient are positive). Since $q_{i,i}$ tend to 1 as t grows we can assume that all $m_{i_v} = 1$.

To find the leading part of $A_{m_{i_1}\cdots}$ note that there are $(k-1)!$ rotationally different ways to arrange k different elements in a circle. Each such arrangement can be rotated to k different positions and the coefficient in Φ_1 of each such rotation is :

$$\frac{1}{k}(\tilde{q}_{+,-})^2(\tilde{q}_c)^{k-2}(1+O(1/t)).$$

Hence the coefficient

$$A_{i_1,\ldots}(s_{i_1}s_{i_2}\ldots.s_{i_k}) = \frac{1}{2}(k-1)!(\tilde{q}_{+,-})^2(\tilde{q}_c)^{k-2}(1+O(1/t)).$$

Now consider $B_{m_{i_1},\ldots}$. To find the leading part we need to consider arrangements with s_1 at either end of the arrangement or in between some other terms. In the first case there are $(k-1)!$ ways to arrange them so that s_1 is the first index and same number of arrangements where it is the last. Each will give a coefficient in Φ_2:

$$\frac{\alpha^2}{2}\frac{D_+D_-}{t}(\tilde{q}_{+,-})(\tilde{q}_c)^{k-2}(1+O(1/t)).$$

There are $k! - 2(k-1)!$ arrangements of the second type, each giving a coefficient in Φ_2:

$$\frac{\alpha^2}{2}\frac{D_+^2}{t}(\tilde{q}_{+,-})^2(\tilde{q}_c)^{k-3}(1+O(1/t)).$$

Hence,

$$B_{i_1,\ldots}(s_{i_1}s_{i_2}\ldots.s_{i_k}) = \frac{\alpha^2}{2}(k-1)!(\tilde{q}_c)^{k-2}(\tilde{q}_{+,-})^2 \cdot \Big[\frac{2D_+D_-}{t(\tilde{q}_{+,-})} + (k-2)\frac{D_+^2}{t(\tilde{q}_c)}\Big](1+O(1/t)).$$

And

$$C_{i_1,\ldots}(s_{i_1}s_{i_2}\ldots.s_{i_k}) = \frac{1}{2}(k-1)!(\tilde{q}_{+,-})^2(\tilde{q}_c)^{k-2} \cdot \Big\{1+\frac{\alpha^2}{t}\Big[\frac{2D_+D_-}{\tilde{q}_{+,-}} + (k-2)\frac{D_+^2}{\tilde{q}_c}\Big]\Big\}(1+O(1/t)).$$

The last expression will give an upper bound for the critical point of G only in the case when

$$[\frac{2D_+D_-}{\tilde{q}_{+,-}} + (k-2)\frac{D_+^2}{\tilde{q}_c}] < 0, \tag{5.5}$$

or, equivalently, when

$$|D_-| > \frac{k-2}{2} \cdot \frac{\tilde{q}_{+,-}}{\tilde{q}_c} D_+.$$

In this case it will give the upper bound, call it α_k, as follows:

$$\alpha_k = \left|\frac{2D_+D_-}{-(d_1-\epsilon)\gamma_n} + (k-2)\frac{D_+^2}{-\gamma_n}\right|^{-1} \tag{5.6}$$

This will be improvement over α_2 i.e. lower bound derived from analysis of the two index terms only if

$$\left|\frac{2D_+D_-}{(d_1-\epsilon)\gamma_n} + (k-2)\frac{D_+^2}{\gamma_n}\right| > \left|\frac{2D_+D_-}{(d_1-\epsilon)\gamma_n}\right| \tag{5.7}$$

This can only happen simultaneously with (5.5) if both summands on the left have the same sign, which is not true here. Therefore, in the case of perturbed exchangeable vector the lowest upper bound on the critical point is obtained from the two index analysis. □

6 General form of the coefficient for the term of the form $s_i^{m_i} s_j^{m_j}$.

In the statement of Lemma 3.2 we define A_{m_i,m_j} to be the coefficient of $s_i^{m_i} s_j^{m_j}$ in the power series expansion of $\Phi_1(S)$. In (3.4) we estimate A_{m_i,m_j}, for fixed m_i, m_j as $t \to \infty$. However, to apply Lemma 3.1 to show coefficient positivity we must also consider A_{m_i,m_j} when m_i and m_j are larger than t. We do this in the next lemma.

Lemma 6.1. *For $m_i, m_j \geq 1$*

$$A_{m_i,m_j} = \sum_{p=0}^{min(m_i,m_j)-1} \binom{m_i-1}{p}\binom{m_j-1}{p} \cdot \frac{1}{p+1}(q_{i,i})^{m_i-p-1}(q_{j,j})^{m_j-p-1}(q_{i,j})^{2(p+1)} \tag{6.1}$$
$$= \sum_{p=0}^{min(m_i,m_j)-1} V(m_i,m_j,p)\frac{q_{i,j}^2}{p+1}(1+O(1/t)),$$

where

$$V(m_i,m_j,p) = (q_{i,i}^{m_i-(p+1)})(q_{j,j}^{m_j-(p+1)})(q_{i,j}^{2p})\binom{m_i-1}{p}\binom{m_j-1}{p}. \tag{6.2}$$

Proof By (1.6), to find A_{m_i,m_j} we sum over all terms in the trace of $(QS)^{m_i+m_j}$ that involve s_i, m_i times and s_j, m_j times. Since the summands of the coefficient come from the trace of a matrix each of them will have the form $q_{i_1,i_2}q_{i_2,i_3}....q_{i_{m-1},i_m}$ with $i_1 = i_m = i$ or $i_1 = i_m = j$. It is convenient to consider the circular arrangement of the factors $s_{(\cdot)}$, since once such arrangement is fixed each of m of its rotations result in the same coefficient.

Consider arrangement of the s_i and s_j into blocks. By blocks we mean an unbroken string of s_i or s_j factors of length one or greater. For example, $s_j^3 s_i^4 s_j$ contains an s_i block of length 4 but $s_i s_j^2 s_i^3$ does not.

To account for all possible arrangements of indices of $s_i^{m_i}s_j^{m_j}$, consider any fixed circular arrangement. Suppose this arrangement has $p+1$ separate blocks of s_i, it then must also have $p+1$ blocks of s_j. We will have $0 \leq p < \min\{m_i, m_j\}$. For each p there exist

$$\binom{m_i-1}{p}\binom{m_j-1}{p} \tag{6.3}$$

ways to make this separation into blocks. Once the blocks are defined there exist exactly one circular arrangement of blocks(if we agree whuch index to always place first).

Each arrangement can be rotated one position clockwise exactly $m = m_i + m_j$ times. A given arrangement will repeat itself after k one-positional rotations, where $m = kv_1$ for some integer v_1. Obviously also $p + 1 = kv_2$ for some integer v_2. Hence this arrangement can be rotated v_1 times giving rise to a new arrangement. This cancels the v_1 factor in the denominator of (1.6), leaving k. The coefficients in Φ_1 of all such rotations are identical and equal to

$$\frac{1}{m}\{q_{i,j}^{p+1} q_{i,i}^{m_i-p-1} q_{j,j}^{m_j-p-1}\}.$$

To see this note that after any such rotation the resulting string of factors will either begin and end with the same index or will begin with s_i, end with s_j or vice versa. Using the symmetry of Q, see that:

If it begins and ends with, say, s_i, the coefficient in trace$(QS)^m$ is:

$$q_{i,i}(q_{i,i})^{m_i-p-2}(q_{j,j})^{m_j-p-1}(q_{i,j})^{p+1},$$

and is it begins with, say, s_i and ends withs_j, the coefficient in trace$(QS)^m$ is:

$$q_{j,i}(q_{i,i})^{m_i-p-1}(q_{j,j})^{m_j-p-1}(q_{i,j})^{p},$$

and the same with i, j interchanged. Using the fact that $q_{i,j} = q_{j,i}$ again, see that this equals to the expression in figure parentheses above,although the order of q factors varies with the rotation; for example starting with an arrangement giving rise to a coefficient

$$\frac{1}{m} q_{i,i}^{u_1} q_{i,j} q_{j,j}^{v_1} \ldots\ldots q_{i,j} q_{j,j}^{v_{p+1}} q_{j,i}$$

rotating it one position results in a coefficient

$$\frac{1}{m} q_{j,i} q_{i,i}^{u_1} q_{i,j} q_{j,j}^{v_1} \ldots . q_{i,j} q_{j,j}^{v_{p+1}}$$

and so on. Also note that each fixed arrangement with a fixed statring point gives rise to two coefficient summands, each corresponding to the direction - clockwise or counterclockwise - of the order of q-factors. For example, the last arrangement "'read"' in the opposite direction will give the coefficient

$$\frac{1}{m} q_{j,j}^{v_{p+1}} q_{j,i} \ldots . q_{j,j}^{v_1} q_{j,i} q_{i,i}^{u_1} q_{i,j}.$$

Therefore the coefficient of the whole sum is multiplied by 2.

Finally we note that each of the v_1 rotated block arrangements give rise to v_2 repetitions, according to which block is taken as the starting point of the circular arrangement. Therefore we divide by v_2. Thus the resulting denominator is: $v_2 k = p + 1$. □

Recall that B_{m_i,m_j} is the coefficient of $s_i^{m_i} s_j^{m_j}$ in the power series expansion of $\Phi_2(S)$. Let

$$B_{m_i,m_j} = \sum_{p=0}^{min(m_i,m_j)-1} B_{m_i,m_j,p}$$

Lemma 6.2. *For $m_i, m_j \geq 1$*

$$B_{m_i,m_j} = \sum_{p=0}^{min(m_i,m_j)-1} \frac{\alpha^2}{2t} V(m_i, m_j, p) \{ 2D_i D_j + [\frac{p}{m_i - p} D_j^2 + \frac{p}{m_j - p} D_i^2] \} (1 + O(1/t)). \quad (6.4)$$

Proof By (2.3), (1.7), (2.13) and (2.14) we obtain the coefficient in $P(S)$ arising from a particular arrangement, say $s_{k_1} s_{k_2} s_{k_3} \ldots s_{k_{m_i+m_j}}$, where k_u

is either i or j equals to:

$$\frac{1}{t}\sum_{r=1}^{n} D_r q_{(r,k_1)} q_{(k_1,k_2)} q_{(k_2,k_3)} \cdots$$
$$\cdots q_{(k_{m_i+m_j-1},k_{m_i+m_j})} q_{(k_{m_i+m_j},k_{m_i+m_j})} D_{k_{m_i+m_j}} (1+O(1/t))$$

Using 3.8 see that (6.5) equals to:

$$\frac{1}{t} D_{k_1} q_{(k_1,k_2)} q_{(k_2,k_3)} \cdots$$
$$\cdots q_{(k_{m_i+m_j-1},k_{m_i+m_j})} q_{(k_{m_i+m_j},k_{m_i+m_j})} D_{k_{m_i+m_j}} (1+O(1/t))$$

To find $B_{m_i,m_j,p}$, assume that one group of factors - either s_i or s_j - is broken into $p+1$ blocks for some p . Since we are dealing with a term in two indices only, one of the two cases is possible:

1. Both groups of factors are broken into $p+1$ blocks. Then $0 \leq p < \min\{m_i, m_j\}$. There will be $\binom{m_i-1}{p}\binom{m_j-1}{p}$ such arrangements. That means either starting with s_i, ending with s_j block, or vice versa. Both will result in exactly $2p+1$ of $q_{i,j}$ factors and have the coefficient in Φ_2:

$$\sum_{p=0}^{min(m_i,m_j)-1} \binom{m_i-1}{p}\binom{m_j-1}{p} \frac{\alpha^2}{2t}\Big[D_i(q_{i,i}^{m_i} q_{j,j}^{m_j}) \frac{q_{i,j}^{(2p+1)}}{(q_{i,i}q_{j,j})^{(p+1)}} q_{j,j} D_j$$
$$+D_j(q_{i,i}^{m_i} q_{j,j}^{m_j}) \frac{q_{i,j}^{(2p+1)}}{(q_{i,i}q_{j,j})^{(p+1)}} q_{i,i} D_i\Big](1+O(1/t))$$
$$= \sum_{p=0}^{min(m_i,m_j)-1} \frac{\alpha^2}{2t} D_i D_j V(m_i,m_j,p)(\frac{q_{i,j}}{q_{i,i}} + \frac{q_{i,j}}{q_{j,j}})(1+O(1/t)) \quad (6.5)$$

2. Now restrict p to $1 \leq p < \min\{m_i, m_j\}$. The group of s_i factors is broken into $p+1$ blocks and that of s_j factors into p blocks, or vice versa. (If $m_i < m_j$ then there is also a case when there are m_i

of s_i factors and $m_i + 1$ of s_j factors. The resulting coefficient in Φ_2 is of the form $B_{m_i,m_j,m_i}(1/t)$.) There will be $\binom{m_i-1}{p}\binom{m_j-1}{p-1}$ arrangements of the first kind and $\binom{m_i-1}{p-1}\binom{m_j-1}{p}$ of the second. This will result in exactly $2p$ of $q_{i,j}$ factors, moreover the D factors will have the same index in each of the arrangements - same index as the factors broken into $p+1$ blocks. The corresponding coefficient in Φ_2 is, then:

$$\frac{\alpha^2}{2t} \sum_{p=1}^{min(m_i,m_j)-1} (q_{i,i}^{m_i} q_{j,j}^{m_j}) q_{i,j}^{2p} \cdot \Big[\binom{m_i-1}{p}\binom{m_j-1}{p-1} \frac{D_i^2 q_{i,i}}{(q_{i,i})^{(p+1)}(q_{j,j})^p} + \binom{m_i-1}{p-1}\binom{m_j-1}{p} \frac{D_j^2 q_{j,j}}{(q_{i,i})^p (q_{j,j})^{(p+1)}}\Big](1+O(1/t)) \tag{6.6}$$

Using now the identity:

$$\binom{m-1}{p-1} = \frac{p}{m-p}\binom{m-1}{p}$$

see that (6.6) equals to:

$$\sum_{p=1}^{min(m_i,m_j)-1} \frac{\alpha^2}{2t} V(m_i,m_j,p)\Big[\frac{p}{m_i-p} D_j^2 + \frac{p}{m_j-p} D_i^2\Big](1+O(1/t)) \tag{6.7}$$

since the expression in square parentheses is 0 when $p = 0$.

Now denote as follows:

$$A_{m_i,m_j,p} = V(m_i,m_j,p)\frac{q_{i,j}^2}{p+1}(1+O(1/t)) \tag{6.8}$$

And

$$B_{m_i,m_j,p} = \frac{\alpha^2}{2t} V(m_i, m_j, p) \{ [\frac{p}{m_i - p} D_j^2 + \frac{p}{m_j - p} D_i^2] + 2D_i D_j \} (1 + O(1/t)) \qquad (6.9)$$

Let

$$C_{m_i,m_j,p} = A_{m_i,m_j,p} + B_{m_i,m_j,p} \qquad (6.10)$$

Hence,

$$C_{m_i,m_j} = \sum_{p=0}^{\min(m_i,m_j)-1} C_{m_i,m_j,p} \qquad (6.11)$$

And

$$C_{m_i,m_j,p} = V(m_i, m_j, p) \Big\{ \frac{(q_{i,j})^2}{p+1} + \frac{\alpha^2}{2t} \Big[2D_i D_j q_{i,j} + \frac{p}{m_j - p} D_i^2 + \frac{p}{m_i - p} D_j^2 \Big] \Big\} (1 + O(1/t)) \qquad (6.12)$$

Denote the expression in figure parentheses by $R_{m_i,m_j,p}$. We are interested in the sign of C_{m_i,m_j}, which will coincide with the sign of $C_{m_i,m_j,p}$, if they are identical for all p. Since the sign of $C_{m_i,m_j,p}$ is the same as the sign of $R_{m_i,m_j,p}$ we will concentrate on the later. Assume that $m_i m_j$ is bounded by linear multiple of t^2, i.e.

$$\sqrt{m_i m_j} \leq Nt \quad \text{for some fixed} \quad N$$

$$
\begin{aligned}
R_{m_i,m_j,p} &= \Big\{\frac{1}{p+1}\frac{|\Gamma^{-1}_{i,j}|^2}{t^2}+\frac{\alpha^2}{2}\Big[-\frac{2|D_iD_j||\Gamma^{-1}_{i,j}|}{t^2} \\
&\qquad +\big(\frac{D_i^2}{t}\frac{p}{m_j-p}+\frac{D_j^2}{t}\frac{p}{m_i-p}\big)\Big]\Big\}(1+O(1/t)) \\
&\geq \Big\{\frac{1}{p+1}\frac{|\Gamma^{-1}_{i,j}|^2}{t^2}+\frac{\alpha^2}{2}\Big[-\frac{2|D_iD_j||\Gamma^{-1}_{i,j}|}{t^2} \\
&\qquad +\frac{2|D_iD_j|}{t}\frac{p}{\sqrt{m_im_j}}\Big]\Big\}(1+O(1/t)) \\
&\geq \frac{1}{t^2}\Big[\frac{|\Gamma^{-1}_{i,j}|^2}{p+1}+\frac{\alpha^2}{2}|2D_iD_j|(-|\Gamma^{-1}_{i,j}|+\frac{p}{N})\Big](1+O(1/t))
\end{aligned}
\tag{6.13}
$$

The representation in (6.13) easily leads to the proof of the following

Lemma 6.3. *Let*

$$
C_{m_i,m_j}=\sum_{p=0}^{\min m_i,m_j} C_{m_i,m_j,p}.
$$

Then for any $N\in\mathbf{R}^+$ there exists $\beta>0$, independent of m_i, m_j and p such that for all $\alpha\leq\beta$

$$
C_{m_i,m_j}>0
$$

for all t sufficiently large and whenever

$$
\sqrt{m_im_j}\leq Nt.
$$

To show the positivity of $R_{m_i,m_j,p}$ for all p we minimize the function

$$
\frac{|\Gamma^{-1}_{i,j}|^2}{p+1}+\alpha^2|D_iD_j|\frac{p}{N}
$$

and show that the value at minimum is positive. The minimum of function of the continuous argument is achieved at

$$
\frac{|\Gamma^{-1}_{i,j}|}{\alpha}\sqrt{\frac{N}{|D_iD_j|}}-1. \tag{6.14}
$$

Hence for the discrete p the minimum will occur at

$$\lfloor \frac{|\Gamma_{i,j}^{-1}|}{\alpha}\sqrt{\frac{N}{2|D_iD_j|}}\rfloor - 1$$

or

$$\lceil \frac{|\Gamma_{i,j}^{-1}|}{\alpha}\sqrt{\frac{N}{2|D_iD_j|}}\rceil - 1.$$

Substituting the value of p from (6.14) into (6.13) get:

$$\frac{2\alpha|\Gamma_{i,j}^{-1}|}{\sqrt{N}}\sqrt{|D_iD_j|} - \alpha^2|D_iD_j|(|\Gamma_{i,j}^{-1}| + \frac{1}{N}) \tag{6.15}$$

This is positive whenever

$$\alpha < \frac{|\Gamma_{i,j}^{-1}|N}{(|\Gamma_{i,j}^{-1}|N+1)}\frac{2\sqrt{N}}{\sqrt{|D_iD_j|}}.$$

Hence the lemma is shown for

$$\beta = \frac{|\Gamma_{i,j}^{-1}|N}{(|\Gamma_{i,j}^{-1}|N+1)}\frac{2\sqrt{N}}{\sqrt{|D_iD_j|}}.$$

Which is defined for any N and is independent of m_i, m_j and p.

Remark 6.1. *Lemma 6.3 provides an alternative proof to the Lemma 5.2 of [5]. (6.13) can also be used to obtain some results in the proof of Lemma 4.2 [5], namely:*

$$\sum |C_{m_i,m_j}| \le K q_{i,i}^{m_i} q_{j,j}^{m_j} \sum_{p=0}^{\min(m_i,m_j-1)} t^{-2p}\binom{m_i-1}{p}\binom{m_j-1}{p}$$

where for all sufficiently large t K is a constant independent of m_i, m_j, p and bounded for fixed values of α. Noting by (2.6) that $q_{i,i} < \exp(-(1-\delta)\frac{\Gamma_{i,i}^{-1}}{t})$, proceed with the proof following (4.20) in [5].

7 The connection between the associated and infinitely divisible Gaussian processes.

The interest in the research of Gaussian vectors with infinitely divisible squares is motivated by the intimate connection that exists between such vectors and the class of associated Gaussian processes. The results presented are taken from [3], Ch13.2. We start with

Definition 7.1. *Let (Ω, P^x) be a probability space, $\mathcal{G}_t$ for $t \geq 0$ a σ-field filtration on Ω, $\mathcal{G} = \cup_{t \geq 0} \mathcal{G}_t$. Let X_t be a family of random variables on Ω adapted to $\mathcal{G}_t$ and θ_t a family of shift operators, such that $\theta_{t+s} = \theta_t \circ \theta_s$ and $X_t \circ \theta_s = X_{t+s}$. A collection $X = (\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \theta_t, P^x)$ is called a Borel right process with transition semigroup $\{P_t, t \geq 0\}$ if the following conditions are satisfied:*

1. *X is a right continuous simple Markov process with transition semigroup $\{P_t, t \geq 0\}$.*

2. *For all $f \in C_b(\Omega_\triangle)$ ($\triangle$ being a cemetery state of the process), all $\alpha > 0$, $U^\alpha f(X_t) := \int_0^\infty e^{-\alpha s} P_s f(X_t) ds$ is right continuous in t.*

3. *$\{\mathcal{G}_t, t \geq 0\}$ is augmented and right continuous.*

In addition, if X is strongly symmetric then α-potential densities $u^\alpha(x, y)$ with respect to some σ-finite reference measure m on $B(S)$ exist and are symmetric, i.e. $u^\alpha(x, y) = u^\alpha(y, x)$. Borel right process is local if in (2) the continuity in t is required for only such t for which $X_t \in S$.

We are now ready to define the object of further discussion:

Definition 7.2. *Let S be a locally compact space with a countable base. A Gaussian process $\{G_x; x \in S\}$ is said to be associated with a strongly* *symmetric transient local Borel right process X on S, with respect to reference measure m, if the covariance of G, $\Gamma(x,y) := E(G_x G_y)$ is the 0-potential density of X with respect to m for all $(x,y) \in S$.*

The following (necessary but not sufficient) property of an associated Gaussian vector is the direct consequence of the property of 0-potential density of Borel right process:

$$\Gamma_{i,j} \leq \Gamma_{i,i} \wedge \Gamma_{j,j}. \tag{7.1}$$

contrasting with the regular property of the strictly positive definite covariance matrix:

$$\Gamma_{i,j}^2 \leq \Gamma_{i,i}\Gamma_{j,j}. \tag{7.2}$$

Although the above regularity property is too weak to provide much information about G in general; in case $n = 3$ we have the following

Corollary 7.1. *Let $G = (G_1, G_2, G_3)$ be a Gaussian vector satisfying (7.1). Then the upper bound for the critical point of G is positive .*

This follows immediately from Theorem3.1 by writing the matrix Γ in terms of Γ^{-1} and above mentioned regularity condition: Suppose Γ^{-1} has a zero entry $\Gamma^{-1}_{i,j}$ while Γ satisfies (2). In $n = 3$ case this implies positivity of the i-th and j-th row sums. I.e.:

If

$$\Gamma^{-1} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & g_{23} \\ 0 & g_{32} & g_{33} \end{pmatrix}$$

then by (2), $g_{21} < g_{11}$ and $g_{32} < g_{33}$. Hence in the case of a 3-dimensional Gaussian vector the presence of zero entry in Γ^{-1} matrix satisfying (7.1) never means negativity of any coefficient for $t \to \infty$.

The remarkable connection between the class of associated processes and that of Gaussian processes with infinitely divisible squares is given by the following

Theorem 7.1. *Let S be a locally compact space with a countable base. Let $\{G_x; x \in S\}$ be a Gaussian process with strictly positive definite covariance matrix $\Gamma(x, y)$. The following are equivalent:*

1. *G is associated with a strongly symmetric transient Borel right process on S.*
2. *$\{(G_x + c)^2; x \in S\}$ is infinitely divisible for all $c \in \mathbf{R}$.*
3. *Let ξ be a standard Normal R.V. independent of G. $\{(G_x + b\xi)^2; x \in S \cup \{\delta\}, G_\delta \equiv 0\}$ is infinitely divisible for some $b \neq 0$. If this holds for some $b \neq 0$ it holds for all $b \in \mathbf{R}$.*

Remarkably, the proof not only establishes the equivalence (1) through (3), but gives the explicit decomposition of the squares in (2) into two infinitely divisible variables, namely:

$$\left\{L^x_{\tau(t)} + \frac{1}{2}G^2_x; x \in S\right\} \stackrel{law}{=} \left\{\frac{1}{2}\big(G_x + \sqrt{2t}\big)^2; x \in S\right\}$$

Hence all Gaussian vectors previously considered are the non-associated Gaussian vectors whose inverse covariance matrices are M-matrices. The question that may be asked at this point is: could these vectors be considered as associated with some processes? If yes, with which?

To begin, recall the construction of the corresponding Borel process for the given Gaussian process with strictly positive definite covariance matrix Γ such that Γ^{-1} is an M-matrix with non-negative row sums. Define $P_t(i, j) := \exp\{-t\Gamma^{-1}\}_{i,j}$. It is trivial to check the sub-Markov contraction semigroup property and P_t can be extended to a Markov semigroup on $S \cup \triangle$ by the standard procedure. We will call i an exit state if $\frac{d}{dt}P_0(i,S) < 0$ and we will call it a non-exit state if $\frac{d}{dt}P_0(i,S) = 0$. Heuristically, i is an exit state if it is possible to go from i to $\triangle$ directly and it is a non-exit state if the only way to get from i to $\triangle$ is via some other state.

$$\frac{d}{dt}P_t(i,S) = \frac{d}{dt}\sum_{j=1}^{n} \exp\{-t\Gamma^{-1}\}_{i,j} = -\sum_{j=1}^{n} D_j \exp\{-t\Gamma^{-1}\}_{i,j}$$

hence

$$\frac{d}{dt}P_t(i,S)|_{t=0} = -D_i$$

So we see that i is a non-exit state iff $D_i = 0$. Hence a state with corresponding row sum positive is an exit state and our matrix by assumption has all non-negative row sums. The result above suggests that perhaps a strictly positive definite covariance matrix can be used to construct a process - not Markovian, but a variation of such - for which the negative row sums will correspond to a state which can be termed an 'entry' state in some sense, i.e. a state at which the process will, in some sense increase, rather than dissipate, as at a state with positive row sum.

Let G be a Gaussian vector with a strictly positive definite covariance matrix Γ such that Γ^{-1} is an M-matrix. Let $q := \inf\{D_i\} \wedge 0$ where infimum is taken over all row sums of Γ^{-1}. Define $\tilde{\Gamma}^{-1} := \Gamma^{-1} - qI$ and note that $\tilde{\Gamma}^{-1}$ is an M-matrix with positive row sums. Let

$$\tilde{P}_t(i,j) = e^{-tq} \exp\{-t\tilde{\Gamma}^{-1}\}_{i,j} = e^{-tq}P_t(i,j).$$

where $P_t(i,j)$ is a contraction semigroup and X_t is the corresponding Markov process. Note that $e^{-tq} \geq 1$ with a strict equality holding iff Γ^{-1} has non-negative row sums and $\tilde{\Gamma}^{-1}$ is an M-matrix with non-negative row sums. So $\tilde{P}_t$ is a semigroup which is not a contraction semigroup in case Γ^{-1} has at least one negative row sum. However since the 'growth factor' e^{-tq} is completely deterministic the process with a semigroup $\tilde{P}_t$, call it $\tilde{X}_t$, can be regarded as a Markov process on $(S, \mathcal{F}_t, \mathcal{F}, \theta_t)$ endowed with a 'filtration' of excessive measures: $m_t(A) := e^{-tq}m(A)$ for all $A \in \mathcal{B}(S)$ and m is the reference measure corresponding to $P_t(i,j)$. Let $\tilde{Y}$ be a process on $S \cup \triangle$ such that $\tilde{X} = \{\tilde{Y}$, killed the first time it hits $\triangle\}$. If $T_\triangle$ denotes the first hitting time of $\triangle$ and u denotes zero potential density of $\tilde{Y}$, then $\Gamma^{-1}_{i,j} = u_{T_\triangle}(i,j)$. Let λ be an exponential random variable. Then $\tau\lambda$ — with $\tau(.)$ denoting inverse local time at $\triangle$ — is a terminal time. Let L^x_t denote the local time of $\tilde{Y}$ at x.

$$\begin{aligned}
E^i(\theta_s \circ L^j_{\tau(\lambda)} 1_{\{\lambda > s\}}) &= E^i(\int_s^{\theta_s \circ \tau(\lambda)} dL^j_t) \\
&= E^i(\int_0^{e^{qs}\tau(\lambda)} d(\theta_{\tau(s)} \circ L^j_t)) \\
&= E^i(\int_0^{e^{qs}\tau(\lambda)} dL^j_{e^{qs}t}) \\
&= E^i(e^{-qs} L^j_{\tau(e^{qs}\lambda)}) \\
&= u_{\tau(\lambda)}(i,j) \qquad (7.3)
\end{aligned}$$

This justifies the use of Kac's moment formula for calculation of the Laplace transform of certain type of local times of X_t.

Theorem 7.2 ([3],Theorem 3.10.1)**.** *Let $\tilde{X}$, λ, $\tau(\lambda)$ be as above and*

$$E^x(\int_0^\infty e^{-\alpha t} dL^y_t) = u^\alpha(x,y).$$

Then

$$E^x(\prod_{k=1}^n L^{y_k}_{\tau(\lambda)}) = \sum_\pi u_{\tau(\lambda)}(x, y_{\pi_1}) \ldots . u_{\tau(\lambda)}(x, y_{\pi_n})$$

where the sum is over all permutations π of $\{1, 2 \ldots n\}$.

In the case where X — and so also Y — is Markov the decomposition of $(G+c)^2$ into two infinitely divisible random variables is possible by the Second Ray-Knight Theorem because of the decomposition: $u_{\tau(\lambda)}(i,j) = u_{T_0}(i,j) + \gamma$. The fact that non-associated Gaussian processes with infinitely divisible squares do not have $(G+c)^2$ infinitely divisible for all c is in itself an evidence that no such decomposition is possible for $\tilde{X}$. It would be interesting to find a decomposition similar to the one above for the processes of the type of Y and analyze the Laplace transform of the corresponding local times to attempt to shed some light on its behavior when λ has small mean which corresponds to the small values of c in $(G+c)^2$.

8 Some numerical examples.

The G-vectors with inverse covariance matrices Γ^{-1} with zero entries and corresponding row sums of different signs indeed exist.
Below is an example for $n = 3$ with one zero entry:

Let

$$\Gamma = \frac{1}{7}\begin{pmatrix} 15 & 4 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 4 \end{pmatrix}; \text{ then } \quad \Gamma^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 8 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Where Γ^{-1} is an M-matrix, so G^2 is infinitely divisible and $\Gamma^{-2}_{1,3} = 2$ and $\Gamma^{-1}_{1,3} = 0$, $D_1 = -1 < 0$, $D_3 = 1 > 0$.

Therefore for a vector with this covariance matrix $\tilde{B}_{i,j} = -2\frac{1}{t^3}$ while $\tilde{A}_{i,j} = \frac{1}{t^4}$ so that $\tilde{C}_{i,j} = \frac{1}{t^4} - 2\alpha^2\frac{1}{t^3}$ is negative for all sufficiently large t and all $\alpha \neq 0$, so the corresponding vector has critical point $\alpha = 0$. Note that Γ does not satisfy (7.1).

Needless to say, examples of such matrices abound in higher dimensions.

The following example illustrates that (7.1) is not a sufficient condition for a vector to be associated:

$$\Gamma = \frac{1}{5}\begin{pmatrix} 8 & 3 & 4 \\ 3 & 5 & 2 \\ 4 & 2 & 4 \end{pmatrix}; \text{ then } \quad \Gamma^{-1} = \frac{1}{12}\begin{pmatrix} 16 & -4 & -14 \\ -4 & 16 & -4 \\ -14 & -4 & 31 \end{pmatrix}.$$

Where Γ satisfies (7.1), Γ^{-1} is an M-matrix and has the first row sum negative – hence our vector is is not associated.

It is possible, of course, for a 3-dimensional vector to satisfy (7.1) and have all positive row sums - i.e. be an associated vector and still have a zero entry in the inverse covariance matrix:

$$\Gamma = \frac{1}{37}\begin{pmatrix} 15 & 4 & 2 \\ 4 & 6 & 3 \\ 2 & 3 & 20 \end{pmatrix}; \text{ then } \quad \Gamma^{-1} = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 8 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

In dimensions 4 and higher the condition (7.1) no longer prevents the Γ^{-1} from having a zero entry at the intersection on the row sums with different signs:

$$\Gamma^{-1} = \begin{pmatrix} 10 & -3 & -3 & 0 \\ -3 & 9 & -2 & -3 \\ -3 & -2 & 9 & -3 \\ 0 & -3 & -3 & \frac{65}{11} \end{pmatrix}, \text{then} \quad \Gamma = \begin{pmatrix} \frac{257}{1400} & \frac{39}{280} & \frac{39}{280} & \frac{99}{700} \\ \frac{39}{280} & \frac{171}{616} & \frac{115}{616} & \frac{33}{140} \\ \frac{39}{280} & \frac{115}{616} & \frac{171}{616} & \frac{33}{140} \\ \frac{99}{700} & \frac{33}{140} & \frac{33}{140} & \frac{143}{350} \end{pmatrix}$$

and satisfies (7.1).

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