Geometric Characterization and Dynamics of Holomorphic Maps

by

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Abstract

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We prove the existence of the canonical Thurston obstruction for sub-hyperbolic semi-rational branched coverings when they are obstructed. Then we geometrically characterize meromorphic maps with exactly two asymptotic values and no critical values. We finish with the proof of non-existence of the invariant line fields for the family $\lambda e^{iz} + \gamma e^{-iz}$.

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Contents

Chapter 1

Introduction

In this thesis, I will present three theorems from my work. Each is part of a collaboration. The first theorem describes the relation between Thurston obstructions and the canonical obstruction for a sub-hyperbolic semi-rational branched covering. The second theorem is a characterization of meromorphic functions with two asymptotic values. The third is a theorem about the non-existence of an invariant line field for a family of entire functions with exactly two critical values that includes the sine family.

In order to state the results of the thesis, we recall some basic notations and give some context and definitions.

Recall that a *surface* is a connected Hausdorff topological manifold of dimension 2 with countable bases.

Definition 1.1. *An orientation-preserving continuous map* $f : X \to Y$ *between surfaces is called a* branched covering *if for any* $y \in Y$*, there is a neighborhood* $V(y)$*, such that for each component* $U(x)$ *of* $f^{-1}(V(y))$*, there are homeomorphisms*

$\phi : U(x) \to \mathbb{D}$ and $\psi : V(y) \to \mathbb{D}$, with $\phi(x) = \psi(y) = 0$ such that $\psi \circ f \circ \phi^{-1}(z) = z^k$, i.e. the following diagram commutes

$$\begin{tikzcd} U(x) \arrow[r, "f"] \arrow[d, "\phi"] & V(y) \arrow[d, "\psi"] \\ \mathbb{D} \arrow[r, "z \mapsto z^k"] & \mathbb{D} \end{tikzcd} .$$

The number $k = k(x)$ is called the local degree *of f at the point x. In particular if $k = 1$ for every $x \in X$, the map f is called a* covering.

If $f : X \to Y$ is a branched covering. The point $x \in X$ such that $k = k(x) > 1$ is called a *critical point* of f and the set $C(f)$, which consists of all the critical points of f, is called the *critical set* of f. The image of a critical point is called a *critical value* of f.

We are most interested in the branched coverings $f : X \to Y$ for which X is simply connected and $Y = S^2$ or S^2 minus finitely many points, and subject the additional condition:

$$f : X \setminus f^{-1}(A) \to S^2 \setminus A$$

is a covering, where A consists of finitely many points. Fix an open disk $D \subset S^2$ that contains only one point a in the set A. If V is a component of $f^{-1}(D \setminus \{a\})$, then $f : V \to D \setminus \{a\}$ is a covering of a ring, and its degree k is independent of the choice of D. The following are possible.

1, If $k = \infty$, then V is simply connected and its boundary consists of a single simple curve in X tending to infinity in both directions. In this case, V defines a *logarithmic singularity* over a, called an *asymptotic tract* of a and the point a is called an *asymptotic value* of f.

2, If $1 < k < \infty$ and there exists a point $x \in X$, such that $\widetilde{V} = V \cup \{x\}$ is an open topological disk. Then $f : \widetilde{V} \to D$ is a branched covering and has local degree of k at x, that is, x is a critical point of f.

3, If $1 < k < \infty$ but there is no point $x \in X$, such that $\widetilde{V} = V \cup \{x\}$ is an open topological disk. Then we can add to X an ideal point x and define the topology on $\widetilde{X} = X \cup \{x\}$ so that it remains a surface. In this way $\widetilde{X}$ is homeomorphic to S^2, $Y = S^2$ and f can be extended to be a branched covering betweens S^2, still denoted as f. Then the cardinality of $f^{-1}(y)$ is a constant number for each $y \in S^2$, say d, and d is called the *degree* of f.

The set of critical values and asymptotic values is called the *singular set* of f and is denoted by $S(f)$.

Rational maps, entire functions and meromorphic functions are holomorphic branched coverings, and the theory of holomorphic dynamics studies the iteration of rational functions, entire function and meromorphic functions.

Any rational map has the form $P(z)/Q(z)$, where $P(z) = a_n z^n + \cdots + a_1 z + a_0$ and $Q(z) = b_m z^m + \cdots + b_1 z + b_0$, neither a_n nor b_m is zero, and $P(z)$ and $Q(z)$ are relatively prime, that is, $(P, Q) = 1$. Then $d = \max\{m, n\}$ is the degree of $R(z)$.

Since for $\lambda \in \mathbb{C}$,

$$\frac{P(z)}{Q(z)} = \frac{\lambda P(z)}{\lambda Q(z)},$$

it follows that any rational map of degree d corresponds to a point

$$[a_d, \cdots, a_0, b_d, \cdots, b_0] \in \mathbb{CP}^{2d+1}.$$

For any two polynomials $P(z) = a_d z^d + \cdots + a_0$ and $Q(z) = b_d z^d + \cdots + b_0$, where both a_d and b_d are not zero, the *resultant* of $P(z)$ and $Q(z)$, denoted by $Res(P, Q)$, is defined by the determinant

$$\begin{vmatrix} a_d & \cdots & a_0 & & \\ & a_d & \cdots a_0 & & \\ & & \vdots & & \\ & & a_d & \cdots & a_0 \\ b_d & \cdots & b_0 & & \\ & b_d & \cdots b_0 & & \\ & & \vdots & & \\ & & b_d & \cdots & b_0 \end{vmatrix}.$$

The following theorem is in Appendix in [Mil].

Theorem 1.1. *Res(P,Q)=0 iff either (1) $P(z)$ and $Q(z)$ share a common root or (2) both $P(z)$ and $Q(z)$ have degree less than d.*

Let $V = \{[a_1, \cdots, a_{d+1}, a_{d+2}, \cdots, a_{2d+2}] \in \mathbb{CP}^{2d+2} \mid Res(P, Q) = 0\}$, where $P(z) = a_1 z^d + \cdots + a_{d+1}$ and $Q(z) = a_{d+2} z^d + \cdots + a_{2d+2}$. Then V is closed subset in $\mathbb{CP}^{2d+1}$, usually called a Zariski closed subset. By Theorem 1.1, the set of rational maps of degree d is embedded into the finite dimensional space $\mathbb{CP}^{2d+2}$ and it is in one-to-one correspondence with the set $\mathbb{CP}^{2d+2} \setminus V$.

A rational map f of degree $d > 1$ is a regular covering at every point except its critical values. The covering property of f forces it to be globally expanding, whereas the presence of the critical points, where the derivative vanishes, makes it locally strongly contracting. The overall behavior of f depends therefore very much on the interplay of these two opposite forces and the orbits of the critical points play an important role in dynamics.

The iteration of a rational map f decomposes $\widehat{\mathbb{C}}$ naturally into the Fatou set (the stable locus) and the Julia set (the chaotic locus). Roughly speaking, the Fatou set is the set where the iterative behavior is relatively tame in the sense that points close to each other behave similarly, while the Julia set is the set where chaotic phenomena take place. By definition, the Fatou set is the set of all points $z \in \widehat{\mathbb{C}}$ such that there is a neighborhood U about z such that $\{f^n|U\}_{n=0}^{\infty}$ is a normal family; that is, every sequence in this family has a convergent subsequence. It is clear that F is an open subset. The complement of the Fatou set F is called the Julia set J; that is, $J = \widehat{\mathbb{C}} \setminus F$. Since F is open, J is compact.

Definition 1.2. *Two maps f and g are said to be* topologically equivalent *if there is a pair of homeomorphisms ϕ and ψ such that $\phi \circ g = \psi \circ f$.*

Definition 1.3. *A* conjugacy $\phi : S^2 \to S^2$ *between two maps f and g is a bijection such that $\phi \circ f = g \circ \phi$.*

An equivalence and conjugacy can be topological, quasiconformal or conformal depending on the quality of ϕ. It is easy to see that two maps are topologically equivalent if they are conjugate.

Recall that for a branched covering f of the S^2, the critical set $C(f)$ of f consists of the points z where the local degree of f is greater than 1. The *post-critical set*, denoted by P_f, is the closure of the set $\cup_{n>0} f^n(C_f)$, i.e.

$$P_f = \overline{\cup_{n>0} f^n(C_f)}.$$

If $\#P_f < \infty$, then f is called *post-critically finite.* If $\#P_f = \infty$ but $\#P'_f < \infty$, then f is called *geometrically finite.* In the latter case, P'_f consists of a finite number of periodic cycles (see Proposition 2.1).

Definition 1.4 (Combinatorial Equivalence)**.** *Suppose f and g are two branched coverings. Then f and g are said to be* combinatorially equivalent *if there exist homeomorphisms $\phi, \psi : S^2 \to S^2$ such that*

(i) the diagram

$$\begin{tikzcd} S^2 \arrow[r, "\phi"] \arrow[d, "f"] & S^2 \arrow[d, "g"] \\ S^2 \arrow[r, "\psi"] & S^2 \end{tikzcd}$$

commutes and

(ii) ϕ is isotopic to ψ rel P_f. That is, there is a continuous map $H(t,x) : [0,1] \times S^2 \to S^2$ satisfying

1) For every $0 \le t \le 1$, $H(t,\cdot) : S^2 \to S^2$ is a homeomorphism;

2) For all $0 \leq t \leq 1$ *and* $x \in P_f$, $H(t,x) = \phi(x) = \psi(x)$;

3) $H(0,x) = \phi(x)$ *and* $H(1,x) = \psi(x)$ *for all* $x \in S^2$.

Definition 1.5. *A rational map f of degree two or more is double covered by an integral torus endomorphism if there is a linear map $L(z) = nz + b$, where n is an integer, and a holomorphic map $\Theta : \mathbb{T} \to \widehat{\mathbb{C}}$ of degree 2, where $\mathbb{T}$ is a torus, such that the following diagram commutes*

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "L"] \arrow[d, "\Theta"] & \mathbb{T} \arrow[d, "\Theta"] \\
\widehat{\mathbb{C}} \arrow[r, "f"] & \widehat{\mathbb{C}}
\end{tikzcd}.$$

The following theorem is in [McM2].

Theorem 1.2. *Let f be a rational map of degree greater than one. Then the following conditions are equivalent.*

1, The post-critical set P_f is disjoint from the Julia set.

2, There is an integer $n > 0$, such that f^n strictly expands the spherical metric on the Julia set.

3, The orbit of any the critical point tends to an attractive or super-attractive cycles (see Definition 2.2).

Definition 1.6. *The map f is* hyperbolic *if any of the equivalent conditions above are satisfied. A hyperbolic map is also sometimes said to be* expanding.

Conjecture 1.1 (Density of Hyperbolicity). *The set of hyperbolic rational maps is an open and dense set in the space of Rat_d of all rational maps of degree d.*

An important aim in holomorphic dynamics is to characterize holomorphic maps among all topological maps with similar dynamical behaviors. In particular, one wants to know if there can be only one such holomorphic map up to conformal conjugation. That is, one asks whether, if two holomorphic maps are topologically conjugate, then the conjugacy is conformal? If they are, the map is called *rigid*. Rigidity also comes up in trying to solve conjecture 1.1. In particular it would be good to know that all non-hyperbolic maps are rigid.

In [T], Thurston gave the first topological characterization of post-critically finite rational maps by using a combinatorial condition called a *Thurston obstruction* (see Definition 2.5). A complete and comprehensive proof of Thurston's Theorem is presented in Douady-Hubbard's work [DH]. Roughly speaking, the set of post-critically finite rational maps (except the rational maps double covered by an integral torus endormorphisms) is in one-to-one correspondence with those homotopy classes of post-critically finite branched self-coverings of two sphere with no Thurston obstructions. This characterization implies that whenever two post-critically finite rational maps are combinatorially equivalent, then they are conformally equivalent. Among all Thurston obstructions that may exist for a branched covering, the *canonical Thurston obstruction*, which is defined in term of the shortest curves in $S^2 \setminus P_f$, is the most interesting. In Pilgrim's work [Pi], the existence of any Thurston obstruction is shown to be equivalent to the existence of the canonical Thurston obstruction.

Thurston's theorem can not be naturally extended to a map with infinite post-critically set or a map with infinite degree because the proof of Thurston's Theorem depends on the finiteness of both the degree and the post-critical set in a crucial way. In Cui-Jiang's work [CJ] (see also [CJS]), a counterexample of a geometrically finite branched covering is found which shows that Thurston's theorem can not hold for the maps with infinite post-critical set. Based on this counterexample, the authors define the concepts of *semi-rational branched coverings* and *sub-hyperbolic semi-rational branched coverings* (see Definition 2.2). They prove that a semi-rational branched covering is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering. This means the characterization of geometrically finite rational maps is reduced to the characterization of the sub-hyperbolic semi-rational maps. To this end in the same paper, they define the CLH-equivalence (combinatorial and locally holomorphic equivalence, see Definition 2.3) among all sub-hyperbolic semi-rational branched coverings. In their work [CJS], they prove a Thurston type rigidity theorem about CLH-equivalence for sub-hyperbolic semi-rational branched coverings. The paper [CJS] was rewritten as [CT] where more explanation was given. A completely different proof of Cui-Jiang-Sullivan's Theorem is given in Jiang-Zhang's work [JZ]. Jiang-Zhang's work provides a new approach to characterize geometrically finite rational maps by following the original idea of Thurston in his work on the characterization of post-critically finite rational maps.

The main idea in Jiang-Zhang's work is summarized as an intermediate step called *bounded geometry* (see Definition 3.3) in the framework given in the survey paper of Jiang [J]. The bounded geometry condition is an analytic condition but is connected with the topological condition, Thurston obstructions, and with the geometric condition, the canonical Thurston obstruction. In the first two parts of this thesis, we will follow this framework to characterize Thurston rigidity in two different families of holomorphic maps.

In the first part of this thesis, Chapter 2, we present our work which originally appeared in [ChJi1] about the existence of the canonical Thurston obstruction for a sub-hyperbolic semi-rational branched covering in the case it has a Thurston obstruction.

Theorem 1.3 (Existence of Canonical Thurston Obstruction)**.** *Suppose f is a sub-hyperbolic semi-rational branched covering. Let Γ_c denote the set of all homotopy classes of non-peripheral curves γ in $\widehat{\mathbb{C}} \setminus Q$ (for Q see 2.3) such that $l(\gamma, x_n) \to 0$ as $n \to \infty$ for any initial $x_0 \in T_f$ (see Definition 2.8). Then we have that either*

- *(a) $\Gamma_c = \emptyset$, and f is CLH-equivalent to a sub-hyperbolic rational map, or*
- *(b) $\Gamma_c \neq \emptyset$ is a Thurston obstruction for f and f is not CLH-equivalent to a rational map. In this case, we call Γ_c the canonical Thurston obstruction for f.*

The main idea of the proof of the theorem above is that we associate each f with a Teichmüller space of a Riemann surface of finite type but not analytic finite type and a Teichmüller space of an analytically finite Riemann surface. We show that the short geodesics on both surfaces are comparable. This implies the existence of the canonical Thurston obstruction.

Entire or meromorphic functions with finitely many singular values are the natural extension of rational maps, since, like rational maps, they have a natural embedding into finitely dimensional spaces. In the Chapters 3 and 4, we investigate two families of entire or meromorphic functions with exactly 2 singular values. One is a family of meromorphic maps with 2 asymptotic values but no critical values and the other one is a family of entire maps with 2 critical values and no asymptotic values.

In the second part of this thesis, Chapter 3, we present our work that originally appeared in [CJK] about the geometric characterization of post-singularly meromorphic functions with two asymptotic values.

Recall that a point v is called an asymptotic value of a branched covering f if there is neighborhood V of v, such that there is a simply connected component U of $f^{-1}(V \setminus \{v\})$ and $f : U \to V \setminus \{v\}$ is a universal covering. Before we present our theorem, we give some examples and definitions.

Example 1.1. *Let $g(z) = e^z$. The range of g is the whole sphere except $\{0, \infty\}$. We can check that $g^{-1}(\mathbb{D}^*) = LH$ and $g^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}) = \mathbb{C} \setminus \overline{LH}$, where*

$$\mathbb{D}^* = \{z \neq 0 \mid |z| < 1\} \text{ and } LH = \{z \in \mathbb{C} \mid \Re(z) < 0\},$$

and $g : LH \to \mathbb{D}^*$ *and* $g : \mathbb{C} \setminus \overline{LH} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ *are holomorphic universal coverings. Therefore* 0 *and* ∞ *are asymptotic values of* g*. In fact, all the maps which have exactly two omitted asymptotic values are in the form*

$$\frac{ae^{\lambda z}+b}{ce^{\lambda z}+d},$$

where $a, b, c, d \in \mathbb{C}, ad - bc = 1$ *and* $\lambda \in \mathbb{C}^*$*. Thus the family of tangent functions*

$$\lambda \tan z = \lambda i \frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = \frac{\frac{\lambda i}{\sqrt{-2\lambda i}}e^{2iz} + \frac{\lambda i}{\sqrt{-2\lambda i}}}{\frac{1}{\sqrt{-2\lambda i}}e^{2iz} - \frac{1}{\sqrt{-2\lambda i}}}$$

also has two asymptotic values.

Definition 1.7. *For any map* f *with two omitted asymptotic values* $\{a,\ b\}$*, the* post-singular *set is defined as*

$$PS_f = \overline{\cup_{n\geq 0} f^n(\{a,b\})} \cup \{\infty\}.$$

If $\#PS_f < \infty$*, then* f *is called* post-singularly finite. *The set of the meromorphic functions with two asymptotic values is denoted by* $\mathcal{M}_2$*.*

In Definition 3.1, we define the space $\mathcal{AV}2$ of topological meromorphic maps with two asymptotic values and prove the following theorem.

Theorem 1.4 (Geometric Property of $\mathcal{M}_2$)**.** *A mapping* $f \in \mathcal{AV}2$ *is, up to affine conjugation, combinatorially equivalent to a unique meromorphic map in* $\mathcal{M}_2$ *if and only if* f *has bounded geometry.*

The sketch of the proof is that: Following Thurston, each map f is associated with a Teichmüller space T_f and a self-map σ_f of T_f which is contracting. The map f is combinatorially equivalent to a map in $\mathcal{M}_2$ if and only if the self-map σ_f has a unique fixed point. In order to prove the existence of the unique fixed point, we choose any point $\tau_0 \in T_f$, and consider $\tau_n = \sigma_f^n(\tau_0)$, the orbit of τ_0. Each τ_n determines a meromorphic map g_n topologically equivalent to f. The topological equivalence defines a topological constraint for all g_n. Using this topological constraint and the bounded geometry hypothesis, we show the set $\{\tau_n\}$ is in a compact set of T_f, which in turn implies the existence of the unique fixed point of σ_f in T_f.

Finally, in the third part of this thesis, Chapter 4, we present our work that originally appeared in [CJZ] about the non-existence of an invariant line field for a family of entire functions with two critical values but no asymptotic values; this family includes the sine family. Before we present our theorem, we first introduce some results and definitions.

An approach to the hyperbolic density conjecture is developed in [MSS] and [McS] , using the quasiconformal map. This approach has the advantage of shifting the focus from a family of maps to the dynamics of a single map, and leads to the *No Invariant Line Fields* conjecture which implies the density of hyperbolic maps conjecture. In order to state the conjecture, we need to the following definition.

Definition 1.8. *Let f be a holomorphic function and $B \subset \mathbb{C}$ be a set. A (non-trivial) f-invariant* line field *supported on B is a measurable Beltrami differential*

$$\mu = \mu(z)\frac{d\overline{z}}{dz}$$

where $\mu(z)$ is a measurable complex function such that

1) for almost every $z \in \mathbb{C}$, $|\mu(z)| = 1$ or 0 and

2) the support $supp(\mu) = \{z \mid |\mu(z)| = 1\} \subset B$ has positive Lebesgue measure.

3) $f^*\mu = \mu$ a.e., that is,

$$\mu(f(z))\frac{\overline{f'(z)}}{f'(z)} = \mu(z)$$

if $f'(z) \neq 0$.

Conjecture 1.2 (No Invariant Line Fields)**.** *A rational map f carries no invariant line field on its Julia set, except when f is double covered by an integral torus endomorphism.*

In the setting of transcendental functions, the situation is less clear. Indeed, it is known [EL1] that there exists ”pathological” entire transcendental functions that support invariant line fields on their Julia sets. So it makes sense to try to find a family of entire functions for which the conjecture holds.

Recall that for an entire function or meromorphic function f, the singular set $S(f)$ is the set of critical values and asymptotic values. The *Eremenko-Lyubich class* $\mathcal{B}$ consists of the set of all entire functions f such that $S(f) \cap \mathbb{C}$ is bounded.

Definition 1.9. *Let f be an entire function. The escaping set I of f is*

$$I = \{z \in \mathbb{C} \mid f^k(z) \to \infty \text{ as } k \to \infty\}.$$

For any map $f \in \mathcal{B}$, we know that $I \subseteq J$ by Theorem 1 in [EL2].

We consider the family $F_{\lambda,\gamma}$ of functions of the form

$$f_{\lambda,\gamma}(z) = \lambda e^{iz} + \gamma e^{-iz}, \quad \lambda, \gamma \in \mathbb{C}^*.$$

It contains the sine family

$$f_\lambda(z) = \lambda \sin z, \quad \lambda \in \mathbb{C}^*.$$

By Theorem A.1 in [McM3], we know that the family $F_{\lambda,\gamma}$ contains all maps quasiconformally conjugate to f_λ. For any $\lambda,\ \gamma$, the set of critical points of $f_{\lambda,\gamma}$, which we denote by C, consists of all points z such that

$$\tan z = i\frac{\lambda - \gamma}{\lambda + \gamma}.$$

Let c_0 be the point in the strip $\{z \in \mathbb{C}^*, 0 \le \Re z < \pi\} \cap C$. Then

$$C = \{c_0 + n\pi \mid n \in \mathbb{Z}\}.$$

While $f_{\lambda,\gamma}$ has infinitely many critical points, it has exactly two critical values

$$\pm\frac{4\lambda\gamma}{\lambda + \gamma}\cos c_0 = \pm\frac{4\lambda\gamma}{\lambda + \gamma}\sqrt{\frac{1}{1 - (\frac{\lambda - \gamma}{\lambda + \gamma})^2}}.$$

Moreover, $f_{\lambda,\gamma}$ has no finite asymptotic values. So we conclude that $f_{\lambda,\gamma}$ is in the *Eremenko-Lyubich class* $\mathcal{B}$ and so $I \subset J$. In the case of the sine family $f_\lambda = \lambda \sin z$, McMullen proved in [McM1] that I has positive Lebesgue measure, which implies that the Julia set J has positive Lebesgue measure.

Our third result is a partially answer to the comment "*If one was able to furnish* *the proof by a direct proof ....*" of the following theorem in [R]: *Any function* $f \in \mathcal{B}$ *supports no invariant line field on* I. We give an explicit and dynamical proof of the absence of line fields on the escaping set for $f = f_{\lambda,\gamma}$ by using the expansion of f on the escaping set I. Our result is

Theorem 1.5 (No Invariant Line Fields)**.** *For any* $\lambda, \gamma \in \mathbb{C}^*$*, the escaping set* I *supports no* $f_{\lambda,\gamma}$*-invariant line fields.*

Chapter 2

Canonical Thurston Obstruction

2.1 Branched Coverings

Recall that a branched covering $f : S^2 \to S^2$ of degree more than 1 is called geometrically finite if $\#P_f = \infty$ and $\#P'_f < \infty$, where P'_f is the set of accumulation points of P_f.

Definition 2.1. *A point p such that $f^n(p) = p$ for some $n \geq 1$ is a* periodic point *for f. The least such n is the* period *of p. If p is not a periodic point, and $f^i(p) = f^j(p)$ for some $i > j > 0$, then p is said to be* eventually periodic.

Proposition 2.1. *Let f be a geometrically finite map. Then every point in P'_f is periodic.*

Proof. Given any $p \in P'_f$, there is a subsequence $\{q_n\}_{n=1}^{\infty}$ of distinct points in P_f such that

$$\lim_{n \to \infty} q_n = p.$$

Since the local degree, $1 \leq \deg_p f \leq d$, $\{f(q_n)\}_{n=1}^{\infty}$ is also a subsequence of infinitely many distinct points in P_f. Since f is continuous,

$$\lim_{n\to\infty} f(q_n) = f(p) \in P'_f.$$

This implies that

$$f(P'_f) \subseteq P'_f.$$

Since P'_f contains only finitely many points, every point in P'_f is periodic or eventually periodic, that is, for every $p \in P'_f$, there are minimal integers $l \geq 0$ and $k \geq 1$ such that

$$f^l(p) = f^{l+k}(p).$$

If $l = 0$, then p is periodic. So we need to prove that $l = 0$ for every $p \in P'_f$.

Suppose there is a $p \in P'_f$ such that $l > 0$. Assume $p_i = f^{i-1}(p)$, $1 \leq i \leq l+k$. Then $O = \{p_{l+1}, \cdots, p_{l+k}\}$ is a periodic cycle with k distinct points.

Suppose $d_s(\cdot,\cdot)$ is the spherical distance on S^2. Let

$$B_\epsilon(x) = \{y \in S^2 \mid d_s(x,y) < \epsilon\}$$

be the disk of radius $\epsilon > 0$ centered at x. Then we have a small number $\epsilon > 0$ such that

$$B_\epsilon(x) \cap B_\epsilon(y) = \emptyset, \quad \forall x \neq y \in P'_f$$

and

$$f(B_\epsilon(x)) \cap B_\epsilon(y) = \emptyset, \quad \forall x \in O,\ y \in P'_f \setminus O.$$

Now let $c \in \Omega_f$ be a critical point such that the subsequence $\{q_n = f^{m_n}(c)\}$

tends to $p = p_1$ as n goes to ∞. Then

$$\lim_{n\to\infty} f^{m_n+i}(c) = p_i, \quad l+1 \leq i \leq l+k.$$

From this and the fact that

$$f(B_\epsilon(x)) \cap B_\epsilon(y) = \emptyset, \quad \forall x \in O,\ y \in P'_f \setminus O,$$

we conclude that

$$\{f^i(c)\}_{i=N}^\infty \subset \cup_{x\in O} B_\epsilon(x)$$

for some large N. (Otherwise, there would be infinitely many points of $\{f^i(c)\}_{i=1}^\infty$ outside $\cup_{x\in P'_f} B_\epsilon(x)$. This would give an extra accumulation point other than those in P'_f.) This contradicts our choice of c which has the property that

$$\lim_{n\to\infty} f^{m_n}(c) = p.$$

The contradiction comes from our assumption that $l > 0$. Therefore, $l = 0$ for all $p \in P'_f$. This completes the proof. □

Definition 2.2. *Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a geometrically finite branched covering of degree $d \geq 2$. We say f is a* semi-rational branched covering *if f is holomorphic in a neighborhood of P'_f. Moreover a semi-rational branched covering is said to be* sub-hyperbolic semi-rational *if each cycle $< p_0, \cdots, p_{k-1} >$ of period $k \geq 1$ in P'_f is either attractive, that is, $0 < |(f^k)'(p_0)| < 1$, or super-attractive, that is, $(f^k)'(p_0) = 0$.*

Definition 2.3. *Suppose f and g are two sub-hyperbolic semi-rational branched coverings. We say that they are* CLH-equivalent *if there exists a pair of homeomorphisms $\phi,\ \varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

1. *ϕ is isotopic to φ rel P_f,*
2. *$\phi \circ f = g \circ \varphi$, and*
3. *$\phi|U_f = \varphi|U_f$ is holomorphic on some open set $U_f \supset P'_f$.*

Definition 2.4. *A function f is called K-quasiregular in a plane domain G if it admits a representation*

$$f = \phi \circ w,$$

where $w : G \to G'$ is a K-quasiconformal homeomorphism and ϕ is non-constant analytic function in G. A function is quasiregular if it is K-quasiregular for some K. Moreover a function f is quasiregular on $\widehat{\mathbb{C}}$ if $f = g \circ w$, where g is rational map and w is a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$.

In order to prove that for any branched covering, there is a quasiregular map in its CLH-equivalent class, we need some preparations.

A *quasicircle* in the Riemann sphere is the image of the unit circle under a quasiconformal map of the plane. An orientation preserving homeomorphism f of the unit circle $S^1 = \{e^{it}\}$ is called to be *quasisymmetric* if there exists a constant M, such that

$$\frac{1}{M} < \left|\frac{f(e^{i(x+t)}) - f(e^{ix})}{f(e^{it}) - f(e^{i(x-t)})}\right| < M$$

for all x and all $|t| < \frac{\pi}{2}$. It is easy to see that the composition of two quasisymmetric maps is still quasisymmetric, and any real analytic map on S^1 is also quasisymmetric.

By Beurling-Ahlofors Extension Theorem in [BA] or [Leh], we know that for any quasisymmetric map h, there exists a quasiconformal self-mapping f of the disk which has boundary values h. Moreover the map f can be chosen to be diffeomorphic, or even real analytic.

Lemma 2.1. *Let $h : S^1 \to S^1$ be an orientation preserving homeomorphism of the unit circle. Assume that h can be extended to a quasiconformal map f on an inner neighborhood B of S^1 (i.e. $\{z \mid 1-\epsilon < |z| < 1\} \subset B$ for some $\epsilon > 0$). Then h is quasisymmetric.*

Proof. Denote by μ the Beltrami coefficient of f. Denote by $\mathbb{D}$ the unit disk. Let $\nu = \mu$ on B and $\nu = 0$ on $\mathbb{D} \setminus B$. Then there is a quasiconformal map g of $\mathbb{D}$ whose Beltrami differential is ν. On the other hand, $f \circ g^{-1}$ is holomorphic on $g(B)$. Therefore $f \circ g^{-1}$ is real analytic on S^1, and in particular quasisymmetric. So $h = (f \circ g^{-1}) \circ g|_{S^1}$ is also quasisymmetric.

□

Lemma 2.2. *Let $h : \gamma_1 \to \gamma_2$ be a homeomorphism between two quasicircles γ_1 and γ_2 in $\widehat{\mathbb{C}}$. If h can be extended to a quasiconformal map on a one side of a neighborhood of γ_1, then h can be extended to a global quasiconformal map of $\widehat{\mathbb{C}}$. Moreover, the extension can be chosen to be a diffeomorphism from $\widehat{\mathbb{C}} \setminus \gamma_1$ onto $\widehat{\mathbb{C}} \setminus \gamma_2$.*

Proof. Fix $i = 1, 2$. By the definition of quasicircles, there is a quasiconformal map ϕ_i of $\widehat{\mathbb{C}}$ such that $\phi_i(\gamma_i) = S^1$. Furthermore, ϕ_i can be chosen to be a diffeomorphism on $\widehat{\mathbb{C}} \setminus \gamma_i$ as follows: Set $\Delta = \phi_i^{-1}(\mathbb{D})$. Let $\psi : \Delta \to \mathbb{D}$ be a conformal map. Then $\phi_i \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$ is a quasiconformal map. Thus its boundary map is quasisymmetric. By the result of Beurling-Ahlfors, $\phi_i \circ \psi^{-1}$ restricted on S^1 admits a diffeomorphism η on $\mathbb{D}$. Now $\eta \circ \psi|_\Delta$ is again a diffeomorphism, whose boundary map is $\phi_i|_{S^1}$.

Set $h_1 = \phi_2 \circ h \circ \phi_1^{-1}$. Then h_1 is quasisymmetric by Lemma 2.1 and thus has a quasiconformal extension to $\widehat{\mathbb{C}}$. Moreover its extension can be chosen to be a diffeomorphism outside S^1. Thus $h = \phi_2^{-1} \circ h_1 \circ \phi_1$ can be extended to be a quasiconformal map of $\widehat{\mathbb{C}}$ and a diffeomorphism outside γ_1. □

Lemma 2.3. *Let $U_i \subset \widehat{\mathbb{C}}$ (i=1,2) be a pair of domains such that each ∂U_i, (i=1,2) consists of $p > 0$ disjoint quasicircles. Let $P \subset U_1$ be a finite (or empty) set. Let $f : U_1 \to U_2$ be an orientation preserving homeomorphism. If $f|_{\partial U_1}$ can be extended to a quasiconformal map on a one sided neighborhood of each curve of ∂U_1, then there is a quasiconformal map in the isotopy class of f rel $\partial U_1 \cup P$.*

Proof. By Lemma 2.2, $f|_{\partial U_1}$ can be extended to a quasiconformal map g on a small neighborhood W of ∂U_1 such that g is differentiable on $W \setminus \partial U_1$. Let $V_1 \subset U_1$ be a domain such that the boundary consists of disjoints quasicircles in W and $U_1 \setminus V_1 \subset W$. Then $g|_{\partial V_1}$ is a diffeomorphism and can be extended to a diffeomorphism isotopic to f rel $\partial U_1 \cup P$. □

Lemma 2.4. *Let f be a sub-hyperbolic semi-rational branched covering with $P'_f \neq \emptyset$;*

then f is a CLH-equivalent to a quasiregular semi-rational map.

Proof. Consider f as a branched covering from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$. There is a unique complex structure χ on $\widehat{\mathbb{C}}$ such that $f : (\widehat{\mathbb{C}}, \chi) \to \widehat{\mathbb{C}}$ is holomorphic. The uniformization theorem provides a conformal homeomorphism $\xi : (\widehat{\mathbb{C}}, \chi) \to \widehat{\mathbb{C}}$. Set $R := f \circ \xi^{-1}$. Then $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic with respect to the standard complex structure, and therefore is a rational map.

Let $U \subset \widehat{\mathbb{C}}$ be a finite union of quasidisks with pairwisely disjoint closures, such that $P'_f \subset U$, $\overline{U} \subset f^{-1}(U)$, ∂U does not contain the critical points of f, and f is holomorphic in a neighborhood of $\overline{U}$. Since $f = R \circ \xi$, one sees that the homeomorphism ξ is holomorphic in a neighborhood of ∂U with respect the standard complex structure. It follows that $\xi(\partial U)$ consists of infinitely many pairwisely disjoint quasicircles.

Set $L = \widehat{\mathbb{C}} \setminus U$. Then by Lemma 2.3 there is a quasiconformal homeomorphism $\eta : L \to \eta(L)$ such that $\eta = \xi$ on $\partial L \cup (P_f \cap L)$ and η is isotopic to ξ rel $\partial L \cup (P_f \cap L)$. Set $\zeta = \eta^{-1} \circ \xi$ on L and $\zeta = id$ on U. Then ζ is isotopic to the identity rel $\overline{U} \cup P_f$, so $f \circ \zeta^{-1}$ is CLH-equivalent to f. But $f \circ \zeta^{-1} = R \circ \eta$ on L, with η quasiconformal and R holomorphic. One sees that $f \circ \zeta^{-1}$ is quasiregular in L. Moreover on $U = \widehat{\mathbb{C}} \setminus L$, the map $f \circ \zeta^{-1}$ is equal to f and therefore is holomorphic. Thus $f \circ \zeta^{-1}$ is quasiregular on the entire sphere. □

Suppose f is a sub-hyperbolic semi-rational branched covering. Recall that $P'_f = \{a_i\}$ is the set of accumulation points of P_f. Then every a_i is periodic by Proposition

2.1. There exists a collection of a finite number of open disks

$$\Lambda = \{D_i\} \tag{2.1}$$

centered at $\{a_i\}$ and a collection of a finite number of annuli $\{A_i\}$ (we call them the *shielding rings*) such that

(i) $\overline{A_i} \cap P_f = \emptyset$;

(ii) $A_i \cap D_i = \emptyset$, but one of the components of ∂A_i is the boundary of D_i;

(iii) $(\overline{D_i \cup A_i}) \cap (\overline{D_j \cup A_j}) = \emptyset$ for $i \neq j$;

(iv) f is holomorphic on $\overline{D_i} \cup A_i$; and

(v) every $f(\overline{D_i} \cup A_i)$ is contained in D_{i+1} for $1 \leq i \leq k-1$ and $f(\overline{D_k} \cup A_k)$ is contained in D_1 where k is the period of a_i.

Set $D = \cup_i D_i$ and

$$P_1 = P_f \setminus D. \tag{2.2}$$

Since the a_i are accumulation points of P_f, it follows that $\sharp P_1$ is finite. Without loss of generality, we assume that 0, 1, and ∞ belong to P_1. Define

$$Q = P_1 \cup \overline{D} \ \text{ and } \ X = \partial Q = P_1 \cup \partial D. \tag{2.3}$$

2.2 Thurston obstructions

Suppose f is a sub-hyperbolic semi-rational branched covering. Let Q be the set we defined in (2.3). Then

$$f : \widehat{\mathbb{C}} \setminus f^{-1}(Q) \longrightarrow \widehat{\mathbb{C}} \setminus Q$$

is a covering map of finite degree. If γ is a simple closed curve in $\widehat{\mathbb{C}} \setminus Q$, then all the components of $f^{-1}(\gamma)$ are simple closed curves in $\widehat{\mathbb{C}} \setminus f^{-1}(Q)$, which is a subset of $\widehat{\mathbb{C}} \setminus Q$. Thus all the components of $f^{-1}(\gamma)$ are simple closed curves in $\widehat{\mathbb{C}} \setminus Q$.

A simple closed curve γ is said to be *non-peripheral* if each component of $\widehat{\mathbb{C}} \setminus \gamma$ contains at least two points of Q. A *multi-curve*

$$\Gamma = \{\gamma_1, \ \cdots, \ \gamma_n\} \tag{2.4}$$

is a set of finitely many pairwise disjoint, non-homotopic, and non-peripheral curves in $\widehat{\mathbb{C}} \setminus Q$. For each multi-curve Γ in (2.4), let

$$\mathbb{R}^{\Gamma} = < \gamma_1, \cdots, \gamma_n >$$

be the real vector space of dimension n with a basis Γ. We define a linear transformation

$$f_{\Gamma} : \mathbb{R}^{\Gamma} \to \mathbb{R}^{\Gamma}$$

as follows: For each $\gamma_j \in \Gamma$, let $\gamma_{i,j,\alpha}$ denote the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\widehat{\mathbb{C}} \setminus Q$ and let $d_{i,j,\alpha}$ be the degree of $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$. Define

$$f_\Gamma(\gamma_j) = \Sigma_i \Big(\Sigma_\alpha \frac{1}{d_{i,j,\alpha}} \Big) \gamma_i.$$

Let A_Γ be the corresponding matrix, that is

$$f_\Gamma \mathbf{v} = A_\Gamma \mathbf{v}, \quad \mathbf{v} \in \mathbb{R}^\Gamma.$$

Since the matrix A_Γ is non-negative, by the Perron-Frobenius Theorem, there exists a maximal non-negative eigenvalue $\lambda(A_\Gamma)$ which is the spectral radius of A_Γ.

A multi-curve Γ is said to be *f-stable* if for any $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic to an element of Γ rel Q.

Definition 2.5. *A stable multi-curve* Γ *is called a* Thurston obstruction *for* f *if* $\lambda(A_\Gamma) \geq 1$.

2.3 Non-negative matrices

Since a Thurston obstruction is determined by a non-negative matrix, we give a brief review of some results in the theory of non-negative matrices. We use [Gan] as a reference.

A non-negative $n \times n$ matrix A is called *irreducible*, if no permutation of the indices places the matrix in a block lower-triangular form. More precisely, there is no permutation matrix P, which is a matrix consisting of 0's and 1's such that each row or each column contains one and only one 1, and such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices. An equivalent definition of irreducibility is that for any $1 \leq i,\ j \leq n$, there exists a $0 \leq q = q(i,j) \leq n$ such that the ij-th entry of A^q is positive.

For the n-dimensional vector space $\mathcal{V}$, we will use the norm

$$\|\mathbf{v}\| = \max_{1\leq i\leq n} |v_i|, \quad \mathbf{v} = (v_1, \cdots, v_n) \in \mathcal{V}. \tag{2.5}$$

For any linear map $L : \mathcal{V} \to \mathcal{V}$, let A be the corresponding matrix for L, and define

$$\|A\| = \sup_{\|\mathbf{v}\|=1} \|A\mathbf{v}\|.$$

The spectral radius $\lambda(A)$ of A can be calculated as

$$\lambda(A) = \lim_{n\to\infty} \sqrt[n]{||A^n||} \geq 0.$$

If A is non-negative, the Perron-Frobenius Theorem implies that $\lambda(A)$ is an eigenvalue of A. Thus it is a maximal eigenvalue of A. If A is irreducible, $\lambda(A)$ is a simple, positive, maximal eigenvalue with a positive eigenvector $\mathbf{v} = (v_1, \cdots, v_n)$, i.e, $v_i > 0$ for all $1 \leq i \leq n$. However, there may exist another eigenvalue $\mu \neq \lambda(A)$ with $|\mu| = \lambda(A)$. For example, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is an irreducible matrix. The spectral radius is 1; it is a simple, positive, maximal eigenvalue with an eigenvector $\mathbf{v}_1 = (1, 1)$. However, -1 is also an eigenvalue with an eigenvector $\mathbf{v} = (1, -1)$. But if A is positive, that is, every entry is a positive number, the Perron-Frobenius theorem states that $\lambda(A)$ is a unique, simple, positive, maximal eigenvalue with a positive eigenvector $\mathbf{v} = (v_1, \cdots, v_n)$, i.e, $v_i > 0$ for all $1 \leq i \leq n$. Here the term "unique" means that all other eigenvalues μ of A satisfy

$$|\mu| < \lambda(A).$$

Definition 2.6. *We say that a multi-curve* Γ *is* irreducible *if the corresponding matrix* A_Γ *of the linear map* $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ *is irreducible.*

For any non-negative matrix A, we can rearrange the order of the basis such that

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix} \tag{2.6}$$

and all the blocks A_{jj} on the diagonal are either irreducible or 0 matrices. It is not hard to calculate that

$$\lambda(A) = \max_j \lambda(A_{jj}).$$

Now we consider $A = A_\Gamma$ as the corresponding matrix of the linear map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ for an f-stable multi-curve $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$. We assume that A is in the form of (2.6). Then we can use Γ_j to denote the subset of curves in Γ corresponding to the j-th block in A. That is, $A_{jj} = A_{\Gamma_j}$. It is clear that

$$\Gamma = \cup_j \Gamma_j.$$

We call $\{\Gamma_j\}$ the irreducible decompositions of Γ. Note that Γ_j may not be f-stable.

Denote

$$\Gamma_{Ob} = \cup_j \Gamma_j$$

where the union runs over all j such that $\lambda(A_{jj}) \geq 1$. We have the following definition to relate every element in Γ to Γ_{Ob} whenever it is not empty.

Definition 2.7. *Suppose Γ is an f-stable multi-curve. For every $\gamma \in \Gamma$, if there exists a $\gamma_{ob} \in \Gamma_{Ob}$ and an integer $k \geq 0$ such that γ is homotopic to a component $f^{-k}(\gamma_{ob})$, then we define the depth of γ with respect to Γ to be the least such integer k. Otherwise, we define the depth as ∞. The set of all elements in Γ with finite depth is denoted by Γ_0. The set of all elements with infinite depth is denoted by Γ_∞.*

Then

$$\Gamma = \Gamma_0 \cup \Gamma_\infty.$$

It is clear that if Γ is a Thurston obstruction, then Γ_0 is non-empty. Moreover, we have

Lemma 2.5. *If Γ is a Thurston obstruction, then Γ_0 is also a Thurston obstruction. In particular, under a permutation of the basis, we can write*

$$A_\Gamma = \begin{pmatrix} A_{\Gamma_\infty} & 0 \\ \bigstar & A_{\Gamma_0} \end{pmatrix} \tag{2.7}$$

where $\lambda(A_{\Gamma_\infty}) < 1$ *and* $\lambda(A_\Gamma) = \lambda(A_{\Gamma_0}) \geq 1$.

Proof. First, for every curve $\gamma \in \Gamma_0$, there exists an integer $k \geq 0$ and an element $\gamma_{ob} \in \Gamma_{Ob}$ such that γ is homotopic to a component of $f^{-k}(\gamma_{ob})$. It follows that any non-peripheral component $\widetilde{\gamma}$ of $f^{-1}(\gamma)$ is homotopic to a component of $f^{-(k+1)}(\gamma_{ob})$. Since Γ is f-stable, then there exists an element $\gamma_i \in \Gamma$ which is homotopic to $\widetilde{\gamma}$. Therefore, any non-peripheral component of $f^{-1}(\gamma)$ is homotopic to an element $\gamma_i \in \Gamma$ whose depth is at most $k+1$. This implies that $\gamma \in \Gamma_0$. Thus Γ_0 is f-stable.

Let us write $\Gamma_\infty = \{\gamma_1, \cdots, \gamma_s\}$. Then $\Gamma_0 = \{\gamma_{s+1}, \cdots, \gamma_n\}$. Since Γ_0 is f-stable, A_Γ must be of the form of (2.7). Furthermore, since $\Gamma_{Ob} \subset \Gamma_0$, we have that

$$\lambda(A_{\Gamma_\infty}) < 1 \quad \text{and} \quad \lambda(A_{\Gamma_0}) = \lambda(A_\Gamma) \geq 1.$$

□

For each disk D_i in Λ, we take a point b_i on the boundary ∂D_i. Set

$$E = P_1 \cup (\cup_i \{a_i, b_i\}). \tag{2.8}$$

Let $p = \sharp E$. It is obvious that every multi-curve Γ in $\widehat{\mathbb{C}} \setminus Q$ is a multi-curve in $\widehat{\mathbb{C}} \setminus E$. It follows that there are only a finite number of possible matrices for all linear transformations f_Γ (refer to [DH, Lemma 1.2]). (There are infinitely many possible multi-curves Γ.) Therefore, we have

Proposition 2.2. *There is a number* $0 < \beta \leq 1$ *depending only on the degree* d *of* f *and the cardinality* p *of* E *such that for any irreducible multi-curve* Γ *in* $\widehat{\mathbb{C}} \setminus Q$

(not necessarily f-stable) with $\lambda(A_\Gamma) \geq 1$, if $\mathbf{v}$ is the unique positive eigenvector of A_Γ corresponding to $\lambda(A_\Gamma) \geq 1$ with $\|\mathbf{v}\| = 1$, then the smallest coordinate of $\mathbf{v}$ is bounded below by β.

Proof. Since there are only finitely many possible matrices for all irreducible multi-curves, there are finitely many simple, positive, maximal eigenvalues. Thus there are finitely many positive eigenvectors $\mathbf{v}$ with $\|\mathbf{v}\| = 1$. This proves the proposition. $\square$

Proposition 2.3. *There exists a positive integer m such that for any non-empty f-stable multi-curve Γ, if it is a Thurston obstruction,*

$$||A^m_{\Gamma_\infty}|| < 1/2.$$

Proof. Since there are only finitely many matrices A_Γ corresponding to all Γ's, there are only finitely many A_{Γ_∞}'s. For each A_{Γ_∞}, $\lambda(A_{\Gamma_\infty}) < 1$. So there is an integer $m > 0$ such that

$$||A^m_{\Gamma_\infty}|| < 1/2.$$

$\square$

Every multi-curve Γ can contain at most $p - 3$ curves, so we have that

Proposition 2.4. *There is a positive integer M depending on p such that for any f-stable multi-curve Γ in $\widehat{\mathbb{C}} \setminus Q$, the depth of every $\gamma \in \Gamma_0$ is less than or equal to M.*

2.4 Teichmüller space and short geodesics.

Suppose f is a sub-hyperbolic semi-rational branched covering. Recall Q and P_1 as defined in (2.2) and (2.3) and the assumption that $0,\ 1,\ \infty \in P_1$. Let $\mathcal{M}(\mathbb{C})$ be the unit ball of the space $L^\infty(\mathbb{C})$. That is, it is the set of all measurable functions μ on $\mathbb{C}$ such that the essential supremum norm $\|\mu\|_\infty < 1$. Each element $\mu \in \mathcal{M}(\mathbb{C})$ is called a Beltrami coefficient since the measurable Riemann mapping theorem [AB] says that the Beltrami equation

$$\phi_{\overline{z}} = \mu\phi_z \tag{2.9}$$

has a unique quasiconformal self-map ϕ^μ of $\widehat{\mathbb{C}}$ fixing 0, 1, and ∞ as a solution and depends holomorphically on $\mu \in \mathcal{M}(\mathbb{C})$. The map ϕ^μ is called the normalized solution of the equation 2.9.

Definition 2.8. *The Teichmüller space T_f is the equivalence class $[\mu]$ for $\mu \in \mathcal{M}(\mathbb{C})$ satisfying $\mu|Q = 0$ a.e., where μ_1 and μ_2 are equivalent if and only if ϕ^{μ_1} is isotopic to ϕ^{μ_2} rel Q. Furthermore, we can define the Teichmüller distance between two points $x = [\mu]$ and $y = [\nu]$ in T_f as*

$$d_T(x, y) = \frac{1}{2} min_{\widetilde{\mu}\in[\mu],\widetilde{\nu}\in[\nu]} \log K[\phi^{\widetilde{\mu}} \circ (\phi^{\widetilde{\nu}})^{-1}]$$

where $K[\phi]$ is the maximal dilation of the quasiconformal map ϕ.

From [Li] (or [JZ]), we know that T_f is the Teichmüller space of Riemann surfaces

$\widehat{\mathbb{C}} \setminus Q$ with boundary ∂Q. It is a complex Banach manifold and the projective map

$$\Phi : \mathcal{M}(\mathbb{C}) \to T_f$$

is a holomorphic split submersion.

Moreover, since f is a quasiregular map, it induces a self-linear map f^* on $M(\mathbb{C})$ by $\mu \to f^*\mu$ by the formula,

$$(f^*\mu)(z) = \frac{\mu_f(z) + \mu(f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)},$$

where

$$\theta(z) = \frac{\overline{f_z}}{f_z} \ \ \textit{and} \ \ \mu_f(z) = \frac{f_{\bar{z}}}{f_z}.$$

Since $f(Q) \subset Q$, it follows that if $[\mu_1] = [\mu_2]$ in T_f, then $[f^*(\mu_1)] = [f^*(\mu_2)]$ in T_f. In this way f^* induces a well-defined map $\sigma_f : T_f \to T_f$.

Since

$$\sigma_f = \Phi \circ f^* \circ \Phi^{-1}$$

where Φ^{-1} means a local holomorphic section of Φ, it follows that

$$\sigma_f : T_f \to T_f$$

is a holomorphic map.

Let $H(\mathbb{D}, T_f)$ be the collection of all holomorphic maps from the unit disk $\mathbb{D}$ to T_f, and $\rho_{\mathbb{D}}(z, w)$ be the hyperbolic distance of any two points z, w in $\mathbb{D}$.

Definition 2.9. *The Kobayashi's pseudo-metric d_K on T_f is defined to be the largest* *pseudo-metric on* T_f *such that*

$$d_K(f(z), f(w)) \leq \rho_{\mathbb{D}}(z, w)$$

for any two points $z, w \in \mathbb{D}$ *and any* $f \in H(\mathbb{D}, T_f)$

The Kobasyshi pseudo-metric on T_f is the same as the Teichmüller metric (see in [GL]). From the definition of Kobayashi pseudo-metric, it follows that any holomorphic self-map of T_f is a contraction on the Kobayashi pseudo-metric. Since σ_f is holomorphic, we have that

$$\mathrm{d}_T(\sigma_f(x), \sigma_f(y)) \leq \mathrm{d}_T(x, y), \quad \forall x, y \in T_f.$$

From [JZ], we also know that

$$\mathrm{d}_T(\sigma_f(x), \sigma_f(y)) < \mathrm{d}_T(x, y), \quad \forall x, y \in T_f. \tag{2.10}$$

We need more definitions and lemmas from [JZ] as follows.

Let Z be a subset of Q with $\sharp(Z) \geq 4$. Let $x = [\mu] \in T_f$ and let $\gamma \in \widehat{\mathbb{C}} \setminus Z$ be a simple closed and non-peripheral curve. We use $l_Z(\gamma, x)$ to denote the hyperbolic length of the unique simple closed geodesic which is homotopic to $\phi^\mu(\gamma)$ in the hyperbolic Riemann surface $\widehat{\mathbb{C}} \setminus \phi^\mu(Z)$. We say γ is a (μ, Z)-simple closed geodesic if $\phi^\mu(\gamma)$ is a simple closed geodesic in $\widehat{\mathbb{C}} \setminus \phi^\mu(Z)$.

Remark 2.1. *From the definition of the Teichmüller space* T_f*, we know that the definition of* $l_Z(\gamma, x)$ *is independent of the choice of* μ *in* x*.*

For $x_0 \in T_f$, let $x_n = \sigma_f^n(x_0)$, $n = 1, \cdots$ be a sequence in T_f. Recall our definition of E in (2.8).

Lemma 2.6. *If there is a real number $a > 0$ such that there is a point $x_0 \in T_f$ and every (x_n, E)-simple closed geodesic $\gamma \subset \widehat{\mathbb{C}} \setminus Q$ has hyperbolic length greater than or equal to a, then the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent in T_f and the limiting point is the unique fixed point of σ_f in T_f.*

Remark 2.2. *This lemma implies that if there exists an $x_0 \in T_f$ such that the length of the shortest geodesics on all the x_n have a uniform lower bound, then f has no Thurston obstructions.*

Lemma 2.7. *There exists an $\eta > 0$ such that for any point $x = [\mu] \in T_f$ with $\mu(z) = 0$ on $\cup_i A_i$ and for any (x, E)-simple geodesic $\gamma \subset \widehat{\mathbb{C}} \setminus E$ with $l_E(\gamma, x) < \eta$, we have $\gamma \subset \widehat{\mathbb{C}} \setminus Q$. Moreover, for any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$l_E(\gamma, x) > (1 - \epsilon) l_Q(\gamma, x)$$

whenever $l_E(\gamma, x) < \delta$.

Remark 2.3. *The above lemma implies that for any $x = [\mu] \in T_f$ with $\mu(z) = 0$ for all $z \in \cup_i A_i$, all sufficiently short geodesics in $\widehat{\mathbb{C}} \setminus \phi^\mu(E)$ are homotopic to the sufficiently short geodesics in $\widehat{\mathbb{C}} \setminus \phi^\mu(Q)$. More precisely, we can find a constant $\delta_0 > 0$ such that*

$$\frac{1}{e} l_Q(\gamma, x) < l_E(\gamma, x) < l_Q(\gamma, x) \quad \textit{whenever } l_E(\gamma, x) < \delta_0.$$

Suppose $x = [\mu] \in T_f$ and $Z \subset Q$. Define

$$w_Z(\gamma, x) = -\log l_Z(\gamma, x).$$

Consider the set

$$L_{Z,x} = \{w_Z(\gamma, x)\}$$

where γ ranges over all the non-peripheral simple closed curves in $\widehat{\mathbb{C}} \setminus Q$. Define

$$w_Z(x) = \sup\{w_Z(\gamma, x)\}$$

and

$$w_Z(\Gamma, x) = \max_{\gamma \in \Gamma} w_Z(\gamma, x).$$

The following lemma is a general result for hyperbolic Riemann surfaces (refer to [DH, JZ]). We just state it in our case.

Lemma 2.8. *Let $Z \subset Q$ be a finite subset with $\sharp Z \geq 4$ and $\gamma \subset \widehat{\mathbb{C}} \setminus Q$ be a non-peripheral simple closed curve. Then the function*

$$x \mapsto w_Z(\gamma, x) : T_f \to \mathbb{R}$$

is Lipschitz with Lipschitz constant 2.

Let

$$A = \max\{-\log\log(2\sqrt{2} + 3), -\log \delta_0\}$$

where δ_0 is the number in Remark 2.3. From Keen's Collar Lemma in [Keen], we know that $\log(2\sqrt{2} + 3)$ is the magic number in the theory of hyperbolic Riemann surfaces such that for any hyperbolic Riemann surface S, any two simple closed geodesics γ and γ' in S are disjoint whenever the hyperbolic lengths of γ and γ' are less than $\log(2\sqrt{2}+3)$. This implies that for any point $x \in T_f$, there are at most $p-3$ curves γ with $l_E(\gamma, x) \leq \log(2\sqrt{2}+3)$.

For any $J > 0$, let (a, b) be the lowest interval in $\mathbb{R} \setminus L_{E,x}$ such that $a \geq A$ and $b - a = J$. For any $x = [\nu] \in T_f$, define

$$\Gamma_{J,x} = \{\gamma \mid \gamma \text{ is a simple closed geodesic on } R_x \text{ and } w_E(\gamma, x) \geq b\}.$$

Then $\Gamma_{J,x}$ is a multi-curve consisting of the geodesics which are sufficiently short on $\widehat{\mathbb{C}} \setminus \phi^\mu(E)$. This is equivalent saying that $\Gamma_{J,x}$ contains all the simple closed curves in $\widehat{\mathbb{C}} \setminus \phi^\mu(Q)$ which are homotopic to sufficiently short simply closed geodesics on $\widehat{\mathbb{C}} \setminus \phi^\mu(Q)$. There are at most $p-3$ elements in $\Gamma_{J,x}$ for any x and they are pairwise disjoint.

For any $x \in T_f$, let $D = d_T(x, \sigma_f(x))$.

Lemma 2.9. *If* $J \geq \log d + 2D + 1$ *and* $\Gamma_{J,x} \neq \emptyset$, *then* $\Gamma_{J,x}$ *is an* f*-stable multi-curve.*

See Lemma 7.3 in [JZ].

2.5 Upper bound for Γ_∞

We keep the notation of the previous sections. Suppose $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$ for all $n \geq 1$. Then we have a sequence $\{x_n\}_{n=0}^\infty$ in T_f.

For all $n > 0$ and all $z \in \cup_i A_i$, since $f(\cup_i A_i) \subset \cup_i D_i$, where A_i are the shielding rings we constructed, we have $\mu_n(z) = 0$, where $[\mu_n] = x_n$.

Recall the definition of $E = P_1 \cup \cup_i \{a_i, b_i\}$ in (2.8) and m in Proposition 2.3. Let

$$P_2 = E \cup f^m(E) \cup (\cup_{1 \leq j \leq m} f^j(\Omega_f)) \subset Q.$$

The following lemma is also from [JZ].

Lemma 2.10. *There exists an* $\epsilon_0 > 0$*, such that for any* $x = [\mu] \in T_f$ *with* $\mu(z) = 0$*, for all* $z \in \cup_i A_i$*, and for any* (μ, P_2)*-simple closed geodesic* γ'*, if* $l_{P_2}(\gamma', x) < \epsilon_0$*, then there is a* (μ, E)*-simple closed geodesic* γ *such that* γ' *is homotopic to* γ *in* $\widehat{\mathbb{C}} \setminus P_2$*.*

The following lemma is also a general result in the theory of hyperbolic Riemann surfaces and the reader can find a proof in [DH].

Lemma 2.11. *Let* X *be a hyperbolic Riemann surface,* $P \subset X$ *is a finite subset, and* $\sharp P < p$*. Let* $X' = X \setminus P$ *and* $L < \log(3 + 2\sqrt{2})$*. Let* γ *be a simple closed geodesic on* X*, and let* $\gamma'_1, \cdots, \gamma'_k$ *be all the geodesics on* X' *homotopic to* γ *in* X *whose hyperbolic length on* X' *is less than* L*. Set* $l = l_X(\gamma)$ *and* $l'_i = l_{X'}(\gamma'_i)$*. Then:*

(1) $k \leq p + 1$*;*

(2) for all i*,* $l'_i \geq l$*;*

(3) $\frac{1}{l} - \frac{1}{\pi} - \frac{(p+1)}{L} < \sum_{i=1}^{k} \frac{1}{l'_i} < \frac{1}{l} + \frac{(p+1)}{\pi}$.

The next proposition is essential for our proof of Theorem 1.3.

Proposition 2.5. *Let* m *be the constant in Proposition 2.3. Let* $x_0 \in T_f$ *and* $x_n = \sigma_f^n(x_0)$ *for* $n > 0$. *There exists a constant* $C(J) > 0$ *depending on* p, d, ϵ_0, $D = d_T(x_0, x_1)$ *and* $J \geq m(\log d + 2D + 1)$ *such that if* $w_E(x_0) > C(J)$, *then* $\Gamma = \Gamma_{J,x_0} \neq \emptyset$ *is a stable multi-curve. Moreover, if* $\Gamma_\infty \neq \emptyset$, *then*

$$w_E(\Gamma_\infty, x_m) \leq w_E(\Gamma_\infty, x_0).$$

Proof. If $w_E(x_0) \geq A + (p-3)J$, then Γ_{J,x_0} is non-empty, since R_{x_0} has at most $(p-3)$ simple closed geodesics with hyperbolic length less than e^{-A} (they are not homotopic to each other). From Lemma 2.9, $\Gamma = \Gamma_{J,x_0}$ is also f-stable.

Suppose $\Gamma_\infty \neq \emptyset$ and A_Γ is in the form of (2.7). From Proposition 2.3, $||A_{\Gamma_\infty}^m|| < 1/2$.

For each $\gamma_j \in \Gamma_{J,x_0}$, let $\gamma_{i,j,\alpha}$ be any component of $f^{-m}(\gamma_j)$ homotopic to γ_i in $\widehat{\mathbb{C}} \setminus Q$. Then $\gamma_{i,j,\alpha}$ is also homotopic to γ_i in $\widehat{\mathbb{C}} \setminus E$. Let $g = \phi^\mu \circ f^m \circ (\phi^\nu)^{-1}$, where $[\mu] = x_0$ and $[\nu] = x_m$. Then g is a rational map and

$$g : \widehat{\mathbb{C}} \setminus \phi^\nu(f^{-m}(P_2)) \to \widehat{\mathbb{C}} \setminus \phi^\mu(P_2)$$

is a holomorphic covering map. Therefore

$$l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m) = d_{i,j,\alpha} l_{P_2}(\gamma_j, x_0),$$

where $d_{i,j,\alpha}$ is the degree of $f^m : \gamma_{i,j,\alpha} \to \gamma_j$. We get

$$\sum_\alpha \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} = \Big(\sum_\alpha \frac{1}{d_{i,j,\alpha}}\Big) \frac{1}{l_{P_2}(\gamma_j, x_0)} = b_{ij} \frac{1}{l_{P_2}(\gamma_j, x_0)},$$

where b_{ij} is the ij-entry of A_Γ^m.

Since $E \subset P_2$, the inclusion

$$\iota : \widehat{\mathbb{C}} \setminus P_2 \hookrightarrow \widehat{\mathbb{C}} \setminus E$$

decreases the hyperbolic distances. So we have that $l_{P_2}(\gamma_j, x_0) > l_E(\gamma_j, x_0)$ for any γ_j. It follows that

$$\sum_\alpha \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} < b_{ij} \frac{1}{l_E(\gamma_j, x_0)}.$$

From the definitions of P_2 and E, we know that $E \subset f^{-m}(P_2)$. Let $C = C(d, m, p) = \sharp(f^{-m}(P_2) \setminus E)$, where $p = \sharp E$.

We claim that for any $(\nu, f^{-m}(P_2))$-simple closed geodesic γ which is homotopic to γ_i in $\widehat{\mathbb{C}} \setminus E$, either γ is homotopic to some $\gamma_{i,j,\alpha}$ in $\widehat{\mathbb{C}} \setminus f^{-m}(P_2)$ or

$$l_{f^{-m}(P_2)}(\gamma, x_m) > \min\{e^{-(A+PJ)}, \epsilon_0\},$$

where ϵ_0 is the constant in Lemma 2.10.

We prove the claim. In fact, if γ is not homotopic in $\widehat{\mathbb{C}} \setminus f^{-m}(P_2)$ to some $\gamma_{i,j,\alpha}$, then $f^m(\gamma)$ is a (μ, P_2)-simple closed geodesic which is not homotopic to any γ_j in $\widehat{\mathbb{C}} \setminus P_2$. Then there are two cases: either (1) $f^m(\gamma)$ is homotopic in $\widehat{\mathbb{C}} \setminus P_2$ to some (μ, E)-simple closed geodesic ξ which does not belong to Γ_{J,x_0}, and we have

$$l_{P_2}(f^m(\gamma), x_0) > l_E(f^m(\gamma), x_0) = l_E(\xi, x_0) > e^{-a} > e^{-(A+PJ)}$$

or (2) $f^m(\gamma)$ is not homotopic in $\widehat{\mathbb{C}} \setminus P_2$ to any (μ, E)-simple closed geodesic, and by Lemma 2.10, we have

$$l_{P_2}(f^m(\gamma), x_0) > \epsilon_0.$$

Thus we have

$$l_{f^{-m}(P_2)}(\gamma, x_m) \geq l_{P_2}(f^m(\gamma), x_0) > \min\{e^{-(A+PJ)}, \epsilon_0\}.$$

This proves the claim.

From the left hand of the inequality given by (3) in Lemma 2.11, for each $\gamma_i \in \Gamma$, we have

$$\frac{1}{l_E(\gamma_i, x_m)} - \frac{1}{\pi} - \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}} \leq \sum_{j,\alpha} \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} \leq \sum_j b_{ij} \frac{1}{l_E(\gamma_j, x_0)}.$$

Suppose $\Gamma_\infty = \{\gamma_1, \ \cdots, \ \gamma_s\} \subset \Gamma$. Then for each $\gamma_i \in \Gamma_\infty$, from the form (2.7) of A_Γ,

$$\frac{1}{l_E(\gamma_i, x_m)} \leq \sum_{j=1}^{s} b_{ij} \frac{1}{l_E(\gamma_j, x_0)} + \frac{1}{\pi} + \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}.$$

Let

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{l_E(\gamma_1, x_m)} \\ \vdots \\ \frac{1}{l_E(\gamma_s, x_m)} \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} \frac{1}{l_E(\gamma_1, x_0)} \\ \vdots \\ \frac{1}{l_E(\gamma_s, x_0)} \end{pmatrix}.$$

Since $\|A_\infty^m\| < 1/2$,

$$\|\mathbf{v}_1\| < \frac{1}{2}\|\mathbf{v}\| + \frac{1}{\pi} + \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}.$$

Define

$$C(J) = \max\Big\{2\Big(\frac{1}{\pi} + \frac{C+1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}\Big), A + (p-3)J\Big\}.$$

If $w_E(\Gamma_\infty, x_0) \geq C(J)$, then we have

$$w_E(\Gamma_\infty, x_m) < w_E(\Gamma_\infty, x_0).$$

□

Lemma 2.12. *Let* $J \geq m(\log d + 2D + 1)$. *Suppose* $w_E(x_0) < C(J)$ *and suppose* $\Gamma = \Gamma_{J,x_k} \neq \emptyset$ *for some* $k \geq 0$. *Let* $E(J) = C(J) + 2mD$. *If* $\Gamma_\infty \neq \emptyset$, *then for all* n,

$$w_E(\Gamma_\infty, x_n) < E(J).$$

Moreover, if $w_E(\gamma, x_k) \geq E(J)$, *then* $\gamma \in \Gamma_0$.

Proof. We prove the first inequality by contradiction. Suppose there is an $n > 0$ such that $w_E(\Gamma_\infty, x_n) \geq C(J) + 2mD$. Suppose n_0 is the first integer having this property. Then we have $w_E(\Gamma_\infty, x_{n_0-m}) \geq C(J)$. Now by Proposition 2.5 and the fact that n_0 is the first integer such that $w_E(\Gamma_\infty, x_{n_0}) \geq C(J) + 2mD$, we have

$$w_E(\Gamma_\infty, x_{n_0}) \leq w_E(\Gamma_\infty, x_{n_0-m}) < C(J) + 2mD.$$

This is a contradiction.

If $w_E(\gamma, x_k) \geq E(J) > C(J) \geq A + (p-3)J$, then $\gamma \in \Gamma_{J,x_k} = \Gamma$ since there are at most $p-3$ simple closed curves in R_{x_k} such that $w_E(\gamma, x_k) > A$. But $\gamma \notin \Gamma_\infty$ because of the first conclusion and the assumption. Therefore, $\gamma \in \Gamma_0$. □

2.6 Lower bound for Γ_0

In order to get the lower bound for Γ_0, we need the following definition.

Definition 2.10. *Let κ be a real number. A sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers is called κ-quasi-nondecreasing if for all $n_1 < n_2$ we have $a_{n_2} - a_{n_1} \geq \kappa$. A sequence is called quasi-nondecreasing if it is κ-quasi-nondecreasing for some κ.*

It is easy to check that the following two properties are true.

Property 2.1. *Suppose $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences. If $\{a_n\}_{n=0}^{\infty}$ is κ-quasi-nondecreasing and if $|a_n - b_n| < r$ for all n, then $\{b_n\}$ is $(\kappa - 2r)$-quasi-nondecreasing.*

Property 2.2. *Suppose $\{a_n\}$ is quasi-nondecreasing and unbounded. Then $a_n \to +\infty$ as $n \to +\infty$.*

Recall that any $x = [\mu] \in T_f$ represents a complex structure on $\widehat{\mathbb{C}} \setminus Q$, which makes $\widehat{\mathbb{C}} \setminus Q$ a hyperbolic Riemann surface R_x. For any simple closed geodesic γ on R_x, let $A(\gamma, x)$ be the Riemann surface, conformally isomorphic to an annulus, obtained by taking the unit disk $\mathbb{D}$ modulo a $\mathbb{Z}$-subgroup of the fundamental group of R_x generated by γ. It is a covering space of R_x. The core curve of $A(\gamma, x)$ is a geodesic of length $l_Q(\gamma, x)$ and

$$\mathrm{mod}(A(\gamma, x)) = \frac{\pi}{l_Q(\gamma, x)}, \tag{2.11}$$

where $\mathrm{mod}(A)$ means the modulus of an annulus A.

If γ is a simple closed geodesic of hyperbolic length l on the Riemann surface R_x, then there is an embedding annulus $a(\gamma, x)$ of modulus $m(l)$ which is continuous and decreasing and satisfies

$$\frac{\pi}{l} - 1 < m(l) < \frac{\pi}{l}.$$

Thus for all $x \in T_f$, we have

$$\operatorname{mod}(A(\gamma, x)) - 1 < \operatorname{mod}(a(\gamma, x)) < \operatorname{mod}(A(\gamma, x)). \tag{2.12}$$

We need the following technical lemma:

Lemma 2.13. *If* $t \geq 1$*, then* $\log(t+1) - 1 < \log t$.

Proof. For $t \geq 1$,

$$\log(t+1) - \log t = \log(\frac{t+1}{t}) \leq \log 2 < 1.$$

□

If $w_Q(\gamma, x) \geq \log \frac{2}{\pi} = -0.451582705\cdots$, then we have $\operatorname{mod}(A(\gamma, x)) - 1 \geq 1$. By taking logarithms on all terms of Inequality (2.12) and by applying Lemma 2.13 and Equation (2.11), we have

$$\log \pi - 1 + w_Q(\gamma, x) < \log \operatorname{mod}(a(\gamma, x)) < \log \pi + w_Q(\gamma, x).$$

It follows that, if $w_Q(\gamma, x) \geq \log \frac{2}{\pi}$, then

$$|\log \operatorname{mod}(a(\gamma, x)) - w_Q(\gamma, x)| < \log \pi. \tag{2.13}$$

Given a multi-curve Γ, we denote vectors of moduli

$$(\mathrm{mod}(A(\gamma, x))) \text{ and } (\mathrm{mod}(a(\gamma, x)))$$

by $\mathrm{mod}(A(\Gamma, x))$ and $\mathrm{mod}(a(\Gamma, x))$ respectively. Define

$$\underline{\mathrm{mod}}(A(\Gamma, x)) = \min_{\gamma\in\Gamma}\{\mathrm{mod}(A(\gamma, x))\}$$

and

$$\underline{\mathrm{mod}}(a(\Gamma, x)) = \min_{\gamma\in\Gamma}\{\mathrm{mod}(a(\gamma, x))\}.$$

Lemma 2.14. *Let β be the constant in Proposition 2.2. Let Γ be an irreducible multi-curve. Suppose the leading eigenvalue of the matrix A_Γ is greater than or equal to 1. Then for any $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$, $n > 0$,*

(1) $\underline{mod}(A(\Gamma, x_n)) \geq \beta \underline{mod}(a(\Gamma, x_0))$ and

(2) $\underline{mod}(a(\Gamma, x_n)) \geq \beta \underline{mod}(a(\Gamma, x_0)) - 1$.

Proof. Since for any n, $f^n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering, we can similarly define the linear map $f_\Gamma^n : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$. Let B be the corresponding matrix for the linear map f_Γ^n with the basis Γ. It is easy to see that $B \geq A_\Gamma^n$.

Let $\mathbf{v}$ be the unique positive eigenvector of A_Γ with $\|\mathbf{v}\| = \mathbf{1}$. Let $\mathbf{1}$ denote the vector whose coordinates are all equal to 1. Then

$$\mathrm{mod}(a(\Gamma, x_0)) \geq \underline{\mathrm{mod}}(a(\Gamma, x_0)\mathbf{1} \geq \underline{\mathrm{mod}}(a(\Gamma, x_0))\mathbf{v}.$$

For any $n \geq 1$, let $\gamma_{i,j,\alpha}^n$ be the components of $f^{-n}(\gamma_j)$ homotopic to γ_i, and $a_{i,j,\alpha}^n$ be the components of $f^{-n}(a(\gamma_j, x_0))$ homotopic to γ_i. Then

$$\mathrm{mod}(a^n_{i,j,\alpha}) = \mathrm{mod}(a(\gamma_j, x_0))/d^n_{i,j,\alpha},$$

where $d^n_{i,j,\alpha} = \deg f^n|_{\gamma^n_{i,j,\alpha}}$. Since $a^n_{i,j,\alpha}$ are disjoint annuli homotopic to the curve γ_i, we have

$$\sum_{\alpha,j} \mathrm{mod}(a^n_{i,j,\alpha}) \leq \mathrm{mod}(A(\gamma_i, x_n)).$$

(One can obtain this inequality by lifting them to the covering space $A(\gamma_i, x_n)$ of R_{x_n} and then by using Grötzsch's inequality.) Consequently we get

$$\begin{aligned} \mathrm{mod}(A(\Gamma, x_n)) &\geq \mathrm{mod}(a(\Gamma, x_n)) \geq B\ \mathrm{mod}(a(\Gamma, x_0)) \\ &\geq A^n_\Gamma\ \mathrm{mod}(a(\Gamma, x_0)) \geq A^n_\Gamma\ \underline{\mathrm{mod}}(a(\Gamma, x_0))\mathbf{v} \\ &\geq \underline{\mathrm{mod}}(a(\Gamma, x_0))\mathbf{v} \geq \beta\ \underline{\mathrm{mod}}(a(\Gamma, x_0))\mathbf{1} \end{aligned}$$

Hence for all $\gamma \in \Gamma$, we have $\mathrm{mod}(A(\gamma, x_n)) \geq \beta\ \underline{\mathrm{mod}}(a(\Gamma, x_0))$. The second conclusion follows form the first one and Inequality (2.12). □

Lemma 2.15. *If* $a,\ b > 0$, $\beta > 0$, *and* $e^a \geq \beta e^b - 1$, *then* $a - b \geq \log \beta - 1$.

Proof. If $\beta \exp b - 1 \geq 1$, then by Lemma 2.13, we have

$$\log(\beta \exp b - 1) \geq \log(\beta e^b) - 1 \geq \log \beta + b - 1.$$

Hence by the assumption, we have $a - b \geq \log \beta - 1$.

If $\beta \exp b - 1 < 1$, then $b < \log 2 - \log \beta$. Since $a > 0$,

$$a - b > 0 - b = -b > \log \beta - \log 2 > \log \beta - 1.$$

□

For any $x \in T_f$ and any multi-curve Γ, define

$$\underline{w}(\Gamma, x) = \min_{\gamma\in\Gamma} w_Q(\gamma, x).$$

Lemma 2.16. *Suppose Γ is an irreducible multi-curve and suppose the leading eigenvalue of the matrix A_Γ is greater than or equal to 1. For any $x_0 \in T_f$, if $\underline{w}(\Gamma, x_0) \geq \log(3/\beta) + \log \pi$, then the sequence $\{\underline{w}(\Gamma, x_n)\}_{n=0}^{\infty}$, where $x_n = \sigma_f^n(x_0)$, is $(\log\beta - 1 - 2\log\pi)$-quasi-nondecreasing.*

Proof. For $\underline{w}(\Gamma, x_0) \geq \log(3/\beta) + \log\pi > \log\frac{2}{\pi}$, by Inequality (2.13), we have

$$\log \underline{\mathrm{mod}}(a(\Gamma, x_0)) \geq \log(\frac{3}{\beta}).$$

That is, $\underline{\mathrm{mod}}(a(\Gamma, x_0)) \geq 3/\beta$. So $\beta\underline{\mathrm{mod}}(a(\Gamma, x_0)) - 1 \geq 2$. By Lemma 2.14, we have that for all $n \geq 0$,

$$\underline{\mathrm{mod}}(a(\Gamma, x_n)) \geq 2. \tag{2.14}$$

Now consider the sequence $y_n = \log \underline{\mathrm{mod}}(a(\Gamma, x_n))$. Choose arbitrarily $n_2 > n_1 \geq 0$, and let $a = y_{n_2}$, $b = y_{n_1}$ and $n = n_2 - n_1$. By Lemma 2.14, we have $e^a \geq \beta e^b - 1$. Applying Lemma 2.15, we have $a - b \geq \log\beta - 1$, so the sequence $\{y_n\}$ is a $(\log\beta - 1)$-quasi-nondecreasing.

By Inequalities (2.12) and (2.14), we have $\underline{\mathrm{mod}}(A(\Gamma, x)) \geq 2$. This implies that $\log\pi + \underline{w}(\Gamma, x_n) \geq \log 2$. That is, $\underline{w}(\Gamma, x_n) \geq \log(2/\pi)$. Since $\mathrm{mod}(a(\gamma, x_n))$ is continuous and decreasing with $l_Q(\gamma, x_n)$, we obtain

$$\underline{\mathrm{mod}}(a(\Gamma, x_n)) = \mathrm{mod}(a(\gamma, x_n)) \quad \text{and} \quad \underline{w}(\Gamma, x_n) = w_Q(\gamma, x_n)$$

for the same $\gamma \in \Gamma$. This further implies that

$$|y_n - \underline{w}(\Gamma, x_n)| < \log \pi.$$

From Property 2.1, we finally have that $\underline{w}(\Gamma, x_n)$ is $(\log \beta - 1 - 2\log \pi)$-quasi-nondecreasing. $\square$

Lemma 2.17. *Let $k \geq 1$ be an integer. For any $x_0 \in T_f$, let $x_n = \sigma_f^n(x_0)$ for $n > 0$. Let $D = d_T(x_0, x_1)$. If γ_1, γ_2 are non-peripheral curves in $\widehat{\mathbb{C}} \setminus Q$ such that some component of $f^{-k}(\gamma_1)$ is homotopic to γ_2, then*

$$w_Q(\gamma_2, x_0) \geq w_Q(\gamma_1, x_0) - k(\log d + 2D).$$

Proof. Let $Y = f^{-k}(R_{x_0})$. Then $Y \subset R_{x_k}$ is a Riemann surface and $f^k : Y \to R_{x_0}$ is a holomorphic covering map of degree d^k. Then

$$l_Y(\gamma_2) \leq d^k l_Q(\gamma_1, x_0).$$

Since the inclusion map $\iota : Y \hookrightarrow R_{x_k}$ decreases the hyperbolic lengths,

$$l_Q(\gamma_2, x_k) \leq d^k l_Q(\gamma_1, x_0).$$

It follows that

$$w_Q(\gamma_2, x_k) > w_Q(\gamma_1, x_0) - k \log d.$$

Since σ_f decreases the Teichmüller distance d_T,

$$d_T(x_i, x_{i+1}) \leq d_T(x_0, x_1) = D.$$

The map $\gamma \mapsto w_Q(\gamma, x)$ for any $x \in T_f$ is a Lipschitz function with Lipschitz constant 2 (see Lemma 2.8), so we have

$$w_Q(\gamma_2, x_0) \geq w_Q(\gamma_2, x_k) - 2kD \geq w_Q(\gamma_1, x_0) - k(2D + \log d).$$

□

Lemma 2.18. *Suppose* Γ *is an irreducible multi-curve. Then for all* $\gamma_i,\ \gamma_j \in \Gamma$ *and all* $x \in T_f$,

$$|w_Q(\gamma_i, x) - w_Q(\gamma_j, x)| \leq (p-3)(\log d + 2D).$$

Proof. Since Γ is irreducible, there is an integer $q \leq \sharp\Gamma \leq p-3$ such that γ_i is homotopic to a preimage of $f^{-q}(\gamma_j)$. By Lemma 2.17, we see that $w_Q(\gamma_i, x) \geq w_Q(\gamma_j, x) - (p-3)(\log d + 2D)$. By exchanging i and j, we complete the proof. □

Proposition 2.6. *Suppose* Γ *is an* f*-stable multi-curve satisfying* $\Gamma = \Gamma_0$. *Let* $x_0 \in T_f$ *and* $x_n = \sigma_f^n(x_0)$, $n > 0$. *Let* $D = d_T(x_0, x_1)$. *Suppose* $min_\gamma w_Q(\gamma, x_0) \geq \log(3/\beta) + \log\pi$, *where* β *is the number in Proposition 2.2. Write* $\Gamma = \Gamma' \sqcup \Gamma''$, *where* $\Gamma' = \Gamma_{Ob}$ *is the union of the irreducible component* Γ_j *of* Γ *for which* $\lambda(A_{\Gamma_j}) \geq 1$. *Then*

(1) for all $\gamma \in \Gamma'$, $\{w_Q(\gamma, x_n)\}_{n\geq 0}$ *is* κ*-quasi-nondecreasing, where* $\kappa = \log\beta - 1 - 2\log\pi - 2(p-3)(\log d + 2D)$;

(2) for all $\gamma \in \Gamma''$ *and all* $n \geq 0$,

$$w_Q(\gamma, x_n) \geq min_{\gamma' \in \Gamma'}\{w_Q(\gamma', x_n)\} - M(\log d + 2D),$$

where M *is the constant in Proposition 2.4.*

(3) Suppose $min_{\gamma \in \Gamma} w_Q(\gamma, x_0) \geq J_A - 1$, *where*

$$J_A = max\{\log(3/\beta) + \log \pi, A\} + \kappa + M(\log d + 2D) + 1.$$

Then for all $\gamma \in \Gamma$ *and for all* $n \geq 0$, *we have*

$$w_Q(\gamma, x_n) \geq A.$$

Proof. Let Γ_j be an irreducible component of Γ for which $\lambda(\Gamma_j) \geq 1$. By the assumption that $\underline{w}(\Gamma, x_0) \geq \log(3/\beta) + \log \pi$, we have $\{\underline{w}(\Gamma_j, x_n)\}$ is $\log \beta - 1 - 2\log \pi$-quasi-nondecreasing.

Since Γ_j is an irreducible multi-curve, by Lemma 2.17 and Property 2.1, we have for each $\gamma \in \Gamma_j$, the sequence $\{w_Q(\gamma, x_k)\}_{k=0}^{\infty}$ is a $\kappa = \log \beta - 1 - 2\log \pi - 2(p-3)(\log d + 2D)$-quasi-nondecreasing. This completes (1).

By Lemma 2.16 and (1), we have for all $\gamma \in \Gamma''$ and all $n \geq 0$,

$$w_Q(\gamma, x_n) \geq \min_{\gamma' \in \Gamma'}\{w_Q(\gamma', x_n)\} - M(\log d + 2D).$$

This is (2).

(3) follows from (1) and (2) immediately. □

Proposition 2.7. *Suppose* Γ *is an* f*-stable multi-curve satisfying* $\Gamma = \Gamma_0$. *Let* $x_0 \in T_f$ *and* $x_n = \sigma_f^n(x)$, $n > 0$, *and* $D = d_T(x_0, x_1)$. *Suppose* $min_{\gamma\in\Gamma} w_E(\gamma, x_0) \geq J_A$. *Write* $\Gamma = \Gamma' \sqcup \Gamma''$, *where* $\Gamma' = \Gamma_{Ob}$ *is the union of the irreducible component* Γ_j *of* Γ *for which* $\lambda(A_{\Gamma_j}) \geq 1$. *Then*

1) For all $\gamma \in \Gamma$, $w_E(\gamma, x_n) \geq A$ *for any* $n \geq 0$.

2) For all $\gamma \in \Gamma'$, $\{w_E(\gamma, x_n)\}_{n\geq 0}$ *is* $(\kappa - 2)$*-quasi-nondecreasing.*

3) For all $\gamma \in \Gamma''$ *and all* $n \geq 0$,

$$w_E(\gamma, x_n) \geq min_{\gamma'\in\Gamma'}\{w_E(\gamma', x_n)\} - 2 - M(\log d + 2D).$$

Proof. From Lemma 2.7, we have, for any $x \in T_f$,

$$w_Q(\gamma, x) \leq w_E(\gamma, x) \leq w_Q(\gamma, x) + 1$$

if $w_E(\gamma, x) \geq A$. If $\min_\gamma w_E(\gamma, x_0) \geq J_A$, then $\min_\gamma w_Q(\gamma, x_0) \geq J_A - 1$, and by Proposition 2.6, for any $n \geq 0$ and $\gamma \in \Gamma$, $w_Q(\gamma, x_n) \geq A$. Consequently, $w_E(\gamma, x_n) \geq A$, which gives us 1).

From 1), we have $|w_E(\gamma, x_n) - w_Q(\gamma, x_n)| < 1$ for all $n \geq 0$. Then by Property 2.1 and Proposition 2.6, we have 2) and 3). □

2.7 Existence of the canonical Thurston obstruction

This section is devoted to proving Theorem 1.3.

For any choice $x_0 \in T_f$, we can find a $J \geq J_A$ such that $w_E(x_0) < C(J)$. Without loss of generality, we assume that $J = J_A$. Since $C(J)$ is an increasing function of J, we have $w_E(x_0) < C(J)$ for all $J \geq J_A$. Let $x_n = \sigma_f^n(x_0)$, $n > 0$, and $D = d_T(x_0, x_1)$.

Suppose that f is not equivalent to a rational map. By Lemma 2.6, the sequence $\{w_E(x_n)\}_{n\geq 0}$ is unbounded. Thus there exists γ_k and x_{n_k} with $w_E(\gamma_k, x_{n_k}) \to \infty$, as $k \to \infty$.

Fix $J > J_0 = J_A + |A|$. Then $w_E(\gamma_k, x_{n_k}) > E(J) = C(J) + 2mD$ for some k. So by Lemma 2.12, the set of finite depth curves in $\Gamma_{J,x_{n_k}}$, denoted by $\Gamma_{J,x_{n_k},0}$, is nonempty.

Moreover, if for some n_0, $\gamma \in \Gamma_{J,n_0,0}$, $w_E(\gamma, x_{n_0}) > a + J \geq J_A$, then $w_E(\gamma, x_n) \geq A$ for all $n \geq n_0$ by Proposition 2.7. This implies that $\Gamma_J = \cup_n \Gamma_{J,x_n,0}$ and $\mathcal{G} = \cup_{J\geq J_0}\Gamma_J$ are multi-curves, since $\gamma \in \Gamma_J$ satisfies $w_E(\gamma, x_n) \geq A$ for all n sufficiently large.

Since $w_E(\gamma_k, x_{n_k}) \to \infty$, as $k \to \infty$, given any fixed $J \geq J_0$, $w_E(\gamma_k, x_{n_k}) \geq E(J)$ for infinitely many k. Hence $\gamma_k \in \Gamma_J \subset \mathcal{G}$ infinitely often. Since $\mathcal{G}$ is finite, for some $\gamma \in \mathcal{G}$, we have $\gamma_k = \gamma$ for infinitely many k. Hence the set

$$\Gamma_u = \{\gamma \mid \{w_E(\gamma, x_n)\}_{n\geq 0} \text{ is unbounded}\}$$

is nonempty.

Proposition 2.8. $\Gamma_u = \cap_{J\geq J_0}\Gamma_J$.

Proof. The inclusion $\cap_{J>J_0}\Gamma_J \subset \Gamma_u$ is clear. To see the other inclusion, let $\gamma \in \Gamma_u$.

Given J, there exists some n such that $w(\gamma, x_n) > E(J)$. By Lemma 2.12, $\gamma \in \Gamma_{J,x_n,0}$. Thus $\cap_{J>J_0}\Gamma_J \supset \Gamma_u$. This proves the proposition. $\square$

Proposition 2.9. $\Gamma_u = \Gamma_{J_c}$ *for some* $J_c \geq J_A$.

Proof. The proof is by contradiction. Since $\Gamma_u = \cap_{J\geq J_0}\Gamma_J$, for all $J \geq J_0$, if $\Gamma_u \neq \Gamma_J$, then there exists a curve γ_J such that $\gamma_J \in \Gamma_J \subset \mathcal{G}$ but $\gamma_J \notin \Gamma_u$. Since $\mathcal{G}$ is finite, this implies that there is some $\gamma \in \mathcal{G}$ such that $\gamma = \gamma_J \in \Gamma_J$ for infinitely many J, but also $\gamma \notin \Gamma_u$. This is a contradiction, since $\gamma \in \Gamma_J$ for infinitely many J implies that the sequence $\{w_E(\gamma, x_n)\}$ is unbounded. The contradiction proves the proposition. $\square$

Now consider $\Gamma_u = \Gamma_{J_c} = \cup_n\Gamma_{J_c,x_n,0}$.

For each k such that $\Gamma = \Gamma_{J_c,x_k,0}$ is nonempty, applying Proposition 2.7, we know that if $\gamma' \in \Gamma'$, then the sequence $\{w_E(\gamma', x_n)\}_{n\geq 0}$ is both unbounded and quasi-nondecreasing, so $w_E(\gamma', x_n) \to \infty$, as $n \to \infty$. 3) of Proposition 2.7 implies that $w_E(\gamma, x_n) \to \infty$, as $n \to \infty$, for all $\gamma \in \Gamma''$. Hence

$$\Gamma_u = \{\gamma \mid w_E(\gamma, x_n) \to \infty \text{ as } n \to \infty\}.$$

Proposition 2.10. $\Gamma_u = \Gamma_{J_c,x_{n_c},0}$ *for some* $n = n_c$.

Proof. Since $\Gamma_u = \cup_n\Gamma_{J_c,x_n,0}$, the inclusion $\Gamma_{J_c,x_n,0} \subset \Gamma_u \subset \mathcal{G}$ holds for all n.

Since there are finitely many elements in $\mathcal{G}$, there exists an n_c such that for all $\gamma \in \Gamma_u$,

$$w_E(\gamma, x_{n_c}) > E(J_c).$$

By Lemma 2.12, we have $\gamma \in \Gamma_{J_c,x_{n_c},0}$. Thus $\Gamma_u = \Gamma_{J_c,n_c,0}$. $\square$

From Proposition 2.10, Γ_u is a Thurston obstruction. Furthermore, Γ_u depends only on f and is independent of the initial point x_0, since for any γ, the map $x \mapsto w_E(\gamma, x)$ is a Lipschitz map with Lipschitz constant 2 (see Lemma 2.8) and since σ_f decreases the Teichmüller distance d_T.

Finally, since

$$w_Q(\gamma, x) \leq w_E(\gamma, x) \leq 1 + w_Q(\gamma, x),$$

if $w_E(\gamma, x) \geq A$ (see Remark 4.3), we have that

$$\begin{aligned}\Gamma_c &= \{\gamma \mid w_Q(\gamma, x_n) \to \infty \text{ as } n \to \infty\} \\ &= \{\gamma \mid w_E(\gamma, x_n) \to \infty \text{ as } n \to \infty\} = \Gamma_u.\end{aligned}$$

Therefore, Γ_c is a Thurston obstruction. This completes the proof of Theorem 1.3.

Chapter 3

Geometric Characterization of Topological Functions with Two Asymptotic Values

3.1 The space $\mathcal{AV}2$

Recall that the any meromorphic map in $\mathcal{M}_2$ is a holomorphic map with exactly two asymptotic values and no singular values. We now want to consider the topological structure of functions in $\mathcal{M}_2$ and define $\mathcal{AV}2$ to be the set of maps with the same topology.

Definition 3.1. *For any two distinct points* $(a, b) \in S^2$*, let* $f_{a,b}$ *be a universal covering map* $f_{a,b} : X \to S^2 \setminus \{a, b\}$*. Any such map is called a* 2-asymptotic value or 2AV map *and the space of these maps is denoted by* $\mathcal{AV}2$*.*

The relation between the spaces $\mathcal{M}_2$ and $\mathcal{AV}2$ is summarized as the following theorem.

Theorem 3.1. *If* $g(z) \in \mathcal{M}_2$ *then* $g(z) \in \mathcal{AV}2$ *and, conversely, if* $g \in \mathcal{AV}2$ *is meromorphic then* $g \in \mathcal{M}_2$*.*

Proof. Any $g(z) \in \mathcal{M}_2$ is a universal cover of $\hat{\mathbb{C}}$ so belongs to $\mathcal{AV}2$. Conversely, if $g \in \mathcal{AV}2$ is meromorphic, it has two asymptotic values and no other singular values so by definition it is in $\mathcal{M}_2$. □

If $f \in \mathcal{AV}2$, its set of singular values $S_f = \{a, b\}$ from the fact that f is a universal covering. We define the post-singular set just as we did for functions in $\mathcal{M}_2$ in Definition 1.7.

Definition 3.2. *For* $f = f_{a,b} \in \mathcal{AV}2$*, the* post-singular set PS_f *is defined by*

$$PS_f = \overline{\bigcup_{x \in S_f} \cup_{n \geq 0} f^n(x)} \cup \{\infty\}$$

Note that by our notation, $S^2 \setminus X$ is the point at infinity and it has no forward orbit although it may be an asymptotic value. We therefore include it in PS_f.

Post-composition of $f_{a,b}$ with an affine transformation T results in another map in $\mathcal{AV}2$. In what follows, therefore, we will always assume $a = 0$ and the second asymptotic value, λ is determined by the condition $f(0) = 1$.

We will be concerned only with functions in $\mathcal{AV}2$ such that PS_f is finite. Such functions are called *post-singularly finite.*

3.2 A Topological Constraint

Suppose $f \in \mathcal{AV}2$ is post-singularly finite. By our normalization, one asymptotic value is 0 with $f(0) = 1$ and the other asymptotic value is $\lambda \neq 0$. We will define a topological invariant for f that is based on a choice of a path from 0 to $f(1)$ that passes through 1. This choice gives us a partial marking of the space $(\mathbb{R}^2 \cup \{\infty\}) \setminus PS_f$.

We assume we have a standard complex structure on $\mathbb{R}^2$ and denote it by $\mathbb{C}$. Let γ be the straight line segment in $\mathbb{C}$ connecting 0 and 1. We assume it doesn't pass through any other points of PS_f. If it does, we modify it slightly to avoid such points. There are finitely many of them at most. Now consider the image $f(\gamma)$ which is a curve in $\mathbb{C}$ connecting 1 to $f(1)$. It spirals around 0 and is closed only in the special case that $f(1) = 1$. By construction, it avoids any points of P_f. If it is closed, it has a winding number with respect to 0, and denote $\sigma = f(\gamma)$. If not, follow $f(\gamma)$ by the straight line segment σ_0 in $\mathbb{C}$ joining $f(1)$ to 1 to obtain a closed curve $\sigma = f(\gamma) \cup \sigma_0$. This closed curve has a winding number about 0. Again, If the line segment goes through a point of PS_f, modify it slightly to avoid such points. That is, close $f(\gamma)$ in the simplest way to determine the number of times σ spirals around 0.

We use η_f to denote the winding number of σ with respect to 0 on $S^2 \setminus \{\lambda\}$. Since the winding number is topological invariant. we have the following lemma:

Lemma 3.1. *Suppose $\tilde{f}$ is another post-singularly finite map in $\mathcal{AV}2$ and $\psi, \phi : S^2 \to S^2$, are homeomorphisms fixing $0, 1, \infty$ such that $\phi \circ f = \tilde{f} \circ \psi$. Then $\eta_{\tilde{f}} = \eta_f$.*

For the post-singularly finite maps in $\mathcal{AV}2$, we can define combinatorially equivalent equivalence class in the same way as Definition 1.4 for the post-critically branched coverings.

3.3 Teichmüller space T_f.

For each $f \in \mathcal{AV}2$, we associate a Teichmüller space T_f in the same way as Definition 2.8 by replacing Q by PS_f. That is

$$T_f = M(\mathbb{C})/\sim,$$

where $\mu_1 \sim \mu_1$ if ϕ^{μ_1} is isotopic to ϕ^{μ_2} rel. PS_f. In fact, the space T_f is the Teichmüller space modeled on (S^2, PS_f); that is, it may be identified with the Teichmüller space of the sphere with p punctures, $T_{0,p}$, where $p = \#PS_f$.

Since $f(PS_f) \subset PS_f$, it follows that f, like branched coverings, induces a holomorphic map σ_f on T_f, which contracts the Teichmüller metric on T_f; that is

Lemma 3.2. *For any two points τ and τ' in T_f,*

$$d_T\Big(\sigma_f(\tau), \sigma_f(\tau')\Big) \leq d_T(\tau, \tau').$$

Lemma 3.3. *A post-singularly finite f in $\mathcal{AV}2$ is combinatorially equivalent to a meromorphic map in $\mathcal{M}_2$ if and only if σ_f has a fixed point in T_f.*

Proof. If f is equivalent to a map $g \in \mathcal{M}_2$, there exist ϕ, $\psi : (S^2, PS_f) \to (\widehat{\mathbb{C}}, P_g)$ isotopic rel PS_f and such that the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"] & \mathbb{C} \arrow[d, "g\in\mathcal{M}_2"] \\ S^2 \arrow[r, "\phi"] & \widehat{\mathbb{C}} \end{tikzcd}$$

commutes. This means that if τ is the represented by ϕ, then $\sigma_f(\tau)$ is represented by ϕ'. Since ϕ and ϕ' are isotopic rel PS_f, it follows that $\sigma_f(\tau) = \tau$.

Suppose σ_f has a fixed point $\tau = [\phi]$. Consider commutative the diagram

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "f"] & \mathbb{C} \arrow[d, "g\in\mathcal{M}_2"] \\
S^2 \arrow[r, "\phi"] & \widehat{\mathbb{C}}
\end{tikzcd}$$

where $[\psi] = \sigma_f(\tau)$. Since ψ represents the same point of T_f as ϕ, there exists an isomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing ∞ such that $\phi \circ h = \psi$ on PS_f and they are isotopic rel PS_f. Then $g \circ h$ is a meromorphic map equivalent to f, as we see by considering the following diagram

$$\begin{tikzcd}
X \arrow[r, "h^{-1}\circ\psi"] \arrow[d, "f"] & \mathbb{C} \arrow[d, "g\circ h"] \\
S^2 \arrow[r, "\phi"] & \widehat{\mathbb{C}}
\end{tikzcd}$$

□

Remark 3.1. *If $\#PS_f = 3$, then T_f consists of only one points, so it is the fixed point. By the lemma above, f is combinatorially equivalent to a map $\mathcal{M}_2$. So we assume $\#PS_f \geq 4$ in the rest of this chapter.*

3.4 Bounded geometry.

From Lemma 3.3, Theorem 1.4 is proved if we can find a fixed point of σ_f in T_f. Since the map f is of infinite degree, the preimage of any simple closed curve is either an open curve homeomorphic to $\mathbb{R}$ or a union of infinitely many simple closed curves. Thurston obstructions do not make sense. Therefore, we use an analytic condition called *bounded geometry*, which is equivalent to the non-existence of Thurston obstructions for the case of maps with finite degree, to obtain the existence of a fixed point. In order to give the definition of bounded geometry, we need some context.

Let the base point of T_f be the hyperbolic Riemann surface $R = \hat{\mathbb{C}} \backslash PS_f$ equipped with the standard complex structure $[0] \in T_f$. For τ in T_f, denote by R_τ the hyperbolic Riemann surface R equipped with the complex structure τ.

Recall that a simple closed curve $\gamma \subset R$ is called non-peripheral if each component of $\hat{\mathbb{C}} \setminus \gamma$ contains at least two points of PS_f. Let γ be a non-peripheral simple closed curve in R. For any $\tau \in T_f$, let $l_\tau(\gamma)$ be the hyperbolic length of the unique closed geodesic homotopic to γ in R_τ.

For any $\tau_0 \in T_f$, let $\tau_n = \sigma_f^n(\tau_0)$, $n \geq 1$.

Definition 3.3. *[Hyperbolic version] We say f has* bounded geometry *if there is a constant $a > 0$ and a point $\tau_0 \in T_f$ such that $l_{\tau_n}(\gamma) \geq a$ for all $n \geq 0$ and all non-peripheral simple closed curves γ in R.*

The iteration sequence $\tau_n = \sigma_f^n \tau_0 = [\phi_n]$ determines a sequence of subsets of $\hat{\mathbb{C}}$

$$P_n = \phi_n(PS_f), \quad n = 0, 1, 2, \cdots .$$

From definition of T_f, the sets P_n do not depend on the choice of ϕ_n in the equivalent class. Moreover, since all the maps ϕ_n fix $0, 1, \infty$, it follows that $0, 1, \infty \in P_n$.

Definition 3.4 (Spherical Version)**.** *We say f has* bounded geometry *if there is a* *constant* $b > 0$ *and a point* $\tau_0 \in T_f$ *such that*

$$d_{sp}(p_n, q_n) \geq b$$

for all $n \geq 0$ *and* $p_n, q_n \in P_{n,\tau_0}$, *where*

$$d_{sp}(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}$$

is the spherical distance on $\hat{\mathbb{C}}$.

Note that $d_{sp}(z, \infty) = \frac{|z|}{\sqrt{1+|z|^2}}$. Away from infinity the spherical metric and Euclidean metrics are equivalent. Precisely, in any bounded $K \subset \mathbb{C}$, there is a constant $C > 0$ which depends only on K such that

$$C^{-1} d_{sp}(x, y) \leq |x - y| \leq C d_{sp}(x, y) \quad \forall x, y \in K.$$

The following simple lemma justifies using the term "bounded geometry" in both of the definitions above.

Lemma 3.4. *Consider the hyperbolic Riemann surface* $\hat{\mathbb{C}} \setminus S$ *equipped with the standard complex structure where* X *is a finite subset such that* $0, 1, \infty \in S$. *Let* $a > 0$ *be a constant. If every simple closed geodesic in* $\hat{\mathbb{C}} \setminus S$ *has hyperbolic length greater than* a, *then the spherical distance between any two distinct points in* S *is bounded below by a bound* b *which depends only on* a *and* $m = \#(S)$.

Proof. If $m = 3$ there are no non-peripheral simple closed curves so in the following argument we always assume that $m \geq 4$. Let $X = \{x_1, \cdots, x_{m-1}, x_m = \infty\}$ and let

$|\cdot|$ denote the Euclidean metric on $\mathbb{C}$.

Suppose $0 = |x_1| \leq \cdots \leq |x_{m-1}|$. Let $M = |x_{m-1}|$. Then $|x_2| \leq 1$, and we have

$$\prod_{2\leq i\leq m-2} \frac{|x_{i+1}|}{|x_i|} = \frac{|x_{m-1}|}{|x_2|} \geq M.$$

Hence

$$\max_{2\leq i\leq m-2} \left\{\frac{|x_{i+1}|}{|x_i|}\right\} \geq M^{\frac{1}{m-3}}.$$

Let

$$A_i = \{z \in \mathbb{C} \;\Big|\; |x_i| < z < |x_{i+1}|\}$$

and let $\text{mod }(A_i) = \frac{1}{2\pi}\log\frac{|x_{i+1}|}{|x_i|}$ be its modulus. Then for some integer $2 \leq i_0 \leq m_0 - 2$ it follows that

$$\text{mod }(A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$

Denote the extremal length of the core curve γ_{i_0} in $A_{i_0} \subset \hat{\mathbb{C}} \setminus X$ by $\|\gamma_{i_0}\|$. By properties of extremal length $\|\gamma_{i_0}\| \leq \frac{\pi}{\text{mod }(A_{i_0})}$. Since extremal length is defined by taking a supremum over all metrics and the area of $\hat{\mathbb{C}} \setminus X = 2\pi(m-2)$ for every hyperbolic metric,

$$\|\gamma_{i_0}\| \geq \frac{l^2_{\tau_n}(\gamma)}{2\pi(m-2)} \geq \frac{a^2}{2\pi(m-2)}.$$

On the other hand

$$\|\gamma_{i_0}\| = \frac{1}{\text{mod }(A_{i_0})} \leq \frac{2\pi(m-3)}{\log M}.$$

Combining these two inequalities, we have

$$M \leq M_0 = e^{\frac{4\pi(m-2)(m-3)}{a^2}}.$$

Thus the spherical distance between ∞ and any finite point in X has a positive lower bound b which depends only on a and m.

Next we show that the spherical distance between any two finite points in X has a positive lower bound depending only on a' and m. By the equivalence of the spherical and Euclidean metrics in a bounded set in the plane, it suffices to prove that $|x-y|$ is greater than a constant b for any two finite points in X.

First consider the map $\alpha(z) = 1/z$ which is a hyperbolic isometry from X to $\alpha(X)$. It preserves the set $\{0, 1, \infty\}$ so that $0, 1, \infty \in \alpha(X)$. For any $2 \leq i \leq m-1$, the above argument implies that $1/|x_i| \leq M_0$ and hence $|x_i| \geq 1/M_0$. Similarly, for any $x_i \in X$, $2 \leq i \leq m-1$, consider the map $\beta(z) = z/(z-x_i)$. It maps $\{0, \infty, x_i\}$ to $\{0, 1, \infty\}$ so that $\beta(X)$ contains $\{0, 1, \infty\}$ and it is also a hyperbolic isometry. For any $2 \leq i \leq m-1$, the above argument implies that $|x_j|/|x_j - x_i| \leq M_0$ which in turn implies that $|x_j - x_i| \geq 1/M_0^2$ proving the lemma. □

3.5 Geometric characterization of $\mathcal{M}_2$

This section is devoted to proving Theorem 1.4

3.5.1 Necessity

The proof of necessity is straightforward and we give it here. The proof of sufficiency needs preliminary material and we defer it to the next section.

Proof of necessity. If f is combinatorially equivalent to a map $g(z) \in \mathcal{M}_2$ with homeomorphisms ϕ, ψ, which are isotopic to each other rel PS_f, implying that $[\phi] = [\psi]$. Since g is holomorphic, it follows that $[\psi] = \sigma_f([\phi])$. Suppose $\tau_0 = [\phi]$, then τ_0 is a (unique) fixed point of σ_f. Using the hyperbolic metric on $\hat{\mathbb{C}} \setminus PS_f$ induced by τ_0, the non-peripheral curves have definite lengths. Since τ_0 is fixed under Thurston iteration, these lengths are fixed and f satisfies the hyperbolic definition of bounded geometry. □

3.5.2 Sufficiency

Suppose f is a post-singularly finite map in $\mathcal{AV}2$. For any $\tau = [\mu] \in T_f$, let $T^*_\tau T_f$ be the cotangent spaces of T_f at τ and ϕ^μ be a representative normalized quasiconformal map fixing $0, 1, \infty$. By Proposition 3.1 in [DH], we have $T^*_\tau T_f$ coincides with the space $\mathcal{Q}_\mu$, the space of holomorphic quadratic differentials on $\widehat{\mathbb{C}} \setminus \phi^\mu(PS_f)$ with at most simple poles on $\phi^\mu(PS_f)$. Since all the possible poles are simple, so the quadratic differentials in $\mathcal{Q}_\mu$ are integrable. The norm on the space $\mathcal{Q}_\mu$ is defined as

$$||q|| = \int_{\hat{\mathbb{C}}} |\phi(z)| dz d\overline{z}$$

for any $q \in \mathcal{Q}_\mu$.

For any $\tau \in T_f$, set $\tilde{\tau} = \sigma_f(\tau) = [\tilde{\mu}]$ and denote by ϕ^μ and $\phi^{\tilde{\mu}}$ the normalized quasiconformal maps with Beltrami coefficients μ and $\tilde{\mu}$, respectively. By definition of σ_f, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\phi^{\tilde{\mu}}"] \arrow[d, "f"] & \mathbb{C} \arrow[d, "g=g_{\mu,\tilde{\mu}}\in\mathcal{M}_2"] \\
S^2 \arrow[r, "\phi^{\mu}"] & \widehat{\mathbb{C}}
\end{tikzcd}$$

where $\tilde{\mu} = f^*\mu$.

For each $g_{\mu,\tilde{\mu}}$, we define a push-forward operator $(g_{\mu,\tilde{\mu}})_* : \mathcal{Q}_{\tilde{\mu}} \to \mathcal{Q}_\mu$ between cotangent spaces: for any $\tilde{q} = \tilde{\phi}(w)dw^2 \in \mathcal{Q}_{\tilde{\mu}}$, a co-tangent vector in $T^*_{\tilde{\tau}}T_f$, define

$$q = (g_{\mu,\tilde{\mu}})_*(\tilde{q}) = \phi(z)dz^2$$

The coefficient $\phi(z)$ of q is

$$\phi(z) = (\mathcal{L}\tilde{\phi})(z) = \sum_{g(w)=z} \frac{\tilde{\phi}(w)dw^2}{(g'(w))^2} \tag{3.1}$$

where $\mathcal{L}$ is the standard transfer operator and $\tilde{\phi}$ is the coefficient of $\tilde{q}$.

Let $(d\sigma_f)_\tau : T^*_{\tilde{\tau}}T_f \to T^*_\tau T_f$ be the co-tangent map of σ_f. Then by Proposition 3.2 in [DH], we have that the map $(d\sigma_f)_\tau$ coincides with the map $(g_{\mu,\tilde{\mu}})_* : \mathcal{Q}_\mu \to \mathcal{Q}_{\tilde{\mu}}$; that is $\|(\sigma_f)_*\| = \|d\sigma_f\| = \|g_*\|$.

Lemma 3.5. *The map $(g_{\mu,\tilde{\mu}})_*$ is contracting; that is*

$$||q|| < ||\tilde{q}||$$

Proof. Let $g = g_{\mu,\tilde{\mu}}$. Suppose there is a q such that $||\widetilde{q}|| = ||q||$, where $q = g_*(\widetilde{q})$ and let Z be the set of poles of $\tilde{q}$. Then, since g has no critical points, the poles of q

must be contained in $g(Z)$. Formula (3.1) together with the equalities

$$\int_{\hat{\mathbb{C}}} |\phi(z)| dz\, d\overline{z} = \int_{\hat{\mathbb{C}}} |\tilde{\phi}(w)|\, dw d\overline{w} = \int_{\hat{\mathbb{C}}} \Big| \frac{\tilde{\phi}(w)}{(g'(w))^2} \Big|\, dz d\overline{z}$$

imply that at every point the argument $\frac{\tilde{\phi}(w)}{(g'(w))^2}$ is the same; that is,

$$\frac{\tilde{\phi}(w)}{(g'(w))^2} = a_{w,w'} \frac{\tilde{\phi}(w')}{(g'(w'))^2}$$

for a positive real number $a_{w,w'}$. Thus, again by formula (3.1) we obtain

$$g_* q = \phi(g(w)) = a\tilde{q}(w)$$

and therefore

$$g^{-1}(g(Z)) \subset Z \cup S(f).$$

But this is a contradiction because $g^{-1}(g(Z))$ is an infinite set and $Z \cup S(f)$ is a finite set. □

As an immediate corollary we have

Corollary 3.1. *For any two points τ and τ' in T_f,*

$$d_T\Big(\sigma_f(\tau), \sigma_f(\tau')\Big) < d_T(\tau, \tau').$$

Furthermore,

Lemma 3.6. *If σ_f has a fixed point in T_f, then this fixed point must be unique. This is equivalent to saying that if a post-singularly finite f in $\mathcal{AV}2$ is combinatorially equivalent to a normalized meromorphic map $g(z) \in \mathcal{M}_2$ then $g(z)$ is unique.*

We can now finish the proof of the sufficiency of the main theorem by using bounded geometry.

Proof of sufficiency. By Theorem 3.1,

$$g_n = \phi^{\mu_n} \circ f \circ (\phi^{\mu_{n+1}})^{-1} \tag{3.2}$$

is in $\mathcal{M}_2$.

By our normalization of f we have $S(f) = \{0, \lambda\}$, $f(0) = 1$ so that $\{0, 1, \lambda, \infty\} \subset PS_f$. Recall that $P_n = \phi^{\mu_n}(PS_f)$ for $n = 0, 1, \ldots$. Since ϕ^{μ_n} fixes $\{0, 1, \infty\}$ for all $n \geq 0$, $\{0, 1, \infty\} \subset P_n$. Furthermore, from equation (3.2), g_n is normalized so that $g_n(0) = 1$ and

$$g_n(z) = \frac{\alpha_n e^{\beta_n z}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n z} + \frac{1}{\alpha_n}}.$$

If we set $\lambda_n = \phi^{\mu_n}(\lambda)$ then 0 and λ_n are omitted points for g_n. Thus

$$\lambda_n = \frac{\alpha_n}{\alpha_n - \frac{1}{\alpha_n}} \in P_n.$$

Again, from equation (3.2),

$$g_n(1) = \phi^{\mu_n}(f(1)) = \frac{\alpha_n e^{\beta_n}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n} + \frac{1}{\alpha_n}} \in P_n.$$

Therefore, we see that

$$\{0, 1, \lambda_n, g_n(1), \infty\} \subseteq P_n.$$

We assume that $\#PS_f \geq 4$ for the rest of the discussion. Recall that f has bounded geometry means that there is a constant $b > 0$ such that

$$d_{sp}(p_n, q_n) \geq b, \quad \forall n \geq 0, \quad \forall p_n, q_n \in P_n.$$

Lemma 3.7. *If f has bounded geometry with base point τ_0 and bound b, then $\{g_n\}_{n=0}^{\infty}$ lies in a compact subset $K_{\tau_0,b}$ of $\mathcal{M}_2$.*

Proof. Since $\lambda_n \neq 0, 1, \infty \in P_{n,\tau_0}$, there are constants $0 < c < C < \infty$ depending on τ_0 and b such that

$$c \leq |\alpha_n| \leq C, \quad \forall n \geq 0.$$

Similarly since $g_n(1) \neq 1$, there is a positive constant, which we again denote by c, and again depending on τ_0 and b such that

$$|\beta_n| \geq c.$$

Let $\beta_n = x_n + y_n i$. Because $g_n(1) \neq 0, \lambda_n \in P_n$, there is another constant which we again denote by C such that

$$|x_n| \leq C.$$

Now set $\sigma_n = \phi^{\mu_{n+1}}(\sigma)$. Then the winding number η_n of the closed curve $g_n(\sigma_n)$ is determined by y_n. By lemma 3.1, $\eta_n = \eta_f$ is constant, so there is another constant which we again denote by C such that

$$|y_n| \leq C.$$

Therefore, taking c, C with $0 < c < C < \infty$, respectively small and big enough depending on τ_0 and b, we see that

$$c \leq |\beta_n| \leq C, \quad \forall n \geq 0.$$

Finally we conclude that $\{g_n\}_{n=0}^{\infty}$ lies in a compact subset of $\mathcal{M}_2$. $\square$

Any integrable quadratic differential $q_{n+1} \in T^*_{\tau_{n+1}} T_f$ has, at worst, simple poles at $P_{n+1} = \phi^{\mu_{n+1}}(PS_f)$. Since $T^*_{\tau_{n+1}} T_f$ is a finite dimensional linear space, we have a quadratic differential $q_{n,max} \in T^*_{\tau_{n+1}} T_f$ with $\|q_{n,max}\| = 1$ such that

$$0 < a_n = \sup_{||q_n||=1} \|(g_n)_* q_n\| = \|(g_n)_* q_{n,max}\| < 1.$$

Moreover, by the bounded geometry condition, the potential simple poles of

$$\{q_{n,max}\}_{n=1}^{\infty}$$

lie in a compact set and hence these quadratic differentials lie in a compact subset of the space of quadratic differentials on $\hat{\mathbb{C}}$ with, at worst, simple poles at $m = \#(P_f)$ points.

Let

$$a_{\tau_0} = \sup_{n \geq 0} a_n.$$

Let $\{n_i\}$ be a sequence of integers such that the subsequence $a_{n_i} \to a_{\tau_0}$ as $i \to \infty$. By compactness, $\{g_{n_i}\}_{i=0}^{\infty}$ has a convergent subsequence, (for which we use the same notation) that converges to a meromorphic map $h \in \mathcal{M}_2$. Taking a further subsequence if necessary, we obtain a convergent sequence of sets $P_{n_i} = \phi^{\mu_{n_i}}(P_f)$

with limit set X. By bounded geometry, $\#(X) = \#(P_f)$ and $d_{sp}(p, q) \geq b$ for any $p, q \in X$. Thus we can find a subsequence $\{q_{n_i,max}\}$ converging to an integrable quadratic differential q of norm 1 whose only poles lie in X and are simple. Now by lemma 3.5, we have

$$a_{\tau_0} = ||h_* q|| < 1.$$

Thus we have proved that there is an $0 < a_{\tau_0} < 1$, depending only on b and f, such that

$$\|(\sigma_f)_*\| \leq \|d\sigma_f)\| \leq a_{\tau_0}.$$

Let l_0 be a curve connecting τ_0 and τ_1 in T_f and set $l_n = \sigma_f^n(l_0)$ for $n \geq 1$. Then $l = \cup_{n=0}^{\infty} l_n$ is a curve in T_f connecting all the points $\{\tau_n\}_{n=0}^{\infty}$. For each point $\tilde{\tau}_0 \in l_0$, we have $a_{\tilde{\tau}_0} < 1$. Taking the maximum gives a uniform $a < 1$ for all points in l_0. Since σ is holomorphic, a is an upper bound for all points in l. Therefore,

$$d_T(\tau_{n+1}, \tau_n) \leq a\, d_T(\tau_n, \tau_{n-1})$$

for all $n \geq 1$. Hence, $\{\tau_n\}_{n=0}^{\infty}$ is a convergent sequence with a unique limit point τ_∞ in T_f and τ_∞ is a fixed point of σ. □

Chapter 4

No Invariant Line Fields on Escaping sets of Some Entire Functions

This chapter is devoted to proving Theorem 1.5.

Notations: The following notations will be used in the rest of this thesis. For any $w_0 = a + ib$, we use

$$B_r(w_0) = \{w \in \mathbb{C} \mid |w - w_0| < r\}$$

to denote the disk of radius $r > 0$ centered at w_0, and use

$$S_r(w_0) = \{w = \xi + i\vartheta \mid |\xi - a| < r,\ |\vartheta - b| < r\}$$

where $r > 0$, to denote the square centered at w_0 and with side length $2r > 0$. For any point $z \in \mathbb{C}$, we use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary part of z. For any measurable set $X \subset \mathbb{C}$, let $m(X)$ denote the Lebesgue measure of X.

Theorem 1.5 is proved by contradiction. Suppose there exists $\lambda, \gamma \in \mathbb{C}^*$ such that the escaping set $I = I_{\lambda,\gamma}$ of $f = f_{\lambda,\gamma}$, supports an f-invariant line field μ. Note that f is a periodic function of period 2π and μ is f-invariant, so we have

Lemma 4.1. *$\mu(z) = \mu(z + 2\pi)$ for almost every $z \in \mathbb{C}$.*

Proof. If z is not a critical point, then

$$\mu(z + 2\pi) = \mu(f(z + 2\pi))\frac{\overline{f'(z + 2\pi)}}{f'(z + 2\pi)} = \mu(f(z))\frac{\overline{f'(z)}}{f'(z)} = \mu(z).$$

Then the lemma follows. □

The backward orbits of the critical points

$$\Lambda = \bigcup_{k\geq 0} f^{-k}(C)$$

where C is the critical set of f, is a countable set and thus has zero Lebesgue measure. Since $\mu(z)$ is measurable and

$$\operatorname{supp}(\mu) = \{z \mid |\mu(z)| = 1\} \subset I$$

has positive Lebesgue measure, by the Lebesgue Density Theorem, we can find a point $z_0 \in I \setminus \Lambda$ such that

i) $|\mu(z_0)| = 1$,

ii) for any $0 < \epsilon, \eta < 1$, there is a $\delta > 0$ such that for any $0 < r < \delta$, one has

$$\frac{m(\{z \in B_r(z_0) \mid |\mu(z) - \mu(z_0)| > \epsilon\})}{m(B_r(z_0))} < \eta. \tag{4.1}$$

(refer to Corollary 2.15 on page 20 of [McM2].)

For such a point z_0, let $z_k = f^k(z_0)$ for $k > 0$. Then we have

Lemma 4.2. *$|\Im(z_k)| \to \infty$ and $|f'(z_k)| \to \infty$, as $k \to \infty$.*

Proof. Let $\lambda = r_1 e^{i\alpha}$ and $\gamma = r_2 e^{i\theta}$. For any $z = x + iy$, we have

$$f(z) = \lambda e^{iz} + \gamma e^{-iz}$$

$$= r_1 e^{-y}\cos(x+\alpha) + r_2 e^{y}\cos(x-\theta) + i(r_1 e^{-y}\sin(x+\alpha) - r_2 e^{y}\sin(x-\theta)).$$

It follows that

$$|f(z)|^2 = |\lambda e^{iz} + \gamma e^{-iz}|^2$$

$$=(r_1 e^{-y}\cos(x+\alpha) + r_2 e^{y}\cos(x-\theta))^2 + (r_1 e^{-y}\sin(x+\alpha) - r_2 e^{y}\sin(x-\theta))^2$$

$$=r_1^2 e^{-2y} + r_2^2 e^{2y} + 2r_1 r_2\cos(\alpha+\theta).$$

So from this equation, we have that

$$|\lambda e^{iz} + \gamma e^{-iz}| \le r_1 e^{-y} + r_2 e^{y} \le (r_1 + r_2)e^{|\Im(z)|} \tag{4.2}$$

This implies that

$$|z_{k+1}| \le (r_1 + r_2)e^{|\Im(z_k)|}.$$

Since $z_0 \in I$, $|z_{k+1}| = |f(z_k)| \to \infty$ as $k \to \infty$. Thus, $|\Im(z_k)| \to \infty$ as $k \to \infty$. Since $f'(z) = \lambda i e^{iz} - \gamma i e^{-iz}$,

$$|f'(z)| = |\lambda e^{iz} + (-\gamma)e^{-iz}|$$

Then we have that

$$|f'(z)|^2 = |f(z)|^2 - 4r_1 r_2\cos(\alpha+\theta).$$

This implies the second assertion. The proof of Lemma 4.2 is completed. $\square$

We fix a point $z_0 \in I \setminus \Lambda$ satisfying i), ii) and let $z_k = f^k(z_0)$, $k > 0$.

Lemma 4.3. *There is a sequence of positive numbers $L_k \to \infty$ such that the pull back of $B_{L_k}(z_k)$ along the orbit $z_k, z_{k-1}, \cdots, z_0$ is univalent; that is, there is a simply connected domain U_k containing z_0, such that $f^k : U_k \to B_{L_k}(z_k)$ is a holomorphic isomorphism.*

Proof. By Lemma 4.2, there exists a positive integer N such that

$$|\Im(z_k) - \frac{1}{2}\log|\frac{\gamma}{\lambda}|\,| > 2\pi$$

and $|f'(z_k)| > 4$ for $k \geq N$. Take such an N and let it be fixed in the following. Now let us prove that f is univalent on $S_\pi(z_k)$ for all $k \geq N$. To see this, let us first factorize f into

$$f(z) = J \circ A \circ E(z)$$

where

$$E(z) = e^{iz}, \quad A(z) = z\sqrt{\frac{\lambda}{\gamma}}, \quad \text{and} \quad J(z) = \sqrt{\gamma\lambda}(z + z^{-1}).$$

For $k \geq N$ and $z,\ w \in S_\pi(z_k)$, $E(z)$ is univalent on the square $S_\pi(z_k)$. It maps $S_\pi(z_k)$ into either the disk of $D_{\sqrt{|\gamma/\lambda|}}(0)$ or the outside of the disk of $D_{\sqrt{|\gamma/\lambda|}}(0)$. Then $A(z)$ maps $E(S_\pi(z_k))$ either into the unit disk $D_1(0)$ or the outside of the the unit disk $D_1(0)$. Since $J(z)$ is $\sqrt{\gamma\lambda}$ times Joukowski transformation $z \to z + z^{-1}$, $J(z)$ is an univalent function on the unit disk and on the outside of the unit disk. This implies that f is univalent on $S_\pi(z_k)$.

For $z_0 \notin \Lambda$, there is a $0 < \delta < \pi/2$, such that the pull back of $B_\delta(z_N)$, along the orbit of $z_N, \cdots, z_0$, is univalent. For $0 \leq k \leq N$, let C_k denote the component of $f^{-(N-k)}(B_\delta(z_N))$ containing z_k, and L_k be the largest number such that $B_{L_k}(z_k) \subset C_k$ for $k = 0, \cdots, N$.

Now we define $L_k = |f'(z_{k-1})|\delta/4$ for $k > N$. We have the following claim:

Claim. f is univalent in $B_\delta(z_k)$ and

$$B_\delta(z_{k+1}) \subset B_{L_{k+1}}(z_{k+1}) \subset f(B_\delta(z_k))$$

for all $k \geq N$.

Proof of the claim. Since $0 < \delta < \pi/2$ and f is univalent on $S_\pi(z_k)$ for $k \geq N$, we know that for $k \geq N$, when restricted on $B_\delta(z_k)$, f is univalent. By the Koebe 1/4-theorem (refer to Theorem 1.3 on page 2 of [CG]) and the definition of L_k, $f(B_\delta(z_k))$ contains the disk $B_{L_{k+1}}(z_{k+1})$ for all $k \geq N$. Since $|f'(z_k)| > 4$ for $k \geq N$, the disk $B_{L_{k+1}}(z_{k+1})$ contains $B_\delta(z_{k+1})$ and thus $f(B_\delta(z_k))$ contains $B_\delta(z_{k+1})$ for all $k \geq N$. The claim has been proved.

Now for all k, let U_k be the component of $f^{-k}(B_{L_k}(z_k))$ containing z_0. From the definition of L_k, it follows that $f^k : U_k \to B_{L_k}(z_k)$ is univalent for every k.

Note that $L_k \to \infty$ because $|f'(z_k)| \to \infty$ as $k \to \infty$. This completes the proof of the lemma. □

Let N_0 be a large integer, and $B_k = B_{2N_0}(z_k)$. By Lemma 4.3, we have $B_k \subset B_{L_k}(z_k)$, when k is large enough. Let $V_k \subset U_k$ be the domain such that $f^k(V_k) = B_k$. Let $R_k = \max_{z \in \partial V_k} |z - z_0|$ and $r_k = \min_{z \in \partial V_k} |z - z_0|$. Since $N_0/L_k \to 0$, by the Koebe

Distortion Theorem (refer to Theorem 1.6 on page 3 of [CG]), when restricted on V_k, the map f^k behaves more and more like the linear map $z \mapsto z \cdot (f^k)'(z_0)$. Thus we have

Lemma 4.4.

$$R_k \to 0, \quad \frac{r_k}{R_k} \to 1 \text{ and } \sup_{z \in V_k} \left| \frac{(f^k)'(z)}{(f^k)'(z_0)} - 1 \right| \to 0.$$

as $k \to \infty$.

Since $r_k/R_k \to 1$, by the Lebesgue Density Theorem, we have

Lemma 4.5. *For any* $\epsilon > 0$ *and* $0 < \eta < 1$, *there is an* $N \geq 0$ *such that for all* $k \geq N$,

$$\frac{m(\{z \in V_k \mid |\mu(z) - \mu(z_0)| > \epsilon\})}{m(V_k)} < \eta.$$

Proof. Let

$$P_k = \{z \in V_k \mid |\mu(z) - \mu(z_0)| > \epsilon\}$$

and

$$Q_k = \{z \in B_{R_k}(z_0) \mid |\mu(z) - \mu(z_0)| > \epsilon\}.$$

For $B_{r_k}(z_0) \subset V_k \subset B_{R_k}(z_0)$, by Lemma 4.4, it follows that $P_k \subset Q_k$ and

$$\lim_{k \to \infty} \frac{m(B_{R_k}(z_0))}{m(V_k)} = 1. \tag{4.3}$$

Then we have

$$\frac{m(P_k)}{m(V_k)} \leq \frac{m(Q_k)}{m(V_k)} = \frac{m(Q_k)}{m(B_{R_k}(z_0))} \frac{m(B_{R_k}(z_0))}{m(V_k)}.$$

From the inequality (4.1) and the equation (4.3), we have

$$\lim_{k\to\infty} \frac{m(P_k)}{m(V_k)} = 0.$$

Then the lemma follows. □

For $k \geq 0$, let

$$T_k = S_\pi(z_k).$$

It is clear that T_k is well contained in $B_k = B_{2N_0}(z_k)$ for all $k \geq 0$.

Now for $z \in V_k$, let $\zeta = f^k(z)$. Since μ is f-invariant, we have

$$\mu(\zeta) = \mu(f^k(z)) = \mu(z)\frac{\overline{(f^k)'(z)}}{(f^k)'(z)}$$

and

$$\mu(z_k) = \mu(f^k(z_0)) = \mu(z_0)\frac{\overline{(f^k)'(z_0)}}{(f^k)'(z_0)}.$$

Since $m(T_k)/m(B_k)$ is a positive number independent of k, by Lemmas 4.4, 4.5 and the Koebe Distortion Theorem (refer to Theorem 1.6 on page 3 of [CG])), we have that

Lemma 4.6. *For any $\epsilon > 0$ and $0 < \eta < 1$, there is an $N \geq 0$ such that for all*

$k \geq N$,

$$\frac{m(\{\zeta \in T_k \mid |\mu(\zeta) - \mu(z_k)| > \epsilon\})}{m(T_k)} < \eta,$$

Since $f(z) = \lambda e^{iz} + \gamma e^{-iz}$ is dominated by λe^{iz} when $\Im z$ tends to $-\infty$, and by γe^{-iz} when $\Im z$ tends to ∞. Without loss of generality, suppose $\Im(z_k) < 0$ in the following, then $f(z)$ is dominated by λe^{iz}. (If $\Im(z_k) > 0$, we just consider γe^{-iz}).

Let

$$A_k = \{z \mid a_k < |z| < b_k\},$$

with

$$a_k = |\lambda| e^{-\Im(z_k) - \pi} \text{ and } b_k = |\lambda| e^{-\Im(z_k) + \pi}.$$

Since $f(z)$ is dominated by the λe^{iz}, it follows that the domain $f(T_k)$ looks very nearly like the slitted annulus A_k with the slit connecting the two boundary components of A_k. More precisely, as $k \to \infty$, we have

$$\frac{m(f(T_k))}{m(A_k)} \to 1 \quad \text{and} \quad \frac{m(f(T_k) \cap A_k)}{m(A_k)} \to 1.$$

So by Lemma 4.6, the lines associated to the $\mu(z)$ for $z \in T_k$ are almost parallel (here the line at each point z is given by the longest axis of the infinitesimal ellipse represented by $\mu(z)$). The action of f on these lines is the rotation given by the argument of $f'(z)$. When $|\Im(z)|$ is large, the argument of $f'(z)$ is almost $-\Re(z) + \arg(\lambda)$. Since T_k is a square with side length 2π, as z moves from the left side of T_k to the right side of T_k, the argument of $f'(z)$ takes almost each angle between 0 and 2π evenly. Note that $f(T_k)$ is like the ring A_k. It follows that on each radial line segment which is contained in $f(T_k)$, the lines associated to μ are almost parallel, and when z runs through the concentric circles in $f(T_k)$, the direction of the lines associated to μ variously continuously and take all the angles between 0 and 2π evenly.

Let

$$G_k = \{z \mid z = w - 2\pi, \text{ for some } w \in T_k\}.$$

Then

$$\frac{m(G_k \cap T_k)}{m(T_k)} \to 1 \text{ as } k \to \infty.$$

By Lemma 4.1, the line field is 2π period. This is impossible because a radial line can not be invariant under horizontal shifting. Therefore the proof of the *No Invariant Line Field* theorem is completed.

Chapter 5

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