75-1178

PRIGOZY, Stephen Israel, 1931-
RESISTANCE n-PORT NETWORKS AND GRAPHS.

The City University of New York, Ph.D., 1974
Engineering, electrical

Xerox University Microfilms, Ann Arbor, Michigan 48106

# RESISTANCE n-PORT NETWORKS AND GRAPHS

by

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A dissertation submitted to the Graduate Faculty
in Engineering in partial fulfillment of the
requirements for the degree of Doctor of Philosophy,
The City University of New York.

1974

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

RESISTANCE n-PORT NETWORKS AND GRAPHS

by

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This dissertation comprises a study of various topics, all related to the synthesis of n-port networks containing only resistances (and no ideal transformers). A summary of results is given below.

We proved that paramountcy is a necessary and sufficient condition for a fourth-order singular resistance or conductance matrix to be realizable. In addition, we included canonic networks and formulas for carrying out the synthesis.

We presented some unproved results of R.M. Foster on a system of planar canonic n-terminal, n-port networks, which have all of their accessible terminals arranged on a simple closed curve which encircles a planar network of resistances. We explained how to derive the network formulas by relating the network to the conditions for the existence of the converse of the star-mesh transformation. Foster gave two synthesis formulas, one for radial resistances and one for shunt conductances. We showed that by altering the network numbering scheme that the formulas could be unified into a single one, which is valid for all n. In addition, an algorithm for properly ordering the network ports, and a synthesis algorithm, were given.

The concept of the realizability boundary of a conductance matrix was introduced next. Any set of off-diagonal elements of an nth-order conductance matrix can be realized with some set of main-diagonal elements, since any dominant conductance matrix is known to be realizable. The set of minimal n-tuples of main-diagonal elements defines the realizability boundary of the matrix off-diagonal elements.

Networks which are able to realize matrices which are on the realizability boundary, are termed minimal networks, and are characterized by not having shunts across any ports. We also included in our treatment the charaterization of networks by their port-tree structures, discussion of matrix sign patterns, network degeneracy, and other relevant background topics.

The above concepts are used to solve the 6-terminal, 4-port conductance case. This was accomplished by enumerating all of the feasible minimal networks and indicating how they are used to locate the realizability boundary. Relevant synthesis procedures are illustrated.

The final results are concerned with degenerate networks, that is, networks which require an equation among elements of the conductance matrix to be satisfied. The major result is the characterization of degeneracy in (n+p)-terminal networks, for all n, and $p > 1$. Previous results by Biorci and Civalleri covered (n+2)-terminal networks.

ACKNOWLEDGEMENTS

I should like to thank my advisor, Professor Louis Weinberg, for his guidance, encouragement and assistance. Thanks also to Provost Egon Brenner for his sympathetic support, and to my family for its endurance.

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# CHAPTER 1. INTRODUCTION

This dissertation comprises a study of various topics, all related to the synthesis of n-port networks containing only resistances (and no ideal transformers).

Motivation for the study derives from the utility of the results in analog systems, state variable synthesis, and single-element-kind and RLC network synthesis. There is also an intellectual satisfaction in contributing some results to a classic problem.

The general problem of resistance n-port network synthesis may be stated as follows: determine the necessary and sufficient conditions on an nth-order, real symmetric matrix for it to be realizable as the open-circuit resistance or short-circuit conductance matrix of an n-port network containing only resistances. In addition, it is desirable that the sufficiency part of the conditions be proved by specifying a synthesis algorithm that can be used to realize the network. Hence we also desire to develop such algorithms.

For convenience we shall refer to the open-circuit resistance matrix and short-circuit conductance matrix as the resistance matrix and conductance matrix, respectively.

Many research workers have attacked the problem; however, though many partial contributions have been made, no general solution has yet been found. A panel discussion (PD 1) in 1962 and Weinberg's book (WE 1) published in the same year give summaries of the properties of resistance networks known up to that time. In addition, they suggest possible approaches for future attacks on the problem. In Chapter 2 we present a historical summary, which provides a good background for the balance of the dissertation. The difficulty of the problem is underscored by the fact that no major breakthroughs have appeared in the literature in recent years. It seems unlikely that a general solution will be forthcoming.

In this thesis we have targeted our research on several specific areas which are described below.

It is known that the necessary and sufficient condition for a matrix of order $n \leq 3$ to be realizable is that it be a paramount matrix. In the summary in Chapter 2 we include solutions for the second- and third-order conductance matrices. The increase in complexity in going from n=2 to n=3 is evident from inspection of the equations, and illustrates the difficulties to be expected as n becomes greater than three. Some of our effort is directed toward obtaining results for n=4. For $n \geq 4$ it is known that paramountcy is a necessary but not sufficient condition.

In Chapter 3 we prove the new result that any fourth-order paramount singular matrix is realizable by a 4-port resistance network. A large part of the chapter is concerned with a particular singular matrix called an indefinite matrix. We give canonic networks for resistance and conductance matrices of order 4 and of ranks 1,2, and 3. These networks are canonic in the sense that they can realize any matrix of the specified type and they contain the minimum possible number of resistances. We know that this result cannot be generalized, as Nambiar (NA 1) has shown that a particular fifth-order paramount indefinite matrix is not realizable as a resistance matrix. It is, however, realizable as a conductance matrix, which serves to indicate that the conditions for realizability for a resistance matrix are different from those for a conductance matrix.

When a matrix is realizable by a planar network, then the use of duality provides us with realizations for both resistance and conductance matrices. The important area of the realization of planar networks is the subject of Chapter 4. Foster (WE 3) has given a procedure for realizing planar n-terminal networks, but has not given a proof. We prove some of Foster's results, and in so doing intertwine the converse of the star-mesh transformation and results of Campbell (CA 1) , Sharpe and Spain (SH 1) , and Puckett (PU 1). Some of these basic and useful concepts, although published, have disappeared from the contemporary scene. For example, they are not included in modern textbooks. As a result, many students and engineers are unaware of them. We also give a new and compact method for stating the conditions for the existence of the converse of the star-mesh transformation and an algorithm for properly ordering the ports for a planar realization.

The synthesis of nth-order conductance matrices by n-port networks with (n+1) terminals may be considered solved, even though a specific set of necessary and sufficient conditions is not known. What is known are synthesis algorithms that yield a network if the matrix is realizable and characterize a matrix as unrealizable if the procedure cannot be carried out. We recall that in a complete network with (n+1) terminals, the number of branches is equal to the number of independent matrix elements, namely, $n(n+1)/2$. It is always possible to express the network branch conductances in terms of the matrix elements. If the conductance matrix cannot be realized by an (n+1)-terminal network, it can possibly be realized by an (n+p)-terminal network, with $p>1$. However, when p is greater than one, the number of branches in the complete network exceeds the number of independent conductance matrix elements. In general, an explicit solution for the branch conductances does not exist, unless we eliminate (or set equal to constants) some of the branches from the complete network, so that the number of branch variables remaining is equal to the number of independent matrix elements. An algorithm for choosing the missing branches is not known at present, and it is unlikely that the general resistance n-port problem will be solved by this technique.

A new approach to the (n+p)-terminal network problem is described in Chapter 5, where we make use of results due to Cederbaum and Foster. Cederbaum (CE 2) has given an example of a fourth-order paramount matrix that is not realizable as either a resistance matrix or a conductance matrix. Foster (WE 1) has shown that any dominant conductance matrix is realizable. Since we can make a conductance matrix with any fixed set of off-diagonal elements dominant, by suitably increasing the diagonal elements, we conclude that some sets of diagonal elements will permit a matrix to be realized, while others will not. Therefore, given a fixed set of off-diagonal elements, there must exist a continuum of n-tuples of diagonal elements which are on the borderline of realizability. We define this continuum as the realizability boundary. Networks which realize conductance matrices on the realizability boundary are called minimal networks, and, as we shall see, are characterized by not having shunt conductances at the ports. In explaining our results we include other pertinent background material.

In Chapter 6, we apply the principles discussed in Chapter 5 to the solution of the (n+2)-terminal 4-port case. Using a procedure derived from Guillemin (GU 3) we find and catalog all possible minimal networks. For any given conductance matrix, a maximum of 22 networks is required to be tested. In addition, certain ports of some minimal networks have constant driving-point conductances. We illustrate how to predict this effect, and suggest a practical application of this result.

In obtaining a minimal network, a number of network branches are set equal to zero, that is, removed from the complete network. When certain combinations of branches are missing from an (n+p)-terminal network, solution of the resulting network equations yields an equation among some of the elements of the conductance matrix. We term such networks degenerate. If the equation of degeneracy is either linear or nonlinear, we characterize the network as having either a linear degeneracy or a nonlinear degeneracy, respectively. Degenerate networks cannot be used for realization of a general matrix. Therefore, it is useful to identify them and eliminate them from consideration as possible network structures for solving the general synthesis problem.

Biorci and Civalleri (BI 8) have studied and given conditions for degeneracy in (n+2)-terminal networks. No results have been published on degeneracy in (n+p)-terminal networks, for $p>2$. In Chapter 7, we review Biorci and Civalleri's results and indicate an error which they made. We then give the conditions for degeneracy in (4+p)-terminal networks for p=3 and p=4 , followed by the conditions for degeneracy in (n+p)-terminal networks. These conditions may be quite important in that they provide insight to aid in the complete solution of the n-port problem. We also give the conditions for identifying a class of networks, which although not nonlinear degenerate, does not have explicit solutions for the branch conductances.

Chapter 8, the concluding chapter, gives a summary of the contributions of this dissertation, and suggests lines for future research.

We also include an appendix in which we give a list of pertinent definitions.

# CHAPTER 2. SUMMARY OF PREVIOUS RESULTS

## 2.1 INTRODUCTION

In this chapter we begin to use terms which are employed in synthesis. We assume that the reader is familiar with most of the terms, and thus do not define them in the text, so that the flow of the exposition is not disturbed. Instead, items which are underlined in this chapter are defined in the appendix.

However, we give below some conventions which we have adopted for use in presenting network diagrams and graphs.

1) Resistance branch. A branch of a network which represents a physical resistance or conductance. On the network diagrams it is represented by a line connected between two vertices. It is generally labelled with a capital letter and a single subscript, $R_j$ for resistance and $G_j$ for conductance. In some later chapters we use $c_{ij}$ to label the conductances, for reasons which will become clear. We also use a double subscript notation for branch resistances and conductances in the planar network problem.

2) Port branch. A branch connected between a pair of terminals, which designates the terminal pair as a port of the network. On the network diagrams, the port branch is represented by a directed branch and a numbered circle. The number is the port number, and the direction of the arrow indicates the orientation of the port, that is, the assumed positive direction for current and for voltage rise.

## 2.2 SUMMARY OF PREVIOUS RESULTS

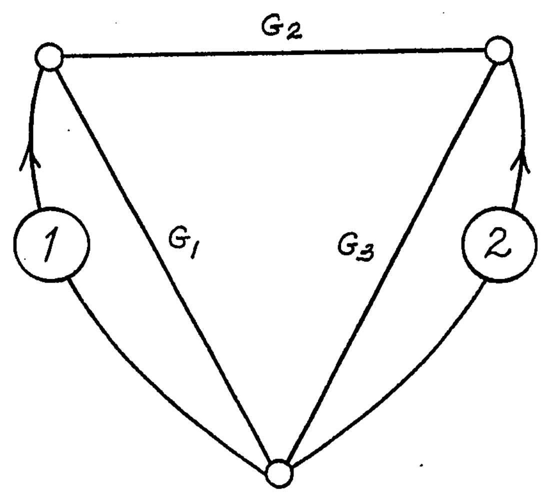
Cederbaum discovered some of the basic properties of resistance networks. He showed that a necessary condition for a matrix to be realizable by a resistance network is that the matrix be paramount (CE 4). He also proved that paramountcy is a generalization of the fact that a transformerless resistance network must have voltage ratios which are less than or equal to one (CE 1).

For matrices of order $n \leq 3$, Tellegen (TE 1) proved that paramountcy is a sufficient condition for realizability. He accomplished this by finding canonic networks which realize all second- and third-order paramount matrices (WE 1). Figures 2-1 and 2-2 show the canonic networks for synthesizing the 2-port and 3-port networks for conductance matrices. The dual networks of these networks will realize the corresponding resistance matrices.

It should be noted that the fact that paramountcy is not sufficient for $n > 3$ was proved by Cederbaum (CE 2). Using examples of paramount matrices suggested by Weinberg and Foster, he specified a matrix that was realizable as a conductance matrix but not as a resistance matrix, a second matrix that was realizable as a resistance matrix but not as a conductance matrix, and a third matrix that was not realizable as either.

Certain special types of matrices have a role in resistance network synthesis. One of these is the dominant matrix. Foster (WE 1) has shown that an nth-order dominant conductance matrix is always realizable by a 2n-terminal network. No solution is known for the corresponding case of the resistance matrix. In fact, a solution for the resistance matrix is not known even for $n=4$. If the matrix is hyperdominant, then it is of the same character as the node-to-datum conductance matrix, and therefore may always be realized as a conductance matrix, by an (n+1)-terminal network, with the ports arranged as a star tree. Again, no general solution is available for the resistance matrix. However, a solution due to Foster (FO 1) for the fourth-order resistance matrix is given.

Foster also studied a class of planar n-terminal canonic networks. His results were reported by Weinberg (WE 3). The networks are defined to have the terminals arranged on a simple closed curve, with the resistances inside the curve. Such networks are capable of realizing nth-order hyperdominant indefinite matrices. No proofs of Foster's results have been given.

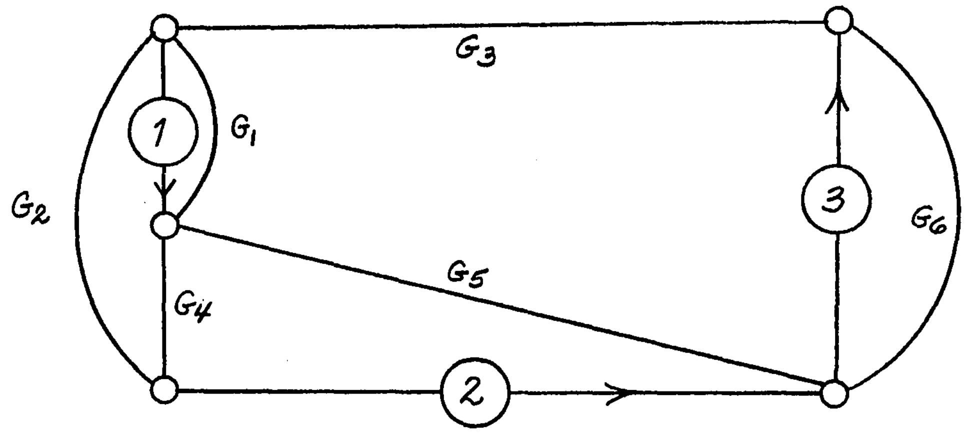



$G_1 = g_{11} + g_{12}$

$G_2 = -g_{12}$

$G_3 = g_{22} + g_{12}$

Fig. 2-1. 2-port conductance network




$$G_1 = g_{11} + \frac{g_{13}g_{22}(g_{12} - g_{13})}{g_{13}g_{22} - g_{23}(g_{12} + g_{22} - g_{23})}$$

$$G_2 = \frac{(g_{12}-g_{23})(g_{12}g_{23}-g_{13}g_{22})}{g_{13}g_{22}-g_{23}(g_{12}+g_{22}-g_{23})}$$

$$G_3 = \frac{g_{13}g_{22}-g_{12}g_{23}}{g_{22}-g_{23}}$$

$$G_4 = g_{12} + \frac{g_{22}g_{23}(g_{22}-g_{23})}{g_{13}g_{22}-g_{23}(g_{12}+g_{22}-g_{23})}$$

$$G_5 = \frac{g_{13}g_{22}-g_{12}g_{23}}{g_{23}}$$

$$G_6 = g_{33} + \frac{(g_{22}-g_{23})(g_{23}-g_{13})}{g_{12}-g_{13}}$$

Fig. 2-2. 3-port conductance network

Foster conjectured that a particular fifth-order hyperdominant indefinite matrix was unrealizable as a resistance matrix. Nambiar (NA 1) proved that this was indeed true, and also gave the necessary and sufficient conditions for an nth-order indefinite matrix to be the resistance matrix of an n-port resistance network. In addition, Nambiar presented the following open question (NA 2): "Given a real, symmetric, indefinite matrix of order n and rank n-1, find a finite procedure to decide the realizability (without transformers) of the matrix as a short-circuit conductance matrix." This question is still unanswered so that an interesting situation exists here: the resistance case is solved but the conductance case is not. We recall that the reverse situation is true for a dominant matrix.

Guillemin (GU 2) showed that if an nth-order matrix is uniformly tapered, it may always be realized as a conductance matrix by an (n+1)-terminal network with the ports arranged as an oriented linear tree.

An algebraic connection between resistance and conductance matrices and the topological properties of network graphs was made by Cederbaum (CE 5). He showed that an nth-order matrix, which can be decomposed into the matrix triple product $TDT^t$, where D is a diagonal matrix with positive diagonal elements, can be realized as a conductance matrix by a network of rank n, if T is a cut-set matrix, or it can be realized as a resistance matrix by a network of nullity n, if T is a circuit matrix. Cederbaum gives an algorithm for carrying out the decomposition, if it exists. T is a matrix with special properties called a $\theta$-matrix.

Brown and Lee (BR 3,4) give some necessary conditions on an nth-order matrix for it to be realizable as a resistance matrix by a network of nullity n. In addition, they have given the complete solution for a network of nullity n when n=4. Lee (LE 2) and Harrison (HA 1) have given computer algorithms for realizing resistance matrices by networks which have the ports arranged as star trees.

Another topological approach for conductance matrices realizable by (n+1)-terminal networks has been developed. This was first suggèsted by Guillemin (GU 2), and then amplified by Biorci and Civalleri (BI 1,2), and Cederbaum, Halkias, and Kim (CE 6). The method makes use of the fact that the graph of the port structure of an (n+1)-terminal network is a tree that determines the sign pattern of the conductance matrix. In other words, they found that the port structure of a given realizable conductance matrix can be determined from the sign pattern of the matrix. Once the port tree is known, a linear transformation to a star tree yields the network elements. The major difficulty with this procedure is that a matrix with k zero entries above the main diagonal has $2^k$ possible sign patterns. This is true because it is necessary to assume a sign for each zero entry. It may be necessary to try all possible sign patterns to determine if a realization exists. Thus if k is large, Cederbaum's decomposition method is preferred. If the method using the sign pattern fails, then the conductance matrix cannot be realized by an (n+1)-terminal network. A computer program for implementing the above was written in 1967 (HO 1). Boesch (BO 1,2) reconstructs the port tree by use of the path-to-ground matrix, which he determines algebraically from the matrix sign pattern. Zero matrix entries cause similar difficulties with this procedure. Thus the realization of conductance matrices by (n+1)-terminal networks is a solved problem.

If a conductance matrix cannot be realized by an (n+1)-terminal network, it can possibly be realized by an (n+p)-terminal network, where $p \geqq 2$. In this case the port subgraph forms p disjoint subtrees. The existence of matrices requiring networks with more than (n+2) terminals was demonstrated by Lempel and Cederbaum (LE 1), who showed a class of matrices that requires networks with (2n-1) terminals.

The first theoretical work on the characterization of n-port resistance networks with (n+p) terminals, with $p \geqq 2$, was done by Cederbaum in 1958 (CE 4). Briefly, Cederbaum first used the fact that

$Z_m = LRL^t$, where $Z_m$ is a resistance matrix of an m-port network, L is a fundamental circuit matrix, and R is a diagonal matrix with positive diagonal elements. The conductance matrix of the m-port network is $Y_m = (LRL^t)^{-1}$. Now suppose that we consider only n of the ports, $n < m$, to form an n-port network from the m-port network, and short-circuit the unused ports. Then the conductance matrix of the n-port network can be obtained from $Y_m$ by deleting the rows and columns of $Y_m$ corresponding to the m-n short-circuited ports. Thus an nth-order conductance matrix is realizable if it is a principal submatrix of an mth-order realizable matrix $Y_m$. A solution, if one exists, is not necessarily unique, as there may be many matrices which can solve a particular problem. Unfortunately, there is no known method for constructing a matrix $Y_m$ to arrive at a solution. However, the theory provides insight into the problem.

Guillemin (GU 3) proposed the first practical method for attacking the problem. He showed that the realization of an (n+2)-terminal network can be reduced to finding two networks with identical graphs and then connecting them in parallel. One network is derived directly from an expanded conductance matrix, the other from a matrix of rank 1. The expanded matrix consists of the conductance matrix with an additional row and column of zeros. Each network will have some positive and some negative branch conductances; and it is therefore required that we be able to choose the parameters of the rank 1 matrix in such a manner that the final parallel combination contains only non-negative conductances. However, there is no algorithm for choosing the parameters of the rank 1 matrix. Guillemin demonstrated trial-and-error solutions to several problems. He also indicated that the procedure may be generalized to the synthesis of (n+p)-terminal networks.

Several other methods (derived from or similar to Guillemin's method) have appeared in the literature (BR 1, LU 1,2,3, JA 1, RE 1, WA 1). These methods, like Guillemin's method, are also heuristic.

The problem that has not been resolved by any of these procedures arises because an (n+2)-terminal network contains n+1 more branches than the number of independent conductance matrix elements. Swaminathan and Frisch (SW 1) give some necessary conditions for the realization of conductance matrices by (n+p)-terminal networks.

Biorci and Civalleri (BI 8) showed that if n+1 branches of an (n+2)-terminal network are set equal to zero, the resulting set of nonlinear equations can be inverted to give explicit equations for the branch conductances, provided that the network is not degenerate. Briefly, as indicated previously, a network is degenerate when an equation among some elements of the matrix is required to be satisfied. We discuss degenerate networks in detail in Chapter 7.

# CHAPTER 3. REALIZATION OF FOURTH-ORDER SINGULAR MATRICES

## 3.1 INTRODUCTION

As previously noted in the introductory chapters, paramountcy is known to be a necessary and sufficient condition for a matrix to be realizable by a resistance network, when $n \leq 3$. Paramountcy is also known to be necessary for $n > 3$; however, it is no longer sufficient. In this chapter, we extend the class of realizable paramount matrices by formulating necessary and sufficient conditions for the realization of fourth-order singular matrices. For convenience, we first consider indefinite matrices and then discuss singular matrices that are not indefinite.

A strong stimulus for research on indefinite matrices was provided by Foster (FO 3) in 1961. He presented a fifth-order hyper-dominant indefinite matrix and conjectured that it was unrealizable as the resistance matrix of a 5-port network. (As previously pointed out (WE 1, p.366) such matrices of any order are realizable as conductance matrices.)

Nambiar (NA 1) followed up on this and proved that Foster's matrix was unrealizable. This was a startling result to Foster at the time. He remarked (FO 3): "If this contention is correct, there are certain consequences, among them the following: 1) some of the preconceived notions as to duality may have to be revised, and 2) the necessary and sufficient conditions for physical realizability of a matrix will differ according to whether it is to be realized as an admittance or as an impedance matrix".

Nambiar (NA 1) also proved a necessary and sufficient condition for an nth-order indefinite matrix of rank n-1, to be the resistance matrix of an n-port network. In addition, he (NA 2) presented an open problem, approximately the dual of the preceding problem which he had solved. "Given a real, symmetric, indefinite matrix of order n and rank n-1, find a finite procedure to decide the realizability (without transformers) of the matrix as a short-circuit conductance matrix." The problem remains unsolved.

We will review some of Nambiar's proofs and results. We will then prove that any fourth-order paramount, indefinite matrix of rank 3, is realizable as both a resistance matrix and a conductance matrix. We will also derive canonic networks with element value formulas. This will be followed by a similar treatment of fourth-order indefinite matrices of ranks 2 and 1. Finally, we indicate how to realize any paramount, singular, nonindefinite fourth-order matrix.

## 3.2 NAMBIAR'S RESULTS

Nambiar initially develops his work by the use of a theorem due to Puckett (PU 1).

Theorem 3-1. If $R = [r_{ij}]$ is the resistance matrix of an n-port resistance network, and if the ports of the network form an oriented circuit, then the matrix is indefinite.

Proof: Connect a unit-current generator at each port, (as shown in Figure 3-1). Application of Kirchhoff's current law at each terminal shows that the current entering the network through each terminal is zero. Therefore, all of the port voltages are zero. Substituting in the matrix equation RI = E yields:

$$\begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{3-1}$$

Multiplying out equation (3-1) gives:

$$\sum_{j=1}^{n} r_{ij} = 0, \qquad \text{for } i = 1,2,\dots,n \tag{3-2}$$

Equation (3-2) is the definition of an indefinite matrix.

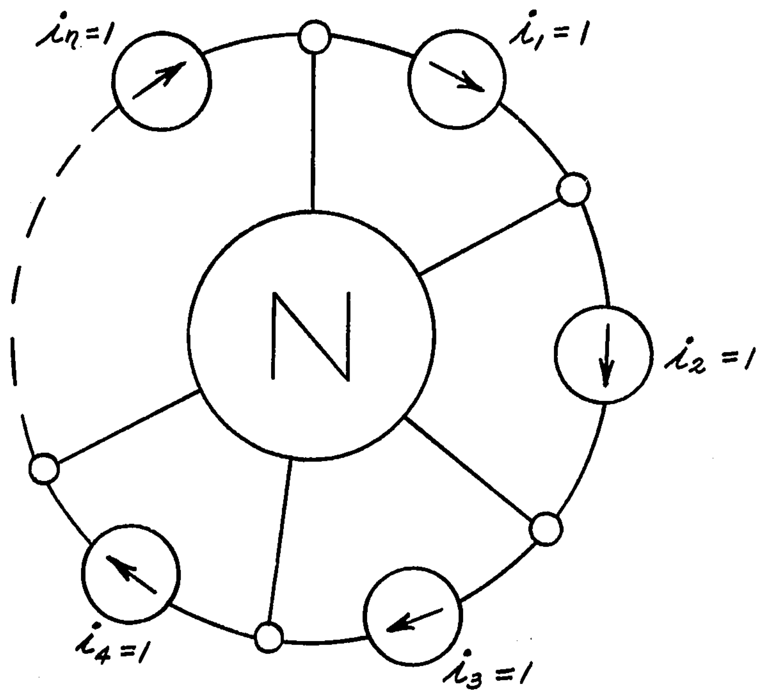



Fig. 3-1. N-port network driven by unit current generators

Nambiar also shows that the matrix in (3-1) has rank n-1.

He then asks the converse question: given that the resistance matrix R of an n-port network is indefinite and of rank n-1, can it be concluded that the ports form an oriented circuit?

The answer is in the affirmative. To demonstrate this, connect a unit-current generator at each port as in Figure 3-1. Equation (3-1) applies, and the vector of port voltages equals zero. Thus no current enters the network through any terminal. Then by Kirchhoff's current law there must be an even number of current generators connected to each terminal, half of them oriented toward the terminal, and the others oriented away from the terminal. Therefore, the set of port branches must be equal to a disjoint union of oriented circuits, where each oriented circuit is represented by its set of port branches. However, the qualification that the rank of R equals n-1, restricts the ports to a single circuit; otherwise, as Nambiar shows, the rank would be lower. We can now state the following theorem:

<ins>Theorem 3-2.</ins> (Nambiar) A network with an nth-order indefinite resistance matrix R, of rank n-1, must have its ports form an oriented circuit.

Suppose that we form an (n-1)-port network from the n-port network, by setting to zero the value of the current in the current generator of a port. The resistance matrix of the (n-1)-port network is obtained by deleting the row and column of R corresponding to the suppressed port. The remaining ports are in the form of an <ins>oriented linear tree</ins>. It is known from the work of Guillemin (GU 2) that the conductance matrix of a resistance network whose ports form an oriented linear tree must be presentable in the <ins>uniformly tapered</ins> form. (By presenting a matrix in a particular form, we mean that some rows and corresponding columns of the matrix may be multiplied by -1, or a <ins>matrix permutation</ins> be affected, to achieve the form.) Thus Theorem 3-3 follows.

Theorem 3-3. (Nambiar) Let R' be a nonsingular matrix obtained by deleting one row and the corresponding column of a given indefinite matrix R of order n and rank n-1. Let G' be the inverse matrix of R'. The necessary and sufficient condition that R be realizable as a resistance matrix is that G' be presentable in the uniformly tapered form.

Since it is known that every third-order paramount matrix is realizable (TE 1), then every third-order paramount indefinite matrix is certainly realizable. In addition, we know from the preceding introduction that certain fifth-order paramount indefinite matrices are not realizable as resistance matrices. However, no general results have been published for fourth-order indefinite matrices.

## 3.3 CHARACTERIZATION OF FOURTH-ORDER INDEFINITE MATRICES

It is convenient to separate fourth-order indefinite matrices into two types.

Type A: matrices with all off-diagonal elements nonpositive. (These are hyperdominant matrices.)

Type B: matrices which contain some positive off-diagonal elements.

Of course, a necessary condition for realizability is that these matrices be paramount (CE 4). All type A matrices are paramount since hyperdominance implies paramountcy (WE 1,p.372). However, type B matrices are not necessarily paramount. We first establish a new necessary condition for paramountcy of a type B matrix.

Lemma 3-1. A necessary condition for the paramountcy of a fourth-order indefinite matrix R is that there be at most one positive off-diagonal element in any row (or column), and that this element be less than or equal to the magnitude of every other element in the row (or column).

Proof. Since R is indefinite, the sum of the elements of each row or column is zero. Therefore, from row k we obtain $r_{kk} + r_{ka} + r_{kb} + r_{kc} = 0$, where a,b,c,k are the column indices. Hence we have $r_{kk}$ dependent upon the off-diagonal elements.

$$r_{kk} = -r_{ka} - r_{kb} - r_{kc} \tag{3-3}$$

Suppose that $r_{ka}$ and $r_{kb}$ are negative and $r_{kc}$ is positive and greater than the magnitude of say $r_{kb}$. Then from (3-3), $r_{kk} < |r_{ka}|$, thus violating paramountcy. However, if $r_{kc}$ is less than or equal to the magnitudes of both $r_{ka}$ and $r_{kb}$, then from (3-3), $r_{kk} \geq |r_{ka}|$, $r_{kk} \geq |r_{kb}|$, $r_{kk} \geq |r_{kc}|$, not violating paramountcy (but not assuring it either). Now suppose that $r_{ka}$ is negative and both $r_{kb}$ and $r_{kc}$ are positive; then again from (3-3), $r_{kk} < |r_{ka}|$, a violation of paramountcy. If $r_{ka}$, $r_{kb}$, and $r_{kc}$ are all positive, then from (3-3), $r_{kk} < 0$, again a violation of paramountcy. Thus all cases are considered and the lemma is proved.

Using Lemma 3-1 we can show that at most two independent off-diagonal elements of a type B indefinite matrix can be positive. Since these elements cannot appear in the same row or column, if p, q, s, t is any permutation of the indices 1, 2, 3, 4, and if $r_{pq}$ is positive, then $r_{st}$ is the only other possible positive element. (Note that since the matrix is symmetric, that is $r_{ij} = r_{ji}$, the positive elements occur in pairs.)

The fact that Lemma 3-1 is not a sufficient condition for paramountcy is demonstrated by the following example.

<u>Example 3-1.</u> Equation (3-4) gives M, an indefinite matrix, which satisfies Lemma 3-1.

$$M = \begin{bmatrix} 5 & -3 & 1 & -3 \\ -3 & 5 & -4 & 2 \\ 1 & -4 & 8 & -5 \\ -3 & 2 & -5 & 6 \end{bmatrix} \tag{3-4}$$

Testing (3-4) for paramountcy, we find that:

$$r_{11}r_{22} - r_{12}^2 < |r_{11}r_{23} - r_{12}r_{13}| , \quad 16 < 17.$$

This is a violation of paramountcy, since a principal minor is less than the absolute value of a nonprincipal minor built from the same rows. Therefore, type B matrices cannot be assumed to be paramount, and must be tested (WE 1,p.274).

## 3.4 REALIZATION OF TYPE A INDEFINITE MATRICES

Foster has given results (WE 3) on a class of canonic planar networks. The network terminals are arranged on a simple closed curve, and the resistive elements form a planar network which is located inside the closed curve. As a consequence of the topology of these networks, the transfer elements of their resistance and conductance matrices are nonpositive. Thus they possess the characteristics necessary for realizing type A indefinite matrices.

Figure 3-2 shows one of Foster's 4-port planar networks. We could use Foster's formulas to derive the branch values, as is discussed in Chapter 4. However, in order to illustrate the solution to a set of multilinear network equations, we formulate the equations for the resistance matrix elements, and then solve them for the branch resistance values. Using conventional analysis techniques (WE 1), we compute the six transfer resistances $r_{ij}$ of the resistance matrix R.

$$
\begin{aligned}
&\text{a)} \quad r_{12} = -(R_1R_2/T + R_6) \\
&\text{b)} \quad r_{13} = -(R_2R_4/T) \\
&\text{c)} \quad r_{14} = -(R_2R_3/T + R_5) \\
&\text{d)} \quad r_{23} = -(R_1R_4/T) \qquad\qquad (3\text{-}5) \\
&\text{e)} \quad r_{24} = -(R_1R_3/T) \\
&\text{f)} \quad r_{34} = -(R_3R_4/T) \\
&\text{g)} \quad T = R_1 + R_2 + R_3 + R_4
\end{aligned}
$$

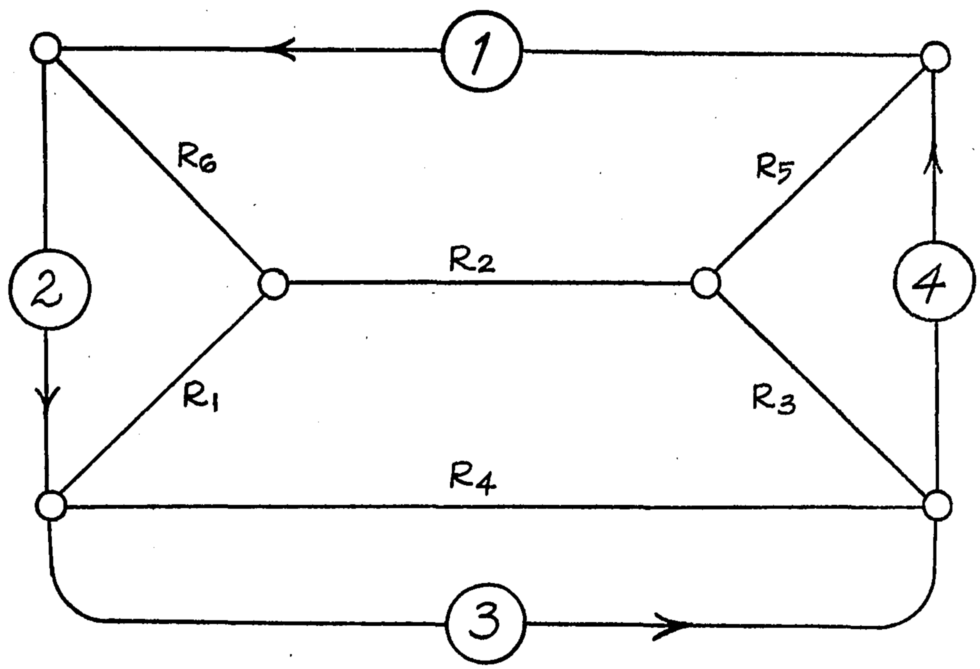



Fig. 3-2. Canonic network for realizing fourth-order hyperdominant indefinite resistance matrices

Examination of equations (3-5) reveals a set of simultaneous multilinear equations, for which a general solution algorithm does not exist. However, the equations are solvable, as is demonstrated below.

Adding (3-5b,d,f) yields:

$$r_{13} + r_{23} + r_{34} = -R_4(R_1 + R_2 + R_3)/T \qquad (3\text{-}6)$$

Multiplying (3-5d) times (3-5f) and dividing by (3-5e) yields:

$$r_{23}r_{34}/r_{24} = -R_4^2/T \qquad (3\text{-}7)$$

Adding (3-6) to (3-7) we obtain:

$$r_{13}+r_{23}+r_{34}+r_{23}r_{34}/r_{24} = -R_4(R_1+R_2+R_3+R_4)/T = -R_4 \qquad (3\text{-}8)$$

Thus $\boxed{R_4 = -N/r_{24}}$ (3-9)

where $\boxed{N = r_{13}r_{24} + r_{23}r_{24} + r_{24}r_{34} + r_{23}r_{34}}$ . (3-10)

Dividing (3-5e) by (3-5f) yields:

$$R_1/R_4 = r_{24}/r_{34} \qquad (3\text{-}11)$$

Substituting (3-9) in (3-11) we obtain:

$$\boxed{R_1 = -N/r_{34}} \qquad (3\text{-}12)$$

Similarly we obtain:

$$\boxed{R_2 = -Nr_{13}/r_{23}r_{34}} \qquad (3\text{-}13)$$

$$\boxed{R_3 = -N/r_{23}} \qquad (3\text{-}14)$$

Equation (3-5c) yields:

$$R_5 = -(r_{14} + R_2R_3/T) \qquad (3\text{-}15)$$

From (3-5b,d,e) we obtain:

$$R_2R_3/T = \frac{(R_1R_3/T)(R_2R_4/T)}{(R_1R_4/T)} = -r_{13}r_{24}/r_{23} \qquad (3\text{-}16)$$

Substituting (3-16) into (3-15) yields:

$$\boxed{R_5 = (r_{14}r_{23} - r_{13}r_{24})/-r_{23}} \qquad (3\text{-}17)$$

And similarly:

$$\boxed{R_6 = (r_{12}r_{34} - r_{13}r_{24})/-r_{34}} \qquad (3\text{-}18)$$

Next we wish to prove that any type A matrix is realizable by the network of Figure 3-2. To accomplish this, we have to show that the branch resistances are non-negative. From equation (3-10) and the fact that $r_{ij} \leq 0$, for $i \neq j$, it is clear that $N \geq 0$. Therefore, $R_1$ through $R_4$ are all non-negative. Since the denominators of (3-17) and (3-18) are both non-negative, it is required that $r_{13}r_{24} \leq r_{14}r_{23}$ and $r_{13}r_{24} \leq r_{12}r_{34}$, in order for $R_5$ and $R_6$ to be non-negative. Examination of the matrix shown below indicates the positions of the product pairs $r_{13}r_{24}$, $r_{14}r_{23}$, and $r_{12}r_{34}$.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{12} & r_{22} & r_{23} & r_{24} \\ r_{13} & r_{23} & r_{33} & r_{34} \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix}$$

If R is permuted by exchanging pairs of rows and corresponding columns (equivalent to renumbering the ports), the above product pairs change location, but retain their original pairing. For example, if we permute rows and columns 1 and 2, then $r_{13}r_{24}$ becomes $r_{14}r_{23}$, etc. Therefore, the matrix can be permuted so that the smallest product pair is $r_{13}r_{24}$. This ensures that $R_5$ and $R_6$ are non-negative. Thus all of the $R_k$ , $k = 1, \ldots, 6$, are non-negative and hence satisfy the necessary and sufficient conditions for realizability.

Since the network in Figure 3-2 is planar, its dual will realize any fourth-order type A matrix as a conductance matrix. As previously mentioned, it is known that such conductance matrices are always realizable. However, the known realization for n=4 is in general nonplanar, whereas this one is planar. Figure 3-3 shows the network. The element value formulas are the same as those of the resistance matrix case, except that $G_k$ is substituted for $R_k$, and $g_{ij}$ is substituted for $r_{ij}$.

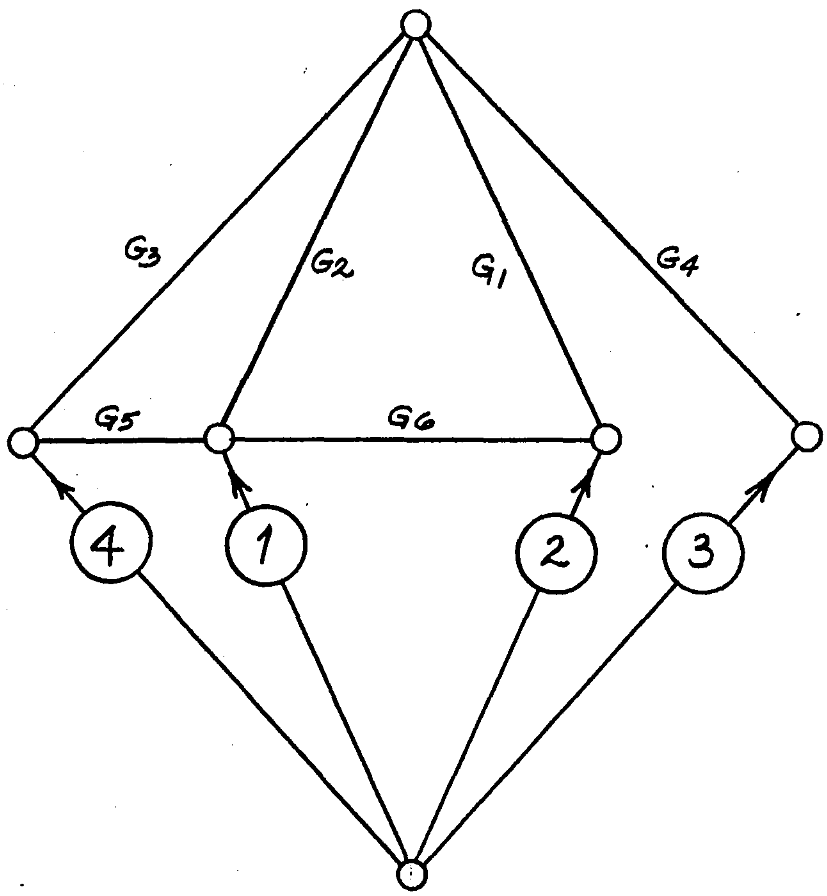



Fig. 3-3. Canonic network for realizing fourth-order hyperdominant indefinite conductance matrices

We can now state a theorem based upon the preceding material.

Theorem 3-4. Any fourth-order, type A indefinite matrix is realizable as both a resistance and a conductance matrix.

Example 3-2. As an example to illustrate the use of the formulas and networks for the type A matrices, consider the following matrix M, which can be a resistance matrix R=M, and a conductance matrix G=M.

$$M = \begin{bmatrix} 6 & -3 & -1 & -2 \\ -3 & 12 & -5 & -4 \\ -1 & -5 & 12 & -6 \\ -2 & -4 & -6 & 12 \end{bmatrix} \tag{3-19}$$

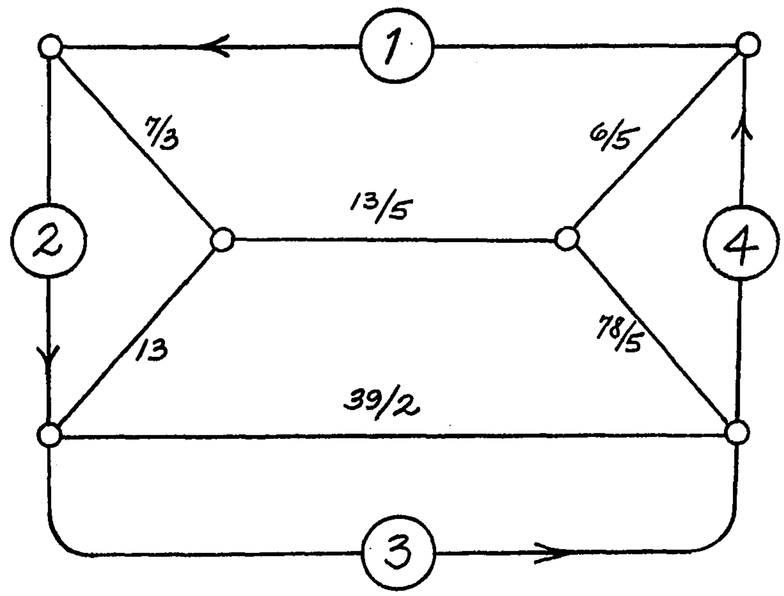
Application of the formulas (3-9,10,12,13,14,17,18) gives the realizations shown in Figure 3-4.

## 3.5 REALIZATION OF TYPE B INDEFINITE MATRICES

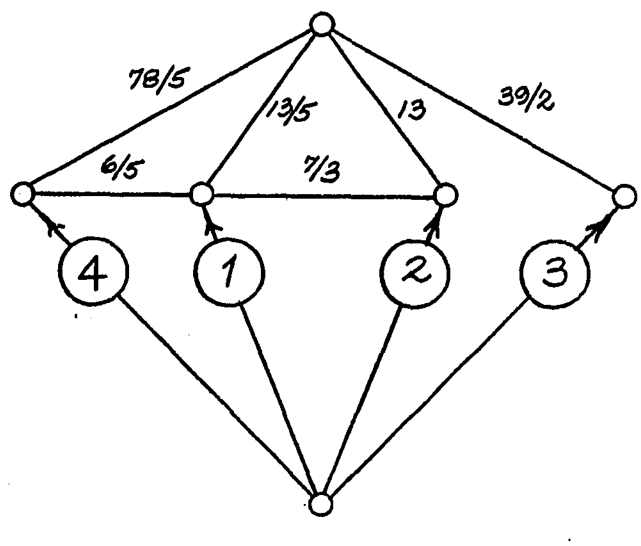
In Section 3.3, we found that Lemma 3-1 implied that in a type B matrix there are at most two independent positive off-diagonal elements, and that their subscripts are all different. Without loss of generality, let the positive elements be $r_{13}$ and $r_{24}$. If a type B matrix has some other elements positive, it can always be permuted to this form, corresponding to renumbering the ports. It is also possible that only one element is positive; in that case, let it be $r_{24}$.

Clearly the network of Figure 3-2 is unsatisfactory for realizing a type B matrix, since as we have indicated, this network can only realize nonpositive off-diagonal elements. However, if we keep the same graph of resistances but reconnect the ports as in Figure 3-5, the new configuration is able to realize matrices with positive $r_{13}$ and $r_{24}$.

Figure 3-6 shows the type B ports and the type A ports (indicated by primes) on the same diagram. From the diagram it is




Realization of R = M (values in ohms)




Realization of G = M (values in mhos)

Fig. 3-4. Examples of realizations of fourth-order hyperdominant indefinite matrices

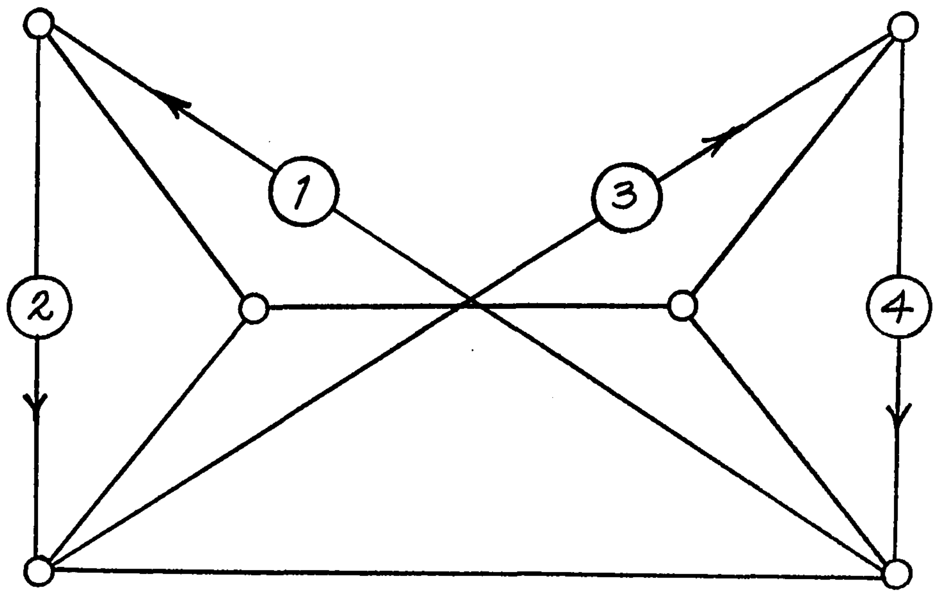



Fig. 3-5. Port rearrangement for realizing type B matrices

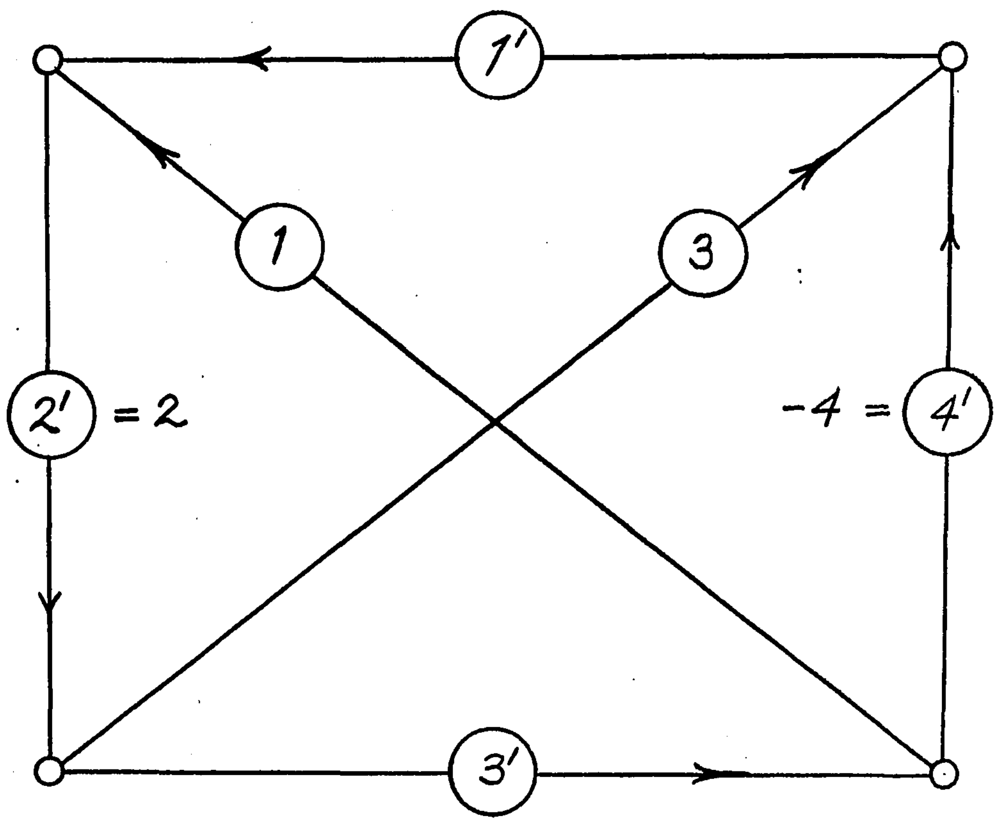



Fig. 3-6. Port graph showing the two sets of ports

evident that each of the primed port voltages can be expressed as a linear combination of the unprimed port voltages. In other words, there exists a linear transformation that transforms a resistance matrix in the unprimed system to a resistance matrix in the primed system. We make use of this to derive the branch resistance formulas for the type B (unprimed) system.

Assume that a type B matrix $R = [r_{ij}]$ is given. The corresponding matrix in the primed system is $R'$. From the matrix definitions:

$$E = RI \tag{3-19}$$

$$E' = R'I' \tag{3-20}$$

From Figure 3-6 we obtain:

$$e'_1 = e_1 + e_4$$

$$e'_2 = e_2$$

$$e'_3 = e_3 + e_4$$

$$e'_4 = - e_4$$

where the e's are the port voltages of the two systems. Writing the equations in matrix form, we obtain:

$$\begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \\ e'_4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{3-21}$$

or $E' = AE$ (3-22)

Similarly : $A^t I' = I$ (3-23)

Premultiplying (3-19) by A yields:

$$AE = ARI \tag{3-24}$$

Substituting (3-22) and (3-23) into (3-24) yields:

$$E' = [ARA^t]I' \tag{3-25}$$

And using (3-20), we obtain: $R' = ARA^t$ (3-26)

Applying the congruence transformation of (3-26) to R, and using identities derived from the fact that the sum of the elements in a row or column of R is zero, we obtain:

$$R' = \begin{bmatrix} (r_{22}+r_{33}+2r_{23}) & (r_{12}+r_{24}) & (r_{13}-r_{24}) & (r_{24}+r_{34}) \\ (r_{12}+r_{24}) & r_{22} & (r_{23}+r_{24}) & -r_{24} \\ (r_{13}-r_{24}) & (r_{23}+r_{24}) & (r_{11}+r_{22}+2r_{12}) & (r_{14}+r_{24}) \\ (r_{24}+r_{34}) & -r_{24} & (r_{14}+r_{24}) & r_{44} \end{bmatrix} \tag{3-27}$$

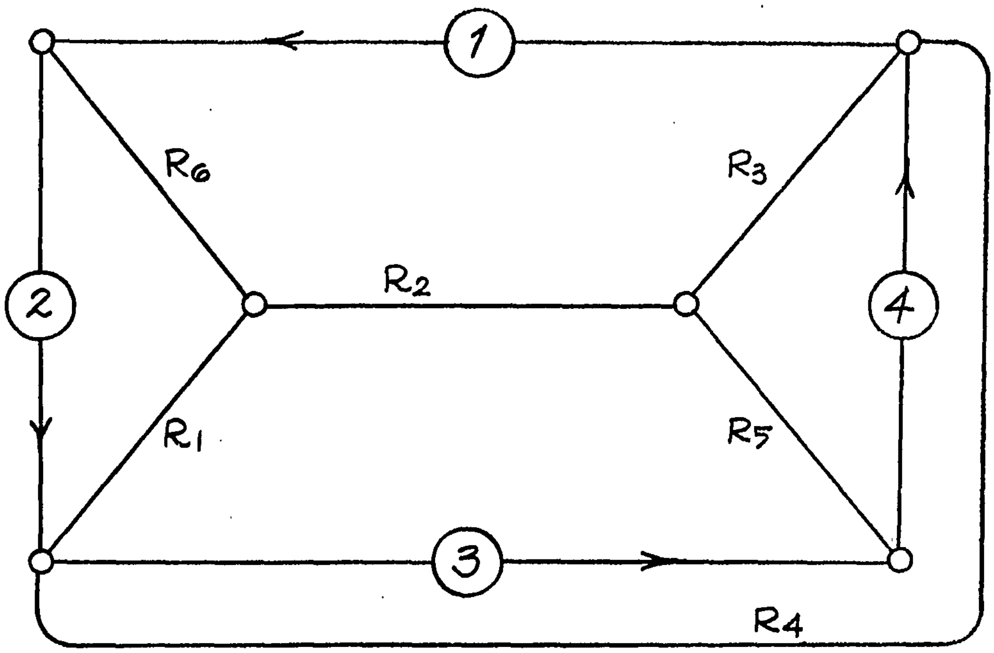
Equation (3-27) yields:

$$\begin{aligned} &\text{a)}\quad r'_{12} = r_{12} + r_{24} \\ &\text{b)}\quad r'_{13} = r_{13} - r_{24} \\ &\text{c)}\quad r'_{14} = r_{24} + r_{34} \\ &\text{d)}\quad r'_{23} = r_{23} + r_{24} \\ &\text{e)}\quad r'_{24} = - r_{24} \\ &\text{f)}\quad r'_{34} = r_{14} + r_{24} \end{aligned} \tag{3-28}$$

where the $r'_{ij}$ are now the $r_{ij}$ of equations (3-9,10,12,13,14,17,18). In order to obtain the type B network branch resistance formulas, we substitute $r'_{ij}$ for $r_{ij}$ in the type A network formulas. The resulting formulas, equations (3-29), and the type B network, redrawn so as to indicate that it is a planar network, are shown in Figure 3-7.

Next we must prove that the network of Figure 3-7 is able to realize all type B paramount matrices. We can show this by two possible procedures: 1) prove that all of the $R_k$ of (3-29) are non-negative, or 2) prove that R' of (3-27) satisfies the conditions for realizability of a type A matrix. We arbitrarily choose the second procedure.

Since R' is a type A matrix, it is hyperdominant, and thus the




$$\text{a)}\quad R_1 = \frac{r_{14}r_{23}-r_{13}r_{24}}{-(r_{24}+r_{23})}$$

$$\text{b)}\quad R_2 = \frac{(r_{14}r_{23}-r_{13}r_{24})(r_{24}-r_{13})}{(r_{24}+r_{14})(r_{23}+r_{24})}$$

$$\text{c)}\quad R_3 = \frac{r_{14}r_{23}-r_{13}r_{24}}{-(r_{24}+r_{14})} \qquad (3\text{-}29)$$

$$\text{d)}\quad R_4 = \frac{r_{14}r_{23}-r_{13}r_{24}}{r_{24}}$$

$$\text{e)}\quad R_5 = \frac{r_{23}r_{34}+r_{24}(r_{13}+r_{23}+r_{34})}{-(r_{24}+r_{23})}$$

$$\text{f)}\quad R_6 = \frac{r_{12}r_{14}+r_{24}(r_{12}+r_{13}+r_{14})}{-(r_{24}+r_{14})}$$

Fig. 3-7. Canonic network and formulas for realizing type B resistance matrices

$r'_{ij}$ of (3-28) must be nonpositive. As R satisfies Lemma 3-1, and its positive off-diagonal elements satisfy the criteria established in the second paragraph of this section, it is clear that $r'_{12}$, $r'_{14}$, $r'_{23}$, $r'_{24}$, and $r'_{34}$ are nonpositive. However, if $r_{13}$ and $r_{24}$ are both positive, then in order for $r'_{13}$ to be nonpositive, $r_{13} \leq r_{24}$. If this is not the case, we can always satisfy that condition by permuting some rows and corresponding columns of R. For example, the permutation from (1,2,3,4) to (2,1,4,3) will exchange $r_{13}$ and $r_{24}$, thus satisfying the condition.

In addition to the above, we recall from Section 3.4 that two additional conditions on R' must be satisfied, namely:

$$\begin{aligned} &\text{a)} \quad r'_{13}r'_{24} \leq r'_{14}r'_{23} \\ &\text{b)} \quad r'_{13}r'_{24} \leq r'_{12}r'_{34} \end{aligned} \tag{3-30}$$

Substituting (3-28) into (3-30) yields:

$$\begin{aligned} &\text{a)} \quad r_{23}r_{34} \geq -r_{24}(r_{13} + r_{23} + r_{34}) \\ &\text{b)} \quad r_{12}r_{14} \geq -r_{24}(r_{12} + r_{13} + r_{14}) \end{aligned} \tag{3-31}$$

Since R is symmetric and the sum of the elements in any row or column of R is equal to zero, we can make the following substitutions:

$$\begin{aligned} &\text{a)} \quad -(r_{13} + r_{23} + r_{34}) = r_{33} \\ &\text{b)} \quad -(r_{12} + r_{13} + r_{14}) = r_{11} \end{aligned} \tag{3-32}$$

Substituting (3-32) into (3-31) yields:

$$\begin{aligned} &\text{a)} \quad r_{23}r_{34} \geq r_{24}r_{33} \\ &\text{b)} \quad r_{12}r_{14} \geq r_{24}r_{11} \end{aligned} \tag{3-33}$$

Now the following substitutions are made:

$r_{24} = -(r_{44}+r_{14}+r_{34})$ and $r_{23} = -(r_{33}+r_{13}+r_{34})$ in (3-33a), and $r_{14} = -(r_{11}+r_{12}+r_{13})$ and $r_{24} = -(r_{22}+r_{12}+r_{23})$ in (3-33b).

This gives:

$$\text{a)} \quad r_{33}r_{44} - r_{34}^2 \geq - r_{33}r_{14} + r_{1+}r\,\bar{r}_{12}r_{13}$$
$$\text{b)} \quad r_{11}r_{22} - r_{12}^2 \geq - r_{11}r_{23} + r_{12}r_{13} \qquad (3\text{-}34)$$

The left-hand sides of (3-34) are both principal minors of R; whereas the right-hand sides are nonprincipal minors of R built from the same rows as the principal minors. Thus, if R is paramount the inequalities of (3-34) are always satisfied, and the paramountcy of the type B matrix R is the necessary and sufficient condition for its realizability. In addition, since the network of Figure 3-7 is planar, its dual will realize any type B paramount matrix as a conductance matrix. The dual network is shown in Figure 3-8. We can now state an important theorem based upon the preceding material.

Theorem 3-5. Any paramount, fourth-order, indefinite matrix of rank 3 is realizable as both a resistance matrix and a conductance matrix.

Proof: The proof follows from the fact that a paramount, fourth-order, rank 3, indefinite matrix is either of type A or of type B; and we have shown that both types are realizable.

## 3.6 OTHER CANONIC NETWORKS

It is well known that in a resistance network, any node which is not connected to a port may be removed by the star-mesh transformation (RO 1), yielding an equivalent network. If we operate upon the networks of Figures 3-2 and 3-7 in this manner, both transform into the same equivalent network, shown in Figure 3-9.

The resistances in the network of Figure 3-9 form a complete 4-terminal graph. From general results on n-terminal graphs discussed in Chapter 4, which are traceable to Campbell (CA 1), we obtain the following formula for the branch resistances.

$$Z_{j,j+1}^{i,i+1} = \frac{- \text{sign}\,\boxed{(j+1) - j}\; R_4^4}{R_{j,j+1}^{i,i+1}} \qquad (3\text{-}35)$$

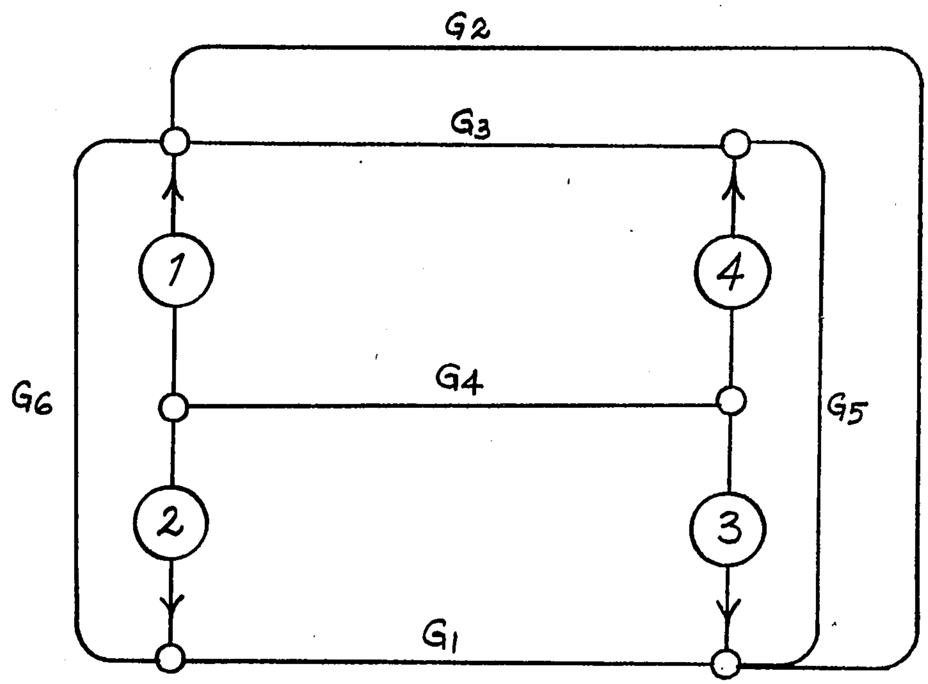
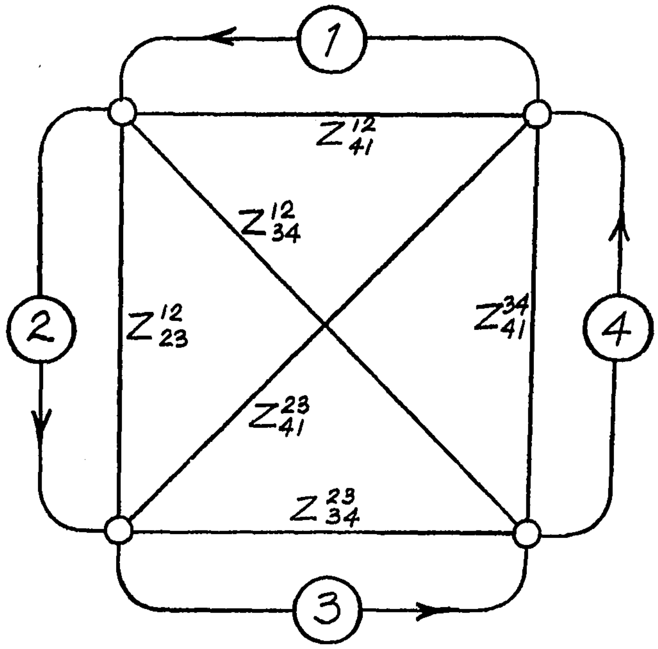



Fig. 3-8. Canonic network for realizing type B conductance matrices




$$
\begin{aligned}
&\text{a)} \quad Z^{12}_{23} = -R^{4}_{4}/R^{12}_{23} \\
&\text{b)} \quad Z^{12}_{34} = -R^{4}_{4}/R^{12}_{34} \\
&\text{c)} \quad Z^{12}_{41} = R^{4}_{4}/R^{12}_{41} \\
&\text{d)} \quad Z^{23}_{34} = -R^{4}_{4}/R^{23}_{34} \\
&\text{e)} \quad Z^{23}_{41} = R^{4}_{4}/R^{23}_{41} \\
&\text{f)} \quad Z^{34}_{41} = R^{4}_{4}/R^{34}_{41}
\end{aligned}
\qquad (3\text{-}36)
$$

Fig. 3-9. Canonic network and formulas for realizing any rank 3, fourth-order indefinite resistance matrix

In equation (3-35), $Z^{i,i+1}_{j,j+1}$ is the branch resistance of the branch connected from the junction of ports i and i+1 to the junction of ports j and j+1; $R^{i,i+1}_{j,j+1}$ is the cofactor of the resistance matrix R obtained by deleting rows i and i+1 and columns j and j+1; $R^4_4$ is a first cofactor of R (they are all equal, since R is an equicofactor matrix); when j=4, we replace j+1 with 1 (in this case, the sign $[(j+1)-j]$ becomes negative). The formulas are given on Figure 3-9 as equation (3-36).

Since the network of Figure 3-9 can realize both type A and type B resistance matrices, it is a single canonic network for all fourth-order paramount indefinite resistance matrices. And its dual network, shown in Figure 3-10, is able to realize any fourth-order paramount indefinite conductance matrix of rank 3. Of course, the $Y^{i,i+1}_{j,j+1}$ terms shown in the figure, no longer identify with the port junctions as in the resistance case.

## 3.7 QUASI-INDEFINITE MATRICES

It is possible to disguise a realizable indefinite matrix by multiplying some of its rows and corresponding columns by -1, equivalent to changing the direction of the respective ports. The matrix will still remain singular; however, its rows and columns will no longer sum to zero, and its sign pattern will be altered. In addition, the indefinite character of the matrix can be destroyed by increasing some of the main-diagonal elements. We define a matrix with either or both of the above characteristics as quasi-indefinite. We define the indefinite matrix which can be extracted from the quasi-indefinite matrix as the kernel matrix. This is summarized by the following theorem.

Theorem 3-6. Any fourth-order quasi-indefinite matrix M, which contains a paramount kernel matrix of rank 3, is realizable as both a resistance matrix and a conductance matrix.

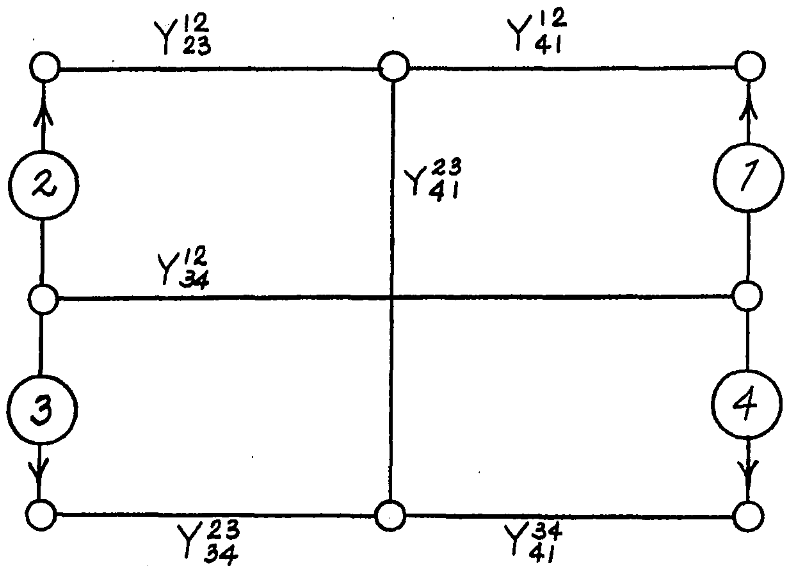



Fig. 3-10. Canonic network for realizing any rank 3, fourth-order indefinite conductance matrix

(The formulas for the $Y^{ab}_{cd}$ are the same as the formulas for the $Z^{ab}_{cd}$ in Fig. 3-9, with G substituted for R.)

Proof: If M is quasi-indefinite, its sign pattern (if different from that of an indefinite matrix) can be converted to one of an indefinite type, by multiplying some rows and corresponding columns by -1. This corresponds merely to changing the direction of the respective ports.

Suppose that M has some main-diagonal elements which are greater than required for indefiniteness. That is, the sum of the elements in the ith row is greater than zero. Let the sum be $s_i$. In the case of a resistance matrix, this means that the driving-point resistance of port i is greater by $s_i$ ohms, than that provided by the canonic network. This can always be realized by inserting a resistance of $s_i$ ohms in series with port i. Dually, if M is a conductance matrix, the driving-point conductance at port i is $s_i$ mhos greater than that provided by the canonic network. This can be realized by shunting port i with a conductance of $s_i$ mhos. Q.E.D.

A simple procedure for accomplishing the realization of a quasi-indefinite matrix M, is (if necessary) to convert M to a type A or type B sign pattern, by multiplying some rows and corresponding columns by -1, and then to express M as the sum of an indefinite matrix and a diagonal matrix with non-negative diagonal elements. We then realize the indefinite part with one of the canonic networks, and then add elements to the ports to satisfy the diagonal matrix. The following example will illustrate some of the above points.

Example 3-3. Suppose we wish to realize the following matrix M as a resistance matrix and as a conductance matrix.

$$M = \begin{bmatrix} 8 & 3 & 1 & 5 \\ 3 & 9 & 6 & 2 \\ 1 & 6 & 15 & 7 \\ 5 & 2 & 7 & 14 \end{bmatrix}$$

We want to test M to determine if it can be converted to a sign pattern of a type A or type B matrix. Since a type B matrix is required to have $r_{24} \geq 0$, if we multiply rows and columns 2 and 4 (or rows and columns 1 and 3) by -1, then the sign of $r_{24}$ does not change. The matrix, now designated as $M_1$, is shown below and is seen to have the sign pattern of a type B matrix.

$$M_1 = \begin{bmatrix} 8 & -3 & 1 & -5 \\ -3 & 9 & -6 & 2 \\ 1 & -6 & 15 & -7 \\ -5 & 2 & -7 & 14 \end{bmatrix}$$

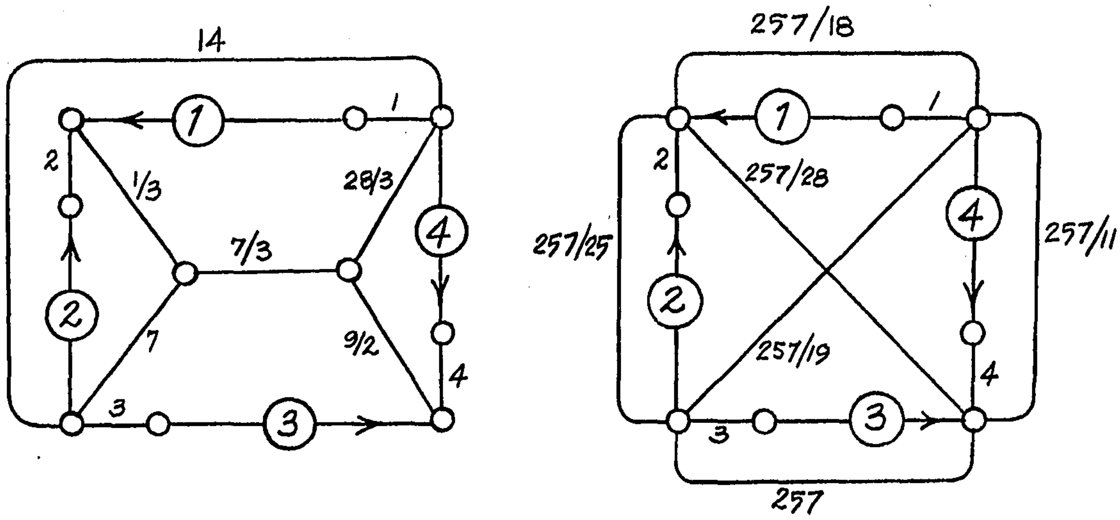
$M_1$ is not an indefinite matrix, as the sum of the elements in each row is not zero. However, it can be decomposed into $M_1 = M_2 + M_3$, as shown below.

$$M_2 = \begin{bmatrix} 7 & -3 & 1 & -5 \\ -3 & 7 & -6 & 2 \\ 1 & -6 & 12 & -7 \\ -5 & 2 & -7 & 10 \end{bmatrix} \qquad M_3 = \mathrm{diag}(1,2,3,4)$$

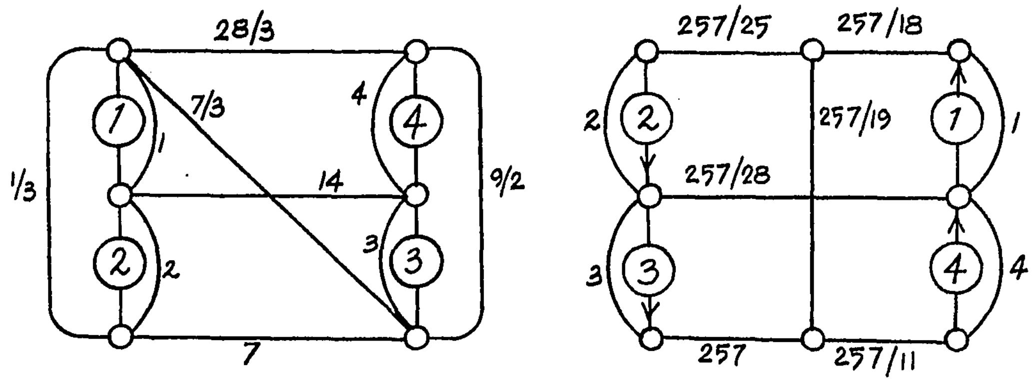
$M_2$ is a paramount indefinite type B matrix, realizable by one of the canonic networks. $M_3$ is realizable by series resistances for the resistance matrix case or shunt conductances for the conductance matrix case. Figure 3-11 gives two different realizations for each case. The reader should note the reversal of the arrows on ports 2 and 4, corresponding to the multiplication of rows and columns 2 and 4 by -1.

## 3.8 REALIZATION OF FOURTH-ORDER MATRICES OF RANKS 1 AND 2

In the proof of Theorem 3-1, we showed that the port subgraph of a network which realizes an indefinite resistance matrix R, must be comprised of disjoint circuits. In a network which contains a multicircuit port subgraph, if we set to zero the current generators of all of the ports except those of one particular circuit, the resulting resistance matrix of this new network, (which we obtain from R by deleting the rows and columns corresponding to the zero current




Realizations of M as a resistance matrix (values in ohms)




Realizations of M as a conductance matrix (values in mhos)

Fig. 3-11. Examples of realizations of a fourth-order quasi-indefinite matrix generator ports) is an indefinite matrix, by Theorem 3-1. Since the matrix is indefinite, the sum of the elements in each row is zero; and its presence therefore reduces the rank of R by one. The identical treatment may be applied to the other port circuits of R. Therefore, if the number of disjoint port circuits is c, the rank of R, $r_m = n - c$. The rank of a graph is defined as $r_g = v - 1$, where v is the number of nodes. The number of nodes of a port subgraph is $v = n - c + p$, where p is the number of separate parts of the port subgraph. Thus, $r_g = n - c + p - 1$. If the ports form a connected subgraph, then p=1, and the rank of the port subgraph is equal to the rank of the matrix. This was proved by Nambiar (NA 3).

In the fourth-order case, there is but one way of arranging the ordered four ports in two disjoint circuits. This is shown in Figure 3-12, which depicts the port subgraph. From the preceding discussion it is clear that the rank of the resistance matrix associated with the port subgraph is equal to 4-2=2. Since we showed that submatrices corresponding to each port circuit are indefinite, then the resistance matrix must have the form shown below.

$$R = \begin{bmatrix} a & -a & r_{13} & r_{14} \\ -a & a & r_{23} & r_{24} \\ r_{13} & r_{23} & b & -b \\ r_{14} & r_{24} & -b & b \end{bmatrix}$$

The fact that the rows and columns of R must each sum to zero, forces the condition: $r_{13} = -r_{14} = r_{24} = -r_{23} = c$. Therefore, the final form of R is:

$$R = \begin{bmatrix} a & -a & c & -c \\ -a & a & -c & c \\ c & -c & b & -b \\ -c & c & -b & b \end{bmatrix}$$

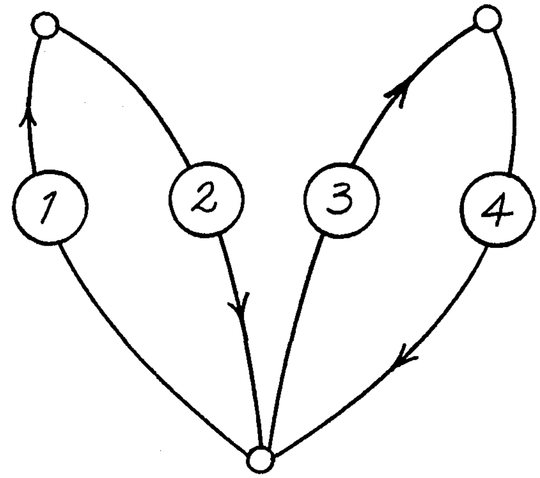
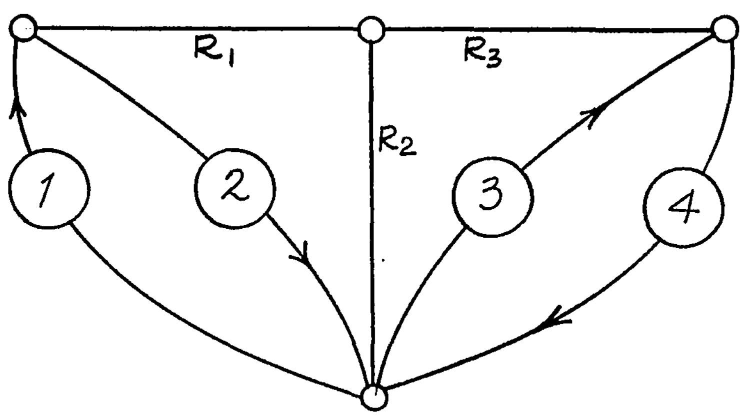



Fig. 3-12. Rank 2 port graph




$R_1 = a - c$
$R_2 = c$
$R_3 = b - c$

Fig. 3-13. Network and formulas for realizing rank 2, fourth-order indefinite resistance matrices

A network which realizes R is given in Figure 3-13, with the branch resistance formulas. Since this network is planar, its dual, shown in Figure 3-14, will realize R as a conductance matrix.

Using arguments similar to the above, we can show that the form of a fourth-order indefinite matrix of rank 1, is as shown below.

$$P = \begin{bmatrix} a & -a & a & -a \\ -a & a & -a & a \\ a & -a & a & -a \\ -a & a & -a & a \end{bmatrix}$$

Networks realizing P as a resistance matrix and as a conductance matrix are shown in Figures 3-15 and 3-16, respectively.

These rank 1 and rank 2 indefinite matrices may be contained in quasi-indefinite matrices, and should be treated as explained in Section 3.7.

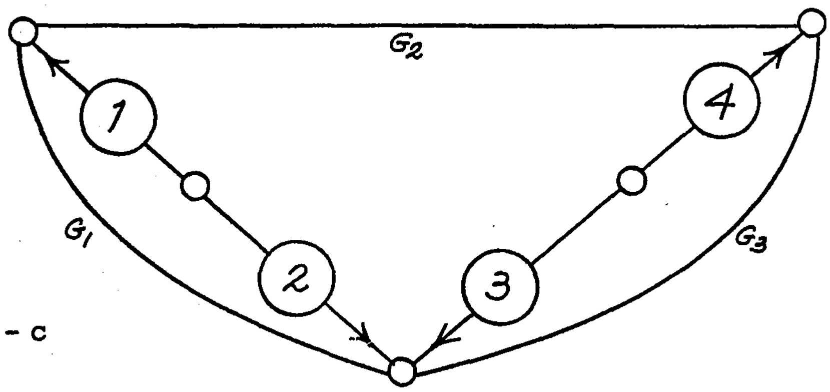
## 3.8 UNIFYING THEOREM

We can unify the preceding material by stating the following theorem.

Theorem 3-7. Any fourth-order, paramount, indefinite (or quasi-indefinite) matrix is realizable as both a resistance matrix and a conductance matrix.

## 3.9 NAMBIAR'S OPEN PROBLEM

In Section 3.1 we stated Nambiar's open problem (NA 2). It is clear that the solution to this problem for the case of n=4, is given by Theorem 3-5. The network of Figure 3-10 is a canonic network for both the type A and type B matrices. As predicted by Nambiar (NA 2), the ports of the network form an oriented cut-set, joining two connected networks of resistances.




$G_1 = a - c$
$G_2 = c$
$G_3 = b - c$

Fig. 3-14. Network and formulas for realizing rank 2, fourth-order indefinite conductance matrices

$R = a$

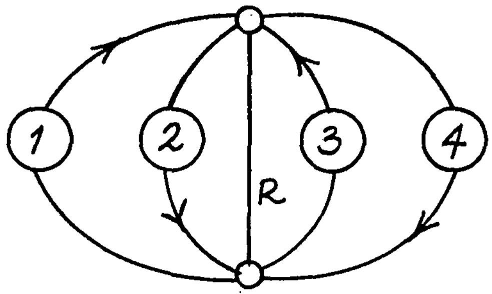



Fig. 3-15. Network and formula for realizing rank 1, fourth-order indefinite resistance matrices

$G = a$

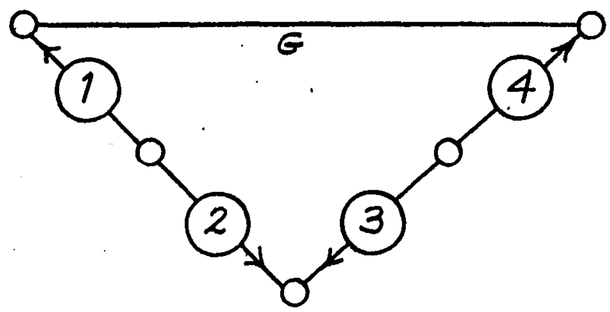



Fig. 3-16. Network and formula for realizing rank 1, fourth-order indefinite conductance matrices

## 3.10 REALIZATION OF NON-INDEFINITE FOURTH-ORDER SINGULAR MATRICES

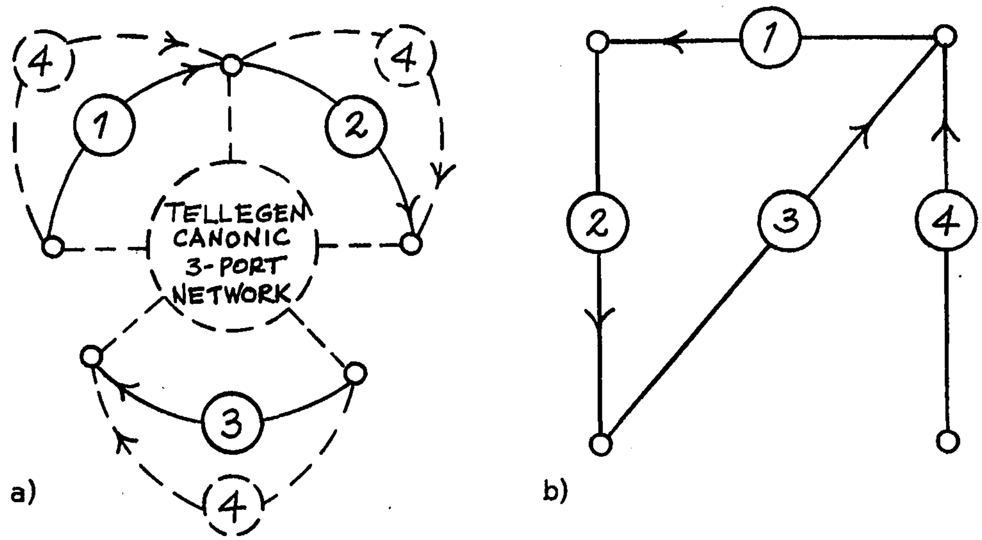
It is possible to obtain fouth-order singular matrices which are not indefinite and cannot be transformed into indefinite matrices by multiplying some rows and corresponding columns by -1. These singular matrices correspond to rank 3 and rank 2 port-graphs, where the set of ports does not form a disjoint union of circuits. Figure 3-17 shows the port-graphs for all possible resistance matrix cases.

It should be indicated at this point, that there can exist quasi-singular matrices, in the same respect as quasi-indefinite matrices bear to indefinite matrices. These quasi-singular matrices should be treated using the procedures espoused in Section 3.7; that is, using matrix permutation to convert to a standard sign pattern and main-diagonal reduction to manifest the singularity.

The port-graph of Figure 3-17a has 2 ports in parallel, which corresponds to a resistance matrix of rank 3, with two identical rows. If we delete one of the two identical rows and its corresponding column, say the row and column of R corresponding to port 4, the remaining third-order paramount matrix is realizable by one of Tellegen's canonic networks (TE 1). This is indicated by dashed lines on the port-graph. Note that the ports are divided into two trees; this is the most general representation. In certain cases, due to the character of the particular resistance matrix, simpler networks are possible; for example, if the third-order matrix is dominant.

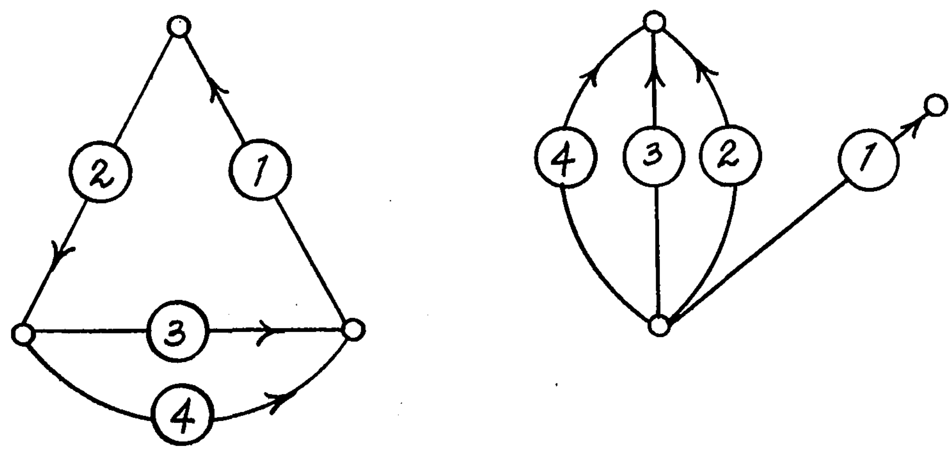
In Figure 3-17b, ports 1, 2, and 3 form an oriented circuit, therefore the submatrix of R corresponding to these 3 ports is indefinite. (It may be necessary to multiply some rows and corresponding columns by -1, to put this in evidence.) Thus:

$$\begin{aligned} r_{11} + r_{12} + r_{13} &= 0 \\ r_{12} + r_{22} + r_{23} &= 0 \qquad (3\text{-}37) \\ r_{13} + r_{23} + r_{33} &= 0 \end{aligned}$$




a) b)

Rank 3 port graphs




c) d)

Rank 2 port graphs

Fig. 3-17. Port graphs for rank 3 and rank 2 non-indefinite fourth-order singular matrix realizations

It is also clear from the port-graph, since $E_1 + E_2 + E_3 = 0$ and thus $E_1/I_4 + E_2/I_4 + E_3/I_4 = 0$, that :

$$r_{14} + r_{24} + r_{34} = 0 \tag{3-38}$$

Thus matrices which are characterized by equations (3-37) and (3-38) may be realized by a network having the port structure of Figure 3-17b. Applying (3-37) and (3-38) to $R = r_{ij}$ , we obtain:

$$R = \begin{bmatrix} -(r_{12}+r_{13}) & r_{12} & r_{13} & r_{14} \\ r_{12} & -(r_{12}+r_{23}) & r_{23} & r_{24} \\ r_{13} & r_{23} & -(r_{13}+r_{23}) & -(r_{14}+r_{24}) \\ r_{14} & r_{24} & -(r_{14}+r_{24}) & r_{44} \end{bmatrix} \tag{3-39}$$

Figure 3-18 shows the relationship between the port structure of the system under consideration, Figure 3-17b (indicated by unprimed numbers), and the port structure of an indefinite matrix network (indicated by primed numbers). The port systems are related by the linear transformation previously discussed. This leads to the congruent transformation:

$$R' = ARA^t \tag{3-40}$$

If R, as described above by (3-39), is paramount, then R' will be a paramount indefinite matrix of type A or type B, and can be realized by the techniques developed in Sections 3.4-3.6.

The A matrix is defined by equation (3-22) and is equal to:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3-41}$$

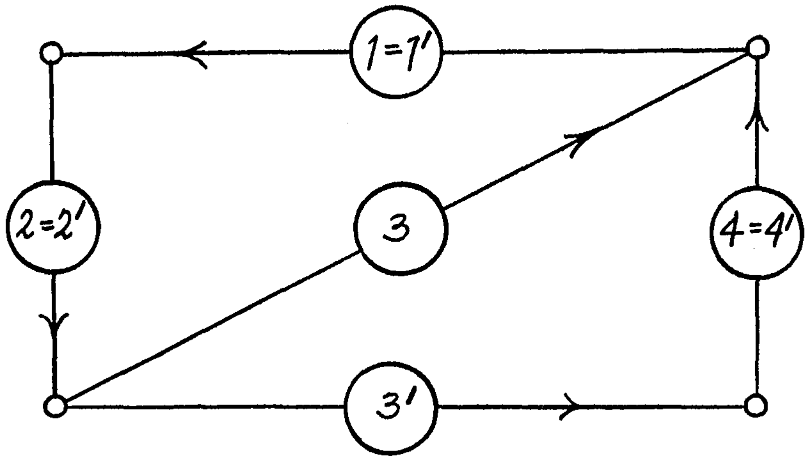
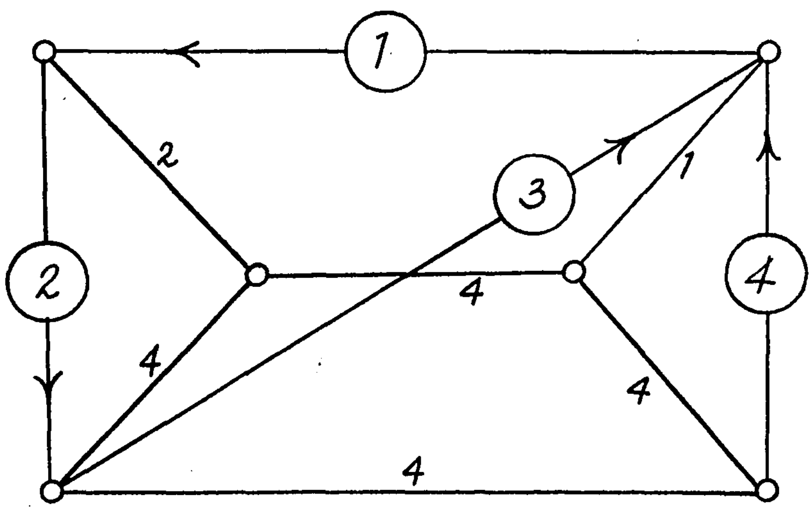



Fig. 3-18. Port structures of the primed and unprimed systems




(Values in ohms)

Fig. 3-19. Realization of R of Example 3-4 as a resistance matrix

Substituting (3-39) and (3-41) into (3-40) yields:

$$R' = \begin{bmatrix} -(r_{12}+r_{13}) & r_{12} & (r_{13}-r_{14}) & r_{14} \\ r_{12} & -(r_{12}+r_{23}) & (r_{23}-r_{24}) & r_{24} \\ (r_{13}-r_{14}) & (r_{23}-r_{24}) & \begin{matrix} 2(r_{14}+r_{24})+r_{44} \\ -(r_{13}+r_{23}) \end{matrix} & -(r_{44}+r_{14}+r_{24}) \\ r_{14} & r_{24} & -(r_{44}+r_{14}+r_{24}) & r_{44} \end{bmatrix} \tag{3-42}$$

We illustrate the above with an example.

Example 3-4. Given the following non-indefinite singular matrix R; realize it as a resistance matrix.

$$R = \begin{bmatrix} 6 & -3 & -3 & -2 \\ -3 & 5 & -2 & -1 \\ -3 & -2 & 5 & 3 \\ -2 & -1 & 3 & 4 \end{bmatrix} \tag{3-43}$$

From equation (3-42) and equation (3-43) we obtain:

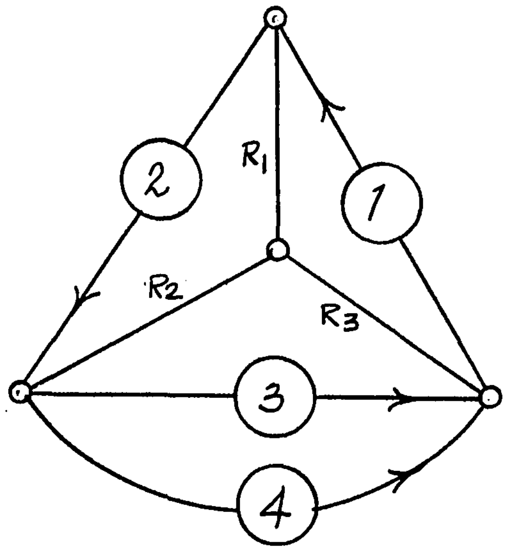
$$R' = \begin{bmatrix} 6 & -3 & -1 & -2 \\ -3 & 5 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -2 & -1 & -1 & 4 \end{bmatrix} \tag{3-44}$$

R' is a type A indefinite matrix and can be realized by the network of Figure 3-2 and associated formulas. The final realization of R is shown in Figure 3-19.

The rank 2 cases of Figure 3-17c and d are relatively simple, and the results are shown in Figures 3-20 and 3-21, respectively.

All of the preceding cases are realizable as planar networks and thus their dual networks will realize the associated conductance matrices.

We have covered all possible cases of fourth-order singular matrices, allowing us to state a final theorem.



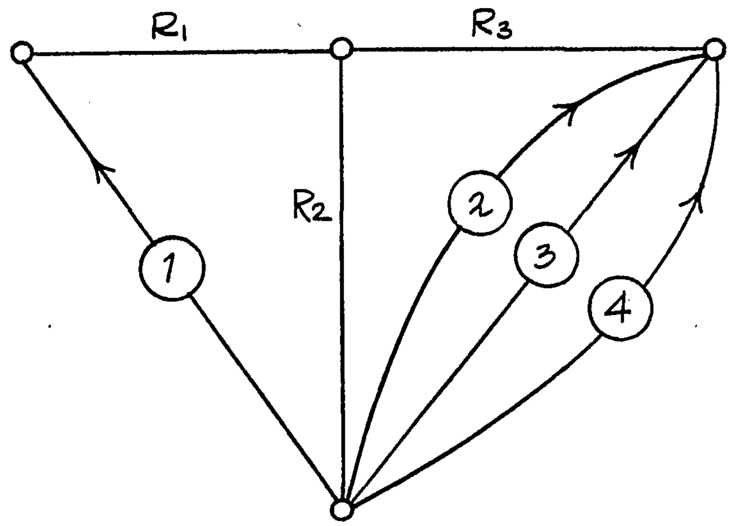

$$R = \begin{bmatrix} -(r_{12}+r_{13}) & r_{12} & r_{13} & r_{13} \\ r_{12} & -(r_{12}+r_{23}) & r_{23} & r_{23} \\ r_{13} & r_{23} & -(r_{13}+r_{23}) & -(r_{13}+r_{23}) \\ r_{13} & r_{23} & -(r_{13}+r_{23}) & -(r_{13}+r_{23}) \end{bmatrix}$$

$R_1 = -r_{12}$

$R_2 = -r_{23}$

$R_3 = -r_{13}$

Fig. 3-20. Realization of a rank 2 singular resistance matrix




$$R = \begin{bmatrix} r_{11} & r_{12} & r_{12} & r_{12} \\ r_{12} & r_{22} & r_{22} & r_{22} \\ r_{12} & r_{22} & r_{22} & r_{22} \\ r_{12} & r_{22} & r_{22} & r_{22} \end{bmatrix}$$

$$R_1 = r_{11} - r_{12}$$
$$R_2 = r_{12}$$
$$R_3 = r_{22} - r_{12}$$

Fig. 3-21. Realization of a rank 2 singular resistance matrix

## 3.11 FINAL THEOREM

Any fourth-order, paramount , singular matrix is realizable as both a resistance matrix and a conductance matrix.

## 3.12 CONCLUSION

Previously, paramountcy was shown to be the necessary and sufficient condition for realizability of a third-order matrix as a resistance matrix and a conductance matrix. We have shown that this class can now be extended to include all fourth-order singular matrices. We have given canonic networks for all possible cases. In addition, we have given the fouth-order solution for Nambiar's open problem.

# CHAPTER 4. PLANAR NETWORKS

## 4.1 INTRODUCTION

Realization of resistance and conductance matrices by planar networks is an important problem. One practical reason is that planar networks are useful in the manufacture of integrated and thick-film circuits. A planar circuit realization allows the interconnection of components by a single metalization (conductive) layer, whereas a nonplanar circuit would require several layers. Another motivation for studying planarity is its relation to duality. If a resistance matrix is realizable by a planar network, then a dual network will realize the same matrix as a conductance matrix, and vice versa.

An additional impetus for the further study of planar networks was provided by R. M. Foster in 1964. He proposed a system of canonic planar resistance networks, defined on n terminals. We can define n ports on these networks, as is discussed later. The only published record of his research is an account by Weinberg (WE 3) based on Foster's notes. Below we give the salient features from the reference.

The topology of the planar canonic networks is defined as follows. The externally accessible terminals of the network are to be arranged on a simple closed curve, within which is a planar network of resistances. this definition may seem unduly restrictive, for we can certainly generate other planar networks which do not satisfy these criteria. For example, in Figure 4-1 the network is planar, that is, there are no crossed branches; however, ports 1, 2, and 3 are embedded in the network and are not externally accessible. That is, were the network constructed on a printed circuit board (plane), all of the terminals defining the ports could not be brought out to a connector without some crossings, which would render the system nonplanar. This is illustrated in Figure 4-2. It should be clear that networks which are in accord with the definition avoid this problem. We will expand this point later when we discuss

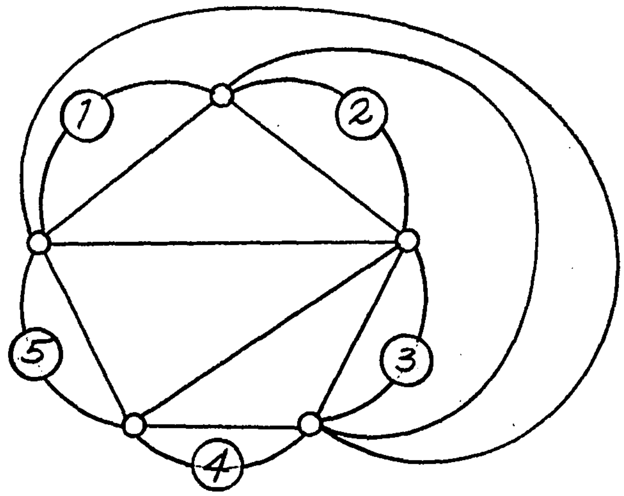



Fig. 4-1. Non-Foster planar network

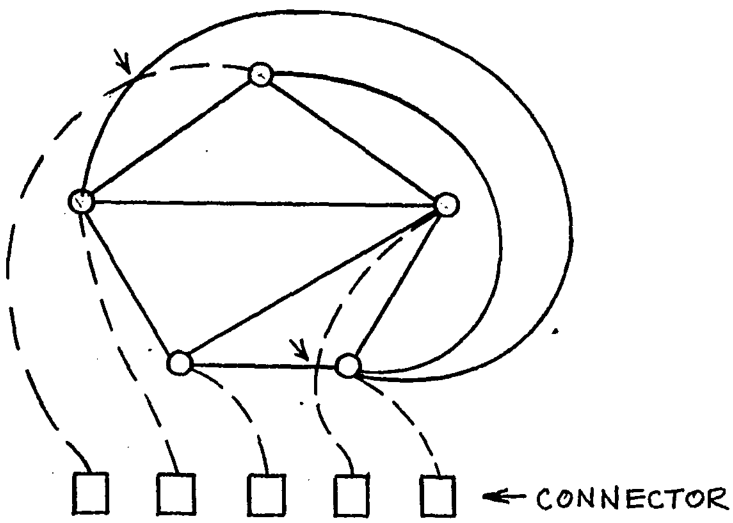



Fig. 4-2. Non-Foster planar network with terminals attached to a connector

the actual port configurations employed.

It is required that the order of the accessible terminals around the closed curve be fixed. The implication of this should be investigated. Suppose that we are given a resistance or conductance matrix to be realized on a Foster network. Is the order of the ports significant, that is, must the ports be renumbered in some manner to guarantee realizability? We discuss this point later, and show that in general there is a unique port numbering required for realizability, and we give an algorithm for finding it.

If the network is not required to be planar, then a canonic network is a complete n-terminal graph. In fact, our derivations begin with such a network, and we modify it until it is transformed into a Foster network.

The canonic network graph must have $b = n(n-1)/2$ branches corresponding to the number of independent matrix elements. We note that when n is odd, $b = kn$, where k is an integer; whereas when n is even, $b = (k+\frac{1}{2})n$. As we indicate below, this causes the networks for n even to have a slightly different form from those for n odd.

There are two basic types of canonic networks, and they are duals of one another. If n is odd, they are formed in the following manner:

Type 1. Connect the n accessible terminals with n branches to form a circuit. From the n terminals, connect a set of n radial branches inward to form a second set of n terminals. Now connect these n terminals with n branches, to form a circuit. Continue in this manner until $b = n(n-1)/2$ branches have been placed, with the proviso that if the last set of branches is radial, they connect to a common interior node.

Type 2. Connect the n accessible terminals with n radial branches directed inward to a second set of n terminals. Then continue as in type 1.

If n is even, we follow the same procedure, except in the first step, where we keep only one or the other of each pair of opposite branches. This is made clear by reference to Figure 4-3, which shows the networks through n=8.

Two different port structures are defined for the networks. One of them, for realizing resistance matrices, has n ports arranged in a circuit, with each port connected between two adjacent accessible terminals. The port structure for realizing conductance matrices has the n ports arranged on a star tree, with one end of each port connected to an accessible terminal, and the other ends connected to a common node. The 4-port versions are illustrated in Figure 4-4.

Now that we have stated the port structures, the Foster definition of planarity assumes significance, as it ensures that the system of resistances and ports, taken together, is planar. Stated in an equivalent manner: Given a planar network of resistances; place an additional terminal outside of the network. Connect each accessible network terminal to the outside terminal with a branch. If the graph of the new network is planar, then the system is in accord with Foster.

A fact that is made use of by Puckett (PU 1) and Sharpe and Spain (SH 1), is that the resistance and conductance matrices associated with the above mentioned port structures, are indefinite. Recognizing that the conductance matrix of a star tree port structure, is the node-to-datum admittance matrix, indicates that the conductance matrix is also hyper-dominant. Foster indicates that the resistance matrix is also hyper-dominant. This can be seen clearly by the use of duality. The dual of a type 1, (2) network with the ports arranged in a circuit, is a type 2, (1) network with the ports on a star tree. Both networks realize the same matrix. We take advantage of this fact later, to simplify some derivations.

The accessible terminals are numbered in natural order, around the closed curve, beginning with 1. Star tree port k is defined as connecting between terminal k and the common terminal of the star. Circuit port k is connected between terminals k and k+1.

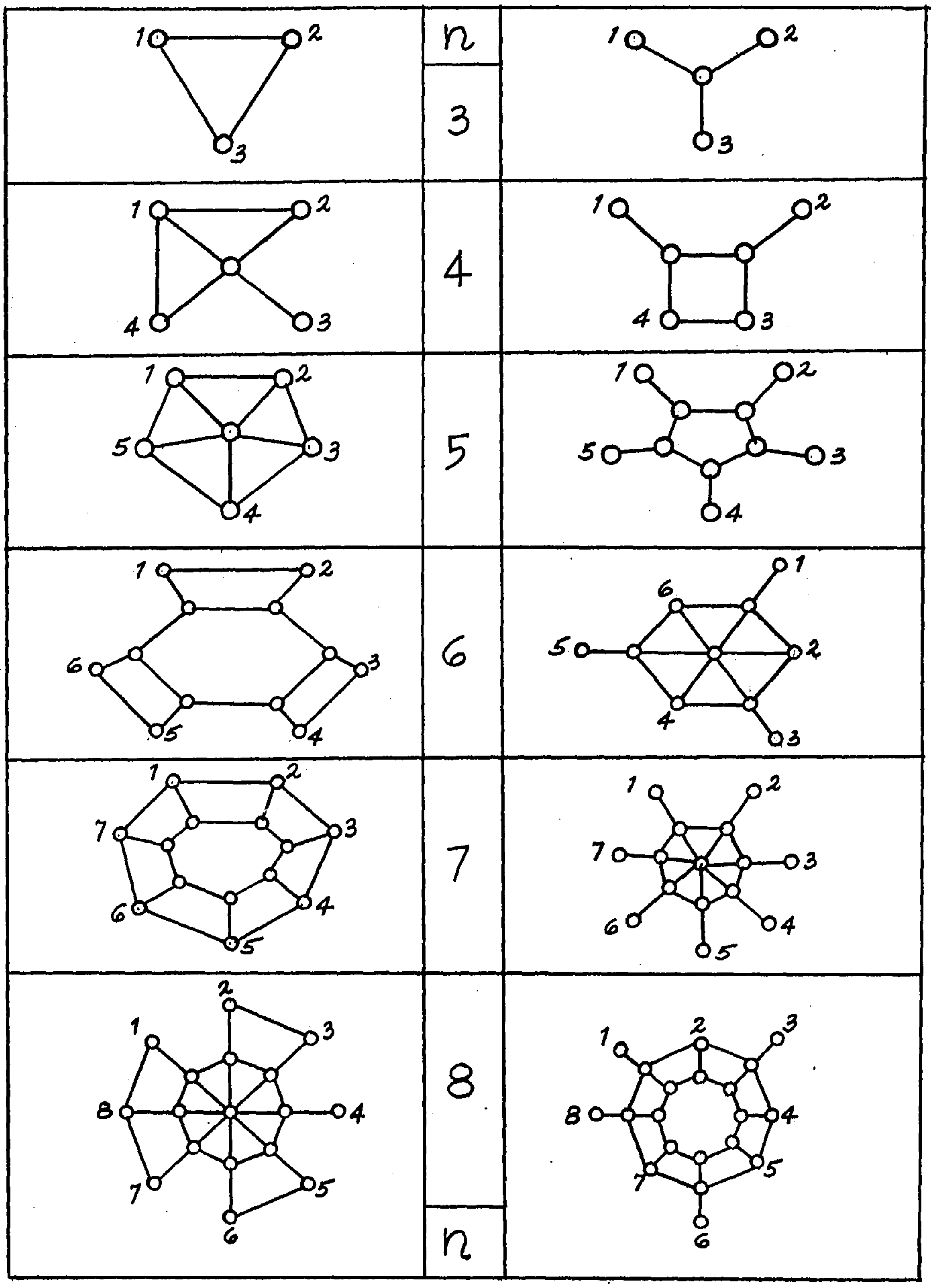
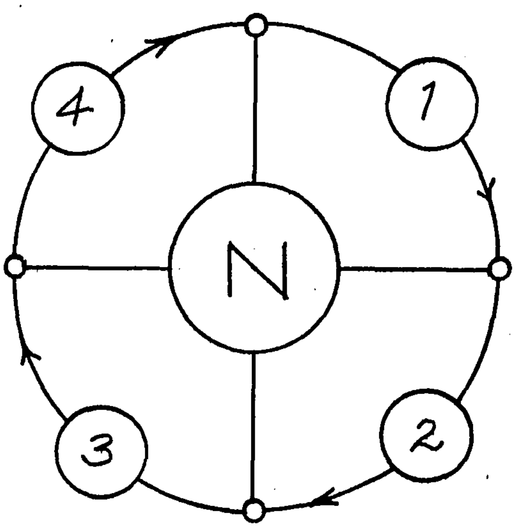
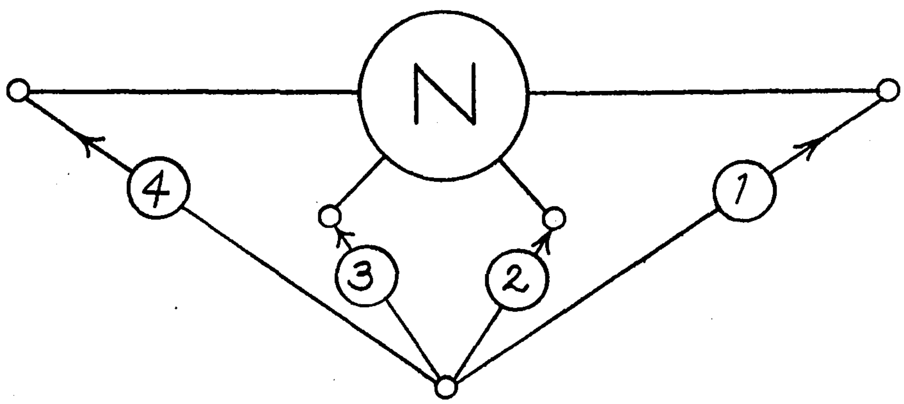



Fig. 4-3. Foster's planar networks through order n=8




a) Port structure for realizing a resistance matrix




b) Port structure for realizing a conductance matrix

Fig. 4-4. Port structures for Foster planar networks
(4-port examples)

In the reference, necessary and sufficient conditions, without proofs, are given for a resistance (or conductance) matrix to be realizable by a planar canonic network. We discuss these below and indicate some difficulties.

From the complete cycle of n terminal indices, we select a subset of 2m indices, arranged as a sequence in the same cyclic order and denoted as $(i_1, i_2, \ldots, i_m, \ldots, i_{2m})$, where $m = 1, 2, \ldots, n/2$, an integer. Form the following determinant from the resistance (or conductance) matrix.

$$\begin{vmatrix} -r_{i_m, i_{m+1}} & -r_{i_m, i_{m+2}} & \cdots & -r_{i_m, i_{2m}} \\ -r_{i_{m-1}, i_{m+1}} & -r_{i_{m-1}, i_{m+2}} & \cdots & -r_{i_{m-1}, i_{2m}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -r_{i_1, i_{m+1}} & -r_{i_1, i_{m+2}} & \cdots & -r_{i_1, i_{2m}} \end{vmatrix} \qquad (4\text{-}1)$$

A necessary and sufficient condition on the resistance (or conductance) matrix for it to be realizable by a planar network is that every determinant of the above type be greater than or equal to zero, for all m, and for every choice of indices in the subset.

We can demonstrate that the method of selection of index subsets is not precise. For example, suppose that n=4, then $m_{max} = 4/2 = 2$. A $2m_{max}$ subset is of length four, and leads to the sequence $(i_1, i_2, i_3, i_4) = (1, 2, 3, 4)$ as the only possible choice of indices. Calculating the determinant yields:

$$\begin{vmatrix} -r_{i_2, i_3} & -r_{i_2, i_4} \\ -r_{i_1, i_3} & -r_{i_1, i_4} \end{vmatrix} = \begin{vmatrix} -r_{23} & -r_{24} \\ -r_{13} & -r_{14} \end{vmatrix} = r_{14}r_{23} - r_{13}r_{24} \geqq 0$$

Now refer to Figure 3-2 of Chapter 3, which shows one of the canonic networks for n=4. As all of the off-diagonal elements of R are non-positive, the numerators of $R_5$ and $R_6$ (equations (3-17) and (3-18)) must be nonpositive. This gives:

$$r_{14}r_{23} - r_{13}r_{24} \geq 0$$

$$r_{12}r_{34} - r_{13}r_{24} \geq 0$$

However, the stated conditions, interpreted literally, only provide the first of the two required conditions. We find that in order to obtain the second condition, we must use the sequence (2, 3, 4, 1). This has the same cyclic order as the first index set, but begins with a different index.

It is pointed out that when m=2, a set of second order determinants which are related to the conditions for the existence of the converse of the star-mesh transformation, is obtained.

Rosen (RO 1) has given the formulas for the star-mesh transformation. However, in general the converse of the star-mesh transformation does not exist, as the number of branches in the mesh is n(n-1)/2 and the number of branches in the star is a smaller number n, except for the case n=3, where they are equal. Bedrosian and others (BE 1, SH 2, CA 2) have shown that if a number of conditions on the matrix, equal to the difference between the number of branches in the mesh and the number of branches in the star, n(n-3)/2, are satisfied, then the converse of the star-mesh transformation can be successfully carried out. Bedrosian gives a somewhat complex method for determining the required conditions, derived from his dissertation results. Later we give a simple algorithm for writing these conditions directly. As we shall see, these conditions are a primary key to proving Foster's formulas and conditions, as they provide a procedure for converting the nonplanar mesh to the planar star.

Finally, a synthesis method for realizing the networks is given in the reference. We assume that we are given a hyperdominant indefinite resistance matrix R. A consecutive cyclic minor of R, $R_{p,q,m}$ is then defined as follows.

$$R_{p,q,m} = \begin{vmatrix} -r_{p,q} & -r_{p,q+1} & \cdots & -r_{p,q+m-1} \\ -r_{p-1,q} & -r_{p-1,q+1} & \cdots & -r_{p-1,q+m-1} \\ \cdots & \cdots & \cdots & \cdots \\ -r_{p-m+1,q} & -r_{p-m+1,q+1} & \cdots & -r_{p-m+1,q+m-1} \end{vmatrix} \tag{4-2}$$

For n odd and $m = m_{max}$, a radial resistance $R_k$ is extracted from each terminal k. If n is even, either one or the other of each opposite pair is extracted. $R_k$ is given by:

$$R_k = \frac{R_{k-1,k,m}}{R_{k-2,k+1,m-1}} \tag{4-3}$$

The resistance matrix of the network remaining after the removal of the radial resistances, is then converted to the equivalent conductance matrix G. (Later we discuss exactly how this is accomplished.) m is then reduced by one, and a shunt conductance $G_k$ is extracted at each port k. $G_k$ is given by:

$$G_k = \frac{G_{k,k+1,m}}{G_{k-1,k+2,m-1}} \tag{4-4}$$

We continue this procedure, reducing m by one at each step, until the network is realized. If we started with the conductance matrix, we would obtain the dual network.

It is further pointed out that if the denominator of $R_k$ or $G_k$ is zero, then the numerator is also. As we shall see later, this case leads to indeterminate forms which have limiting values.

The formulas for $R_k$ and $G_k$ as given in the reference are different because ports and terminals do not correspond to one another. This difference is disturbing in the light of our knowledge of duality. Surely, if we constructed the dual of one of the planar networks, then a radial resistance in the first network should become a shunt conductance in the dual network, of the same numerical value. Then must the formulas be different? We can, in fact, make the formulas for $R_k$ and $G_k$ the same by renumbering the ports and choosing dual locations for $R_k$ and $G_k$ as shown in Figure 4-5. In order to avoid confusion, we use a double subscript notation. $R_{k,k+1}$ is a resistance connected to the junction of circuit ports k and k+1. $G_{k,k+1}$ is a conductance spanning star ports k and k+1. $G_{k,k+1}$ is in the same relative position as $G_k$; therefore, its formula is the same as the formula for $G_k$. However, $R_{k,k+1}$ is in the same relative position as $R_{k+1}$; therefore, its formula becomes identical in form to that of $G_{k,k+1}$, which can be seen by substituting k+1 for k in equation (4-3). Thus the combined formula is:

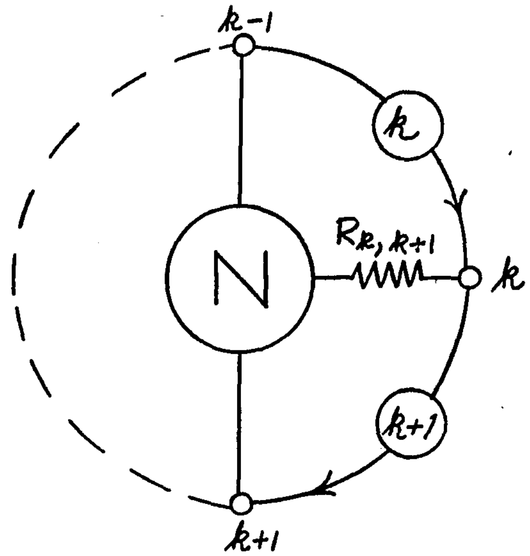
$$R_{k,k+1} = G_{k,k+1} = \frac{A_{k,k+1,m}}{A_{k-1,k+2,m-1}} \tag{4-5}$$

where A equals R or G.

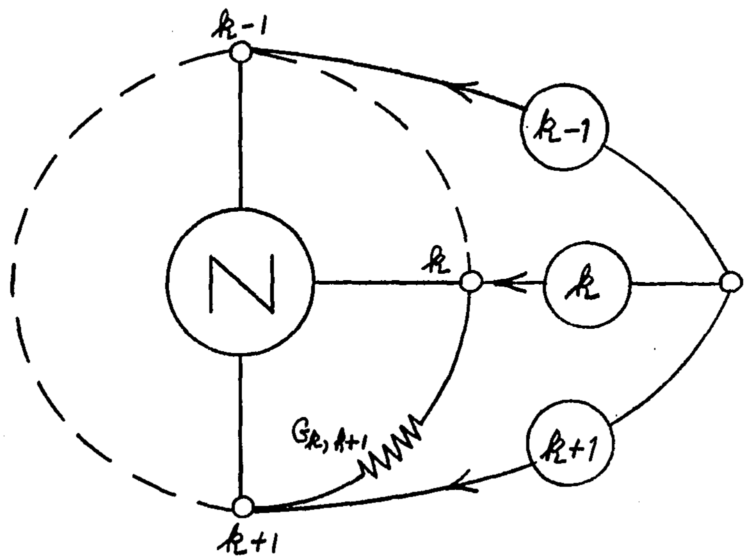
A procedure similar to one described by Puckett (PU 1) is used to convert a resistance matrix to an equivalent conductance matrix. Our numbering scheme for the ports and terminals is different from that employed by Puckett, in order to correspond with the system used in arriving at formula (4-5).

Assume that we begin with a resistance matrix R and that we wish to convert it to the equivalent conductance matrix G. The procedure is general, but for definiteness we illustrate it for n=5. We start with R, a hyperdominant indefinite matrix of rank n-1.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} \\ r_{12} & r_{22} & r_{23} & r_{24} & r_{25} \\ r_{13} & r_{23} & r_{33} & r_{34} & r_{35} \\ r_{14} & r_{24} & r_{34} & r_{44} & r_{45} \\ r_{15} & r_{25} & r_{35} & r_{45} & r_{55} \end{bmatrix} \tag{4-6}$$




a) Resistance matrix case




b) Conductance matrix case

Fig. 4-5. Positions of extracted elements with respect to ports and terminals

Since R is singular, its inverse does not exist. If we supress any port by permanently open-circuiting it (say port 5), the resulting resistance matrix $R_d$ (obtained by deleting row 5 and column 5 of R) of the four-port network shown in Figure 4-6 is:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{12} & r_{22} & r_{23} & r_{24} \\ r_{13} & r_{23} & r_{33} & r_{34} \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix} \qquad (4-7)$$

Since $R_d$ is nonsingular, its inverse $G_d$ exists and is given by:

$$\hat{R} = 1/R_5^5 \begin{bmatrix} R_{15}^{15} & R_{25}^{15} & R_{35}^{15} & R_{45}^{15} \\ R_{25}^{15} & R_{25}^{25} & R_{35}^{25} & R_{45}^{25} \\ R_{35}^{15} & R_{35}^{25} & R_{35}^{35} & R_{45}^{35} \\ R_{45}^{15} & R_{45}^{25} & R_{45}^{35} & R_{45}^{45} \end{bmatrix} \qquad (4-8)$$

where $R_{kl}^{ij}$ is the cofactor of R with rows i and j and columns k and l deleted.

The port structure shown in Figure 4-6 is not the desired one. In Figure 4-7 the network is shown with the desired port structure (ports are primed) and the original port structure (shown dashed). It is well known (WE 1) that when the ports form trees, a linear transformation exists for changing the port voltages and currents from one tree to another. Such a situation is in evidence here. However, since we have shown how this is accomplished in Chapter 3, we omit the details and just give the results.

$$G'_d = P^t G_d P \qquad (4-9)$$

where $G'_d$ is the conductance matrix of the primed port structure and P is the transformation matrix in $E = PE'$, and is given below.

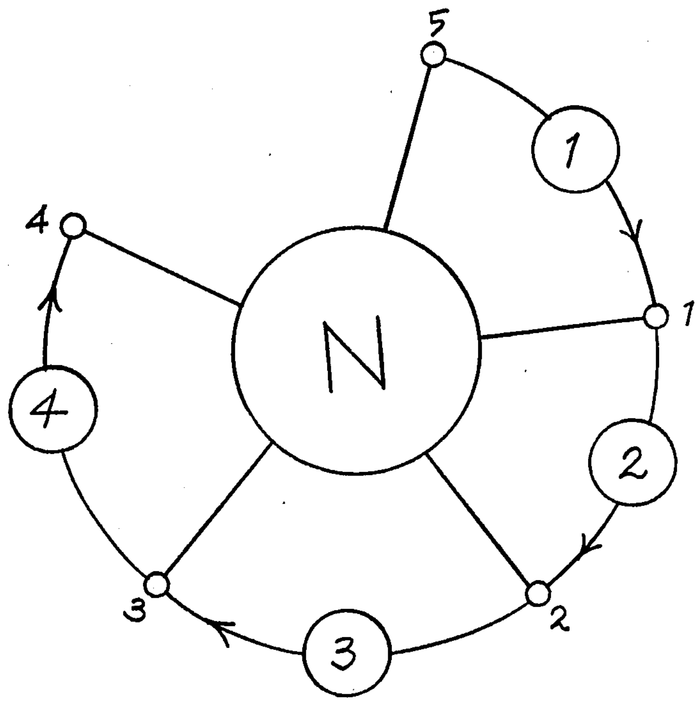



Fig. 4-6. 5-port network with port 5 supressed

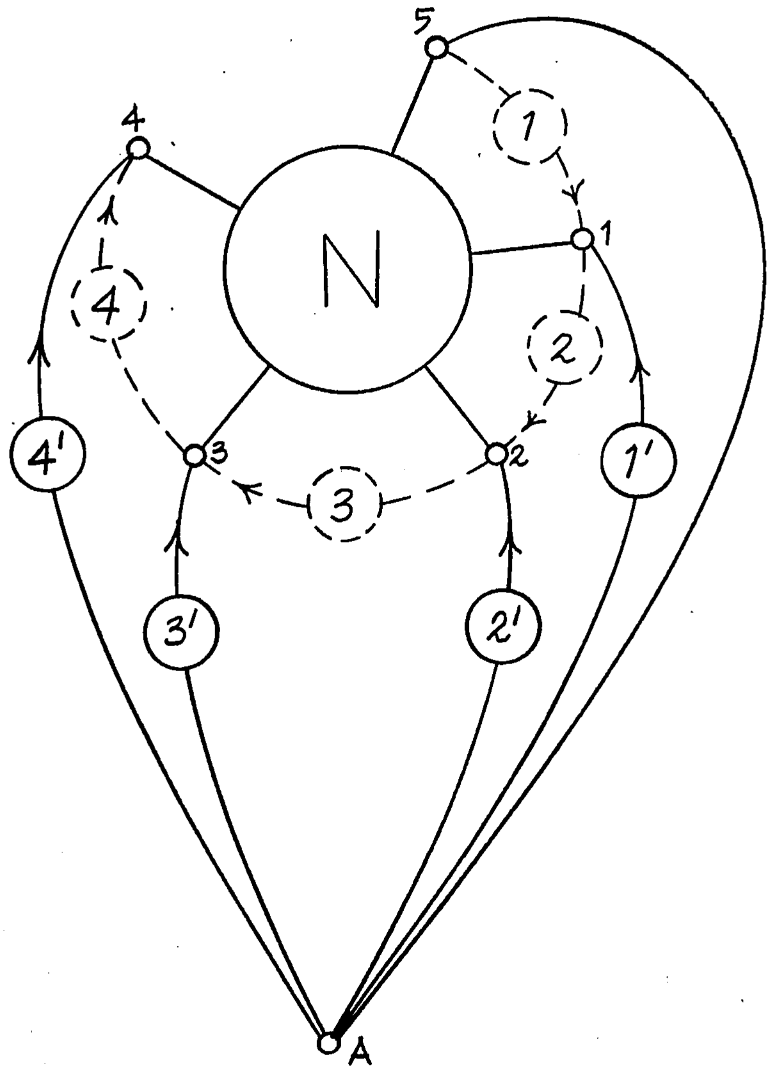



Fig. 4-7. Diagram showing the two sets of ports

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{4-10}$$

As Puckett points out, we do not have to carry out the matrix multiplication to obtain $G'_d$, as a simple algorithm exists for obtaining its elements. We have modified the algorithm to correspond with our numbering system. To find an element in $G'_d$, take the corresponding element in $G_d$ and add it to the lower adjacent diagonal element in $G_d$, and subtract the element to the right of and the element below the corresponding element in $G_d$. Stated as an equation:

$$g'_{ij} = g_{ij} + g_{i+1,j+1} - g_{i,j+1} - g_{i+1,j} \tag{4-11}$$

(Assume $G_d$ to be bordered by zeros at the bottom and on the right, thus any elements with indices greater than n are equal to zero.)

Applying (4-11) to (4-8) yields:

$$G'_d = \frac{1}{R^5_5}\left[\begin{array}{c|c|c|c} R^{15}_{15} + R^{25}_{25} - 2R^{15}_{25} & R^{15}_{25} + R^{25}_{35} - R^{15}_{35} - R^{25}_{25} & R^{15}_{35} + R^{25}_{45} - R^{15}_{45} - R^{25}_{35} & R^{15}_{45} - R^{25}_{45} \\ \hline R^{15}_{25} + R^{25}_{35} - R^{15}_{35} - R^{25}_{25} & R^{25}_{25} + R^{35}_{35} - 2R^{25}_{35} & R^{25}_{35} + R^{35}_{45} - R^{25}_{45} - R^{35}_{35} & R^{25}_{45} - R^{35}_{45} \\ \hline R^{15}_{35} + R^{25}_{45} - R^{15}_{45} - R^{25}_{35} & R^{25}_{35} + R^{35}_{45} - R^{25}_{45} - R^{35}_{35} & R^{35}_{35} + R^{45}_{45} - 2R^{35}_{45} & R^{35}_{45} - R^{45}_{45} \\ \hline R^{15}_{45} - R^{25}_{45} & R^{25}_{45} - R^{35}_{45} & R^{35}_{45} - R^{45}_{45} & R^{45}_{45} \end{array}\right] \tag{4-12}$$

We now remove the short-circuit between terminal 5 and terminal A and insert port 5, as shown in Figure 4-8. The G matrix is then formed by adding a fifth row and column to $G'_d$ such that the matrix becomes an indefinite one (rows and columns sum to zero). Thus we obtain:

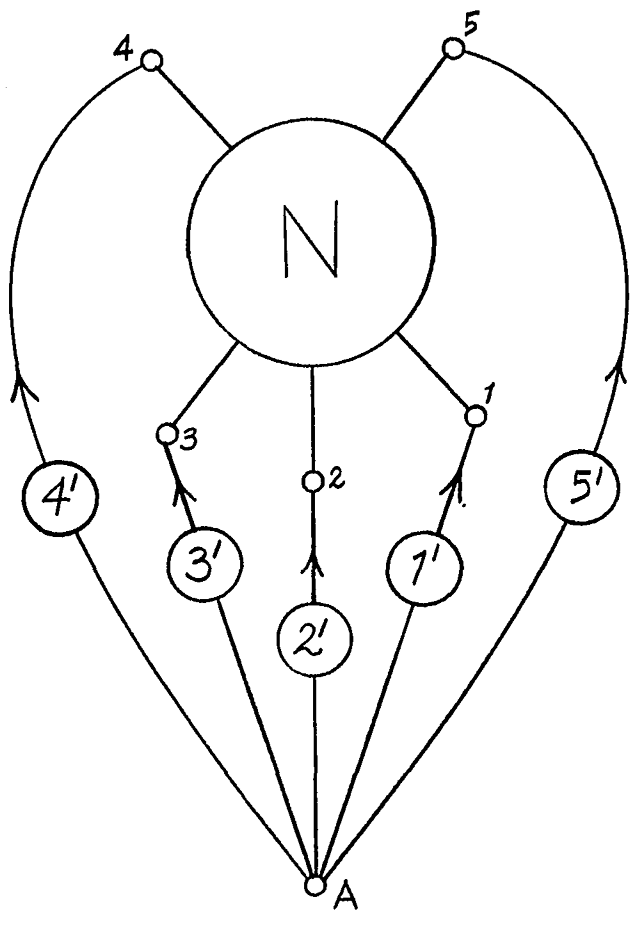



Fig. 4-8. 5-port network with ports arranged so as to realize a conductance matrix

$$G=\frac{1}{R_{5}^{5}}\left[\begin{array}{c|c|c|c|c}
\begin{matrix}R_{15}^{15}+R_{25}^{25}\\-2R_{25}^{15}\end{matrix} & \begin{matrix}R_{25}^{15}+R_{35}^{25}\\-R_{35}^{15}-R_{25}^{25}\end{matrix} & \begin{matrix}R_{35}^{15}+R_{45}^{25}\\-R_{45}^{15}-R_{35}^{25}\end{matrix} & R_{45}^{15}-R_{45}^{25} & R_{25}^{15}-R_{15}^{15}\\
\hline
\begin{matrix}R_{25}^{15}+R_{35}^{25}\\-R_{35}^{15}-R_{25}^{25}\end{matrix} & \begin{matrix}R_{25}^{25}+R_{35}^{35}\\-2R_{35}^{25}\end{matrix} & \begin{matrix}R_{35}^{25}+R_{45}^{35}\\-R_{45}^{25}-R_{35}^{35}\end{matrix} & R_{45}^{25}-R_{45}^{35} & R_{35}^{15}-R_{25}^{15}\\
\hline
\begin{matrix}R_{35}^{15}+R_{45}^{25}\\-R_{45}^{15}-R_{35}^{25}\end{matrix} & \begin{matrix}R_{35}^{25}+R_{45}^{35}\\-R_{45}^{25}-R_{35}^{35}\end{matrix} & \begin{matrix}R_{35}^{35}+R_{45}^{45}\\-2R_{45}^{35}\end{matrix} & R_{45}^{35}-R_{45}^{45} & R_{45}^{15}-R_{35}^{15}\\
\hline
R_{45}^{15}-R_{45}^{25} & R_{45}^{25}-R_{45}^{35} & R_{45}^{35}-R_{46}^{45} & R_{45}^{45} & -R_{45}^{15}\\
\hline
R_{25}^{15}-R_{15}^{15} & R_{35}^{15}-R_{25}^{15} & R_{45}^{15}-R_{36}^{15} & -R_{45}^{15} & R_{15}^{15}
\end{array}\right] \tag{4-13}$$

A similar procedure is employed to convert from a G matrix to an R matrix.

In this introduction we analyzed the synthesis procedures due to Foster and expanded related points in some detail. We note the exceptional insight which Foster possesses in order to postulate a generalized solution for this difficult problem.

The tasks which remain are to:

a) correct and prove the stated necessary and sufficient realizability conditions, and

b) show the derivation of the extraction formula given by (4-5).

We shall begin our analysis by deriving results for the fifth-order case. These will allow us to establish the techniques required to proceed further, as well as to familiarize the reader with the procedures. Since the odd-order and even-order cases have different characteristics, we then establish results for the sixth-order case. The greater complexity of the sixth-order case will allow us to achieve insight that will be used to extend the results.

Prior to initiating the above, we digress a bit to introduce a useful formula, which is due to Campbell (1911).

## 4.2 CAMPBELL'S FORMULA FOR A COMPLETE GRAPH

It is well known (WE 1) that in a network N which comprises a complete n-terminal graph, has an n-port star tree port structure (for example, Fig. 4-8), and has a conductance matrix $G = [g_{ij}]$, that the conductance of the branch connecting terminals i and j, $c_{ij}$ is equal to $-g_{ij}$.

In a 1911 paper, Campbell (CA 1) gave the branch conductance formulas for N in terms of the cofactors of its resistance matrix R. A modern version of his formula follows.

<u>Theorem 4-1</u>. Given an n-terminal complete graph network with its ports arranged in an <u>oriented circuit</u>, and having a resistance matrix R. Ports and terminals are numbered as in Figure 4-5a. The conductance of the branch between terminals i and j is given by the following formula.

$$c_{ij} = \frac{-(\mathrm{sign}\,[(j+1)-j])R^{i,i+1}_{j,j+1}}{R^{n}_{n}} \quad ,\text{for all } i<j,\ j=2,\ldots,n \tag{4-14}$$

where $R^{ab}_{cd}$ is the cofactor of R with rows a and b and columns c and d deleted; if j is equal to n then j+1 is replaced with 1.

<u>Proof</u>: For use in the proof we introduce an extension of Jean's theorem due to Sharpe and Spain (SH 1).

<u>Theorem 4-2.</u> (Extension of Jean's theorem)

Given an indefinite matrix R. The following relation between its cofactors holds.

$$R^{ip}_{jk} = \mathrm{sign}(i-p)\mathrm{sign}(j-k)\left[R^{in}_{jn} + R^{pn}_{kn} - R^{pn}_{jn} - R^{in}_{kn}\right]. \tag{4-15}$$

Substitute the terms from (4-8) into (4-11); this yields:

$$g'_{ij} = 1/R^n_n \left[ R^{in}_{jn} + R^{i+1,n}_{j+1,n} - R^{in}_{j+1,n} - R^{i+1,n}_{jn} \right] \qquad (4\text{-}16)$$

Now apply equation (4-15) to (4-16), which gives:

$$g'_{ij} = \text{sign}\left[(i+1)-i\right] \text{sign}\left[(j+1)-j\right] R^{i,i+1}_{j,j+1}/R^n_n \qquad (4\text{-}17)$$

Since $1 \leq i \leq n-1$, sign $\left[(i+1)-i\right]$ is always positive. However, sign $\left[(j+1)-j\right]$ becomes negative when $j=n$ and $j+1$ is replaced by 1. Thus:

$$g'_{ij} = \text{sign}\left[(j+1)-j\right] R^{i,i+1}_{j,j+1}/R^n_n \qquad (4\text{-}18)$$

And since $c_{ij} = -g'_{ij}$, the theorem is proved.

The utility of this theorem can be illustrated, for example, by its application to the network of Figure 3-9 in Chapter 3. As another example, the theorem allows us to write equation (4-13) in the following compact form.

$$G = 1/R^5_5 \begin{bmatrix} R^{12}_{12} & R^{12}_{23} & R^{12}_{34} & R^{12}_{45} & -R^{12}_{51} \\ R^{12}_{23} & R^{23}_{23} & R^{23}_{34} & R^{23}_{45} & -R^{23}_{51} \\ R^{12}_{34} & R^{23}_{34} & R^{34}_{34} & R^{34}_{45} & -R^{34}_{51} \\ R^{12}_{45} & R^{23}_{45} & R^{34}_{45} & R^{45}_{45} & -R^{45}_{51} \\ -R^{12}_{51} & -R^{23}_{51} & -R^{34}_{51} & -R^{45}_{51} & R^{51}_{51} \end{bmatrix} \qquad (4\text{-}19)$$

## 4.3 SYNTHESIS OF FIFTH-ORDER FOSTER NETWORKS

We will now derive the element value formulas for a fifth-order Foster network. Let us begin with the network given by Figure 4-9, which has a resistance matrix $R = \left[r_{ij}\right]$. If we examine the circuit diagram in Figure 4-9, we find that each radial resistance $R_{ij}$ affects only the transfer resistance $r_{ij}$ and the driving point resistances $r_{ii}$ and $r_{jj}$. Thus if we extract the five radial resistances $R_{ij}$ to obtain the

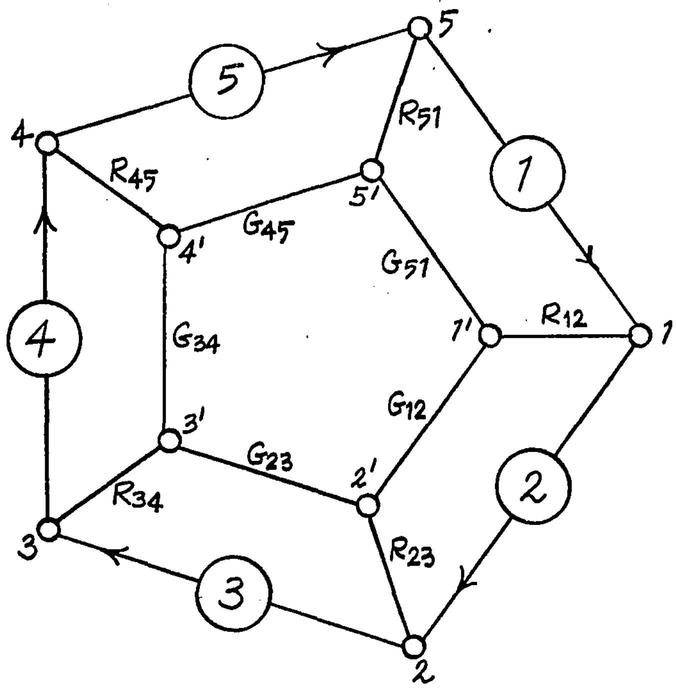



Fig. 4-9. Fifth-order Foster resistance network

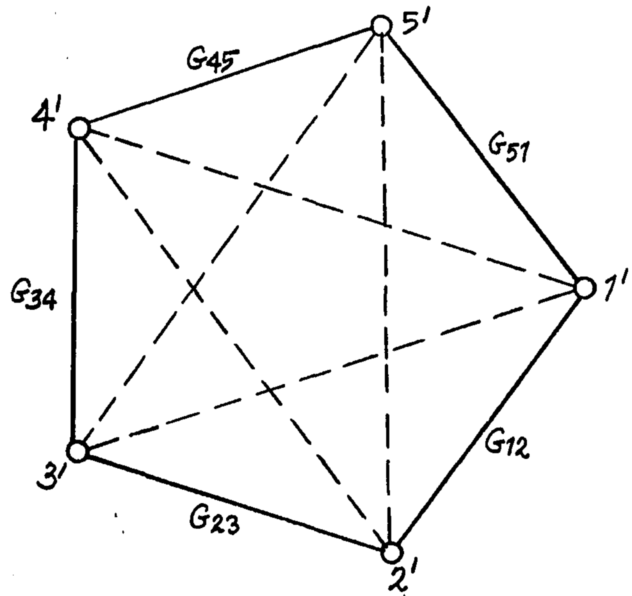



Fig. 4-10. Fifth-order network remaining after extraction of radial resistances

network shown in Figure 4-10, the resistance matrix P of this network is given simply as:

$$P = \begin{bmatrix} (r_{11}-R_{12}-R_{15}) & (r_{12}+R_{12}) & r_{13} & r_{14} & (r_{15}+R_{15}) \\ (r_{12}+R_{12}) & (r_{22}-R_{12}-R_{23}) & (r_{23}+R_{23}) & r_{24} & r_{25} \\ r_{13} & (r_{23}+R_{23}) & (r_{33}-R_{34}-R_{23}) & (r_{34}+R_{34}) & r_{35} \\ r_{14} & r_{24} & (r_{34}+R_{34}) & (r_{44}-R_{34}-R_{45}) & (r_{45}+R_{45}) \\ (r_{15}+R_{15}) & r_{25} & r_{35} & (r_{45}+R_{45}) & (r_{55}-R_{45}-R_{15}) \end{bmatrix} \qquad (4\text{-}20)$$

The formulas for the values of the extracted $R_{ij}$ are not known at this point, but can be derived by consideration of the topology of the 5-branch mesh shown in Figure 4-10. This mesh can also be considered as a degenerate complete graph, degenerate in the sense that all of the cross branches (shown dashed) have zero conductance. Therefore, the equivalent G matrix for this network will have the transfer conductances corresponding to the cross branches equal to zero, that is:

$$g_{ij} = 0, \quad \text{for } j>i+1, \text{ (mod n)} \qquad (4\text{-}21)$$

To obtain the formulas for the $R_{ij}$, we are going to make use of the conditions that the cross branches are zero. First we transform the resistance matrix P into the corresponding conductance matrix G, using Campbell's formula. Thus we obtain equation (4-19) with $R^{ab}_{cd}$ replaced by $P^{ab}_{cd}$. Applying (4-21) to this matrix yields the following set of equations for the elements of G which correspond to the cross branches.

$$\begin{aligned} &\text{a)} \quad g_{13} = P^{12}_{34}/P^{5}_{5} = 0 \\ &\text{b)} \quad g_{14} = P^{12}_{45}/P^{5}_{5} = 0 \\ &\text{c)} \quad g_{24} = P^{23}_{45}/P^{5}_{5} = 0 \\ &\text{d)} \quad g_{25} = -P^{23}_{15}/P^{5}_{5} = 0 \end{aligned} \qquad (4\text{-}22)$$

e) $g_{35} = -P^{34}_{15}/P^{5}_{5} = 0$ (4-22)

Equations (4-22) reduce to:

a) $P^{12}_{34} = 0$

b) $P^{12}_{45} = 0$

c) $P^{23}_{45} = 0$ (4-23)

d) $P^{23}_{15} = 0$

e) $P^{34}_{15} = 0$

As an example, the expanded version of (4-23a) is:

$$P^{12}_{34} = \begin{vmatrix} r_{13} & (r_{23}+R_{23}) & (r_{33}-R_{34}-R_{23}) \\ r_{14} & r_{24} & (r_{34}+R_{34}) \\ (r_{15}+R_{15}) & r_{25} & r_{35} \end{vmatrix} = 0$$

Equations (4-23) are a set of 5 simultaneous equations in the 5 unknown $R_{ij}$. Therefore the solutions to the equations give us the desired formulas for the radial resistances $R_{ij}$ in terms of the matrix elements $r_{ij}$. Unfortunately, the equations are multilinear and difficult to solve. However, we can use our knowledge of duality to assist us in this task.

Find the dual of the network in Figure 4-9; this is shown in Figure 4-11. Note that $R_{ij}$ in the first network are given in ohms and in the second network in mhos, and vice versa for $G_{ij}$. Thus corresponding (dual) branches in the two networks are numerically equal. Therefore the G matrix of Figure 4-11 is identical to the R matrix of Figure 4-9.

By use of the star-mesh transformation the network of resistances of Figure 4-11 can be transformed to the equivalent complete graph network of Figure 4-12. In the star-mesh transformation, "mesh" has an idiomatic meaning different from that given in the appendix; namely, "mesh" refers to a complete graph. Figure 4-14 shows an n-star and

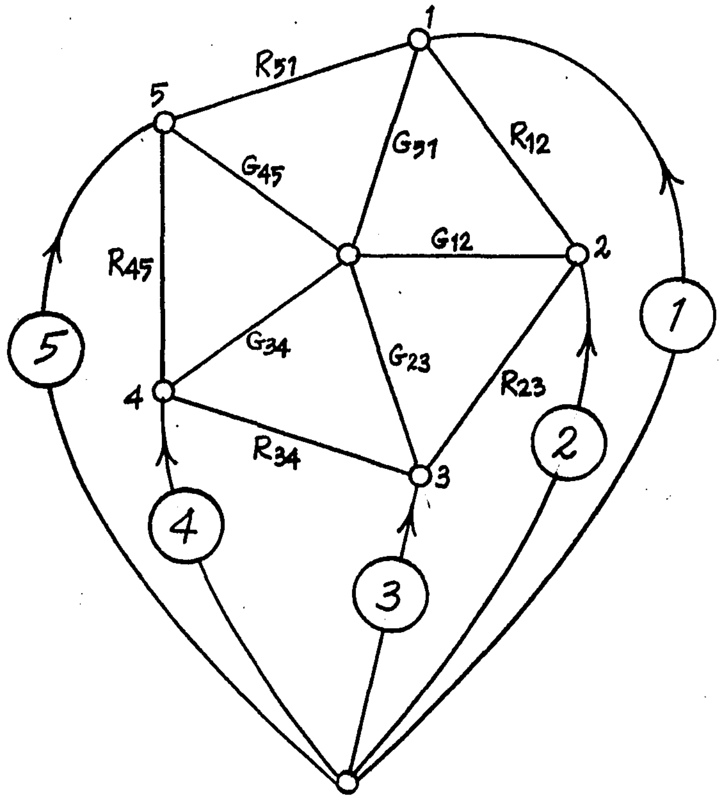



Fig. 4-11. Dual network
($R_{ij}$ in mhos, $G_{ij}$ in ohms.)

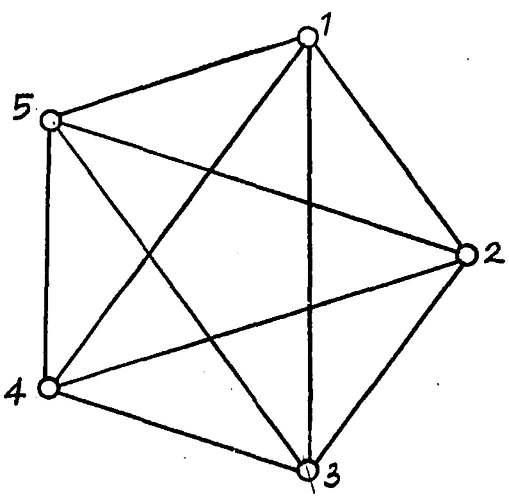



Fig. 4-12. Star-mesh transformation of the resistances of Fig. 4-11

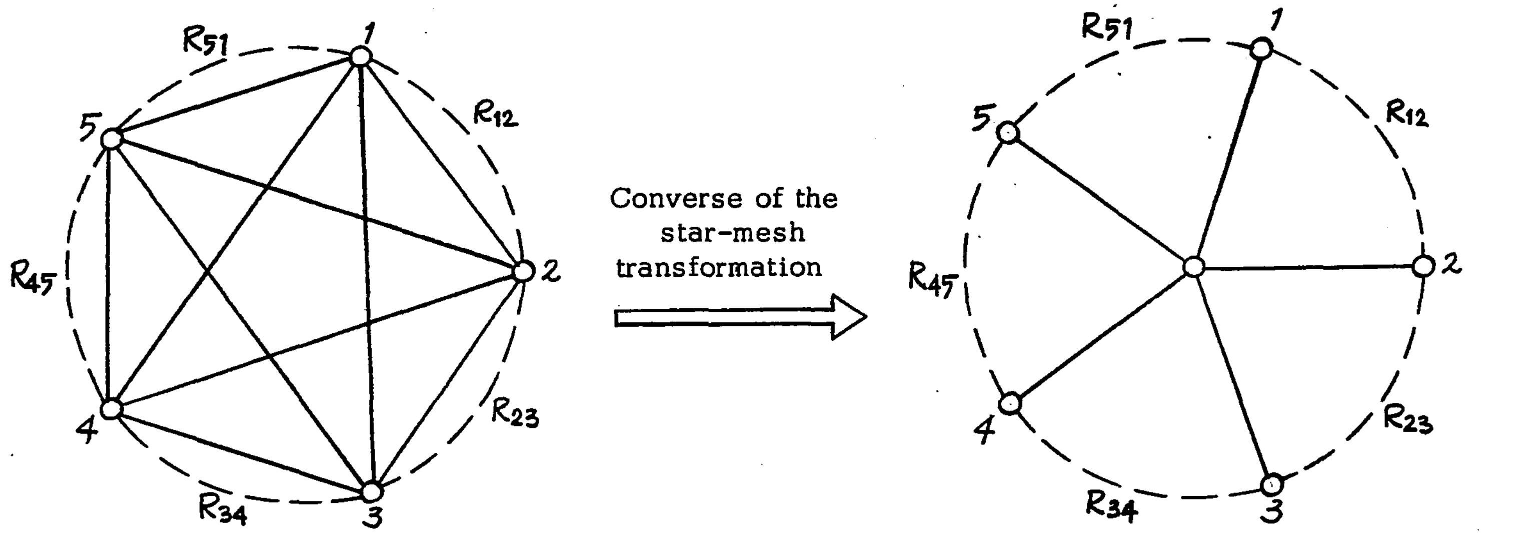
an n-mesh.

Suppose that from the network of Figure 4-12 we remove a shunt conductance $R_{i,i+1}$ from between each pair of adjacent terminals i and i+1, as shown in Figure 4-13a with dashed lines, in such a manner that the network which remains (solid lines) satisfies the conditions for the converse of the star-mesh transformation. Then the remaining complete graph of Figure 4-13a can be transformed by the converse of the star-mesh transformation to the star of Figure 4-13b, which can then be combined with the $R_{i,i+1}$ to yield the desired network of Figure 4-11.

When we remove a shunt conductance $R_{i,i+1}$ from between terminals i and i+1, we affect only the transfer conductance $g_{i,i+1}$ and the driving-point conductances $g_{ii}$ and $g_{i+1,i+1}$. Therefore, the conductance matrix of the solid line elements of Figure 4-13a is identical to P as given by equation (4-20).

The general conditions to be satisfied for the existence of the converse of the star-mesh transformation are that n(n-3)/2 second order minors of the conductance matrix be equal to zero. The derivation of these conditions can be accomplished by comparing the conductance matrices of the mesh and the star. For the present we merely state the conditions as applied to P for the fifth-order case under consideration.

$$\begin{aligned} &a) \quad \begin{vmatrix} p_{13} & p_{14} \\ p_{23} & p_{24} \end{vmatrix} = 0 \\ &b) \quad \begin{vmatrix} p_{14} & p_{15} \\ p_{24} & p_{25} \end{vmatrix} = 0 \\ &c) \quad \begin{vmatrix} p_{24} & p_{25} \\ p_{34} & p_{35} \end{vmatrix} = 0 \\ &d) \quad \begin{vmatrix} p_{25} & p_{12} \\ p_{35} & p_{13} \end{vmatrix} = 0 \end{aligned} \qquad (4\text{-}24)$$




a) b)

Fig. 4-13. Obtaining the extraction formulas by satisfying the conditions for the converse of the star-mesh transformation

$$\text{e)}\quad \begin{vmatrix} p_{35} & p_{13} \\ p_{45} & p_{14} \end{vmatrix} = 0 \qquad (4\text{-}24)$$

Substituting from (4-20) into (4-24) yields:

$$\text{a)}\quad \begin{vmatrix} r_{13} & r_{14} \\ (r_{23}+R_{23}) & r_{24} \end{vmatrix} = 0$$

$$\text{b)}\quad \begin{vmatrix} r_{14} & (r_{15}+R_{51}) \\ r_{24} & r_{25} \end{vmatrix} = 0$$

$$\text{c)}\quad \begin{vmatrix} r_{24} & r_{25} \\ (r_{34}+R_{34}) & r_{35} \end{vmatrix} = 0 \qquad (4\text{-}25)$$

$$\text{d)}\quad \begin{vmatrix} r_{25} & (r_{12}+R_{12}) \\ r_{35} & r_{13} \end{vmatrix} = 0$$

$$\text{e)}\quad \begin{vmatrix} r_{35} & r_{13} \\ (r_{45}+R_{45}) & r_{14} \end{vmatrix} = 0$$

Solving equations (4-25) for the $R_{i,i+1}$ we obtain:

$$\text{a)}\quad R_{23} = \frac{\begin{vmatrix} r_{13} & r_{14} \\ r_{23} & r_{24} \end{vmatrix}}{r_{14}}$$

$$\text{b)}\quad R_{51} = \frac{\begin{vmatrix} r_{14} & r_{15} \\ r_{24} & r_{25} \end{vmatrix}}{r_{24}} \qquad (4\text{-}26)$$

$$\text{c)}\quad R_{34} = \frac{\begin{vmatrix} r_{24} & r_{25} \\ r_{34} & r_{35} \end{vmatrix}}{r_{25}}$$

d)
$$R_{12} = \frac{\begin{vmatrix} r_{25} & r_{12} \\ r_{35} & r_{13} \end{vmatrix}}{r_{35}}$$

e)
$$R_{45} = \frac{\begin{vmatrix} r_{35} & r_{13} \\ r_{45} & r_{14} \end{vmatrix}}{r_{13}} \qquad (4\text{-}26)$$

As a result of the duality principle, equations (4-26) are also the solutions to equations (4-23) and thus give the values of the radial resistances in Figure 4-9. This is verified by substitution in (4-23).

## 4.3a STAR-MESH AND CONVERSE OF THE STAR-MESH TRANSFORMATIONS

We observe a most interesting characteristic of hyperdominant indefinite matrices, from the above example, namely, the second-order minors (eq. (4-24)) which define the conditions for the converse of the star-mesh transformation, are the complementary minors of the (n-2)-order minors (eq. (4-23)) which define the cross branches of the complete graph. Since an n-terminal complete graph has n(n-3)/2 cross branches, we see that there is a one-to-one correspondence between cross branches and conditions for the converse of the star-mesh transformation. This leads to a general theorem on hyperdominant indefinite matrices.

Theorem 4-3. Given an nth-order hyperdominant indefinite matrix R. If the (n-2)-order minors defined by:

$$R^{i,i+1}_{j,j+1}, \quad \text{for all } j > i+1,\ i = 1,2,\ldots,n-2, \text{ and when } j=n,\ j+1=1,\ i \neq j+1, \qquad (4\text{-}27)$$

are all equal to zero, then all of the complementary second-order minors are also zero, and vice versa.

Proof: Let R be a resistance matrix with the prescribed (n-2)-order minors equal to zero. Convert R to the equivalent G matrix using

Puckett, etc. Since each of the zero (n-2)-order minors corresponds to a cross branch, the network realization is an n-polygon. The dual network is an n-star and R is its conductance matrix, therefore, R must satisfy the conditions for the converse of the star-mesh transformation, namely, the complementary second-order minors all equal to zero.

We can now state the conditions for the existence of the converse of the star-mesh transformation in a new compact form.

Theorem 4-4. Given an nth-order hyperdominant indefinite conductance matrix G. If the $n(n-3)/2$ second-order minors

$$C^{i,i+1}_{j,j+1} = 0, \quad \begin{array}{l} \text{for all } j > i+1, \ i = 1,2,\ldots,n-2, \text{ and} \\ \text{when } j=n, \ j+1=1, \text{ and } i \neq j+1, \end{array} \tag{4-28}$$

where $C^{ab}_{cd}$ is the second-order minor of G defined by the elements at the intersection of rows a and b and columns c and d, then G is realizable by an n-star network.

We now state the formulas for the mesh branch conductances $G_{ij}$ in terms of the star branch conductances $G_i$, for the star-mesh transformation, given by Rosen (RO 1). Refer to Figure 4-14.

$$G_{ij} = G_i G_j / G, \qquad \text{where } G = \sum_{k=1}^{n} G_k \tag{4-29}$$

If we want to proceed in the opposite sense, that is, obtain the conductances of the star branches $G_i$ in terms of the mesh branches $G_{ij}$, then of course the conditions for the converse of the star-mesh transformation must be satisfied. Assuming that they are, we proceed as follows. Compute $g_{ii}$, the driving-point conductance of the ith port of the star.

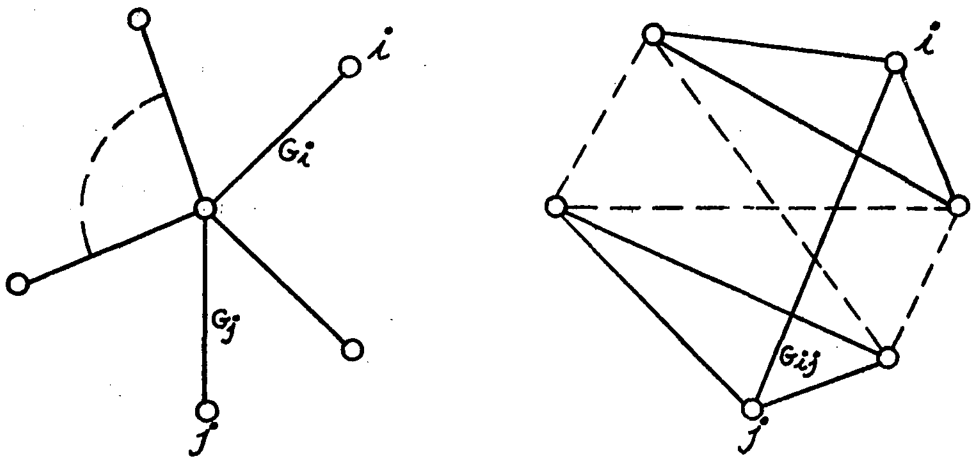
$$g_{ii} = G_i(G-G_i)/G = G_i - G_i^2/G \tag{4-30}$$

Substituting (4-29) into (4-30) to eliminate G yields:

$$g_{ii} = G_i(1-G_{ij}/G_j) \tag{4-31}$$

Similarly:

$$g_{jj} = G_j(1-G_{ij}/G_i) \tag{4-32}$$




a) n-star b) n-mesh

Fig. 4-14. Diagrams showing the conductances of the n-star and n-mesh

Substituting (4-32) into (4-31) to eliminate $G_j$ gives:

$$G_i = \frac{g_{ii}g_{jj} - G_{ij}^2}{g_{jj} - G_{ij}} \tag{4-33}$$

And from the driving-point conductance of port k of the mesh we obtain:

$$g_{kk} = \sum_{p=1}^{n} G_{kp}, \quad \text{for } p \neq k. \tag{4-34}$$

Using equations (4-28) we can obtain identities of the form:

$$G_{ij}G_{i+1,j+1} = G_{i,j+1}G_{i+1,j} \tag{4-35}$$

Employing these identities and (4-33) we can develop alternate expressions for $G_i$. For example,

$$G_i = g_{ii} + \frac{G_{i,i+1}G_{i,i-1}}{G_{i-1,i+1}} \tag{4-36}$$

The example in Bedrosian's thesis (BE 1) is given in this form, although the general formula is not derived.

#### 4.3b FORMULAS FOR THE SHUNT CONDUCTANCES

We have derived formulas for the radial resistances; in order to complete the synthesis of this fifth-order case, we must obtain formulas for the shunt conductances $G_{ij}$.

We substitute equations (4-26), the formulas for the $R_{ij}$, into equation (4-20), the P matrix. P is then completely specified in terms of the elements of the resistance matrix $r_{ij}$. P is then converted to the equivalent G matrix using Campbell's formulas, which yield:

$$G = 1/P_5^5 \begin{bmatrix} P_{12}^{12} & P_{23}^{12} & 0 & 0 & -P_{51}^{12} \\ P_{23}^{12} & P_{23}^{23} & P_{34}^{23} & 0 & 0 \\ 0 & P_{34}^{23} & P_{34}^{34} & P_{45}^{34} & 0 \\ 0 & 0 & P_{45}^{34} & P_{45}^{45} & -P_{51}^{45} \\ -P_{51}^{12} & 0 & 0 & -P_{51}^{45} & P_{51}^{51} \end{bmatrix} \tag{4-37}$$

(We recall that the zeros occur as a result of (4-23).) The shunt conductances can now be obtained directly from the G matrix, using:

$$G_{ij} = -g_{ij} \tag{4-38}$$

Applying (4-38) to (4-37) in terms of the elements of R yields:

$$\begin{aligned}
&\text{a)}\quad G_{12} = r_{14}r_{35}/G_o \\
&\text{b)}\quad G_{23} = r_{14}r_{25}/G_o \\
&\text{c)}\quad G_{34} = r_{13}r_{25}/G_o \\
&\text{d)}\quad G_{45} = r_{13}r_{24}/G_o \\
&\text{e)}\quad G_{51} = r_{24}r_{35}/G_o \\
&\text{f)}\quad G_o = -(r_{13}r_{14}r_{25}+r_{13}r_{24}r_{35}+r_{13}r_{14}r_{35}+r_{14}r_{24}r_{35}+r_{14}r_{25}r_{35})
\end{aligned} \tag{4-39}$$

## 4.4 NECESSARY AND SUFFICIENT CONDITIONS FOR REALIZABILITY

Now that we have formulas for all of the elements of the network, we can investigate the necessary and sufficient conditions required for realizing a matrix by the network. These can be inferred from the fact that the $R_{ij}$ and the $G_{ij}$ must be non-negative in order to be realizable.

If the $R_{ij}$ are non-negative, we would expect that the $G_{ij}$ would automatically be non-negative, as we recall that they are also the element values of the star branches of Figure 4-11, and the entire procedure was predicated on forcing the existence of the converse of the star-mesh transformation. Examination of (4-39) shows that this is indeed so. Thus the necessary and sufficient conditions for realizability are contained in the elements removed so as to satisfy the conditions for the converse of the star-mesh transformation. In this case the elements are the radial resistances $R_{ij}$. Examining equation (4-26) we note that the numerators of the $R_{ij}$ must be less than or equal to zero, as the denominators consist of the nonpositive matrix elements $r_{ij}$. If we compare the numerators of the $R_{ij}$ of (4-26) with the conditions for the converse of the star-mesh transformation given by (4-24), we see that they are identical if the p's are replaced with r's. We can then state the necessary and sufficient conditions, for this fifth-order case, in the following compact form.

Theorem 4-5. Given a fifth-order hyperdominant indefinite matrix M. M is the resistance or conductance matrix of a Foster canonic planar network of resistances if and only if:

$$\text{sign}\left[(j+1)-j\right] C^{i,i+1}_{j,j+1} \leq 0, \text{ for } i=1,2,3,\ j > i+1, \text{ and when } j=5,\ j+1=1, \text{ and } i \neq j+1, \tag{4-40}$$

where $C^{ab}_{cd}$ is a second-order minor of M with elements at the intersection of rows a and b and columns c and d.

Let us now investigate Foster's necessary and sufficient conditions derived from the determinant of equation (4-1). $m_{max} = 2$, therefore, $(i_1, i_2, i_3, i_4)$ defines a $2m_{max}$ index sequence of the five possible indices. A set of five possible sequences and their corresponding determinant conditions are given below.

$$\begin{array}{lll}
\text{a)} & (1, 2, 3, 4) & \begin{vmatrix} -r_{23} & -r_{24} \\ -r_{13} & -r_{14} \end{vmatrix} \geq 0 \\
\text{b)} & (2, 3, 4, 5) & \begin{vmatrix} -r_{34} & -r_{35} \\ -r_{24} & -r_{25} \end{vmatrix} \geq 0 \\
\text{c)} & (3, 4, 5, 1) & \begin{vmatrix} -r_{45} & -r_{14} \\ -r_{35} & -r_{13} \end{vmatrix} \geq 0 \\
\text{d)} & (4, 5, 1, 2) & \begin{vmatrix} -r_{15} & -r_{25} \\ -r_{14} & -r_{24} \end{vmatrix} \geq 0 \\
\text{e)} & (5, 1, 2, 3) & \begin{vmatrix} -r_{12} & -r_{13} \\ -r_{25} & -r_{35} \end{vmatrix} \geq 0
\end{array} \tag{4-41}$$

By comparing equations (4-41) with (4-26) we see that each Foster condition is equivalent to the condition which requires the nonpositivity of the numerators of the $R_{ij}$. However, the reference does not tell us precisely how to choose the index sequences; just that they should be "arranged in the same cyclic order as before". Suppose that we examine condition (4-41c), which is generated by the following sequence (3, 4, 5, 1). Suppose we use the same indices, but arrange them in a different sequence, say (1,3,4,5). This sequence has the same cyclic order as the previous one, however, the resulting condition is now:

$$\begin{vmatrix} -r_{34} & -r_{35} \\ -r_{14} & -r_{15} \end{vmatrix} = r_{15}r_{34} - r_{14}r_{35} \geq 0 \qquad (4\text{-}42)$$

What is the significance of (4-42)? From conditions (4-41b) and (4-41d) we obtain:

$$r_{14}r_{25}/r_{15} \leq r_{24} \leq r_{25}r_{34}/r_{35} \qquad (4\text{-}43)$$

Taking the two outer terms of (4-43), cancelling the common factor $r_{25}$ and cross-multiplying yields: $r_{15}r_{34} \geq r_{14}r_{35}$, and this is identical to (4-42). Thus (4-42) is a redundant condition. However, if we had chosen (4-42) instead of (4-41c), the conditions would no longer be sufficient. Therefore, for $m = m_{max}$ and n odd, since precisely one index is eliminated from the index set to form the sequence, we should state that the sequence should begin with the index which occurs after the eliminated index. This rule avoids the anomalies described.

The reader can easily verify that equation (4-5), the formula for extracting the radial resistances $R_{k,k+1}$, is valid for this case.

## 4.5 ZEROS IN THE DENOMINATORS OF EXTRACTED ELEMENTS

In the reference it is stated that if the denominator of say $R_{k,k+1}$ is zero, then so must the numerator be. Suppose that we let the denominator $r_{14}$ of $R_{23}$ (4-26a) be zero, then the numerator equals $r_{13}r_{24}$. This is not necessarily zero; however, if not, then it is positive and thus violates the realizability condition. On the other hand, if $r_{13}$ or $r_{24}$ is also set equal to zero, then the realizability condition is satisfied. We must be prudent as to just how we make the substitution in the formulas, as we are confronted with indeterminate forms. For example, if $r_{13} = r_{14} = 0$, then direct substitution in (4-26a) gives $R_{23} = 0/0$. However, if we let $r_{13} = r_{14} = e$, and let e approach zero, in the limit we obtain:

$$R_{23} = (er_{24}/e - r_{23})_{e\to 0} = r_{24} - r_{23}$$

$$R_{15} = -r_{15}$$

$$R_{34} = r_{24}r_{35}/r_{25} - r_{34}$$

$$R_{12} = -r_{12}$$

$$R_{45} = (er_{35}/e - r_{45})_{e\to 0} = r_{35} - r_{45}$$

Checking the formulas for the shunt conductances $G_{ij}$ (4-39), we find that $G_o = 0$, thus we follow a similar procedure which yields:

$$1/G_{12} = -(2r_{24}+r_{25}+r_{24}r_{25}/r_{35})$$

$$1/G_{23} = 1/G_{34} = -(2r_{24}r_{35}/r_{25} + r_{24}+r_{35})$$

$$1/G_{45} = -(2r_{35}+r_{25}+r_{25}r_{35}/r_{24})$$

$$1/G_{51} = 0$$

Since $1/G_{51} = 0$, the network takes the degenerate form shown in Figure 4-15.

## 4.6 EXAMPLE 4-1

We now present a numerical example to illustrate some of the previous points. Given the following resistance matrix R, realize R by the network of Figure 4-9.

$$R = \begin{bmatrix} 10 & -2 & -1 & -1 & -6 \\ -2 & 7 & -3 & -1 & -1 \\ -1 & -3 & 9 & -4 & -1 \\ -1 & -1 & -4 & 11 & -5 \\ -6 & -1 & -1 & -5 & 13 \end{bmatrix}$$

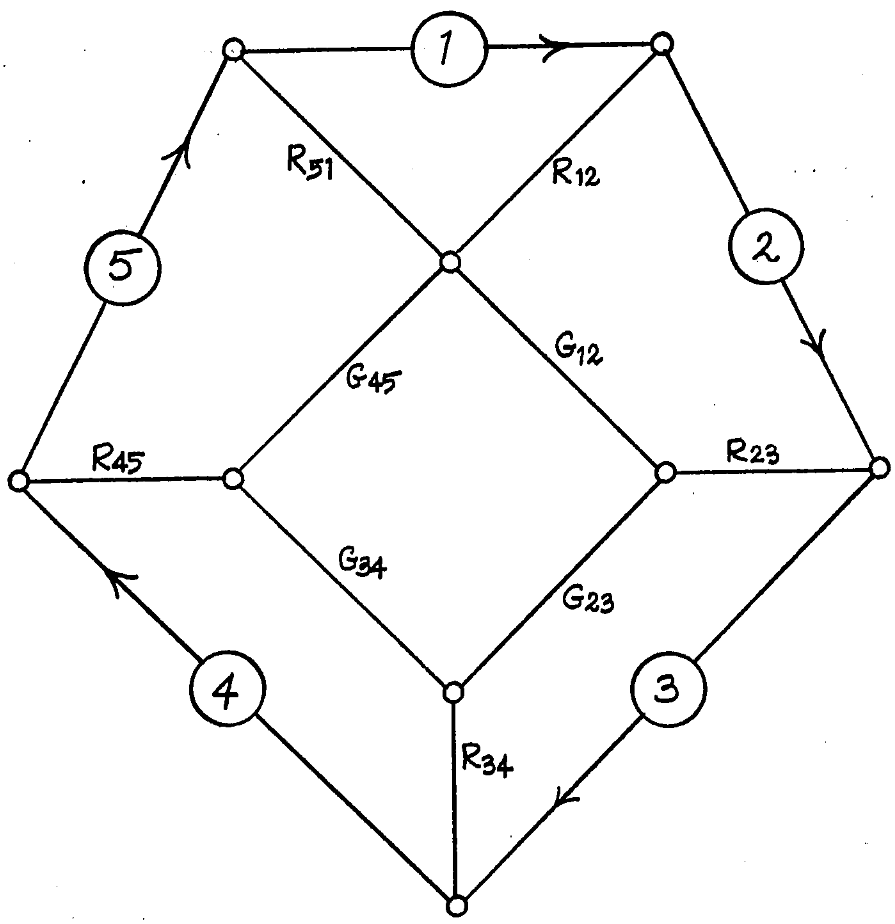



Fig. 4-15. Degenerate 5-port resistance network

Using (4-5) or (4-26) we extract the radial resistances $R_{ij}$.

$$R_{23} = \frac{\begin{vmatrix} -1 & -1 \\ -3 & -1 \end{vmatrix}}{-1} = 2 \text{ ohms}$$

$$R_{51} = \frac{\begin{vmatrix} -1 & -6 \\ -1 & -1 \end{vmatrix}}{-1} = 5 \text{ ohms}$$

$$R_{34} = \frac{\begin{vmatrix} -1 & -1 \\ -4 & -1 \end{vmatrix}}{-1} = 3 \text{ ohms}$$

$$R_{12} = \frac{\begin{vmatrix} -1 & -2 \\ -1 & -1 \end{vmatrix}}{-1} = 1 \text{ ohm}$$

$$R_{45} = \frac{\begin{vmatrix} -1 & -1 \\ -5 & -1 \end{vmatrix}}{-1} = 4 \text{ ohms}$$

The P matrix (4-20) is now computed.

$$P = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

P is now converted to the equivalent conductance matrix G, using the Puckett procedure or Campbell's formula, which yields:

$$G = \begin{bmatrix} .4 & -.2 & 0 & 0 & -.2 \\ -.2 & .4 & -.2 & 0 & 0 \\ 0 & -.2 & .4 & -.2 & 0 \\ 0 & 0 & -.2 & .4 & -.2 \\ -.2 & 0 & 0 & -.2 & .4 \end{bmatrix}$$

Now using formula (4-5) we extract the shunt conductances $G_{ij}$. Thus $G_{12} = G_{1,2,1}/G_{5,3,0} = -g_{12}/G_{5,3,0}$. From elementary theory we know that $G_{12} = -g_{12}$, therefore, we conclude that $G_{5,3,0} = 1$, or in general $A_{p,q,0} = 1$. Thus $G_{12} = G_{23} = G_{34} = G_{45} = G_{51} = .2$ mhos.

## 4.7 SOME REMARKS ON REALIZABILITY

From Nambiar's results (NA 1) it is known that the following fifth-order hyperdominant indefinite matrix is not realizable as a resistance matrix.

$$R = \begin{bmatrix} 7 & -1 & -2 & -2 & -2 \\ -1 & 7 & -3 & -2 & -1 \\ -2 & -3 & 9 & -1 & -3 \\ -2 & -2 & -1 & 8 & -3 \\ -2 & -1 & -3 & -3 & 9 \end{bmatrix} \qquad (4\text{-}44)$$

If we test R with the Foster realizability conditions, we find that condition (4-41b), $r_{34}r_{25} \geq r_{24}r_{35}$, is violated $(1 \not\geq 6)$. The other conditions are satisfied.

The realizability conditions for the complete-graph, 5-terminal, nonplanar network can be derived from the equivalent G matrix of (4-19). Since, as is well known, $c_{ij}$ the conductance of the branch connecting terminal i and terminal j, is equal to $-g_{ij}$, then each off-diagonal element of G must be nonpositive. Checking all 10 off-diagonal elements we find that $g_{24}$ is positive and hence the matrix is unrealizable. (All of the other elements are negative.) If we change $r_{34}$ in matrix (4-44) from -1 to -52/9 (call the new matrix R'), then $g_{24} = 0$, and R' is realizable on the complete 5-terminal network, less branch $c_{24}$. The absence of branch $c_{24}$ allows the network to be drawn as a planar graph, similar to the network of Figure 4-1. However, the Foster network realizability condition is still not satisfied $(52/9 \not\geq 6)$. Therefore, R' is an example of a matrix which may be realized by a planar graph but which cannot be realized by a Foster canonic planar network. That is, the network is not planar according to our definition.

## 4.8 ORDERING THE PORTS

Suppose that we are given a matrix and it is desired that this matrix be realized on a Foster network. We duly test the matrix with the appropriate conditions and find that one or more of them is violated. Can we conclude that the matrix is unrealizable? The answer is— not necessarily, as some other permutation of rows and columns (equivalent to renumbering the ports) may lead to a possible realization. There are n! possible permuatations of a matrix; for large n it would be quite a chore to test all of them.

We can make use of the fact that the Foster networks are radially symmetric, to reduce the required number of permutations to be tested. For example, for n=5, suppose that 1-2-3-4-5 is the correct port order. Then as a result of the network symmetry, 2-3-4-5-1, 3-4-5-1-2, 4-5-1-2-3, or 5-1-2-3-4 would work equally as well, as they are in the same cyclic order. This reduces the number of permutations to (n-1)!. This is clearly true for odd values of n. For n even, since we extract only one of each opposite pair of elements in the first step of the synthesis procedure, the degree of symmetry is not as high. However, since we permit the removal of either one of the pair, this in effect restores the symmetry.

Further reduction in permutations is afforded by eliminating mirror images, which account for half the permutations. Thus if 1-2-3-4-5 is a realizable port order, then 5-4-3-2-1 will be also. Therefore, (n-1)!/2 is the minimum number of permutations to be considered; this is still a large number.

Fortunately, a theorem due to Nambiar (NA 1) and some results of Guillemin (GU 2) can be combined into an algorithm which allows us to obtain the correct order of the ports dirctly. The theorem referred to is stated as theorem 3-3 in Chapter 3. Nambiar proves that if we delete one row and corresponding column from an nth-order indefinite matrix of rank n-1, and then find its inverse G, then R is realizable as a resistance matrix if and only if G can be presented in the uniformly tapered form. Guillemin, the originator of the uniformly tapered form, gives a simple method for rearranging the ports to test the matrix. From the partial results of each of the above, we construct our algorithm.

<u>Algorithm 4-1.</u> (Port rearrangement algorithm)

Step 1. Delete the nth row and column of R; call this matrix M.

Step 2. Let $G = M^{-1}$. If any $g_{ij}$ is negative, R is not realizable; stop algorithm.

Step 3. Permute the rows and corresponding columns of G such that $g_{1,n-1}$ becomes the smallest off-diagonal element.

Step 4. Permute the rows and columns of G such that the first row of G is arranged as follows:

$$g_{11} \geq g_{12} \geq g_{13} \geq \dots g_{1,n-2} \geq g_{1,n-1}$$

Step 5. Permute R in the same manner as G. R is now arranged in the correct port order for realization on a Foster network.

It is always possible to arrange a matrix in this manner; and, at most n matrix permutations are required, and only the final one need be tested for realizability.

As an example of the port ordering algorithm, let R be the following fourth-order matrix.

$$R = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 10 & -4 & -5 \\ -2 & -4 & 12 & -6 \\ -3 & -5 & -6 & 14 \end{bmatrix}$$

Testing R with the realizability conditions, we find that the condition $r_{12}r_{34} \geq r_{13}r_{24}$ is violated $(6 \not\geq 10)$. Therefore, we use Algorithm 4-1 to reorder the ports.

Delete row and column 4, and obtain the inverse of the resulting third-order matrix.

$$G = 1/556 \begin{array}{ccc} 1 & 2 & 3 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} 104 & 20 & 24 \\ 20 & 68 & 26 \\ 24 & 26 & 59 \end{array}\right]} \end{array}$$

Permute rows and columns 2 and 3.

$$G = 1/556 \begin{array}{ccc} 1 & 3 & 2 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} 104 & 24 & 20 \\ 24 & 59 & 26 \\ 20 & 26 & 68 \end{array}\right]} \end{array}$$

G is now arranged in accordance with step 4 of the algorithm; therefore, the correct port order is 1-3-2-4, and R as properly permuted is:

$$R = \begin{bmatrix} 6 & -2 & -1 & -3 \\ -2 & 12 & -4 & -6 \\ -1 & -4 & 10 & -5 \\ -3 & -6 & -5 & 14 \end{bmatrix}$$

Checking the previously violated realizability condition, we find that it is now satisfied, and R is realizable.

## 4.9 SIXTH-ORDER FOSTER NETWORKS

From our previous results on the fifth-order case, we see that it is computationally easier to work with the Foster network which has a star center rather than a polygon center. Accordingly we choose the network shown in Figure 4-16. As can be forseen, the complexity of the sixth-order case is increased over the fifth-order case, as we now have an additional set of components to be extracted, that is, $m_{max}$ is increased to 3.

We proceed in a manner similar to that used for the fifth-order case. This is diagrammed in Figure 4-17. We begin with a complete 6-terminal graph and extract the three radial resistances and six shunt conductances (shown dashed), in such a way as to satisfy the

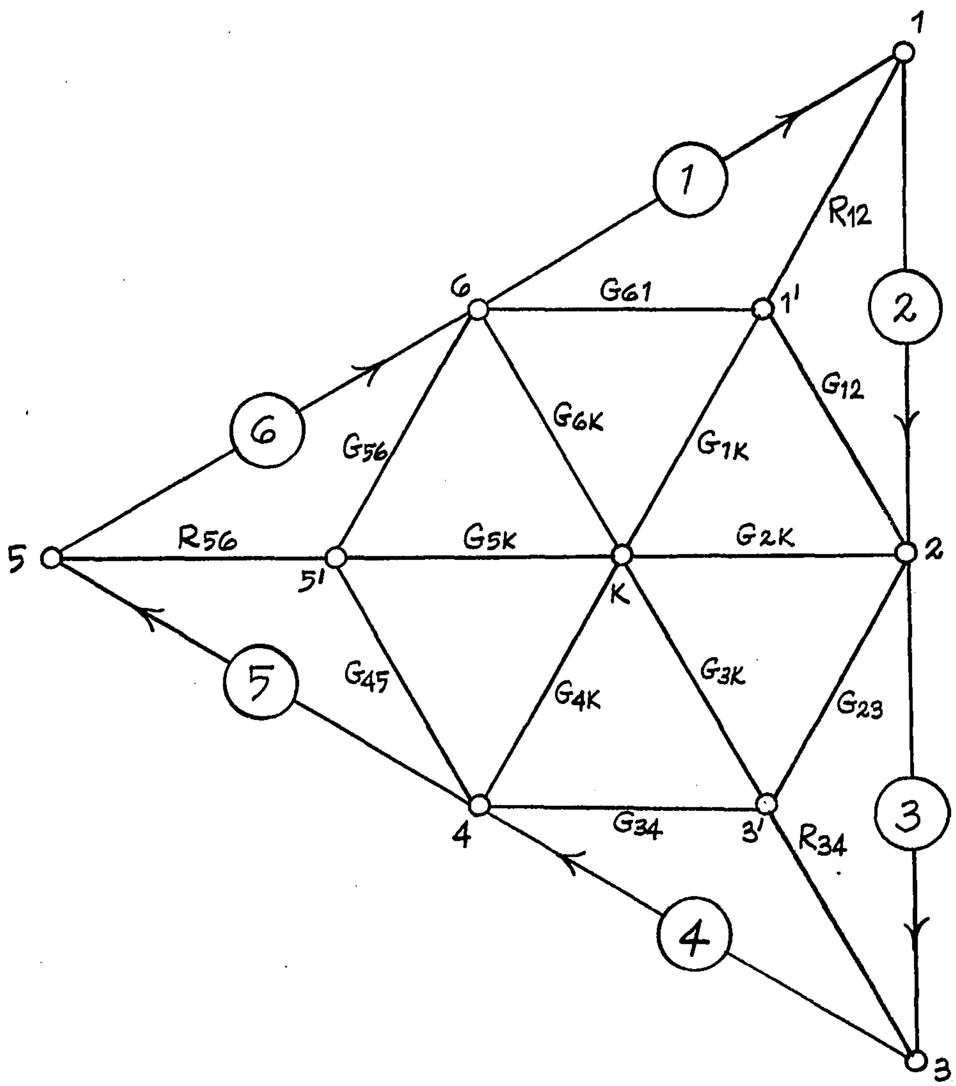
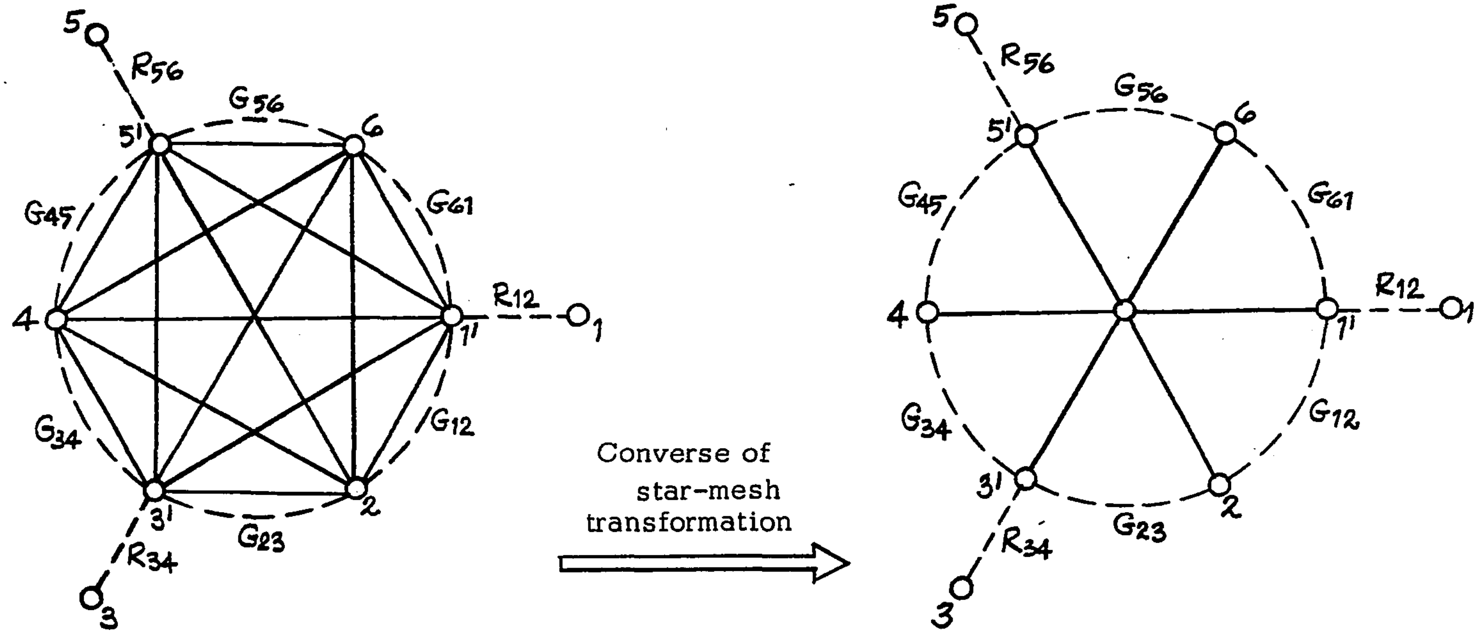



Fig. 4-16. Sixth-order Foster network




a) b)

Fig. 4-17. Obtaining the extraction formulas by satisfying the conditions for the converse of the star-mesh transformation

$n(n-3)/2 = 9$ conditions for the existence of the converse of the star-mesh transformation, so that the mesh of Figure 4-17a (shown solid) may be transformed into the star of Figure 4-17b, thus completing the realization.

Let us assume that we wish to realize R, a resistance matrix, by the network of Figure 4-16. The network realization passes through three stages, the original stage, the stage after removal of the radial resistances, and the stage after removal of the shunt conductances. We have tabulated designations for the various matrices employed in Table 4-1.

| | Resistance Matrix | Conductance Matrix |
|---|---|---|
| Original network | R | G |
| Network with $R_{ij}$'s removed | P | M |
| Network with $R_{ij}$'s and $G_{ij}$'s removed | Q | Y |

Table 4-1

Our aim then is to force Y to satisfy the conditions for the converse of the star-mesh transformation. Applying Theorem 4-4 gives us the conditions.

$$\text{a)}\; C^{12}_{45} = \begin{vmatrix} y_{14} & y_{15} \\ y_{24} & y_{25} \end{vmatrix} = 0 \qquad \text{b)}\; C^{34}_{61} = \begin{vmatrix} y_{36} & y_{46} \\ y_{13} & y_{14} \end{vmatrix} = 0 \qquad \text{c)}\; C^{23}_{56} = \begin{vmatrix} y_{25} & y_{26} \\ y_{35} & y_{36} \end{vmatrix} = 0$$

$$\text{d)}\; C^{23}_{61} = \begin{vmatrix} y_{26} & y_{36} \\ y_{12} & y_{13} \end{vmatrix} = 0 \qquad \text{e)}\; C^{12}_{34} = \begin{vmatrix} y_{13} & y_{14} \\ y_{23} & y_{24} \end{vmatrix} = 0 \qquad \text{f)}\; C^{23}_{45} = \begin{vmatrix} y_{24} & y_{25} \\ y_{34} & y_{35} \end{vmatrix} = 0$$

$$\text{g)}\; C^{34}_{56} = \begin{vmatrix} y_{35} & y_{36} \\ y_{45} & y_{46} \end{vmatrix} = 0 \qquad \text{h)}\; C^{45}_{61} = \begin{vmatrix} y_{46} & y_{56} \\ y_{14} & y_{15} \end{vmatrix} = 0 \qquad \text{i)}\; C^{12}_{56} = \begin{vmatrix} y_{15} & y_{16} \\ y_{25} & y_{26} \end{vmatrix} = 0$$

(4-45)

Examining (4-45) we see that each of the conditions d) through i) contains precisely one term of the form $y_{i,i+1}$, a transfer conductance between adjacent ports. Thus extraction of the six $G_{i,i+1}$ shunt conductances will satisfy these six conditions. Examination of conditions (4-45a,b,c) reveals that none of the terms in the conditions is of the form $y_{i,i+1}$. Therefore, we conclude that these conditions are to be satisfied by the extraction of the three radial resistances $R_{i,i+1}$, (although the relationship between the resistances and the conditions is not evident at this point).

Since we are going to extract the radial resistances first, let us begin with conditions (4-45a,b,c). Since these conditions are independent of the removal of the shunt conductances (as noted above), we can express them in terms of the M matrix elements, yielding:

$$\text{a)}\ C_{45}^{12}=\begin{vmatrix} m_{14} & m_{15} \\ m_{24} & m_{25} \end{vmatrix}=0 \qquad \text{b)}\ C_{61}^{34}=\begin{vmatrix} m_{36} & m_{46} \\ m_{13} & m_{14} \end{vmatrix}=0 \qquad \text{c)}\ C_{56}^{23}=\begin{vmatrix} m_{25} & m_{26} \\ m_{35} & m_{36} \end{vmatrix}=0 \tag{4-46}$$

Using the Puckett procedure the M matrix can be expressed in terms of the P matrix, as shown on the following page, as equation (4-47).

Substituting the $m_{ij}$ elements from (4-47) into (4-46a) yields:

$$C_{45}^{12} = \begin{vmatrix} P_{46}^{16}+P_{56}^{26}-P_{46}^{26}-P_{56}^{36} & P_{56}^{16}-P_{56}^{26} \\ P_{46}^{26}+P_{56}^{36}-P_{46}^{36}-P_{56}^{26} & P_{56}^{26}-P_{56}^{36} \end{vmatrix}=0 \tag{4-48}$$

Multiplying (4-48) out, cancelling terms and rearranging gives:

$$C_{45}^{12}=\begin{vmatrix} P_{46}^{16} & P_{56}^{16} \\ P_{46}^{26} & P_{56}^{26} \end{vmatrix} + \begin{vmatrix} P_{46}^{26} & P_{56}^{26} \\ P_{46}^{36} & P_{56}^{36} \end{vmatrix} - \begin{vmatrix} P_{46}^{16} & P_{56}^{16} \\ P_{46}^{36} & P_{56}^{36} \end{vmatrix}=0 \tag{4-49}$$

$$M=\frac{1}{P^{6}_{6}}\left[\begin{array}{c|c|c|c|c|c}
P^{16}_{16}+P^{26}_{26}-2P^{16}_{26} & P^{16}_{26}+P^{26}_{36}-P^{26}_{26}-P^{16}_{36} & P^{16}_{36}+P^{26}_{46}-P^{26}_{36}-P^{16}_{46} & P^{16}_{46}+P^{26}_{56}-P^{26}_{46}-P^{16}_{56} & P^{16}_{56}-P^{26}_{56} & P^{16}_{26}-P^{16}_{16} \\ \hline
P^{16}_{26}+P^{26}_{36}-P^{26}_{26}-P^{16}_{36} & P^{26}_{26}+P^{36}_{36}-2P^{26}_{36} & P^{26}_{36}+P^{36}_{46}-P^{36}_{36}-P^{26}_{46} & P^{26}_{46}+P^{36}_{56}-P^{36}_{46}-P^{26}_{56} & P^{26}_{56}-P^{36}_{56} & P^{16}_{36}-P^{16}_{26} \\ \hline
P^{16}_{36}+P^{26}_{46}-P^{26}_{36}-P^{16}_{46} & P^{26}_{36}+P^{36}_{46}-P^{36}_{36}-P^{26}_{46} & P^{36}_{36}+P^{46}_{46}-2P^{36}_{46} & P^{36}_{46}+P^{46}_{56}-P^{46}_{46}-P^{36}_{56} & P^{36}_{56}-P^{46}_{56} & P^{16}_{46}-P^{16}_{36} \\ \hline
P^{16}_{46}+P^{26}_{56}-P^{26}_{46}-P^{16}_{56} & P^{26}_{46}+P^{36}_{56}-P^{36}_{46}-P^{26}_{56} & P^{36}_{46}+P^{46}_{56}-P^{46}_{46}-P^{36}_{56} & P^{46}_{46}+P^{56}_{56}-2P^{46}_{56} & P^{46}_{56}-P^{56}_{56} & P^{16}_{56}-P^{16}_{46} \\ \hline
P^{16}_{56}-P^{26}_{56} & P^{26}_{56}-P^{36}_{56} & P^{36}_{56}-P^{46}_{56} & P^{46}_{56}-P^{56}_{56} & P^{56}_{56} & -P^{16}_{56} \\ \hline
P^{16}_{26}-P^{16}_{16} & P^{16}_{36}-P^{16}_{26} & P^{16}_{46}-P^{16}_{36} & P^{16}_{56}-P^{16}_{46} & -P^{16}_{56} & P^{16}_{16}
\end{array}\right] \qquad (4\text{-}47)$$

Each of the terms of (4-49) is a minor of $P_d^{-1}$, where $P_d$ is the P matrix with the sixth row and column deleted. Therefore, by Jacobi's theorem (AI 1), the complementary signed minors to those of (4-49) in $P_d$ must also sum to zero. This yields:

$$\begin{vmatrix} p_{13} & p_{23} & p_{33} \\ p_{14} & p_{24} & p_{34} \\ p_{15} & p_{25} & p_{35} \end{vmatrix} + \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{14} & p_{24} & p_{34} \\ p_{15} & p_{25} & p_{35} \end{vmatrix} + \begin{vmatrix} p_{12} & p_{22} & p_{23} \\ p_{14} & p_{24} & p_{34} \\ p_{15} & p_{25} & p_{35} \end{vmatrix} = 0 \tag{4-50}$$

Summing the terms of (4-50) gives:

$$\begin{vmatrix} (p_{11}+p_{12}+p_{13}) & (p_{12}+p_{22}+p_{23}) & (p_{13}+p_{23}+p_{33}) \\ p_{14} & p_{24} & p_{34} \\ p_{15} & p_{25} & p_{35} \end{vmatrix} = 0 \tag{4-51}$$

Adding rows 1, 2, and 3 together, and using the fact that P is indefinite (the sum of the rows and columns of P is zero), yields:

$$\begin{vmatrix} -p_{16} & -p_{26} & -p_{36} \\ p_{14} & p_{24} & p_{34} \\ p_{15} & p_{25} & p_{35} \end{vmatrix} = 0 \tag{4-52}$$

This result indicates that a generalization of Jean's theorem to cofactors of order other than (n-2) is feasible. However, in order not to digress, we do not pursue the matter.

Since (4-52) is equal to zero, we can remove the minus signs and rearrange the rows and columns to yield:

$$\begin{vmatrix} p_{34} & p_{35} & p_{36} \\ p_{24} & p_{25} & p_{26} \\ p_{14} & p_{15} & p_{16} \end{vmatrix} = 0 \tag{4-53}$$

We recall from the previous discussion, that extracting a radial resistance $R_{i,i+1}$ affects only one transfer resistance, namely, $p_{i,i+1}$. In (4-53) there are two such elements, $p_{34}$ and $p_{61}$. However, these correspond to opposite elements, and since we are removing $R_{34}$ but not $R_{61}$, then only $p_{34}$ is affected. Therefore, we can express (4-53) in terms of the elements of the R matrix and $R_{34}$, which yields:

$$\begin{vmatrix} (r_{34}+R_{34}) & r_{35} & r_{36} \\ r_{24} & r_{25} & r_{26} \\ r_{14} & r_{15} & r_{16} \end{vmatrix} = 0 \qquad (4\text{-}54)$$

Solving (4-54) for $R_{34}$ we obtain:

$$R_{34} = \frac{\begin{vmatrix} r_{34} & r_{35} & r_{36} \\ r_{24} & r_{25} & r_{26} \\ r_{14} & r_{15} & r_{16} \end{vmatrix}}{-\begin{vmatrix} r_{25} & r_{26} \\ r_{15} & r_{16} \end{vmatrix}} = \frac{R_{3;4;3}}{R_{2,5,2}} . \qquad (4\text{-}55)$$

This verifies the extraction formula (4-5) as shown. We obtain the expressions for the other two radial resistances by the same procedure.

$$R_{12} = \frac{\begin{vmatrix} r_{12} & r_{13} & r_{14} \\ r_{26} & r_{36} & r_{46} \\ r_{25} & r_{35} & r_{45} \end{vmatrix}}{-\begin{vmatrix} r_{36} & r_{46} \\ r_{35} & r_{45} \end{vmatrix}} \qquad R_{56} = \frac{\begin{vmatrix} r_{56} & r_{15} & r_{25} \\ r_{46} & r_{14} & r_{24} \\ r_{36} & r_{13} & r_{23} \end{vmatrix}}{-\begin{vmatrix} r_{14} & r_{24} \\ r_{13} & r_{23} \end{vmatrix}} \qquad (4\text{-}56)$$

## 4.10 EXTRACTION OF THE SHUNT CONDUCTANCES

We will now derive the values of the shunt conductances $G_{i,i+1}$ to be extracted. As we previously discussed, conditions (4-45d-i) are to be satisfied by the removal of these elements. Let us choose condition (4-45d) which will provide us with the value of $G_{12}$.

$$C_{61}^{23} = \begin{vmatrix} y_{12} & y_{13} \\ y_{26} & y_{36} \end{vmatrix} = 0 \qquad \text{(4-45d) (repeated)}$$

We recall that the shunt conductance $G_{12}$ affects only the transfer conductance $y_{12}$ in (4-45d); therefore, we can express (4-45d) in terms of $G_{12}$ and the elements of the M matrix, which yields:

$$C_{61}^{23} = \begin{vmatrix} (m_{12}+G_{12}) & m_{13} \\ m_{26} & m_{36} \end{vmatrix} = 0 \qquad (4\text{-}57)$$

Solving for $G_{12}$, we obtain:

$$G_{12} = \frac{\begin{vmatrix} m_{12} & m_{13} \\ m_{26} & m_{36} \end{vmatrix}}{-m_{36}} = \frac{M_{1,2,2}}{M_{6,3,1}} \qquad (4\text{-}58)$$

This again verifies the extraction formula (4-5), for the m=2 level of extraction. The other shunt conductance formulas are obtained similarly.

## 4.11 NECESSARY AND SUFFICIENT CONDITIONS FOR REALIZABILITY

Returning to (4-55) and (4-56); it is not difficult to show that the determinants in these formulas correspond to some of the necessary and sufficient conditions generated by (4-1), since the determinants are already expressed in terms of the R matrix elements, and it is required that the radial resistances be non-negative.

However, this is not the case for the shunt conductance formulas, which we have expressed in terms of the elements of the M matrix, for example, $G_{12}$ which is given by (4-58). In order to verify the realizability conditions, we have to show that they guarantee the nonnegativity of $G_{12}$ and the other shunt conductances. Therefore, we must express $G_{12}$ in terms of the R matrix elements.

Examining (4-58) we see that its numerator must be greater than or equal to zero since the $m_{ij}$ are nonpositive. That is:

$$\begin{vmatrix} m_{12} & m_{13} \\ m_{26} & m_{36} \end{vmatrix} \geq 0 \qquad (4\text{-}59)$$

Applying the same procedure to (4-59) as the one which we used on (4-46a) yields:

$$-\begin{vmatrix} p_{13} & p_{35} & p_{36} \\ p_{14} & p_{45} & p_{46} \\ p_{15} & p_{55} & p_{56} \end{vmatrix} \geq 0 \qquad (4\text{-}60)$$

From Table 4-1 we see that the $p_{i,i+1}$'s are affected only by the removal of the $R_{i,i+1}$'s, and since we do not remove $R_{45}$, the only off-diagonal term involved is $p_{56}$. In addition, the term $p_{55}$ is also a function of $R_{56}$. Thus we can express (4-60) in terms of the elements of the R matrix and $R_{56}$, as is shown below.

$$-\begin{vmatrix} r_{13} & r_{35} & r_{36} \\ r_{14} & r_{45} & r_{46} \\ r_{15} & (r_{55}-R_{56}) & (r_{56}+R_{56}) \end{vmatrix} \geq 0 \qquad (4\text{-}61)$$

From (4-56) we have $R_{56}$ expressed in terms of the elements of the R matrix; therefore, by substitution in (4-61) we are able to obtain the condition solely in terms of the $r_{ij}$. However, when this is done, the result is a complex expression which does not resemble any of the realizability conditions. Therefore, we suspect that the use of some of the simpler Foster conditions for m=2, those conditions which correspond to those for the existence of the converse of the star-mesh transformation, will enable us to show that (4-61) is satisfied. The assumed conditions for m=2 and n=6, along with the generating sequences, are given below.

a) (1,2,4,5) $\begin{vmatrix} r_{24} & r_{25} \\ r_{14} & r_{15} \end{vmatrix} \geq 0$

b) (3,4,6,1) $\begin{vmatrix} r_{46} & r_{14} \\ r_{36} & r_{13} \end{vmatrix} \geq 0$

c) (2,3,5,6) $\begin{vmatrix} r_{35} & r_{36} \\ r_{25} & r_{26} \end{vmatrix} \geq 0$

d) (6,1,2,3) $\begin{vmatrix} r_{12} & r_{13} \\ r_{26} & r_{36} \end{vmatrix} \geq 0$

e) (1,2,3,4) $\begin{vmatrix} r_{23} & r_{24} \\ r_{13} & r_{14} \end{vmatrix} \geq 0$

f) (2,3,4,5) $\begin{vmatrix} r_{34} & r_{35} \\ r_{24} & r_{25} \end{vmatrix} \geq 0$

g) (3,4,5,6) $\begin{vmatrix} r_{45} & r_{46} \\ r_{35} & r_{36} \end{vmatrix} \geq 0$

h) (4,5,6,1) $\begin{vmatrix} r_{56} & r_{15} \\ r_{46} & r_{14} \end{vmatrix} \geq 0$

i) (5,6,1,2) $\begin{vmatrix} r_{16} & r_{26} \\ r_{15} & r_{25} \end{vmatrix} \geq 0$ (4-62)

We now make the substitution of (4-56) in (4-61), and after simplification we obtain:

$$-\left\{\begin{vmatrix} r_{13} & r_{14} & r_{15} \\ r_{23} & r_{24} & r_{25} \\ r_{35} & r_{45} & r_{55} \end{vmatrix} + \begin{vmatrix} r_{13} & r_{14} & r_{15} \\ r_{23} & r_{24} & r_{25} \\ r_{36} & r_{46} & r_{56} \end{vmatrix}\right\} \frac{\begin{vmatrix} r_{46} & r_{14} \\ r_{36} & r_{13} \end{vmatrix}}{\begin{vmatrix} r_{23} & r_{24} \\ r_{13} & r_{14} \end{vmatrix}} \geq 0 \quad (4\text{-}63)$$

The two second-order determinants of (4-63) are seen to correspond to the conditions (4-62b and e), and are both non-negative. Thus the quantity inside of the braces, $X_{12}$, must be nonpositive. Expanding $X_{12}$ along the bottom rows of the determinants, we obtain:

$$X_{12} = (r_{35}+r_{36})\begin{vmatrix} r_{14} & r_{15} \\ r_{24} & r_{25} \end{vmatrix} -(r_{45}+r_{46})\begin{vmatrix} r_{13} & r_{15} \\ r_{23} & r_{25} \end{vmatrix} -(r_{55}+r_{56})\begin{vmatrix} r_{13} & r_{14} \\ r_{23} & r_{24} \end{vmatrix} \leq 0 \tag{4-64}$$

The first and third determinants of (4-64) correspond to conditions (4-62a and e), respectively, except that the row order is reversed, which changes the signs. The second determinant sign can be obtained by combining the first and the third, and its sign is also negative. examination of the coefficient of each determinant reveals that the coefficient signs are - + +, reading from left to right. This makes the signs of the three terms of (4-64), + - -. Thus we cannot yet make any statement with regard to the sign of (4-64). From (4-62b) we obtain:

$$r_{36} \leq r_{13}r_{46}/r_{14} \tag{4-65}$$

Combining (4-62b and g) yields:

$$r_{35} \leq r_{13}r_{45}/r_{14} \tag{4-66}$$

We now substitute (4-65) and (4-66) into the first term of (4-64) in a "worst case" manner, that is, the $\leq$ signs are replaced with = signs, so that the one positive term in (4-64) takes on its maximum value. Multiplying out the result and simplifying gives:

$$X_{12_{max}} =(r_{55}+r_{56}-r_{15}r_{45}/r_{14}-r_{15}r_{46}/r_{14})\begin{vmatrix} r_{13} & r_{14} \\ r_{23} & r_{24} \end{vmatrix} \leq 0 \tag{4-67}$$

The determinant in (4-67), which corresponds to condition (4-62e), is nonpositive, whereas the coefficient is positive. Therefore, the inequality holds true, as long as the particular second-order realizability conditions are satisfied. A similar treatment of the other shunt conductances, which have symmetrical equations, yields similar results. Thus the conditions of (4-62) are necessary for a realization.

We are tempted to try a similar procedure on the numerators of the radial resistances (4-55 and 4-56), for if they could be shown to be nonpositive by the use of (4-62), then (4-62) would constitute the necessary and sufficient conditions for realizability. However, carrying out this exercise shows that this is not the case; therefore, the numerators of the $R_{ij}$ must be included among the necessary and sufficient conditions. Thus the Foster conditions, generated by (4-1), are verified.

For the case of n=7, shown in Figure 4-3, exactly the same situation prevails, as no new levels of components have been added. The application of the techniques above proves the necessary and sufficient conditions similarly. For n greater than 7, additional levels of components are added, requiring a more complex analysis, which has not yet been undertaken. The generalization of these conditions appears to be a difficult problem. However, it is possible to generalize the extraction formula (4-5), which we proceed to do.

## 4.12 GENERALIZATION OF THE EXTRACTION FORMULA

We have proved the validity of the extraction formula (4-5), through n=6, by solving the particular cases. Let us assume that we want to prove that the extraction formula is general for $M_{k,k+1}$, where $M_{k,k+1}$ can be either a resistance $R_{k,k+1}$ or a conductance $G_{k,k+1}$ as defined in Figure 4-5. Let $T = [t_{ij}]$ be the resistance or conductance matrix of of a planar canonic network <u>prior</u> to the extraction of the $M_{k,k+1}$, and let $S = [s_{ij}]$ be the resistance or conductance matrix of the network <u>after</u> extraction of the $M_{k,k+1}$. We recall that extraction of the $M_{k,k+1}$ affects only the off-diagonal element $s_{k,k+1}$. Thus:

$$s_{k,k+1} = t_{k,k+1} + M_{k,k+1} \qquad (4\text{-}68)$$

We next proceed to recognize some patterns which exist and then extend them to the general case. From the cases which we have previously studied (n=5, n=6), we can establish that there exists a one-to-one correspondence between the conditions for the converse of the star-mesh transformation, which we must satisfy, and the $M_{k,k+1}$ elements which we extract to satisfy them. For example, in the sixth-order case, $C^{12}_{45}$ was satisfied by extraction of $R_{34}$, and $C^{23}_{61}$ was satisfied by extraction of $G_{12}$. In general, how can we relate the condition $C^{i,i+1}_{j,j+1}$ to the $M_{k,k+1}$ which must be extracted to satisfy it?

First we establish the level m which corresponds to the particular condition. If we write the subscripts and superscripts of the condition $C^{i,i+1}_{j,j+1}$ in cyclic order, grouping them in the longest possible uninterrupted sequences, leaving a blank when a number from the set $\{1,2,\ldots,n\}$ is omitted, we find that the number of interior blanks in the sequence is related to m. For example, the sequence for $C^{12}_{45}$ would be written as (1,2,-,4,5,-), and the sequence for $C^{23}_{61}$ as (-,6,1,2,3,-). The first sequence corresponds to m=3, the second to m=2. And in general we see that if the number of interior blanks is x, that the condition corresponds to m=x+2. Thus we can establish the value of m to which a particular condition corresponds. Next we must find precisely which $M_{k,k+1}$ on the previously established m level, is the appropriate one.

Equation (4-53) which resulted from satisfying condition

$$C^{12}_{45} = \begin{vmatrix} y_{24} & y_{25} \\ y_{14} & y_{15} \end{vmatrix} = 0$$

is repeated below.

$$\begin{vmatrix} p_{34} & p_{35} & p_{36} \\ p_{24} & p_{25} & p_{26} \\ p_{14} & p_{15} & p_{16} \end{vmatrix} = 0 \qquad \begin{array}{c}(4\text{-}53)\\ \text{(repeated)}\end{array}$$

Note that the second-order minor in the lower-left corner of (4-53), indicated by dashed lines, has the identical subscript arrangement as $C^{12}_{45}$, and the subscripts in the additional row and column increase in natural order. The dimension of (4-53) is 3 and is equal to m. In addition, the upper-left-hand element has subscripts of the form k,k+1. These subscripts are the key to the extracted element $M_{k,k+1}$. In this example, $M_{k,k+1} = R_{34}$.

Thus given a general converse of the star-mesh transformation condition $C^{i,i+1}_{j,j+1}$, write it in the following form:

$$C^{i,i+1}_{j,j+1} = \begin{vmatrix} y_{i+1,j} & y_{i+1,j+1} \\ y_{ij} & y_{i,j+1} \end{vmatrix} \quad ,\text{where } i+1 < j+1. \qquad (4\text{-}69)$$

Then find m from the subscripts and superscripts as detailed above. Next form the mth-order determinant $S_{i+m-1,j,m}$ using the arrangement of subscripts in (4-69) as the lower-left-hand section, and adding rows above and columns to the right as shown below.

$$S_{i+m-1,j,m} = \begin{vmatrix} s_{i+m-1,j} & s_{i+m-1,j+1} & \cdots & s_{i+m-1,j+m-1} \\ s_{i+m-2,j} & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_{i+1,j} & s_{i+1,j+1} & & \cdot \\ s_{ij} & s_{i,j+1} & \cdot & s_{i,j+m-1} \end{vmatrix} = 0$$

(4-70)

Recalling that the upper-left-hand element of (4-70) has subscripts of the form k,k+1, make the following substitutions: $k = i+m-1$, and $k+1 = j$. This yields:

$$S_{k,k+1,m} = \begin{vmatrix} s_{k,k+1} & s_{k,k+2} & \cdots & s_{k,k+m} \\ s_{k-1,k+1} & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ s_{k-m+2,k+1} & s_{k-m+2,k+2} & & \cdot \\ s_{k-m+1,k+1} & s_{k-m+1,k+2} & \cdots & s_{k-m+1,k+m} \end{vmatrix} = 0$$

(4-71)

Substitute (4-68) into (4-71), and use the fact that $t_{ij} = s_{ij}$ for all ij except ij = k,k+1, and then solve for $M_{k,k+1}$.

$$M_{k,k+1} = \frac{\begin{vmatrix} t_{k,k+1} & t_{k,k+2} & \cdots & t_{k,k+m} \\ t_{k-1,k+1} & t_{k-1,k+2} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ t_{k-m+1,k+1} & \cdot & \cdot & t_{k-m+1,k+m} \end{vmatrix}}{-\begin{vmatrix} t_{k-1,k+2} & t_{k-1,k+3} & \cdot & t_{k-1,k+m} \\ t_{k-2,k+2} & t_{k-2,k+3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ t_{k-m+1,k+2} & \cdot & \cdot & t_{k-m+1,k+m} \end{vmatrix}} = \frac{T_{k,k+1,m}}{T_{k-1,k+2,m-1}} \tag{4-72}$$

We see that (4-72) is equivalent to (4-5) except for the signs of the determinant elements and the negative sign of the denominator. Foster's determinant (4-2) has a negative sign on each element. However, $M_{k,k+1}$ is always formed by the quotient of an odd-order determinant and an even-order determinant or vice versa; thus if we change all of the element signs of (4-2) and make the quotient negative, the result is the same. Thus the extraction formula (4-5) is proved for any n.

## 4.13 EXAMPLE 4-2

We illustrate with an example some of the previous concepts related to the extraction formulas. Figure 4-18 depicts one of the planar n-port networks for n=8. Assume that we are given a conductance matrix G to be realized by the network. The diagram in Figure 4-19 gives the matrix nomenclature for the various stages of the network, and indicates which elements are removed at each stage. Figure 4-20 shows in tabular form the details involved in arriving at the formulas for four typical elements, which are circled on Figure 4-18, $G_{45}$, $R_{34}$, $Y_{23}$, $Z_{12}$.

An interesting observation to be noted in Figure 4-20 is that whenever we transfer from a determinant condition, say B=0, through

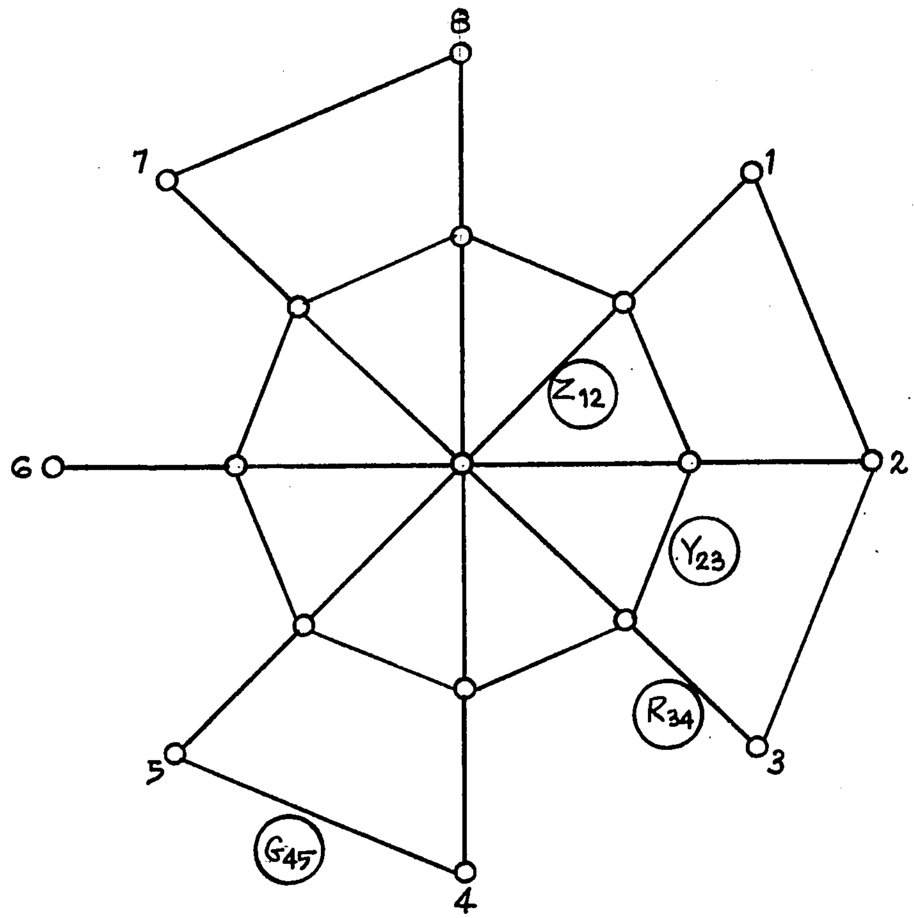



Fig. 4-18. 8-terminal network

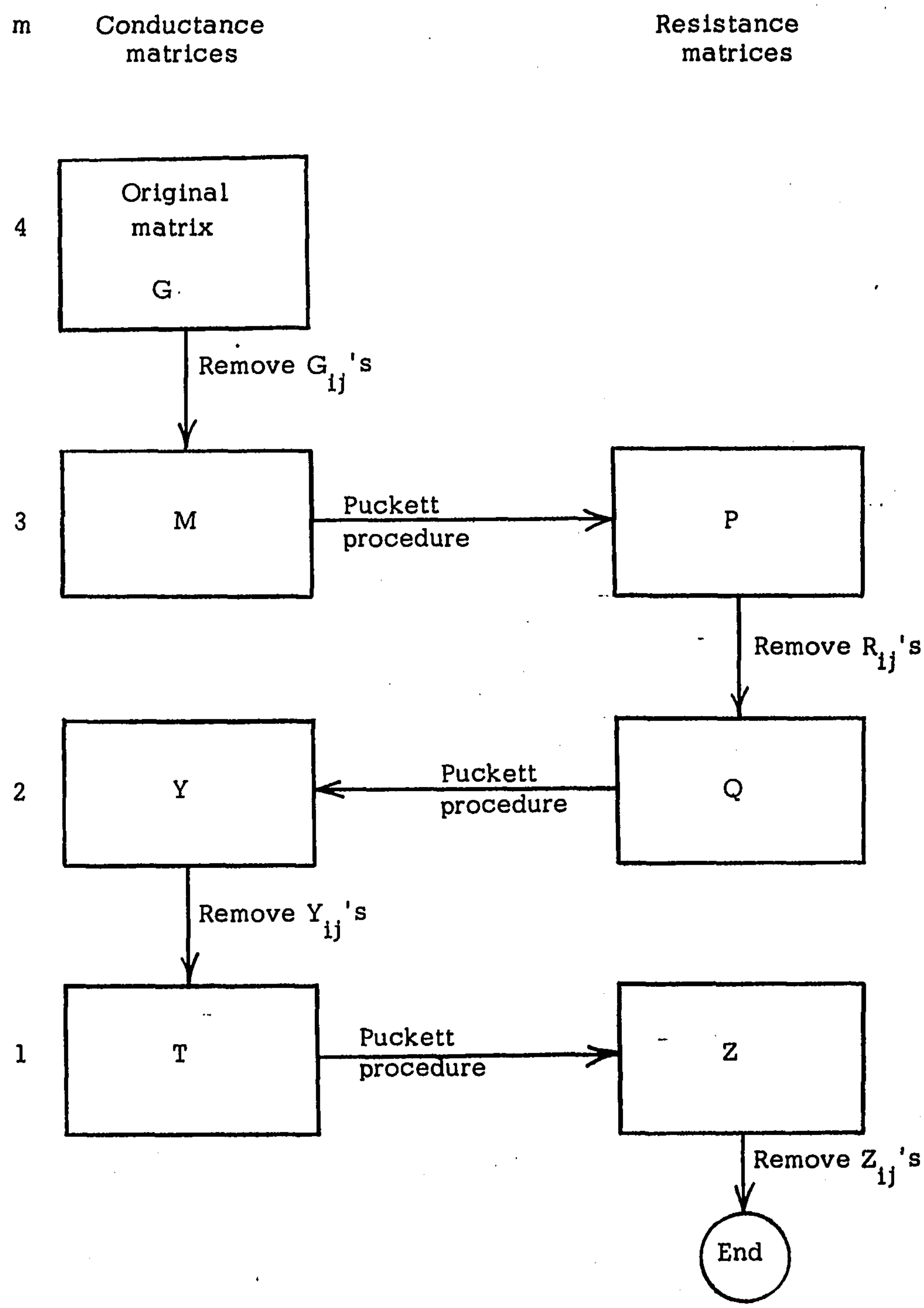



Fig. 4-19. Flow diagram of synthesis procedure

| $m=4$ | $m=3$ | $m=2$ | $m=1$ |
|---|---|---|---|
| $C^{12}_{56}$ | $C^{12}_{45}$ | $C^{12}_{34}$ | $-$ |
| $\begin{vmatrix} t_{25} & t_{26} \\ t_{15} & t_{16} \end{vmatrix} = 0$<br>$\downarrow\ t_{ij} = y_{ij}$<br>$\begin{vmatrix} y_{25} & y_{26} \\ y_{15} & y_{16} \end{vmatrix} = 0$<br>$\downarrow$ PUCKETT<br>$\begin{vmatrix} g_{35} & g_{36} & g_{37} \\ g_{25} & g_{26} & g_{27} \\ g_{15} & g_{16} & g_{17} \end{vmatrix} = 0$<br>$\downarrow\ p_{ij} = g_{ij}$<br>$\begin{vmatrix} p_{35} & p_{36} & p_{37} \\ p_{25} & p_{26} & p_{27} \\ p_{15} & p_{16} & p_{17} \end{vmatrix} = 0$<br>$\downarrow$ PUCKETT<br>$\begin{vmatrix} m_{45} & m_{46} & m_{47} & m_{48} \\ m_{35} & m_{36} & m_{37} & m_{38} \\ m_{25} & m_{26} & m_{27} & m_{28} \\ m_{15} & m_{16} & m_{17} & m_{18} \end{vmatrix} = 0$<br>$\downarrow\ m_{45} = g_{45} + G_{45}$<br>$G_{45} = \dfrac{\begin{vmatrix} g_{45} & g_{46} & g_{47} & g_{48} \\ g_{35} & g_{36} & g_{37} & g_{38} \\ g_{25} & g_{26} & g_{27} & g_{28} \\ g_{15} & g_{16} & g_{17} & g_{18} \end{vmatrix}}{-\begin{vmatrix} g_{36} & g_{37} & g_{38} \\ g_{26} & g_{27} & g_{28} \\ g_{16} & g_{17} & g_{18} \end{vmatrix}}$<br>$G_{45} = \dfrac{G_{4,5,4}}{G_{3,6,3}}$ | $\begin{vmatrix} t_{24} & t_{25} \\ t_{14} & t_{15} \end{vmatrix} = 0$<br>$\downarrow\ t_{ij} = y_{ij}$<br>$\begin{vmatrix} y_{24} & y_{25} \\ y_{14} & y_{15} \end{vmatrix} = 0$<br>$\downarrow$ PUCKETT<br>$\begin{vmatrix} g_{34} & g_{35} & g_{36} \\ g_{24} & g_{25} & g_{26} \\ g_{14} & g_{15} & g_{16} \end{vmatrix} = 0$<br>$\downarrow\ g_{34} = P_{34} + R_{34}$<br>$R_{34} = \dfrac{\begin{vmatrix} p_{34} & p_{35} & p_{36} \\ p_{24} & p_{25} & p_{26} \\ p_{14} & p_{15} & p_{16} \end{vmatrix}}{-\begin{vmatrix} p_{25} & p_{26} \\ p_{15} & p_{16} \end{vmatrix}}$<br>$R_{34} = \dfrac{P_{3,4,3}}{P_{2,5,2}}$ | $\begin{vmatrix} t_{23} & t_{24} \\ t_{13} & t_{14} \end{vmatrix} = 0$<br>$\downarrow\ t_{23} = y_{23} + Y_{23}$<br>$Y_{23} = \dfrac{\begin{vmatrix} y_{23} & y_{24} \\ y_{13} & y_{14} \end{vmatrix}}{-y_{14}}$<br>$Y_{23} = \dfrac{Y_{2,3,2}}{Y_{1,4,1}}$ | $Z_{12} = \dfrac{\lvert z_{12} \rvert}{-1}$<br>$Z_{12} = \dfrac{Z_{1,2,1}}{Z_{8,3,0}}$ |

Fig. 4-20. Details for example 4-2

the Puckett procedure, to an equivalent determinant condition, say L=0, we add a row on to the top and a column on to the right of the first determinant B, to obtain the second determinant L, (of course, we also change to the proper alphabetics). The procedure terminates when the upper-left-hand element of a particular determinant acquires subscripts $k,k+1$, at which stage we extract $M_{k,k+1}$.

In the case of n even and $m=m_{max}$, (in our example, m=4), we see that in addition to $g_{k,k+1} = g_{45}$, $g_{k,k+1} = g_{81}$, indicating that we could have satisfied condition $C^{12}_{56}$ by removing $G_{81}$ instead of $G_{45}$.

## 4.14 NECESSARY AND SUFFICIENT CONDITIONS FOR REALIZABILITY

We are unable to obtain a general proof for the necessary and sufficient conditions proposed by Foster. However, since we have proved the extraction formula (4-5), we can use it to formulate necessary and sufficient conditions through a synthesis procedure. That is, we carry out the synthesis procedure outlined in the introduction (Sec. 4-1), after first applying the port ordering algorithm to the matrix; if the extracted elements are all non-negative, then the matrix is realizable; if any element is negative, then the matrix is unrealizable. We can state this result as a theorem.

Theorem 4-6. Given an nth-order hyperdominant indefinite matrix S, which has been ordered according to Algorithm 4-1; S is realizable on a Foster planar canonic network if and only if the following Algorithm 4-2 is successfully completed.

Algorithm 4-2.

Step 1. $m_{max}$ = integer(n/2).

Step 2. $m = m_{max}$.

Step 3. If n is even, go to step 6.

Step 4. Extract n elements:

$$M^{(m)}_{k,k+1} = \frac{S_{k,k+1,m}}{-S_{k-1,k+2,m-1}}, \quad \text{for } k=1,2,\dots,n$$

Step 5. Go to step 7.

Step 6. Extract n/2 elements:

$$M^{(m)}_{k,k+1} = \frac{S_{k,k+1,m}}{-S_{k-1,k+2,m-1}}, \qquad \text{for } k=1,2,\ldots,n/2.$$

Step 7. If any $M^{(m)}_{k,k+1} < 0$, terminate algorithm; matrix is unrealizable.

Step 8. $m \Rightarrow m-1$.

Step 9. If $m=0$, go to step 12.

Step 10. $S \Rightarrow S$ op Puckett.

Step 11. Go to step 4.

Step 12. End, matrix is realizable.

Definitions.

a) $\Rightarrow$ means "is replaced by".

b) S op Puckett, means "take S, modify it by extracting the $M^{(m)}_{k,k+1}$ elements, perform the Puckett procedure".

c) $S_{k,k+1,m}$ is defined by (4-71).

d) $S_{k-1,k+2,0} = 1$.

## 4.15 CONCLUSION

A system of canonic planar n-port networks on n-terminals, devised by R.M. Foster, was described. We have shown how to derive the network formulas, by satisfying the conditions for the existaence of the converse of the star-mesh transformation. A set of necessary and sufficient conditions, proposed by Foster, was proved through n=7. In addition, the formulas he gave for extracting the network elements, were unified into a single one. Algorithms for properly ordering the ports and synthesizing the networks were given. Also some theorems on hyperdominant indefinite matrices and a compact statement of the conditions for the existence of the converse of the star-mesh transformation were presented.

# CHAPTER 5. INTRODUCTION TO THE REALIZATION OF NON-SINGULAR FOURTH-ORDER CONDUCTANCE MATRICES

## 5.1 INTRODUCTION

In Chapter 3 we have considered the realization of fourth-order indefinite matrices. An indefinite matrix is an example of a highly restricted type of matrix, the restrictions being that each of the main-diagonal elements equals the negative of the sum of the off-diagonal elements in the same row. Our search for a realization was directed toward discovering one or more networks which matched this matrix characteristic. When all such constraints (except paramountcy) are removed, the search for network realizations becomes more difficult. It is unlikely that a single network can be found to satisfy all non-singular matrices.

Consider the following matrix M.

$$M = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 1 & 5 & 4 & 5 \\ 2 & 4 & 6 & 6 \\ 3 & 5 & 6 & 53/7 \end{bmatrix} \qquad (5\text{-}1)$$

M is a paramount matrix of a type called irreducible (none of the main-diagonal elements can be reduced without destroying paramountcy). Cederbaum proved (CE 2) that M is not realizable as either a resistance matrix or a conductance matrix.

Suppose that we increase the main-diagonal elements of M so that M becomes dominant. By using a method due to Foster (WE 2) we can realize the dominant M matrix as a conductance matrix by an 8-terminal network. In fact Foster's method can realize any nth-order dominant conductance matrix by a 2n-terminal network. Thus any set of off-diagonal elements of a conductance matrix is realizable with some set of main-diagonal elements. We point out that this is not the case with resistance matrices, for Nambiar (NA 1) has shown that a particular fifth-order hyperdominant matrix is unrealizable.

A question which naturally arises is — what are the lower limits on the main-diagonal elements, such that M is realizable as a conductance matrix?

Before proceeding, let us state the problem in a more precise form. Let G be a conductance matrix. Let the off-diagonal elements of G, $g_{ij}$ $(i \neq j)$, be considered as constants. Let the main-diagonal elements of G, $g_{ii}$ be considered as parameters. Thus the $g_{ii}$ can be presented as an n-tuple $D = (g_{11}, g_{22}, \ldots, g_{nn})$, whose elements are positive real numbers. A minimal n-tuple will have the property that the reduction of any element of the n-tuple, will render G (with that n-tuple defining its main-diagonal elements) unrealizable as a conductance matrix.

The set of all minimal n-tuples defines the realizability boundary for the particular set of off-diagonal elements $g_{ij}$. That is, those n-tuples located above the realizability boundary allow the realization of the G matrix, whereas those n-tuples located below the realizability boundary define unrealizable G matrices.

Thus our primary task is to discover the networks which can realize matrices whose main-diagonal elements fall on the realizability boundary. We term such networks minimal networks. One characteristic of a minimal network is that none of its ports is shunted by a finite branch conductance. If this were not so, say, port k was shunted by a conductance $y_k$, we could then remove $y_k$ and reduce $g_{kk}$ to $g_{kk} - y_k$. Once we have a minimal network realization, we can adapt the network to matrices with larger main-diagonal elements by shunting appropriate ports.

We make extensive use of these concepts in Chapter 6, where the subject of the realization of 4-port networks on n+2 terminals is considered.

## 5.2 CLASSIFICATION OF NETWORKS FOR REALIZATION OF CONDUCTANCE MATRICES

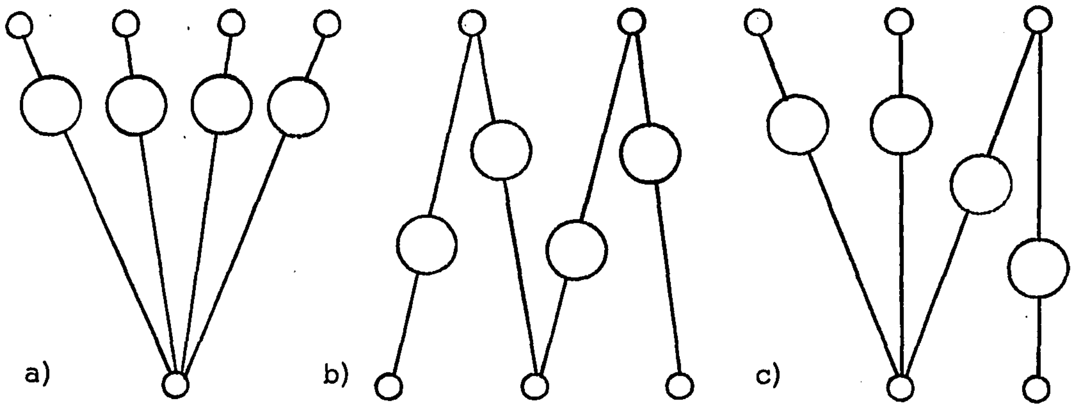
In order to present our results in an organized fashion, we discuss some of the criteria used in classifying networks. The number of terminals contained in a network is considered an indicator of complexity. The least complex networks for realizing non-singular conductance matrices contain n+1 terminals. The n ports of the network thus form a tree of the network graph. As indicated in Chapter 2, the synthesis of networks on n+1 terminals is solved. Therefore we will present the fourth-order results and then concentrate our efforts on the more complex networks—those containing more than n+1 terminals.

The maximum number of terminals we need consider is 2n. For if a network contains additional nodes above 2n, then these nodes will not be connected to ports, and they may be removed by application of the star-mesh transformation (RO 1). It is known (LE 1) that certain matrices require 2n-1 terminals for a realization. It is not known, however, whether 2n terminals are required in any particular case.

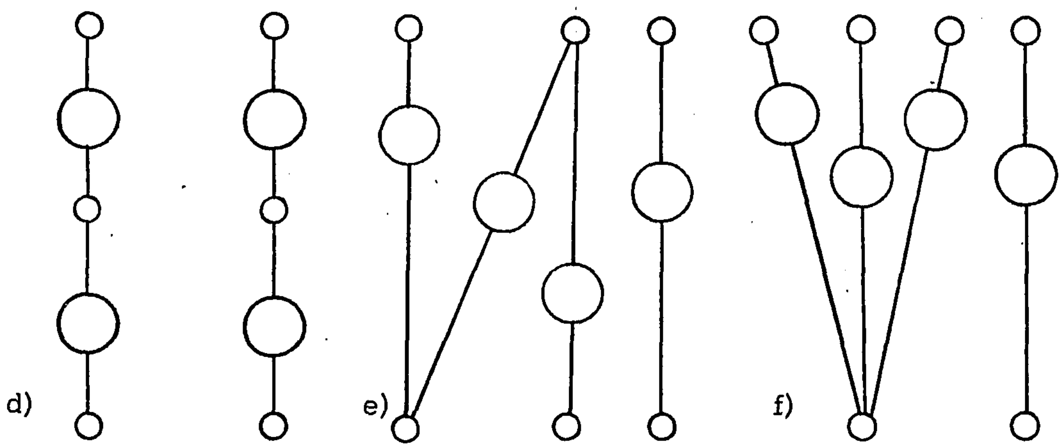
If the number of terminals of a network is equal to n+p, then the ports must form p disjoint subtrees of the network graph, in order that the ports span all of the terminals. Thus we have an additional sub-classification—the arrangement of the port subtrees. Figure 5-1 shows all of the possible unlabeled port structures for 4-port networks.

## 5.3 (n+1)-TERMINAL 4-PORT NETWORKS

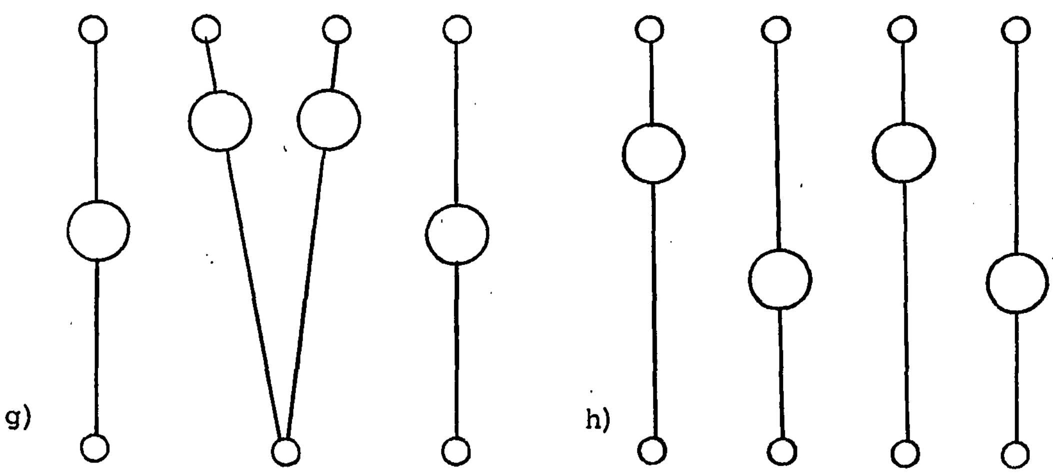
In a network which contains n+1 terminals, the maximum number of branches is equal to n(n+1)/2. This number is also equal to the number of independent elements in an nth-order condcutance matrix. The implication of this is that, given such a network, we can analyze it by standard methods (WE 1) and compute the n(n+1)/2 elements of the conductance matrix in terms of the network branch condcutances. Since the number of branch conductances is then equal to the number of equations, we can invert the equations and find the formulas for the




5-terminal port graphs




6-terminal port graphs




7-terminal port graph

8-terminal port graph

Fig. 5-1. All possible unlabelled fourth-order port graphs branch conductances in terms of the conductance matrix elements, or obtain them using a linear transformation from some known network. Obtaining these formulas provides us with a synthesis procedure.

For the case of n=4, there are only three such networks. These are illustrated in Figure 5-2. Network 5-2a has the star-tree port structure and can realize any fourth-order hyperdominant conductance matrix. The formulas which are given in (WE 1), are shown in the figure. Network 5-2b is the linear-tree port structure which can realize any fourth-order uniformly tapered condcutance matrix. The formulas shown on the figure were given by Guillemin (GU 2). The branch conductance formulas for the network of Figure 5-2c are not stated explicitly in the literature. However, Brown and Reed (BR 6) give the necessary and sufficient conditions for realizability for this port configuration. The branch nomenclature used in 5-2c was proposed by Guillemin (GU 1). Its utility becomes evident below.

We can derive the branch conductance formulas for the network of Figure 5-2c, by relating this network, through a linear transformation, to the network of Figure 5-2b, whose formulas are known. The reader will recall that we used this procedure in Chapter 3, in order to derive the formulas for the network of Figure 3-7. Since we have given the derivation in Chapter 3, we do not repeat it here; instead we give the results. Let G be a conductance matrix to be realized by the network of 5-2c. Let G' be the linear transformation of the G matrix to the network of 5-2b. Then

$$G' = A^tGA \tag{5-2}$$

where $E = AE'$. The A matrix relating these two systems is given below:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{5-3}$$

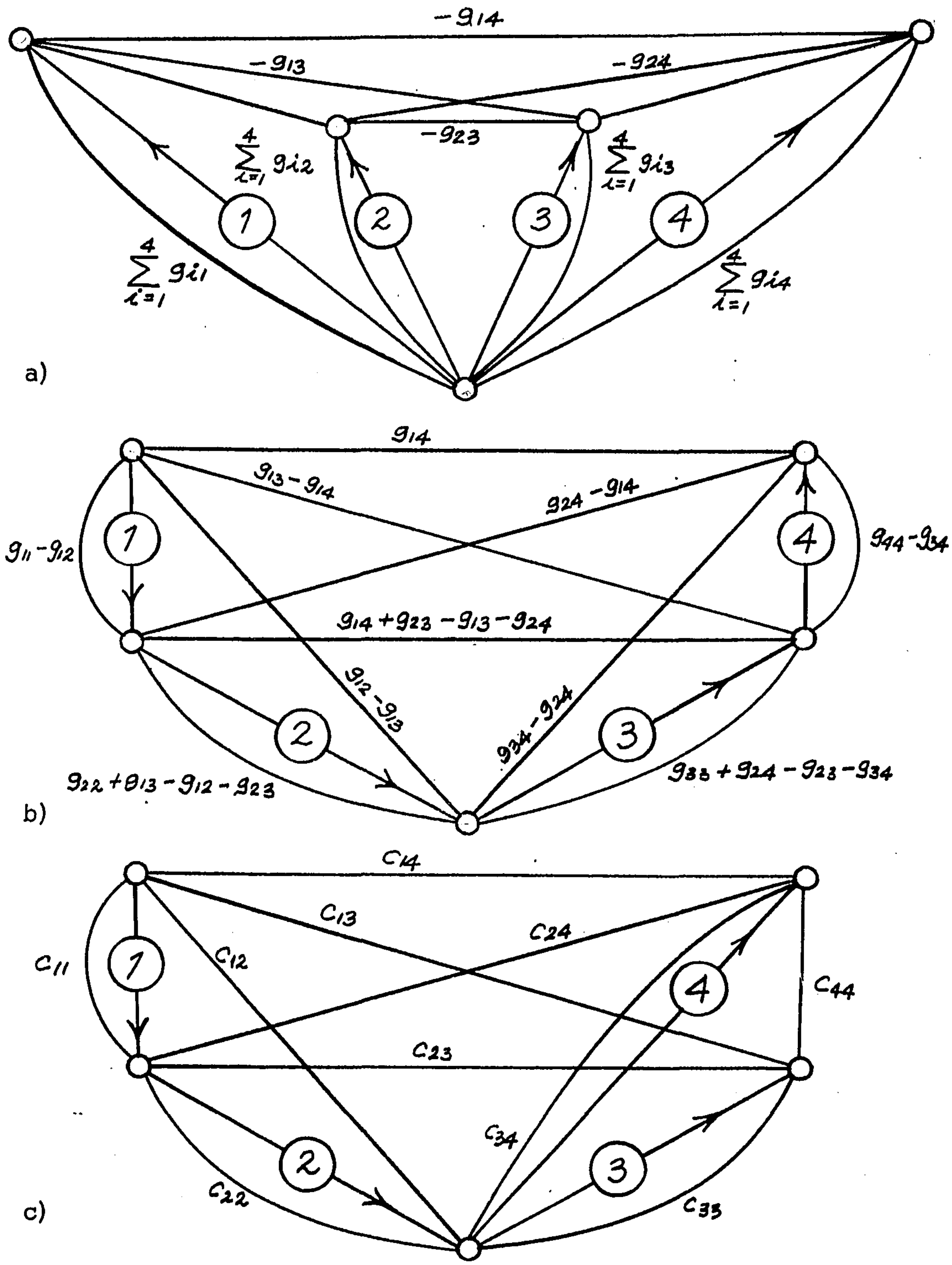



Fig. 5-2. The three (n+1)-terminal 4-port networks

The network of Figure 5-2b has its ports arranged as a linear tree. Guillemin (GU 1) has shown that the matrix of branch conductances $C = [c_{ij}]$ of a linear tree port structure network, corresponding to a conductance matrix G', may be obtained by the transformation $C = TG'T$, where T is given below:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{5-4}$$

Applying this and the previous results, we can find C, the matrix of branch conductances of the network of Figure 5-2c: $C = TA^tGAT$. Carrying out the matrix multiplications, we obtain:

$$C = [c_{ij}] = \begin{bmatrix} (g_{11}-g_{12}) & (g_{12}-g_{13}-g_{14}) & g_{13} & g_{14} \\ & \begin{matrix}(g_{22}+g_{13}+g_{14}+ \\ -g_{12}-g_{23}-g_{24})\end{matrix} & (g_{23}-g_{13}) & (g_{24}-g_{14}) \\ & & (g_{33}+g_{34}-g_{23}) & (g_{44}+g_{34}-g_{24}) \\ & & & -g_{34} \end{bmatrix} \tag{5-5}$$

Thus using Figure 5-2c and the above matrix we see, for example that branch $c_{44}$ of the network is equal to $-g_{34}$ mhos, where $g_{34}$ is the (3,4) entry of the given G matrix.

## 5.4 NECESSARY AND SUFFICIENT CONDITIONS FOR REALIZABILITY

Our ultimate aim is two-pronged: first, to find the necessary and sufficient conditions for a fourth-order matrix to be realizable as a conductance matrix by a four-port resistance network, and second, to synthesize the network.

In some cases the prongs are independent. That is, we can find the first, without finding the second, and vice versa. For example, for the third-order case, the necessary and sufficient condition for realizability is that the matrix be paramount. Paramountcy can be tested for independently of any synthesis procedure.

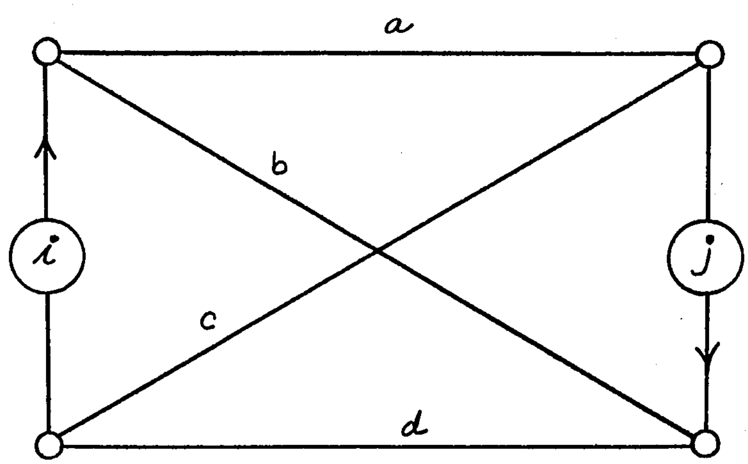
Now for general cases, where n is greater than three, and paramountcy is known to be no longer sufficient, the realizability conditions must be related to some synthesis procedure, as no general matrix property is known. As an example of this, we have Cederbaum's decomposition procedure (CE 5).

As another illustration, in any network for which we have the formulas for the branch conductances, the necessary and sufficient conditions for realizability by the network are that all of the branch conductances be non-negative. Thus in the network of Figure 5-2c, the necessary and sufficient conditions for realizability are that $c_{ij} \geq 0$, for $j = 1,2,3,4$, and for all $i \leq j$.

## 5.5 SIGN PATTERNS

The sign of any nonzero off-diagonal matrix element may be positive or negative. Since there are $x = n(n-1)/2$ independent off-diagonal elements in a general nth-order matrix, the number of different possible sign patterns is $2^x$. As mentioned in Chapter 2, synthesis procedures for (n+1)-terminal networks were developed employing the sign patterns of conductance matrices. The sign pattern was shown to have a one-to-one correspondence with the port-tree structure.

If the number of network terminals is n+p, where $p \geq 2$, the ports are then separated into p subtrees. In this case, the port structure no longer completely determines the matrix sign pattern (and vice versa). An example of this is given by Figure 5-3, which represents an (n+2)-terminal network in which all ports except i and j have been short-circuited. Note that port i is in one subtree and port j in the other. Computing $g_{ij}$, we obtain:




(a,b,c,d are the branch conductances)

Fig. 5-3. Equivalent (n+2)-terminal network with all ports except i and j short-circuited

$$g_{ij} = \frac{ad - bc}{a+b+c+d} \tag{5-6}$$

Equation (5-6) shows that the transfer conductance $g_{ij}$ can be positive, negative, or zero, depending upon the relative values of the branch conductances. Thus it is evident that the sign of $g_{ij}$ is independent of the port-tree configuration when ports i and j are in different subtrees.

However, the sign matrix and the port structure are interrelated. Some of the matrix signs determine the relative port orientations in each port subtree. Assume that it is desired to realize a conductance matrix G on an (n+p)-terminal network. Let G be partitioned into p parts corresponding to the p subtrees.

$$G = \begin{bmatrix} G_{11} & & & G_{1p} \\ & & G_{kk} & \\ G_{p1} & & & G_{pp} \end{bmatrix}$$

Suppose that the kth subtree of the network contains m ports. Now form a new m-port network by short-circuiting all ports except those in the kth subtree. The conductance matrix for the m-port network is formed by deleting the rows and columns in G corresponding to the short-circuited ports. The new matrix is $G_{kk}$, a principal submatrix of G. Based upon the previous discussion, the port structure (a tree) of the m-port network, must correspond to the sign pattern of $G_{kk}$. These results can be stated as a theorem.

<u>Theorem 5-1</u>. Given an nth-order conductance matrix G. If G is realizable by an (n+p)-terminal network, then the structure of each port subtree is determined by the sign pattern of the corresponding principal submatrix of G.

## 5.6 SIGN PATTERNS OF FOURTH-ORDER MATRICES

For the case of n=4, we have x=6, and the maximum number of sign patterns is then $2^6 = 64$. However, multiplying some rows and corresponding columns of the matrices by -1, (equivalent to changing the port orientations) each of the 64 sign patterns may be converted into one of three patterns. The three patterns are:

1) all off-diagonal elements negative,
2) all off-diagonal elements positive,
3) 5 positive and 1 negative off-diagonal elements.

We illustrate the above with an example. Assume that a conductance matrix has the sign pattern shown below:

$$\begin{bmatrix} + & - & - & + \\ - & + & - & + \\ - & - & + & - \\ + & + & - & + \end{bmatrix}$$

If we multiply row 3 and column 3 by -1, all of the signs become positive with the exception of $g_{12}$. Thus the matrix sign pattern is of the third type listed above.

In addition, we note that in the third type of sign pattern, the single negative off-diagonal element may be located at any of the six off-diagonal locations, by suitably permuting the matrix (equivalent to renumbering the ports).

For the 5-terminal, 4-port case, reference to Figure 5-2 shows that each of the three networks corresponds to one of the three sign patterns. Network 5-2a realizes matrices with all off-diagonal elements negative, and thus corresponds to sign pattern 1). Network 5-2b realizes matrices with all off-diagonal elements positive, and corresponds to sign pattern 2). And network 5-2c realizes matrices with 5 positive and 1 negative off-diagonal elements, and thus corresponds to sign pattern 3).

It is well known (WE 1) that the network of Figure 5-2a can realize any hyperdominant conductance matrix. It can also be demonstrated that any paramount matrix with all off-diagonal elements negative must be hyperdominant. Thus we may exclude this sign pattern from further consideration.

If a conductance matrix with all positive off-diagonal elements is uniformly tapered (GU 1), then it is realizable by the network shown in Figure 5-2b. However, if it is not uniformly tapered, it may be realizable on a higher-order network, that is, a network with more terminals. A similar comment applies to sign pattern 3) and the network shown in Figure 5-2c. Therefore, in Chapter 6, where we discuss (n+2)-terminal 4-port networks, it is only necessary for us to consider sign patterns 2) and 3).

## 5.7 PLANARITY

Planarity is important for several reasons. The first is a practical engineering concern. Suppose that it is desired to manufacture n-port networks as integrated circuits. If the realization is a planar network, then only a single metalization layer (and mask) is required. If the network is nonplanar, then several would be required.

A second reason is related to the realization of resistance matrices. It is well known (WE 1) that any planar network N has at least one dual network N'. If N realizes a matrix M as a conductance matrix, then N' will realize M as a resistance matrix, and vice versa. Thus any planar conductance matrix solutions also solve associated resistance matrix problems. In Chapter 3 we have given examples of the use of this principle.

However, all three of the 5-terminal, 4-port networks of Figure 5-2 are nonplanar, since the graph of each network contains as a subgraph, $K_5$, the complete 5-terminal graph. Thus we cannot state any dual properties in relation to resistance matrices. However, removing any branch which is not in parallel with a port, from any of the networks of Figure 5-2, converts it to a planar network.

## 5.8 DEGENERATE NETWORKS

We mentioned above the possibility of deleting a branch from a basic network. Let us now consider the implication of such a deletion, choosing the network of Figure 5-2c for illustration. Suppose branch $c_{12}$ is deleted from the network. Then $c_{12} = 0$, and from (5-5) we have $g_{12} = g_{13} + g_{14}$. If a matrix is to be realized by the new 9-branch network, this equation must be satisfied by the conductance matrix. We define a network which requires such an equation to be satisfied by some of the matrix elements as a degenerate network. Since the equation is a linear one, the network is said to possess a linear degeneracy. The branch conductance formulas for (n+1)-terminal networks consist of linear combinations of the conductance matrix elements. This is established by the fact that the formulas for the branch conductances of the star-tree port structure network are linear, and any other tree structure network can be derived from the star-tree network through a linear transformation. Therefore, any missing branch in an (n+1)-terminal network will cause a linear degeneracy.

In Chapter 6, in our study of the (n+2)-terminal, 4-port networks, we find and catalog all minimal networks. It is clearly beneficial to have the number of minimal networks be as small as possible. Therefore, it is important to be able to recognize the degenerate networks and to eliminate them, since a degenerate network is not useful for synthesis of a general network, because of the constraint imposed on the matrix by the equation of degeneracy. For example, a degenerate network may realize some particular conductance matrix G, but will then be unable to realize G+g, where g is a real symmetric matrix of arbitrarily small numbers.

Biorci and Civalleri (BI 7,8) have studied and reported some results on degeneracy in (n+2)-terminal networks. Since Chapter 7 is devoted entirely to the subject of degeneracy, we discuss Biorci and Civalleri's results in detail there. However, we indicate below a few items which are relevant to the material in Chapter 6.

Biorci and Civalleri (BI 8) show that one of several conditions which cause a degeneracy in an (n+2)-terminal network, is that the missing branches of the network form an even circuit, that is, a circuit of even length. This means that at least four branches must be missing in order to cause this type of degeneracy. This type of degeneracy is characterized by having a nonlinear equation of degeneracy, and is thus termed a nonlinear degeneracy. For example, $g_{12}g_{34} = g_{13}g_{24}$ is the equation of degeneracy of an example in Chapter 6.

In proving their theorem relating to this condition, Biorci and Civalleri made an error; certain circuits of four missing branches, rather than leading to nonlinear equations of degeneracy, instead lead to identities. Thus the number of useable equations is reduced and is insufficient to provide a unique solution for the branch conductances. This is discussed in detail in Chapter 7.

Very little use seems to have been made of the previous results on degenerate networks. A search of the literature finds no reference to Biorci and Civalleri's work, or any new research on the subject. The conditions for degeneracy in (n+p)-terminal networks, for $p \geq 2$, are not known. We have obtained some results on degeneracy in (n+p)-terminal networks, which are reported in Chapter 7.

## 5.9 BRANCH CONDUCTANCE FORMULAS FOR (n+2)-TERMINAL NETWORKS

An (n+2)-terminal complete graph, n-port network has (n+2)(n+1)/2 branch conductances. Given such a network, we can analyze it by standard methods (WE 1) and compute the n(n+1)/2 short-circuit conductance elements $g_{ij}$ of its conductance matrix G. If we wish to invert the equations, that is, solve for the branch conductances in terms of the $g_{ij}$, we find that a unique solution does not exist, as there are n+1 more unknowns (branches) than equations. Biorci and

and Civalleri (BI 8) have shown that if n+1 branches are set equal to zero, then the set of n+1 additional equations which results from this, plus the original set of equations, can always be inverted, provided that the network is not degenerate. If the network is degenerate, then the number of independent equations is reduced and is insufficient to provide a unique solution. Thus degeneracy and the missing branches are important concepts in obtaining the formulas for the branch conductances. We show how this is accomplished and give examples in Chapter 6.

We define solutions as described above as rigid solutions. That is, in cases where the number of branches is equal to the number of independent matrix elements $g_{ij}$, the solution is generally unique. We say generally unique, because, as Biorci and Civalleri (BI 8) point out, the branch conductance formulas occasionally involve quadratic functions, which may have two real solutions.

In addition, Biorci and Civalleri (BI 8) have shown that if n branch conductances of an (n+2)-terminal network are set equal to zero and one branch conductance is set equal to a parameter k, the equations may be inverted so as to express the branch conductances in terms of the $g_{ij}$ and k. We define this type of solution as a flexible solution, since by varying k, a continuously equivalent set of network realizations of the conductance matrix G may be obtained. We make use of this technique in Chapter 6 to construct the minimal networks.

## 5.10 BIORCI'S CONJECTURE

In 1962 Biorci (PD 1) suggested investigating the following conjecture: given a network of positive resistors with n ports and m branches, where m is larger than n(n+1)/2, which is the number of independent elements of the short-circuit conductance matrix, there exists another network with n ports and m-1 branches having the same short-circuit conductance matrix as the first network. If the statement is true it follows that, if a short-circuit condcutance matrix is realizable with m branches, it is also realizable with $n(n+1)/2$ branches.

No counterexample to Biorci's conjecture has been found; nor has a general proof of its truth been discovered. In Chapter 6, we show that general realizations of (n+2)-terminal, 4-port networks satisfy the conjecture.

## 5.11 CONCLUSION

In this chapter we have introduced the concepts of the realizability boundary and minimal networks, discussed the problems associated with the synthesis of (n+p)-terminal networks, and presented a varied collection of background material. In Chapter 6, we proceed to the realization of 6-terminal, 4-port networks.

# CHAPTER 6. REALIZATION OF NONSINGULAR FOURTH-ORDER CONDUCTANCE MATRICES BY 6-TERMINAL NETWORKS

## 6.1 INTRODUCTION

We recall from Chapter 5, that the 6-terminal, 4-port network has three possible port-tree structures. These are shown in Figures 5-1d,e,f. We designate them respectively as the 2-2 linear tree, the 1-3 linear tree, and the 1-3 star tree. In order to be thorough, we consider all of these structures. So as to avoid repetition, we cover the 2-2 linear tree case in detail, whereas for the other two cases, we report the results, including only such detail as necessary.

We begin with a synthesis procedure suggested by Guillemin (GU 3) in 1961. Although the procedure is incomplete, it provides insight and is a convenient starting point.

## 6.2 GUILLEMIN'S METHOD

As the networks under consideration have 6 terminals and 4 ports, the ports will be divided into two subtrees. Guillemin suggested adding a fifth port to the tree structure in such a manner that the new port structure becomes a single 5-port linear tree. We illustrate this in Figure 6-1, for the case of the 2-2 linear tree. Cases where the subtrees are not linear, for example, Figure 5-1f, may be accommodated through the use of a linear transformation, as is done in Section 6.17.

Now suppose we are given a conductance matrix G which is known to be unrealizable by a 5-terminal network. We expand the G matrix to the fifth order, in order to take account of the additional port which we have added. We carry out this expansion by adding a row and column of zeros to G, in the same relative location in the matrix, as the new port bears to the 2-2 linear tree. In this case the row and column of zeros will be the third and will partition G in a manner corresponding to the division of the ports in the 2-2 linear tree.

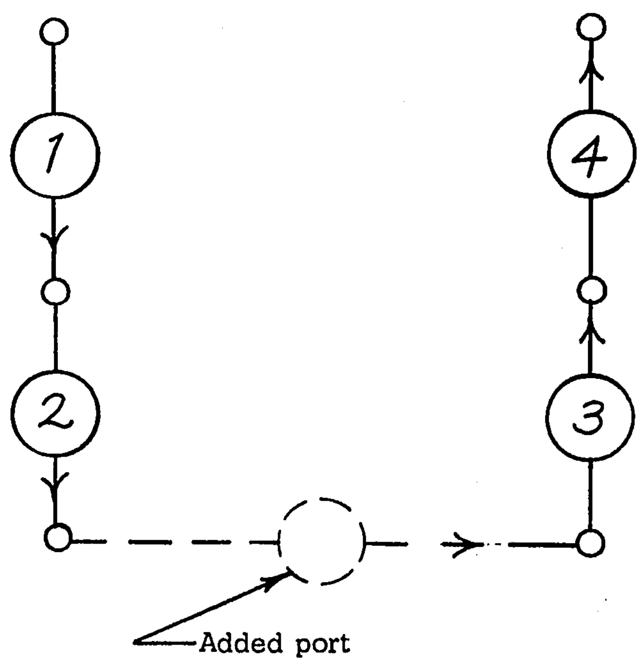



Fig. 6-1.    Port graph of 2-2 tree structure

The expanded matrix is called $G_{exp}$ and is shown below.

$$G_{exp} = \begin{bmatrix} g_{11} & g_{12} & 0 & g_{13} & g_{14} \\ g_{12} & g_{22} & 0 & g_{23} & g_{24} \\ 0 & 0 & 0 & 0 & 0 \\ g_{13} & g_{23} & 0 & g_{33} & g_{34} \\ g_{14} & g_{24} & 0 & g_{34} & g_{44} \end{bmatrix} \qquad (6\text{-}1)$$

Since G is nonsingular and of rank 4, then $G_{exp}$ is of rank 4 (and of order 5). Guillemin proves that adding an arbitrary real symmetric matrix B, of rank 1, with nonzero elements, to $G_{exp}$, gives $G' = G_{exp} + B$, a matrix of rank 5; and applying pivotal condensation (WE 1, p.51) to G' about the added port (corresponding to open-circuiting the added port) gives the original G matrix.

The implication of this is that if we choose the arbitrary B matrix in such a manner that G' is uniformly tapered, then G is realizable. We merely realize G' by a known method, also due to Guillemin (GU 2), and then consider the added port to be open-circuited. The resulting network is a 6-terminal realization of G. The method is not restricted to the fourth-order, but is general; however, we couch our discussion in terms of the fourth-order problem. Guillemin indicated that the method is also applicable to (n+p)-terminal networks, where $p>2$.

As mentioned in Chapter 5, Guillemin (GU 2) has shown that the linear transformation TGT = C applied to the conductance matrix G of a network with a linear port-tree structure, yields C the matrix of branch conductances of the network. T is of the form shown in equation (5-4). We apply the transformation to $G' = G_{exp} + B$.

$$TG'T = T(G_{exp} + B)T = TG_{exp}T + TBT = h + b = C \qquad (6\text{-}2)$$

As stated by Guillemin, h is the branch conductance matrix of a complete graph network associated with $G_{exp}$, and b is the branch conductance matrix of a complete graph network associated with B. Each network will contain both positive and negative conductances. A proper realization of G will result when the two networks are connected in parallel and the resulting branch conductances of the combined network are non-negative.

Applying the transformation $TG_{exp}T = h$ to equation (6-1) yields:

$$h = \left[h_{ij}\right] = \begin{bmatrix} (g_{11}-g_{12}) & g_{12} & -g_{13} & (g_{13}-g_{14}) & g_{14} \\ & (g_{22}-g_{12}) & (g_{13}-g_{23}) & \begin{matrix}(g_{23}-g_{13}+ \\ g_{14}-g_{24})\end{matrix} & (g_{24}-g_{14}) \\ & & g_{23} & (g_{24}-g_{23}) & -g_{24} \\ & & & (g_{33}-g_{34}) & g_{34} \\ & & & & (g_{44}-g_{34}) \end{bmatrix}$$

(6-3)

We observe that the $h_{ij}$ are linear combinations of the $g_{ij}$. We also note that $h_{11}$, $h_{22}$, $h_{44}$, $h_{55}$ are linearly related to the driving-point conductances $g_{ii}$, and that each is non-negative because G is a paramount matrix and hence, $g_{ii} \geq |g_{ik}|$, for all k. (Note also that $h_{44}$ corresponds to port 3 and $h_{55}$ to port 4.)

Guillemin (GU 1) gives a convenient form for b, the branch conductance matrix derived from the rank 1 arbitrary matrix B. The fifth-order b matrix is derived by first constructing the rank 1, B matrix with 5 arbitrary nonzero variables. Then the transformation b = TBT is applied, and a change of variables is made, which yields:

$$b = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ & \frac{-x_1x_2}{a} & \frac{-x_1x_3}{a} & \frac{-x_1x_4}{a} & \frac{-x_1x_5}{a} \\ & & \frac{-x_2x_3}{a} & \frac{-x_2x_4}{a} & \frac{-x_2x_5}{a} \\ & & & \frac{-x_3x_4}{a} & \frac{-x_3x_5}{a} \\ & & & & \frac{-x_4x_5}{a} \end{bmatrix} \qquad (6\text{-}4)$$

$$a = \sum_{i=1}^{5} x_i$$

The $x_i$ are the five arbitrary variables, subject to the constraint that $a \neq 0$, in order that the elements of b remain finite.

The matrix of branch conductances of the combined networks C, is formed by adding b to h. For proper realization, each of the 15 $c_{ij}$ must be greater than or equal to zero. We thus have 15 inequalities of the form $h_{ij} + b_{ij} \geq 0$. The five $x_i$ are available as arbitrary parameters, whose proper choice we hope will satisfy the 15 inequalities. This situation is nondeterministic in the sense that no known algorithm exists for choosing the $x_i$, so that we can determine whether it is possible for a network with this port structure to realize the given G matrix. Guillemin (GU 3) recognized this as evidenced by his statement: "Additional study, however, needs to be directed toward developing a systematic procedure that will clearly indicate whether or not a solution exists in a given situation".

Another difficulty is that even if we manage to find a set of $x_i$ which satisfies all fifteen inequalities, we do not in general end up with a 10-branch network as predicted by Biorci's conjecture, unless five of the inequalities happen to be satisfied by equality. Guillemin gave a third-order example with three different trial-and-error solutions, none of which has the minimum number of branches.

## 6.3 6-TERMINAL MINIMAL NETWORKS

We recall from Chapter 5 that minimal networks are those whose ports are not shunted by any conductances. Figure 6-2 shows the general 11 branch minimal network for the 2-2 linear port tree configuration. The branches on the figure are labeled with the corresponding elements of the C matrix. We can write an equation for each missing port branch.

$$\begin{aligned}
&\text{a)} \quad c_{11} = b_{11} + h_{11} = 0 \\
&\text{b)} \quad c_{22} = b_{22} + h_{22} = 0 \\
&\text{c)} \quad c_{44} = b_{44} + h_{44} = 0 \\
&\text{d)} \quad c_{55} = b_{55} + h_{55} = 0
\end{aligned} \qquad (6\text{-}5)$$

We recall again from Chapter 5 that we desire our minimal network to have a flexible rather than a rigid solution, in order that a continuously equivalent set of network realizations be available, so as to enable the determination of the realizability boundary. Therefore, we let:

$$b_{33} = k \qquad (6\text{-}6)$$

where k is a parameter. As k varies, $c_{33} = h_{33} + k$, varies similarly. Clearly we could have chosen any other branch as the variable one. We can visualize this in the following manner. Assume that $c_{33}$ is a variable conductance and that we vary k by varying $c_{33}$. All of the other $c_{ij}$ branch conductances will have to then vary in order to ensure that the off-diagonal elements of the G matrix remain constant. We can then construct the other $c_{ij}$ from variable conductances and mechanically link them to $c_{33}$ in a fashion such that the $g_{ij}$, $i \neq j$, always remain fixed. Once all of the conductances are linked, it makes no difference which one we vary, and thus any one of them could have been the original variable one.

From equations (6-4,5,6) we obtain the following set of equations:

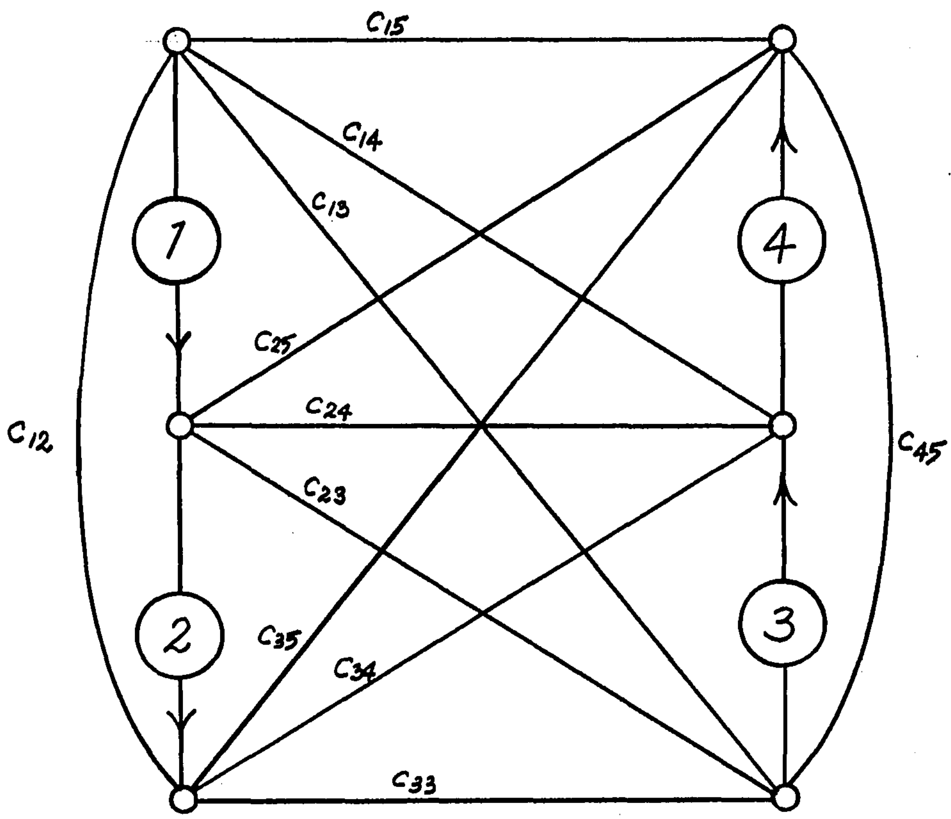



Fig. 6-2. General minimal network for the 2-2 linear tree port configuration

a) $h_{11} = -x_1$

b) $h_{22} = x_1 x_2/a$

c) $h_{44} = x_3 x_4/a$

d) $h_{55} = x_4 x_5/a$ (6-7)

e) $k = -x_2 x_3/a$

f) $a = x_1+x_2+x_3+x_4+x_5$

Solving (6-7) for the $x_i$ (where for convenience we introduce a new parameter $\theta$, defined below in (6-8g)), yields:

a) $x_1 = -h_{11}$

b) $x_2 = -k\theta h_{11}/h_{22}$

c) $x_3 = kh_{11}/h_{22}$

d) $x_4 = \theta h_{11}h_{44}/h_{22}$ (6-8)

e) $x_5 = kh_{11}h_{55}/h_{22}h_{44}$

f) $a = k\theta(h_{11}/h_{22})^2$

g) $\theta = \dfrac{k(1+h_{55}/h_{44}) - h_{22}}{k(1+h_{11}/h_{22}) - h_{44}}$

Substituting (6-8) into (6-4) we obtain the b matrix in terms of the $h_{ii}$, k, and $\theta$ (where we see in (6-8g) that $\theta$ itself is a function of $h_{ii}$ and k).

$$b = \begin{bmatrix} -h_{11} & -k\theta h_{11}/h_{22} & kh_{11}/h_{22} & \theta h_{11}h_{44}/h_{22} & \dfrac{kh_{11}h_{55}}{h_{22}h_{44}} \\ & -h_{22} & h_{22}/\theta & h_{22}h_{44}/k & \dfrac{h_{22}h_{55}}{kh_{44}} \\ & & k & \theta h_{44} & \dfrac{kh_{55}}{h_{44}} \\ & & & -h_{44} & \dfrac{-kh_{55}}{\theta h_{44}} \\ & & & & -h_{55} \end{bmatrix} \tag{6-9}$$

6.4 SIGN PATTERNS

For the present, let us assume that the G matrix has sign pattern 2), as explained in Chapter 5, namely, all off-diagonal elements positive. From equation (6-3) we obtain the sign pattern of the h matrix.

$$\text{sign } h = \left[\begin{array}{cc|ccc} + & + & - & \pm & + \\ & + & \pm & \pm & \pm \\ & & + & \pm & - \\ \hline & & & + & + \\ & & & & + \end{array}\right] \tag{6-10}$$

In order to make it possible to counter the potential negative signs in (6-10), we make both k and θ positive, to establish the following sign pattern for the b matrix.

$$\text{sign } b = \left[\begin{array}{cc|ccc} - & - & + & + & + \\ & - & + & + & + \\ & & + & + & + \\ \hline & & & - & - \\ & & & & - \end{array}\right] \tag{6-11}$$

We see, that as far as the signs are concerned, it is possible for the sum $b + h = C$, not to contain any negative elements. We recall that the branches shunting the ports are absent, and thus the eleven possible remaining branches give rise to 11 inequalities of the form $b_{ij} + h_{ij} \geq 0$. Is there some preferred arrangement of the rows and columns of G, which will satisfy some of these inequalities automatically? Suppose that we permute G, that is, exchange sets of rows and corresponding columns, (equivalent to renumbering the ports) so that $g_{14}$ is the smallest off-diagonal element. The sign pattern of the h matrix then becomes:

$$\text{sign } h = \left[\begin{array}{cc|ccc} + & + & - & \overset{\triangle}{+} & \overset{\triangle}{+} \\ & + & \pm & \pm & \overset{\triangle}{+} \\ & & \overset{\triangle}{+} & \pm & - \\ \cline{3-5} & & & + & + \\ & & & & + \end{array}\right] \qquad (6\text{-}12)$$

Now we see that 4 of the inequalities (designated on the matrix by triangles), in the 3x3 submatrix indicated by the dashed lines, are always satisfied, since the corresponding values of $b_{ij}$ and $h_{ij}$ are both positive. Thus $c_{14}$, $c_{15}$, $c_{25}$, $c_{33}$ are all greater than zero and are always present in the minimal network.

The minimal network then appears as shown in Figure 6-3. The solid lines represent branches which are always present in the network. The dashed lines represent the remaining 7 branches. They are designated by integers, to facilitate coding the network graphs, which we discuss later in this section. Comparing Figures 6-2 and 6-3 makes clear the correspondence between the $c_{ij}$ and integer designations.

As noted in Chapter 5, we postulated that the driving-point conductances $g_{ii}$ of the G matrix are to be considered as parameters; and since the $h_{ii}$ of the h matrix are linearly related to the $g_{ii}$ (see equation (6-3)), our problem is still indeterminate, as far as finding the elements of the b matrix. However, if we allow 4 of the 7 dashed

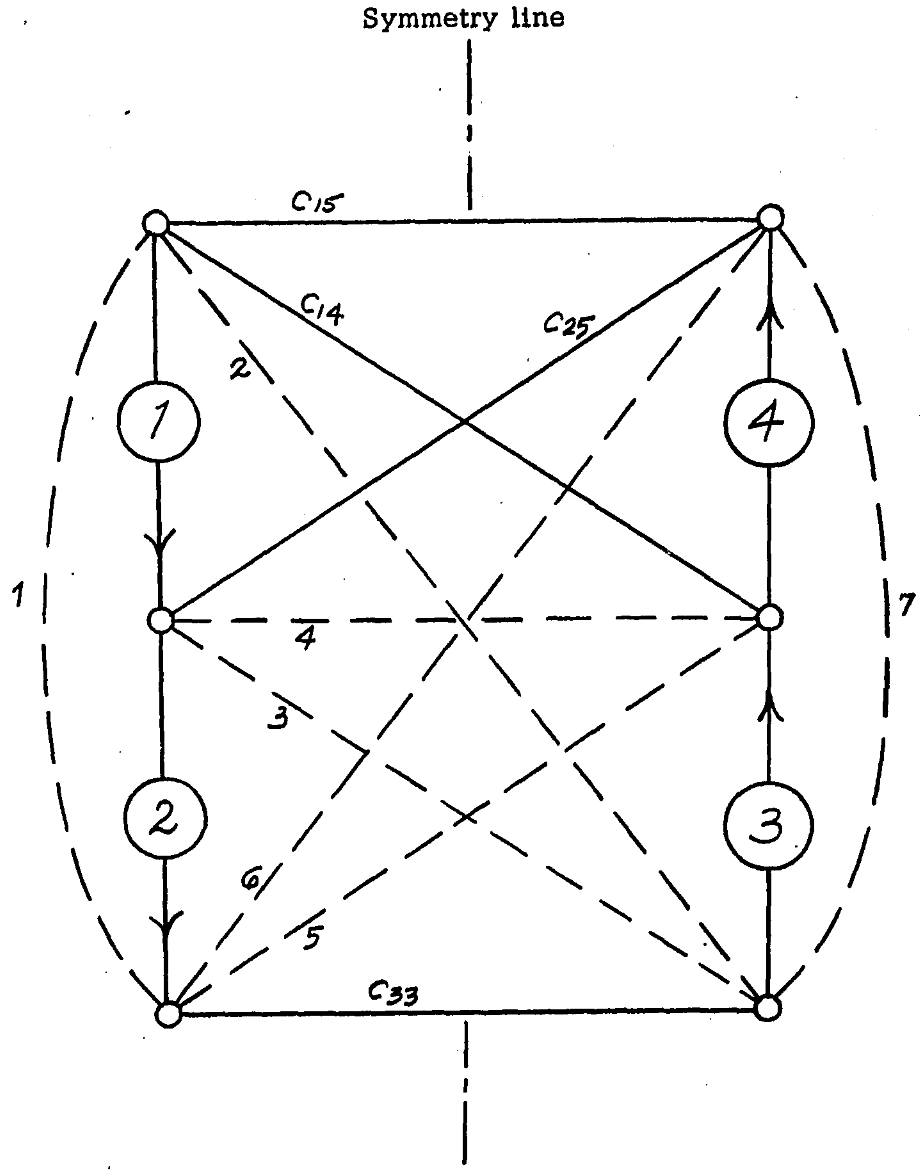



Fig. 6-3. 2-2 tree minimal network (solid-line branches are always present)

line branches to equal zero, we obtain 4 equations in the $h_{ii}$. These can be solved for the $h_{ii}$ as functions of the $h_{ij}$, $i \neq j$, and k. Since the $h_{ij}$ are functions of the off-diagonal elements of G, $g_{ij}$, which are constants in any particular case, the elements of the b matrix and the $b + h = C$ matrix, become a function of only the single parameter k, thus yielding the flexible solution described in Section 5.9. That is, each value of k yields a different network; the G matrices of these networks all have the same off-diagonal elements, but each network has a different 4-tuple of driving-point conductances $g_{ii}$. Thus by varying k, we can generate a continuum of 4-tuples which defines the realizability boundary for the particular set of off-diagonal elements and the particular network configuration chosen. In order to find a minimal realizability boundary for a particular set of off-diagonal elements of G, it is necessary to examine all possible minimal networks. One or possibly a combination of several will yield the boundary.

We can of course realize values of $g_{ii}$ larger than those of a minimal network, by merely shunting port i with the appropriate conductance. If we shunt all four ports of a 7-branch minimal network, the result is an 11-branch network, which would appear to contradict Biorci's conjecture, which predicts a 10-branch network. However, by properly choosing k it is always possible to satisfy one specified driving-point conductance $g_{ii}$, without requiring a shunt, thus yielding a 10-branch network. We discuss this in Section 6.11 and give examples.

We can establish the number of minimal networks MN, as each minimal network has 4 of the 7 dashed-line branches of Figure 6-3, missing. Thus $MN = \binom{7}{4} = 35$ minimal networks.

To identify the minimal networks, we code each network with a 4-digit number. Each digit represents the number of a missing branch, as shown in Figure 6-3. Certain pairs of these minimal networks are mirror images of one another with respect to the symmetry line shown in the figure. For example, networks 1234 and 4567 are such a pair. We need consider only one network of each such pair, since if we use the reverse matrix permutation on one of the networks, that is, reverse the order of the port numbers by exchanging approprite pairs of rows and corresponding columns, the second network is in effect restored. For example, if we use matrix permutation (4,3,2,1) with network 1234, it is equivalent to using matrix permutation (1,2,3,4) with network 4567.

## 6.5 DEGENERACY AMONG THE OFF-DIAGONAL ELEMENTS OF G

The missing branches of network 1267 form an even circuit of 4 branches, and according to the material discussed in Section 5.8, should cause the network to possess a nonlinear degeneracy. The network is shown in Figure 6-4 with solid lines. The subgraph of missing branches is shown by dashed lines, designated with both the digit code and the $c_{ij}$. The missing branches yield the following set of equations:

$$\begin{aligned}
&\text{a)}\quad c_{12} = h_{12}+b_{12} = h_{12} - k\theta h_{11}/h_{22} = 0\\
&\text{b)}\quad c_{13} = h_{13}+b_{13} = h_{13} + kh_{11}/h_{22} = 0\\
&\text{c)}\quad c_{35} = h_{35}+b_{35} = h_{35} + kh_{55}/h_{44} = 0\\
&\text{d)}\quad c_{45} = h_{45}+b_{45} = h_{45} - kh_{55}/\theta h_{44} = 0
\end{aligned}\tag{6-13}$$

Solve equations (6-13) for the $h_{ij}$. Then dividing (6-13a) by (6-13b) yields:

$$\theta = -h_{12}/h_{13} \tag{6-14}$$

Similarly, dividing (6-13c) by (6-13d) yields:

$$\theta = -h_{35}/h_{45} \tag{6-15}$$

Equating (6-14) and (6-15), we obtain:

$$h_{12}h_{45} = h_{13}h_{35} \tag{6-16}$$

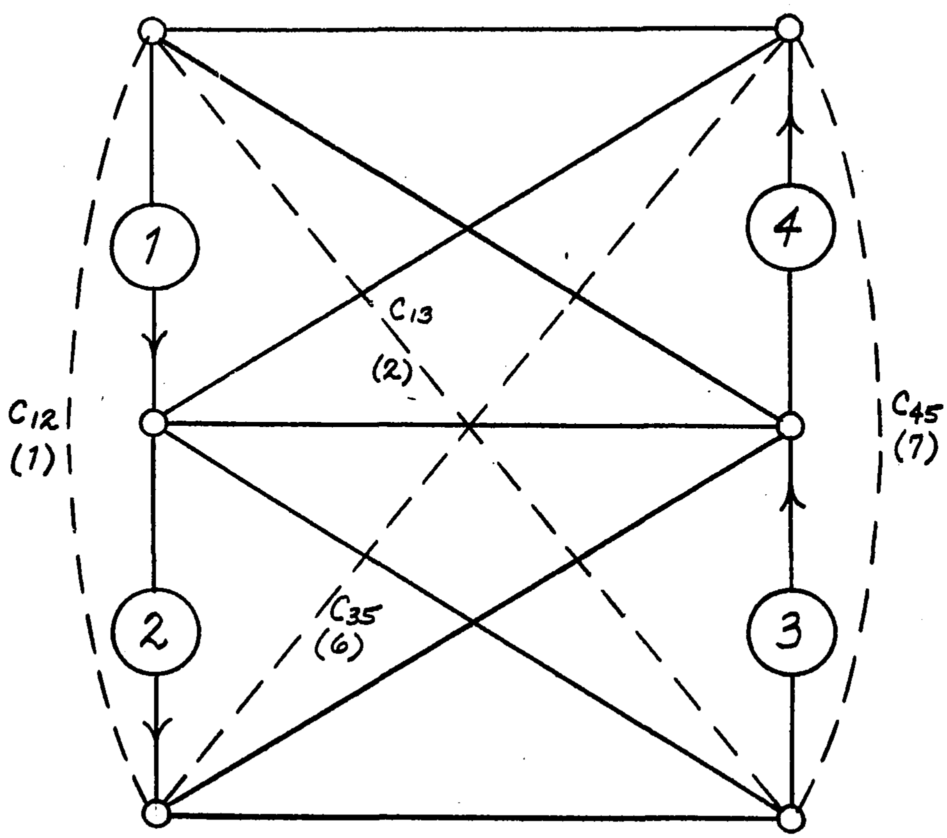



Fig. 6-4. Degenerate network 1267

Substituting for the $h_{ij}$ in (6-16) from (6-3) yields:

$$g_{12}g_{34} = g_{13}g_{24} \tag{6-17}$$

Equation (6-17) is the equation of a nonlinear degeneracy. Thus the network can realize a G matrix, only if G satisfies (6-17). The reader can verify this by assigning arbitrary values to the branch conductances of network 1267, computing the appropriate $g_{ij}$, and substituting in (6-17).

## 6.6 A NECESSARY CONDITION ON THE CONDUCTANCE MATRIX G

If we replace the equality signs in equations (6-13) with $\geq$ signs, representing the general case, then manipulation of the inequalities results in the following inequality:

$$g_{12}g_{34} \geq g_{13}g_{24} \tag{6-18}$$

This is then a necessary condition on G for it to be realizable by a network with a 2-2 linear tree-port structure. Swaminathan and Frisch (SW 1) obtained this condition in an example in their paper.

There are 4! = 24 possible permutations (ways of numbering the ports) of a fourth-order matrix. The fact that we make the smallest off-diagonal element of G equal to $g_{14}$, combined with the constraint imposed by (6-18), in general, yields only two useable permutations. They are the reverse of one another, that is, if (a,b,c,d) is a valid permutation, then (d,c,b,a) will be also. There may be several other useable permutations in special cases, for example, the case where the two smallest off-diagonal elements of G are equal.

## 6.7 CATALOG OF THE 2-2 LINEAR TREE MINIMAL NETWORKS

Of the 35 possible minimal networks, previously mentioned in Section 6.4, 16 of them possess the mirror-image symmetry referred to in that section. The only degenerate minimal network is 1267, discussed in Section 6.5. Eliminating these from consideration leaves a total of 35-16-1 = 18 minimal networks. The networks are divided into three classes, ND, D1, D2, which are defined and discussed in Section 6.10. Figure 6-5 shows the networks. In addition, Figure 6-6 shows all of the complementary graphs of the networks. As will be discussed later, the topology of the complementary graphs reveals characteristics of the networks without requiring any calculations. Before proceeding further, we will give an example to illustrate some of the previous points.

## 6.8 EXAMPLE 6-1

Let us take the matrix of equation (5-1) and realize the off-diagonal elements on a minimal network. Arranging the matrix by suitable permutations, as described above, we obtain:

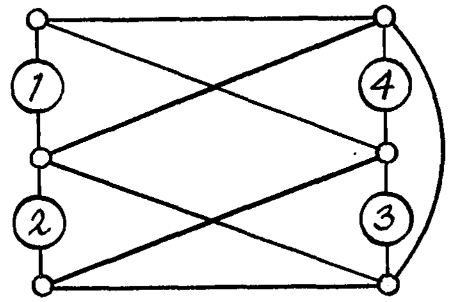
$$G = \begin{bmatrix} g_{11} & 4 & 5 & 1 \\ 4 & g_{22} & 6 & 2 \\ 5 & 6 & g_{33} & 3 \\ 1 & 2 & 3 & g_{44} \end{bmatrix} \tag{6-19}$$

From equation (6-3) we compute the h matrix.

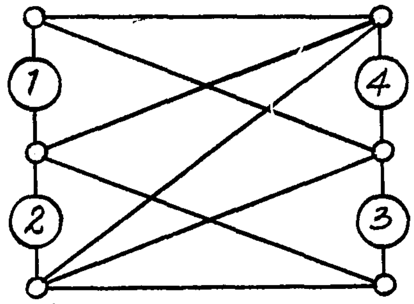
$$h = \begin{bmatrix} h_{11} & 4 & -5 & 4 & 1 \\ & h_{22} & -1 & 0 & 1 \\ & & 6 & -4 & -2 \\ & & & h_{44} & 3 \\ & & & & h_{55} \end{bmatrix} \tag{6-20}$$

In order to compute the b matrix, we must select a particular minimal network and solve for the $h_{ii}$. As an example, we choose network 1256, the missing branches of which yield the following equations:

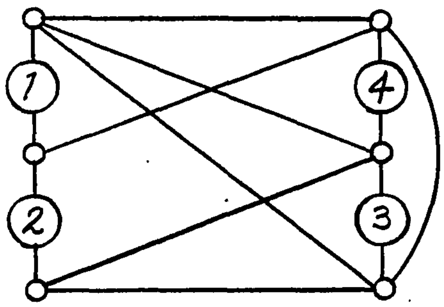
$$\begin{array}{lll} \text{a)} & c_{12} = 0 & -\theta k h_{11}/h_{22} + h_{12} = 0 \\ \text{b)} & c_{13} = 0 & k h_{11}/h_{22} + h_{13} = 0 \\ \text{c)} & c_{34} = 0 & \theta h_{44} + h_{34} = 0 \\ \text{d)} & c_{35} = 0 & k h_{55}/h_{44} + h_{35} = 0 \end{array} \tag{6-21}$$



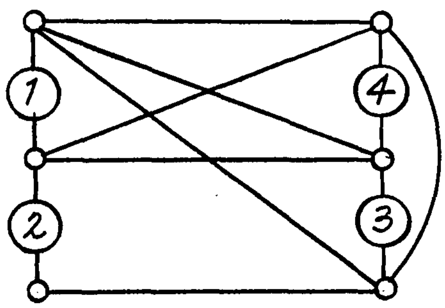

a) 1246



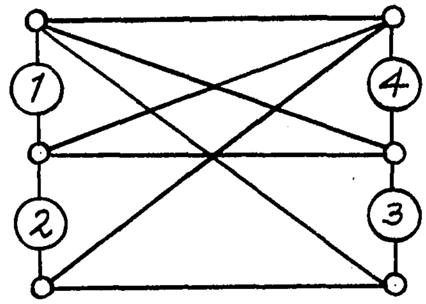

b) 1247



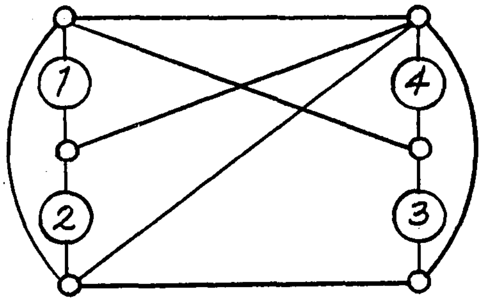

c) 1346 (non-planar)



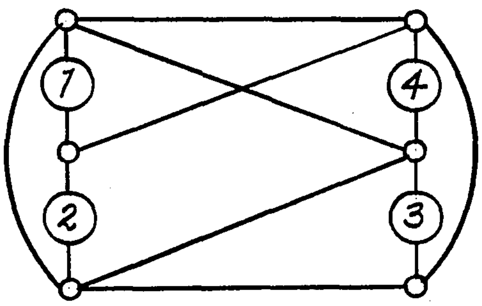

d) 1356 (non-planar)



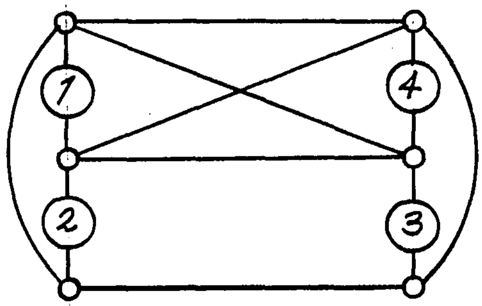

e) 1357 (non-planar)




f) 2345



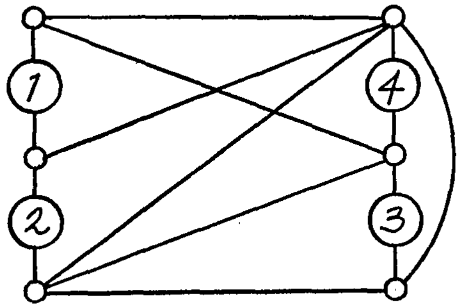

g) 2346



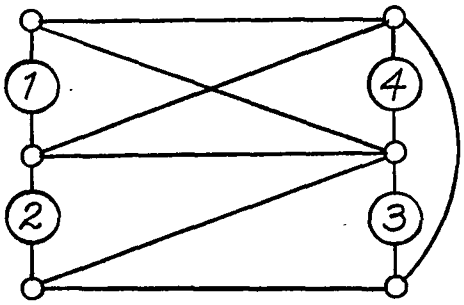

h) 2356 (non-planar)

ND Networks

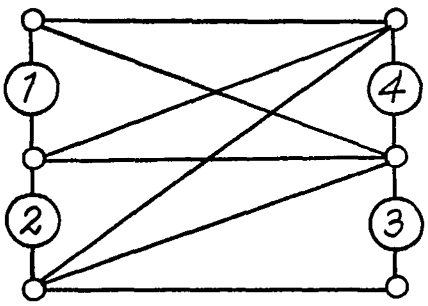
Fig. 6-5



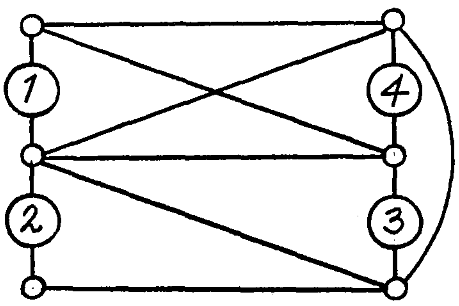

i) 1234



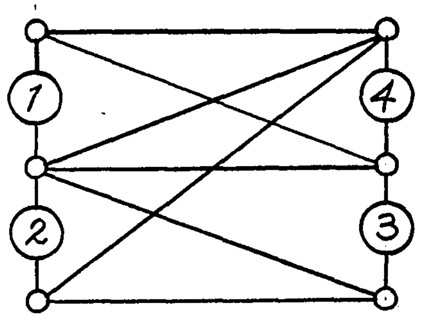

j) 1236



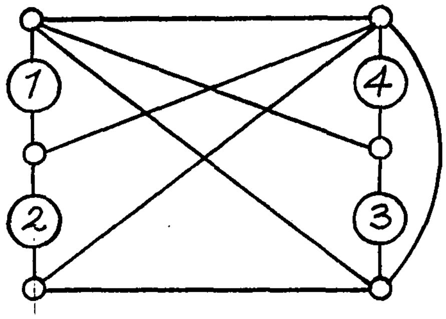

k) 1237



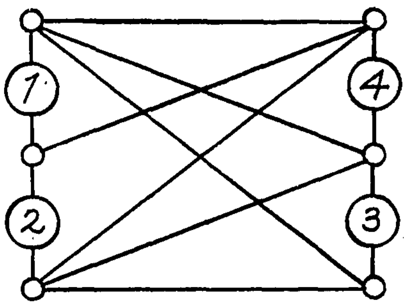

l) 1256



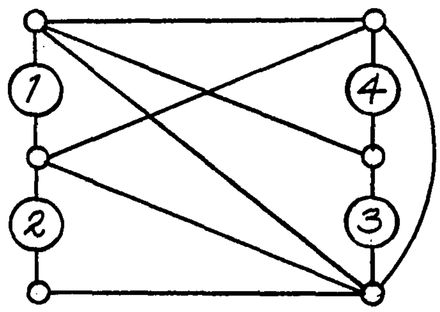

m) 1257




n) 1345



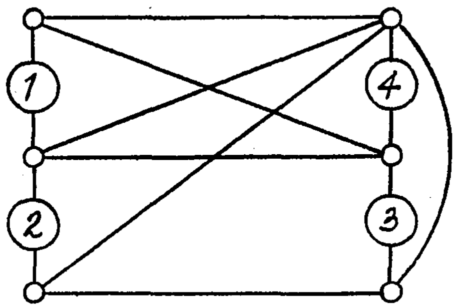

o) 1347



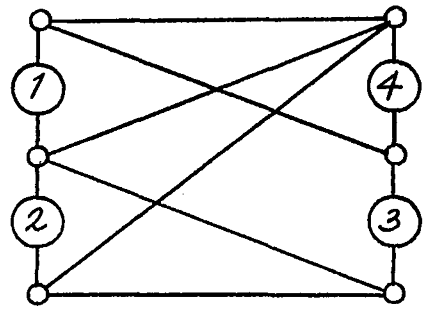

p) 1456

D1 Networks

Fig. 6-5




q) 1235




r) 1245

D2 Networks

Fig. 6-5

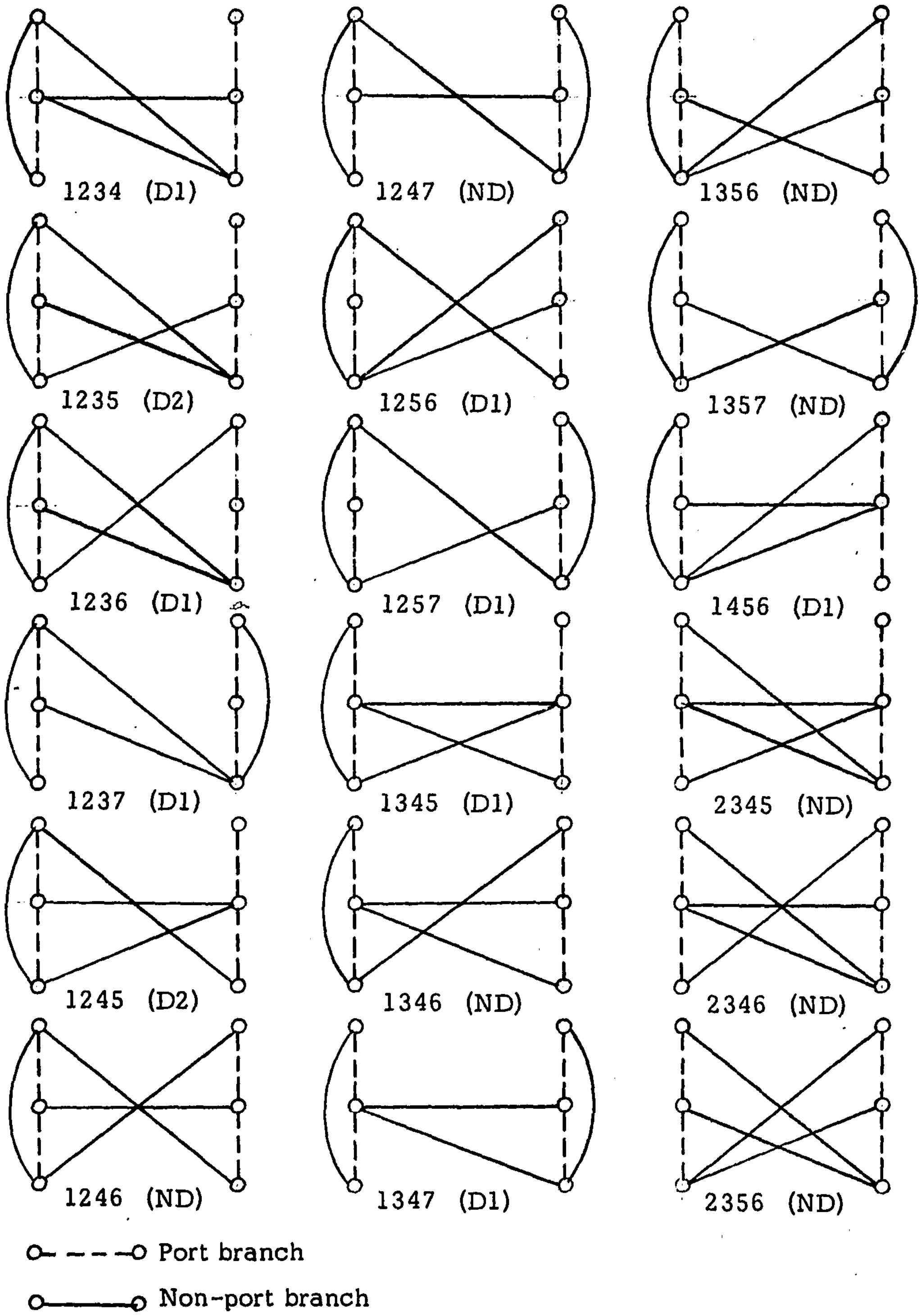



Fig. 6-6. Complementary graphs

Solving (6-21) for the $h_{ii}$ and using (6-8g) gives:

$$\begin{aligned}
&\text{a)}\quad h_{11} = h_{13}(h_{12}+h_{34}+h_{35})/k - (h_{12}+h_{34}) \\
&\text{b)}\quad h_{22} = k(1+h_{12}/h_{13}) - (h_{12}+h_{34}+h_{35}) \\
&\text{c)}\quad h_{44} = h_{13}h_{34}/h_{12} \\
&\text{d)}\quad h_{55} = -h_{13}h_{34}h_{35}/kh_{12} \\
&\text{e)}\quad \theta = -h_{12}/h_{13}
\end{aligned} \tag{6-22}$$

In the minimal network we call the branches which are always present, the <u>base</u> branches, and the others, the <u>nonbase</u> branches. We must ensure that the nonbase branches are non-negative. Since there are three of them, we obtain three necessary and sufficient conditions for the realizability of the off-diagonal elements of G by the particular network. The conditions arise from the three nonbase branch inequalities. In our example these yield:

$$\begin{aligned}
&\text{a)}\quad c_{23} \geq 0 \qquad && h_{22}/\theta \geq -h_{23} \\
&\text{b)}\quad c_{24} \geq 0 \qquad && h_{22}h_{44}/k \geq -h_{24} \\
&\text{c)}\quad c_{45} \geq 0 \qquad && kh_{55}/\theta h_{44} \leq h_{45}
\end{aligned} \tag{6-23}$$

We now substitute the numerical values for $h_{ij}$, $i \neq j$, from (6-20) into (6-22) and obtain:

$$\begin{aligned}
&\text{a)}\quad h_{11} = 10/k + 1 \\
&\text{b)}\quad h_{22} = k/5 + 2 \\
&\text{c)}\quad h_{44} = 5 \\
&\text{d)}\quad h_{55} = 10/k
\end{aligned} \tag{6-24}$$

Similarly substituting into (6-23) yields:

$$\begin{aligned}
&\text{a)}\quad k/4 + 5/2 \geq 1 \\
&\text{b)}\quad 1 + 10/k \geq 0 \\
&\text{c)}\quad 5/2 \leq 3
\end{aligned} \tag{6-25}$$

Examination of (6-25) reveals that the conditions are satisfied when the range of k is: $0 < k < \infty$. And using the expressions for the $h_{ii}$

in (6-3) we solve for the driving point conductances $g_{ii}$.

$$
\begin{aligned}
&\text{a)} \quad g_{11} = 5 + 10/k \\
&\text{b)} \quad g_{22} = 6 + k/5 \\
&\text{c)} \quad g_{33} = 8 \\
&\text{d)} \quad g_{44} = 3 + 10/k
\end{aligned}
\qquad (6\text{-}26)
$$

We can then realize the following G matrix by network 1256.

$$
G = \begin{bmatrix} (5+10/k) & 4 & 5 & 1 \\ 4 & (6+k/5) & 6 & 2 \\ 5 & 6 & 8 & 3 \\ 1 & 2 & 3 & (3+10/k) \end{bmatrix} \qquad (6\text{-}27) \quad (0<k<\infty)
$$

Figure 6-7 gives plots of the $g_{ii}$ vs. k, for network 1256 (shown by solid lines). As k is varied over its range, the 4-tuple ( 5+10/k, 6+k/5, 8, 3+10/k), defines the realizability boundary for network 1256 and the off-diagonal elements of G. If we test the entire set of 18 minimal networks, we find that networks 1236, 1237, and 1257 also realize the off-diagonal elements of (6-27). However, in this case, the 4-tuples representing the driving-point conductances of those three networks, are all greater than that of network 1256. Therefore, network 1256 provides the smallest driving-point conductances possible, on a 2-2 linear tree port structure network, for the set of off-diagonal elements in (6-27). That is, network 1256 defines the realizability boundary.

If we consider the 4-tuples to define 4-dimensional surfaces, then the lowest surface of all of the networks represents the realizability boundary. It is possible that the surfaces corresponding to several networks may intersect, in which case the realizability boundary would be defined by more than one minimal network. Such is not the case in our example, in which the surfaces do not intersect, but are contained in one another. To illustrate this, the $g_{ii}$ vs. k of network 1237 are plotted on Figure 6-7 (with dashed lines).

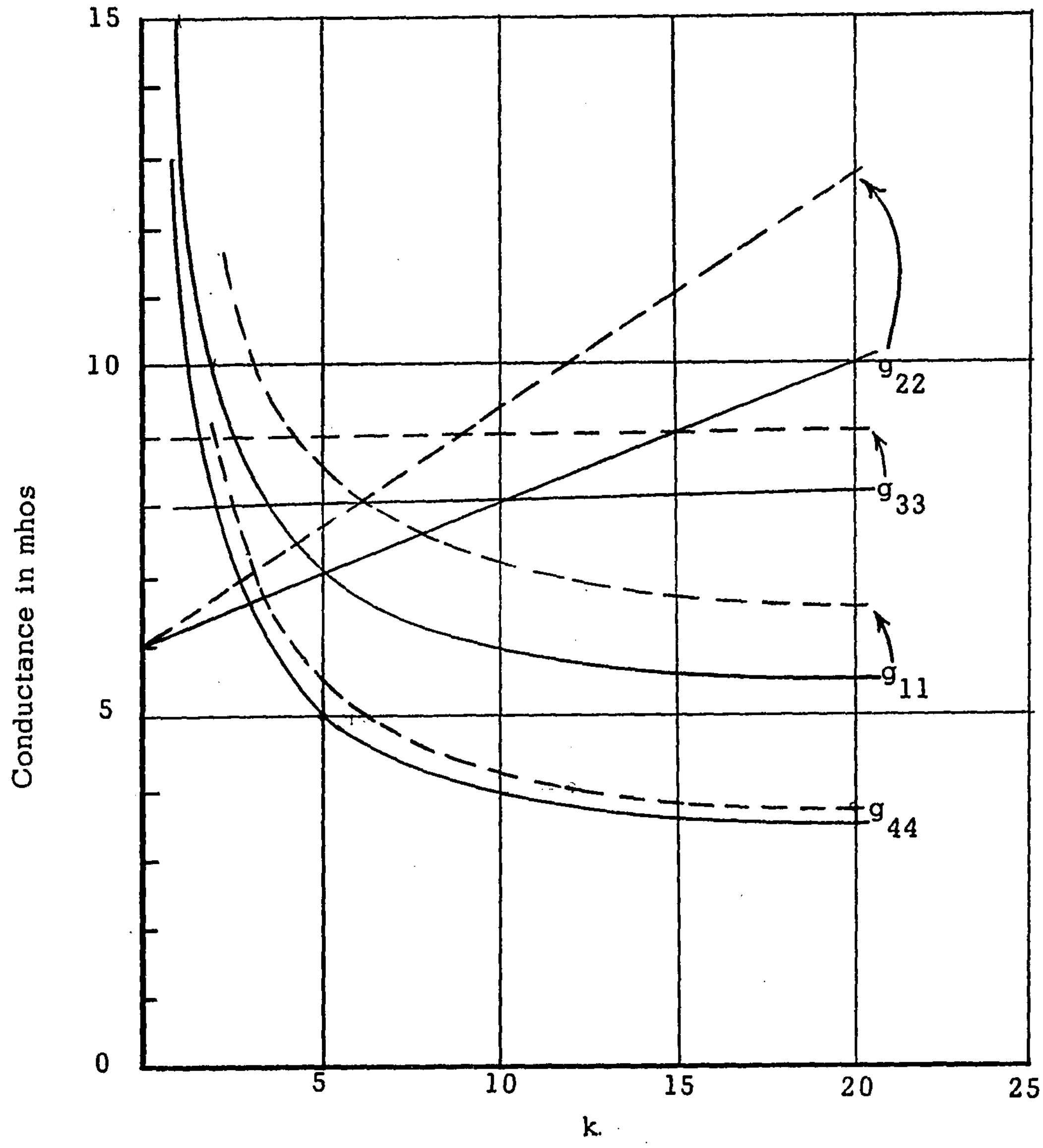



Fig. 6-7. Plot of driving-point conductances vs. k for network 1256 (solid lines) and network 1237 (dashed lines) and the off-diagonal elements of matrix (6-27)

Minimal networks which do not realize a particular set of off-diagonal elements of a G matrix, are evidenced by some of the nonbase branches becoming negative; that is, some of the necessary and sufficient conditions are not satisfied.

We can present a geometric interpretation of the realizability boundary by solving (6-26a) for k and substituting in (6-26b,d) to yield:

$$\begin{aligned} &a)\quad g_{22} = 6 + 2/(g_{11}-5) \\ &b)\quad g_{33} = 8 \\ &c)\quad g_{44} = g_{11} - 2 \end{aligned} \tag{6-28}$$

Equations (6-28a,c) are plotted in Figure 6-8. The intersection of the inclined plane and the curved surface (with the constraint that $g_{33}=8$) defines the realizability boundary. The volume above the inclined plane and to the right of the curved surface (with $g_{33} = 8$) contains the points which represent all of the 4-tuples realizable with the off-diagonal elements of G on a 2-2 linear tree port structure.

The formulas for the 7 branch conductances are obtained from C = b + h., and are given below for the example.

$$\begin{aligned} &a)\quad c_{14} = 4 + 20/k \\ &b)\quad c_{15} = 1 + 10/k \\ &c)\quad c_{23} = 3/2 + k/4 \\ &d)\quad c_{24} = 1 + 10/k \\ &e)\quad c_{25} = 1 + 2/5k + 4/k^2 \\ &f)\quad c_{33} = k + 6 \\ &g)\quad c_{45} = \tfrac{1}{2} \qquad\qquad (0 < k < \infty) \end{aligned} \tag{6-29}$$

## 6.9 FIGURE OF MERIT

In order to make comparisons between networks, it is convenient to establish a figure of merit for a network which realizes a set of off-diagonal elements of a G matrix. As we are primarily interested in the minimum values of $g_{ii}$ attainable (i.e., the lowest surface), an

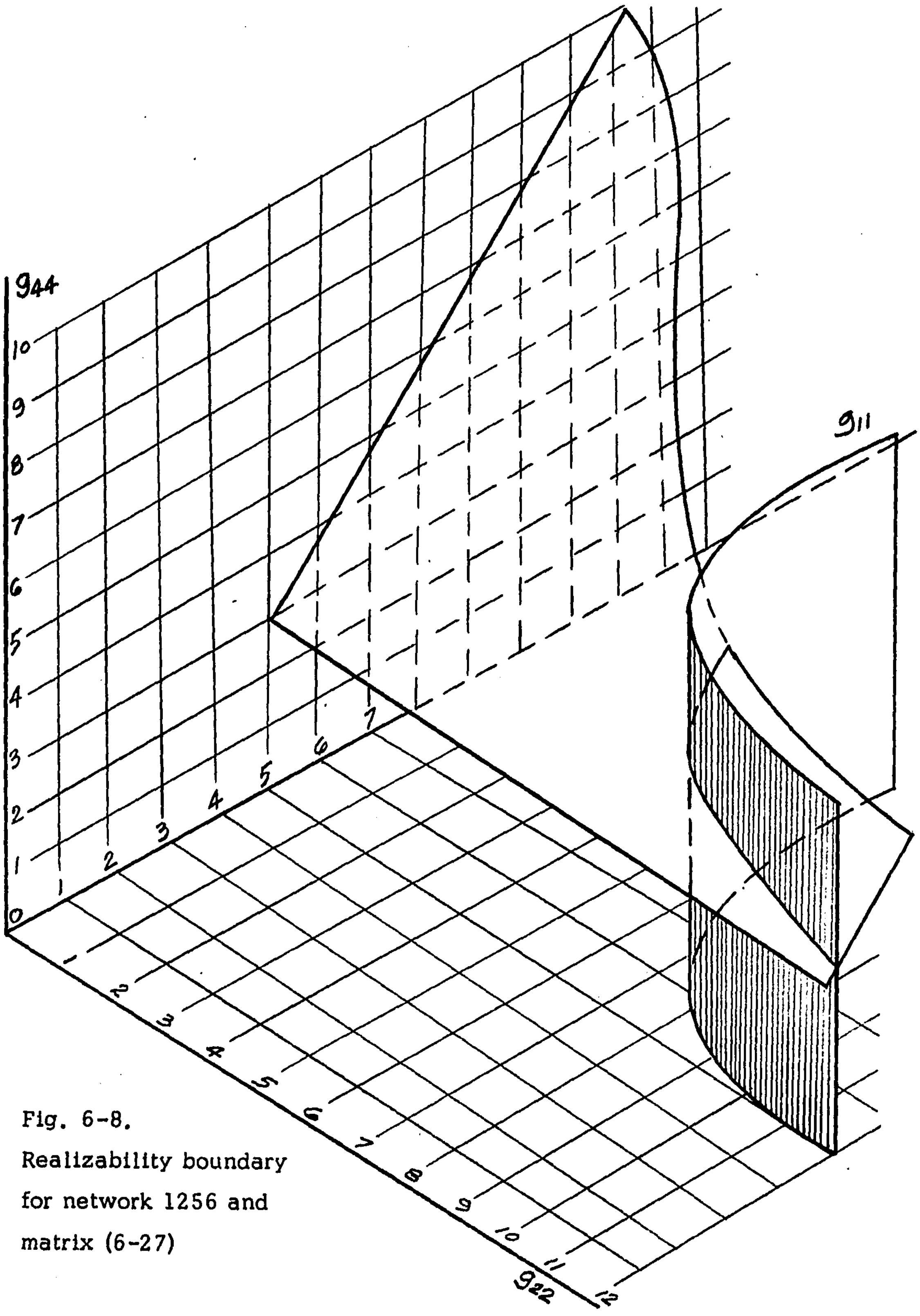



Fig. 6-8.
Realizability boundary
for network 1256 and
matrix (6-27)

appropriate figure of merit is the minimum value of the trace of G.

$$\text{Figure of merit} = M = \left[\sum_{i=1}^{n} g_{ii}(k)\right]_{\min.} \qquad (6\text{-}30)$$

In the previous example 6-1, $M = (22 + k/5 + 20/k)_{\min.}$. The minimum occurs at k=10 and gives M=26.0. A lower bound on M is the trace of equation (5-1) which equals 21.6. Table 6-1 gives M for the four possible realizable networks.

| Network | M |
|---|---|
| 1236 | 37.7 |
| 1237 | 30.5 |
| 1256 | 26.0 |
| 1257 | 26.8 |

Table 6-1

## 6.10 CHARACTERISTICS OF MINIMAL NETWORKS: ND, D1, AND D2 NETWORKS

Examination of equations (6-26) reveals the somewhat surprising result that $g_{33}$ is equal to a constant. Is this a chance occurence? The answer is no, and is explained by the following. From the material in Section 5.8, we recall that Biorci and Civalleri showed that if the missing branches in a network form a circuit of even length, a nonlinear degeneracy exists. Up to this point, we have only considered even circuits comprising nonport branches. However, in a minimal network, by definition, the branches shunting the ports are all missing. Therefore, it is possible that an even circuit of missing branches, consisting of say 3 nonport branches and one port branch, exists. Thus we have a new type of degeneracy, a degeneracy with respect to a port, which we define as a port degeneracy. Biorci and Civalleri do not make this distinction. Inspection of the graph of missing branches (called the complementary graph) of network 1256, shown in Figure 6-9, makes clear that the missing branches $c_{12}$, $c_{13}$, $c_{34}$, $c_{44}$, form an even circuit. Thus as predicted, a port degeneracy exists with respect to port 3. The equation of degeneracy is (6-22c). However, this type of degeneracy is not restrictive as we can always shunt the port to obtain a different value of the driving-point conductance. In Figure 6-5, networks which exhibit this characteristic with respect to a single port are designated D1. In these 4-port networks, it is possible for two port degeneracies to occur in one network; these are designated D2. Networks without port degeneracies are designated ND. All of the networks realizing the matrix of example 6-1 are in class D1. All of the nonplanar minimal networks are in class ND.

Minimal networks with port degeneracies may have practical applications, for example, as a variable matching network in a balanced mixer circuit, as shown in Figure 6-10. Here k is varied to obtain the best conductance match to the crystals, without affecting the local oscillator or r.f. input ports.

## 6.11 REALIZATIONS BY NONMINIMAL NETWORKS

Suppose that we wish to realize a G matrix whose main-diagonal elements are larger than required for realization by a minimal network. We could shunt each port of a minimal network with the appropriate conductance so as to match the specified main diagonal of the G matrix. However, in general this would yield a network with 11 branches. According to Biorci's conjecture, discussed in Section 5.10, we should be able to realize the matrix by a network with 10 branches. We can always use only 10 branches, by adjusting the value of k until one specified driving-point conductance is satisfied without benefit of a port shunt; then we need add shunts to only the other three ports, thus yielding a 10-branch network. For example, take the following matrix:

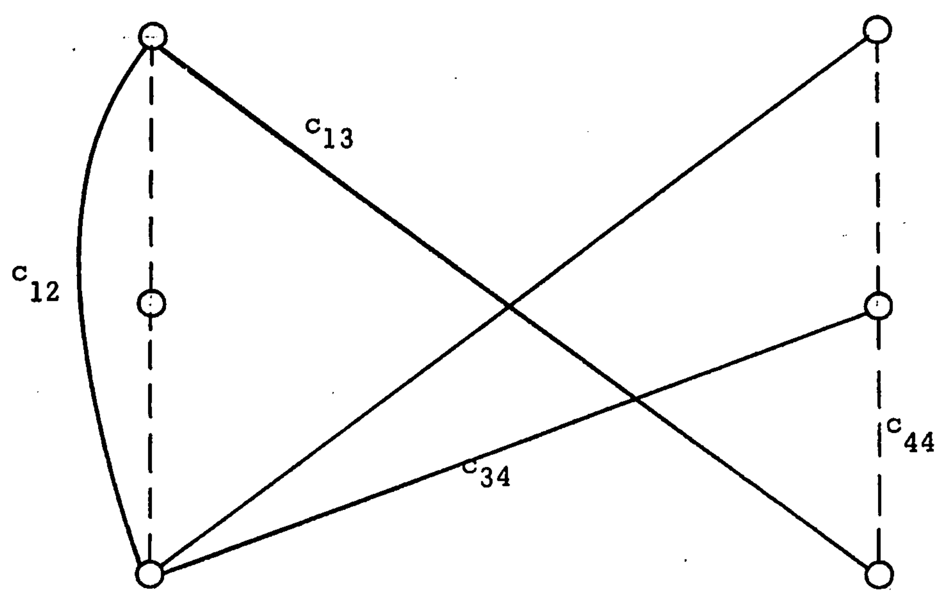



Fig. 6-9. Complementary graph of network 1256

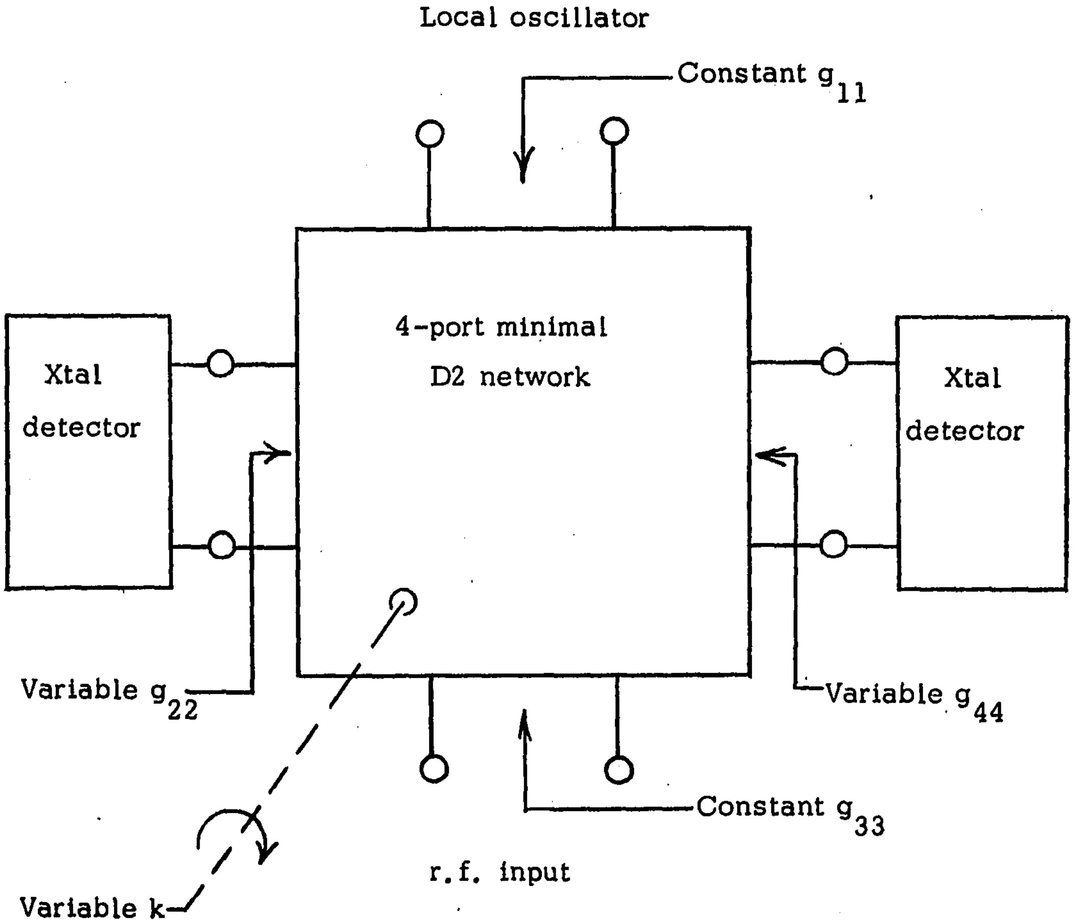



Fig. 6-10. Balanced mixer employing minimal network

$$G = \begin{bmatrix} 12 & 4 & 5 & 1 \\ 4 & 15 & 6 & 2 \\ 5 & 6 & 18 & 3 \\ 1 & 2 & 3 & 7 \end{bmatrix} \qquad (6\text{-}31)$$

Equation (6-31) is our previous example with larger than minimal values specified for the main-diagonal elements. In fact, this is a permuted version of the well known Slepian-Weinberg matrix (WE 2). The off-diagonal elements of G correspond to those in example 6-1; therefore, the solid line curves of Figure 6-7 are applicable to this example. On the curves, choose a value of k so that one of the $g_{ii}$ of (6-31) is satisfied by equality, and the other $g_{ii}$ are less than or equal to the values specified in the matrix. In general, there may be several solutions. One exists when k=2.5; then $g_{44}=7$, which matches the specified matrix value. Correspondingly, $g_{11}=9$, $g_{22}=6.5$, and $g_{33}=8$, necessitating a shunt on each of these ports, in order to satisfy the requirements of (6-31). The complete 10-branch realization is shown in Figure 6-11.

Another realization exists when k=45; here $g_{11}=5.22$, $g_{22}=15$, $g_{33}=8$, and $g_{44}=3.22$. In this case, port 2 matches the matrix, and ports 1,3, and 4 require shunts.

The network of Figure 6-11 is planar (i.e., can be drawn without crossed branches), so that its dual will realize G as a resistance matrix.

## 6.12 SCREENING THE NETWORKS

Returning to equation (6-12), we note that the signs of only three elements, $h_{23}$, $h_{24}$, and $h_{34}$, are not definite. The signs are a function of the relative magnitudes of the various $g_{ij}$ as shown in equation (6-3). There are five possible sign patterns for these three elements, which are given below:

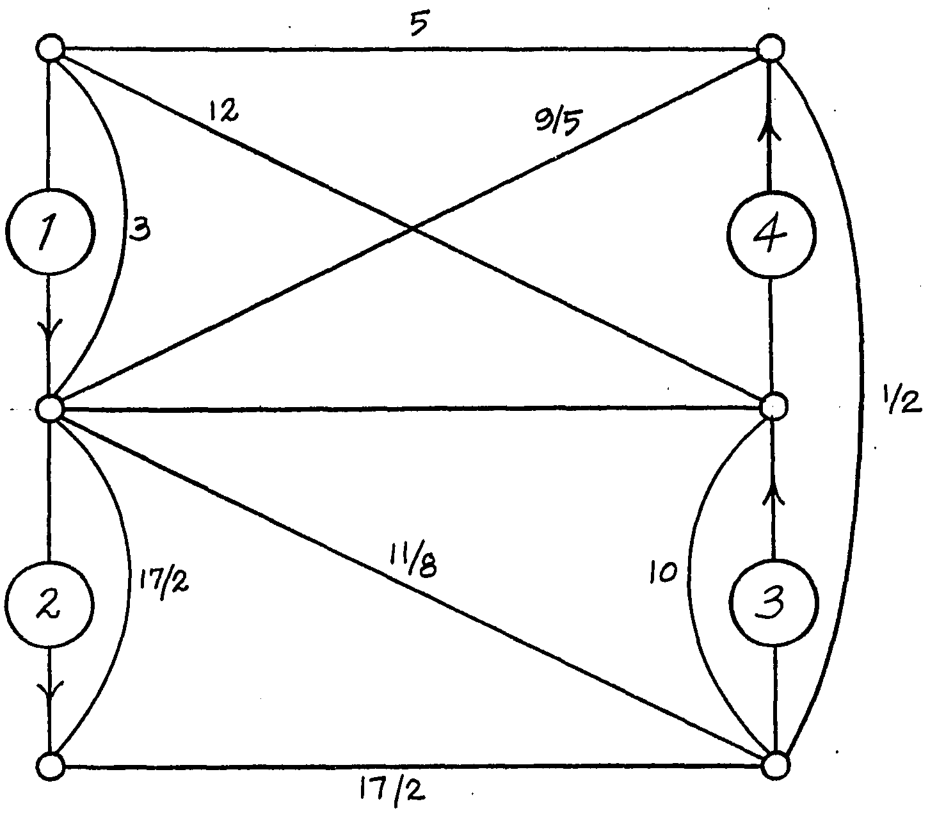



Fig. 6-11. Example of non-minimal network

a) + − +

b) + − −

c) − − +

d) − + −

e) − − −

In these sign patterns the + sign is understood to mean non-negative. The corresponding elements of the b matrix, $b_{23}$, $b_{24}$, and $b_{34}$, are all positive. Since the branch conductance $c_{ij} = b_{ij} + h_{ij}$, then if any of the three $h_{ij}$ is non-negative, the corresponding branch definitely appears in the network. Thus each of the above sign patterns can only be realized by a subset of the 18 minimal networks. Table 6-2 lists the sign patterns and the corresponding network subsets.

| + − + | + − − | − − + | − + − | − − − |
|---|---|---|---|---|
| 1246 | 1246 | 1234 | 1235 | All |
| 1247 | 1247 | 1236 | 1236 | |
| | 1245 | 1237 | 1237 | |
| | 1256 | 1247 | 1256 | |
| | 1257 | 1246 | 1257 | |
| | 1456 | 1346 | 1356 | |
| | | 1347 | 1357 | |
| | | 2346 | 2356 | |

Table 6-2

Thus except for the last sign pattern, eight is the largest number of networks required to be searched.

## 6.13 NECESSARY AND SUFFICIENT CONDITIONS FOR REALIZABILITY

As the minimal networks contain seven branches, and the four base branches are always present, the non-negativity of the remaining three branch conductances, are the necessary and sufficient conditions for realizing the off-diagonal elements of G by the network. We state this as a theorem.

<u>Theorem 6-1</u>. Given a fourth-order conductance matrix G and and a 4-port minimal network N; the off-diagonal elements of G are realizable by N if and only if $c_{ij} = b_{ij} + h_{ij} \geq 0$, for ij = ab, cd, ef, where $c_{ab}$, $c_{cd}$, and $c_{ef}$ are the conductances of the nonbase branches.

If Theorem 6-1 is satisfied, then additionally we want to be able to realize a particular 4-tuple of input conductances $g_{ii}$. This is predicated on being able to select a value of k such that all of the $g_{ii}$ of the matrix are greater than or equal to the $g_{ii}(k)$ of the minimal network. Thus we present the following theorem.

<u>Theorem 6-2</u>. Given a 4-port minimal network N and a conductance matrix G which satisfy Theorem 6-1, G is realizable by N (with appropriate port shunts) if and only if there exists a positive number k such that $(g_{ii})_G \geq (g_{ii}(k))_N$, for i = 1,2,3,4.

## 6.14 OTHER MATRIX SIGN PATTERNS

As mentioned in Chapter 5, in addition to considering the sign pattern of G with all off-diagonal elements positive, we also have to deal with sign pattern 3), which has 5 positive and one negative off-diagonal elements. We can realize a negative off-diagonal element $g_{ij}$, as long as port i is in one subtree of ports and port j is in the other, as illustrated in Figure 5-3 and the accompanying text. If we make $g_{14}$ the negative element, which satisfies that requirement, then our synthesis procedures are the same as the preceding, with the following addition: the additional condition $c_{14} = b_{14} + h_{14} \geq 0$, must be satisfied. This condition combined with the other inequalities, yields an upper bound on k.

$$k \leq -h_{12}h_{45}/h_{15} = -g_{12}g_{34}/g_{14} \tag{6-32}$$

## 6.15 THE 1-3 LINEAR TREE PORT STRUCTURE

As previously remarked, we eliminate repetitive details and present the results.

$$G_{exp} = \begin{bmatrix} g_{11} & 0 & g_{12} & g_{13} & g_{14} \\ 0 & 0 & 0 & 0 & 0 \\ g_{12} & 0 & g_{22} & g_{23} & g_{24} \\ g_{13} & 0 & g_{23} & g_{33} & g_{34} \\ g_{14} & 0 & g_{24} & g_{34} & g_{44} \end{bmatrix} \tag{6-33}$$

$$h = \left[\begin{array}{c|cccc} g_{11} & -g_{12} & (g_{12}-g_{13}) & (g_{13}-g_{14}) & g_{14} \\ & g_{12} & -(g_{12}-g_{13}) & -(g_{13}-g_{14}) & -g_{14} \\ \hline & & (g_{22}-g_{23}) & (g_{23}-g_{24}) & g_{24} \\ & & & \begin{array}{c}(g_{33}+g_{24}+ \\ -g_{23}-g_{34})\end{array} & (g_{34}-g_{24}) \\ & & & & (g_{44}-g_{34}) \end{array}\right] \tag{6-34}$$

$$b = \left[\begin{array}{c|cccc} -h_{11} & h_{11}\theta & h_{11}h_{33}/k & \dfrac{h_{11}h_{44}\theta}{h_{33}} & \dfrac{h_{11}h_{33}h_{55}}{kh_{44}} \\ & k & h_{33}/\theta & kh_{44}/h_{33} & \dfrac{h_{33}h_{55}}{\theta h_{44}} \\ \hline & & -h_{33} & -k\theta h_{44}/h_{33} & \dfrac{-h_{33}h_{55}}{h_{44}} \\ & & & -h_{44} & \dfrac{-h_{33}^2 h_{55}}{k\theta h_{44}} \\ & & & & -h_{55} \end{array}\right] \tag{6-35}$$

$$\theta = \frac{k - h_{33}(1 + h_{55}/h_{44})}{k(1 + h_{44}/h_{33}) - h_{11}} \tag{6-36}$$

Since the six elements of the b matrix (6-35) below the dashed line are all negative (as k, $\theta$, and $h_{ii}$ are all positive), the corresponding elements of the h matrix must be positive. That is, $h_{ij} > 0$, for ij = 33, 34, 35, 44, 45, and 55. Examination of (6-34) reveals that these expressions for the $h_{ij}$ are the uniformly tapered conditions for a third-order matrix, with the $\geqq$ signs replaced with $>$ signs. Swaminathan and Frisch (SW 1) give these conditions with the $\geqq$ signs; thus their conditions are not as strong as possible. Of the 24 possible permutations of the G matrix, in general only 8 of them will satisfy these conditions.

In the 2x4 submatrix of h (indicated by the dashed lines) the second row is the negative of the first row. Depending upon the relative magnitudes of $g_{12}$, $g_{13}$, and $g_{14}$, the 2x2 submatrix:

$$\begin{bmatrix} h_{13} & h_{14} \\ h_{23} & h_{24} \end{bmatrix}$$

can have the following sign patterns:

a) $\begin{matrix} + & + \\ - & - \end{matrix}$ b) $\begin{matrix} - & - \\ + & + \end{matrix}$ c) $\begin{matrix} + & - \\ - & + \end{matrix}$ d) $\begin{matrix} - & + \\ + & - \end{matrix}$

The corresponding elements of the b matrix, $b_{13}$, $b_{14}$, $b_{23}$, and $b_{24}$, are all positive. Thus depending upon which of the four sign patterns is present, either branch $c_{13}$ or $c_{23}$ must be present in the network, and either branch $c_{14}$ or $c_{24}$ must be present in the network. From this situation we derive four base networks, each corresponding to one of the sign patterns. The networks are shown in Figure 6-12.

Examination of (6-34) and (6-35) shows that branches $c_{22}$ and $c_{15}$ are always present in the minimal networks. Figure 6-13 shows these branches as solid lines and the remaining 9 branches, numbered from 1 to 9, as dashed lines. As the minimal network contains 7

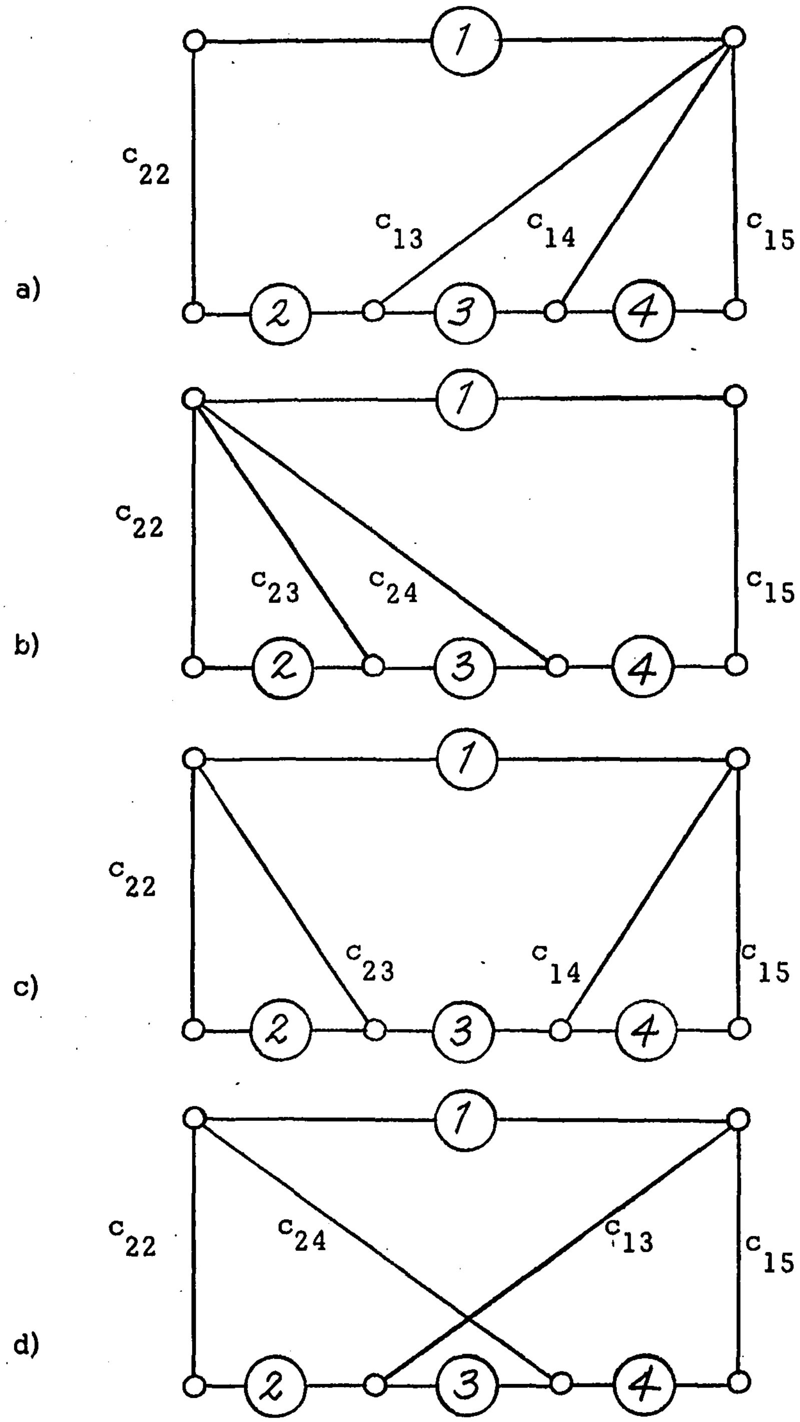



Fig. 6-12. The four base networks

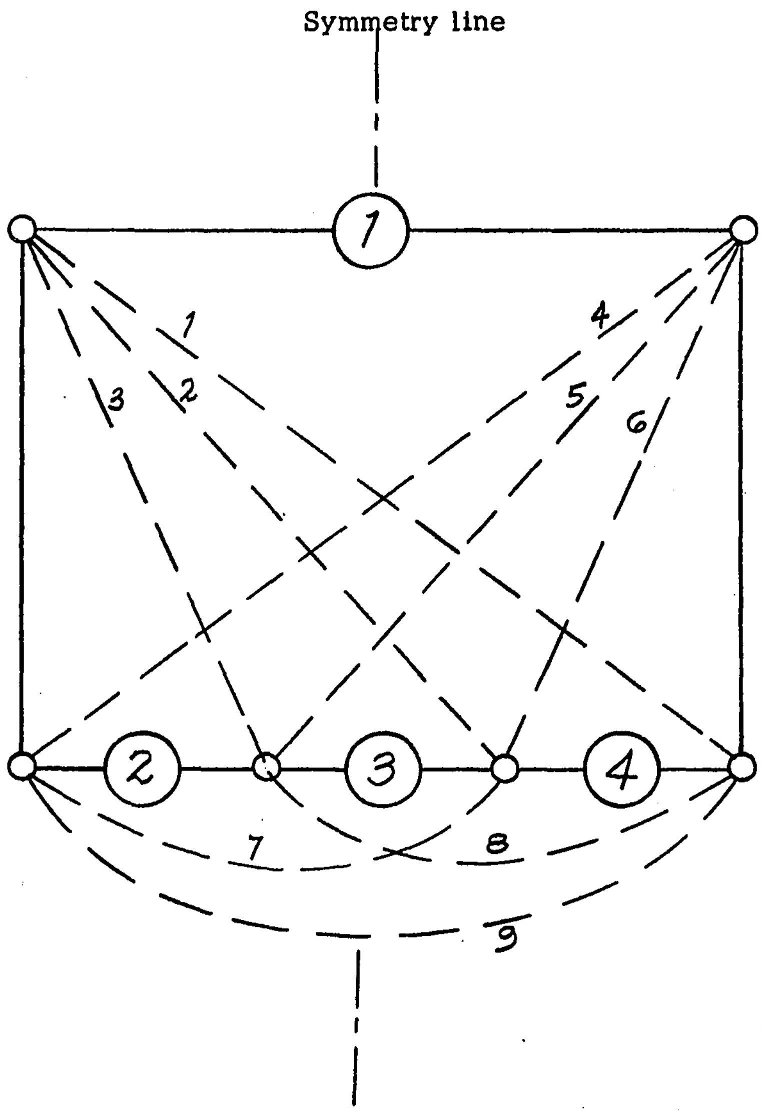



Fig. 6-13. 1-3 linear tree minimal network

branches, then 4 of the 9 numbered branches in Figure 6-13 are missing from any minimal network. Thus there are $\binom{9}{4} = 126$ possible networks. As before, we code them by listing the numbers of the missing branches in increasing order. Therefore, for example, the network of Figure 6-14 has the code 1245.

We can eliminate networks to be considered in the following ways:

1) The complementary graphs of networks 4589, 1279, and 2356 contain even circuits of 4 branches, and thus the networks are degenerate.
2) A network containing 267 in its code has ports 3 and 4 in series; a network containing 358 in its code has ports 2 and 3 in series. These networks possess a linear degeneracy; for, if ports i and j are in series, then $g_{ik} = g_{jk}$ for all k.
3) A network with 123 or 456 in its code has port 1 in series with a nonport branch. The positions of port 1 and the nonport branch may be exchanged without altering the G matrix of the network. The result is that the four ports then form a linear tree, that is, the network becomes an (n+1)-terminal network.
4) A network with 189 or 479 in its code has port 4 in series with a nonport branch or port 2 in series with a nonport branch, respectively. In either case, the positions of the port and the nonport branch may be exchanged without affecting the G matrix of the network. The result of this is that the network becomes converted to a 2-2 linear tree port structure, and is covered by the previous material.
5) A network with 35 or 26 in its code is not allowable because of the previous sign pattern discussion. That is, either branch 3 or branch 5 must be present in the network, and

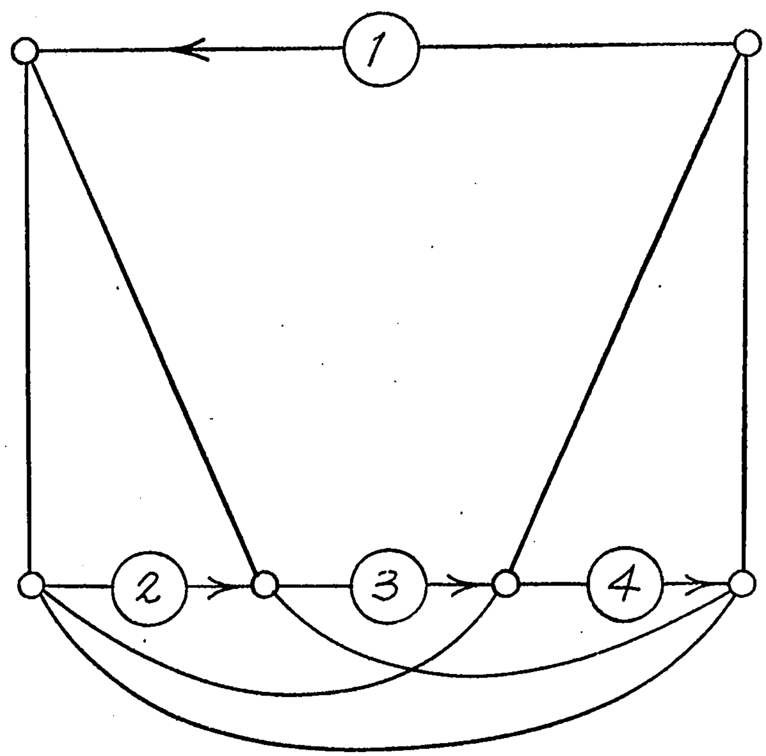



Fig. 6-14. Network 1245

either branch 2 or branch 6 must be present in the network.

6) Mirror image symmetrical networks are discarded. These are taken into account by considering the reverse of any useable matrix permutation. The symmetry line is shown on Figure 6-13. Note that the two base networks, Figures 6-12a and 6-12b, are mirror images. We arbitrarily omit 6-12b.

The above criteria reduce the number of useable minimal networks to 34. These are grouped under the appropriate sign pattern in Table 6-3. From the table we see that at most 22 networks must be tested.

| − − | | − + | | + − | |
|---|---|---|---|---|---|
| + + | | + − | | − + | |
| 1457 | 1579 | 1245 | 2579 | 1346 | 3679 |
| 1458 | 1678 | 1257 | 4578 | 1367 | 4678 |
| 1459 | 1679 | 1258 | 5789 | 1368 | 4689 |
| 1467 | 4578 | 1259 | | 1369 | 6789 |
| 1468 | 4678 | 1457 | | 1467 | |
| 1469 | 4689 | 1458 | | 1468 | |
| 1478 | 5678 | 1459 | | 1469 | |
| 1567 | 5679 | 1478 | | 1478 | |
| 1568 | 5689 | 1578 | | 1678 | |
| 1569 | 5789 | 1579 | | 1679 | |
| 1578 | 6789 | 2578 | | 3678 | |

Table 6-3

6.16 EXAMPLE 6-2

As an example of the above procedures, let us use the matrix (6-19), which was employed in example 6-1. This yields the following h matrix.

$$h = \begin{bmatrix} h_{11} & -4 & -1 & 4 & 1 \\ & 4 & 1 & -4 & -1 \\ & & h_{33} & 4 & 2 \\ & & & h_{44} & 1 \\ & & & & h_{55} \end{bmatrix} \tag{6-37}$$

We see from (6-37) that the critical sign pattern is $\begin{smallmatrix} - & + \\ + & - \end{smallmatrix}$, and we test the first network listed in the second column of Table 6-3, network 1245, shown in Figure 6-14.

Solving the equations for the missing branches yields:

$$\begin{aligned} &\text{a)} \quad h_{11} = -h_{12}/\theta \\ &\text{b)} \quad h_{33} = k\theta h_{13}/h_{12} \\ &\text{c)} \quad h_{44} = -\theta h_{13}h_{24}/h_{12} \\ &\text{d)} \quad h_{55} = \theta h_{24}h_{25}/k \\ &\text{e)} \quad \theta = \frac{k - h_{12}}{k(1+h_{13}/h_{12}) - (h_{24}+h_{25})} \end{aligned} \tag{6-38}$$

Substituting the values from (6-37) into (6-38), we obtain:

$$\begin{aligned} &\text{a)} \quad h_{11} = 5 \\ &\text{b)} \quad h_{33} = k/5 \\ &\text{c)} \quad h_{44} = 4/5 \\ &\text{d)} \quad h_{55} = 16/5k \\ &\text{e)} \quad \theta = 4/5 \end{aligned} \tag{6-39}$$

This is an interesting case, for if we examine the complementary network graph, we see that there are no circuits of four branches which include a single port, so that we would not expect $h_{11}$ and $h_{44}$ to be constants. However, the particular combination of the elements of the G matrix causes $\theta$ (which is normally a function of k) to be a constant, thus making $h_{11}$ and $h_{44}$ constants.

Applying (6-34) to (6-39) and solving for the driving-point conductances yields:

$$\begin{aligned} &\text{a)} \quad g_{11} = 5 \\ &\text{b)} \quad g_{22} = 3 + k/5 \\ &\text{c)} \quad g_{33} = 7.8 \\ &\text{d)} \quad g_{44} = 6 + 16/5k \end{aligned} \qquad (6\text{-}40)$$

Computing the figure of merit M, we obtain:
$M = (21.8 + k/5 + 16/5k)_{\text{min.}}$. The minimum occurs at k=4, yielding M = 23.4. This is a significant improvement over example 6-1, and gives the following G matrix (permuted so as to facilitate comparison with equation (5-1)).

$$G = \begin{bmatrix} 3.8 & 1 & 2 & 3 \\ 1 & 5 & 4 & 5 \\ 2 & 4 & 6.8 & 6 \\ 3 & 5 & 6 & 7.8 \end{bmatrix} \qquad (6\text{-}41)$$

6.17 THE 1-3 STAR-TREE PORT STRUCTURE

As shown in Figure 5-1f, we have to consider one more 6-terminal port structure, the 1-3 star-tree. After a linear transformation to a 1-3 linear tree structure, the derivation of the h and b matrices follows along lines similar to the preceding material. The results are presented below.

$$h = \begin{bmatrix} g_{11} & -g_{12} & -g_{13} & (g_{12}+g_{13}-g_{14}) & g_{14} \\ & g_{12} & g_{13} & -(g_{12}+g_{13}-g_{14}) & -g_{14} \\ & & -g_{23} & (g_{22}+g_{23}-g_{24}) & g_{24} \\ & & & (g_{33}+g_{23}-g_{34}) & g_{34} \\ & & & & (g_{44}-g_{34}-g_{24}) \end{bmatrix} \tag{6-42}$$

$$b = \begin{bmatrix} -h_{11} & \theta h_{11} & \theta h_{11}h_{44}/h_{34} & h_{11}h_{34}/k & \theta h_{11}h_{55}/h_{34} \\ & k & kh_{44}/h_{34} & h_{34}/\theta & kh_{55}/h_{34} \\ & & -\theta kh_{44}/h_{34} & -h_{34} & -\theta kh_{55}/h_{34} \\ & & & -h_{44} & -\theta kh_{44}h_{55}/h_{34}^2 \\ & & & & -h_{55} \end{bmatrix} \tag{6-43}$$

$$\theta = \frac{h_{34} - k}{h_{11} - k(1 + h_{44}/h_{34} + h_{55}/h_{34})} \tag{6-44}$$

We next examine the sign patterns of the b and h matrices.

$$\text{sign } b = \begin{bmatrix} - & + & + & + & + \\ & + & + & + & + \\ & & - & - & - \\ & & & - & - \\ & & & & - \end{bmatrix} \tag{6-45}$$

$$\text{sign } h = \begin{bmatrix} + & - & - & \pm & + \\ & + & + & \mp & - \\ & & + & + & + \\ & & & + & + \\ & & & & + \end{bmatrix} \tag{6-46}$$

From (6-45) and (6-46) it is clear that branches $c_{22}$, $c_{23}$, and $c_{15}$ (indicated by triangles on (6-46)), are always present in the network. Figure 6-15 shows the general minimal network. The solid-line branches are always present; the dashed-line branches are numbered for the usual coding scheme.

From the port structure or equation (6-42) we see that $g_{23}$ is negative. Thus this port structure is useful for realizing sign pattern 3), described in Chapter 5, which contains 5 positive and one negative signs. Therefore, this port structure cannot be used if all of the off-diagonal elements of G are positive.

From (6-46) we see that all of the signs of the h matrix are definite except those of $h_{14}$ and $h_{24}$, which in addition, are opposite to one another. If $h_{14}$ is positive, then $c_{14}$ is definitely present in the network, since $b_{14}$ is positive; and a similar statement can be made for $h_{24}$ and $c_{24}$. This aids in reducing the number of networks to be considered.

From Figure (6-15) we see that there are 8 possible missing branches. Since four of them must be used in a minimal network of seven branches, the number of possible networks is $\binom{8}{4} = 70$. Using the above sign patterns and the techniques described in the last section, degeneracy, reduction to lower order networks, etc., we can reduce the number of useful networks to 35. These are listed in Table 6-4 under the appropriate $h_{14}$, $h_{24}$ sign pattern.

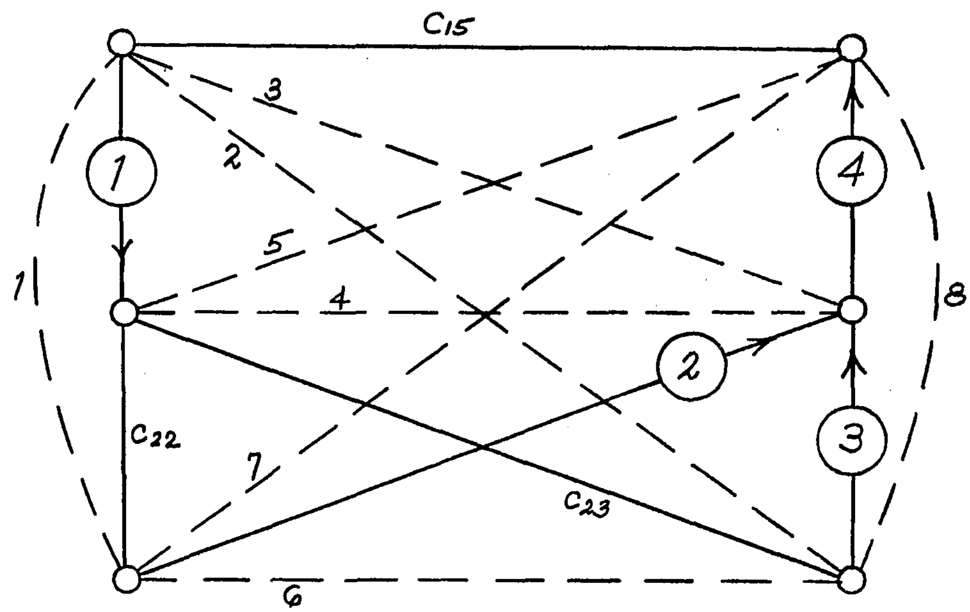



Fig. 6-15. 1-3 star tree minimal network

| + - | | - + | |
|---|---|---|---|
| 1245 | 1478 | 1256 | 2358 |
| 1246 | 1568 | 1257 | 2367 |
| 1247 | 2456 | 1258 | 2378 |
| 1248 | 2457 | 1356 | 2567 |
| 1256 | 2458 | 1357 | 3567 |
| 1257 | 2467 | 1358 | 3568 |
| 1258 | 2478 | 1368 | 3678 |
| 1456 | 2567 | 1378 | |
| 1457 | 4567 | 1568 | |
| 1458 | 4568 | 2356 | |
| 1468 | 4678 | 2357 | |

Table 6-4

From the table, it is clear that at most, 22 networks are required to be tested in a given situation.

Since $g_{23}$ must be the negative element, the number of matrix permutations to be considered is reduced from 24 to 4. For example, if (1,2,3,4) is a proper permutation, then (1,3,2,4), (4,3,2,1), and (4,2,3,1) are also.

## 6.18 CONCLUSION

We have enumerated all of the minimal 4-port networks on 6 terminals, and given procedures for using the networks in synthesis. We have also shown how to obtain the realizability boundary for a G matrix realized on a minimal network. In total there are 87 minimal networks; however, as indicated, the maximum number required to be searched in any individual example on one particular port structure, is 22. Assuming that the matrix sign pattern was such that any of the three port structures could realize it, then a maximum of 62 networks would be required to be tested.

We can compute the literal equations for the branch conductances, network driving-point conductances, figure of merit, etc., for all of the minimal networks, and store them in the memory of a digital computer. We can then have the computer do the searching for the network with the lowest value of M, choose k, and print out the solution. Although we have not yet done this, there do not seem to be any serious difficulties.

We state a final theorem.

Theorem 6-3. Let MN be the set of all 4-port minimal networks on 6 terminals. Given a fourth-order paramount conductance matrix G, with sign pattern 2) or 3); if G is not realizable on a network $N \in MN$ (possibly with some ports shunted), then G is not realizable on 6 terminals.

# CHAPTER 7. DEGENERACY IN n-PORT, (n+p)-TERMINAL NETWORKS

## 7.1 INTRODUCTION

Biorci and Civalleri (BI 8) have studied and reported some results on (n+2)-terminal degenerate networks. As previously discussed in Section 5.8, a degenerate network is one which requires an equation to be satisfied among some of the elements of the conductance matrix G. Thus a degenerate network is useful for realizing matrices whose number of independent elements is less than the maximum possible. But if we wish to realize the most general type of matrix, that is, one in which there is no equation relating any of the matrix elements, then we need a nondegenerate network.

As seen in Chapter 6, there are a considerable number of minimal networks, even for the fourth-order, 6-terminal case. As we go to higher order (n+p)-terminal networks, and as p increases, the number of minimal networks will similarly increase. Therefore, one motivation for finding the degenerate networks, is to be able to eliminate them from consideration, so as to simplify the synthesis of general matrices.

Many of Biorci and Civalleri's results are given in terms of the complementary graph of the network, that is, a graph comprising the same terminals as the original graph, and the branches missing from the original graph (with respect to a complete graph). The ports are not members of the complementary graph; however, we occasionally show the ports with dashed lines on the complementary graphs, in order to make clear some characteristics of the graphs.

We now present some of Biorci and Civalleri's definitions and theorems. Although the balance of the chapter is essentially a generalization of their Theorem 7-1, we include some other theorems, in order that the reader may become more familiar with the subject of degeneracy.

Definition 7-1. If the number of branches in a network is less than the number of independent elements of the conductance matrix, the network is said to possess a trivial degeneracy.
We gave an example of this in Section 5.8 of Chapter 5.

Definition 7-2. A network is said to possess a nonlinear degeneracy if some elements of the conductance matrix $G = g_{ij}$ are constrained by the equation: $\ldots h_{ab}h_{cd} = h_{ef}h_{gh}\ldots$, where the h's are linear combinations of the $g_{ij}$.
Equation (6-16) is an equation of a nonlinear degeneracy, for the 4-port network of Figure 6-4.

Theorem 7-1. (Biorci and Civalleri)
If the complementary graph of an (n+2)-terminal network contains a circuit with an even number of branches, the network possesses a nonlinear degeneracy.

This theorem can be proved by use of the properties of the b matrix (see equation (6-4)), which cause the missing branches to produce equations of the type:

$$-(x_{i-1}x_j/a) + h_{ij} = 0, \qquad \text{where } x_o = -a \tag{7-1}$$

The presence of an even circuit in the complementary graph leads to a set of equations of the type of (7-1), which can be solved for the ratios $x_a/x_b = f_1(h)$ and $x_a/x_b = f_2(h)$. This is a consequence of the fact that two branches (of the complementary graph) incident on a common terminal are associated with two elements of the b matrix, which appear in the same row or column of b, and hence have common factors which cancel. The resulting degenerate equation is $f_1(h)=f_2(h)$.

Biorci and Civalleri missed an important point in their analysis. Under certain circumstances $f_1(h) = f_2(h)$ is itself an identity, and thus does not lead to a nonlinear degeneracy. In Section 7.7 we give an example of this as a counterexample to Theorem 7-1, and state the general rule which distinguishes these cases.

<u>Theorem 7-2.</u> (Biorci and Civalleri)
If at least one disjoint part of the complementary graph of an (n+2)-terminal network contains two or more circuits, the network possesses a nonlinear degeneracy.

If the disjoint part is not a hinged graph, then Theorem 7-2 follows from Theorem 7-1, since at least one even circuit is present in the disjoint part. If the disjoint part is a hinged graph, then the proof follows along similar lines to that of Theorem 7-1.

<u>Theorem 7-3.</u> (Biorci and Civalleri)
If each disjoint part of the complementary graph of an (n+2)-terminal network contains at least one circuit, then the network possesses a trivial degeneracy.

This theorem follows from definition 7-1, as a network which satisfies the theorem has less branches than independent matrix elements.

<u>Theorem 7-4.</u> (Biorci and Civalleri)
If the complementary graph of an (n+p)-terminal network contains all of the nonport branches, which are incident on a terminal at the junction of two of the ports of any port subtree, then the network possesses a linear degeneracy.

This is the case of two ports in series. If ports i and j are the two ports, then $g_{ki} = g_{kj}$, for all $k \neq i,j$, a linear degeneracy.

No previous results have been published on nonlinear degeneracy in (n+p)-terminal networks, where $p > 2$. We proceed to develop the theory of degenerate networks for the 4-port, (4+p)-terminal networks. We then extend the results to the general (n+p)-terminal case.

## 7.2 DEGENERACY IN (n+3)-TERMINAL, 4-PORT NETWORKS

As motivation for this effort, Lempel and Cederbaum (LE 1) give an example of a fourth-order G matrix which requires 7 terminals for its realization. They also show how to construct a G matrix of any order, which requires 2n-1 terminals for a realization. In these cases, G is the inverse matrix of R, where R is a matrix defined by $r_{ii} = |r_{ik}|$, for all i and a particular k. The authors state that they do not know of other G matrix types which require 2n-1 (or even 2n) terminals for realization. Of course this may be due to the fact that no simple synthesis procedures are available. In addition, it is pointed out that if G is defined by $g_{ii} = |g_{ik}|$, for all i and a particular k, if G is realizable, it is realizable by an (n+1)-terminal network (CE 2).

A 7-terminal, 4-port network has its ports divided into three subtrees. As shown in Figure 5-1g, there is only one such configuration, the 1-2-1 linear tree. In Chapter 6, in order to realize 2-tree port structure networks, we employed a procedure derived from Guillemin. Guillemin (GU 3) indicated that his method was a general one, that is, not restricted to 2-tree port structures. If we desire to synthesize a network on a p-tree port structure, Guillemin suggests that we expand the G matrix with p-1 rows and columns of zeros, to correspond with the desired port-tree structure, and then compute the h matrix in the usual way. As for the b matrix, which must now be derived from a rank p-1 matrix, Guillemin forms this by summing p-1 matrices of the b type, each of rank 1. The matrix of branch conductances C is then equal to b + h as in the previous procedure.

A search of the literature fails to reveal any attempt to employ this technique in solving actual problems. Perhaps the reason is that when we initiate the procedure, we find that the equations which arise as a result of the missing branches, consist of sums of nonlinear terms in the arbitrary b matrix parameters. No general solution for simultaneous equations of this type is known, and thus a general synthesis procedure is extremely complex. However, we are able to employ the procedure to derive some new results on degenerate networks.

## 7.3 THE 1-2-1 LINEAR TREE PORT STRUCTURE

Figure 7-1 shows the complete graph and port arrangement for the 7-terminal, 4-port, 1-2-1 linear tree port structure network. The G matrix is expanded to correspond with the port structure and the h matrix is formed as explained in Chapter 6, yielding:

$$h = \begin{bmatrix} g_{11} & -g_{12} & (g_{12}-g_{13}) & g_{13} & -g_{14} & g_{14} \\ & g_{12} & -(g_{12}-g_{13}) & -g_{13} & g_{14} & -g_{14} \\ & & (g_{22}-g_{23}) & g_{23} & -g_{24} & g_{24} \\ & & & (g_{33}-g_{23}) & (g_{24}-g_{34}) & (g_{34}-g_{24}) \\ & & & & g_{34} & -g_{34} \\ & & & & & g_{44} \end{bmatrix} \qquad (7\text{-}2)$$

As a consequence of summing two b matrices of rank 1, the b matrix for this case has terms of the following form:

$$b_{ij} = -(x_{i-1}x_j/a + y_{i-1}y_j/b)\ , \quad \text{for } i \leq j, \qquad (7\text{-}3)$$

where: $a = \sum_{i=1}^{n+p-1} x_i \neq 0\,, \quad b = \sum_{i=1}^{n+p-1} y_i \neq 0,\ \ x_o = -a,\ y_o = -b.$

We now observe that if we set a branch conductance equal to zero, say $c_{ij} = 0$, we obtain the equation:

$$-(x_{i-1}x_j/a + y_{i-1}y_j/b) + h_{ij} = 0 \qquad (7\text{-}4)$$

As previously noted, the solutions for the x's and y's of a set of equations of this type would in general be difficult to obtain. However, due to certain characteristics of the h matrix we are able to establish some results on degeneracy.

The pertinent characteristics of the h matrix are as follows:

1) A port subtree of length one causes either two rows or two

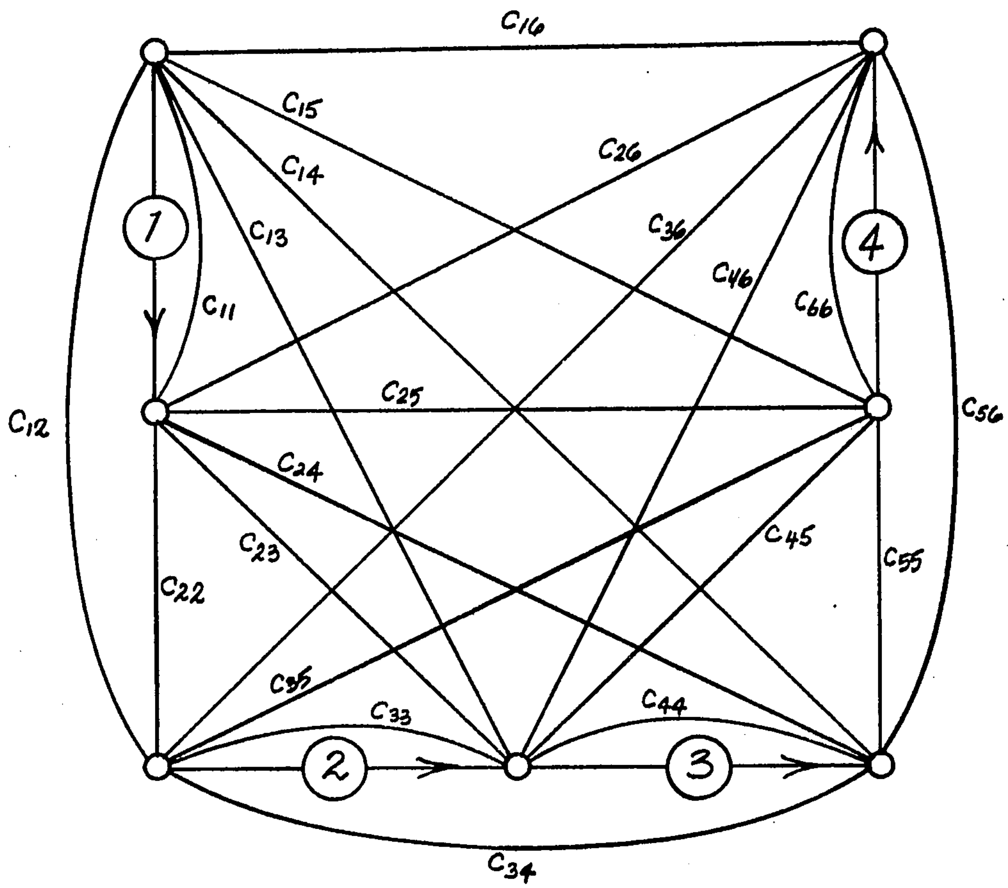



Fig. 7-1. Complete 7-terminal, 1-2-1 linear tree network

columns of the h matrix to be the negatives of one another. This results from the rows and columns of zeros which isolate the row and column corresponding to the port subtree of length one in the expanded matrix. For example, in equation (7-2), rows 1 and 2, and columns 5 and 6, have this property, resulting from the two port subtrees of length one in the graph.

2) In a port subtree of length m, where $m > 1$, if a branch is connected between each of the m+1 terminals of the port subtree and any single terminal in another subtree, then the elements of the h matrix corresponding to these branches, sum to zero. And it follows that the sum of the first m of these terms is equal to the negative of the last term, and the sum of the last m terms is equal to the negative of the first term. As an example, see $h_{12}$, $h_{13}$, and $h_{14}$ of equation (7-2).

These two characteristics allow us to establish equalities among members of certain sets of equations of the type of (7-4), (which correspond to sets of branches in the complementary graph). This is illustrated by the following example.

## 7.4 EXAMPLE 7-1

Set the following set of 6 branch conductances equal to zero: $\{c_{14}, c_{15}, c_{24}, c_{25}, c_{34}, c_{35}\}$. This yields the following set of equations:

$$
\begin{array}{lll}
\text{a)}\ c_{14} = 0 & x_4 + y_4 = -g_{13} & \\
\text{b)}\ c_{15} = 0 & x_5 + y_5 = g_{14} & \\
\text{c)}\ c_{24} = 0 & -(x_1x_4/a + y_1y_4/b) = g_{13} & \\
\text{d)}\ c_{25} = 0 & -(x_1x_5/a + y_1y_5/b) = -g_{14} & (7\text{-}5) \\
\text{e)}\ c_{34} = 0 & -(x_2x_4/a + y_2y_4/b) = -g_{23} & \\
\text{f)}\ c_{35} = 0 & -(x_2x_5/a + y_2y_5/b) = g_{24} &
\end{array}
$$

Rearranging a) and c) and dividing c) by a) yields:

$$x_1/a = \frac{g_{13} + y_1y_4/b}{g_{13} + y_4} \tag{7-6}$$

Similarly operating on b) and d) yields:

$$x_1/a = \frac{g_{14} - y_1y_5/b}{g_{14} - y_5} \tag{7-7}$$

Equating the right-hand sides of (7-6) and (7-7), multiplying out and cancelling common factors, we obtain:

$$y_5/y_4 = -g_{14}/g_{13} \tag{7-8}$$

Dividing b) by a) and using (7-8) yields:

$$x_5/x_4 = y_5/y_4 \tag{7-9}$$

Dividing f) by e) after rearranging, gives:

$$x_5/x_4 = \frac{y_2y_5/b + g_{24}}{y_2y_4/b - g_{23}} \tag{7-10}$$

Equating the right-hand sides of (7-9) and (7-10), multiplying out and cancelling common factors, yields:

$$y_5/y_4 = -g_{24}/g_{34} \tag{7-11}$$

Equating the right-hand sides of (7-8) and (7-11), we obtain:

$$g_{13}g_{24} = g_{14}g_{23} \tag{7-12}$$

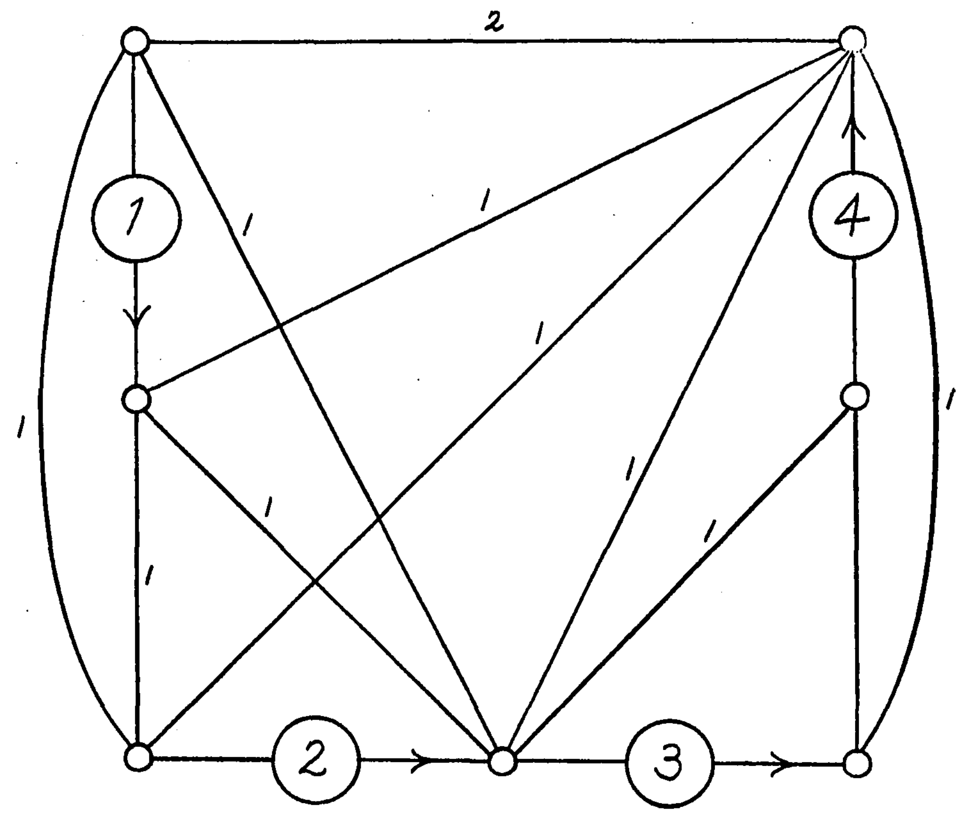
Equation (7-12) is the equation of a nonlinear degeneracy. That is, any G matrix realizable by a network which has the branches of (7-5) missing from its graph, must satisfy equation (7-12). The following numerical example serves to illustrate this point.

## 7.5 EXAMPLE 7-2

Figure 7-2 shows a network with the set of branches of equation (7-5) missing from its graph. Computing the pertinent $g_{ij}$, yields:

$$g_{13} = 4/47,\ g_{23} = 26/47,\ g_{14} = 4/47,\ g_{24} = 26/47. \tag{7-13}$$

Equations (7-13) clearly satisfy equation (7-12).




(values in mhos)

Fig. 7-2. Example of a degenerate network

## 7.6 THE COMPLEMENTARY GRAPH

Figure 7-3 shows the complementary graph, that is, the graph of missing branches, of example 7-1. As explained in Section 7.1, we indicate the ports with dashed lines, so as to show their locations; however, they are not to be construed as being part of the complementary graph. From the figure, we see that the missing branches form a <u>complete 3,2 bipartite graph</u>. Our natural inclination is to investigate other possible complete 3,2 bipartite complementary graphs, in order to discover if they also cause degeneracies. Figure 7-4 shows four complete 3,2 bipatite complementary graphs. Of the four graphs, three lead to degeneracies (the equations are listed adjacent to the graphs), and one does not. Later on in this section we explain why this is so. However, prior to that, we introduce some terminology, in order to simplify the discussion.

<u>Definition 7-3</u>. A k,2 complete bipartite graph is said to <u>cover</u> a port subtree, if each terminal of the port subtree is incident on a valence 2 vertex of the bipartite graph.

In Figure 7-4b, the 3,2 complete bipartite graph covers the port subtree of length 2 (ports 2 and 3).

<u>Definition 7-4</u>. A k,2 complete bipartite graph is said to <u>straddle</u> a port subtree of length <u>one</u>, if each of the two terminals of the port is incident on a valence k vertex of the bipartite graph.

In Figure 7-4d, the 3,2 complete bipartite graph straddles port 1.

We now examine the graphs of Figure 7-4. We observe that each graph which causes a degeneracy, covers a port subtree, but does not straddle a port. On the other hand, the nondegenerate graph straddles a port. If we check all possible 3,2 complete bipartite graphs, we observe that the following is true:

<u>Theorem 7-5</u>. A 7-terminal, 4-port network is nonlinear degenerate if its complementary graph contains a 3,2 complete bipartite subgraph, which covers at least one port subtree, and does not straddle a port.

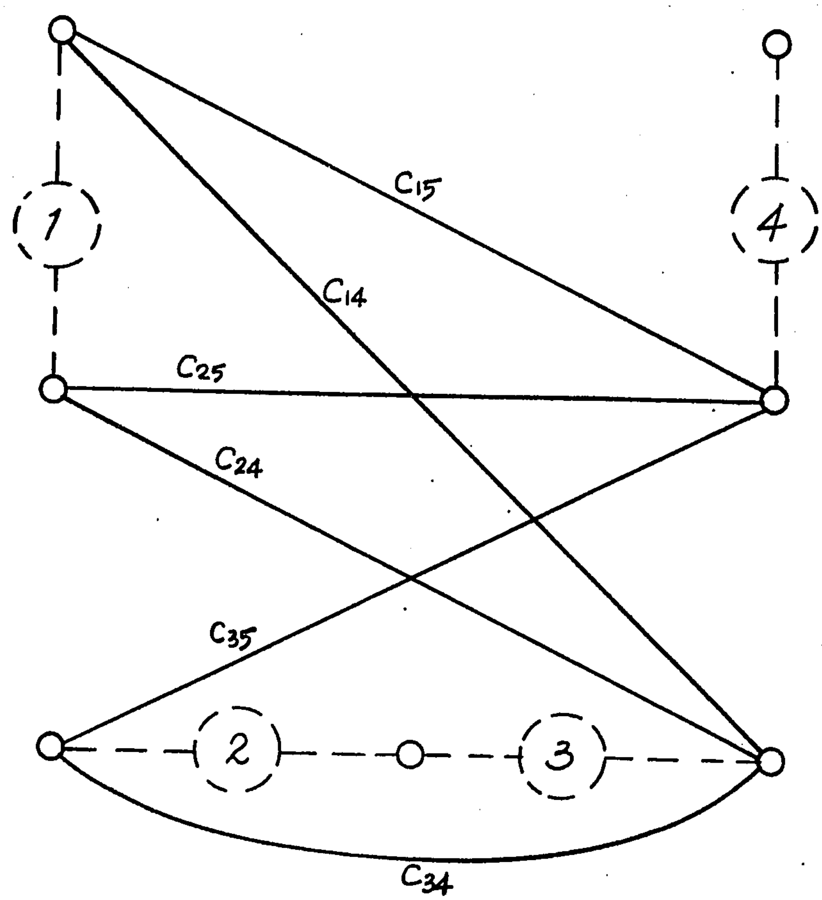
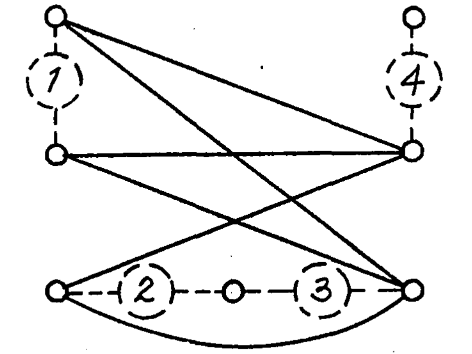
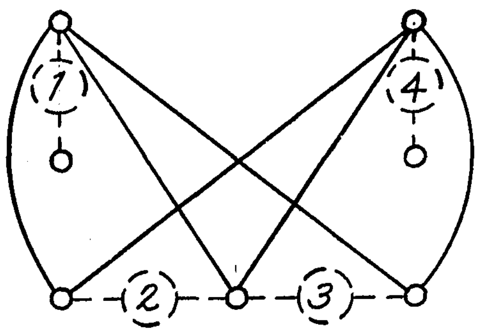



Fig. 7-3. 3,2 Bipartite complementary graph

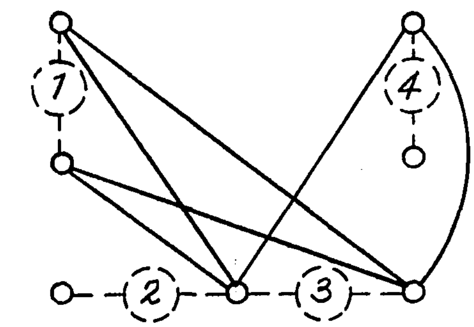
Equation of degeneracy

a) 

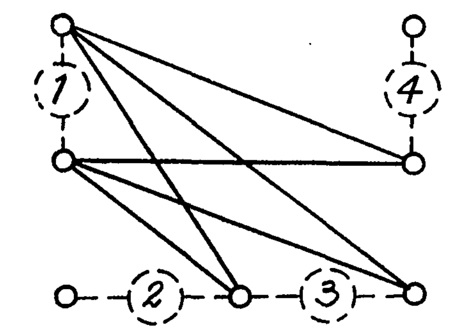

$g_{13}g_{24} = g_{14}g_{23}$

b) 


$g_{12}g_{34} = g_{13}g_{24}$

c) 


$g_{12}g_{34} = g_{13}g_{24}$

d) 


Not degenerate

Fig. 7-4. Several complementary graphs

Rather than an observational proof, we would prefer an analytical explanation, in order that we may be able to extend our results further.

If we examine equations (7-5a,b,c,d), which result from port 1 being covered, we see that the right-hand sides of the equations form two pairs of equal magnitudes. When we combine the results of operating on these equations with (7-5e,f), whose right-hand sides do not have equal magnitude elements, we obtain the degenerate equation.

The case of the graph of Figure 7-4b, falls into what we designated as characteristic 2), in Section 7.3. Here the right-hand sides of two pairs of equations sum to zero. Thus by adding some of the equations, we can form a set similar in form to (7-5) and obtain a degenerate equation.

However, if the graph straddles a port, as in Figure 7-4d, the right-hand sides of the equations are dictated by characteristic 1), where two rows of the h matrix are the negatives of one another. Thus the right-hand sides of the missing branch equations form three pairs of equal-magnitude elements. Substituting the results of operating upon any two pairs of equations into the third pair yields <u>identity</u> relations rather than degenerate equations. As a result, these complementary graphs indicate that the networks do not have unique solutions for the branch conductances in terms of the matrix elements. This result extends down to the (n+2)-terminal case as well, as we illustrate with the following example.

## 7.7 EXAMPLE 7-3

Figure 7-5 shows a 6-terminal, 4-port complementary network graph. Although the graph is an even circuit of four branches, it is also a k,2 complete bipartite graph, where k=2, and it straddles port 1. From equation (7-1) applied to the missing branches, we obtain:

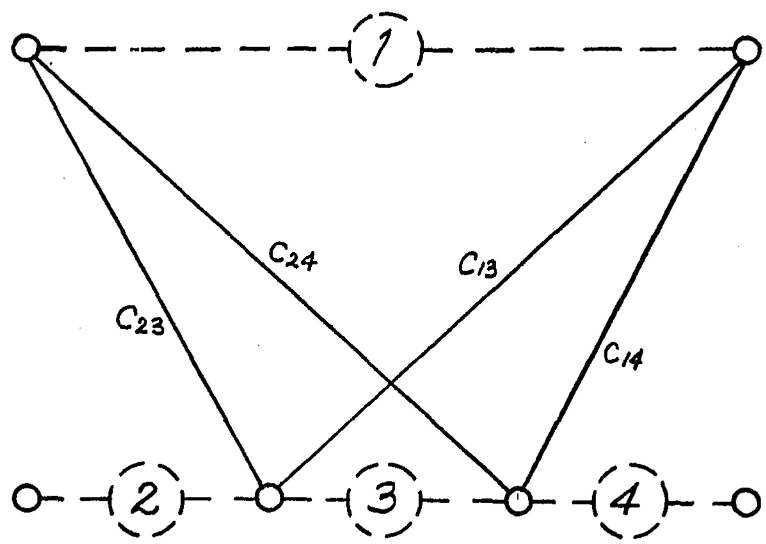



Fig. 7-5. 2,2 complete bipartite complementary graph

$$\begin{array}{lll} \text{a)} & c_{13} = 0 & x_3 = -h_{13} \\ \text{b)} & c_{14} = 0 & x_4 = -h_{14} \\ \text{c)} & c_{23} = 0 & x_1 x_3/a = h_{23} \\ \text{d)} & c_{24} = 0 & x_1 x_4/a = h_{24} \end{array} \tag{7-14}$$

Dividing c) by a) yields:

$$x_1/a = -h_{23}/h_{13} \tag{7-15}$$

Dividing b) by d) yields:

$$x_1/a = -h_{24}/h_{14} \tag{7-16}$$

Equating the right-hand sides of (7-15) and (7-16), we obtain:

$$h_{23}/h_{13} = h_{24}/h_{14} \tag{7-17}$$

Equation (7-17) appears to be the equation of a nonlinear degeneracy. However, if we substitute for the $h_{ij}$'s from equation (6-34), we see that $h_{23}/h_{13} = h_{24}/h_{14} = -1$, an identity. Thus the network does not possess a nonlinear degeneracy; and, in addition, a unique solution for the branch conductances does not exist. Therefore, example 7-3 is a counterexample to Biorci and Civalleri's Theorem 7-1. Aside from this type of complementary graph, their theorem is correct. We can now state the following theorem.

Theorem 7-6. If the complementary graph of an (n+p)-terminal network, where $p \geqq 2$, contains one k,2 complete bipartite subgraph, where $k \geqq p$, and that subgraph straddles a port, then the network does not possess a nonlinear degeneracy, and, in addition, does not have a unique solution for the branch conductances in terms of the matrix elements.

## 7.8 DEGENERACY IN (n+4)-TERMINAL, 4-PORT NETWORKS

We now attempt to extend the previous results to the 8-terminal, 4-port case. The port structure is shown in Figure 5-1h, and the h matrix, formed in the usual way from the expanded conductance matrix, is given below.

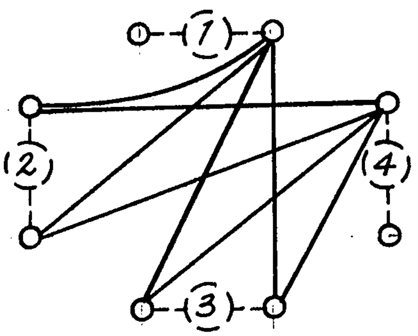
$$h = \begin{bmatrix} g_{11} & -g_{12} & g_{12} & -g_{13} & g_{13} & -g_{14} & g_{14} \\ & g_{12} & -g_{12} & g_{13} & -g_{13} & g_{14} & -g_{14} \\ & & g_{22} & -g_{23} & g_{23} & -g_{24} & g_{24} \\ & & & g_{23} & -g_{23} & g_{24} & -g_{24} \\ & & & & g_{33} & -g_{34} & g_{34} \\ & & & & & g_{34} & -g_{34} \\ & & & & & & g_{44} \end{bmatrix} \qquad (7\text{-}18)$$

The b matrix for this case is derived by adding three rank 1 matrices. A typical equation for a missing branch $c_{ij}$, has the following form:

$$-(x_{i-1}x_j/a + y_{i-1}y_j/b + z_{i-1}z_j/c) + h_{ij} = 0 \qquad (7\text{-}19)$$

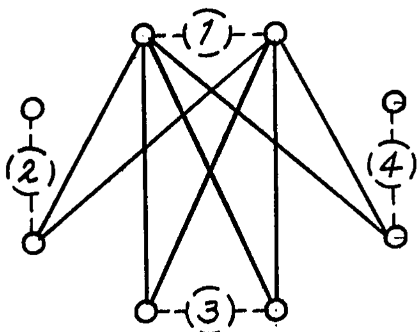
where $\quad c = \sum_{i=1}^{n+p-1} z_i \neq 0, \qquad z_o = -c$

The introduction of the additional term in equation (7-19) requires that we have a set of eight equations in order to be able to solve for the equation of a nonlinear degeneracy. And as the reader may suspect, the complementary graph in that case is a 4,2 complete bipartite graph. As in the previous case, not every 4,2 complete bipartite graph causes a nonlinear degeneracy. Figure 7-6 gives three examples of 4,2 complete bipartite complementary graphs for the 8-terminal, 4-port network. One of them is degenerate, the other two are not. The degenerate graph, Figure 7-6a, covers <u>two</u> port subtrees. The graph in Figure 7-6b does not cause a degeneracy, as port 1 is straddled by the bipartite graph. Thus this case is one for which there is no unique solution for the branch conductances in terms of the matrix elements. The third case, shown in Figure 7-6c, does not straddle a port, but covers only one subtree, Equation of degeneracy



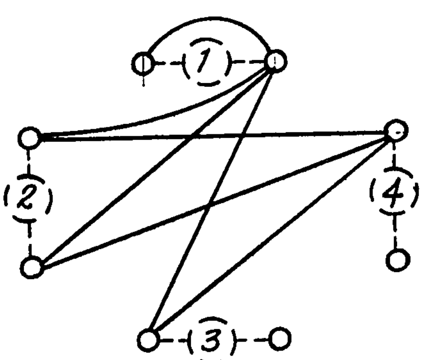

a)

$$g_{13}g_{24} = g_{14}g_{23} \quad .$$




b)

Not degenerate




c)

Not degenerate

Fig. 7-6. Several 8-terminal complementary graphs and is not degenerate. These results can be verified by inspection of the entries of the h matrix, as was done in the previous cases. We can thus state the following theorem.

Theorem 7-7. An 8-terminal, 4-port network possesses a nonlinear degeneracy if its complementary graph contains a 4,2 complete bipartite subgraph, which covers two port subtrees, but does not straddle a port.

## 7.9 GENERALIZATION TO (n+p)-TERMINAL, n-PORT NETWORKS

We can extend the results of the 4-port case to the n-port case. For large n, it is possible that a port subtree can have a length greater than p-1, say k-1. Then a k,2 complete bipartite graph would be required in order to cover the port subtree, where $k > p$. In addition, from our previous results on 6,7, and 8-terminal, 4-port networks, we see that p-2 port subtrees are required to be covered in an (n+p)-terminal network, in order to cause a nonlinear degeneracy; and so again k may be larger than p. For example, Figure 7-7 shows two complementary complete bipartite graphs, for an (n+3)-terminal, 6-port network. The 3,2 complete bipartite graph of Figure 7-7a straddles port 1, and thus does not cause a degeneracy. Whereas the 5,2 complete bipartite graph of Figure 7-7b, covers one port subtree and does not straddle a port, and thus leads to a nonlinear degeneracy. We can thus formulate the general result, which is given by the following theorem.

Theorem 7-8. An (n+p)-terminal, n-port network, where $p \geq 2$, possesses a nonlinear degeneracy, if its complementary graph contains a k,2 complete bipartite subgraph, where $k \geq p$, and the subgraph covers p-2 port subtrees, and does not straddle a port.

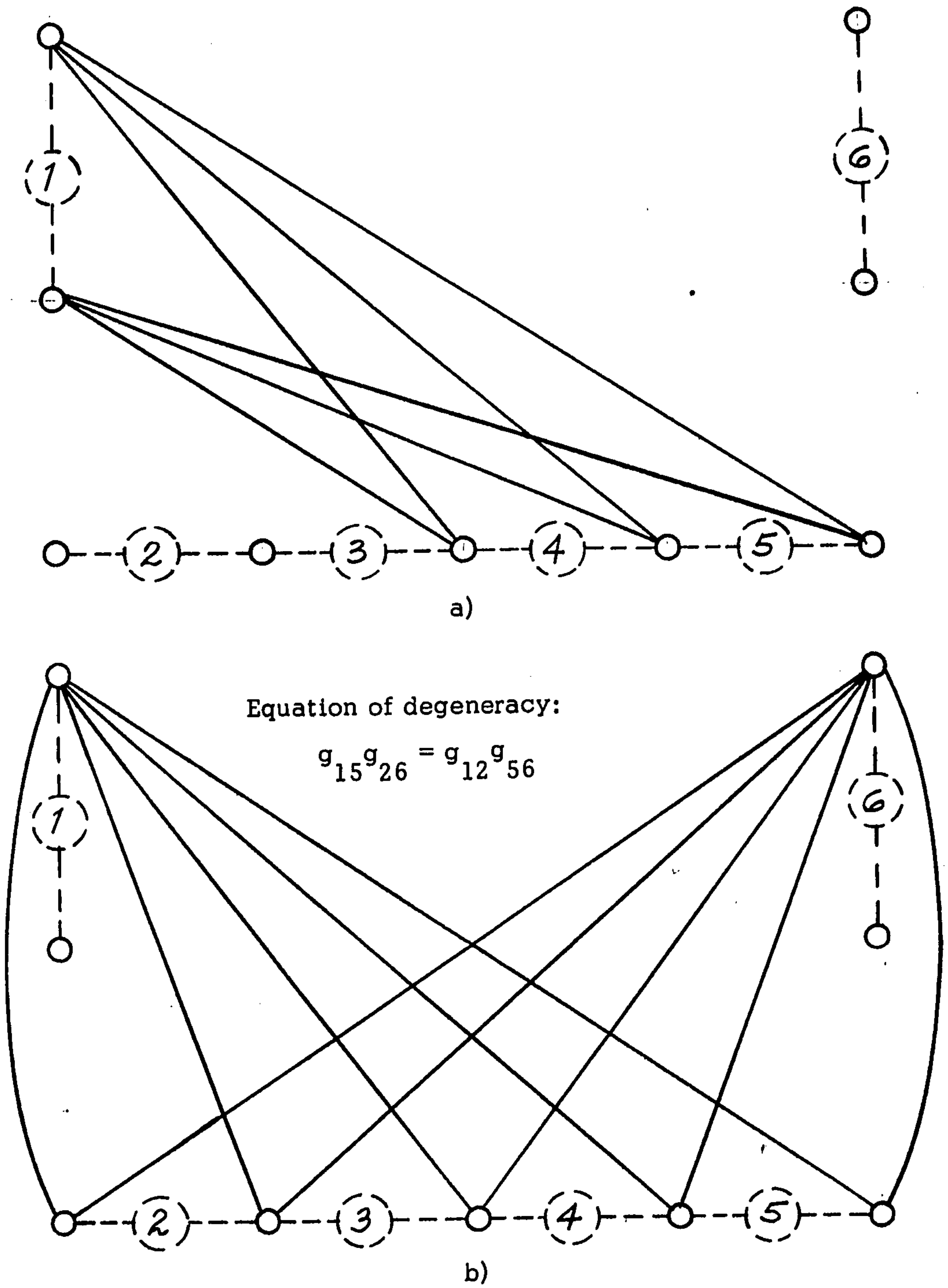



Fig. 7-7. Complementary graphs for (n+3)-terminal, 6-port networks

## 7.10 CONCLUSION

Biorci and Civalleri have done pioneering work in recognizing and reporting on degeneracy in (n+2)-terminal networks. We have provided new results by giving the conditions for nonlinear degeneracy, first in (4+p)-terminal, 4-port networks, and then in (n+p)-terminal, n-port networks. In addition, we have identified a class of networks which is not nonlinear degenerate, but for which a unique solution for the branch conductances in terms of the conductance matrix elements, does not exist. We believe that these results will provide new insight and assistance in the solution of the general n-port synthesis problem.

# CHAPTER 8. SUMMARY OF CONTRIBUTIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In this chapter we summarize the contributions of this thesis and suggest areas for possible future research.

In Chapter 3, we proved that paramountcy is a necessary and sufficient condition for a fourth-order singular resistance or conductance matrix to be realizable by a resistance network. In addition, we included canonic networks and formulas for carrying out the synthesis.

Part of these results solve the fourth-order case of an open problem posed by Nambiar, namely, to find the necessary and sufficient realizability conditions for an nth-order indefinite conductance matrix of rank n-1. The general solution to this problem is a suggested topic for further study; even the fifth-order solution would be a significant contribution.

The equations for the matrix elements of a network consist of terms containing products of the branch conductances. Thus to invert the equations, that is, to express the branch conductances in terms of the matrix elements, requires the solution of a set of multilinear equations. We have given an example of a solution to a set of these equations; however, a general solution algorithm is needed, and it is suggested that research effort be directed here.

In Chapter 4, we presented some unproved results of R.M. Foster on a system of canonic planar n-terminal, n-port networks, which have all of their accessible terminals arranged on a simple closed curve which encircles a planar network of resistances. We explained how to derive the network formulas by relating the network to the conditions for the existence of the converse of the star-mesh transformation. Foster gave two synthesis formulas, one for radial resistances and one for shunt conductances. We showed by altering the network numbering scheme that the formulas could be unified into a single one. We then proved that the formula is valid for all n. In addition, an algorithm for properly ordering the network ports, and a synthesis algorithm, were given. It was shown how to prove the necessary and sufficient conditions suggested by Foster, through n=7. A general proof of these conditions would constitute an area for future research.

Chapter 5 introduced the concept of the realizability boundary of a conductance matrix. Any set of off-diagonal elements of an nth-order conductance matrix can be realized with some set of main-diagonal elements, since any dominant conductance matrix is known to be realizable. The set of minimal n-tupies of main-diagonal elements defines the realizability boundary of the matrix off-diagonal elements. Networks which are able to realize matrices which are on the realizability boundary, are termed minimal networks, and are characterized by not having shunts across any ports. We also included in our treatment the characterization of networks by their port-tree structures, discussion of matrix sign patterns, network degeneracy, and other relevant background topics.

We made use of the concepts introduced in Chapter 5, to solve the 6-terminal, 4-port conductance case, in Chapter 6. We accomplished this by enumerating all of the feasible minimal networks and indicating how they are used to locate the realizability boundary. We then illustrate how to carry out synthesis procedures. The extension of these results to higher order (n+p)-terminal networks is a suggested topic for additional study.

Chapter 7 is devoted to results on degenerate networks. Biorci and Civalleri published results on degeneracy in (n+2)-terminal networks. We pointed out an error which they made. The major result in this chapter is the characterization of degeneracy in (n+p)-terminal networks, for all n, and $p > 1$. We showed that if the complementary network graph contained a k,2 complete bipartite sub-graph of certain specifications, that the network is nonlinear degenerate. In addition, we identified a class of networks, which although not nonlinear degenerate, does not possess a unique solution for the branch conductances in terms of the matrix elements. The implications of these results on the general n-port resistance network problem should be explored further.

## APPENDIX

The theory of resistance networks is allied with certain types of matrices. These are defined below with examples.

Paramount matrix. A real, symmetric, nth-order matrix, in which each principal minor of order r is greater than or equal to the absolute value of all other rth-order minors built from the same rows (or columns) for $r = 1, 2, \ldots, n-1$. For a matrix to be the resistance or conductance matrix of a resistance network, it is necessary that it be paramount.

Dominant matrix. A real, symmetric, nth-order matrix, each of whose main-diagonal elements is greater than or equal to the sum of the absolute values of all of the other elements in the same row (or column). A dominant matrix is paramount, and may be realized as the conductance matrix of a resistance network.

Hyperdominant matrix. A dominant matrix with all off-diagonal elements nonpositive.

Indefinite matrix. A real, symmetric, nth-order matrix, such that the sum of all of the elements in each row (and each column) is equal to zero. An indefinite matrix is singular and has the property that all of its first cofactors are equal (equicofactor).

Uniformly tapered matrix. A real, symmetric, nth-order matrix, with non-negative elements, satisfying:

$$g_{ij} + g_{i-1,j+1} \geq g_{i-1,j} + g_{i,j+1} \quad \text{, for all } i \leq j,$$

and $g_{p,n+1} = g_{0,r} = 0$.

A uniformly tapered matrix may always be realized as the conductance matrix of an (n+1)-terminal, n-port network, with the ports arranged on an oriented linear tree.

$M_1$ and $M_2$ shown below, are matrices which illustrate some of the above definitions.

$$M_1 = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 5 & -4 \\ -2 & -4 & 6 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 7 & 5 \\ 2 & 5 & 6 \end{bmatrix}$$

$M_1$ and $M_2$ are both paramount. In addition, $M_1$ is dominant, hyperdominant, and indefinite. $M_2$ is uniformly tapered.

For convenience, we represent a resistance network by a linear graph, a set of nodes (vertices), interconnected by a set of branches (edges). Nodes which are externally accessible are designated terminals. A specific pair of terminals, to which a voltage or current generator may be connected, or an output voltage or current measured, is referred to as a port.

Resistance networks are characterized by measurements at the ports. This information is compiled in the form of a matrix. Two frequently used forms are given below.

Short-circuit conductance matrix. (conductance matrix) A characterization of an n-port resistance network, with all ports operating under short-circuit conditions. If $E]$ is the column vector of voltages applied to the n-port, and $I]$ is the column vector of response currents, and $[G] = [g_{ij}]$ is the conductance matrix, then:

$[G] . E] = I]$.

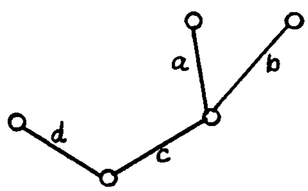
Open-circuit resistance matrix. (resistance matrix) A characterization of an n-port resistance network, with all ports operating under open-circuit conditions. If $I]$ is the column vector of currents applied to the n-port, and $E]$ is the column vector of response voltages, and $[R] = [r_{ij}]$ is the resistance matrix, then:

$[R] \, I] = E]$.

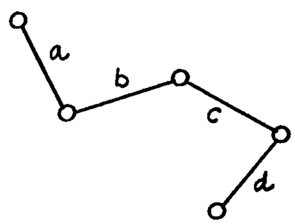
When the input and the response are at the same port, say port i, then the corresponding matrix element, $r_{ii}$ or $g_{ii}$, is called a driving-point resistance or conductance, respectively. When the input is at port i and the output at port j, with $i \neq j$, the corresponding matrix element, $r_{ij}$ or $g_{ij}$, is called a transfer resistance or conductance, respectively.

The characteristics of resistance networks are related to the topological properties of their network graphs. Accordingly, we introduce some pertinent terms.

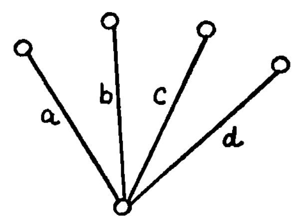
A tree (spanning tree) is a connected graph with no circuits. A linear tree is a tree which has no more than two branches incident on a node. An oriented linear tree is a linear tree with directed branches, all oriented in the same direction. A star tree is a tree such that one end of every branch is incident on a single node. Illustrated below are several trees.



Tree



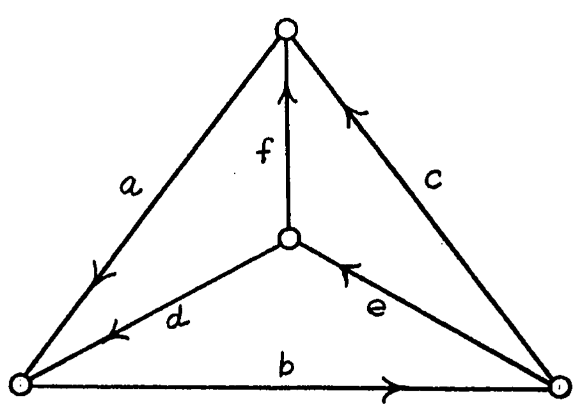
Linear tree



Star tree

A subtree of a graph G is a connected subgraph of G with no circuits. For example, branches a and b and their associated nodes form subtrees of the above graphs.

A graph which consists of a simple closed path is called a circuit or loop. In the graph P below, the graph formed from branches a, b, and c and their associated nodes, is a circuit.



Graph P

If we select a particular tree of a graph G, the graph comprising the branches which are not in the tree (and their associated nodes), is called a cotree. Each branch of the cotree is a link or chord, and has the property that when it is added separately to the tree, it forms a circuit. The family of all of these sets of circuits is called a fundamental set of circuits. For example, in graph P, where the arrows indicate the assumed branch direction, choose tree $\{a,b,d\}$. The cotree is then $\{c,e,f\}$, and the fundamental set of circuits is $\{\{a,b,c\}, \{b,d,e\}, \{a,d,f\}\}$. This can be described by an (lxb) matrix called a circuit matrix or a loop matrix L, where l is the number of links and b is the number of branches. An entry $l_{ij}$ of L is ±1 if branch j is in circuit i, +1 if the branch is traversed in the direction of the arrow and -1 if not, assuming that the circuit is traversed in the same direction as the link arrow. $l_{ij} = 0$ if branch j is not in circuit i. The circuit matrix corresponding to the above example is:

$$L = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ c \\ e \\ f \end{array}$$

A cut-set of a graph is a minimal set of branches which when deleted from the network graph cause the remaining subgraph to be divided into exactly two separate parts. $\{b,c,e\}$ is a cut-set of P. If we choose a tree of a graph and then find a family of cut-sets such that each cut-set contains exactly one different tree branch, the resulting family of sets is called a fundamental set of cut-sets. For example, in graph P, if we choose the tree $\{a,b,d\}$, the fundamental set of cut-sets is $\{\{b,c,e\}, \{d,e,f\}, \{a,c,f\}\}$. This can be described by a (cxb) matrix called a cut-set matrix C, where c is the number of cut-sets and b is the number of branches. An entry $c_{ij}$ of

C is $\pm 1$ if branch j is in cut-set i, being +1 if the branch direction is the same as the direction of the tree branch, -1 otherwise. $c_{ij} = 0$ if branch j is not in cut-set i. The cut-set matrix corresponding to the above example is:

$$C = \begin{array}{c c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} & \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} & \begin{array}{c} a \\ b \\ d \end{array} \end{array}$$

A <u>θ-matrix</u> or unimodular matrix is one (not necessarily square) in which each entry and every subdeterminant is +1, -1, or 0. L and C are examples of θ-matrices.

A <u>complete graph</u> is one which contains precisely one branch between every pair of nodes. P is an example of a complete graph.

An <u>a,b complete bipartite graph</u> comprises two sets of nodes, one set contains <u>a</u> nodes, the other <u>b</u> nodes. There is exactly one branch connecting each node in one set with each node in the other set. There are no other branches.

The <u>nullity</u> $\mu$ of a connected graph with b branches and v nodes is $\mu = b - v + 1$, and is equal to the number of fundamental circuits. In graph P, $\mu = 3$.

The <u>rank</u> r of a connected graph with v nodes is equal to $r = v - 1$, and is equal to the number of fundamental cut-sets. In graph P, $r = 3$.

The <u>rank of a matrix</u> is equal to the maximum of the orders of the nonzero minors of the matrix. The <u>degeneracy</u> or <u>nullity of a matrix</u> is equal to the order (or dimension in the case of a nonsquare matrix) minus the rank.

A <u>planar graph</u> is one which may be mapped onto a plane or sphere with no branches crossing one another. Graph P is an example of a planar graph. A <u>mesh</u> of a graph is a circuit containing no branches inside of it. The outer circuit of a graph is considered a mesh, enclosing the area outside of the graph. A planar graph possesses at least one dual graph. The dual may be constructed by placing one node of the dual inside of each mesh (including the outer one), and joining these by branches such that each branch of the original graph is crossed once by a dual branch.

Two graphs $G_1$ and $G_2$ are defined to be isomorphic if there is a one-to-one correspondence between the nodes and branches of $G_1$ and $G_2$ which preserves the incidence relations. That is, if a branch exists in $G_1$, a corresponding branch exists in $G_2$, between corresponding nodes, and vice versa.

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