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TIME-OPTIMAL CONTROL OF LINEAR SYSTEMS WITH CONTROLLER OF PARTIALLY SPECIFIED STRUCTURE

by

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$\underline{x}(t)$	state variable vector
$\underline{y}(t)$	output vector
$\underline{u}(t)$	input vector
t_o	arbitrary initial time
t_f	final or terminal time
T	elapsed time ($T = t_f - t_o$)
t_{f_o}	shortest terminal time in optimal w.r.a.s. solution
T_o	elapsed time in optimal w.r.a.s. solution
t_f^*	minimum terminal time in optimal solution
T^*	minimum elapsed time in optimal solution
$\Vert\underline{k}\Vert_q$	q^{th} norm of the vector k
$h_a(t)$	impulse response of augmented plant
$h_j(t)$	impulse response of given, single output plant
$H(t)$	impulse response matrix
$H(s)$	Laplace transform of $H(t)$
A_{jk}	j-k^{th} feedback coefficient
y_d	output terminal point
$\underline{e}_d$	$\underline{y}_d - C(t_f)\,\phi(t_f, t_o)\,\underline{x}(t_o)$
$\phi(t, \tau)$	Fundamental matrix
$\underline{\lambda}$	parameters in functional analysis technique
$\underline{\lambda}^*$	set of parameters which minimize $\Vert\underline{k}\Vert_q$ subject to $\underline{\lambda}'^* \cdot \underline{e}_d = 1$
$\underline{k}^*(t)$	$\underline{\lambda}^{*'} \cdot H(t)$

$(s_i)_o$	previous value of $\dfrac{\partial t_f}{\partial \lambda_i}$
$(s_i)_n$	present value of $\dfrac{\partial t_f}{\partial \lambda_i}$
$(s_i)_{nw}$	newest value of $\dfrac{\partial t_f}{\partial \lambda_i}$
$\delta(t)$	unit impulse function

ABSTRACT

This research is concerned with the time-optimal control of an n^{th} order, linear, time-varying plant with one constrained and (r-1)-unconstrained inputs and m-constrained outputs. Magnitude, integral, and quadratic constraints are considered. The controller for the constrained input is found in the absence of the output limitations using the functional analysis approach. The controller for the unconstrained inputs consists of a linear feedback arrangement. The values of the feedback coefficients in this specified portion of the controller are adjusted to satisfy the output constraints. For a given set of coefficients, the constrained input, found in the above manner, is optimal with respect to the assumed structure, i.e. optimal w.r.a.s. Some necessary conditions for the existence of an optimal w.r.a.s. solution are derived and lower bounds on the elapsed time are given. These conditions are used to significantly reduce the computation time in the numerical search which is required to find the set of feedback coefficients, the optimal w.r.a.s. input, and the minimum elapsed time. The conditions for an optimal w.r.a.s. solution to the given problem to be optimal are developed. The stability of the resultant plants under these conditions is discussed. When the Functional Analysis approach is used to solve certain time-optimal problems, the curve of the elapsed time versus the lambda's in $(T,\underline{\lambda})$ - space is shown to be a maximum at $(T^*, \underline{\lambda}^*)$. If $\|k^*\|_q$ is a non-decreasing function of T, this point is a unique maximum. The sufficient condition for this to be true for time invariant plants is given. It is shown that initially quiescent systems and regulator type systems satisfy this condition. Standard hill climbing techniques which are well suited to implementation on a digital computer can be used to find T^* and $\underline{\lambda}^*$.

I. INTRODUCTION

1.1 Motivation

One can postulate a dynamic process which must be controlled and which, in general, has many input and output variables that have various physical constraints placed on them. When we attempt to control the process in a time-optimal[†] fashion, the solution often presents formidable computational problems and, therefore, requires a large economic effort (i.e. costly computers). This is not always justified since it may be possible to use a far simpler and hence less costly control scheme which satisfies all constraints and initial and terminal conditions in a slightly longer elapsed time than the time-optimal case. We have thus traded simplicity and lower cost for a greater running (i.e. elapsed) time.

1.2 Mathematical Formulation of the Problem

In order to put the problem which is solved in this thesis in proper perspective, it is necessary to consider as background several other problems which have been solved by other investigators.

Let the completely output controllable[37,38] process (i.e. the plant) be described by the following system of state equations

$$\dot{\underline{x}}(t) = E(t)\ \underline{x}(t) + F(t)\ \underline{u}(t) \qquad (1\text{-}1a)$$

$$\underline{y}(t) = C(t)\,\underline{x}(t) + D(t)\ \underline{u}(t) \qquad (1\text{-}1b)$$

[†]A time-optimal control is one that permits all constraints and the initial and terminal conditions to be satisfied in the shortest time.

where $\underline{x}$, $\underline{u}$, and $\underline{y}$ are respectively the system state variable, input, and output vectors; E(t), F(t), C(t), and D(t) are respectively $(n \times n)$, $(n \times r)$, $(m \times n)$, and $(m \times r)$ time-varying matrices; and the dot signifies differentiation with respect to time.

Problem 1

Consider the linear plant shown in Figure 1-1a. (In Eqs. 1-1, E(t) and F(t) are respectively $(n \times 1)$ and $(m \times 1)$ time-varying matrices.) The single input has a constraint placed on it which is expressed in the form

$$\|u\|_p \triangleq \left[\int_{t_o}^{t_f} |u(t)|^p \, dt\right]^{1/p} \leq L \qquad (1\text{-}2)$$

where t_o and t_f are the initial and final process times and L is a positive real number.[†] The problem is to find the input, u(t), which drives the output vector from $\underline{y}(t_o) \triangleq y_o$ to $\underline{y}(t_f) \triangleq \underline{y}_d$ (i.e. satisfies the initial and terminal conditions) and which satisfies the constraint in Eq.(1-2) in the shortest elapsed time

$$T_1^* = t_{f1}^* - t_o$$

Problem 2

Consider the linear plant shown in Figure 1-1b. Here, each of the r-inputs is constrained as in Eq. (1-2), the value of p being selected according to the type of physical limitation on each input. (See footnote.) The problem is the same as that of Problem 1 except that it is now

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[†]When $p = \infty$, the maximum magnitude of the input is constrained; when $p = 2$, a quadratic constraint is placed on u; and if $p=1$, a fuel constraint is placed on u provided the input represents the rate of change of the fuel.

necessary to find r-control actions (i.e. the inputs) which permit all constraints, initial, and final conditions to be satisfied in the shortest elapsed time

$$T_2^* = t_{f_2}^* - t_o$$

Problems 1 and 2 have been previously studied[8,13,21,22]. For example, it is well known that when $p = \infty$, a bang-bang control results. A detailed description of one approach to the general solution of these problems is given in Section 2.2.

Problem 3

We again consider the single input plant in Figure 1-1a. In addition to the input constraint in Eq.(1-2), the outputs of the plant are also constrained, i.e.

$$f_i(y_i) \geq L_{2i} \qquad (i = 1, 2, \ldots m)$$
$$\leq L_{2i} \tag{1-3}$$

where the f_i can have any functional form and the sign of the inequality is determined by the problem†. We seek the input, u(t), which permits all

† For example, if we are dealing with an aircraft, it might be required to keep below a given altitude. Then

$$f_i(y_i) \triangleq y_i \leq L_{2i}$$

On the other hand, however, an orbiting satelite might be required to remain above a certain altitude so that we would then have

$$f_i(y_i) = y_i \geq L_{2i}$$

Also, if one of the y_i's represented the rate of production in a given process and it was required to produce at least a certain amount of the product, then Eq. (1-3) would become

$$\int_{t_o}^{t_f} y_i \, dt \geq L_{2i}$$

constraints, initial, and terminal conditions to be satisfied in the shortest elapsed time, T_3^*. It is important to note that because of the additional output constraints, $T_3^* \geq T_1^*$.

<u>Problem 4</u>

This problem is exactly the same as Problem 2 except that now the outputs of the plant in Fig. 1-1b are constrained as in Eq.(1-3) and we seek the r-inputs which permit all constraints initial, and terminal conditions to be satisfied in the shortest elapsed time, T_4^*. (Note that $T_4^* \geq T_2^*$).

Problems 3 and 4 with magnitude constraints on the outputs have been previously studied[24-30]. In general, it is found that the control actions which result are more complicated† compared to the ones found for Problems 1 and 2. In addition, the sensing operations which are required to realize the former controls are also complicated.

In this thesis, we wish to solve Problems 3 and 4 using control actions of <u>no greater complexity</u> than those found in the solution of Problems 1 and 2.

With the above as background, the problem which is solved in this dissertation can now be stated.

<u>Problem 5</u>

Consider the linear plant which is shown in Figure 1-1b and is described by Eqs.(1-1). Constraints are placed on the r-inputs and the m-outputs (i.e. Eq.(1-2) and Eq.(1-3) respectively). It is required that

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†That is, they involve complicated switching functions.

Figure 1-1 N^{th} order, linear time-varying plant with m-outputs and
(a) one-input
(b) r-inputs

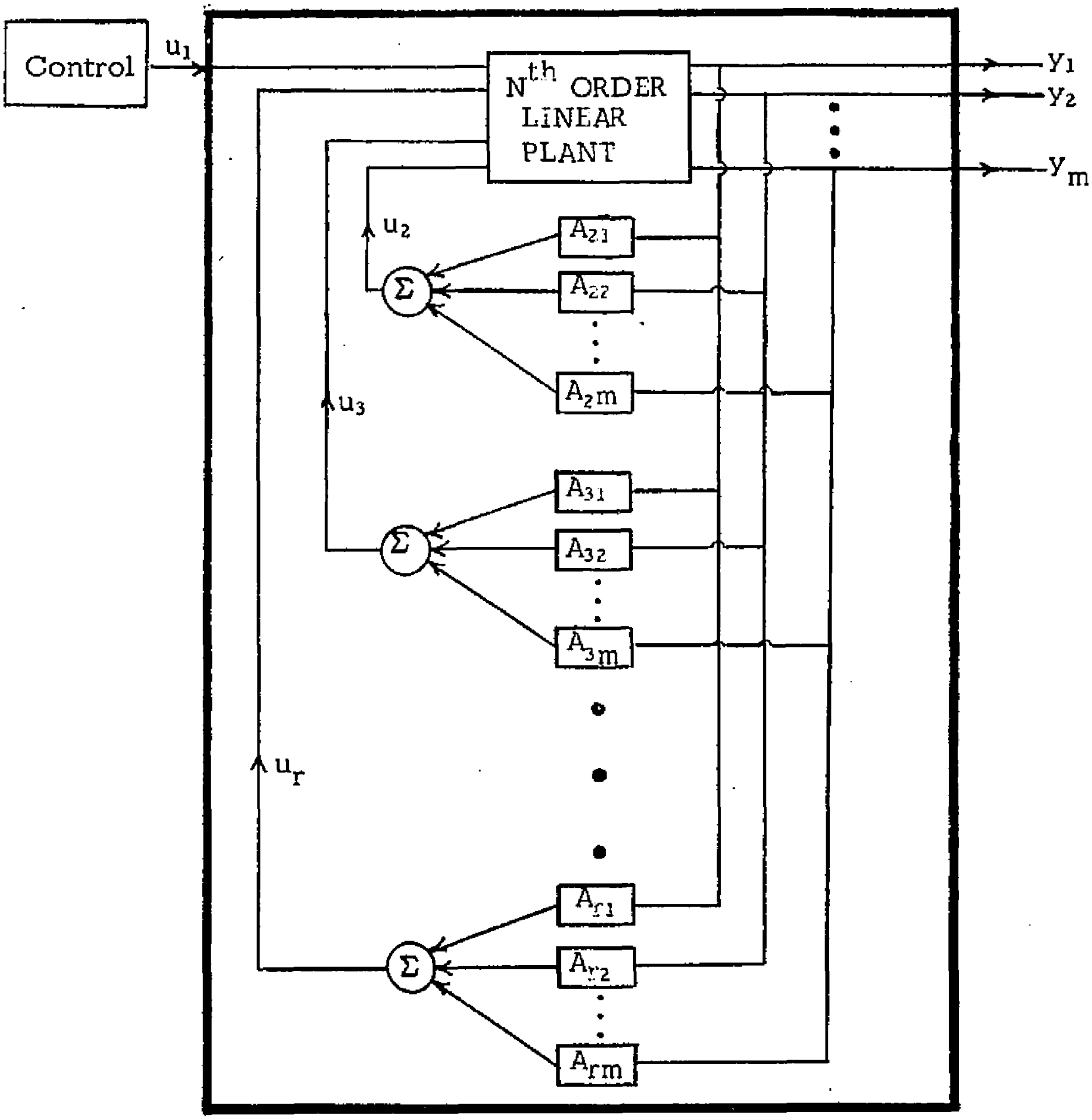

Figure 1-2 Augmented Plant

$u_1(t)$ have the <u>same form</u> as the optimal control which is used in Problem 1 and that $u_2, u_3, \ldots u_r(t)$ be easily realized. The problem is to find $u_1(t)$ and to specify the remaining (r-1)-inputs such that all of the constraints and the initial and final conditions are satisfied in the shortest elapsed time, T_O.

It is observed that this is Problem 4 with the additional requirements that one of the inputs have the form of the optimal control in Problem 1 and that the remaining control actions be easily realized. It is important to note that $T_O \geq T_4^*$ or T_3^*.

1.3 General Approach

The general approach to the solution of Problem 5 which is to be outlined below does not depend on the functional form of the output constraints in Eq.(1-3). In order to be specific, however, we will require that these constraints have only the following forms:

$$\begin{aligned} \|y_i\|_{p_i} &\triangleq \left[\int_{t_O}^{t_f} |y_i(t)|^{p_i}\,dt\right]^{1/p_i} \geq L_{p_i} \qquad [p_i = 1 \text{ or } 2] \\ |y_i(t)| &\leq L_{p_i} \qquad [p_i = \infty] \end{aligned} \tag{1-4}$$

Consistent with the requirement that the remaining (r-1)-plant inputs be easily realized, a reasonable choice for these inputs is a linear combination of the plant outputs, i.e.

$$u_j(t) = \sum_{k=1}^{m} A_{jk}\, y_k(t) \qquad [j = 2, 3, \ldots r] \tag{1-5}$$

where the A_{jk} are a set of unspecified feedback coefficients. (See Figure 1-2) An important advantage of choosing these inputs in this manner is the fact that the single input, m-output augmented system, consisting of the given plant and the feedback as shown in Figure 1-2, is completely output controllable.†

The solution of Problem 5 is now approached in the following manner:

The input to the augmented plant is found in terms of the feedback coefficients, A_{jk}, without regard to the output constraints. For any set of A's, this input is chosen so that it satisfies the constraint in Eq.(1-3) and permits the initial and terminal conditions to be satisfied in the shortest elapsed time. To this end, the method of functional analysis[21,22] which is discussed in Section 2.2 is employed. As the feedback coefficients provide several degrees of freedom, they are then adjusted to meet the output constraints in Eq.(1-4) while making the elapsed time as small as possible.

It is apparent from Figure 1-2 that the given plant is controlled by two separate devices. The first one consists of the controller which produces the input $u_1(t)$ (found in the above manner). The form of this controller is, of course, dependent upon the particular system. The second one is specified and consists of the simple feedback arrangement shown in Figure 1-2. It is clear that once the set of feedback coefficients

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†This follows from the work of E.G. Gilbert who showed that when a composite system is made up of subsystems S_a and S_b, where S_b consists of the feedback elements, the necessary and sufficient condition for such a system to be controllable is that the cascade connection, S_a S_b be controllable.

is chosen, $u_1(t)$ is optimal in the sense that it permits all constraints, initial, and terminal conditions to be satisfied in the shortest elapsed time. We will henceforth refer to an input found in the above manner as being "optimal w.r.a.s.", i.e. optimal with respect to the assumed structure (of the controller).

In the above, it was tacitly assumed that the set of A's which permit the output constraints to be satisfied in the shorted elapsed time do not cause the constraints on u_2 through u_r to be violated. If, however, one or more of these constraints is violated, a different set of feedback coefficients must be selected so that every input constraint is satisfied. Under these conditions, the elapsed time would be greater than that found for the first set of A's.

In this dissertation, however, we will usually assume that either the set of A's and the optimal w.r.a.s. input found in the above manner do not cause the constraints on the remaining (r-1)-inputs to be violated or that these constraints are not known at all.

The question of how to make a fair comparison between the optimal w.r.a.s. and time-optimal solutions then arises. In addition, it is pertinent to ask what one means by a time-optimal solution for a problem involving inputs which have either loose or unknown constraints.

Since the specified portion of the controller (i.e. the feedback) produces input signals which are proportional to the constrained outputs (Eq.1-5), these inputs have the same class of limitation placed on them also†. In this dissertation, the term "time-optimal solution" will be understood to refer to the solution which is obtained when optimal signals are applied to <u>all</u> of the inputs and the output constraints are neglected. The signal applied to input one satisfies the condition in Eq.(1-2). The remaining (r-1)-signals satisfy the following set of <u>derived</u> constraints

$$\|u_j\|_{p_2} \triangleq \left[\int_{t_o}^{tf} |u_j|^{p_2}\, dt\right]^{1/p_2} \leq L_{p_{2j}} \tag{1-6}$$

where p_2 is chosen so that the type of input limitation is the same as the output constraint and

$$L_{p_{2j}} \triangleq \sum_{k=1}^{m} A_{jk}\, L_{p_2k} \tag{1-7}$$

It is important to realize that under the above assumptions, it is necessary to solve Problem 2 in order to find the optimal time. Then, a measure of performance of the optimal w.r.a.s. approach is obtained by comparing T_O with T_2^*. It should be noted that, in general, this is a very pessimistic comparison.

1.4 <u>Scope of the Dissertation</u>

Since the form of the control action is to be found in the absence of the output constraints, it can be realized using established techniques.[45-49] These techniques are not considered for this dissertation.

† For example, if all of the outputs have their magnitudes limited, the (r-1)-inputs do also.

In the next chapter, a detailed presentation of the Functional Analysis approach[21,22] to solving optimal control problems is given. This is the method used in the dissertation. The chapter also includes a brief historical survey of time-optimal control.

Besides solving Problem 5 of Section 1.2, one of the objectives of this work is to develop numerical procedures for solving time-optimal control problems using the Functional Analysis approach. To this end, Chapter III presents a theorem which permits a digital computer to be used to solve the subsidiary mathematical problem which arises when this approach is used, i.e. the minimization of a norm with respect to several parameters.[†] It is shown that if the norm in question is a non-decreasing function of the terminal time and the minimizing parameters, these parameters also <u>maximize</u> the elapsed time. The sufficient condition for this to be so is given for time-invariant plants. It is shown that initially quiescent and regulator type systems satisfy this condition.

In Chapter IV, the solution to Problem 5 is obtained for general linear systems. As there are essential differences in the procedures used in obtaining solutions for single and multiple output plants, the chapter is divided into two parts. In the first, systems with integral or quadratic constraints on the single output and magnitude, integral, or quadratic constraints on the inputs are treated. Lower bounds on the elapsed time are derived and some necessary conditions for the existence of a solution

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[†] These parameters are called λ's and are similar to the Lagrange multipliers which are used in variational calculus.

to Problem 5 are presented. It is shown that if $y(t_o) = 0$, the magnitude of the output never exceeds the magnitude of $y(t_f) \triangleq y_d$. Using this result, it is immediately apparent whether or not a given output magnitude constraint can be satisfied. In the second part of the fourth chapter, systems with magnitude, integral, or quadratic constraints on the multiple outputs and inputs are treated. Some necessary conditions for the existence of a solution are given. It is also shown that when the input to the augmented plant has a quadratic constraint on it, the subsidiary mathematical problem associated with the Functional Analysis approach is reduced to the problem of finding the real root of a transcendental equation. In both parts the numerical search procedure for finding the feedback coefficients is given and illustrated with numerous examples.

In Chapter V, it is shown that for a class of problems, the optimal w.r.a.s. and optimal solutions are identical. The conditions for this to be so are developed and several illustrations are given. It is demonstrated that the augmented plants which result are generally unstable if the given plant is stable.

The final chapter is devoted to a detailed discussion of the numerical procedures used to solve the problems in Chapter IV. The practical aspects of the search procedures used in obtaining the optimal w.r.a.s. solution are given.

As a final note one may observe that this thesis (and the proposed solution to Problem 5) represents a marriage between certain modern and classical control techniques.

1.5 Contributions

The following is a summary of the contributions made in this thesis:

1. It is demonstrated that when the Functional Analysis approach is used to solve certain time-optimal control problems, the curve of T, the elapsed time, versus the λ's in $(T, \underline{\lambda})$ space is a maximum at the point $(T^*, \underline{\lambda}^*)$. If $\|\underline{k}^*\|_q$ is a non-decreasing function of t_f, this point is a unique maximum. The sufficient condition for this to be so is that $\underline{e}_d$ be independent of t_f. Initially quiescent and regulating type systems satisfy this condition. From these results, standard hill climbing techniques which are well suited to implementation on a digital computer can be used to find $(T^*, \underline{\lambda}^*)$.

2. It is shown that application of the optimal w.r.a.s. input (Eq. 4-5) implies that the <u>single output</u> of a linear plant can never overshoot the terminal point, y_d. This result is easily applied to determine whether it is possible to satisfy a given magnitude constraint on the output. If the output has an integral or a quadratic constraint on it, the necessary conditions for the existence of a solution to Problem 5 (Eqs. 4-7 and 4-22) and lower bounds on the elapsed time (Eqs. 4-13 and 4-17) are derived. Necessary conditions for the existence of a solution to Problem 5 are derived for multiple output plants (Eqs. 4-37 and 4-42). These results are used to significantly reduce the computation time which is required to find the set of feedback coefficients and the elapsed time.

3. It is shown that a class of problems exists having the property that the proposed control scheme gives the time-optimal solution. The conditions that the systems in these problems must satisfy are developed and several examples included. (Chapter V)

1.6 Suggestions for Future Study

The work done on the given problem has, by no means, exhausted the subject. Other possible areas of study have also been uncovered. The following is a list of topics for future study:

1. This work dealt solely with continuous systems. An obvious extension would be to repeat the entire problem for discrete plants.
2. Since the form of the optimal w.r.a.s. input is known (although the switching times are not in the case of an input magnitude constraint) use of an analog computer might facilitate the solution of a particular problem. This possibility should be explored.
3. As was previously mentioned, every plant input has some physical limitation on it. If these limitations are not exceeded using the proposed control scheme, it would be interesting to learn whether the use of bias batteries which cause the inputs to operate on the boundaries of their constraints might reduce the elapsed time.
4. The question of whether $\|k^*\|_q$ can decrease as t_f increases is still unanswered. It would be useful to know the specific conditions under which this behavior is possible, or for that matter if it is possible at all.

II. BACKGROUND

2.1 Historical Background

In recent years control theory has advanced quite rapidly. One of the primary reasons for this is that control engineers have had to deal with increasingly complex systems. As problems have become more complex and the demands on system performance more stringent, it has become necessary to develop new techniques for solving such problems. As a result, the methods of classical control theory† have been augmented and in some instances replaced by those of modern or optimal control theory.

In essence, the philosophy of the modern approach to control problems is that a functional on the system state and input variables is selected or pre-specified. Physical constraints are then imposed on these variables and a control law is found which minimizes or maximizes the functional by some variational method.

Historically, the optimal control problem first arose as the "time-optimal" problem (due to Feldbaum[1]). In it one seeks the set of control actions, $\underline{u}(t)$ (i.e. the system inputs), which drives the system state variable vector, $\underline{x}(t)$, from an initial point, $\underline{x}(t_o)$, to a final point, $\underline{x}(t_f)$, in the shortest time. In addition, it is assumed that these control actions have amplitude, integral, or quadratic constraints placed on them.

† The classical approach to the design and synthesis of automatic control systems falls into two categories: (1) a trial and error design procedure using z and Laplace transforms; (2) an infinite time integral of the error is used as a measure of performance.

In the early 1950's Bushaw[2] investigated the time-optimal regulator problem (i.e. problems in which $\underline{x}(t_f) = 0$). He used geometrical constructions in the phase plane to obtain solutions for second order systems with a magnitude constraint on the single input. Since that time, more general methods have been developed for solving optimal control problems.

In particular, Desoer[3] used the calculus of variations to formulate the time-optimal problem. Variational techniques were also used by Friedland[4], Lee[5], Leitmann[6], and Miele[7]. The latter reference contains an extensive bibliography in addition to a tutorial presentation of the subject with several practical examples.

Pontryagin employed variational calculus to develop the Maximum Principle[8] which differs from the classical problem of Mayer[†] in that it permits the control actions to be constrained. For very general dynamic systems the Maximum Principle provides an elegent and often powerful method of formulating and solving the following optimal control problem:

Given an n^{th} order system characterized by the set of differential equations

$$\dot{x}_i(t) = f_i(\underline{x}, \underline{u}, t) \qquad (i = 1, 2, \ldots n)$$

where x_i is the i^{th} component of the state variable vector $\underline{x}$, and let the

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[†] In the Mayer problem, one attempts to determine a function which minimizes a given function evaluated at the end points. In addition, the given function contains some variables whose final values are unspecified (e.g. t_f, the final time). Every control signal has unrestricted bounds.

initial state of the system, $x_i(t_o) = X_o^i$, be specified. The problem is to find $\underline{u}$ which minimizes or maximizes a given performance functional

$$I = \int_{t_o}^{t_f} F(\underline{x}, \underline{u}, t)\, dt$$

subject to certain physical constraints on $\underline{u}$ or $\underline{x}$ (or both).

To solve this problem, a new state variable is first introduced as

$$\dot{x}_{n+1} = F(\underline{x}, \underline{u}, t) \qquad\qquad x_{n+1}(t_o) = 0$$

so that the problem is exactly the same except that now, $x_{n+1}(t_f)$ is to be minimized or maximized. The next step involves defining a set of (n+1) adjoint differential equations†, i.e.

$$\dot{p}_i(t) = -\sum_{j=1}^{n+1} p_j(t) \frac{\partial f}{\partial x_i} \qquad\qquad (i = 1, 2, \ldots n+1)$$

$$p_i(t_f) = -b_i$$

where the b_i are a set of known constants. A Hamiltonian function is then defined as

$$H(\underline{x}, \underline{p}, \underline{u}, t) \triangleq \sum_{j=1}^{n+1} p_j f_j$$

The Maximum Principle states that the vector input, $\underline{u}$, which minimizes (maximizes) $x_{n+1}(t_f)$ is found by maximizing (minimizing) H with respect to $\underline{u}$ subject to the 2n+1 differential equations and their respective initial and terminal conditions.

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† The adjoint variables in the Maximum Principle formulation correspond to the Lagrange multipliers often used in the variational formulation of optimal control problems.

To solve this problem a two point boundary value problem is encountered since the terminal conditions of the adjoint variables and the initial conditions of the state variables are specified. In all but a few instances a digital computer must be used. As a result a good deal of research has been devoted to finding efficient numerical techniques for solving such problems[9-13]. For the most part, these techniques involve a trial and error procedure of guessing a set of $p_i(t_o)$ and solving the $(2n+1)$ differential equations to see if the given $p_i(t_f)$ are satisfied. If they are not a new set of $p_i(t_o)$ is guessed and the procedure is repeated. One major difficulty with such a technique is that a small change in $p_i(t_o)$ may cause a very large change in $p_i(t_f)$. Despite the computational difficulties which may be encountered, the advent of high speed digital computers has made its acceptance and use an established fact.

Tutorial presentations of the Maximum Principle have been given by Kopp[14] and Tou[15].

Another method of solving general optimal control problems, Dynamic Programming[16], was developed by Richard Bellman. The basic idea of this method is that a very difficult problem is imbedded into a class of simpler and more solvable problem. This concept, known as invariant imbedding, is specialized to optimization problems through the "Principle of Optimality"[16] which states: "an optimal control strategy has the property that, whatever the initial state and the initial decision, the remaining decisions must form an optimal control strategy with respect to the state resulting from the decision." This means that if we start with a multistage decision process[†], it is possible to replace the original multi-stage optimization by a sequence of single stage decision processes which are, in general, easier to solve.

Mathematically suppose that $M(\underline{x},\tau)$ is the smallest (or largest) value of some given performance index which can be attained from state $\underline{x}$ with time τ remaining. If $\underline{x}$ is changed from $\underline{x}_1$ to $\underline{x}_2$ by the transformation

$$\underline{x}_2 = g(\underline{x}_1, \underline{u}_1)$$

then using the Principle of Optimality gives

$$M(\underline{x}_2,\tau) = \underset{\text{w.r.t. } \underline{u}_1}{\text{min (or max)}} \{ M[g(\underline{x}_1, \underline{u}_1), \tau-1] \}$$

Performing the minimization (or maximization) with respect to $\underline{u}_1$ gives $\underline{u}_1^o$, the optimal control (policy) at state $\underline{x}_1$ with τ seconds remaining. The last equation states that the smallest (or largest) performance index which is attainable with τ seconds left is found by replacing $\underline{x}_2$ by $\underline{x}_1$ in M and then doing the best that can be done with respect to $\underline{u}_1$. It is noted that $M(\underline{x}_n,0)$ is the smallest (or largest) final value of the performance index which is attainable at the end of the n-stage decision process. A tutorial account of Dynamic Programming is given by Tou[19].

In general, a digital computer must be used to solve problems employing the Dynamic Programming method. A flooding technique[17,18] can be used whereby a number of optimal trajectories are obtained working backwards from $M(\underline{x},0)$. The trajectory which minimizes (or maximizes)

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† A multistage decision process is one that requires many consecutive decisions to reach the final goal.

$M(\underline{x}, \tau)$ for each value of τ is then chosen and the corresponding values of $\underline{u}$ form the optimal control. Since it is necessary to store all of the values of $\underline{x}$, $\underline{u}$, and $M(\underline{x}, \tau)$, it is quite possible to exceed the capacity of the computer memory when a multi-dimensional problem is attempted. Using a technique of polynomial approximations which involves expanding $M(\underline{x}, \tau)$ in a series of Legendre polynomials, the memory requirements can be significantly reduced[19].

The decision as to whether the Maximum Principle or Dynamic Programming is the better method is not clear cut. As is often the case, certain problems are best handled using the first technique while others are more easily solved using the second one.

In 1963, Kranc and Sarachik[21,22] introduced in this country another method of solving a class of optimal control problems based on certain results of functional analysis. As this is the method used in this dissertation, a more detailed treatment is required. This is deferred until the next section.

Recently, an interest has arisen in solving problems in which both input and state vectors of a system are constrained[23-30]. The physical motivation has come from a need to limit output quantities such as altitude or velocity of a rocket or airplane. While the literature is extensive when a magnitude constraint is placed on the system state vector, no work has been done on problems involving input constraints <u>and</u> integral or quadratic limitations on the state. However, a quadratic constraint on only the output is found in problems where the system mean square error is to be minimized[31].

When the states are magnitude constrained, the approach used to obtain a solution is to find the optimal inputs, consistent with the input constraints in the absence of the state constraints. These inputs are permitted to operate until one or more of the states reaches the boundary of its constraint. The inputs are then modified in such a manner as to keep these states (and any others which may reach their boundaries at some later time) either within or on the constraint boundaries. Obviously, the changes in the inputs must be consistent with both the input constraints and the terminal conditions. The resulting control actions often involve complicated switching surfaces and are, in general, difficult to realize in practice. There is, also, no differential condition or local test for determining when or where to leave the constraint boundaries and as a result it is necessary to use a trial and error procedure for locating these break away points. To date, one of the best treatments of this area is given by McIntyre and Paiewonsky[24] while a complete survey of the optimal control field including a comprehensive bibliography can be found in another paper written by Paiewonsky[32].

2.2 Mathematical Background

In this thesis certain techniques of functional analysis first applied to modern control problems in this country by Kranc and Sarachik[21,22] are used. Therefore, a detailed description of their work follows.

2.2.1 Minimum Norm Problem. Let the plant described by Eqs.(1-1) and repeated here for convenience be specified:

$$\underline{\dot{x}}(t) = E(t)\,\underline{x}(t) + F(t)\,\underline{u}(t) \tag{2-1a}$$

$$\underline{y}(t) = C(t)\,\underline{x}(t) + D(t)\,\underline{u}(t) \tag{2-1b}$$

Furthermore suppose that the initial state of the plant at time t_o is

$$\underline{x}(t_o) \triangleq \underline{X}_o$$

The problem is to find the set of inputs $\underline{u}$ which drives the output vector to $\underline{y}(t_f) \triangleq \underline{y}_d$ and which minimizes the following norm:

$$\|\underline{u}\|_p \triangleq \left[\sum_{i=1}^{r} L_i^{-p} \left(\|u_i\|_{p_i}\right)^p\right]^{1/p} \qquad (p \geq 1) \tag{2-2}$$

where the L_i's are a set of positive real numbers, t_f is a specified terminal time, and

$$\|u_i\|_{p_i} \triangleq \left[\int_{t_o}^{t_f} |u_i(t)|^{p_i}\, dt\right]^{1/p_i} \qquad (i = 1, 2, \ldots r) \tag{2-3}$$

The solution of this "minimum norm problem" proceeds in the following manner:

The vector output of a plant which is specified by Eq.(2-1) is known to be[40]

$$\underline{y}(t) = C(t)\,\phi(t,t_o)\underline{X}_o + \int_{t_o}^{t} H(t,\tau)\,\underline{u}(\tau)\, d\tau \tag{2-4}$$

where $\phi(t,t_o)$ is a fundamental matrix[41] of the system and the impulse response matrix[42], is given by

$$H(t,\tau) = C(t)\,\phi(t,\tau)\,F(\tau) + D(\tau)\,\delta(t-\tau) \tag{2-5}$$

Equation (2-4) is easier to handle if we let

$$\underline{e}(t) \triangleq \underline{y}(t) - C(t)\,\phi(t,t_o)\,\underline{X}_o \tag{2-6}$$

Thus

$$\underline{e}(t) = \int_{t_o}^{t} H(t,\tau)\,\underline{u}(\tau)\, d\tau \tag{2-7}$$

Using the definition in Eq.(2-6), the problem, equivalent to the one stated above, is to the one stated above, is to find the vector input which has a minimum norm, $\|\underline{u}\|_p$, and which satisfies the m-equations

$$e_{d_j} = \int_{t_o}^{t_f} \underline{h}_j(t_f, \tau)\,\underline{u}(\tau)\, d\tau \qquad (j=1,2,\ldots m) \qquad (2\text{-}8)$$

where e_{d_j} is the j^{th} component of

$$\underline{Y}_d - C(t_f)\,\phi(t_f, t_o)\,\underline{X}_o = \underline{e}_d \qquad (2\text{-}8a)$$

and $\underline{h}_j(t_f, \tau)$ is the j^{th} row of the impulse response matrix (Eq. 2-5).

As it has been assumed that the plant is completely output controllable (see Section 1.2), at least one vector input exists which satisfies the m-relations in Eq.(2-8). What is desired here is to find the one which minimizes $\|\underline{u}\|_p$.

The right hand side of Eq.(2-8) can be considered as a linear functional on $\underline{h}_j(t_f, \tau)$ and is denoted by $f(\underline{h}_j)$. Therefore,

$$e_{d_j} = f(\underline{h}_j) \qquad (2\text{-}9)$$

Now choose an m-vector $\underline{\lambda}$ with elements $\lambda_1, \lambda_2, \ldots \lambda_m$ and form the dot product $\underline{\lambda}' \cdot \underline{e}_d$, where the prime denotes transpose†. Making use of Eq.(2-9) gives

$$\underline{\lambda}' \cdot \underline{e}_d = \sum_{j=1}^{m} \lambda_j\, e_{d_j} = \sum_{j=1}^{m} \lambda_j\, f(\underline{h}_j) \qquad (2\text{-}10)$$

† These λ's are similar to the Lagrange multipliers which are employed in variational calculus.

Since $f(\cdot)$ is a linear functional, the right hand side of Eq.(2-10) becomes

$$\sum_{j=1}^{m} \lambda_j f(\underline{h}_j) = f\left[\sum_{j=1}^{m} \lambda_j \underline{h}_j\right] = f\left[\underline{\lambda}' \cdot H(t_f, \tau)\right] \tag{2-11}$$

Defining

$$\underline{k}(t_f, \tau) \triangleq \underline{\lambda}' \cdot H(t_f, \tau) \equiv [k_1, k_2, \ldots k_r] \tag{2-12a}$$

$$k_j \triangleq \sum_{i=1}^{m} \lambda_i h_{ij}(t_f, \tau) \tag{2-12b}$$

where h_{ij} is the i,j^{th} element of $H(t_f, \tau)$, Eq. (2-11) becomes

$$\underline{\lambda}' \cdot \underline{e}_d = f(\underline{k}) \tag{2-13}$$

It is important to note that Eq.(2-9) is implied if Eq.(2-11) [and hence (2-13)] is satisfied for <u>every</u> arbitrary set of λ's.

Using the definition of the functional f, Eq.(2-13) can be rewritten as

$$\underline{\lambda}' \cdot \underline{e}_d = \int_{t_o}^{t_f} \underline{k}(t_f, \tau)\, \underline{u}(\tau)\, d\tau = \int_{t_o}^{t_f} \sum_{i=1}^{r} k_i(t_f, \tau) u_i(\tau) d\tau$$

$$\underline{\lambda}' \cdot \underline{e}_d = \int_{t_o}^{t_f} \sum_{i=1}^{r} L_i\, k_i(t_f, \tau)\, L_i^{-1}\, u_i(\tau)\, d\tau \tag{2-14}$$

where the last form is written for convenience.

Taking magnitudes of both sides of Eq.(2-14) and applying a version of Hölder's Inequality which is valid for sums (see Appendix I), we have

$$|\underline{\lambda}' \cdot \underline{e}_d| \le \|\underline{u}\|_p \, \|\underline{k}\|_q \tag{2-15}$$

where $\|\underline{u}\|_p$ is defined in Eq.(2-2),

$$\|\underline{k}\|_q \triangleq \left[\sum_{i=1}^{r} L_i^q \left(\|k_i\|_{q_i}\right)^q\right]^{1/q} \tag{2-16}$$

$$\|k_i\|_{q_i} \triangleq \left[\int_{t_o}^{t_f} |k_i(t_f, \tau)|^{q_i} \, d\tau\right]^{1/q_i} \tag{2-17}$$

and

$$\frac{1}{q} + \frac{1}{p} = 1 \tag{2-18a}$$

$$\frac{1}{q_i} + \frac{1}{p_i} = 1 \qquad (i=1,2,\ldots r) \tag{2-18b}$$

Therefore

$$\|\underline{u}\|_p \geq \frac{|\underline{\lambda}' \cdot \underline{e}_d|}{\|\underline{k}\|_q} \tag{2-19}$$

Since Eq.(2-13) must be satisfied for any set of λ's, so too must Inequality (2-19). It is observed that since the value of the right hand side of (2-19) is dependent upon the set of λ's chosen, this inequality is never violated if

$$\|\underline{u}\|_p \geq \max_{\lambda_i} \frac{|\underline{\lambda}' \cdot \underline{e}_d|}{\|\underline{k}\|_q} \tag{2-20}$$

Then the value of the minimum norm becomes

$$(\|\underline{u}\|_p)_{min} = \max_{\lambda_i} \frac{|\underline{\lambda}' \cdot \underline{e}_d|}{\|\underline{k}\|_q} = \max_{\substack{\text{w.r.t.}\,\lambda_i \\ \text{subject} \\ \underline{\lambda}' \cdot \underline{e}_d = 1}} \frac{1}{\|\underline{k}\|_q} = \frac{1}{\min\limits_{\substack{\lambda_i \\ \lambda' \cdot \underline{e}_d = 1}} \|\underline{k}\|_q} \tag{2-21}$$

Denoting by $\underline{\lambda}^*$ the set of λ's which minimizes $\|\underline{k}\|_q$ Eq. (2-21) becomes

$$(\|u\|_p)_{min} = \frac{1}{\|\underline{k}\|_q\big|_{\underline{\lambda}=\underline{\lambda}^*}} = \frac{1}{\|k^*\|_q} = \frac{|\underline{\lambda}^{*'} \cdot \underline{e}_d|}{\|\underline{k}^*\|_q} \tag{2-22}$$

In order for Eq.(2-22) to be valid, the inequality

$$\|\underline{u}\|_p \, \|\underline{k}^*\|_q \geq |f(\underline{k}^*)| = |\underline{\lambda}^{*'} \cdot \underline{e}_d| = 1$$

must be satisfied by the sign of equality, i.e.

$$\|\underline{u}\|_p = \frac{1}{\|\underline{k}^*\|_q} \tag{2-23}$$

This is true if and only if (see Appendix I) the elements of the input vector are

$$u_i(\tau) = KL_i^q \left(\| k_i^* \|_{q_i} \right)^{q-q_i} \left| k_i^*(t_f,\tau) \right|^{q_i-1} \mathrm{SGN}\left[k_i^*(t_f,\tau) \right] \qquad (i=1,2,\ldots r) \qquad (2\text{-}24)$$

where K is a positive constant, SGN (ξ) = +1 for $\xi > 0$, SGN (ξ) = −1 for $\xi < 0$, and the SGN is an undefined function whose magnitude is less than one when $\xi = 0$. K is evaluated by using Eq.(2-24) in Eq.(2-22). Thus

$$K = \frac{1}{\left(\| \underline{k}^* \|_q \right)^q}$$

and the inputs which satisfy the terminal conditions (Eq.2-8) and minimize the norm in Eq.(2-2) become

$$u_i(\tau) = \frac{L_i^q}{\left(\| \underline{k}^* \|_q \right)^q} \left(\| k_i^* \|_{q_i} \right)^{q-q_i} \left| k_i^*(t_f,\tau) \right|^{q_i-1} \mathrm{SGN}\left[k_i^*(t_f,\tau) \right] \qquad (i=1,2,\ldots r) \qquad (2\text{-}25)$$

2.2.2 Constrained Inputs. Suppose that the problem in the previous section is now changed so that each of the plant inputs has a separate constraint on it expressed in the form

$$\| u_i \|_{p_i} \triangleq \left[\int_{t_o}^{t_f} \left| u_i(\tau) \right|^{p_i} d\tau \right]^{1/p_i} \le L_i \qquad (i=1,2,\ldots r) \qquad (2\text{-}26)$$

Sarachik and Kranc[22] refer to this as a problem with "Multi-Norm" constraints.

It is observed that the p_i and L_i may be different from each i implying that each element of $\underline{u}$ may have a different type of constraint placed on it†.

This problem is solved utilizing the results of the previous section. Consider the following norm:

$$\|\underline{u}\| = \max_i \left[\frac{\|u_i\|_{p_i}}{L_i} \right] \tag{2-27}$$

A little thought will convince the reader that when the single condition

$$\|\underline{u}\| \leq 1 \tag{2-28}$$

is satisfied, the r-conditions in Eq.(2-26) are also satisfied.

Kirillova[43] has shown that in the limit as $p \rightarrow \infty$, the solution with the p^{th} norm of $\underline{u}$ (Eq. 2-21) constrained in the following manner

$$\|\underline{u}\|_p = \left[\sum_{i=1}^{r} L_i^{-p} (\|u_i\|_{p_i})^p \right]^{1/p} \leq 1 \qquad (p \geq 1) \tag{2-29}$$

approaches the solution with $\underline{u}$ constrained as in Eq.(2-28).

From Eq.(2-18a), $q = 1$ when $p = \infty$. Hence, the inputs (i.e. the optimal) which satisfy the terminal conditions in Eq.(2-8) and the constraints in Eq.(2-26) are

$$u_i(\tau) = \frac{L_i}{\|\underline{k}^*\|_1} (\|k_i^*\|_{q_i})^{1-q_i} \, |k_i^*(t_f,\tau)|^{q_i-1} \, \mathrm{SGN}\,[k_i^*(t_f,\tau)] \qquad (i=1,2\ \ldots r) \tag{2-30}$$

where the starred quantities are obtained by finding λ^*'s which give

$$\|\underline{k}^*\|_1 = \begin{matrix} \text{Min w.r.t. } \lambda_i \\ \text{subject to} \\ \underline{\lambda}' \cdot \underline{e}_d = 1 \end{matrix} \left[\sum_{i=1}^{r} L_i \|k_i\|_{q_i} \right]$$

† For example, u_i has a magnitude constraint on it, $p_i = \infty$; if it is constrained in area (i.e. a fuel constraint), $p_i = 1$; and if it is constrained in energy, $p_i = 2$.

2.2.3 The Time-Optimal Solution. If the terminal time is permitted to vary and it is required that the terminal conditions (Eq.2-8) and the input constraints (Eq.2-26) be satisfied in the minimum elapsed time, $T^* = t_f^* - t_o$, a time-optimal, multi-norm problem results. The solution of this problem is again found from the minimum norm solution of Section 2.2.1.

Combining Eq.(2-23) (with p = ∞ and q = 1) and Eq.(2-28) gives

$$\frac{1}{\|\underline{k}^*\|_1} \le 1$$

or

$$\|\underline{k}^*\|_1 \ge 1 \tag{2-32}$$

When $\|\underline{k}^*\|_1$ is a continuous function of t_f the minimum terminal time, t_f^*, (and hence the minimum elapsed time) is achieved for the smallest value of t_f which makes

$$\|\underline{k}^*\|_1 = 1 \tag{2-33}$$

Under this condition, the time-optimal inputs are found from Eq.(2-30) to be

$$u_i(\tau) = L_i \left(\|k_i^*\|_{q_i}\right)^{1-q_i} \left|k_i^*(t_f^*, \tau)\right|^{q_i - 1} \mathrm{SGN}\,[k_i^*(t_f^*, \tau)]$$

$$(i = 1, 2, \ldots r) \quad (2\text{-}34)$$

where the λ^*s are obtained from Eq.(2-31) and t_f^* is the minimum value of t_f which satisfies Eq.(2-33).

It is easily verified by direct substitution that the time-optimal inputs in Eq.(2-34) satisfy the constraints in Eq.(2-26) with the sign of equality. Clearly then, equality of the input constraints is a necessary condition for a time-optimal solution to exist. In addition, Sarachik and

Kranc[22] showed that the necessary and sufficient condition for an optimal solution to exist is that Eq.(2-32) be satisfied for some t_f with the time-optimal inputs being given by Eq.(2-34).

2.2.4 Special Cases

(a) $L_i = L$ and $p_i = p$. If all of the $L_i = L$ and the $p_i = p$ the input norm in Eq.(2-29) becomes

$$\|\underline{u}\|_p = \frac{1}{L}\left[\int_{t_o}^{t_f}\sum_{i=1}^{r} |u_i(\tau)|^p \, d\tau\right]^{1/p} \le 1 \tag{2-35}$$

The time-optimal problem with this input constraint was considered by Kranc and Sarachik[21]. Its solution is easily obtained from the results of Section 2.2.1.

Using the same argument as that preceding Eq.(2-33), t_f^* is the smallest value of t_f which satisfies

$$\|\underline{k}^*\|_q = L\left[\int_{t_o}^{t_f}\sum_{i=1}^{r} |k_i^*(t_f,\tau)|^q \, d\tau\right]^{1/q} = 1 \tag{2-36}$$

The time-optimal inputs which satisfy the constraints in Eq.(2-35) and the terminal condition in Eq.(2-8) are found from Eq.(2-25) to be

$$u_i(\tau) = L^q \,|k_i^*(t_f^*,\tau)|^{q-1} \, \mathrm{SGN}\,[k_i^*(t_f^*,\tau)] \quad (i=1,2,\ldots r) \tag{2-37}$$

where the λ^*'s are obtained by minimizing $\|\underline{k}\|_q$ with respect to λ_i subject to the condition $\underline{\lambda}' \cdot \underline{e}_d = 1$ and t_f^* is found from

$$\|\underline{k}^*\|_q = L\left[\int_{t_o}^{t_f}\sum_{i=1}^{r} |k_i^*(t_f,\tau)|^q \, d\tau\right]^{1/q} = 1 \tag{2-38}$$

It is important to note that the minimization of $\|\underline{k}\|_q$ is by no means a trivial problem.

(b) Single Output Plants . When a linear system has only one output, a significant simplification in finding the time-optimal inputs and t_f^* results. The reason for this is that it is not necessary to minimize $\|\underline{k}\|_q$ with respect to the single lambda since this parameter is already known from the condition, $\underline{\lambda}' \cdot \underline{e}_d = 1$. Using this fact, the time-optimal inputs which satisfy the multinorm constraints in Eq.(2-26) become

$$u_i(\tau) = L_i(\|h_i\|_{q_i})^{1-q_i} \, |h_i(t_f^*, \tau)|^{q_i-1} \, \mathrm{SGN}\left[\frac{h_i(t_f^*, \tau)}{e_d}\right] \tag{2-39}$$

where t_f^* is the smallest value of t_f satisfying

$$\sum_{i=1}^{r} L_i \|h_i\|_{q_i} = |e_d| \tag{2-40}$$

If the input constraint in Eq.(2-35) is used in place of the multinorm constraint, Eq.(2-39) further simplifies to

$$u_i(\tau) = L^q \left|\frac{h_i(t_f^*, \tau)}{e_d}\right|^{q-1} \mathrm{SGN}\left[\frac{h_i(t_f^*, \tau)}{e_d}\right] \tag{2-41}$$

where t_f^* is found from

$$\left[\int_{t_o}^{t_f}\left[\sum_{i=1}^{r} |h_i(t_f, \tau)|^q\right] d\tau\right]^{1/q} = \frac{|e_d|}{L} \tag{2-42}$$

It is significant to note that when the plant has only one output, the computations which are required to find t_f^* and the time-optimal inputs are routine irrespective of the order of the system. This is to be compared with the computational labor† which is necessary when either Dynamic Programming

† For example, the solution of a two-point boundary value problem is required when the Maximum Principle is employed.

or the Maximum Principle are used to solve the same type of problem.

(c) Time Invariant Plants. If a linear plant is non-time-varying, the results of the previous sections are applicable. The arbitrary initial time, t_o, is replaced by 0, the terminal time becomes T, and the expression (t_f, τ) becomes $(T-\tau)$ in all of the equations.

2.2.5 Example. The following example illustrates the methods outlined in the previous sections:

Consider the linear time-invariant plant shown in Figure 2-1. Defining the state variables as shown, the normal form representation of this plant is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (2E\text{-}1a)$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x} \qquad (2E\text{-}1b)$$

The fundamental matrix of this system can be shown to be

$$\phi(t) = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \qquad (2E\text{-}2)$$

Using Eq.(2-5) and Eq.(2E-2), the impulse response matrix is

$$H(t) = C\phi\,(t)\,F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t^2/2 \\ t \end{bmatrix} \qquad (2E\text{-}3)$$

whereupon

$$h_1(t) = t^2/2 \qquad (2E\text{-}4a)$$

$$h_2(t) = t \qquad (2E\text{-}4b)$$

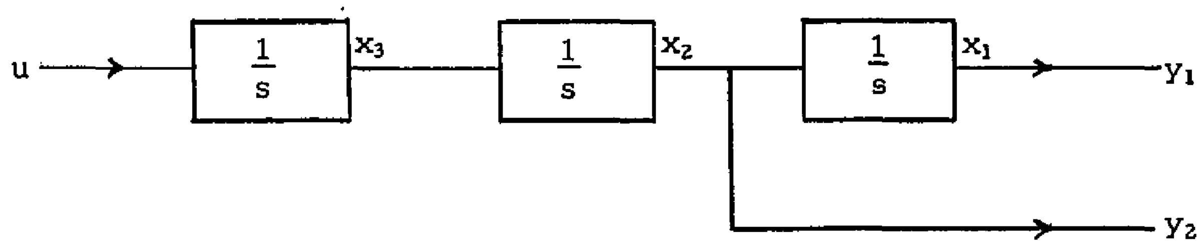


Figure 2-1 Third-order Plant

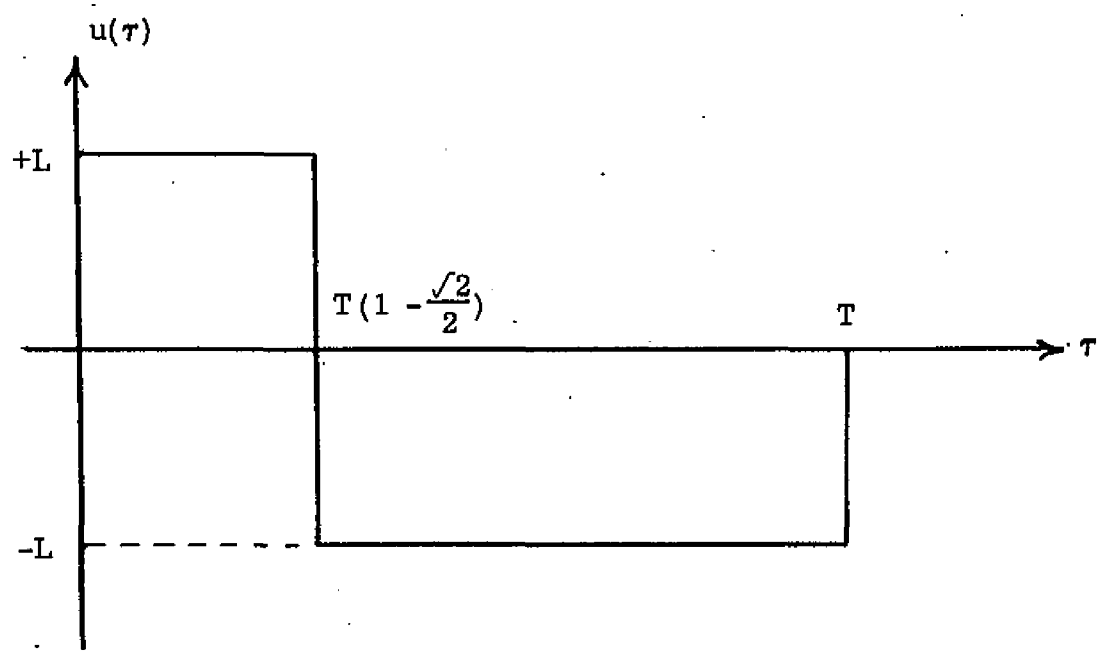


Figure 2-2 Time-optimal Input Used to Control the Plant in Figure 2-1

Now suppose that u is constrained in Magnitude, i.e.

$$|u| \le L \tag{2E-5}$$

and that it is desired to find the input which drives the state variable vector from the initial point

$$\underline{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{2E-6}$$

to the origin in the shortest time while satisfying Eq.(2E-5).

Using Eq.(2-6) at t = T we find that

$$\underline{e}_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

or

$$\underline{e}_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2E-7}$$

From the definition of $\underline{k}(t)$ (Eq.2-12), we have

$$\underline{k}(t) = \underline{\lambda}' \cdot H(t) = \lambda_1 \frac{t^2}{2} + \lambda_2\, t \tag{2E-8}$$

Since $\|\underline{k}\|_q$ must be minimized with respect to λ_1 subject to the condition $\underline{\lambda}' \cdot \underline{e}_d = 1$, we find that

$$\underline{\lambda}' \cdot \underline{e}_d = [\lambda_1\ \lambda_2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 = 1 \tag{2E-9}$$

Using this result together with Eq.(2E-8) in Eq.(2-38) gives

$$\|\underline{k}\|_q = L \left[\int_0^T \left|\frac{t^2}{2} + \lambda_2 t\right|^q dt \right]^{1/q}$$

As u is constrained in magnitude, $p = \infty$ and therefore, $q = 1$. The expression to be minimized becomes

$$\|\underline{k}\|_1 = L \int_0^T \left|\frac{t^2}{2} + \lambda_2 t\right| dt \tag{2E-10}$$

Carrying out the integration, we find that for $\lambda_2 > -\frac{T}{2}$

$$\|\underline{k}\|_1 = L\left[\frac{T^3}{6} + \frac{T^2}{2}\lambda_2 - \frac{4}{3}\lambda_2^3\right] \tag{2E-11}$$

Taking the partial derivative of this expression with respect to λ_2 and equating to zero gives

$$L\left[\frac{T^2}{2} - 4(\lambda_2)^2\right] = 0$$

or

$$\lambda_2^* = -\frac{T}{\sqrt{8}} \tag{2E-12}$$

To find the minimum time, Eq.(2E-12) is substituted into Eq.(2E-11) and the resulting expression is equated to 1. Thus

$$L T^3 \left[\frac{\sqrt{2} - 1}{6\sqrt{2}}\right] = 1 \tag{2E-13}$$

Using Eq.(2E-12) in Eq.(2-37) the time-optimal input is

$$u(\tau) = L \ \mathrm{SGN}\left[\frac{(T-\tau)^2}{2} - \frac{T(T-\tau)}{\sqrt{8}}\right] \tag{2E-14}$$

or graphically as shown in Figure 2-2.

III ON THE RELATIONSHIP BETWEEN t_f^* AND $\underline{\lambda}^*$

3.1 Introduction

In Section 2.2.4 it was noted that it is usually difficult to perform the minimization of $\|\underline{k}\|_q$ with respect to λ_i necessary when the Functional Analysis approach is used in obtaining the optimal solution for multiple output plants. In this chapter, it is shown that the λ's which minimize $\|\underline{k}\|_q$ subject to the condition

$$\underline{\lambda}' \cdot \underline{e}_d = 1$$

maximize the terminal time, t_f, in $(t_f, \underline{\lambda})$-space. If $\|\underline{k}^*\|_q$ is a non-decreasing function of t_f, the point $(t_f^*, \underline{\lambda}^*)$ is a unique maximum in this space. On the other hand, if $\|\underline{k}^*\|_q$ is not a non-decreasing function of t_f, this point is the smallest of the local maxima. From these results, standard hill climbing methods can be used to calculate t_f^* and $\underline{\lambda}^*$. Two such methods are illustrated. In addition, it is shown that for an important class of linear systems, $\|\underline{k}^*\|_q$ is a non-decreasing function of t_f.

3.2 A Theorem on the Relationship Between $\underline{\lambda}^*$ and t_f^*

It is recalled (Section 2.2-4) that in order to find the time-optimal inputs (Eqs.2-37) which drive the outputs from an initial to a final point in output space in the shortest terminal time t_f^*, the expression (Eq.2-38)

$$\|\underline{k}\|_q = \left[\int_{t_o}^{t_f} \sum_{j=1}^{r} |k_j(t_f,\tau)|^q d\tau\right]^{1/q} = \left[\sum_{j=1}^{r} \int_0^{t_f-t_o} |k_j(t_f, t_f-\tau)|^q d\tau\right]^{1/q} \quad (3\text{-}1)$$

must first be minimized with respect to the λ's subject to the condition

$$\underline{\lambda}' \cdot \underline{e}_d = 1 \tag{3-2}$$

t_f^* is then the smallest value of t_f which satisfies Eq.(2-36), i.e.

$$\|\underline{k}\|_q \Big|_{\underline{\lambda}=\underline{\lambda}^*} \triangleq \|\underline{k}^*\|_q = \frac{1}{L} \tag{3-3}$$

For convenience, let

$$\varphi(t_f, \underline{\lambda}) \triangleq \|\underline{k}\|_q \Big|_{\underline{\lambda}' \cdot \underline{e}_d = 1} \tag{3-4}$$

$\varphi(t_f, \underline{\lambda})$ is therefore a function of (m-1)-lambdas. In addition, let

$$\psi(t_f) = \min_{\underline{\lambda}} [\varphi(t_f, \underline{\lambda})] = \varphi(t_f, \underline{\lambda}^*[t_f]) \tag{3-5}$$

so that t_f^* is the smallest value of t_f which satisfies the expression

$$\psi(t_f) = \frac{1}{L} \tag{3-6}$$

It is important to note that the lambdas which minimize $\varphi(t_f, \underline{\lambda})$ are functions of t_f and are therefore written $\underline{\lambda}^*(t_f)$.

Using these definitions, the proof of the following theorem is now given:

<u>THEOREM</u>

(a) If $\psi(t_f)$ is a non-decreasing function of t_f, the point $(t_f^*, \underline{\lambda}^*)$ is a unique maximum† of $t_f(\underline{\lambda})$.

(b) If $\psi(t_f)$ is <u>not</u> a non-decreasing function of t_f, the point $(t_f^*, \underline{\lambda}^*)$ is the smallest of the local maxima of $t_f(\underline{\lambda})$ where $t_f(\underline{\lambda})$ is obtained by solving for t_f in the equation, $\varphi(t_f, \underline{\lambda}) = \frac{1}{L}$.

---

† By maximum, we mean that $t_f < t_f^*$ for $\underline{\lambda} = \underline{\lambda}^*$.

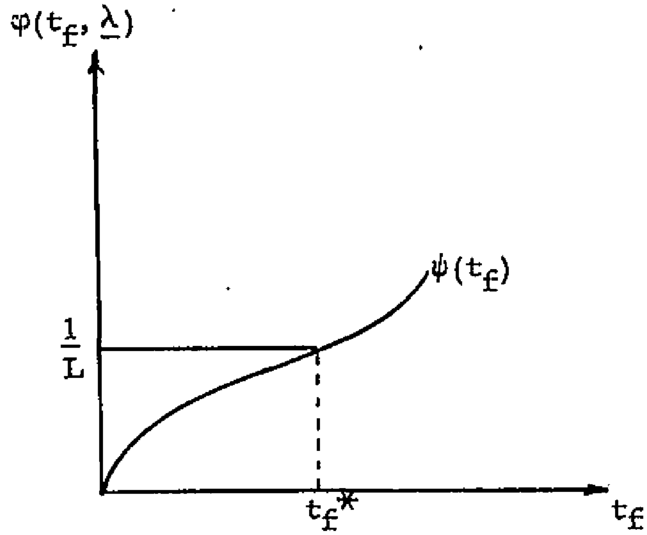
Proof

(a) As $\psi(t_f)$ is assumed to be a non-decreasing function of t_f, there is only one value of t_f which satisfies Eq.(3-6). See Figure 3-1a. Since the $\lambda^*(t_f)$'s minimize $\varphi(t_f,\underline{\lambda})$ at any time, t_f, any other set of lambdas (say $\underline{\lambda}_a$) causes $\varphi(t_f,\underline{\lambda}_a) > \psi(t_f)$, i.e. the curve of $\varphi(t_f,\underline{\lambda}_a)$ versus t_f is always above the curve of $\psi(t_f)$ versus t_f. See Figure 3-1b. From this figure, it is seen that $\varphi(t_f,\underline{\lambda}_a) = \frac{1}{L}$ when $t_f = t_{f_a}$ where $t_{f_a} < t_f^*$. Since $\underline{\lambda}_a$ represents any set of lambda's which is not equal to $\underline{\lambda}^*$, the point $(t_f^*,\underline{\lambda}^*)$ is clearly a maximum in $(t_f,\underline{\lambda})$-space. To show that this point is unique, suppose that another maximum exists at $(t_{f_a}, \underline{\lambda}_a)$. Then for any other set of lambdas in the neighborhood of $\underline{\lambda}_a$ (e.g. $\underline{\lambda}_b$), $\varphi(t_f,\underline{\lambda}_b) > \varphi(t_f,\underline{\lambda}_a)$. (See Figure 3-1b.) That is, in this neighborhood, $\underline{\lambda}_a$ minimizes $\varphi(t_f,\underline{\lambda})$. Now, from Eq.(2-21)

$$(\|\underline{u}\|_p)_{min} = \frac{1}{\underset{\substack{w.r.t.\\ \lambda_i}}{\min} \varphi(t_f,\underline{\lambda})} \qquad (2\text{-}21)$$

it is seen that the lambda stars not only minimize $\varphi(t_f,\underline{\lambda})$ but also minimize the input norm, $\|\underline{u}\|_p$. Since $\varphi(t_f,\underline{\lambda}_a) > \varphi(t_f,\underline{\lambda}^*)$, for any t_f, $\|\underline{u}\|_p\Big|_{\underline{\lambda}=\underline{\lambda}_a} < \|\underline{u}\|_p\Big|_{\lambda=\underline{\lambda}^*}$ which is clearly not permitted. Thus the point $(t_{f_a},\underline{\lambda}_a)$ cannot be a maximum and hence, $(t_f^*,\underline{\lambda}^*)$ is unique.

(b) As $\psi(t_f)$ is not a non-decreasing function of t_f, there may be several values of t_f which satisfy Eq.(3-6). See Figure 3-2.



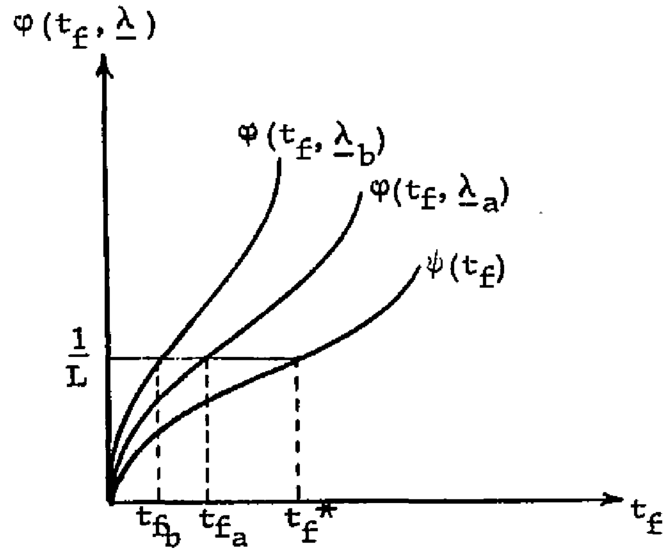


Fig. 3-1 (a) Non-decreasing $\psi(t_f)$ versus t_f

(b) $\psi(t_f)$, $\varphi(t_f, \underline{\lambda}_a)$, and $\varphi(t_f, \underline{\lambda}_b)$ versus t_f .

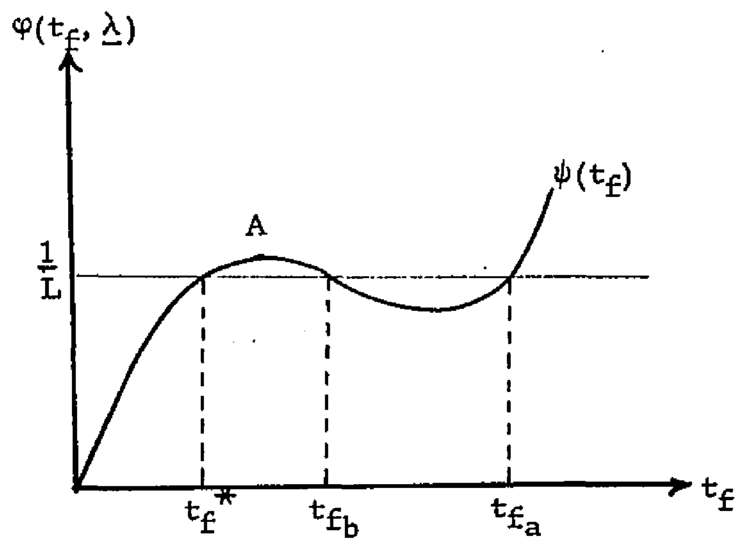


Fig. 3-2 $\psi(t_f)$ versus t_f when $\psi(t_f)$ is <u>not</u> a non-decreasing function of t_f

Using the same argument as in part (a)[†], it is apparent that $(t_f^*, \underline{\lambda}^*(t_f^*))$ and $(t_{f_b}, \underline{\lambda}^*(t_{f_b}))$ are both maxima[††]. The point $(t_f^*, \underline{\lambda}^*)$ is seen to be the smallest of the local maximum in $(t_f, \underline{\lambda})$-space. This completes the proof.

Remark

When $q = 1$ or 2 in Eq.(3-4), the partial derivatives, $\frac{\partial t_f}{\partial \lambda_i}$ $(i = 1, 2, \ldots m-1)$ are continuous in the neighborhood of $(t_f^*, \underline{\lambda}^*)$ provided the system impulse responses do not contain "coast zones". See Fig.3-3a. If "coast zones" are present in the impulse responses, $\varphi(t_f, \underline{\lambda})$ has horizontal sections as illustrated in Figure 3-3b. Discontinuous partial derivatives result if the value of L is such that the equation $\psi(t_f^*) = \frac{1}{L}$ is satisfied on the flat portion of the curve.

When $q = \infty$, $\psi(t_f)$ may be discontinuous. If the situation depicted in Figure 3-4 occurs, where Eq.(3-6) is satisfied on the discontinuous portion of $\psi(t_f)$, the partial derivatives are zero for $t_f = t_f^*$ for a range of values of the lambda's.

The following lemma gives the sufficient condition for $\psi(t_f) \equiv \|\underline{k}^*\|_q$ to be a non-decreasing function of t_f for non-time-varying systems.

---

† In this instance, $\underline{\lambda}_a$ is in the neighborhood of $\underline{\lambda}^*(t_f^*)$.

†† If the value of L is such that $\psi(t_f)$ is tangent to the horizontal line, $\frac{1}{L}$, at point A, a saddle point results. We will assume that this does not occur.

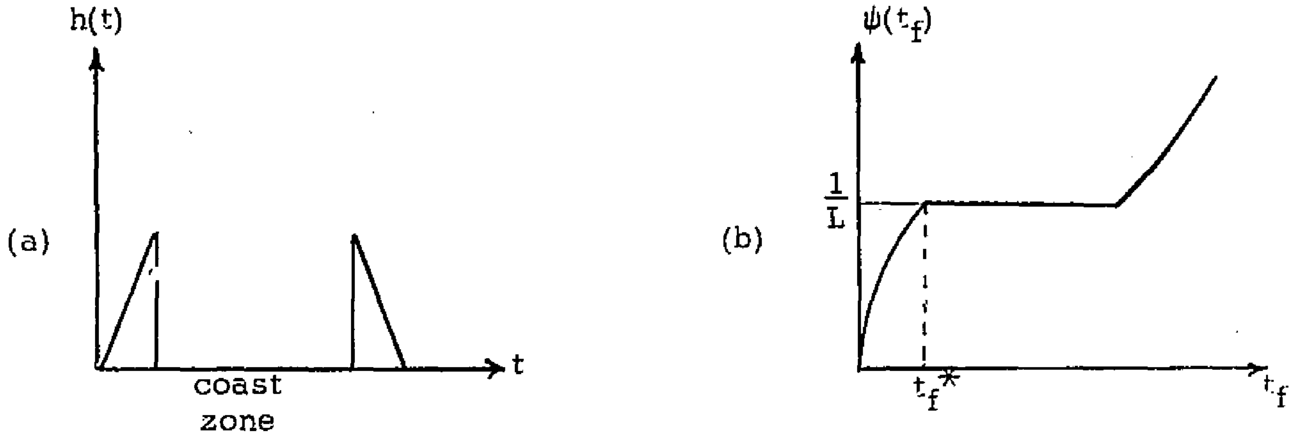


Figure 3-3 (a) Illustration of a "coast zone" in the System Impulse Response

(b) Illustration of a case where $\psi(t_f)$ is constant

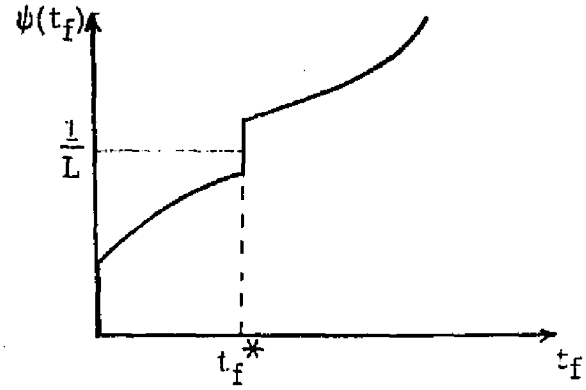


Figure 3-4 Illustration of a discontinuous $\psi(t_f)$

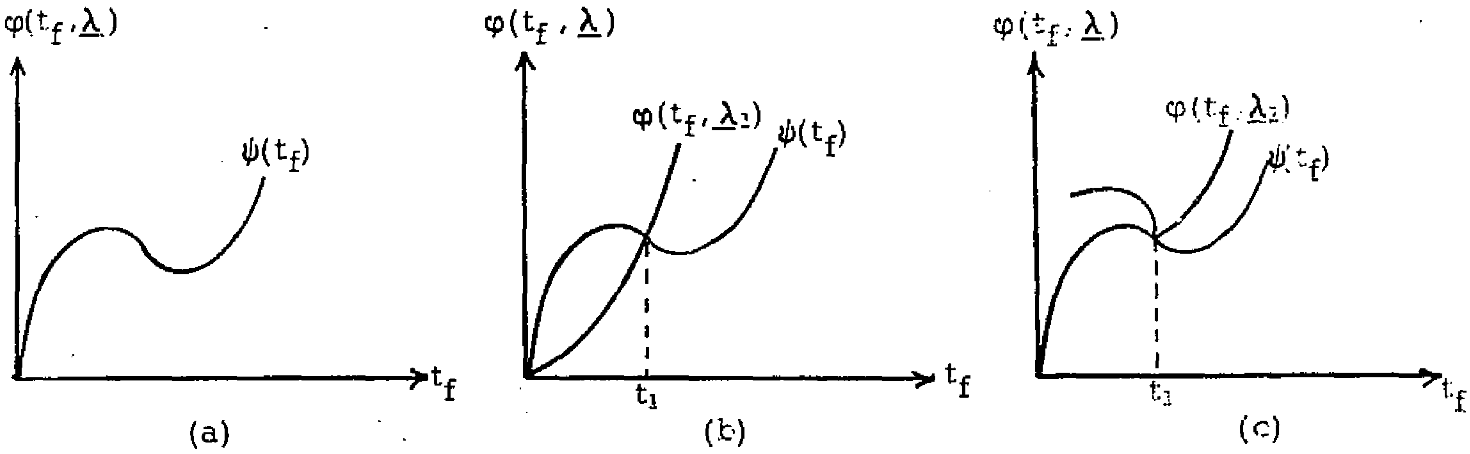


Figure 3-5 (a) $\psi(t_f) = \varphi[t_f, \lambda^*(t_f)]$ versus t_f

(b) Illustration of non-decreasing behavior of $\varphi[t_f, \lambda^*(t_1)]$

(c) Illustration of Inequality (3-9)

Lemma 1.

The sufficient condition for $\psi(t_f)$ to be a non-decreasing function of t_f for non-time-varying systems is that $\underline{e}_d$ be independent of t_f.

Proof (by contradiction)

From the definition of $\underline{k}$ (Eq.2-12a), the expression for $\|\underline{k}\|_q$ valid for non-time-varying systems is

$$\|\underline{k}\|_q = \|\underline{\lambda}' \cdot H(t_f-t)\|_q = \|\sum_{i=1}^{m} \lambda_i \underline{h}_i (t_f-t)\|_q = \|\sum_{i=1}^{m} \lambda_i \underline{h}(t)\|_q$$

$$\|\underline{k}\|_q = \left[\sum_{j=1}^{r} \int_0^{t_f} \left|\sum_{i=1}^{m} \lambda_i h_{ij}(t)\right|^q dt\right]^{1/q} \tag{3-7}$$

where h_{ij} is the $(i,j)^{th}$ element of the impulse response matrix. Making use of Eq.(3-2) to eliminate one of the lambda's in Eq.(3-7), (e.g. λ_1), we obtain

$$\varphi(t_f,\underline{\lambda}) = \left[\sum_{j=1}^{r} \int_0^{t_f} \left|\frac{h_{1j}(t)}{e_{d_1}} + \sum_{i=2}^{m} \lambda_i \left(h_{ij}(t) - \frac{e_{d_i}}{e_{d_1}} h_{1j}(t)\right)\right|^q dt\right]^{1/q} \tag{3-8}$$

For the $\underline{\lambda}^*(t_f)$'s which minimize $\varphi(t_f,\underline{\lambda})$ in Eq.(3-8), suppose that $\psi(t_f)$ is not a non-decreasing function of t_f as shown in Figure 3-5a. Corresponding to $t_f = t_1$, there is a specific set of values of the lambda-stars, i.e. $\underline{\lambda}^*(t_1) = \underline{\lambda}_1$. If these lambda's are used in Eq.(3-8), the integrand of the integral in this equation is not a function of t_f, (since the e_{d_i} $(i=2,3,\ldots m)$ are not functions of t_f) so that $\varphi(t_f,\underline{\lambda}_1)$ is a non-decreasing function of t_f. In addition, it coincides with $\psi(t_f)$ at $t_f = t_1$. This behaviour is depicted in Figure 3-5b.

Now, for any time $t_f \neq t_1$, $\underline{\lambda} = \underline{\lambda}_1$ does not minimize $\varphi(t_f,\underline{\lambda})$ so that

$$\varphi(t_f, \underline{\lambda_1}) \geq \psi(t_f) \tag{3-9}$$

equality occurring only for $t_f = t_1$. See Figure 3-5c.

Clearly, for $t_f < t_1$, $\varphi(t_f, \underline{\lambda_1})$ cannot satisfy inequality (3-9) and be a non-decreasing function of t_f at the same time. Therefore, the original assumption that $\psi(t_f)$ is not a non-decreasing function of t_f leads to a contradiction. It is easily seen that if $\psi(t_f)$ is a non-decreasing function of t_f, both of these conditions can be simultaneously satisfied. This completes the proof.

Remark

If the e_{d_i} are functions of t_f, $\varphi(t_f, \underline{\lambda_1})$ may not be a non-decreasing function of t_f. In this instance the above contradiction is not encountered and $\psi(t_f)$ may not be a non-decreasing function of t_f.

The following lemma defining two classes of systems for which the e_{d_i} are not functions of t_f is now proved:

Lemma 2.

The e_{d_i} are not functions of t_f for

(1) initially quiescent systems

(2) regulator systems without feed through connections.

Proof

(1) Using the definition of $\underline{e}_d$ given in Eq.(2-8a) for non-time-varying systems, we have

$$\underline{e}_d = \underline{Y}_d - C\,\phi(t_f)\,\underline{X}_o$$

where $\phi(t)$ is the fundamental matrix of the system, $\underline{X}_o$ is the vector of initial states, and $\underline{y}_d$ is the vector of the final specified values of the system outputs. Obviously, if $\underline{X}_o \equiv 0$ (i.e. the system is initially quiescent) $\underline{e}_d$ is not a function of t_f.

(2) Consider a class of non-time-varying systems which has no feed-through connections (i.e. D = 0 in Eq.2-1b) and furthermore, suppose that the outputs of this system are identified with the states (i.e. C = I, the identity matrix in Eq.2-1b). Then using Eqs.(2-4) and (2-5) at $t = t_f$ gives

$$\underline{y}(t_f) \equiv \underline{x}(t_f) = \phi(t_f)\underline{X}_o + \int_o^{t_f} \phi(t_f-\tau)\, B\, \underline{u}(\tau)\, d\tau \tag{3-10}$$

Premultiplying both sides of this equation by $\phi(-t_f)$ and recalling that $\phi(-t_f) = \phi^{-1}(t_f)$, we have

$$\phi(-t_f)\,\underline{x}(t_f) = \underline{X}_o + \int_o^{t_f} \phi(-\tau)\, B\, \underline{u}(\tau)\, d\tau$$

For this class of systems, we define

$$\underline{e}_d \triangleq \phi(-t_f)\,\underline{x}(t_f) - \underline{X}_o$$

Under these circumstances, $\underline{e}_d$ is not a function of t_f provided $\underline{x}(t_f) \equiv 0$, i.e. the system is to be driven from some initial state, $\underline{X}_o$, to the origin. Such systems are commonly referred to as regulators. This completes the proof.

Remark

If the system is initially quiescent, the outputs are initially zero. However, the requirement that the outputs be initially zero does not necessarily imply that the system is initially quiescent unless the outputs are linearly independent combinations of the states, i.e. det[C] $\neq$ 0.

3.3 Methods of Finding t_f^* and the λ^*'s

The results of the previous section can be used to find t_f^* and λ^*'s. Since the minimum terminal time has been shown to be the maximum value of t_f when $\underline{\lambda} = \underline{\lambda}^*$ for $\|\underline{k}^*\|_q$ a non-decreasing function of t_f, standard hill climbing techniques[51] are applicable. Two such techniques are now briefly described and illustrated.

3.3.1 The T_{max} Method

The T_{max} Method is a "sectioning" technique[53] whereby only one lambda is varied at a time, all others remaining fixed. For a two-output plant, only one λ (say λ_2) appears in the expression for $\underline{k}(t)$. To start the search λ_2 is initialized and t_f is calculated from the expression

$$\varphi(t_f, \underline{\lambda}) = \frac{1}{L} \tag{3-11}$$

Then λ_2 is incremented by $\Delta\lambda_2$ and t_f is recalculated. If this terminal time is greater than the previous one, λ_2 is again incremented etc. When the current value of t_f is less than the one previously calculated, the maximum value of t_f (and hence t_f^*) has been passed. Therefore, we have succeeded in bracketing λ_2^* between the previous and current values of lambda. A technique of finding a real root of a transcendental equation (e.g. bisection[34]) is now used to locate λ_2^* and t_f^*. It is significant to note that a particularly simple set of calculations is required in this case.

For plants with three or more outputs, two or more lambda's must be found and theoretically, one should be able to extend the above procedure without difficulty. However, other researchers[54] have concluded that this technique, when uncombined with other methods, is not suitable unless one knows in advance that $\varphi(t_f, \underline{\lambda})$ does not contain any ridges, i.e. flat portions of the surface.

Difficulty may arise with the T_{max} method if the λ-increments are too large, i.e. the $\underline{\lambda}^*$ may not be properly bracketed, (even in the two output case). See Figure 3-6. It is observed that even though the maximum has been passed, $t_{f_B} > t_{f_A}$. Since $t_{f_C} < t_{f_B}$, λ_2^* is incorrectly located between λ_{2_B} and λ_{2_C}. This problem can be remedied by reducing the size of the λ-increment. However, such a reduction is accompanied by a corresponding increase in computation time so that this may not always be practical.

A more reliable method of bracketing λ_2^* is to compute the partial derivative of t_f with respect to λ_2 from Eq.(3-4), in addition to t_f, for each value of λ_2. The maximum is passed when the sign of the partial derivative changes.

3.3.2 Steepest Ascent

The second method, particulary useful when the plant has many outputs, is a steepest ascent technique[51]. Taking partial derivatives of $\varphi(t_f, \underline{\lambda})$ with respect to λ_i and equating the resultant expressions to zero, the (m-1)-partial derivatives,

$$s_i \triangleq \frac{\partial t_f}{\partial \lambda_i} \qquad (i=2,3,\ldots m) \qquad (3\text{-}12)$$

can be computed. An initial set of λ's $(\lambda_2, \lambda_3, \ldots \lambda_m)$ is chosen and Eq.(3-11) is used to calculate t_f. These values are then substituted into

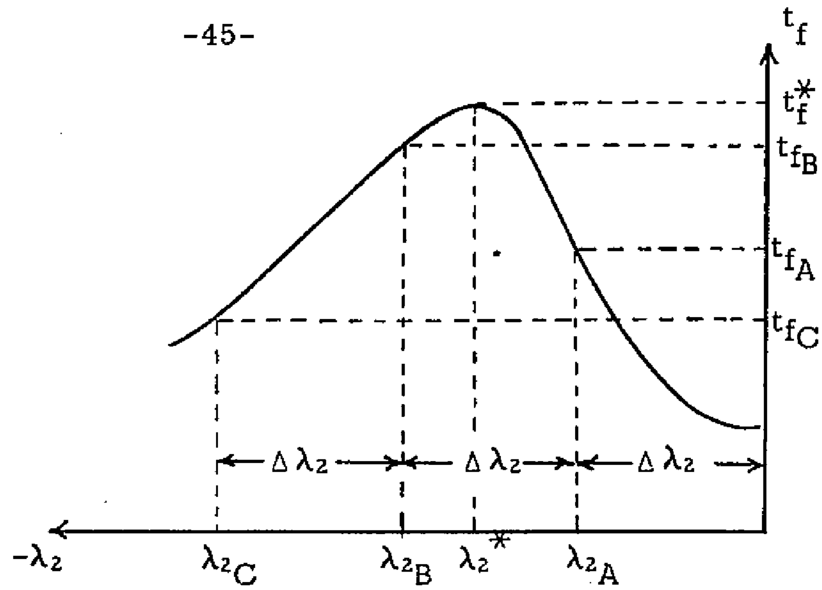


Figure 3-6 Illustration of a Situation which leads to an error in locating t_f^*

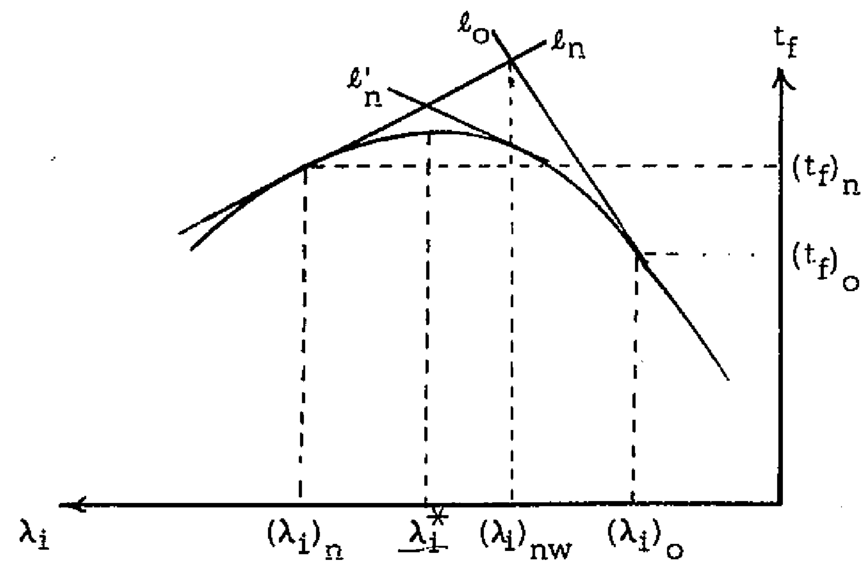


Figure 3-7 Illustration of the method used to find λ_i when $(s_i)_n$ and $(s_i)_o$ differ in sign

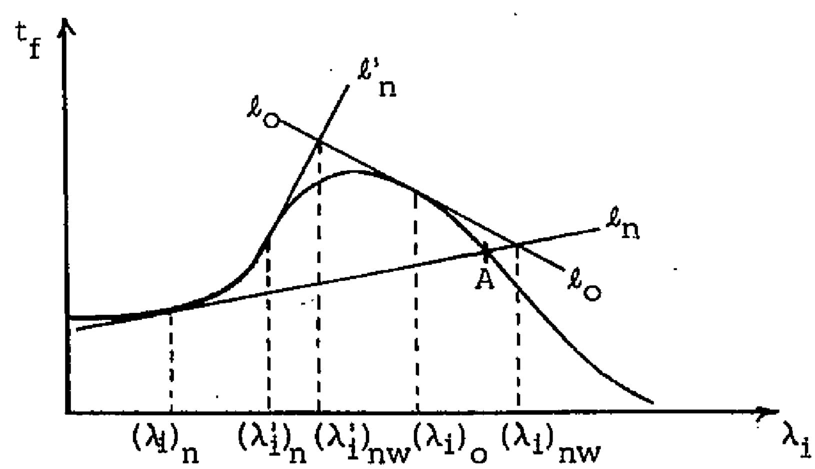


Figure 3-8 Illustration of case where ℓ_o and ℓ_n intersect outside of $[(\lambda_i)_n, (\lambda_i)_o]$

Eq. (3-12) and the s_i are found. Since we have previously shown that t_f passes through a maximum in $(t_f, \underline{\lambda})$-space, moving in a direction defined by these partials such that t_f increases must lead to the maximum. If this is done the new set of λ's is obtained from the expression

$$\frac{\Delta t_f}{(\lambda_i)_{new} - (\lambda_i)_{old}} \approx s_i$$

or solving for $(\lambda_i)_{new}$,

$$(\lambda_i)_n = (\lambda_i)_o + \Delta t \left(\frac{1}{s_i}\right) \qquad (3\text{-}13)$$

where Δt_f is an increment which is specified and the subscripts "n" and "o" mean new and old respectively. These λ's are now used in Eq.(3-11) and the next value of t_f is obtained. Equation (3-12) is again employed and the succeeding set of s_i's is found.

In order to guard against having a divergent or an oscillatory solution, the signs of the new and old values of each of the s_i (i.e. $(s_i)_n$ and $(s_i)_o$) are compared. If the i^{th} such partial derivative does not change sign, Eq.(3-13) is used to find the new λ_i. However, if s_i does undergo a sign change, the next λ_i is found in the manner illustrated in Figure 3-7. It is noted that the abscissa of the intersection of the tangent lines, ℓ_n and ℓ_o, is the new value of λ_i (i.e. $(\lambda_i)_{nw}$). Thus:

$$\ell_n: \quad t_f = (s_i)_n \lambda_i - (s_i)_n (\lambda_i)_n + (t_f)_n$$

$$\ell_o: \quad t_f = (s_i)_o \lambda_i - (s_i)_o (\lambda_i)_o + (t_f)_o$$

where $(t_f)_n$ and $(t_f)_o$ are the values of t_f corresponding to $(\lambda_i)_n$ and $(\lambda_i)_o$ respectively. Equating the right hand sides of these equations and solving for $(\lambda_i)_{nw}$, we have

$$(\lambda_i)_{nw} = \frac{[(s_i)_n(\lambda_i)_n - (s_i)_o(\lambda_i)_o + (t_f)_o - (t_f)_n]}{[(s_i)_n - (s_i)_o]} \qquad (3\text{-}14)$$

$$(i = 2, 3, \ldots m)$$

This lambda together with all of the other new lambda's (computed from either Eq.(3-13) or (3-14)) are again used in Eq.(3-11) to find the next value of t_f, etc. The procedure terminates when the set of λ's and t_f no longer vary. As a matter of practicality, Eq.(3-14) is used to obtain succeeding values of λ_i after the first sign change in s_i. A computer flow diagram for this procedure is given in Figure 3-9.

It is possible for a situation to arise causing the two tangent lines, ℓ_o and ℓ_n, to intersect outside the interval $[(\lambda_i)_n, (\lambda_i)_o]$. This is illustrated in Figure 3-8. If this happens and the procedure is continued the maximum will be incorrectly located at point A. It is, therefore, necessary to make certain that each new value of $(\lambda_i)_{nw}$ (from Eq.3-14) lies within the interval $[(\lambda_i)_n, (\lambda_i)_o]$. If it does not, $(\lambda_i)_n$ is increased to $(\lambda_i')_n$ and the intersection of ℓ_o and ℓ_n' is used to obtain the new value of λ_i (i.e. $(\lambda_i')_{nw}$) as shown in Figure 3-8.

3.4 Examples

Illustrations of the results of the previous section now follow. Using the example of Section 2.2.5, it is necessary to minimize the expression (Eq.2E-11)

$$\|\underline{k}\|_1 = L\left[\frac{t_f^3}{6} + \frac{t_f^2}{2}\lambda_2 - \frac{4}{3}\lambda_2^3\right] \qquad (3E\text{-}1)$$

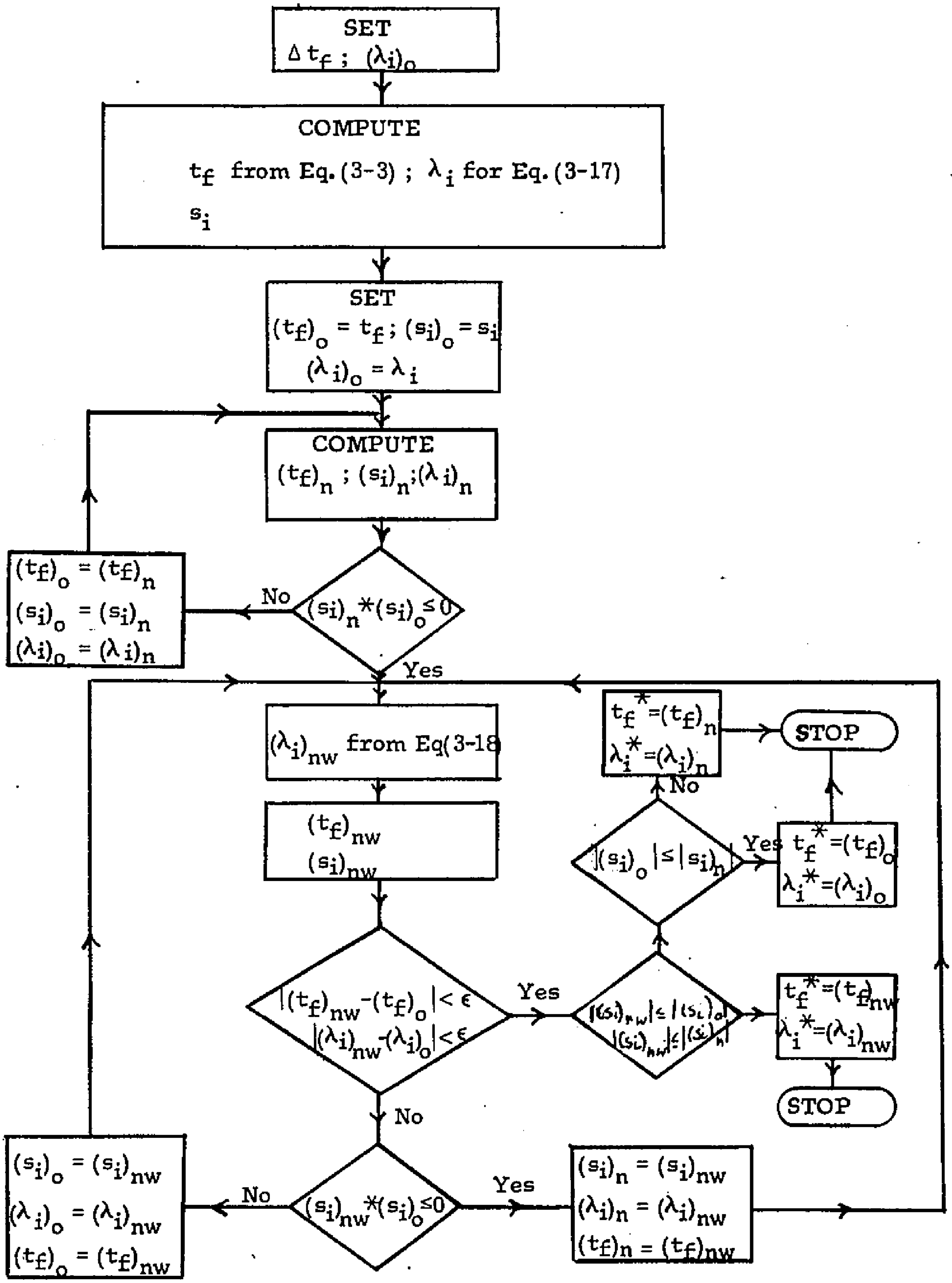


Figure 3-9 Flow diagram for ascent method

with respect to λ_2 and hence to find $\underline{\lambda}^*$ and t_f^*. From Eq.(2E-7), $e_{d_1} = 1$ and $e_{d_2} = 0$. For the purposes of the example,

$$L = \frac{3\sqrt{2}}{4(\sqrt{2}-1)} = 2.56$$

is chosen. Using these values and Eq.(3E-1), Eq.(3-11) and Eq.(3-12) become

$$\frac{t_f^3}{6} + \frac{t_f^2}{2}\lambda_2 - \frac{4}{3}\lambda_2^3 = \frac{1}{2.56} \tag{3E-2}$$

$$s_2 = -\frac{\left[\frac{t_f^2}{2} - 4\lambda_2^2\right]}{t_f\left|\frac{t_f}{2} + \lambda_2\right|} \tag{3E-3}$$

(a) <u>T_{max} Method</u>

The Fortran IV computer program which is used to implement the T_{max} Method is found in Appendix II. After the first sign change in the slope is detected, the program employs bisection to locate λ_2^*.

19 iterations are required to obtain the values

$$\lambda_2^* = -0.70710754$$

$$t_f^* = 2.00000206$$

An accuracy of 76 parts in 10^8 for λ_2^* and 20 parts in 10^8 for t_f^* is found by comparing the above results with the exact answers computed from Eqs.(2E-12) and (2E-13), (i.e. $\lambda_2^* = -\frac{\sqrt{2}}{2}$; $t_f^* = 2.0$). The computer output is summarized in Table 3.1.

(b) <u>Ascent Method</u>

The Fortran IV computer program for the ascent method is located in Appendix III. From Table 3.2, it is seen that 19 iterations are again

Appendix III. From Table 3.2, it is seen that 19 iterations are again required to complete the computation. This time, however, the accuracy of λ_2^* is improved, the error being only 5 parts in 10^8.

It is found that the technique of intersecting tangents works well when the magnitudes of the slopes of ℓ_o and ℓ_n are greater than about 1.5×10^{-4}. For smaller values the procedure converges slowly. To guard against this, the magnitude of the difference of the present and previous values of $(\lambda_2)_{nw}$ is checked. If this quantity falls below a small positive number, ϵ, bisection is used from that point on in the search. Thus, in Table 3.2, the values of λ_2 are obtained using bisection from iteration 15 to the end of the search.

3.5 Summary

In this chapter it was proven that when $\|\underline{k}^*\|_q$ is a non-decreasing function of t_f, the point $(t_f^*, \underline{\lambda}^*)$ is a unique maximum in $(t_f, \underline{\lambda})$-space. If $\|\underline{k}^*\|_q$ is not a non-decreasing function, this point is the smallest of the local maxima. In addition, it was shown that the sufficient condition for $\|\underline{k}^*\|_q$ to be a non-decreasing function for time-invariant systems is that $\underline{e}_d$ be independent of t_f. Initially quiescent systems and regulator type systems were shown to have this property. These results were then illustrated for two types of hill climbing techniques.

It is worthy of mention that this work was done after the examples in Chapter IV had already been programmed and solved on a digital computer. Time did not permit reworking them using the techniques discussed herein.

TABLE 3.1

ITERATION	LAMBDA	T	S
1	-0.00000000	1.32821156	-1.00000000 E-00
2	- .25000000	1.60793682	-1.17062571 E-00
3	- .50000000	1.88001770	- .92747899 E-00
4	- .75000000	1.98924068	.55786243 E-00
5	- .62500000	1.97499238	- .54167162 E-00
6	- .68750000	1.99827618	- .17010152 E-00
7	- .71875000	1.99931652	.12067309 E-00
8	- .70312500	1.99992694	- .37588853 E-01
9	- .71093750	1.99992488	.37850647 E-01
10	- .70703125	2.00000204	- .73614210 E-03
11	- .70898437	1.99998470	.18326024 E-01
12	- .70800781	1.99999803	.87402993 E-02
13	- .70751953	2.00000119	.39885463 E-02
14	- .70727539	2.00000191	.16228347 E-02
15	- .70715332	2.00000204	.44250650 E-03
16	- .70709228	2.00000206	- .14702748 E-03
17	- .70712280	2.00000206	.14768707 E-03
18	- .70720754	2.00000206	.31668103 E-06
19	- .70709991	2.00000206	- .73358685 E-04

TABLE 3.2

ITERATION	LAMBDA	T	S
1	-0.00000000	1.32821156	-1.00000000 E-00
2	- .25000000	1.60793682	-1.17062571 E-00
3	- .46356100	1.84494898	- .99491088 E-00
4	- .71483978	1.99970037	- .78224483 E-01
5	- .62608247	1.97557605	- .53672790 E-00
6	- .67660233	1.99597499	- .25066411 E-00
7	- .69702410	1.99952963	- .91996356 E-01
8	- .70621427	1.99999831	- .85815319 E-02
9	- .71055481	1.99993851	.33992992 E-01
10	- .70827540	1.99999531	.11355978 E-01
11	- .70723794	2.00000196	.12605796 E-02
12	- .70671678	2.00000136	- .37647460 E-02
13	- .70696694	2.00000198	- .13563329 E-02
14	- .70709167	2.00000206	- .15292477 E-03
15	- .70715391	2.00000204	.44822891 E-03
16	- .70711196	2.00000206	.42969777 E-04
17	- .70709104	2.00000206	- .15902769 E-03
18	- .70710150	2.00000206	- .58035122 E-04
19	- .70710673	2.00000206	- .75341705 E-05

IV GENERAL LINEAR PLANTS WITH INPUT AND OUTPUT CONSTRAINTS

4.1 Introduction

In this chapter, the optimal w.r.a.s. solution to problem 5 in Section 1.2 is obtained first for single output and then for multiple output linear plants. A distinction is made between these two types because of essential differences in computational procedures used to obtain their solutions.

For single output plants two necessary conditions for the existence of an optimal w.r.a.s. solution are given. In addition, a lower bound on T_o, the elapsed time, is developed. It is also demonstrated that the single output can never overshoot the terminal point, y_d. This result is used to develop a simple test for the existence of an optimal w.r.a.s. solution when the output has its magnitude constrained. The computer search scheme for finding the feedback coefficients, the optimal w.r.a.s. input, and T_o is illustrated.

For systems with two or more outputs some necessary conditions for the existence of an optimal w.r.a.s. solution are presented. When a quadratic constraint is placed on the input it is shown that the λ's can be found using algebraic techniques. A numerical method of finding these λ's when the magnitude of the input is limited is introduced. A set of examples is included to illustrate these points.

4.2 Optimal W.R.A.S. Solution for Single Output Plants

Consider the augmented, n^{th} order, linear time-varying plant shown in Figure 4-1. For this plant the system of state equations which describes its dynamic behavior becomes

$$\underline{\dot{x}}(t) = E(t)\,\underline{x}(t) + F(t)\,u_1(t) \qquad (4\text{-}1a)$$

$$y(t) = C(t)\,\underline{x}(t) + D(t)\,u_1(t) \qquad (4\text{-}1b)$$

where E(t), C(t), and $\underline{x}(t)$ have previously been defined in Section 1.2; y is the scalar output; $u_1(t)$ is the single input; and F(t) and D(t) are respectively $(n \times 1)$ and (1×1) time-varying matrices. The impulse response (a scalar quantity) of the augmented plant is

$$h_a(t,\tau) = C(t)\,\phi(t,\tau)\,F(\tau) + D(t)\,\delta(t-\tau) \qquad (4\text{-}2)$$

where $\phi(t,\tau)$ is the fundamental matrix of the augmented system. It is noted that since E(t), F(t), C(t), and D(t) are, in general, functions of the feedback coefficients, A_i, so is $h_a(t,\tau)$. Problem 5 can now be restated for this class of systems as follows:

Given the single input, single output plant characterized by the impulse response, $h_a(t,\tau)$, find the set of feedback coefficients and the input, $u_1(t)$, which drives the output from the origin to some final point, $y(t_f) \triangleq y_d$, and which permits the input constraint expressed in the form

$$\|u_1\|_{p_1} \triangleq \left[\int_{t_o}^{t_f} |u_1(t)|^{p_1}\,dt\right]^{1/p_1} \le L_1 \qquad (p_1 = 1, 2, \infty) \qquad (4\text{-}3)$$

and the output constraint

$$f(y) \ge L_2 \;(\text{or} \le L_2) \qquad (4\text{-}4)$$

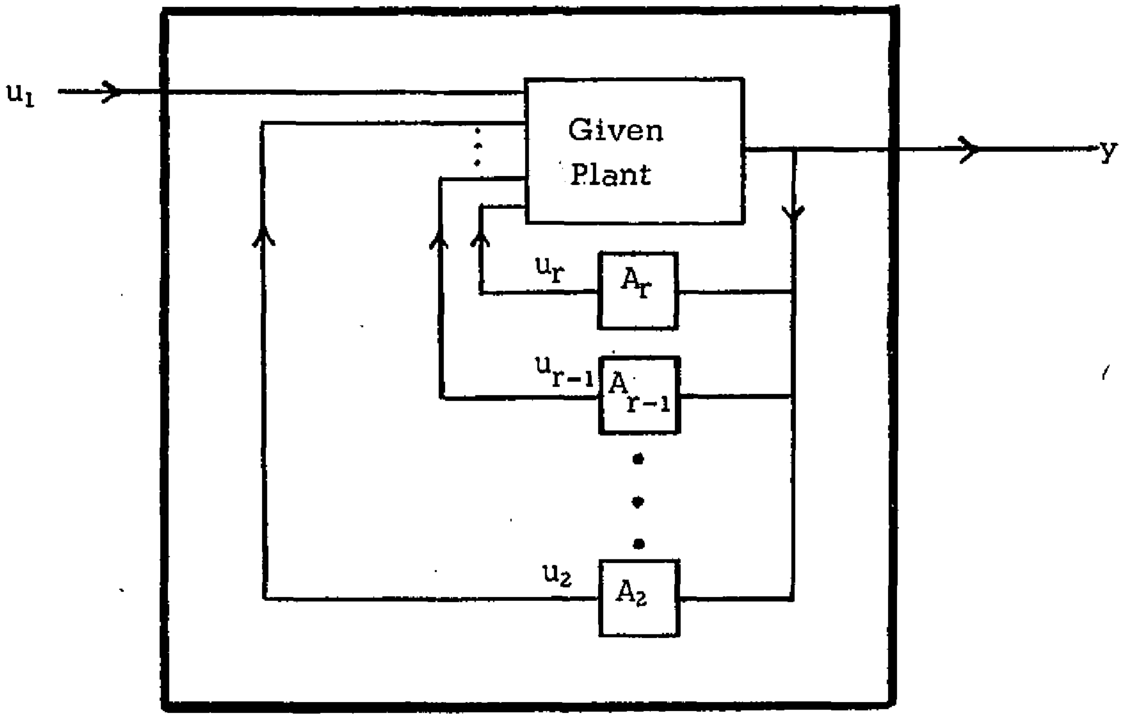


Figure 4-1 Single-Input - Single Output Augmented Plant

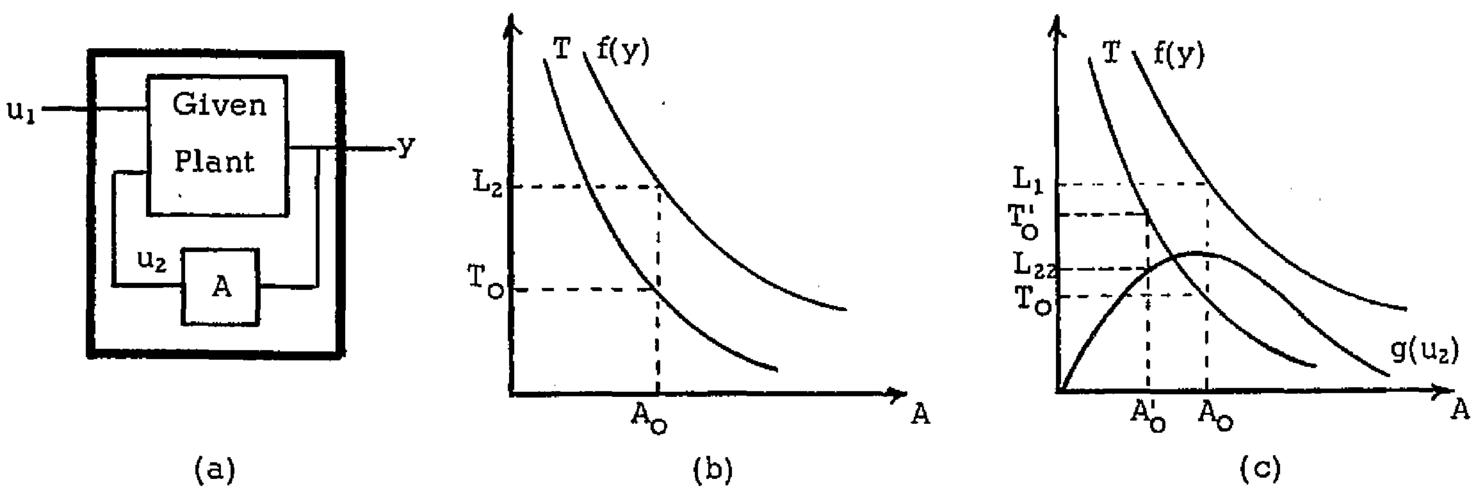


Figure 4-2 (a) Augmented plant with one feedback coefficient and $f(y) \geq L_2$

(b) Illustration of the technique for finding T_o and the best A

(c) Illustration of the technique for finding T_o and the best A when input 2 is constrained, i.e. $g(u_2) \leq L_{22}$

to be satisfied in the shortest elapsed time, $T_o \triangleq t_{f_o} - t_o$.†

4.2.1 The Optimal W.R.A.S. Solution

The method which is used to solve this problem and which is consistent with the discussion in Section 1.3 is as follows:

The output constraint is ignored and the input which satisfies the terminal condition and the input constraint in the shortest elapsed time is found. (Note that this is the solution to Problem 1 in Section 1.2) Using the method of Functional Analysis[21,22] outlined in Section 2.2, the form of the input is given by Eq.(2-41) (repeated here for convenience), i.e.

$$u_1(t) = L_1^{q_1} \left| \frac{h_a(t_{f_o},t)}{y_d} \right|^{q_1-1} \mathrm{SGN}\left[\frac{h_a(t_{f_o},t)}{y_d} \right] \tag{4-5}$$

where

$$\frac{1}{q_1} = 1 - \frac{1}{p_1} \tag{4-6}$$

and t_{f_o} is the least value of t_f which satisfies Eq.(2-42), i.e.

$$\|h_a\|_{q_1} \triangleq \left[\int_{t_o}^{t_{f_o}} |h_a(t_{f_o},t)|^{q_1} dt \right]^{1/q_1} = \frac{|y_d|}{L_1} \tag{4-7}$$

It is observed that this input satisfies Eq.(4-3) with the sign of equality (see Section 2.2). The set of A's is now chosen so that the output constraint is satisfied in the shortest elapsed time. If such a set of A's can be found for specific values of y_d, L_1, and L_2, the input given in Eq.(4-5)

---

† The sign of the inequality is determined by the problem. See footnote on page 3.

has the property of permitting all constraints and the terminal condition to be satisfied in the shortest time. Thus, $u_1(t)$ is the optimal input with respect to the assumed feedback structure, i.e. the "Optimal w.r.a.s." input.

To clarify the above method, consider the augmented plant shown in Figure 4-2a. It is required that the plant output satisfy a constraint, $f(y) \geq L_2$. For the input in Eq.(4-5), $f(y)$ and the elapsed time are calculated and plotted versus A as shown in Figure 4-2b. T_o is the least value of T for which the output constraint is satisfied and is seen to occur for $A = A_o$. For augmented plants with more than one feedback coefficient, a computer search scheme must be employed. Discussion of this deferred until Chapter VI. As a matter of practicability, it is not usually possible to find the feedback coefficients by taking partial derivatives of t_{f_o} with respect to the A's and setting the resulting expressions equal to zero; A's found in this way will not necessarily permit inequality (4-4) to be satisfied.

From Figure 4-1 it is clear that inputs u_2 through u_r belong to the class

$$u_i(t) = A_i\, y(t) \tag{4-8}$$

It is significant to note that if they are not restricted to this class another set of inputs (the optimal) can, in general, be found which satisfies all conditions in a shorter time, T^*, than that found from Eq.(4-7).†

In Section 1.3, it was mentioned that it would usually be assumed that the set of A's and the optimal w.r.a.s. input would not cause any constraints on the inputs in Eq.(4-8) to be violated. If this is not so,

† An exception to this is presented in the next chapter. A class of systems is shown to exist for which the optimal and optimal w.r.a.s. solutions are the same and hence $T_o = T^*$.

the above procedure can be extended to include these additional constraints. Using the plant in Figure 4-2a, this extension is illustrated in Figure 4-2c. Here, input 2 is assumed to have a constraint placed on it of the form, $g(u_2) \le L_{22}$. As shown, $g(u_2)$ is plotted versus A, in addition to f(y) and T. A_o^1 is then the value of A which permits both constraints to be satisfied in the shortest, elapsed time, T_o^1. Note that as expected, $T_o^1 > T_o$.

Satisfying Eq.(4-7) is a necessary condition for solution of Problem 5 to exist for single output plants, i.e. if, for a specific set of A's, no t_{f_o} can be found which satisfies Eq.(4-7), the input given by Eq.(4-5) will not drive the output to y_d.

4.3 A Lower Bound on T_o

In order to obtain a lower bound on the minimum elapsed time, the following theorem is proved:

Theorem

If the optimal w.r.a.s. input in Eq.(4-5) is applied to the system shown in Figure 4-1, the magnitude of the output is less than or equal to $|y_d|$.

Proof

The output of the augmented plant when the input in Eq.(4-5) is applied can be found using the superposition integral. Thus

$$y(t) = \int_{t_o}^{t} h_a(t,\tau)\, u_1(\tau) d\tau = L_1^{q_1} \int_{t_o}^{t} h_a(t,\tau) \left| \frac{h_a(t_{f_o},\tau)}{y_d} \right|^{q_1-1} \mathrm{SGN}\left[\frac{h_a(t_{f_o},\tau)}{y_d}\right] d\tau$$

Taking magnitudes of both sides of this expression and assuming that $h_a(t,\tau)$

is a non-decreasing function of t, we find

$$|y(t)| = |L_1^{q_1}\int_{t_o}^{t} h_a(t,\tau)\left|\frac{h_a(t_{f_o},\tau)}{y_d}\right|^{q_1-1} \mathrm{SGN}\left[\frac{h_a(t_{f_o},\tau)}{y_d}\right]d\tau|$$

$$\le L_1^{q_1}\int_{t_o}^{t} |h_a(t,\tau)|\left|\frac{h_a(t_{f_o},\tau)}{y_d}\right|^{q_1-1} d\tau \le \frac{L_1^{q_1}}{|y_d|^{q_1-1}}\int_{t_o}^{t_{f_o}}|h_a(t_{f_o},\tau)|^{q_1} d\tau$$

$$|y(t)| \le \frac{L_1^{q_1}}{|y_d|^{q_1-1}} (\|h_a\|_{q_1})^{q_1} \qquad (t_o \le t \le t_{f_o}) \tag{4-9}$$

Making use of Eq.(4-7), Eq. (4-9) becomes

$$|y(t)| \le \frac{L_1^{q_1}}{|y_d|^{q_1-1}}\frac{|y_d|^{q_1}}{L_1^{q_1}} = |y_d|$$

Therefore,

$$|y(t)| \le |y_d|$$

Even if $h_a(t,\tau)$ is a decreasing function of t, Inequality (4-10) is still valid. This is easily understood by recalling that Eq.(4-5) is the expression for the time-optimal input - that is, the input that drives the output to y_d in the shortest terminal time, t_{f_o}. Clearly, t_{f_o} is the first time that $y(t) = y_d$ and hence, no overshoot of the terminal value is posssible. This completes the proof.

It is observed from Eq.(4-10) that the output is not permitted to overshoot the final value, y_d. Using this result, a lower bound on t_{f_o}, and hence T_o can now be obtained.

Case 1. An Integral Constraint on the Output

Suppose that the output constraint in Eq.(4-4) takes the form

$$\|y\|_1 \triangleq \int_{t_o}^{t_{f_o}} |y(t)|\, dt \geq L_{2a} \tag{4-11}$$

Using Eq.(4-10), we find

$$\|y\|_1 \leq \int_{t_o}^{t_{f_o}} |y_d|\, dt = |y_d| [t_{f_o} - t_o] = |y_d|\, T_o \tag{4-12}$$

Combining (4-11) and (4-12), we have

$$L_{2a} \leq \|y\|_1 \leq |y_d| T_o$$

or

$$|y_d| T_o \geq L_{2a}$$

whereupon a lower limit on the minimum elapsed time when the output has an integral constraint placed on it is found to be

$$(T_o)_{\substack{\text{lower}\\ \text{limit}}} = \frac{L_{2a}}{|y_d|} \tag{4-13a}$$

Using the definition of the elapsed time, the lower bound on t_{f_o} becomes

$$[t_{f_o}]_{\substack{\text{lower}\\ \text{limit}}} = \frac{L_{2a}}{|y_d|} + t_o \tag{4-13b}$$

Case 2. A Quadratic Constraint on the Output

Suppose that the output constraint in Eq.(4-4) takes the form

$$\|y\|_2 = \left[\int_{t_o}^{t_{f_o}} y^2(t)\, dt\right]^{\frac{1}{2}} \geq L_{2a} \tag{4-14}$$

Using Eq.(4-10) we find

$$\|y\|_2 \leq \left[\int_{t_o}^{t_{f_o}} y_d^2\, dt\right]^{\frac{1}{2}} = |y_d| \sqrt{t_{f_o} - t_o} = |y_d| \sqrt{T_o} \tag{4-15}$$

Combining Eqs.(4-14) and (4-15) yields

$$L_{2e} \leq \|y\|_2 \leq |y_d| \sqrt{T_o} \tag{4-16}$$

Thus a lower bound on the minimum elapsed time when the output has a quadratic constraint placed on it is given by

$$(T_o)_{\substack{\text{lower}\\\text{bound}}} = \left(\frac{L_{2e}}{y_d}\right)^2 \tag{4-17a}$$

and

$$(t_{f_o})_{\substack{\text{lower}\\\text{bound}}} = \left(\frac{L_{2e}}{y_d}\right)^2 + t_o \tag{4-17b}$$

<u>Case 3</u>. <u>A Magnitude Constraint on the Output</u>

If the output constraint in Eq.(4-4) is

$$|y(t)| \leq L_{2m} \qquad (t_o \leq t \leq t_{f_o}) \tag{4-18}$$

it is clear from Eq.(4-10) that L_{2m} must be greater than or equal to $|y_d|$ or else the constraint in Eq.(4-10) is violated. However, if this condition is true, the output constraint can never be violated. Thus, a necessary and sufficient condition for a magnitude constraint on the single output to be satisfied is that $L_{2m} \geq |y_d|$. The case of single output plants with an output magnitude constraint is not considered further.

It should be noted that the bounds derived in Eqs.(4-13) and (4-17) are not, in general infima, i.e. greatest lower bounds, since y(t) is actually only equal to y_d at $t = t_{f_o}$. However, because they are readily calculated, they give an immediate "ball park" estimate of the minimum elapsed time before any lengthy computation is performed. In addition, they do permit the actual computation time to be reduced, somewhat; the explanation of this is to be found in Section 6.3.1.

4.4 Another Necessary Condition for a Solution to Exist

In this section, another condition which must be satisfied in order for a solution to Problem 5 (for single output plants) to exist for a specified L_1, L_2, and y_d is developed.

For the given linear plant shown in Figure 4-1, let us define $h_j(t,\tau)$ $(j=1,2,\ldots r)$ to be the response, $y(t)$, when $u_j(t) = \delta(t-\tau)$ and all of the feedback coefficients are zero. The $h_j(t,\tau)$ characterize the given linear plant only and are obviously independent of the A's.

Using Eq.(4-8) and the superposition integral, the output of the augmented plant is found to be

$$y(t) = \int_{t_o}^{t} h_1(t,\tau)\, u_1(\tau) d\tau + \int_{t_o}^{t} y(\tau)\left[\sum_{j=2}^{r} A_j\, h_j(t,\tau)\right] d\tau \tag{4-19}$$

Recalling that $y(t_{f_o}) \overset{\Delta}{=} y_d$, Eq.(4-19) with $t=t_{f_o}$ becomes

$$y_d = \int_{t_o}^{t_{f_o}} h_1(t_{f_o},\tau)\, u_1(\tau)\, d\tau + \int_{t_o}^{t_{f_o}} y(\tau)\left[\sum_{j=2}^{r} A_j\, h_j(t_{f_o},\tau)\right] d\tau \tag{4-20}$$

Taking magnitudes on both sides of this equation and making use of the Hölder's and Minkowski inequalities,[44] we have

$$
\begin{aligned}
|y_d| &= \left| \int_{t_o}^{t_{f_o}} h_1(t_{f_o},\tau)\, u_1(\tau)\, d\tau + \int_{t_o}^{t_{f_o}} y(\tau) \left[\sum_{j=2}^{r} A_j\, h_j(t_{f_o},\tau)\right] d\tau \right| \\
&\leq \left| \int_{t_o}^{t_{f_o}} h_1(t_{f_o},\tau)\, u_1(\tau)\, d\tau \right| + \left| \int_{t_o}^{t_{f_o}} y(\tau) \left[\sum_{j=2}^{r} A_j\, h_j(t_{f_o},\tau)\right] d\tau \right| \\
&\leq \left| \int_{t_o}^{t_{f_o}} h_1(t_{f_o},\tau)\, u_1(\tau)\, d\tau \right| + \int_{t_o}^{t_{f_o}} \left| y(\tau) \left[\sum_{j=2}^{r} A_j\, h_j(t_{f_o},\tau)\right] \right| d\tau
\end{aligned}
$$

and

$$
|y_d| \leq \left| \int_{t_o}^{t_{f_o}} h_1(t_{f_o},\tau) u_1(\tau) d\tau \right| + \|y\|_{p_2} \left[\int_{t_o}^{t_{f_o}} \sum_{j=2}^{r} |A_j|^{q_2} \, |h_j(t_{f_o},\tau)|^{q_2} d\tau \right]^{1/q_2}. \tag{4-21}
$$

Suppose that the constraint in Eq.(4-4) is assumed to have the form, $\|y\|_{p_2} \geq L_2$. Since a minimum time solution is desired, this constraint becomes an equality, i.e. $\|y\|_{p_2} = L_2$. Therefore, we find that

$$
|y_d| \leq \left| \int_{t_o}^{t_{f_o}} h_1(t_{f_o},\tau)\, u_1(\tau) d\tau \right| + L_2 \left[\int_{t_o}^{t_{f_o}} \sum_{j=2}^{r} |A_j\, h_j(t_{f_o},\tau)|^{q_2}\, d\tau \right]^{1/q_2} \tag{4-22}
$$

If this inequality, together with Eq.(4-7) are not both satisfied for a given set of A's and t_{f_o}, a solution to Problem 5 is not possible. It must be noted, however, that satisfying these two conditions is not sufficient to guarantee that a solution does exist.

4.5 Examples - Single Output Plants

The ideas of the previous sections are now illustrated with several examples.

4.5.1 One Dimensional Search

4E-1.

Consider the second order augmented plant shown in Figure 4-3 and let the following conditions be imposed:

$$y(0) = 0\ ; \quad y(T_o) \stackrel{\Delta}{=} y_d = 2\ ; \quad |u_1| \le 1\ ; \quad \|y\|_2 \ge \frac{\sqrt{3}}{2}$$

We wish to find $u_1(t)$ and the value of A which satisfies these conditions in the shortest time. It is clear that a one-dimensional search is required.

Defining the state variables as shown in Figure 4-3, the normal form description of the augmented plant is

$$\dot{\underline{x}} = \begin{bmatrix} 2 & 0 \\ 1.5A & 0.2A-2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ A-0.25 \end{bmatrix} u_1 \tag{4E-1a}$$

$$y = [1.5 \quad 0.2]\,\underline{x} + [1]\, u_1 \tag{4E-1b}$$

The fundamental matrix of this system can be shown to be

$$\phi(t) = \begin{bmatrix} e^{2t} & 0 \\ \frac{1.5A}{4-0.2A}\left(e^{2t} - e^{-(2-0.2A)t}\right) & e^{-(2-.2A)t} \end{bmatrix} \tag{4E-2}$$

Using Eq.(4-2), the impulse response is found to be

$$h_a(t) = \frac{6}{4-0.2A} e^{2t} + \frac{(.2A-2)(.2A-1)}{(0.2A-4)} e^{(0.2A-2)t} + \delta(t) \tag{4E-3}$$

whereupon, Eq.(4-7) becomes

$$\|h_a\|_1 = 1 + \frac{3}{4-.2A}(e^{2T_o}-1) + \frac{(.2A-1)}{(.2A-4)}[e^{(.2A-2)T_o}-1] = 2 \tag{4E-4}$$

In addition, we find that the output norm is given by

$$\|y\|_2 = \left[\left(\frac{1.5}{4-.2A}\right)^2 (e^{4T_o}-1) + \frac{(1-.2A)^2}{(4-.2A)^2(.4A-4)}(e^{2(.2A-2)T_o}-1) + \frac{30}{A}\,\frac{(1-.2A)}{(4-.2A)^2}(e^{.2AT_o}-1) \right]^{\frac{1}{2}} \tag{4E-5}$$

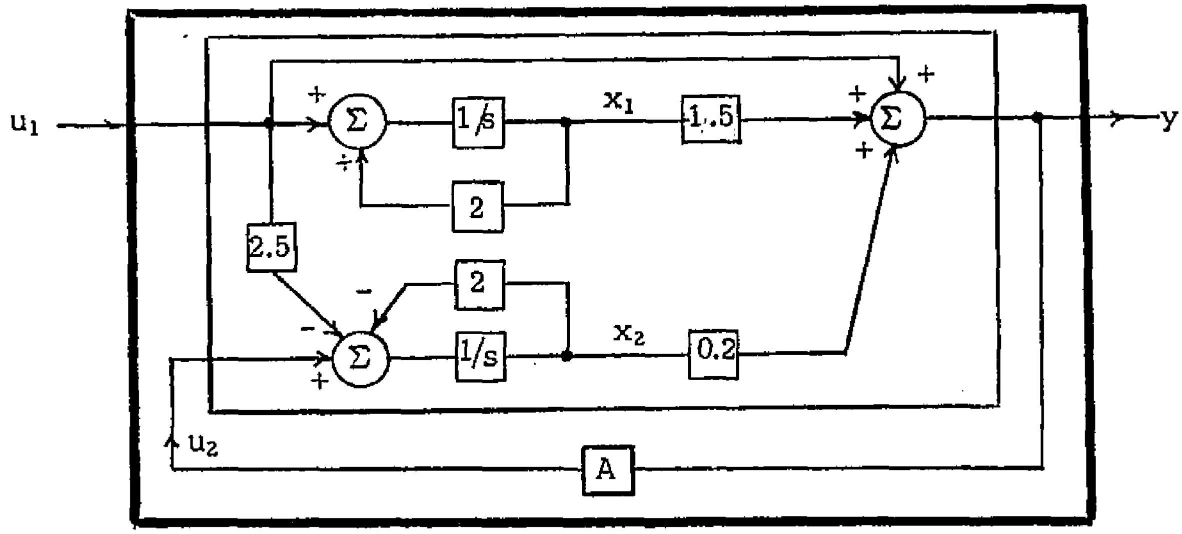


Figure 4-3 Augmented Plant used in Example 4E-1

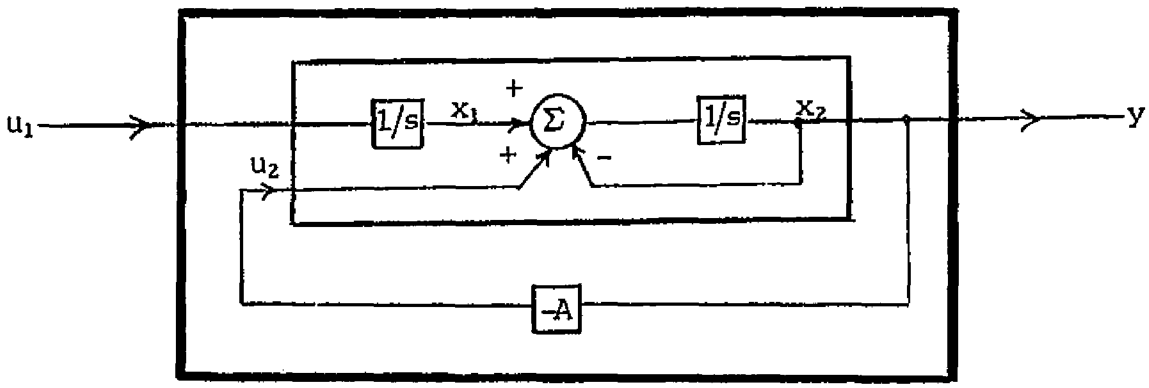


Figure 4-5 Augmented Plant used in Example 4E-2

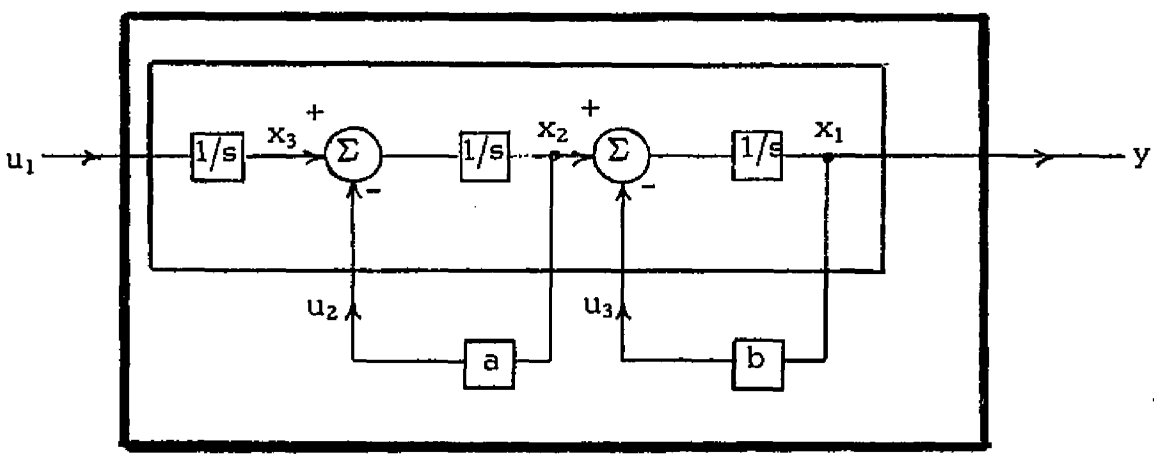


Figure 4-7 Third order plant used in Example 4E-3

T_O is computed for various values of A from Eq.(4E-4). For each pair, (A, T_O), the corresponding value of $\|y\|_2$ is found from Eq.(4E-5). These data are plotted in Figure 4-4. It is observed that the output constraint is satisfied for $T_O = 0.347$ with $A = 5$. Also, from Eq.(4-5), the optimal w.r.a.s. input is the unit step.

As was discussed in Chapter I a comparison between T_O and T^* should be made. Choosing $L_1 = 1$ and $L_2 = \frac{\sqrt{3}}{2}A$ in Eq.(2-40) with $q_1 = 1$ and $q_2 = 2$, and using the impulse responses of the given plant, i.e.

$$h_1(t) = \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} + \delta(t) \qquad (4E\text{-}6a)$$

$$h_2(t) = 0.2\ e^{-2t} \qquad (4E\text{-}6b)$$

the equation for T^* is found to be

$$\frac{3}{4}\left(e^{2T^*}-1\right) - \frac{1}{4}\left(1-e^{-2T^*}\right) + 1 + \frac{\sqrt{3}}{4}\left(1-e^{-4T^*}\right)^{\frac{1}{2}} = 2 \qquad (4E\text{-}7)$$

whereupon, $T^* = 0.347$.

$T_O = T^*$ because this plant belongs to a class of systems having the property that the optimal and optimal w.r.a.s. solutions, and hence the elapsed times are the same. This will be shown in Chapter V.

It is easily verified that inequality (4-22) is volated for $A < 5$ and satisfied for $A \geq 5$.

4E-2

The augmented plant in the previous example is always unstable. This can easily be seen by noting that the Laplace transform of $h_a(t)$ (Eq.4E-3) always has a pole in the right hand portion of the

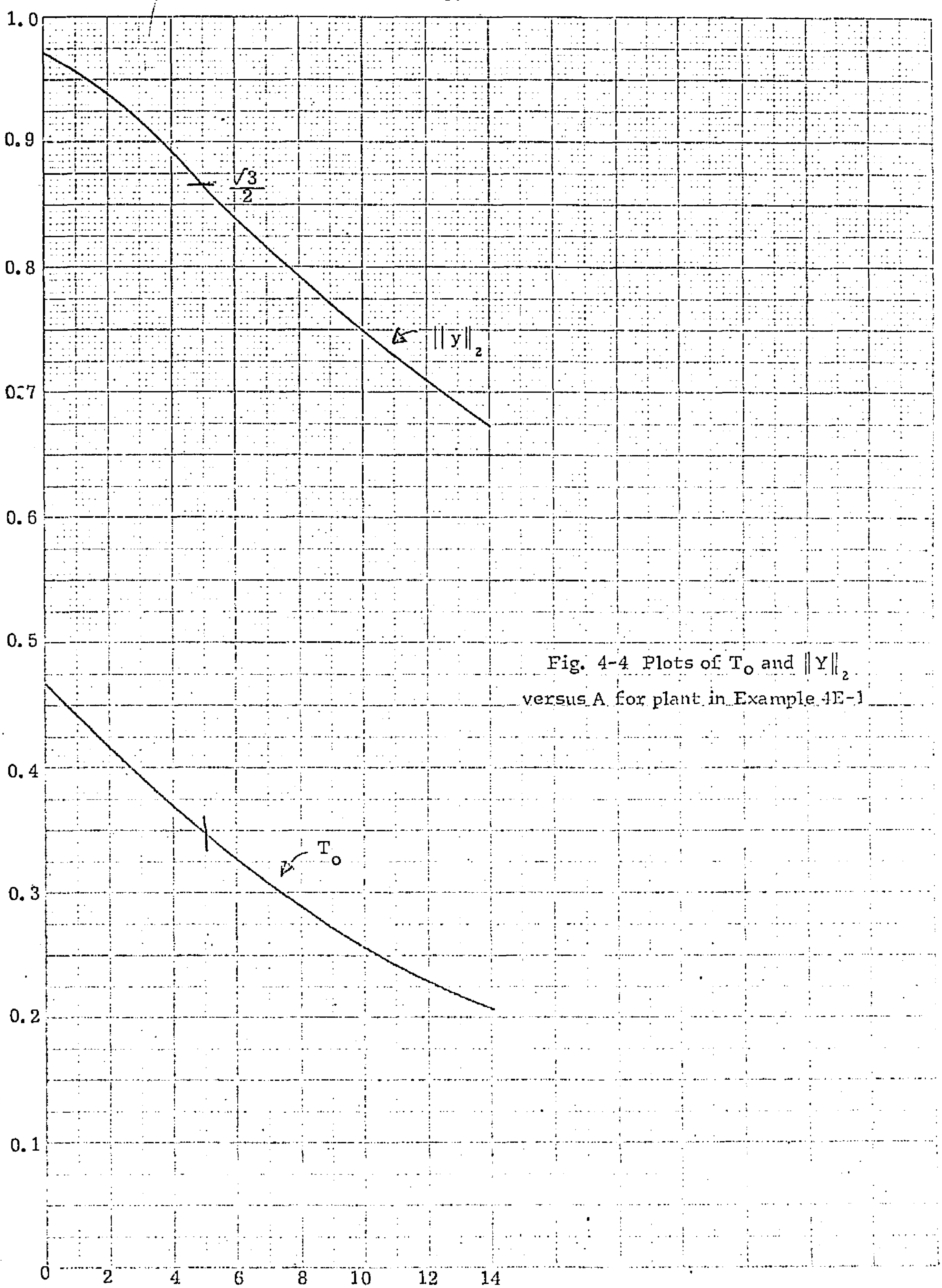


Fig. 4-4 Plots of T_o and $\|Y\|_2$ versus A for plant in Example 4E-1

s-plane regardless of the value of A.

To further investigate the question of stability, consider the plant shown in Figure 4-5. Defining the state variables as shown, the state equations, fundamental matrix, and impulse response of the augmented plant are found to be

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 0 \\ 1 & -(A+1) \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1$$

$$y = [\,0 \quad 1\,]\, \underline{x}$$

$$\phi(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{(A+1)}\left[1-e^{-(1+A)t}\right] & e^{-(1+A)t} \end{bmatrix}$$

$$h_a(t) = \frac{1}{(A+1)}\left[1 - e^{-(1+A)t}\right]$$

Taking the transform of $h_a(t)$, we have

$$\mathcal{L}[h_a(t)] = \frac{1}{s(s+A)} \tag{4E-8}$$

Thus the augmented plant can either be stable or unsable depending on the value of A (the sign of A in this case).

Now suppose that a magnitude constraint is placed on $u_1(t)$, i.e.

$$|u_1(t)| \le L_1$$

T_o is then found from Eq.(4-7), i.e.

$$\|h_a\|_1 = \frac{1}{(1+A)^2}\left[(1+A)T_o - 1 + e^{-(1+A)T_o}\right] = \frac{|y_d|}{L_1}$$

The output norms can be shown to be given by

$$\|y\|_1 = \frac{L_1}{(1+A)^3}\left[\frac{(1+A)^2 T_o^2}{2} - (1+A)T_o + 1 - e^{-(1+A)T_o}\right]$$

and

$$\|y\|_2 = \frac{L_1}{(1+A)^{5/2}}\left[\begin{array}{l}\frac{(1+A)^3 T_o^3}{3} - (1+A)^2 T_o^2 + (1+A)T_o + \\ \frac{1}{2}(1-e^{-2(1+A)T_o}) - 2(1+A)T_o e^{-(1+A)T_o}\end{array}\right]^{\frac{1}{2}}$$

If $y_d = 1.5$ and $L_1 = 0.75$, the problem is to find the optimal w.r.a.s. input and the value of A which satisfy all constraints and the terminal condition is the shortest elapsed time. In order to facilitate the discussion, T_o, $\|y\|_1$, and $\|y\|_2$ are plotted versus A in Figure 4-6. Now consider the following cases:

Case 1. An Integral Constraint on the Output

If $\|y\|_1$ is constrained to be greater than or equal to some number, we have

$$\|y\|_1 \geq L_{2a}$$

Choosing $L_{2a} = 1.0$ it is seen from Figure 4-6 that the output constraint is satisfied in the shortest elapsed time, $T_o = 2.0$, when $A = -1.0$. Also from Eq.(4-5), the optimal w.r.a.s. input is a step of amplitude 0.75.

Making use of Eq.(2-40) with $L_1 = 0.75$ and $L_2 = 1$, $q_1 = 1$ and $q_2 = \infty$, the expression for T^* is

$$\frac{3}{4}(T^* - 1 + e^{-T^*}) + (1) = 1.5$$

whereupon $T^* = 1.426$. Therefore, using the optimal w.r.a.s. procedure increases the elapsed time by about 33%.

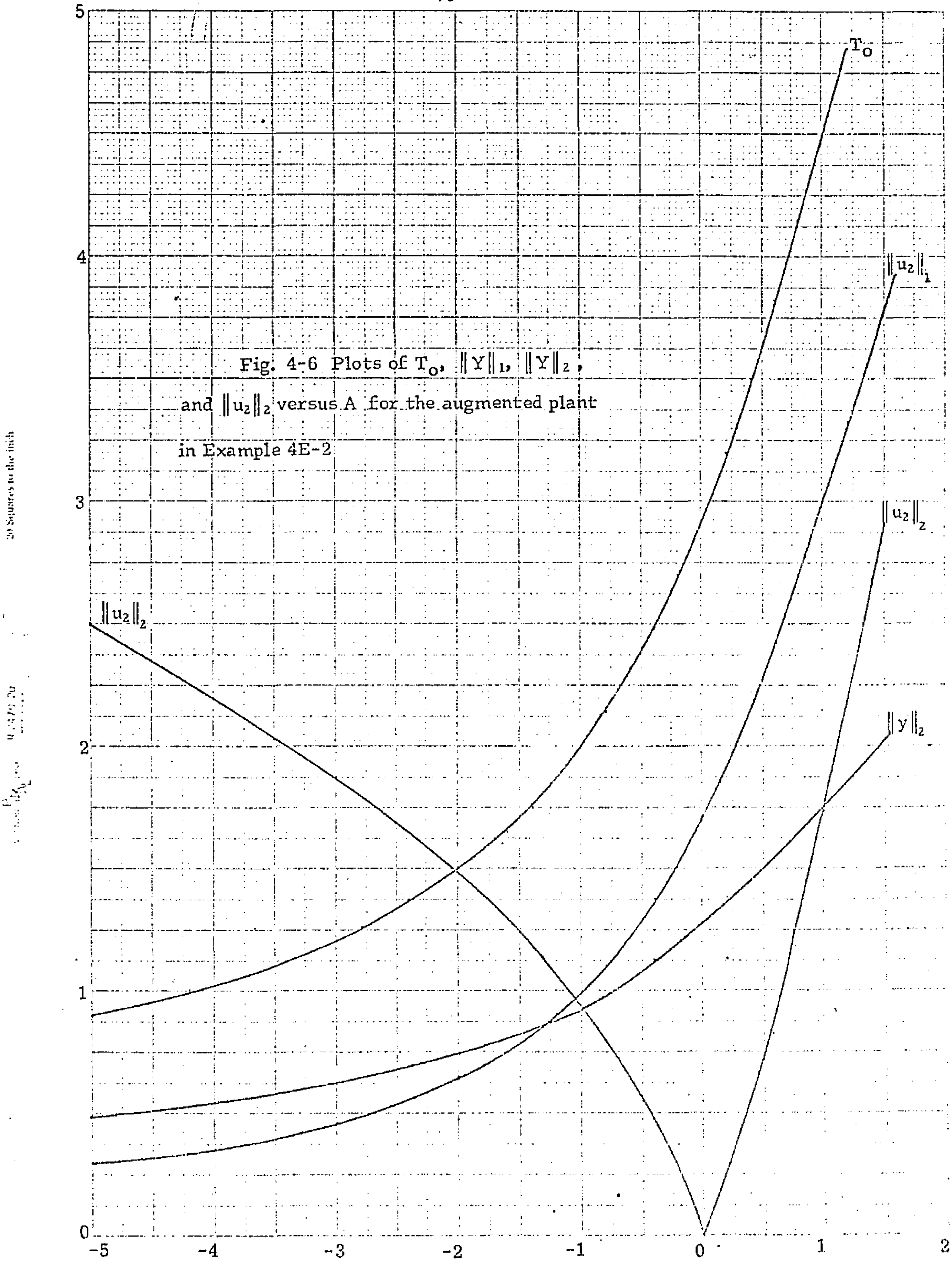


Fig. 4-6 Plots of T_o, $\|Y\|_1$, $\|Y\|_2$, and $\|u_2\|_2$ versus A for the augmented plant in Example 4E-2

We observe (Eq.4E-8) that the transform of the impulse response has a pole in the right hand plane for $A = -1$. The augmented plant is therefore unstable in the sense that its response to a bounded input is unbounded as $t \to \infty$. As the problem actually runs for a finite time, this type of instability may or may not be important. If it is undesirable† to have an unstable augmented plant, Figure 4-6 indicates that the best that can be done under these circumstances is to make A zero. The output constraint is then satisfied with the sign of inequality in an elapsed time of 2.925 minutes and the plant is stable.

The optimal w.r.a.s. solution does not always produce an unstable augmented plant. To illustrate this, suppose that $L_{2a} = 2.5$. Then from Figure 4-6, we find that $A = 0.61$ and $T_O = 3.85$, so that the augmented plant is stable and the output constraint is satisfied with the sign of equality.

Now suppose that in addition to the original output constraint, $L_{2a} = 1.0$, input 2 is constrained as follows:

$$\|u_2\|_2 \leq 0.5$$

Making use of Figure 4-6 again, both constraints can only be satisfied if $[-0.4 \leq A \leq 0.35]$. Under these conditions, $T_O = 2.5$ for $A = -0.4$. The elapsed time has had to be increased in order to satisfy the additional constraint.

† For plants which have poles in the right hand plane, there is a sensitivity problem, i.e. any external disturbance will cause the output trajectory to miss the terminal point.

Case 2. A Quadratic Constraint on the Output

If it is required that $\|y\|_2 \geq L_{2e}$, the quadratic content of the output is constrained. Choosing $L_{2e} = 1.0$, it is seen from Figure 4-6 that this constraint is satisfied in the shortest elapsed time, $T_o = 2.142$ when $A = -0.8$. The optimal w.r.a.s. input is again found to be a step of amplitude 0.75.

Using Eq.(2-40) with $L_1 = 0.75$ and $L_2 = 0.8$, $q_1 = 1$ and $q_2 = 2$, the expression for T^* is

$$0.75\,[T^* - 1 + e^{-T^*}] + 0.8\,[\tfrac{1}{2}(1-e^{-2T^*})]^{\frac{1}{2}} = 1.5$$

whereupon $T^* = 2.132$. Therefore, using the optimal w.r.a.s. procedure increases the elapsed time by less than 1%.

The augmented plant has a pole in the right half plane and is therefore unstable. If a stable plant is required, it is recognized from Figure 4-6 that the best that can be done under these circumstances is to choose $A = 0$. Hence the elapsed time is 2.925 min.

The optimal w.r.a.s. procedure does not always produce an unstable plant when the quadratic content of the output is constrained. This is demonstrated by choosing $L_{2e} = 1.5$ whereupon $A = 0.5$ and $T_o = 3.675$. Since A is positive, the augmented plant is stable.

Now suppose that in addition to the original output constraint, $L_{2a} = 1.0$, input 2 is constrained as follows:

$$\|u_2\|_2 \leq 0.25$$

Making use of Figure 4-6 again, both constraints can only be satisfied if $-0.15 \leq A \leq 0.2$. The minimum elapsed time occurs for $A = -0.15$ and is found to be $T_o = 2.75$.

4.5.2 Two Dimensional Search

4E-3

To illustrate a two-dimensional search, consider the third-order augmented plant shown in Figure 4-7. This form, rather than the one in Figure 4-1 is used for computational convenience, the transformation between the two systems being

$$A_1 = a + b \tag{4E-10a}$$

$$A_2 = ab \tag{4E-10b}$$

Defining the state variables as shown, the normal form description of this plant is

$$\dot{\underline{x}} = \begin{bmatrix} -a & 1 & 0 \\ 0 & -b & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_1 \tag{4E-11a}$$

$$y = [1 \quad 0 \quad 0]\, \underline{x} \tag{4E-11b}$$

The corresponding fundamental matrix is found to be

$$\phi(t) = \begin{bmatrix} e^{-at} & \frac{1}{(a-b)}\left[e^{-bt} - e^{-at}\right] & \frac{1}{ab} + \frac{1}{(a-b)}\left[\frac{e^{-at}}{a} - \frac{e^{-bt}}{b}\right] \\ 0 & e^{-bt} & \frac{1}{b}\left[1-e^{-bt}\right] \\ 0 & 0 & 1 \end{bmatrix} \tag{4E-12}$$

Making use of Eq.(4-2) together with Eqs.(4E-11) and (4E-12), the impulse response is found to be

$$h_a(t) = \frac{1}{ab} + \frac{1}{(a-b)}\left[\frac{e^{-at}}{a} - \frac{e^{-bt}}{b}\right]$$

If a magnitude constraint is imposed on $u_1(t)$, i.e.

$$|u_1(t)| \le L_1$$

T_o is found from Eq.(4-7), i.e.

$$\|h_a\|_1 = \frac{T_o}{ab} + \frac{1}{(a-b)}\left[\frac{(1-e^{-aT_o})}{a^2} - \frac{(1-e^{-bT_o})}{b^2}\right] = \frac{|y_d|}{L_1} \qquad (4E-13)$$

The output norms can be shown to be given by

$$\|y\|_1 = L_1\left[\frac{T_o^2}{2ab} + \frac{1}{(a-b)}\left[\frac{aT_o-(1-e^{-aT_o})}{a^3} - \frac{bT_o-(1-e^{-bT_o})}{b^3}\right]\right] \qquad (4E-14)$$

and

$$\|y\|_2 = L_1\left[\frac{T_o^3}{3(ab)^2} - \frac{T_o^2(a+b)}{(ab)^3} + \frac{(a+b)^2}{(ab)^2}T_o + \frac{2T_o}{(ab)(a-b)}\left[\frac{e^{-aT_o}}{a^3} - \frac{e^{-bT_o}}{b^3}\right] + \frac{2}{(ab)^2(a-b)}\left[\frac{(1-e^{-aT_o})}{a^2} - \frac{(1-e^{-bT_o})}{b^2}\right] + \frac{1}{2(a-b)^2}\left[\frac{(1-e^{-2aT_o})}{a^5} + \frac{(1-e^{-2bT_o})}{b^5} - \frac{4(1-e^{-(a+b)T_o})}{(ab)^2(a+b)}\right]\right]^{\frac{1}{2}} \qquad (4E-15)$$

If $y_d = 1.0$ and $L_1 = 1.0$, the problem is to find the optimal w.r.a.s. input and the values of a and b (and hence A_1 and A_2) which satisfy all constraints and terminal conditions in the shortest time for each of the following cases:

Case 1. An Integral Constraint on the Output

If $\|y\|_1$ is constrained to be greater than or equal to some number, we have

$$\|y\|_1 \geq L_{2a}$$

Letting $L_{2a} = 0.250$ and making use of the search procedure outlined in Chapter VI and implemented with the computer program given in Appendix IV, it is found that

$$T_O = 1.248$$

$$a = -1.50$$

$$b = -1.77$$

or from Eq.(4E-8),

$$A_1 = -3.27$$

$$A_2 = 2.66$$

Using these values in Eq.(4-5), the optimal w.r.a.s. input is seen to be a unit step.

In order to complete T_O with T^*, Eq.(2-40) is used with $L_1 = 1$, $L_2 = 0.25\,|A_2|$, $L_3 = 0.25\,|A_1|$, $q_1 = 1$, and $q_2 = q_3 = \infty$. The impulse responses of the given plant are easily found to be

$$h_1(t) = \frac{t^2}{2}$$

$$h_2(t) = t \qquad (4E-17)$$

$$h_3(t) = 1$$

whereupon the expression for T^* is

$$\frac{(T^*)^3}{6} + 0.665\,T^* + 0.82 = 1 \qquad (4E-18)$$

Solving this equation, we find

$$T^* = 0.270$$

Therefore, the elapsed time is increased 4.6 times when the optimal w.r.a.s. solution is employed.

From the expression for $h_a(t)$, it is noted that the above values of a and b produce an unstable augmented plant. If a stable plant is required, the best that can be done under these circumstances is to choose a =b =0. The output constraint is then satisfied with the sign of inequality in an elapsed time of 1.816 minutes.

The optimal w.r.a.s. solution does not always produce an unstable plant. If L_{2a} is made larger than 0.46, only positive values of a and b will satisfy the more severe output constraint and the augmented plant will be stable.

Case 2. A Quadratic Constraint on the Output

If $\|y\|_2$ is constrained to be greater than or equal to some number, we have

$$\|y\|_2 \geq L_{2e} \qquad (4E\text{-}19)$$

Letting $L_{2e} = 0.400$ and employing the same type of computer search as in case 1 , it is found that

$$T_o = 1.375$$

$$a = -1.20$$

$$b = -1.06$$

or from Eq.(4E-8)

$$A_1 = -2.26$$

$$A_2 = 1.27$$

The optimal w.r.a.s. input is again found to be a unit step. Using Eq. (2-40) with $q_2 = q_3 = 2$ and the impulse responses in Eq.(4E-17), the expression for T^* is

$$\frac{(T^*)^3}{6} + 0.904\,(T^*)^{\frac{1}{2}} + 0.294(T^*)^{3/2} = 1$$

Solving this equation, we find

$$T^* = 0.715$$

Thus, the elapsed time must be increased 1.9 times when the optimal w.r.a.s. solution is employed.

The previous remarks concerning the stability of the augmented plant are also valid for this case with the exception that positive values of a and b are found when the output constraint, L_{2e}, is made greater than 0.51.

The computer output for this example is given in Appendix V.

4.6 Optimal W.R.A.S. Solution of Multiple Output Plants

The optimal w.r.a.s. solution of the n^{th} order, linear time-varying, augmented plant shown in Figure 1-2 is now considered. The normal form description of this plant is given by Eqs.(4-1) with y replaced by the m-vector, $\underline{y}$. In addition C(t) and D(t) are respectively $(m \times n)$ and $(m \times 1)$ time-varying matrices. All other quantities are unchanged.

The impulse response matrix for this system is given by the right hand side of Eq.(4-2) [with C(t) and D(t) defined above] and is an m-vector which is written in the following manner:

$$\underline{H}_a(t,\tau) = \begin{bmatrix} h_{a_1}(t,\tau) \\ h_{a_2}(t,\tau) \\ \vdots \\ h_{a_m}(t,\tau) \end{bmatrix} \qquad (4\text{-}24)$$

where $h_{a_j}(t,\tau)$ $(j=1,2,\ldots m)$ is the j^{th} output when $u_1(t) = \delta(t-\tau)$. Obviously $\underline{H}_a(t,\tau)$ is a function of the feedback coefficients, A_{ij}. Problem 5 can now be restated for this class of systems as follows:

Given the single input, m-output plant characterized by the impulse response matrix, $\underline{H}_a(t,\tau)$, find the set of feedback coefficients and the input, $u_1(t)$, which drives the output vector from the origin to some terminal point in output space, $\underline{y}_d$, and which permits the input constraint (Eq.4-3) and an output constraint

$$\|y_i\|_{p_2} \triangleq \left[\int_{t_o}^{t_f} |y_i(t)|^{p_2}\, dt\right]^{1/p_2} \geq L_{p_2} \qquad [p_2 = 1,2] \quad (4\text{-}25a)$$

$$|y_i(t)| \leq L_{p_2} \qquad [p_2 = \infty] \quad (4\text{-}25b)$$

to be satisfied in the shortest elapsed time, $T_o \triangleq t_{f_o} - t_o$.

4.6.1 The Optimal W.R.A.S. Solution

The method which is used to solve this problem parallels the one employed in Section 4.2.1. The output constraints are ignored and the input which satisfies the terminal conditions and the input constraint in the shortest elapsed time is found. Using the method of Functional Analysis for multiple output plants (see Section 2.2.4a), the form of this input is given by Eq.(2-37), i.e.

$$u_1(t) = L_1^{q_1} \mid k_a^*(t_{f_o},t)\mid^{q_1-1} \text{SGN}[k_a^*(t_{f_o},t)] \tag{4-26}$$

where

$$k_a^*(t,\tau) \triangleq [\lambda_1^*, \lambda_2^* \ldots \lambda_m^*] \cdot \underline{H}_a(t,\tau) = \sum_{i=1}^{m} \lambda_i^* h_{a_i}(t,\tau) \tag{4-27}$$

The subscript "a" indicates that these results are for the augmented plant. It is recalled that the λ^*'s are those λ's which minimize $\|k_a\|_{q_1}$ subject to the condition

$$\underline{\lambda}' \cdot \underline{y}_d = 1 \quad . \tag{4-28}$$

t_{f_o} is then the minimum value of t_f which satisfies Eq.(2-38), i.e.

$$\|k_a^*\|_{q_1} = L_1\left[\int_{t_o}^{t_f} |k_a^*(t_f,\tau)|^{q_1} d\tau\right]^{1/q_1} = 1 \tag{4-29}$$

Finally, the set of feedback coefficients is chosen so that the output constraints are satisfied in the shortest elapsed time.

Employing an argument similar to that found after Eq.(4-7), we conclude that $u_1(t)$ given in Eq.(4-26) is the optimal w.r.a.s. input for multiple output plants. In addition, it is observed that the inputs u_2 to u_r belong to the class

$$u_j(t) = \sum_{i=1}^{m} A_{ji}\, y_i(t) \qquad [j=2,3,\ldots r] \tag{4-30}$$

See Figure 1-2.

If, however, they are not restricted to this class, another set of inputs (the optimal) can, in general, be found which satisfies all conditions in a shorter time than that found from Eq.(4-29).

Usually a closed form expression for $\|k_a\|_{q_1}$ cannot be obtained and a numerical integration is generally required. As the minimization of this norm requires taking partial derivatives with respect to λ_i and equating the resulting expressions to zero, and since these lambda's will, in general, be functions of t_f, it is clear that the mathematical problem of finding the λ^*'s is by no means a trivial one. We conclude from this discussion that the optimal w.r.a.s. solution of multiple output plants generally requires an involved mathematical procedure. This should be compared with the optimal w.r.a.s. solution of single output plants where only the routine calculation of finding the real roots of a transcendental equation is required.

The solution of Eq.(4-29) is a necessary condition for the existence of an optimal w.r.a.s. solution. This is best seen by noting that if, for a specific set of feedback coefficients, a value of T_o cannot be found from Eq.(4-29), the input given by Eq.(4-26) will not drive the vector output to $\underline{y}_d$.

4.7 Another Necessary Condition for the Existence of a Solution

In this section, another condition which must be satisfied in order for an optimal w.r.a.s. solution to Problem 5 to exist is developed.

Making use of the superposition integral together with the optimal w.r.a.s. input (Eq. 4-26), the i^{th} output is found to be

$$y_i(t) = \int_{t_o}^{t} h_{ai}(t,\tau) L_1^{q_1} \left| k_a^*(t_{f_o},\tau) \right|^{q_1 - 1} \mathrm{SGN}\left[k_a^*(t_{f_o},\tau)\right] d\tau \qquad (4\text{-}31)$$

$$(i = 1, 2, \ldots m)$$

Case 1. Integral Constraints on All of the Outputs

Suppose that each of the m-outputs has an integral constraint placed on it which is expressed in the form,

$$\|y_i\|_1 = \int_{t_o}^{t_{f_o}} |y_i(t)|\, dt \geq L_{2_i} a \qquad (i = 1, 2, \ldots m) \qquad (4\text{-}32)$$

Using Eq.(4-31), the norm in this last equation becomes

$$\|y_i\|_1 = L_1^{q_1} \int_{t_o}^{t_{f_o}} \left| \int_{t_o}^{t} h_{ai}(t,\tau) \left|k_a^*(t_{f_o},\tau)\right|^{q_1-1} \mathrm{SGN}[k_a^*(t_{f_o},\tau)]\, d\tau \right| dt$$

or

$$\|y_i\|_1 \le L_1^{q_1} \int_{t_o}^{t_{f_o}} \left[\int_{t_o}^{t} |h_{ai}(t,\tau)|\, \left|k_a^*(t_{f_o},\tau)\right|^{q_1-1} d\tau \right] dt$$

whereupon

$$\|y_i\|_1 < L_1^{q_1} \int_{t_o}^{t_{f_o}} \left[\int_{t_o}^{t_{f_o}} |h_{ai}(t_{f_o},\tau)|\, \left|k_a^*(t_{f_o},\tau)\right|^{q_1-1} d\tau \right] dt \quad (4\text{-}33)$$

Applying Hölder's inequality to the right hand side of this expression, we find

$$\|y_i\|_1 < L_1^{q_1} \int_{t_o}^{t_{f_o}} \|h_{ai}\|_{q_2} \left[\int_{t_o}^{t_{f_o}} \left|k_a^*(t_{f_o},\tau)\right|^{(q_1-1)p_2} d\tau \right]^{1/p_2} dt \quad (4\text{-}34)$$

where $\frac{1}{p_2} + \frac{1}{q_2} = 1$. In order to make use of Eq.(4-29), let us choose $p_2 = \frac{q_1}{q_1-1}$. Then $q_2 = q_1$ and inequality (4-34) becomes

$$\|y_i\|_1 < L_1^{q_1} \int_{t_o}^{t_{f_o}} \|h_{ai}\|_{q_2} \left(\|k_a^*\|_{q_1}\right)^{q_1-1} dt = L_1^{q_1} \|h_{ai}\|_{q_1} \left(\|k_a^*\|_{q_1}\right)^{q_1-1} T_o \quad (4\text{-}35)$$

Substituting Eq.(4-29) into this inequality gives

$$\|y_i\|_1 < L_1 T_o \|h_{ai}\|_{q_1} \quad (4\text{-}36)$$

Combining Inequalities (4-36) and (4-32) yields

$$L_{2_i} a \le \|y_i\|_1 < L_1 T_o \|h_{ai}\|_{q_1}$$

or

$$T_o \|h_{ai}\|_{q_1} > \frac{L_{2_i} a}{L_1} . \qquad (i = 1, 2, \ldots m) \quad (4\text{-}37)$$

Satisfying Inequality (4-37) for all i is a necessary condition for the existence of an optimal w.r.a.s. solution to Problem 5 when all of the outputs have an integral constraint placed on them. The reason for this is that if the inequality is violated for any i, the i^{th} output constraint cannot be satisfied. However, if all of the inequalities are satisfied, the output constraints may or may not be violated. Hence, satisfying (4-37) is a necessary but not a sufficient condition.

Case 2. Quadratic Constraints on All of the Outputs

Suppose that each of the m-outputs has a quadratic constraint placed on it which is expressed in the form

$$(\|y_i\|_2)^2 = \int_{t_o}^{t_{f_o}} [y_i(t)]^2\, dt \geq L_{2_i e} \qquad (i=1,2,\ldots m) \qquad (4\text{-}38)$$

Making use of the output in Eq.(4-31), we have

$$(\|y_i\|_2)^2 = \int_{t_o}^{t_{f_o}} L_1^{\,2q_1} \left|\int_{t_o}^{t} h_{ai}(t,\tau)\,|k_a^*(t_{f_o},\tau)|^{q_1-1} \mathrm{SGN}[k_a^*(t_{f_o},\tau)]d\tau\right|^2 dt$$

or

$$(\|y_i\|)_2^2 \leq L_1^{\,2q_1} \int_{t_o}^{t_{f_o}}\left[\int_{t_o}^{t} |h_{ai}(t,\tau)|\ |k_a^*(t_{f_o},\tau)|^{q_1-1} d\tau\right]^2 dt$$

and

$$(\|y_i\|)_2^2 < L_1^{\,2q_1} \int_{t_o}^{t_{f_o}}\left[\int_{t_o}^{t_{f_o}} |h_{ai}(t_{f_o},\tau)|\ |k_a^*(t_{f_o},\tau)|^{q_1-1}\, d\tau\right]^2 dt \qquad (4\text{-}39)$$

Using Hölder's inequality Eq.(4-39) becomes

$$(\|y_i\|)_2^2 < L_1^{\,2q_1} \int_{t_o}^{t_{f_o}} (\|h_{ai}\|_{q_2})^2 \left[\int_{t_o}^{t_{f_o}} |k_a^*(t_{f_o},\tau)|^{(q_1-1)p_2}\, d\tau\right]^2 dt \qquad (4\text{-}40)$$

Applying the same argument as that following Inequality (4-34), the last expression becomes

$$(\|y_i\|_2)^2 < L_1^{2q_1} T_o (\|h_{ai}\|_{q_1})^2 (\|k_a^*\|_{q_1})^{2(q_1-1)}$$

or making use of Eq.(4-29)

$$(\|y_i\|_2)^2 < L_1^2 T_o (\|h_{ai}\|_{q_1})^2 \tag{4-41}$$

Combining Inequalities (4-38) and (4-41) yields

$$L_{2_i e}^2 \leq (\|y_i\|_2)^2 < L_1^2 T_o (\|h_{ai}\|_{q_1})^2$$

or

$$T_o(\|h_{ai}\|_{q_1})^2 > \left(\frac{L_{2_i e}}{L_1}\right)^2 \qquad (i=1,2,\ldots m) \tag{4-42}$$

Satisfying Inequality (4-42) for all i is a necessary condition for the existence of an optimal w.r.a.s. solution to Problem 5 when all of the outputs have quadratic constraints placed on them. The reason for this is that if the inequality is violated for any i, the i^{th} output constraint cannot be satisfied. However, if all of the inequalities are satisfied, the output constraints may or may not be violated. Hence, satisfying (4-42) is a necessary but not a sufficient condition.

Case 3. Magnitude Constraint on All of the Outputs

Suppose that all of the outputs have their magnitudes limited as follows:

$$|y_i(t)| \leq L_{2_i m} \qquad (i=1,2,\ldots m) \tag{4-43}$$

Taking magnitudes on both sides of Eq.(4-31) we find

$$|y_i(t)| = |L_1^{q_1} \int_{t_o}^{t} h_{ai}(t,\tau)|k_a^*(t_{f_o},\tau)|^{q_1-1} \mathrm{SGN}[k_a^*(t_{f_o},\tau)]\,d\tau|$$

$$\leq L_1^{q_1} \int_{t_o}^{t} |h_{ai}(t,\tau)|\,|k_a^*(t_{f_o},\tau)|^{q_1-1}\,d\tau$$

$$|y_i(t)| < L_1^{q_1} \int_{t_o}^{t_{f_o}} |h_{ai}(t_{f_o},\tau)|\,|k_a^*(t_{f_o},\tau)|^{q_1-1}\,d\tau$$

Making use of Hölder's inequality and the argument following Inequality (4-34), the last expression becomes

$$|y_i(t)| < L_1^{q_1} \, \|h_{ai}\|_{q_1} \, (\|k_a^*\|_{q_1})^{q_1-1}$$

or using Eq.(4-29),

$$|y_i(t)| < L_1 \, \|h_{ai}\|_{q_1} \qquad (i=1,2,\ldots m) \qquad (4\text{-}44)$$

Satisfying Inequality (4-44) for all i is a necessary, but not a sufficient condition for the existence of an optimal w.r.a.s. solution to the given problem when all of the outputs are magnitude constrained.

The discussion of how these necessary conditions are used in a computation scheme is deferred until Chapter VI (Section 6.3.2).

4.8 Finding the λ's When the Quadratic Content of the Input is Constrained

In this section, it is demonstrated that an algebraic technique can be used to find the λ^{*}'s and t_{f_o} when a quadratic constraint is placed on $u_1(t)$, i.e. $p_1 = 2$ in Eq.(4-3).

In order to find t_{f_o}, it is recalled that $\|k_a\|_{q_1}$ must be minimized with respect to λ_i subject to the condition in Eq.(4-28). From Eq.(4-6), if $p_1 = 2$, $q_1 = 2$. Using the definition of $k_a(t)$ (Eq.2-12a) together with Eq.(4-24), the quadratic norm of k_a is

$$(\|k_a\|_2)^2 = \int_{t_o}^{t_f} [\,\underline{\lambda}' \cdot \underline{H}_a(t_f,\tau)]^2 \, d\tau$$

or

$$(\|k_a\|_2)^2 = \int_{t_o}^{t_f} [\lambda_1 h_{a_1}(t_f,\tau) + \lambda_2 h_{a_2}(t_f,\tau) + \ldots \lambda_m h_{am}(t_f,\tau)]^2 d\tau$$

(4-46)

Solving for λ_1 in Eq.(4-28) (assuming that $y_{d_1} \neq 0$) and substituting the result into Eq.(4-46), we have

$$(\|k_a\|_2)^2 = \int_{t_o}^{t_f} \left[\left[\left(\frac{1}{y_{d_1}}, \lambda_2, \lambda_3 \ldots \lambda_m\right) \right] \cdot \begin{bmatrix} h_{a_1}(t_f,\tau) \\ h_{a_2}(t_f,\tau) - \frac{y_{d_2}}{y_{d_1}} h_{a_1}(t_f,\tau) \\ \vdots \\ h_{am}(t_f,\tau) - \frac{y_{d_m}}{y_{d_1}} h_{a_1}(t_f,\tau) \end{bmatrix} \right]^2 d\tau \tag{4-47}$$

Now define

$$(\underline{\lambda}')_\beta \triangleq \left(\frac{1}{y_{d_1}}, \lambda_2, \lambda_3, \ldots \lambda_m\right) \tag{4-48a}$$

$$\underline{H}_a(t_f,\tau)_\beta \triangleq \underline{H}_a(t_f,\tau) - \frac{h_{a_1}(t_f,\tau)}{y_{d_1}} \begin{bmatrix} 0 \\ y_{d_2} \\ y_{d_3} \\ \vdots \\ y_{d_m} \end{bmatrix} \tag{4-48b}$$

Then Eq.(4-47) becomes

$$(\|k_a\|_2)^2 = \int_{t_o}^{t_f} [(\underline{\lambda}')_\beta \cdot \underline{H}_a(t_f,\tau)_\beta]^2 \, d\tau$$

To minimize this norm, we must take partial derivatives with respect to λ_j $(j=2,3,\ldots m)$ and set the resulting expressions equal to zero. Thus

$$\begin{aligned}
\frac{\partial}{\partial \lambda_2}: &\quad 2\int_{t_o}^{t_f} [(\underline{\lambda}')_\beta \cdot \underline{H}_a(t_f,\tau)_\beta](h_{a_2}(t_f,\tau) - \frac{y_{d_2}}{y_{d_1}} h_{a_1}(t_f,\tau)) d\tau = 0 \\
\frac{\partial}{\partial \lambda_3}: &\quad 2\int_{t_o}^{t_f} [(\underline{\lambda}')_\beta \cdot \underline{H}_a(t_f,\tau)_\beta](h_{a_3}(t_f,\tau) - \frac{y_{d_3}}{y_{d_1}} h_{a_1}(t_f,\tau)) d\tau = 0 \\
&\quad \vdots \\
\frac{\partial}{\partial \lambda_m}: &\quad 2\int_{t_o}^{t_f} [(\underline{\lambda}')_\beta \cdot \underline{H}_a(t_f,\tau)_\beta](h_{a_m}(t_f,\tau) - \frac{y_{d_m}}{y_{d_1}} h_{a_1}(t_f,\tau)) d\tau = 0
\end{aligned} \tag{4-49}$$

Multiplying the first of these equations by λ_2, the second by λ_3, etc., and adding gives

$$\int_{t_o}^{t_f}[(\underline{\lambda}')_\beta \cdot \underline{H}_a(t_f,\tau)_\beta]\left[\begin{array}{l}\lambda_2[h_{a_2}(t_f,\tau)-\frac{y_{d_2}}{y_{d_1}}h_{a_1}(t_f,\tau)]+\lambda_3[h_{a_3}(t_f,\tau)-\frac{y_{d_3}}{y_{d_1}}h_{a_1}(t_f,\tau)]\\ +\ldots+\lambda_m[h_{a_m}(t_f,\tau)-\frac{y_{d_m}}{y_{d_1}}h_{a_1}(t_f,\tau)]\end{array}\right]d\tau=0 \tag{4-50}$$

Subtracting this equation from Eq.(4-47) yields

$$(\|k_a\|_2)^2=\int_{t_o}^{t_f}[(\underline{\lambda}')_\beta \cdot \underline{H}_a(t_f,\tau)_\beta]\frac{h_{a_1}(t_f,\tau)}{y_{d_1}}d\tau$$

In order to find t_{f_o} we must solve (Eq.4-29)

$$(\|k_a^*\|_2)^2=\int_{t_o}^{t_{f_o}}[(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]\frac{h_{a_1}(t_{f_o},\tau)}{y_{d_1}}d\tau=\frac{1}{L_1^2} \tag{4-51}$$

where the asterisks indicate that the λ's used are those which minimize $\|k_a\|_2$, i.e. satisfy Eqs.(4-49). Equations (4-49) are now simplified by writing each equality as

$$\int_{t_o}^{t_{f_o}}[(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]h_{aj}(t_{f_o},\tau)d\tau=y_{d_j}\int_{t_o}^{t_{f_o}}[(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau]\frac{h_{a_1}(t_{f_o},\tau)}{y_{d_1}}d\tau$$

and substituting Eq.(4-51) on the right hand side of each of these equations. This gives

$$\begin{aligned}
&\int_{t_o}^{t_{f_o}}[(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]h_{a_1}(t_{f_o},\tau)d\tau=\frac{y_{d_1}}{L_1^2}\\
&\int_{t_o}^{t_{f_o}}[(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]h_{a_2}(t_{f_o},\tau)d\tau=\frac{y_{d_2}}{L_1^2}\\
&\qquad\vdots\\
&\int_{t_o}^{t_{f_o}}[(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]h_{a_m}(t_{f_o},\tau)d\tau=\frac{y_{d_m}}{L_1^2}
\end{aligned} \tag{4-52}$$

Since $\underline{H}_a(t_{f_o},\tau)_\beta$ and therefore all of its elements are independent of the $\lambda^{*'}$s, Eqs(4-52) represent a set of m-equations which are linear in the (m-1)-$\lambda^{*'}$s and non-linear in t_{f_o}. As a consequence, these $\lambda^{*'}$s can be eliminated leaving a single equation in t_{f_o}. Thus the problem of finding t_{f_o} is reduced to finding the real root of a transcendental equation, a routine calculation. Once the terminal time is known, the $\lambda^{*'}$s are found by solving a set of (m-1) simultaneous linear equations. Any of the well known techniques can be used.†

Equations (4-52) can be obtained from the expressions for the terminal conditions as is now demonstrated. Since $q_1 = 2$ the optimal w.r.a.s. input in Eq.(4-26) becomes

$$u_1(t) = L_1^2\, k_a^*(t_{f_o},t) = L_1^2\,[\underline{\lambda}^{*'}\cdot \underline{H}_a(t_{f_o},t)] \tag{4-53}$$

Making use of the superposition integral, the outputs at $t = t_{f_o}$ become

$$\begin{aligned}
y_1(t_{f_o}) &\triangleq y_{d_1} = \int_{t_o}^{t_{f_o}} h_{a_1}(t_{f_o},\tau)\,L_1^2\,[\underline{\lambda}^{*'}\cdot \underline{H}_a(t_{f_o},\tau)]\,d\tau\\
y_2(t_{f_o}) &\triangleq y_{d_2} = \int_{t_o}^{t_{f_o}} h_{a_2}(t_{f_o},\tau)\,L_1^2\,[\underline{\lambda}^{*'}\cdot \underline{H}_a(t_{f_o},\tau)]\,d\tau\\
&\vdots \qquad\qquad\qquad \vdots\\
y_m(t_{f_o}) &\triangleq y_{d_m} = \int_{t_o}^{t_{f_o}} h_{a_m}(t_{f_o},\tau)\,L_1^2\,[\underline{\lambda}^{*'}\cdot \underline{H}_a(t_{f_o},\tau)]\,d\tau
\end{aligned} \tag{4-54}$$

Using Eq.(4-28) and solving for λ_1^* gives

$$\lambda_1^* = \frac{1-\sum_{j=2}^{m}\lambda_j^* y_{d_j}}{y_{d_1}} \qquad (y_{d_1} \neq 0)$$

whereupon

$$\underline{\lambda}^{*'}\cdot \underline{H}_a(t_{f_o},\tau) = (\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta$$

---

† For example, a Gauss-Jordan reduction or a pivotal condensation can be employed[35].

Making use of this result, Eqs. (4-54) become

$$\frac{y_{d_1}}{L_1^2} = \int_{t_o}^{t_{f_o}} [(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]\, h_{a_1}(t_{f_o},\tau)\, d\tau$$

$$\frac{y_{d_2}}{L_1^2} = \int_{t_o}^{t_{f_o}} [(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]\, h_{a_2}(t_{f_o},\tau)\, d\tau \qquad (4\text{-}55)$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{y_{d_m}}{L_1^2} = \int_{t_o}^{t_{f_o}} [(\underline{\lambda}^{*'})_\beta \cdot \underline{H}_a(t_{f_o},\tau)_\beta]\, h_{a_m}(t_{f_o},\tau)\, d\tau$$

Equations (4-52) and (4-55) are identical; therefore, the λ^{*}'s and t_{f_o} can be found by solving the set of equations which results from satisfying the terminal conditions using the optimal w.r.a.s. input and Eq.(4-28).

4.9 Finding the λ's When the Input is Magnitude Constrained

The problem of finding the λ^{*}'s when the input has its magnitude constrained is a considerably more difficult problem. In the expressions which result from taking partials of $\|k_a\|_1$ with respect to the λ_i's, there is no linear dependence upon the λ's. A technique used to solve problems for systems having two outputs and two inputs under these conditions is now presented.

As there are two outputs and the augmented plant has only one input (see Figure 1-2), $k_a(t)$ is a scalar. Using Eqs.(2-12a) and (2-38) with $q = 1$, we find

$$\|k_a\|_1 = \int_{t_o}^{t_f} |\lambda_1 h_{a_1}(t_f,\tau) + \lambda_2 h_{a_2}(t_f,\tau)|\, dt \qquad (4\text{-}56)$$

Making use of Eq.(4-28), λ_1 is eliminated and Eq.(4-56) becomes

$$\|k_a\|_1 = \int_{t_o}^{t_f} \left| \frac{h_{a_1}(t_f,\tau)}{y_{d_1}} + \lambda_2 \left[h_{a_2}(t_f,\tau) - \frac{y_{d_2}}{y_{d_1}} h_{a_1}(t_f,\tau)\right] \right| d\tau$$

Taking the partial derivative of the right hand side of this equation with respect to λ_2 and utilizing the results of Chapter III, λ_2^* must satisfy the following equation:

$$\int_{t_o}^{t_f} \left[h_{a_2}(t_f,\tau) - \frac{y_{d_2}}{y_{d_1}} h_{a_1}(t_f,\tau)\right] \mathrm{SGN}\left[\frac{h_{a_1}(t_f,\tau)}{y_{d_1}} + \lambda_2 \left[h_{a_2}(t_f,\tau) - \frac{y_{d_2}}{y_{d_1}} h_{a_1}(t_f,\tau)\right]\right] d\tau = 0 \tag{4-58}$$

In addition, t_{f_o} is calculated from Eq.(4-29), i.e.

$$\int_{t_o}^{t_f} \left| \frac{h_{a_1}(t_f,\tau)}{y_{d_1}} + \lambda_2 \left[h_{a_2}(t_f,\tau) - \frac{y_{d_2}}{y_{d_1}} h_{a_1}(t_f,\tau)\right] \right| d\tau = \frac{1}{L_1} \tag{4-59}$$

The values of t_f and λ_2 which permit the last two equations to be simultaneously satisfied are t_{f_o} and λ_2^*. It should be noted that this is a considerably more difficult computational problem than that of the previous section. Discussion of the computational procedure which is used to solve the following examples is deferred until Chapter VI (Section 6.2.2).

4.10 Examples - Multiple Output Plants

The results of Sections 4.6 through 4.9 are illustrated in Figure 4-8 for the two-output augmented plant.

For the six examples presented in this section, the optimal w.r.a.s. approach is used to solve the following problem:

Determine the values of the feedback coefficients A and B and find the input, $u_1(t)$, which drives the two outputs from the origin to the terminal point

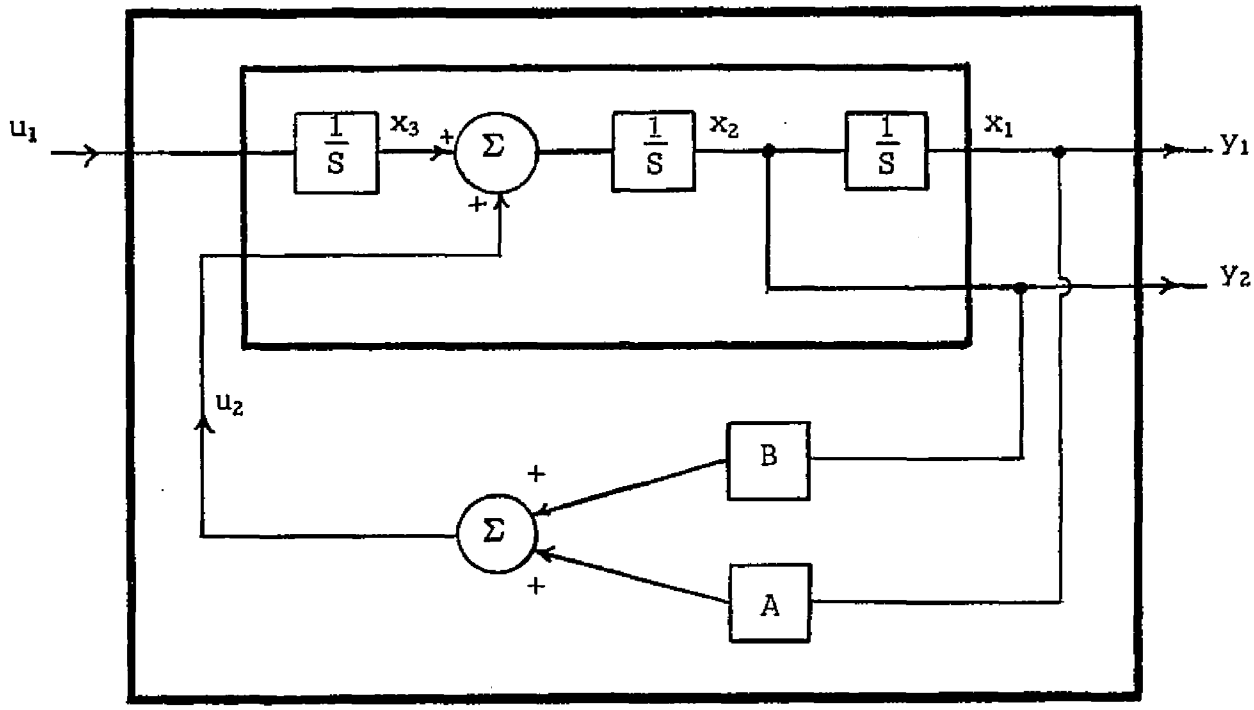


Figure 4-8 Augmented Plant used in Examples 4E-4,5,6,7,8,9

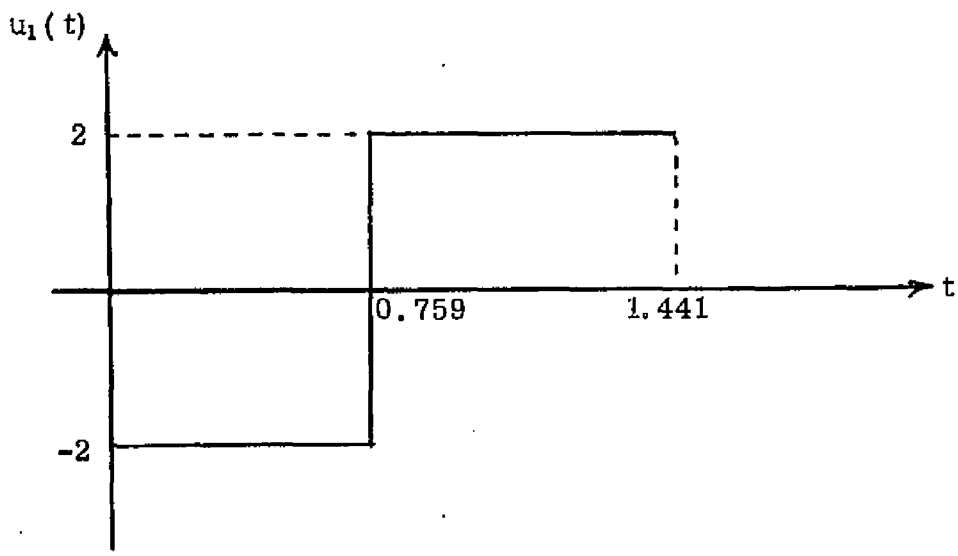


Figure 4-9 Optimal w.r.a.s. input for Example 4E-9

$$\underline{y}(T_o) \triangleq \underline{y}_d = \begin{bmatrix} y_{d_1} \\ y_{d_2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

in the shortest elapsed time, and permits the constraints in Eqs.(4-3) and (4-25) to be satisfied.

In the first three examples $u_1(t)$ has a quadratic constraint on it and the method introduced in Section 4.8 is used. For the last three examples a magnitude constraint is placed on this input and the results of Section 4.9 are utilized.

A comparison between T_o and T^* is made in accordance with the discussion in Section 1.3. Thus Eqs. (1-5) and (1-6) become

$$\|u_2\|_{p_{2i}} \leq L_2 \tag{4E-20}$$

and

$$L_2 = |A|L_{12} + |B|L_{22} \tag{4E-21}$$

where L_{12}, L_{22}, and the p_{2i} are defined in Eq.(4-25), and T^* is calculated from Eq.(2-33) using the impulse responses of the given plant. Using the previously defined terminal conditions and Eqs. (2.12a), (4-28), and (2-31), the expression for T^* becomes

$$L_1\left[\int_o^{T^*} |t^2 + \lambda_{II}^* t|^{q_1} dt\right]^{1/q_1} + L_2\left[\int_o^{T^*} |2t + \lambda_{II}^*|^{q_2} dt\right]^{1/q_2} = 1 \tag{4E-22}$$

where λ_{II}^* is the value of λ_{II} that minimizes $\|\underline{k}\|_1$.

Defining the state variables for the augmented plant as shown in Figure 4-8, the state equations, fundamental matrix and impulse response matrix are respectively

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ A & B & 1 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_1$$

$$\underline{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x}$$

$$\phi(t) = \begin{bmatrix} \dfrac{[(r_1-B)e^{r_1t} - (r_2-B)e^{r_2t}]}{(r_1-r_2)} & \dfrac{[e^{r_1t} - e^{r_2t}]}{(r_1-r_2)} & \dfrac{1 + \dfrac{(r_2e^{r_1t} - r_1e^{r_2t})}{(r_1-r_2)}}{r_1 r_2} \\ \dfrac{A[e^{r_1t} - e^{r_2t}]}{(r_1-r_2)} & \dfrac{[r_1e^{r_1t} - r_2e^{r_2t}]}{(r_1-r_2)} & \dfrac{[e^{r_1t} - e^{r_2t}]}{(r_1-r_2)} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{H}_a(t) = \begin{bmatrix} -\dfrac{1}{A}\left[1 + \dfrac{1}{(r_1-r_2)}\ (r_2 e^{r_1t} - r_1e^{r_2t})\right] \\ \dfrac{1}{(r_1-r_2)}(e^{r_1t} - e^{r_2t}) \end{bmatrix} \qquad (4E\text{-}23)$$

where

$$r_1 = \frac{B}{2} + \left(\frac{B^2}{4} + A\right)^{\frac{1}{2}}$$

$$r_2 = \frac{B}{2} - \left(\frac{B^2}{4} + A\right)^{\frac{1}{2}} \qquad (4E\text{-}24)$$

Substituting the values of y_{d_1} and y_{d_2} into Eq.(4-28), one of the λ^*'s is immediately determined, i.e. $\lambda_2^* = 2$.

4.10.1 A Quadratic Constraint on the Input

If $u_1(t)$ has a quadratic constraint placed on it

$$\|u_1\|_2 \leq L_1 \qquad (4E\text{-}25)$$

and the results of Section 4.8 are applicable. Using the value of λ_2^* found above and Eqs. (4-53) and (4E-23), the optimal w.r.a.s. input is

$$u_1(t) = L_1^2\left[\frac{-2}{A}\left[1+\frac{1}{(r_1-r_2)}\left(r_2e^{r_1(T_O-t)}-r_1e^{r_2(T_O-t)}\right)\right]+\frac{\lambda_2^*}{(r_1-r_2)}\left[e^{r_1(T_O-t)}-e^{r_2(T_O-t)}\right]\right] \tag{4E-26}$$

Utilizing this result in the superposition integral, the two outputs at $t = T_O$ are:

$$y_1(T_O)=L_1^2\left[\begin{array}{c}\frac{2T_O}{A^2}+\frac{1}{(r_1-r_2)^2}\left[\frac{(e^{2r_1T_O}-1)}{r_1^3}+\frac{(e^{2r_2T_O}-1)}{r_2^3}-\frac{4}{AB}(1-e^{BT_O})\right]\\ -\frac{4}{A(r_1-r_2)}\left[\frac{(e^{r_1T_O}-1)}{r_1^2}-\frac{(e^{r_2T_O}-1)}{r_2^2}\right]\\ +\frac{\lambda_2^*}{(r_1-r_2)}\left[\frac{(1-e^{r_1T_O})}{Ar_1}+\frac{(e^{r_2T_O}-1)}{Ar_2}+\frac{1}{(r_1-r_2)}\left(\frac{(e^{2r_1T_O}-1)}{2r_1^2}+\frac{(e^{2r_2T_O}-1)}{2r_2^2}-\frac{(1-e^{BT_O})}{A}\right)\right]\end{array}\right]=\frac{1}{2} \tag{4E-27a}$$

$$y_2(T_O)=\frac{L_1^2}{(r_1-r_2)}\left[\begin{array}{c}\frac{1}{(r_1-r_2)}\left[\frac{(e^{2r_1T_O}-1)}{r_1^2}+\frac{(e^{r_2T_O}-1)}{r_2^2}-\frac{2(1-e^{BT_O})}{A}\right]+\\ \frac{2(1-e^{r_1T_O})}{Ar_1}+\frac{2(e^{r_2T_O}-1)}{Ar_2}+\\ \frac{\lambda_2^*}{(r_1-r_2)}\left[\frac{(e^{2r_1T_O}-1)}{2r_1}+\frac{(e^{2r_2T_O}-1)}{2r_2}+\frac{2(1-e^{BT_O})}{B}\right]\end{array}\right]=0 \tag{4E-27b}$$

Solving for λ_2^* in Eq.(4E-27b), we find

$$\lambda_2^*=\frac{(r_2-r_1)\left[\frac{2(1-e^{r_1T_O})}{Ar_1}-\frac{2(1-e^{r_2T_O})}{Ar_2}+\frac{1}{(r_1-r_2)}\left(\frac{(e^{2r_1T_O}-1)}{r_1^2}+\frac{(e^{2r_2T_O}-1)}{r_2^2}+\frac{2}{A}(e^{BT_O}-1)\right)\right]}{\left[\frac{(e^{2r_1T_O}-1)}{2r_1}+\frac{(e^{2r_2T_O}-1)}{2r_2}+\frac{2}{B}(1-e^{BT_O})\right]} \tag{4E-28}$$

This result is now used to eliminate λ_2^* in Eq.(4E-27a). Thus

$$\frac{2T_o}{A^2}+\frac{1}{(r_1-r_2)}\left[\frac{(e^{2r_1T_o}-1)}{r_1^3}+\frac{(e^{r_2T_o}-1)}{r_2^3}+\frac{4}{AB}(e^{BT_o}-1)\right]-$$

$$-\frac{4}{A(r_1-r_2)}\left[\frac{(e^{r_1T_o}-1)}{r_1^2}-\frac{(e^{r_2T_o}-1)}{r_2^2}\right]-$$

$$\frac{\left[\frac{2(1-e^{r_1T_o})}{Ar_1}+\frac{2(e^{r_2T_o}-1)}{Ar_2}+\frac{1}{(r_1-r_2)}\left(\frac{(e^{2r_1T_o}-1)}{r_1^2}+\frac{(e^{2r_2T_o}-1)}{r_2^2}+\frac{2(e^{BT_o}-1)}{A}\right)\right]^2}{\left[\frac{(e^{2r_1T_o}-1)}{2r_1}+\frac{(e^{2r_2T_o}-1)}{2r_2}+\frac{2}{B}(1-e^{BT_o})\right]}=\frac{1}{2L_1^2} \tag{4E-29}$$

It is observed that the only unknown in this equation is T_o. Hence, it is only necessary to calculate a real root of a transcendental equation in order to find the elapsed time for a given set of feedback coefficients. λ_2^* is then found from Eq.(4E-28).

The expressions for the two outputs at any time, $0 \le t \le T_o$, can be shown to be

$$y_1(t)=L_1^2\left[\begin{array}{l}\frac{2t}{A^2}+\frac{1}{(r_1-r_2)^2}\left[\frac{e^{r_1T_o}(e^{r_1t}-e^{-r_2t})}{r_1^3}+\frac{e^{r_2T_o}(e^{r_2t}-e^{-r_2t})}{r_2^3}\right.\\ \qquad\qquad\left.-\frac{2}{AB}(e^{-Bt}-1)(e^{r_1t+r_2T_o}+e^{r_2t+r_2T_o})\right]\\ \frac{-2}{A(r_1-r_2)}\left[\frac{(1-e^{-r_1t})(e^{r_1T_o}+e^{r_1t})}{r_1^2}-\frac{(1-e^{-r_2t})(e^{r_2T_o}+e^{r_2t})}{r_2^2}\right]+\\ +\frac{\lambda_2^*}{(r_1-r_2)}\left[\begin{array}{l}-\frac{1}{A}\left(\frac{e^{r_1T_o}(1-e^{-r_1t})}{r_1}-\frac{e^{r_2T_o}(1-e^{-r_2T_o})}{r_2}\right)+\\ \frac{1}{(r_1-r_2)}\left(\frac{e^{r_1T_o}(e^{r_1t}-e^{-r_1t})}{2r_1^2}+\frac{e^{r_2T_o}(e^{r_2t}-e^{-r_2t})}{2r_2^2}+\right.\\ \qquad\left.+\frac{(e^{-Bt}-1)}{B}\left(\frac{e^{r_1t+r_2T_o}}{r_1}+\frac{e^{r_1T_o+r_2t}}{r_2}\right)\right)\end{array}\right]\end{array}\right] \tag{4E-30a}$$

$$y_2(t)=\frac{L_1^2}{(r_1-r_2)}\left[\begin{array}{l} \frac{1}{(r_1-r_2)}\left[\frac{e^{r_1T_o}(e^{r_1t}-e^{-r_1t})}{r_1^2}+\frac{e^{r_2T_o}(e^{r_2t}-e^{-r_2t})}{r_2^2}+\right. \\ \qquad\qquad \left.+\frac{2(e^{-Bt}-1)}{B}\left(\frac{e^{r_1T_o+r_2t}}{r_1}+\frac{e^{r_1t+r_2T_o}}{r_2}\right)\right]+ \\ \frac{2(1-e^{r_1t})}{Ar_1}+\frac{2(e^{r_2t}-1)}{Ar_2}+ \\ \frac{\lambda_2^*}{(r_1-r_2)}\left[\frac{e^{r_1T_o}(e^{r_1t}-e^{-r_1t})}{2r_1}+\frac{e^{r_2T_o}(e^{r_2t}-e^{-r_2t})}{2r_2}\right. \\ \qquad\qquad \left.+\frac{(e^{-Bt}-1)}{B}\left(e^{r_1T_o+r_2t}+e^{r_1t+r_2T_o}\right)\right] \end{array}\right] \quad \text{(4E-30b)}$$

As it is not possible to find closed form expressions for $\|y_i\|_1$ or $\|y_i\|_2$ $(i = 1,2)$, a numerical integration procedure must be used. The following results are obtained using an IBM 7040 computer with $L_1 = 2$. A sample data sheet is found in Appendix VI.

4E-4 Both Outputs Limited in Magnitude

In Eq. (3-25b), choose $L_{12} = 0.805$ and $L_{22} = 0.500$. Then employing the computer search, we obtain

$$T_o = 1.146$$
$$\lambda_2^* = -0.884$$
$$A = -10$$
$$B = 3.5$$

Substituting these results into Eq.(4E-26), the optimal w.r.a.s. input is

$$u_1(t) = 0.2\left[1 + e^{1.75(1.146-t)}(0.329 \sin 2.63[1.146-t] - \cos 2.63[1.146-t])\right]$$

Using Eq.(4E-22) with $q_1 = q_2 = 1$ and $L_2 = 9.8$, and carrying out the minimization with respect to λ_{II}, T^* is found by solving the following equations:

$$\|\underline{k}^*\|_1 = 2(T^*)^{3/2}\left[\frac{(T^*)^2}{5} + \frac{\lambda_{II}^* T^*}{2} + \frac{(\lambda_{II}^*)^2}{3}\right]^{\frac{1}{2}} + 9.8\left[(T^*)^2 + (T^*)\lambda_{II}^* + \frac{(\lambda_{II}^*)^2}{2}\right] = 1$$

$$\left[\frac{(T^*)^2}{5} + \frac{(\lambda_{II}^*) T^*}{2} + \frac{(\lambda_{II}^*)^2}{3}\right]^{\frac{1}{2}} = 0.017\,(T^*)^{3/2}\,\frac{(3T^* + 4\lambda_{II}^*)}{(T^* + \lambda_{II}^*)}$$

Then, $T^* = 0.82$. Thus the elapsed time must be increased by a factor 1.4 when the optimal w.r.a.s. approach is used.

4E-5 Both Outputs Have an Integral Constraint

Choose $L_{12} = 0.300$ and $L_{22} = 0.500$ in Eq.(4E-25a) with $p_2 = 1$. Then

$$T_O = 1.430$$

$$\lambda_2^* = -1.144$$

$$A = -5.0$$

$$B = 0.75$$

Using Eq.(4E-26), the optimal w.r.a.s. input is

$$u_1(t) = 0.4\left[1 - e^{0.375(1.43-t)}\Big(0.349 \sin 2.21(1.43-t) + \cos 2.21(1.43-t)\Big)\right]$$

For this example, $q_1 = 2$, $q_2 = \infty$, and $L_2 = 1.875$. The two expressions that must be solved in order to find T^* are

$$\|\underline{k}^*\|_1 = 2(T^*)^{3/2}\left[\frac{(T^*)^2}{5} + \frac{\lambda_{II}^* T^*}{2} + \frac{(\lambda_{II}^*)^2}{3}\right]^{\frac{1}{2}} + 1.875\,[2T^* + \lambda_{II}^*] = 1$$

$$\left[\frac{(T^*)^2}{5} + \frac{\lambda_{II}^* T^*}{2} + \frac{(\lambda_{II}^*)^2}{3}\right]^{\frac{1}{2}} = -.0888(T^*)^{3/2}\,[3T^* + 4\lambda_{II}^*]$$

Therefore, $T^* = 0.527$. The elapsed time is thus increased by a factor of 2.2.

4E-6 Both Outputs Have a Quadratic Constraint

If in Eq.(4E-25a) we let $L_{12} = 0.348$ and $L_{22} = 0.471$ with $p_2 = 2$, then

$$T_O = 1.534$$

$$\lambda_2^* = -1.312$$

$$A = -3.0$$

$$B = 0.5$$

Using these results, the optimal w.r.a.s. input is

$$u_1(t) = 0.667\left[1 - e^{0.25(1.514-t)}\left(0.621 \sin 1.71\,[1.534-t] + \cos 1.71[1.534-t]\right)\right]$$

T^* is found by solving

$$\|\underline{k}^*\|_1 = 2(T^*)^{3/2}\left[\frac{(T^*)^2}{5} + \frac{\lambda_{II}^* T^*}{2} + \frac{(\lambda_{II}^*)^2}{3}\right]^{\frac{1}{2}} + 1.312(T^*)^{\frac{1}{2}}\left[\frac{4}{3}(T^*)^2 + 2T^*\lambda_{II}^* + (\lambda_{II}^*)^2\right]^{\frac{1}{2}} = 1$$

and

$$\left[\frac{\frac{4}{3}(T^*)^2 + 2T^*\lambda_{II}^* + (\lambda_{II}^*)^2}{\frac{(T^*)^2}{5} + \frac{\lambda_{II}^* T^*}{2} + \frac{(\lambda_{II}^*)^2}{3}}\right]^{\frac{1}{2}} = -0.0547\,\frac{[T^* + \lambda_{II}^*]}{T^*[3T^* + 4\lambda_{II}^*]}$$

simultaneously, whereupon $T^* = 0.990$. Thus the elapsed time must be increased by a factor of 1.55 when the optimal w.r.a.s. approach is used.

4.10.2 Input Magnitude Constrained

If $u_1(t)$ is magnitude constrained, i.e.

$$|u_1(t)| \leq L_1 \qquad (0 \leq t \leq T_O)$$

the results of Section 4.9 can be used to obtain a optimal w.r.a.s. solution to the given problem. Since it is not practical to obtain explicity expressions for the output quantities, numerical integration is again employed. The computer program which is used to solve the next three examples is in

Appendix VII and the flow diagram is shown in Figure 6-2.

Using Eqs. (4-26), (4-27) and (4E-24), the general form of the optimal w.r.a.s. input is

$$u_1(t) = L_1 \, \mathrm{SGN} \, [2 h_{a_1}(T_O - 1) + \lambda_2^* h_{a_2}(T_O - t)] \tag{4E-31}$$

4E-7 Both Outputs Limited in Magnitude

Let $L_{12} = 0.435$ and $L_{22} = 0.500$ in Eq.(3-25b), we find

$$T_O = 1.865$$

$$\lambda_2^* = -0.892$$

$$A = 2.0$$

$$B = 1.0$$

Substituting these results into Eq.(4E-31), the optimal w.r.a.s. input is found to be a step of amplitude 2. Making use of Eq.(4E-22) with $q_1 = q_2 = 1$ and $L_2 = 1.37$, and carrying out the minimization with respect to λ_{II}, T^* is computed from the following equations:

$$\frac{2}{3}(T^*)^3 + 1.37(T^*)^2 + \lambda_{II}^* T^* (1.37 + T^*) + 0.685(\lambda_{II}^*)^2 - \frac{2}{3}(\lambda_{II}^*)^3 = 1$$

$$(\lambda_{II}^*)^2 - 0.685\,\lambda_{II}^* = T^*(.685 + 0.5\,T^*)$$

Then, $T^* = 1.03$. Thus, the elapsed time must be increased by a factor of 1.81 when the optimal w.r.a.s. approach is used.

4E-8 Both Outputs Have an Integral Constraint

Letting $L_{12} = 0.330$ and $L_{22} = 0.500$ in Eq.(3-25a), with $p_2 = 1$, we find

$$T_o = 1.602$$

$$\lambda_2^* = -1.057$$

$$A = -1.0$$

$$B = 1.0$$

The optimal w.r.a.s. input is again found to be a step of amplitude 2. For this example, $q_1 = 2$, $q_2 = \infty$, and $L_2 = 0.830$. The two expressions which must be solved in order to find T^* are

$$\frac{2}{3}(T^*)^3 + (T^*)^2 \lambda_{II}^* + 1.660\,T^* + 0.83\,\lambda_{II}^* - \frac{2}{3}(\lambda_{II})^3 = 1$$

$$(\lambda_{II})^2 - 0.5(T^*)^2 = 0.415$$

Then, $T^* = 0.91$. The elapsed time is thus increased by a factor of 1.76 when the optimal w.r.a.s. approach is used.

4E-9 Both Outputs Energy Constrained

Letting $L_{12} = 0.312$ and $L_{22} = 0.507$ in Eq.(4-25a) with $p_2 = 2$, we find

$$T_o = 1.441$$

$$\lambda_2^* = -0.802$$

$$A = -6.0$$

$$B = 1.0$$

Using these results in Eq.(4E-31), the optimal w.r.a.s. input is

$$u_1(t) = 2\,\mathrm{SGN}\left[1 - e^{0.5(1.441-t)}(0.794\sin 2.4[1.441-t] + \cos 2.4[1.441-t])\right]$$

This is a "Bang-Bang" function with a single sign change. See Figure 4-9. T^* is found by solving

$$\frac{2}{3}(T^*)^3 + (T^*)^2 \lambda_{II}^* - \frac{2}{3}(\lambda_{II}^*)^3 + 2.38(T^*)^{\frac{1}{2}}\left[\frac{4}{3}(T^*)^2 + 2(T^*)\lambda_{II}^* + (\lambda_{II}^*)^2\right]^{\frac{1}{2}} = 1$$

$$\left[\frac{4}{3}(T^*)^2 + 2T^*\lambda_{II}^* + (\lambda_{II}^*)^2\right]^{\frac{1}{2}} = 1.189\,\frac{(T^*)^{\frac{1}{2}}[T^* + \lambda_{II}^*]}{[(\lambda_{II}^*)^2 - \frac{(T^*)^2}{2}]}$$

simultaneously. Thus, $T^* = 0.740$ and the elapsed time must be increased by a factor of 1.95 when the optimal w.r.a.s. approach is used.

Examples 4E-4 through 4E-9 indicate that for the plant shown in Figure 4-8, the elapsed time must be increased by a factor of less than two in order to satisfy all input and output constraints and the terminal conditions by using a feedback arrangement as part of the controller.

4.11 Summary and Conclusion

In this chapter, the optimal w.r.a.s. solution of Problem 5 was formulated for single and multiple output systems. Some necessary conditions for the existence of such a solution were developed. For single output plants, a lower bound on the elapsed time was obtained when the output had either an integral or a quadratic constraint placed on it. It was also proven that the magnitude of the output is always less than or equal to $|y_d|$ when the optimal w.r.a.s. input is used. From this result, a simple test for determining whether an output magnitude constraint can be satisfied was given. For multiple output plants, it was demonstrated that when a quadratic constraint is placed on the single input to the augmented system, the problem of finding the λ's is reduced to finding the real root of a transcendeal equation. A procedure for finding the λ's when the input was magnitude constrained was illustrated for a two output plant.

It is significant to note that the procedures for finding the optimal w.r.a.s. input do not depend upon the detailed structure of the specified portion of the controller. As a consequence, the results contained herein are directly applicable to <u>any</u> form of feedback, e.g. feedback signals proportional to either the <u>derivative</u> or integral of the outputs.

V. OPTIMAL CONTROL OF A CLASS OF LINEAR PLANT

5.1 Introduction

In this chapter it is shown that when the quadratic content of the output of a single output plant is constrained, there is a sub-class of Problem 5 for which the optimal w.r.a.s. solution is optimal. It is demonstrated that this situation is only possible when the feedback coefficients of the augmented plant are constants. The conditions which must be satisfied in order for the problem to belong to this class are derived and several examples included. Under these conditions, it is also demonstrated that if the given plant is stable, the augmented plant is generally unstable. Finally, it is shown that the optimal w.r.a.s. solution can never be optimal if an integral constraint is placed on the output.

5.2 Formulation of the Problem

Let a linear, time-varying, n^{th} order plant with r-inputs and one input be described by the following set of state equations:

$$\dot{\underline{x}} = \overline{E}(t)\,\underline{x} + \overline{F}(t)\,\underline{u} \qquad (5\text{-}1a)$$

$$y = \overline{C}(t)\,\underline{x} + \overline{D}(t)\,\underline{u} \qquad (5\text{-}1b)$$

where $\overline{E}(t)$, $\overline{F}(t)$, $\overline{C}(t)$, and $\overline{D}(t)$ are respectively $(n \times n)$, $(n \times r)$, $(1 \times n)$ and $(1 \times r)$ time-varying matrices. Furthermore, suppose that $u_1(t)$ (the first element of the input vector, $\underline{u}$) is constrained so that

$$\|u_1\|_{p_1} = \left[\int_{t_o}^{t_f} |u_1(t)|^{p_1}\, dt\right]^{1/p_1} \leq L_1 \quad (p_1 = 1, 2, \infty) \qquad (5\text{-}2)$$

and that the remaining inputs are restricted to the class

$$u_j(t) = a_j(t)\, y(t) \qquad (j = 2,3, \ldots r) \qquad (5\text{-}3)$$

where the $a_j(t)$ are time-varying feedback coefficients. See Figure 5-1. In addition, it is required that the output satisfy the initial and a terminal condition

$$y(t_o) = 0 \qquad (5\text{-}4a)$$

$$y(t_f) = y_d \qquad (5\text{-}4b)$$

and that its quadratic content be constrained, i.e.

$$(\|y\|_2)^2 = \int_{t_o}^{t_f} y^2(t)\, dt \geq L_{2e}^2 \qquad (5\text{-}5)$$

The problem is to obtain the input, $u_1(t)$, and the values of the feedback coefficients such that both constraints and the initial and terminal conditions are satisfied and the elapsed time, $T = t_f - t_o$ is minimized. Note that this is Problem 5.

As is shown in the sequel each of the inputs in Eq.(5-3) actually has a quadratic constraint on it of the form

$$\|u_j\|_2 \leq b_j \qquad (j = 2,3, \ldots r) \qquad (5\text{-}6)$$

where the b_j's are a set of positive real numbers. If these constraints are assumed to be <u>specified</u> for the given portion of the plant in Figure 5-1, there is a set of r-inputs which permits <u>all</u> input and output constraints and the initial and terminal conditions to be satisfied in the shortest elapsed time, $T^* = t_f^* - t_o$. These are the "time-optimal" inputs. It is noted that when the feedback coefficients are constants and the optimal w.r.a.s. solution is used, the elapsed time, T_o, is generally greater than

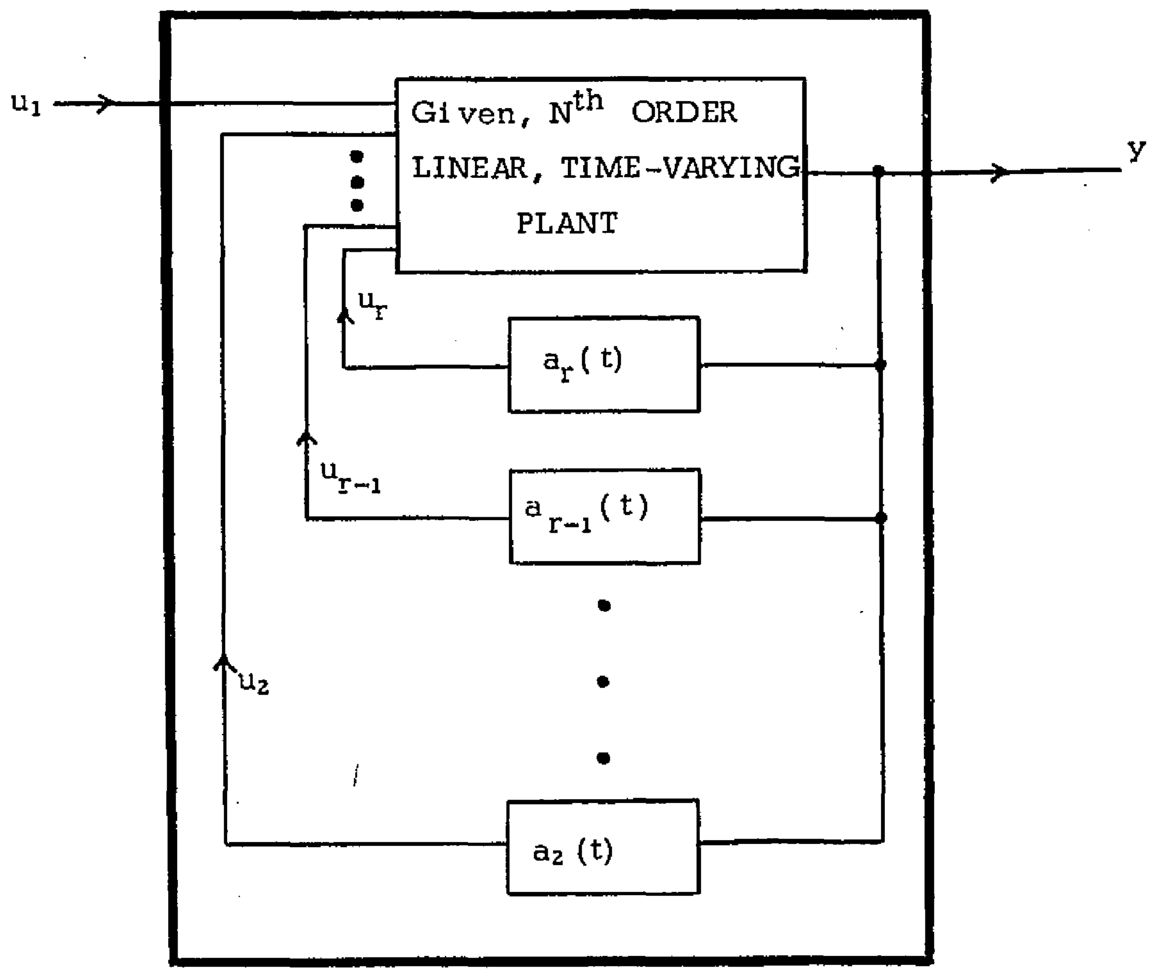


Figure 5-1 Augmented Plant with time-varying feedback coefficients

T^* because the inputs in Eq.(5-3) are not usually optimal. Under certain conditions, however, they are optimal and hence $T_o = T^*$. These conditions are now derived.

5.3 Conditions for a Time-Optimal Solution

The conditions for a time-optimal solution for the above problem are contained in the two assertions which are now proved.

Assertion I

Let the constraints in Eqs.(5-2) and (5-5) and the initial and terminal conditions in Eq.(5-4) be imposed on the linear system described by Eqs.(5-1) and furthermore, suppose that inputs u_2 through u_r are restricted to the class defined by Eq.(5-3). A necessary condition for the above problem to have a time-optimal solution is that the $a_j(t)$'s be positive or negative constants.

Proof: It is first demonstrated that each of the inputs in Eq.(5-3) has an equivalent quadratic constraint on it. Forming the quadratic norms of the inputs u_2 through u_r gives

$$\| u_j \|_2^2 = \int_{t_o}^{t_f} (u_j(t))^2 \, dt = \int_{t_o}^{t_f} a_j^2(t)\, y^2(t)\, dt \qquad (j=2,3,\ldots r) \quad (5\text{-}7)$$

Using Hölder's inequality, this equation becomes

$$\|u_j\|_2^2 = \int_{t_o}^{t_f} a_j^2(t)\, y_j^2(t)\, dt \le \left[\int_{t_o}^{t_f} |a_j^2(t)|^q dt\right]^{1/q} \left[\int_{t_o}^{t_f} |y^2(t)|^p dt\right]^{1/p} \quad (5\text{-}8)$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad (5\text{-}9)$$

As there is a quadratic constraint on the output, p is chosen to be one, whereupon $q = \infty$. In addition, since the left hand side of (5-5) is a monotonic increasing function of t_f, and since a minimum time solution is required, the inequality in (5-5) becomes an equality. Thus

$$\|y\|_2^2 = \int_{t_o}^{t_f} y^2(t)\,dt = L_{2e}^2 \qquad (5\text{-}10)$$

Using these results, Eq.(5-8) becomes

$$\|u_j\|_2^2 = \int_{t_o}^{t_f} a_j^2(t)\,y^2(t)\,dt \le |A_{j\,max}|^2\,L_{2e}^2 \qquad (j=2,3,\ldots r) \qquad (5\text{-}11)$$

where $A_{j\,max}$ is the maximum value of $a_j(t)$ in the interval (t_o, t_f). From this inequality it is seen that each input, u_2 through u_r has an equivalent quadratic constraint placed on it.

Now let it be assumed that the set of derived constraints in (5-11) is specified in advance. Each of the given plant inputs is now constrained. In Section 2.2.3, it was mentioned that a necessary condition for a time-optimal solution to exist for a linear system with multinorm constraints on the inputs only, is that these constraints become equalities. Applying this condition, we require that inequalities (5-2) and (5-11) be satisfied with the sign of equality. In (5-11) it is implied that Hölder's inequality becomes an equality. Using Eq.(A-8) (Appendix I)

$$a_j^2(t) = K_j\,|y^2(t)|^{p-1}\,\mathrm{SGN}\,[y^2(t)] \qquad (j=2,3,\ldots r) \quad (K_j > 0) \qquad (5\text{-}12)$$

Since $p = 1$ and $y^2(t) \ge 0$, Eq. (5-12) becomes

$$a_j^2(t) = K_j$$

or

$$a_j(t) = \pm A_j \qquad (5\text{-}13)$$

where A_j is a positive real number. Thus, $a_j(t) = A_{j\,max} = \pm A_j$, i.e. the derived constraints in (5-11) become equalities only if the feedback coefficients are constants. Using this result in Eq.(5-3) it is seen that

$$y(t) = \frac{u_j(t)}{A_{j\ max}} \qquad (j=2,3,\ldots r) \qquad (5\text{-}14)$$

Taking quadratic norms on both sides of this equation and using Eq.(5-11) (with the sign of equality) yields

$$\|y\|_2 = \frac{\|u_j\|_2}{|A_{j\ max}|} \equiv L_{2e} \qquad (j=2,3,\ldots r) \qquad (5\text{-}15)$$

Satisfying the derived constraints on inputs u_2 through u_r with the sign of equality is equivalent to satisfying the constraint on the output with the sign of equality. Thus a necessary condition for the existence of a time-optimal solution is that the feedback coefficients be positive or negative constants. This completes the proof.

For the system described by Eqs.(5-1), the impulse response matrix is an r-dimensional column vector of the form

$$H(t,\tau) = \begin{bmatrix} h_1(t,\tau) \\ h_2(t,\tau) \\ \vdots \\ h_r(t,\tau) \end{bmatrix} = C(t)\,\phi(t,\tau)\,F(\tau) + D(t)\,\delta(t-\tau) \qquad (5\text{-}16)$$

<u>Assertion II</u>

Given the plant in Figure 5-1 and the restrictions imposed in Assertion I. In addition, let the feedback coefficients, $a_j(t)$, be constants. The necessary and sufficient conditions for a time-optimal solution to the problem in Section 5.2 are that

$$L_1 \|h_1\|_{q_1} + L_2 \sum_{n=2}^{r} |A_n| \, \|h_n\|_2 = |y_d| \tag{5-18}$$

$$u_1(t) = L_1 (\|h_1\|_{q_1})^{1-q_1} \, |h_1(t_f^*, t)|^{q_1-1} \, \mathrm{SGN}[h(t_f^*, t)] \tag{5-19a}$$

$$\frac{L_2 h_j(t_f^*, t) \, \mathrm{SGN}[A_j]}{\|h_j\|_2} = L_2 \sum_{n=2}^{r} |A_n| \int_{t_o}^{t} h_n(t,\tau) \, h_n(t_f^*, \tau) d\tau + \int_{t_o}^{t} h_1(t,\tau) \, u_1(\tau) d\tau \quad (j=2,3,\ldots r) \tag{5-22}$$

Proof: In Section 2.2.3 it was indicated that the necessary and sufficient conditions for the existence of a time-optimal solution for problems involving multinorm constraints on the inputs only are that the conditions in Eqs. (2-33) and (2-34) be satisfied. For single output plants, these conditions become

$$u_n(\tau) = L_n' (\|h_n\|_{q_n})^{1-q_n} \, |h_n(t_f^*, \tau)|^{q_n-1} \, \mathrm{SGN}[h_n(t_f^*, \tau)] \tag{5-17}$$

and

$$\sum_{n=1}^{r} L_n' \, (\|h_n\|_{q_n})^{1-q_n} = |y_d| \tag{5-18}$$

where the L_n' are identical to the L_i's in Eq.(2-26) and t_f^* is the least value of t_f satisfying Eq.(5-18). Making use of the constraints derived for u_2 through u_r, (Eq.5-11) and also Eq.(5-2), we have

$$L_1' = L_1$$

$$L_n' = L_2 \, |A_n| \qquad (n=2,3,\ldots r)$$

$$q_n = 2$$

whereupon, the two previous equations become

$$u_1(t) = L_1\left(\|h_1\|_{q_1}\right)^{1-q_1} \left|h_1(t_f^*, t)\right|^{q_1-1} \mathrm{SGN}[h_1(t_f^*, t)] \qquad (5\text{-}19a)$$

$$u_j(t) = \frac{L_2 |A_j|}{\|h_j\|_2} h_j(t_f^*, t) \qquad (j = 2, 3, \ldots r) \qquad (5\text{-}19b)$$

$$L_1 \|h_1\|_{q_1} + L_2 \sum_{n=2}^{r} |A_n| \, \|h_n\|_2 = |y_d| \qquad (5\text{-}20)$$

By forming the appropriate norms, the inputs in Eq.(5-19) can easily be shown to satisfy the constraints in Eqs.(5-2) and (5-11) with the sign of equality. As was previously demonstrated, this also permits the output constraint to be satisfied (with the sign of equality). Since u_2 through u_r are restricted to the class defined by Eq.(5-3) (with $a_j(t) = A_j$) a time-optimal solution can only be obtained if the inputs in Eq.(5-19b) belong to this class. This requires that

$$y(t) = \frac{u_j(t)}{A_j} = \frac{L_2}{\|h_j\|_2} \mathrm{SGN}[A_j] \cdot h_j(t_f^*, t) \qquad (j = 2, 3, \ldots r) \quad (5\text{-}21)$$

However, the output of a linear plant with r-inputs is known to be

$$y(t) = \sum_{j=1}^{r} \int_{t_o}^{t} h_j(t, \tau)\, u_j(\tau)\, d\tau$$

Substituting Eqs.(5-19a) and (5-21) into this expression yields

$$\frac{L_2}{\|h_j\|_2} \mathrm{SGN}[A_j] \cdot h_j(t_f^*, t) = \int_{t_o}^{t} h_1(t, \tau) u_1(\tau) d\tau + L_2 \sum_{n=2}^{r} \int_{t_o}^{t} \frac{h_n(t, \tau) h_n(t_f^*, \tau) d\tau}{\|h_n\|_2}$$

$$(j = 2, 3, \ldots r) \qquad (5\text{-}22)$$

If a set of A's (call them A^*'s) can be found which permits this relationship to be satisfied for $j = 2, 3, \ldots r$, the inputs u_2 through u_r obtained using feedback are identical to those in Eqs.(5-19b). Therefore, the condition in Eq.(5-22) together with those in Eqs.(5-19a) and (5-20) must be satisfied in order to have a time-optimal solution to the problem in Section 5.2. This completes the proof.

5.4 Conditions for an Optimal W.R.A.S. Solution to be Optimal

Since the specified portion of the controller for the class of systems treated in the preceding chapter consisted of feedback coefficients which were also constants, the optimal w.r.a.s. and time-optimal solutions to Problem 5 can be identical. In order for this situation to occur, the optimal w.r.a.s. input (Eq.4-5) and the time-optimal input in Eq.(5-19a) must be the same. That is

$$\frac{L_1 |h_a(t_f,\tau)|^{q_1-1}}{(\|h_a\|_{q_1})^{q_1-1}} \, \mathrm{SGN}\left[\frac{h_a(t_f,\tau)}{y_d}\right] = \frac{L_1 |h_1(t_f,\tau)|^{q_1-1}}{(\|h_1\|_{q_1})^{q_1-1}} \, \mathrm{SGN}\left[h_1(t_f,\tau)\right]$$

or

$$\frac{|h_a(t_f,\tau)|^{q_1-1}}{(\|h_a\|_{q_1})^{q_1-1}} \, \mathrm{SGN}\left[\frac{h_a(t_f,\tau)}{y_d}\right] = \frac{|h_1(t_f,\tau)|^{q_1-1}}{(\|h_1\|_{q_1})^{q_1-1}} \, \mathrm{SGN}\left[h_1(t_f,\tau)\right] \qquad (5\text{-}23)$$

If Eq.(5-23) is true in addition to Eqs.(5-20) and (5-22), the optimal w.r.a.s. input not only satisfies the initial and terminal conditions and the constraints on $u_1(t)$ and $y(t)$, but also forces u_2 through u_r to be given by Eq.(5-19b). Thus the optimal w.r.a.s. and time-optimal solutions are identical.

Remark

It is noted that a limited class of linear plants can satisfy Eqs. (5-20), (5-22), and (5-23) simultaneously. Moreover, since these plants satisfy these equations for certain values of L_1 and L_2, only a sub-class of Problem 5 can have identical optimal and optimal w.r.a.s. solutions. This is demonstrated in the examples found in Section 5.6.

5.5 Stability

The stability of the augmented plant corresponding to a given system which satisfies Eqs.(5-19a), (5-20), and (5-22) can be investigated. Suppose that a given system is stable† and non time-varying. Then the elements of the impulse matrix (Eq.5-16) consist of linear combinations of terms (modes) of the form

$$t^n e^{-\eta t} \cos(\omega t + \varphi) \tag{5-24}$$

where $\eta \geq 0$, the equality occuring only if $n = 0$. In the s-plane representation this is equivalent to requiring that the elements of $\mathcal{L}[\underline{H}(t)] \triangleq \underline{H}(s)$ contain poles in the left hand plane and/or on the imaginary axis.

Under the conditions of Assertion II the output of the augmented system becomes (Eq.5-21)

$$y(t) = \frac{L_2}{\|h_j\|_2} \text{SGN}[A_j] \cdot h_j(t_f^* - t) \qquad (j = 2, 3, \ldots r)$$

Making use of Eq.(5-24), y(t) is seen to contain terms of the form

$$(t_f^* - t)^n e^{-\eta t_f^*} e^{\eta t} \cos[\omega(t_f^* - t) + \varphi] \tag{5-25}$$

<u>Case 1</u>. $q_1 = 1$

If $u_1(t)$ has a magnitude constraint on it, $q_1 = 1$. Then $u_1(t)$ is "bang-bang" and therefore a bounded input. However, the expression

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†That is, the response of the system to a bounded input is bounded for $0 \leq t < \infty$.

in Eq.(5-25) is bounded† as $t \to \infty$. It is concluded that when a magnitude constraint is placed on $u_1(t)$ the class of stable (and damped) system satisfying Eqs.(5-19a), (5-20), and (5-22) always produces an augmented system which is unstable.

Case 2. $q_1 = \infty$

If $u_1(t)$ has an integral constraint on it, $q_1 = \infty$. Then $u_1(t)$ contains impulses[49]. Taking the Laplace transform of a typical output term (Eq.5-25), it is observed that the transform of the augmented plant impulse response, $H_a(s) \triangleq \frac{Y(s)}{U_1(s)}$, contains poles in the right half of the s-plane†. The response of such a system to a bounded input is therefore unbounded. Thus, when an integral constraint is placed on $u_1(t)$, the class of stable system satisfying Eqs.(5-19a), (5-20), and (5-22) always produces an augmented system which is unstable.

Case 3. $q_1 = 2$

If $u_1(t)$ has a quadratic constraint on it, $q_1 = 2$. Then

$$u_1(t) = L_1 \frac{h_1(t_f^* - t)}{\|h_1\|_{q_1}}$$

Using the Laplace transform, $\underline{H}_a(s)$ contains poles in the right half of the s-plane provided $H_1(s)$ does not contain all of the poles of $H_j(s)$. Thus, when a quadratic constraint is placed on $u_1(t)$, the class of stable system satisfying Eqs.(5-19a), (5-20), and (5-22) produces an unstable augmented

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† This is true unless $\eta \equiv 0$ for <u>every</u> term in $h_j(t)$ in which case the system has no damping and the output is purely sinusoidal.

system if $h_1(t)$ does not contain all of the modes of $h_j(t)$.

In the preceding discussion, the given system was assumed to be stable. If this is not true but instead, $H_j(s)$ contains at least one pole in the left half s-plane, the augmented system is still unstable. However, if all of the poles of the given system are in the right half s-plane the constant "η" in Eq.(5-25)is negative and the augmented plant is stable.

The above results cannot be generalized to include time-varying systems. This is not surprising in view of the lack of a stability theory for general time-varying linear plants.

Finally it is important to note that although it has been shown that most of the augmented plants considered in this chapter exhibit an unbounded response due to a bounded input as $t \to \infty$, none of them exhibit such a response for $t \le t_f$ (t_f finite). They are all, therefore, short term stable. Of course, once the terminal value, y_d, is reached, it is advisable to turn off the input, $u_1(t)$, and to provide a sufficient amount of damping so that the output does not continue to grow without bound.

5.6 Examples

In this section several examples are presented to demonstrate the existence of systems which satisfy the conditions derived in Eqs.(5-19a), (5-20), and (5-22). In addition, it is shown that each of these systems satisfies Eq.(5-23) so that the optimal and optimal w.r.a.s. solutions are identical.

5E.1 Magnitude Constraint on $u_1(t)$

Consider the second order, non time-varying system shown in Figure 5-2. The following set of conditions is imposed:

$$\begin{aligned} \|u_1\|_\infty &\triangleq |u_1(t)| \le L_1 \\ \|u_2\|_2 &\le AL_2 \\ \|y\|_2 &\ge L_2 \\ y(0) &= 0 \\ y(T) &= y_d \end{aligned} \tag{5E-1}$$

where $T = t_f - t_o = t_f$. $u_1(t)$ and $u_2(t) = Ay(t)$ are found so that the above set of conditions are satisfied in the least possible time, $T = T^*$. To permit some generality, an amplifier of gain K is introduced (see Figure 5-2) so that

$$u(t) = K\,u_1(t) \qquad (K > 0) \tag{5E-2}$$

and therefore,

$$|u(t)| = K\,|u_1(t)| \le K\,L_1 \tag{5E-3}$$

K will be set equal to one at the end of the problem because $u_1(t)$ is the actual system input.

Treating u(t) as a system input for the time being, it is easily shown that the appropriate impulse responses are

$$h_1(t) = \delta(t) - \frac{1}{2}e^{-2t} + \frac{3}{2}e^{2t} \tag{5E-4a}$$

$$h_2(t) = \frac{1}{5}e^{-2t} \tag{5E-4b}$$

Using Eqs. (5-19) with $q_1 = 1$, the time-optimal inputs are:

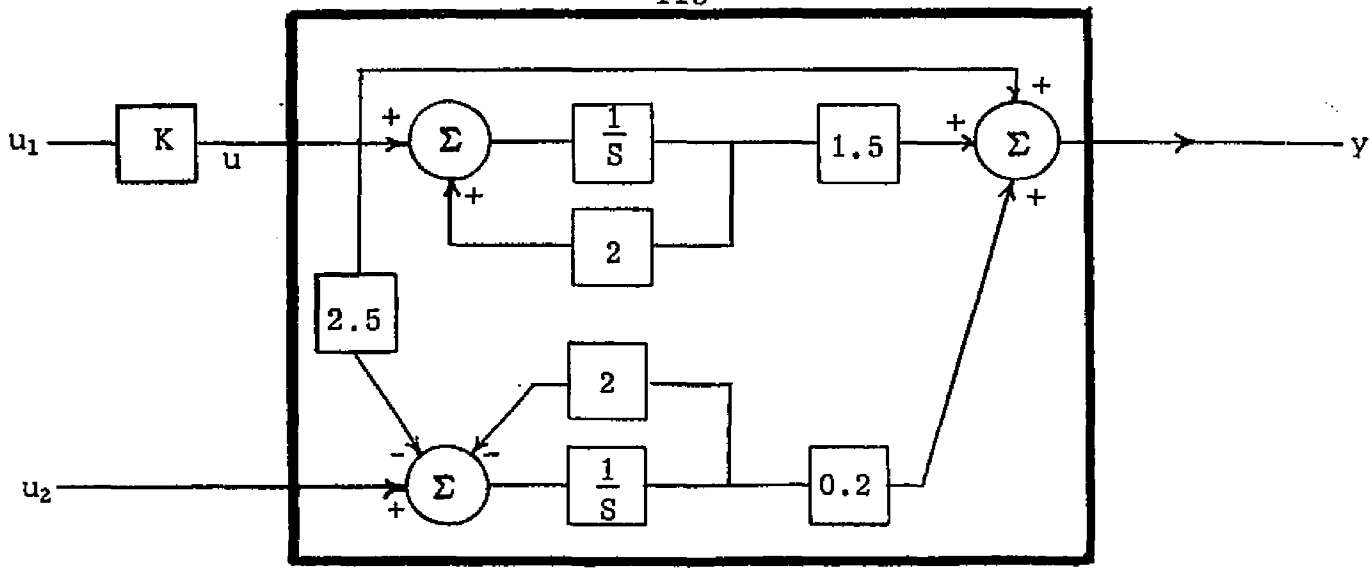

Figure 5-2 Second order system used in Example 5E-1

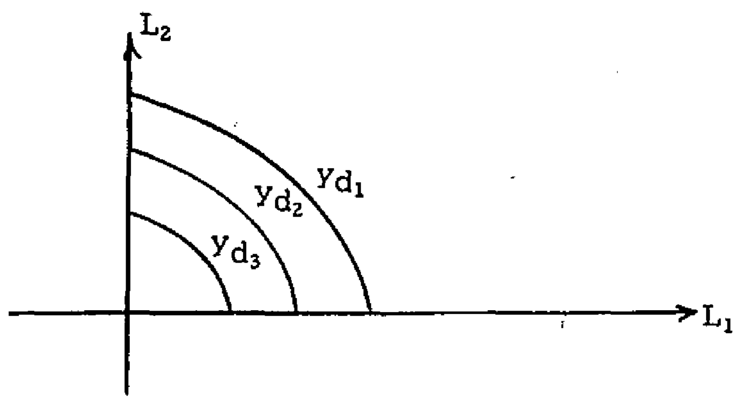

Figure 5-3 Constraint space for Examples 5E-1,2

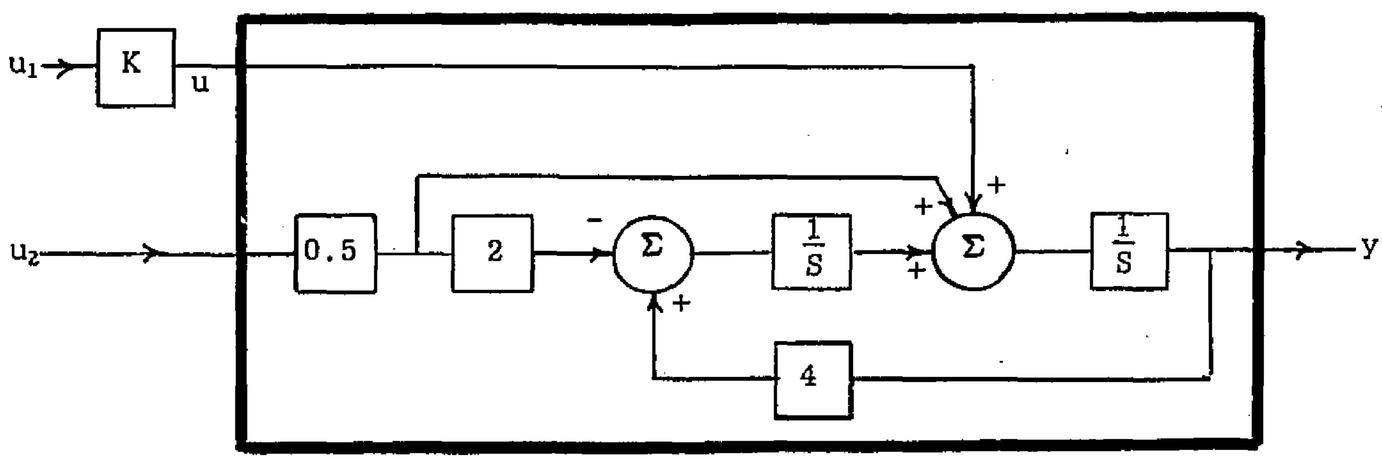

Figure 5-4 Second order system used in Example 5E-2

$$u(t) = KL_1 \; SGN\,[h_1(T^{*}-t)] = KL_1 \; U(t) \tag{5E-5a}$$

$$u_2(t) = \frac{L_2 A}{5\|h_2\|_2} e^{-2(T^{*}-t)} \; U(t) \tag{5E-5b}$$

where U(t) is the unit step function and T^{*} is the minimum value of T satisfying Eq.(5-20). In addition,

$$\|h_2\|_2 = \left[\int_0^{T^{*}} h_2^2(t)\,dt\right]^{\frac{1}{2}} = \left[\int_0^{T^{*}} \frac{e^{-4t}}{25} dt\right] = \frac{1}{10}(1 - e^{-4T^{*}})^{\frac{1}{2}} \tag{5E-6}$$

Therefore, Eq.(5E-5b) becomes

$$u_2(t) = \frac{2AL_2}{(1-e^{-4T^{*}})^{\frac{1}{2}}} e^{-2T^{*}} e^{2t}\, U(t) = \frac{2\,AL_2}{(e^{4T^{*}}-1)^{\frac{1}{2}}} e^{2t}\, U(t) \tag{5E-7}$$

Also,

$$\|h_1\|_1 = \int_0^{T^{*}} \left|\delta(t) - \frac{1}{2}e^{-2t} + \frac{3}{2}e^{2t}\right| dt = 1 - \frac{1}{4}(1 - e^{-2T^{*}}) + \frac{3}{4}(e^{2T^{*}} - 1) \tag{5E-8}$$

Using Eqs.(5E-3), (5E-6), and (5E-8) in Eq.(5-20), we have

$$\frac{KL_1}{4} e^{2T^{*}}\left[3 + e^{-4T^{*}}\right] + \frac{AL_2}{10}\left[1 - e^{-4T^{*}}\right]^{\frac{1}{2}} = |y_d| \tag{5E-9}$$

Finally, Eq.(5-22) must be satisfied. In this example, it is possible to avoid the integration by using Laplace transforms. Thus it is required that

$$U(s)\,H_1(s) + U_2(s)\,H_2(s) = Y(s) = \frac{U_2(s)}{A} \tag{5E-10}$$

The Laplace transforms of u(t), $u_2(t)$, $h_1(t)$, and $h_2(t)$ are given by

$$\begin{aligned} U(s) &= \frac{KL_1}{s} \\ U_2(s) &= \frac{2\,AL_2\, e^{-2T^{*}}}{(1 - e^{-4T^{*}})^{\frac{1}{2}}} \; \frac{1}{(s-2)} \\ H_1(s) &= 1 - \frac{1}{2(s+2)} + \frac{3}{2(s-2)} \\ H_2(s) &= \frac{1}{5(s+2)} \end{aligned} \tag{5E-11}$$

Substituting Eqs. (5E-11) into Eq. (5E-10) gives

$$\frac{KL_1}{s}\left[1 - \frac{1/2}{s+2} + \frac{3/2}{s-2}\right] + \frac{2AL_2\, e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}}\ \frac{1}{5(s-2)(s+2)} = \frac{2L_2\, e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}(s-2)}$$

or

$$KL_1\left[\frac{s(s+1)}{s(s-2)(s+2)}\right] = \frac{2L_2\, e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}}\ \frac{1}{(s-2)}\left[1 - \frac{A}{5(s+2)}\right]$$

or

$$\frac{KL_1(s+1)}{(s-2)(s+2)} = \frac{2L_2\, e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}}\ \frac{(s+2-A/5)}{(s+2)(s-2)}$$

Clearly, this equation is satisfied if

$$A = 5 \tag{5E-12}$$

and

$$K = \frac{2L_2}{L_1}\ \frac{e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}} \tag{5E-13}$$

Using Eq.(5E-12) and Eq.(5E-13) in Eq.(5E-9) gives

$$\frac{L_2}{2}\left[\frac{(3+e^{-4T^*})}{(1-e^{-4T^*})^{\frac{1}{2}}} + (1-e^{-4T^*})^{\frac{1}{2}}\right] = |y_d|$$

Multiplying both sides of this equation by

$$\frac{1}{|y_d|}\,(1-e^{-4T^*})^{\frac{1}{2}}\ ,$$

we find

$$\frac{2L_2}{|y_d|} = (1-e^{-4T^*})^{\frac{1}{2}} \tag{5E-14}$$

Squaring both sides yields

$$e^{-4T^*} = 1 - 4\left(\frac{L_2}{y_d}\right)^2 \tag{5E-15}$$

or

$$e^{-2T^*} = \sqrt{1 - 4\left(\frac{L_2}{y_d}\right)^2} \tag{5E-16}$$

Since $0 \le e^{-4T^*} \le 1$, it follows that

$$\left(\frac{L_2}{y_d}\right)^2 \le \frac{1}{2} \tag{5E-17}$$

As was mentioned previously, K must now be set to one so that u(t) becomes $u_1(t)$. Making use of Eq.(5E-13) and Eq.(5E-14), it is found that

$$\frac{2L_2}{L_1} e^{-2T^*} = (1 - e^{-4T^*})^{\frac{1}{2}} = \frac{2L_2}{y_d}$$

or

$$e^{-4T^*} = \left(\frac{L_1}{y_d}\right)^2 \tag{5E-18}$$

Combining this result with Eq.(5E-15) gives

$$\left(\frac{L_1}{y_d}\right)^2 = 1 - 4\left(\frac{L_2}{y_d}\right)^2$$

or

$$L_1^2 + 4L_2^2 = y_d^2 \tag{5E-19}$$

Equation (5E-19) represents a family of concentric ellipses in $L_1 - L_2$ space with y_d as a parameter. On physical grounds, L_1 and L_2 must be positive so that only the portions of these ellipses which lie in the first quadrant are permitted. See Figure 5-3.

For a given value of $|y_d|$, any combination of L_1 and L_2 which satisfies Eq.(5E-19) is permissible, the minimum time, T^*, is found from Eq. (5E-18) and the value of the feedback coefficient is 5. The time-optimal input is a step of amplitude L_1.

For the system in Figure 5-3, the augmented system impulse response, with A = 5, is found to be

$$h_a(t) = \delta(t) + 2e^{2t} \tag{5E-20}$$

From Eq.(5-23) with $q_1 = 1$, the optimal w.r.a.s. solution is optimal if

$$\mathrm{SGN}\left[\frac{h_a(t)}{y_d}\right] = \mathrm{SGN}\left[h_1(t)\right]$$

As $h_a(t)$ and $h_1(t)$ are always positive, this relationship is satisfied if $y_d \geq 0$ and we conclude that the optimal w.r.a.s. and optimal solutions are identical. As a verification of this, the optimal w.r.a.s. approach was applied to this system with $L_1 = 1$, $L_2 = \frac{\sqrt{3}}{2}$, and $y_d = 2$ (see Example 4E-1). "A" was found to be 5, and T_o was 0.347. Using the same values of y_d, L_1 and L_2 in Eq.(5E-18), $T^* = 0.347 = T_o$.

Finally, it is noted that $h_2(t)$ contains a stable mode. As predicted in Section 5.5, the augmented plant is unstable (see Eq.5E-20).

5E-2 An Integral Constraint on $u_1(t)$

For the second order system shown in Figure 5-4, the following set of conditions is imposed:

$$\|u_1\|_1 \leq L_1$$

$$\|u_2\|_2 \leq AL_2$$

$$\|y\|_2 \geq L_2$$

$$y(0) = 0$$

$$y(T) = y_d$$

$u_1(t)$ and $u_2(t) = Ay(t)$ are again found such that Eqs.(5E-21) are satisfied in the minimum time. The amplifier of gain K is introduced once more for the same reasons as in the previous example. Then

$$\|u\|_1 \leq KL_1 \qquad (5E-22)$$

It is easily shown that the impulse responses are given by

$$h_1(t) = \frac{(e^{2t} + e^{-2t})}{2} \tag{5E-23a}$$

$$h_2(t) = \frac{e^{-2t}}{2} \tag{5E-23b}$$

Then,

$$\|h_1\|_\infty = \frac{e^{2T^*} + e^{-2T^*}}{2} = \frac{e^{2T^*}}{2}[1 + e^{-4T^*}] \tag{5E-24}$$

and

$$\|h_2\|_2 = \frac{1}{4}(1 - e^{-4T^*})^{\frac{1}{2}} \tag{5E-25}$$

The time-optimal inputs are found to be†

$$\begin{aligned} u(t) &= KL_1\,\delta(t) \\ u_2(t) &= \frac{2AL_2\,e^{-2T^*}}{(1 - e^{-4T^*})^{\frac{1}{2}}}\, e^{2t}\, U(t) \end{aligned} \tag{5E-26}$$

Using Eqs.(5E-24) and (5E-25) in Eq.(5E-20), the expression for T^* is

$$\frac{L_1 K}{2} e^{2T^*}\left[1 + e^{-4T^*}\right] + \frac{L_2 A}{4}\left[1 - e^{-4T^*}\right]^{\frac{1}{2}} = |y_d| \tag{5E-27}$$

As in Example 5E-1, the Laplace transform is used to avoid the integration which is required in Eq.(5-22). Thus we have

$$U(s) = KL_1$$

$$U_2(s) = \frac{2\,AL_2}{(1-e^{-4T^*})^{\frac{1}{2}}}\ \frac{1}{(s-2)}$$

$$H_1(s) = \frac{1}{2}\left(\frac{1}{s-2} + \frac{1}{s+2}\right) = \frac{s}{s^2-4}$$

$$H_2(s) = \frac{1}{2}\ \frac{1}{s+2}$$

† This is true because the maximum of $h_1(T^*-t)$ occurs at $t=0$. See reference (49).

Substituting these quantities into Eq.(5E-10) gives

$$\frac{KL_1}{(s+2)(s-2)} + \frac{AL_2 e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}} \frac{1}{(s+2)(s-2)} = \frac{2L_2 e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}} \frac{1}{(s-2)}$$

or

$$KL_1 \left[\frac{s + \dfrac{AL_2 e^{-2T^*}}{KL_1(1-e^{-4T^*})^{\frac{1}{2}}}}{s+2} \right] \frac{1}{(s-2)} = \frac{2L_2 e^{-2T^*}}{(1-e^{-4T^*})^{\frac{1}{2}}} \frac{1}{(s-2)}$$

For the equality to hold, it is required that

$$K = \frac{2L_2 e^{-2T^*}}{L_1(1-e^{-4T^*})^{\frac{1}{2}}} \tag{5E-28}$$

and

$$A = 4 \tag{5E-29}$$

Substituting these results into Eq.(5E-27) gives

$$L_2 \left[\frac{(1+e^{-4T^*})}{(1-e^{-4T^*})^{\frac{1}{2}}} + (1-e^{-4T^*})^{\frac{1}{2}} \right] = |y_d|$$

or

$$\frac{2L_2}{|y_d|} = (1-e^{-4T^*})^{\frac{1}{2}} \tag{5E-30}$$

Squaring both sides of this last equation and solving for e^{-4T^*}, the following is obtained

$$e^{-4T^*} = 1 - 4\left(\frac{L_2}{y_d}\right)^2 \tag{5E-31}$$

Since $0 \le e^{-4T^*} \le 1$, it is required that

$$\left(\frac{L_2}{y_d}\right)^2 \le \frac{1}{4} \tag{5E-32}$$

Setting K = 1 and making use of Eq.(5E-31), Eq.(5E-28) becomes

$$\frac{2L_2}{L_1} e^{-2T^*} = (1 - e^{-4T^*})^{\frac{1}{2}}$$

or

$$4\left(\frac{L_2}{L_1}\right)^2 \left[1 - 4\left(\frac{L_2}{Y_d}\right)^2\right] = 4\left(\frac{L_2}{Y_d}\right)^2$$

Thus,

$$L_1^2 + 4L_2^2 = y_d^2 \qquad (5E-33)$$

The locus of permissible values of L_1, L_2, and y_d again is the first quadrant portion of a family of concentric ellipses. See Figure 5-3. The minimum elapsed time is found from Eq.(5E-31) and the time-optimal control is

$$u_1(t) = L_1 \delta(t)$$

with A = 4.

For the system in Figure 5-4, the augmented plant impulse response, with A = 4, is found to be

$$h_a(t) = e^{2t} \qquad (5E-34)$$

Under the conditions of this problem the optimal w.r.a.s. input is[49]

$$u_1(t) = L_1 \delta(t)$$

which is identical to the time-optimal input found above.

As predicted in Section 5.5, the presence of a stable model in $h_2(t)$ means that the augmented plant is unstable. This is verified in Eq. (5E-34).

5E-3. A Quadratic Constraint on $u_1(t)$

The system in Figure 5-5 has the following set of constraints placed on it:

$$\|u_1\|_2 \le L_1 \qquad [\|u\|_2 \le KL_1]$$
$$\|u_2\|_2 \le AL_2$$
$$\|y\|_2 \ge L_2 \tag{5E-35}$$
$$y(0) = 0$$
$$y(T) = y_d$$

Treating u(t) as the input to the system as before, the impulse responses are given by

$$h_1(t) = 2\,e^{2t} + \delta(t) \tag{5E-36a}$$

$$h_2(t) = \frac{e^{-2t}}{4} \tag{5E-36b}$$

Since both of the inputs have their quadratic contents constrained, it is necessary to evaluate $\|h_1\|_2$ and $\|h_2\|_2$. As $h_1(t)$ contains an impulse, $\|h_1\|_2$ cannot be calculated directly. To circumvent this difficulty let

$$h_1(t) = h_1'(t) + a\,\delta(t)$$

Then:

$$y(t) = \int_0^t h_2(t-\tau)\,u(\tau)\,d\tau + \int_0^t h_2(t-\tau)\,u_2(\tau)\,d\tau$$

$$= \int_0^t h_1'(t-\tau)u(\tau) + a\,\delta(t-\tau)u(\tau)\,d\tau + \int_0^t h_2(t-\tau)u_2(\tau)\,d\tau$$

$$= a\,u(t) + \int_0^t h_1'(t-\tau)u(\tau)\,d\tau + \int_0^t h_2(t-\tau)u_2(\tau)\,d\tau$$

Therefore:

$$y(t) \triangleq y_d = a\,u(T) + \int_0^T h_1'(T-\tau)u(\tau)\,d\tau + \int_0^T h_2(T-\tau)u_2(\tau)\,d\tau$$

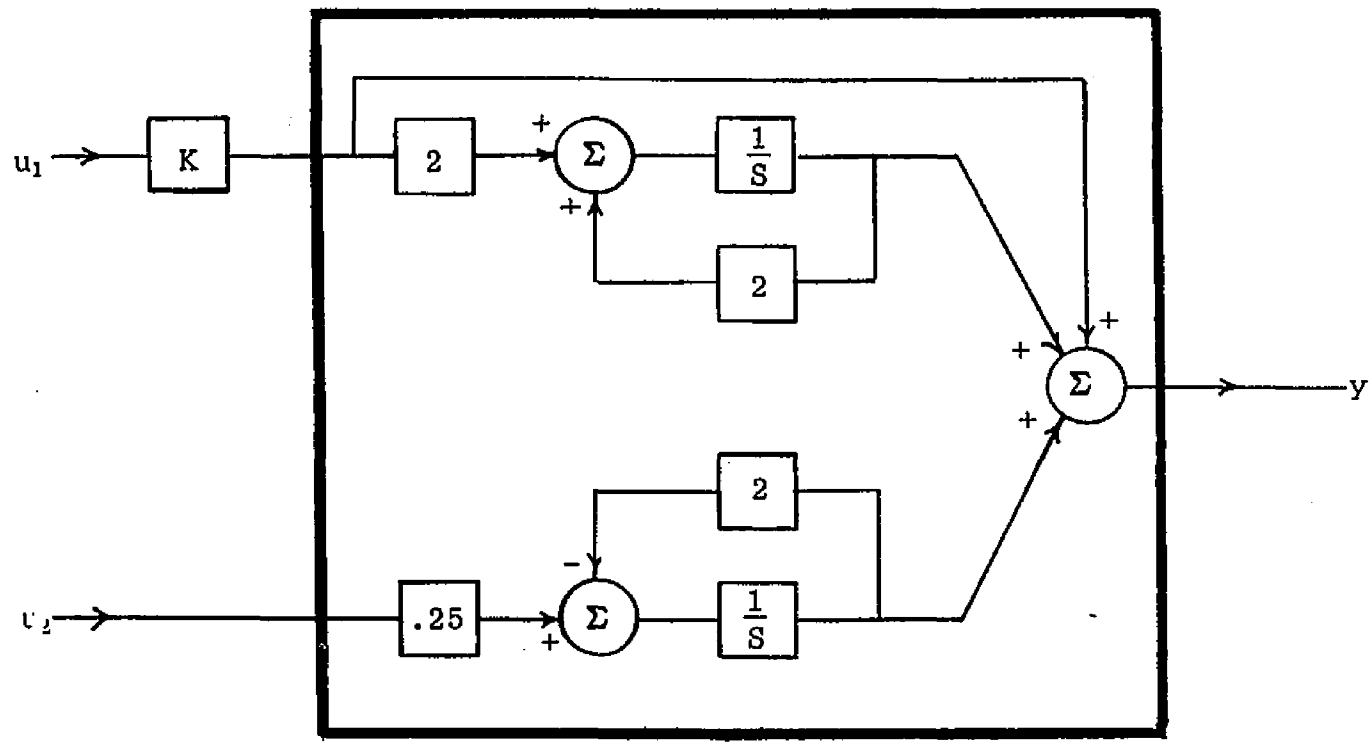

Figure 5-5 Second order system used in Example 5E-3

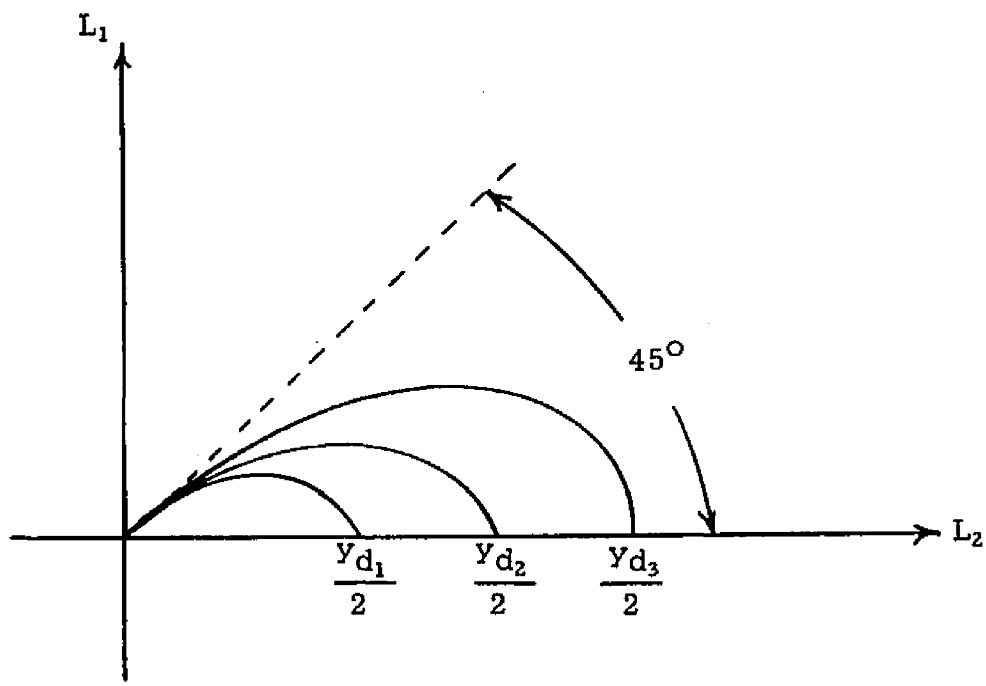

Figure 5-6 Constraint space for Example 5E-3

Now define

$$y'_d \triangleq y_d - a\,u(T) \tag{5E-37}$$

Then

$$y'_d = \int_0^T h'_1(T-\tau)\,u(\tau)d\tau + \int_0^T h_2(T-\tau)u_2(\tau)d\tau$$

It follows that Eqs.(5-19a), (5-20), and (5-22) are still applicable provided that y_d and $h_1(t)$ are replaced by y'_d and $h'_1(t)$ respectively.

Making use of these results, we find

$$\|h'_1\|_2 = \left[\int_0^{T^*} 4e^{4t}dt\right]^{\frac{1}{2}} = e^{2T^*}\left[1 - e^{-4T^*}\right]^{\frac{1}{2}} \tag{5E-38a}$$

$$\|h_2\|_2 = \left[\int_0^{T^*} \frac{e^{-4t}}{16}dt\right]^{\frac{1}{2}} = \frac{1}{8}\left[1 - e^{-4T^*}\right]^{\frac{1}{2}} \tag{5E-38b}$$

From Eqs.(5-19), the time-optimal inputs are

$$u(t) = \frac{2\,KL_1}{[1-e^{-4T^*}]^{\frac{1}{2}}}\, e^{-2t}\, U(t) \tag{5E-39a}$$

$$u_2(t) = \frac{2\,AL_2\, e^{-2T^*}}{[1-e^{-4T^*}]^{\frac{1}{2}}}\, e^{2t}\, U(t) \tag{5E-39b}$$

Using these equations, Eq.(5E-37) becomes

$$y'_d = y_d - \frac{2\,KL_1}{[1-e^{-4T^*}]^{\frac{1}{2}}}\, e^{-2T^*} \tag{5E-40}$$

The minimum elapsed time is then found from

$$KL_1\,\|h'_1\|_2 + AL_2\,\|h_2\|_2 = \left|\, y_d - \frac{2\,KL_1\, e^{-2T_0}}{[1-e^{-4T_0}]^{\frac{1}{2}}}\right|$$

or using Eqs.(5E-38),

$$KL_1[1-e^{-4T^*}]^{\frac{1}{2}}\, e^{2T^*} + \frac{AL_2}{8}[1-e^{-4T^*}]^{\frac{1}{2}} = \left|\, \frac{y_d[1-e^{-4T^*}]^{\frac{1}{2}} - 2KL_1 e^{-2T^*}}{[1-e^{-4T^*}]^{\frac{1}{2}}}\right|$$

Employing Laplace transforms as before, Eq.(5E-10) becomes

$$\frac{2L_2 e^{-2T^*}}{[1-e^{-4T^*}]^{\frac{1}{2}}} \frac{1}{(s-2)} = \frac{2KL_1}{[1-e^{-4T^*}]^{\frac{1}{2}}} \quad \frac{s + \frac{AL_2 e^{-2T^*}}{4KL_1}}{(s+2)} \quad \frac{1}{(s-2)}$$

In order for this to be an equality we require that

$$K = \frac{L_2}{L_1} e^{-2T^*} \tag{5E-42}$$

and

$$A = 8 \tag{5E-43}$$

Using these results in Eq.(5E-41) gives

$$L_2(1-e^{-4T^*})^{\frac{1}{2}} [1+1] = \left| \frac{y_d[1-e^{-4T^*}]^{\frac{1}{2}} - 2L_2 e^{-4T^*}}{[1-e^{-4T^*}]^{\frac{1}{2}}} \right|$$

or

$$2L_2 [1-e^{-4T^*}] = \left| y_d[1-e^{-4T^*}]^{\frac{1}{2}} - 2L_2 e^{-4T^*} \right|$$

Assuming that $y_d[1-e^{-4T^*}]^{\frac{1}{2}} - 2L_2 e^{-4T^*} \geq 0$, T^* is found from the expression

$$e^{-4T^*} = 1 - \left(\frac{2L_2}{y_d}\right)^2 \tag{5E-44}$$

Since $0 \leq e^{-4T^*} \leq 1$, $(\frac{2L_2}{y_d})^2 \leq 1$. Under this condition, it is easily verified that the above assumption is valid.

Setting $K = 1$ and making use of Eq.(5E-43) and Eq.(5E-44), we get

$$\frac{L_2}{L_1} e^{-2T^*} = 1$$

or

$$e^{-4T^*} = \left(\frac{L_1}{L_2}\right)^2 = 1 - \left(\frac{2L_2}{y_d}\right)^2$$

or

$$L_1^2 = L_2^2 \left[1 - \left(\frac{2L_2}{y_d}\right)^2\right]$$

or

$$L_1 = L_2\left[1 - \left(\frac{2L_2}{y_d}\right)^2\right]^{\frac{1}{2}} \qquad (5E\text{-}45)$$

The constraint space for this problem is shown in Figure 5-6.

The time-optimal input is found by substituting Eq.(5E-44) into Eq.(5E-39a). Thus

$$u_1(t) = y_d \frac{L_1}{L_2} e^{-2t}\, U(t)$$

For $A = 8$, the augmented plant impulse response is

$$h_a(t) = \delta(t) + 4e^{2t} = \delta(t) + h'_a(t) \qquad (5E\text{-}46a)$$

where

$$h'_a(t) = 4e^{2t} \qquad (5E\text{-}46b)$$

From Eq.(5-23) with $q_1 = 2$, the optimal w.r.a.s. solution is optimal if

$$\frac{h'_a(t)}{\|h'_a\|_2}\, \mathrm{SGN}\,[y'_d] = \frac{h'_1(t)}{\|h'_1\|_2} \qquad (5E\text{-}47)$$

where $h'_a(t)$ and $h'_1(t)$ are used for the previously mentioned reason. Substituting Eq.(5E-46b) in the left hand side of this expression, we find that

$$\frac{h'_a(t)}{\|h'_a\|_2}\, \mathrm{SGN}\,[y'_d] = \frac{2\,e^{2(t-T^*)}}{[1-e^{-4T^*}]^{\frac{1}{2}}}$$

Using Eq.(5E-38a) and (5E-36a), the right hand side of Eq.(5E-47) becomes

$$\frac{h'_1(t)}{\|h'_1\|_2} = 2\ \frac{e^{2(t-T^*)}}{[1-e^{-4T^*}]^{\frac{1}{2}}}$$

Thus Eq.(5E-47) is an equality and the optimal w.r.a.s. and optimal solutions are the same.

As $H_1(s)$ and $H_2(s)$ do not have common poles and the latter contains a single left half plane pole, the theory of Section 5.5 predicts that the augmented plant is unstable. This is in fact the case (see Eq.5E-46a).

5.7 Single Output Systems with Output Integral Constraints

It is shown in this section that the optimal w.r.a.s. solution to Problem 5 can never be optimal when an integral constraint is placed on the output. To show this, we proceed in the manner used to prove Assertions I and II in Section 5.3.

Since there is an integral constraint on the output, we have

$$\|y\|_1 = \int_{t_o}^{t_f} |y(t)|\, dt \geq L_{2a} \tag{5-24}$$

Employing the same argument as that preceding Eq.(5-10), the output constraint becomes an equality. By virtue of Eq.(5-3), and once again making use of Hölder's inequality, we write

$$\|u_j\|_1 = \int_{t_o}^{t_f} |a_j(t)\, y(t)|\, dt \leq \left[\int_{t_o}^{t_f} |a_j(t)|^q dt\right]^{1/q} \left[\int_{t_o}^{t_f} |y(t)|^p dt\right]^{1/p} \quad (j=2,3,\ldots r) \tag{5-25}$$

Because of the integral constraint, we let $p=1$ whereupon from Eq.(5-9) $q=\infty$. Therefore, Eq.(5-25) becomes

$$\|u_j\|_1 \leq |A_{j\,max}|\, \|y\|_1 = |A_{j\,max}| L_{2a} \quad (j=2,3,\ldots r) \tag{5-26}$$

If an optimal solution is possible, this constraint must be an equality which requires that Hölder's inequality be an equality. This occurs if and only if

$$a_j(t) = k_j\, |y(t)|^{p-1}\, \mathrm{SGN}\, [y(t)] \qquad (k_j \geq 0)$$

or since $p=1$,

$$a_j(t) = k_j \text{ SGN } [y(t)] \tag{5-27}$$

Therefore, in order to have an optimal solution, the feedback coefficients in Figure 5-1 must be positive (or negative) constants if y(t) is greater than (or less than) zero over the entire time interval, (t_o, t_f). If, however, y(t) changes sign during the interval, the $a_j(t)$ are "Bang-Bang" time functions.

In addition to the requirement in Eq.(5-27), an optimal solution is possible only if $u_j(t)$ $(j=2,3,\ldots r)$ has the form[49]

$$u_j(t) = k_j L_2 \delta(t-t_2) \text{ SGN } [h_j(t_f, t_2)] \tag{5-28}$$

where t_2 is the time in the closed interval (t_o, t_f) when $|h_2(t_f, t)|$ is a maximum. Using Eq.(5-3) and Eq.(5-27), we find

$$u_j(t) = k_j y(t) \text{ SGN } [y(t)]$$

or

$$u_j(t) = k_j |y(t)|$$

Therefore,

$$|y(t)| = \frac{u_j(t)}{k_j} \qquad (j=2,3,\ldots r) \tag{5-29}$$

Making use of Eq.(5-28), this last equation becomes

$$|y(t)| = L_2 \delta(t-t_2) \text{ SGN } [h_j(t_f, t_2)] \tag{5-30}$$

Clearly, this can never be an equality if $h_j(t_f, t_2) < 0$. If $h_j(t_f, t_2) \geq 0$, then Eq.(5-30) becomes

$$|y(t)| = L_2 \delta(t_f, t_2)$$

and it follows that

$$|y(t_f)| = 0 \qquad t_2 = t_f$$

$$|y(t_f)| = \infty \qquad t_2 = t_f$$

However, the terminal condition requires that $|y(t_f)| = |y_d|$. Therefore, it is seen that the terminal condition can never be satisfied if the output is given by Eq.(5-30), precisely the form required in order for Eqs.(5-28) and (5-29) to be satisfied. Hence, an optimal solution is not possible when a portion of the controller consists of the feedback arrangement shown in Figure 5-1. As the optimal w.r.a.s. approach uses such a control scheme, we conclude that it can never be the optimal solution if an integral constraint is placed on the output.

5.8 Summary

In this chapter, it was shown that when a quadratic constraint is placed on the output of a single output linear plant, it is possible for the optimal and optimal w.r.a.s. solution to Problem 5 to be identical. The conditions for this to be true were developed and illustrated. Stable plants which satisfy these conditions were shown to produce unstable augmented plants. Finally, it was demonstrated that when the output has an integral constraint placed on it, the optimal w.r.a.s. solution to Problem 5 can never be optimal.

VI. NUMERICAL CONSIDERATIONS

6.1 Introduction

The examples in Chapters III and IV were solved using a digital computer. It is therefore necessary to discuss the more important numerical techniques employed.

6.2 Numerical Search Procedures

This section presents the details of the numerical procedures used to find the feedback coefficients and the minimum elapsed time (i.e. the terminal time) in the examples in Chapter IV. Because of essential differences in the computation procedures used to obtain solutions for single and multiple output plants, a distinction is made between the two.

6.2.1 Single Output Plants

In Chapter IV it was mentioned that it is not generally possible to find the best set of feedback coefficients (i.e. the A's) by taking partial derivatives in Eq.(4-7). The procedure used in this thesis is an (r-1)-dimensional search on the A's. To specifically illustrate the scheme, consider a system with only three inputs (one constrained and two unconstrained) so that it is only necessary to find two feedback coefficients, A_1 and A_2 (i.e. a two-dimensional search is required). A range of A_1 and A_2 in addition to the increments, ΔA_1 and ΔA_2 are selected. For the initial value of A_1, (A_{11}), A_2 is set to its initial value, A_{21}. Using this pair of A's, t_f is found from Eq.(4-7). The output is then calculated from the superposition integral:

$$y(t) = \int_{t_o}^{t} h_a(t,\tau)\, u_1(\tau)\, d\tau$$

where $u_1(\tau)$ is the optimal w.r.a.s. input given in Eq.(4-5). Finally, the value of the output norm (either $\|y\|_1$ or $\|y\|_2$) is computed using the output found in the above equation. The values of A_{11}, A_{21}, t_f, and the appropriate output norm are stored in the computer memory. A_2 is then incremented to A_{22}, where

$$A_{22} = A_{21} + \Delta A_2$$

and the procedure is repeated keeping A_1 at its previous value. A_2 is continually incremented and the cycle repeated until $A_2 = A_{2max}$ (i.e. the upper limit selected for A_2). After running through the calculations with A_2 at A_{2max}, A_1 is incremented to A_{12}, where

$$A_{12} = A_{11} + \Delta A_1$$

A_2 is initialized to A_{21} and again allowed to run through its entire range. The search terminates after the calculations have been performed using A_{1max} and A_{2max}. From data stored in the memory, the computer can be made to choose the set of A's which permits the output constraint to be satisfied for the smallest value of t_f, i.e. t_{f_o}. Alternately, the memory can be printed out and the designer can make the choice. In either instance the optimal w.r.a.s. input is then found from Eq.(4-5). This procedure can, of course, be extended to include systems with more than three inputs.

The flow diagram in Figure 6-1 illustrates a two-dimensional search scheme. The Fortran IV program which implements this scheme and is used to solve Example 4E-3 in Section 4.5.2 is found in Appendix IV.

6.2.2 Multiple Output Plants

As mentioned in Chapter IV, a considerably more difficult computational problem is encountered when the optimal w.r.a.s. solution of multiple output systems is sought. The procedure successfully used in this thesis for two-output plants is to calculate t_f from Eq.(4-58) for an initial value of λ_2. This (λ_2, t_f) pair is then substituted into Eq.(4-59) and the value of

$$\int_{t_o}^{t_f} [h_{a_2}(t_f,\tau) - \frac{yd_2}{yd_1} h_{a_1}(t_f,\tau)] \ \mathrm{SGN}\left[\frac{h_{a_1}(t_f,\tau)}{yd_1} + \lambda_2[h_{a_2}(t_f,\tau) - \frac{yd_2}{yd_1} h_{a_1}(t_f,\tau)]\right] d\tau = 0 \tag{4-58}$$

$$F(t_f,\lambda_2) = \int_{t_o}^{t_f} \left| \frac{h_{a_1}(t_f,\tau)}{yd_1} + \lambda_2\left[h_{a_2}(t_f,\tau) - \frac{yd_2}{yd_1} h_{a_1}(t_f,\tau)\right] \right| d\tau - \frac{1}{L_1} \tag{4-59}$$

$F(t_f,\lambda_2)$ is stored. λ_2 is next incremented by $\Delta\lambda_2$ and the procedure is repeated. When the product of two successive values of $F(t_f,\lambda_2)$ is negative, λ_2^* and therefore t_f^* is bracketed. Modified False Position† is then used to locate the point (λ_2^*, t_f^*). These values are used in Eq.(4-25a) and the two output norms are computed. All of the above computations are

---

† The fairly new technique of Modified False Position[33] differs from the well known false position procedure[34] in that the chord is drawn between the current value of the function and half the value of the function evaluated at a point on the other side of the root. This modification accelerates the convergence of the false position technique.

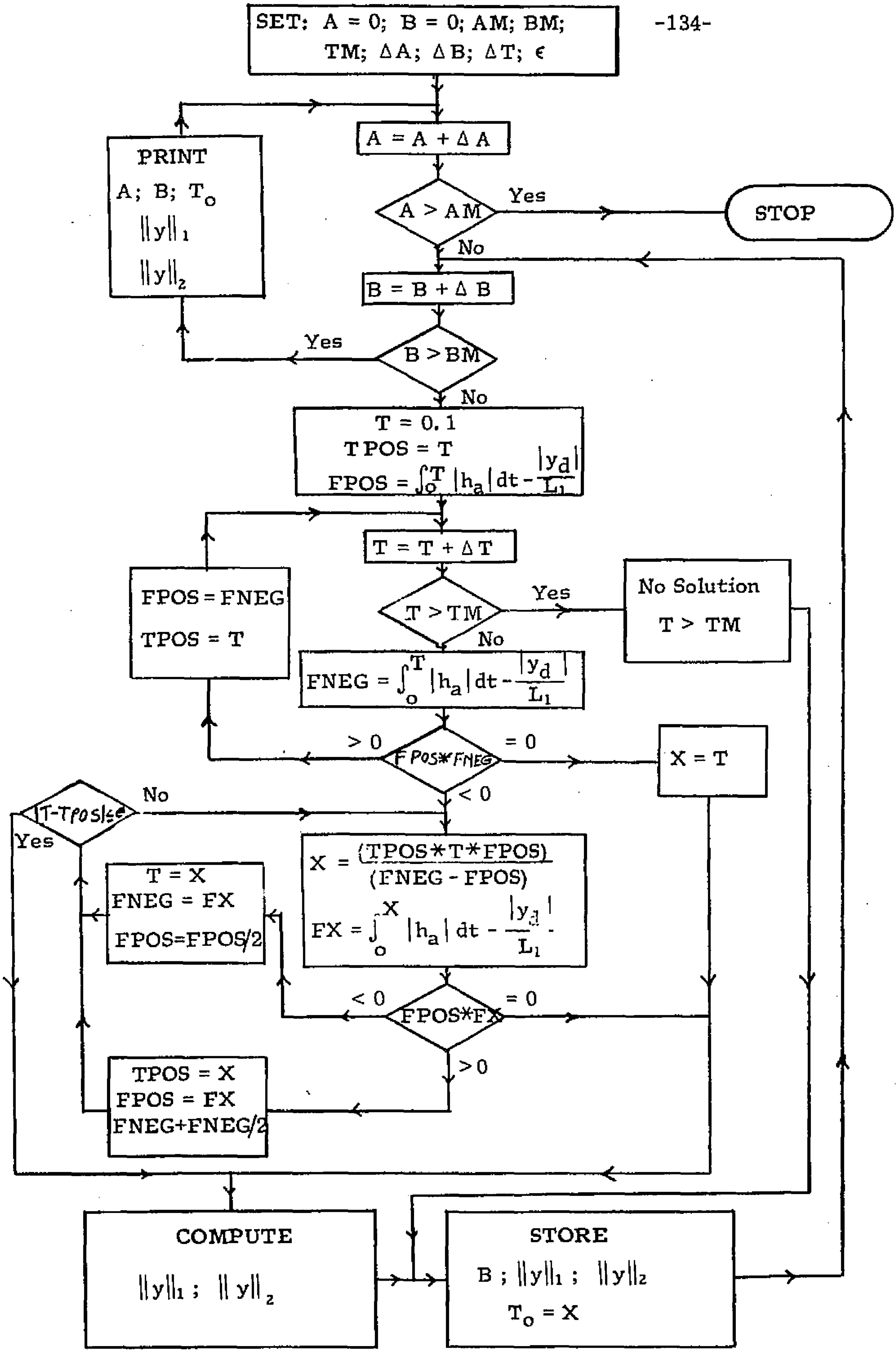

Figure 6-1 Flow Diagram for a 2-Dimensional Search-Single Output System (Example 4E-3)

imbedded in a two-dimensional search scheme on the A's which procedes in the manner already discussed in Section 6.2.1.

Although the procedure is described for a system with two outputs and three inputs, it can be extended to plants with r-inputs and m-outputs. In this instance however, an $[m \times (r-1)]$-dimensional search on the A's is required and m-outputs and their respective norms must be computed.

The flow diagram for a two-dimensional search scheme for a two-output plant is shown in Figure 6-2. The Fortran IV program which implements this scheme and is used to solve Examples 4E-7,8,9 in Section 4.10.2 is in Appendix VI.

6.3 Practical Aspects of the Search

6.3.1 Single Output Plants

If the ranges of both A_1 and A_2 are divided into n different numbers, the two-dimensional search described in Section 6.2.1 requires the calculation of n^2 sets of t_f's and output norms. If there is, à priori, no suggestion of where to search for the best set of A's, two computer runs are necessary. In the first, the increment for each A is chosen allowing the widest region of A-space to be covered in a reasonable amount of computer time. This permits the values of the best A's to be bracketed. These values are then refined by rerunning the problem with ΔA_1 and ΔA_2 reduced, the search being restricted to the region in A-space located in the first computer run. For example, in the first set of computations,

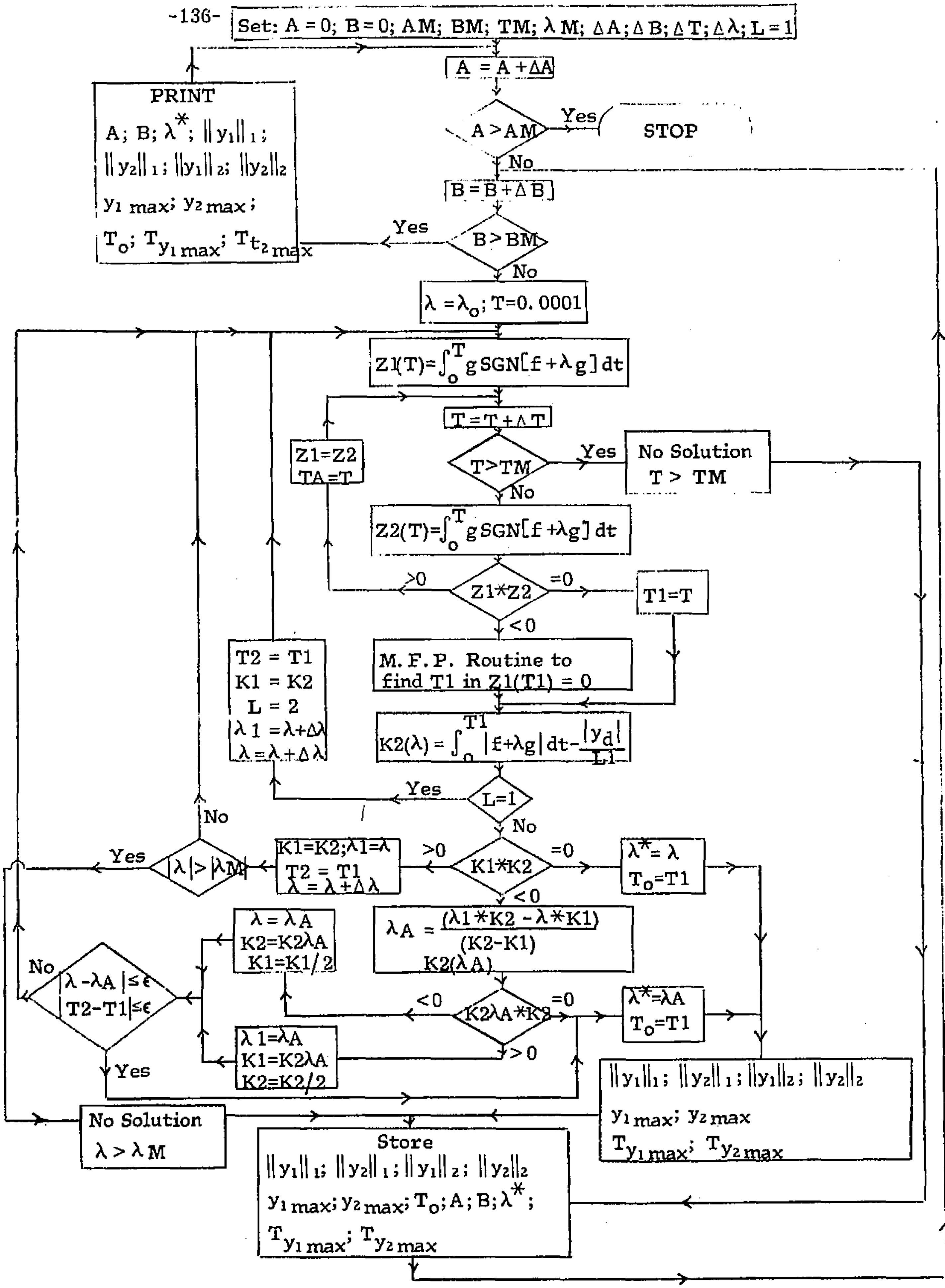

Figure 6-2 Flow Diagram for a Two-Dimensional Search - Two Output System (Examples 4E-7, 8, 9)

only integer values of the A's are used and $\Delta A_1 = \Delta A_2 = 1$. The search is then performed from $-50 \le A_i \le 50$ $(i = 1 \text{ or } 2)$. From this data, the best A's are bracketed. (e.g. $-2 \le A_1 \le -1$ and $-1 \le A_2 \le 0$). The increments are now reduced to 0.01 and the search is repeated in this region. A_1 and A_2 can thus be located to the nearest hundredth.

The results of Sections 4.2 through 4.4 are useful in reducing the computation time. Since finding t_{f_o} in Eq.(4-7) for a given set of A's involves finding a real root of a transcendental equation, it is necessary to start at some initial value of t_f and increment until the root is located between two successive values of t_f. Then some root finding technique, such as modified false position, is used to actually locate the root. Since a lower bound on t_f is available (Eqs.4-13 and 4-17) it is clear that the search to bracket the root can be started at this minimum value. If a lower bound was not known, the initial value of t_f would have to be zero and therefore, the search to bracket the root would take more computer time. The reduction in time per set of A's is probably small†. However, as explained above, this calculation is performed n^2 times in a two-dimensional search. Thus the reduction in time per point must also be multiplied by this factor. Even larger savings are obtained for an R-dimensional search where the factor is n^R.

---

† In Example 4E-3, the reduction is in the order of 80 milliseconds on an IBM 360-40 digital computer.

The two necessary conditions in Section 4.2.1 and 4.4 (Eqs.4-7 and 4-22) which must be satisfied in order for a solution to the given problem to exist are also useful in significantly reducing computing time. As explained above, finding t_{f_O} from Eq.(4-7) involves a search for a root of a transcendental equation. For a given L_1 and y_d, it is possible that no solution to this equation exists (i.e. no real root can be found). By placing an upper limit on the value of t_f used in the search for the root, it is possible to terminate the search after this upper limit is reached, print out some appropriate message, and go on to the next set of A's. However, if a t_f is found which does satisfy this equation, the set of A's and t_{f_O} can then be used to evaluate the right hand side of Inequality (4-22). If it is satisfied the output norm is calculated. However, if the inequality is violated, some suitable message is printed and the next set of A's is used. The calculation of y(t) and the output norm is avoided when either of these terminating tests is satisfied. As is discussed in a subsequent section, these output quantities require the greatest expenditure of computer time per point in the (r-1)-dimensional search. Hence, a significant saving in time is realized by eliminating those sets of A's for which no solution is possible. It is observed that the evaluation of the right hand side of inequality (4-22) is routine.

6.3.2 Multiple Output Plants

In the preceding section it was noted that the feedback coefficient increments should initially be chosen to insure that the values of the best set of A's are bracketed in a reasonable amount of computer time. This is especially true for multiple output plants since the computation time per point is increased over that required for single output systems. This increase occurs because: (1) a set of λ^*'s must be found in addition to t_{f_O} and (2) not one but many outputs and their norms must be calculated.

The results of Section 4.7 may be used, in certain instances, to eliminate the lengthy calculation required to obtain the outputs and their norms. Once the set of λ^*'s and t_{f_O} have been found, the left hand side of the m-inequalities in (4-37) (or 4-42) are computed. If any of these inequalities are violated all of the output constraints can not be satisfied and there is no need to find the outputs and their norms. As the calculation of these output quantities is generally the most time consuming portion of the search per point, a significant saving in computer time is obtained. It is noted that the calculation of the m-inequalities in either (4-37) or (4-42) is straight forward and the additional time required is justified by the possible savings realized when the output constraints are too severe to be satisfied.

Unfortunately, inequality (4-45) cannot be used to save computer time because if this inequality is satisfied, the output magnitude constraints are satisfied. However, if it is volated the output constraints may or may not be satisfied.

In order to avoid large errors in performing the numerical integration in Eq.(4-58), each of the continuous portions of the "Bang-Bang" type kernel is integrated separately†. This requires that the zeros of the argument of the SGN function and hence the position of the discontinuities of the kernel be located before the integration is performed. For this reason, the value of λ_2 is chosen and t_f calculated from Eq.(4-58) and not the other way around. A little thought should convince the reader that if t_f is set instead, a search would be required to find λ_2. This would necessitate the calculation of the zeros of the argument of the SGN function many times for each (λ_2, t_f) pair. Since only one such calculation is required if λ_2 is set and t_f calculated, the computation time is considerably shorter under these conditions.

6.4 Calculation of Outputs and Norms

In Examples 4E-7,8,9 of Section 4.10.2, the outputs and their norms are calculated using numerical integration. In fact, for all but the simplest cases, this is the rule. These outputs are obtained from the superposition integral. Thus

$$y_j(t) = \int_{t_o}^{t} h_{ja}(t,\tau)\, u_1(\tau)\, d\tau \qquad (j=1,2,\ldots m)$$

In order to calculate either $\|y_j\|_1$ or $\|y_j\|_2$, an additional numerical integration is performed. Since this second integration requires appropriately spaced samples††, it is necessary to calculate the m-outputs at a sufficient

† This point, although apparently well understood by experts is nevertheless not touched upon in standard texts on numerical analysis. For example, see references 34 and 35.

†† For instance, if Simpson's Rule[36] is used, (2n+1) equally spaced samples of $y_j(t)$ are required in the interval, (t_o, t_f).

number of points so that the norms are accurate. It is observed, however, that the kernel in the superposition integral is a function of the upper limit, t. Therefore, each time this limit is changed to $t+\Delta t$, it is necessary to perform the entire integration over again (i.e. the previous value of $y_j(t)$ cannot be used to obtain $y_j(t+\Delta t)$ as would be the case if the kernel was independent of the upper limit.)

From this discussion it is clear why the calculation of the output quantities for each set of A's takes such a large portion of the computation time. For if N samples of $y_j(t)$ are required to insure an accurate calculation of the output norms, it is necessary to perform N numerical integrations for each of the m-outputs of the plant or a total of $N \times m$. In point of fact, it was experimentally observed that the calculation of these output quantities for Examples 4E-7,8,9 of Section 4.10.2 took between one half and two-thirds of the overall computation time per point.

6.5 Summary

The numerical search schemes for finding the best set of A's and t_{f_o} for single and multiple output plants was given. It was indicated that when the integral or quadratic constraints on the plant outputs cannot be satisfied, the necessary conditions derived in Chapter IV can be used to eliminate the lengthy computations which are usually required to obtain the output quantities. For single output plants, it was shown how the lower bounds on t_f, derived in Section 4.3 could also be used to reduce computation time. It was pointed out that when discontinuous kernels are integrated, it is important to determine the location of the discontinuities. For this reason, λ_2 is set and t_f calculated in Eq.(4-58) and not the other way around.

APPENDIX I - A GENERALIZATION OF HOLDER'S INEQUALITY

The purpose of this appendix is to extend the usual form of Hölder's Inequality[44]

$$\left| \int_a^b x(t)\, y(t)\, dt \right| \le \left[\int_a^b |x(t)|^p dt \right]^{1/p} \left[\int_a^b |y(t)|^q dt \right]^{1/q} \triangleq \|x\|_p \, \|y\|_q \qquad (p \ge 1,\ q \ge 1) \qquad \text{(A-1)}$$

with $\frac{1}{p} + \frac{1}{q} = 1$, to the situation where x(t) and y(t) are both vectors. To this end, it is shown that

$$\left| \int_a^b \underline{x} \cdot \underline{y}\, dt \right| \le \left[\sum_{i=1}^{n} L_i^{-p} \left(\|x_i\|_{p_i} \right)^p \right]^{1/p} \left[\sum_{i=1}^{n} L_i^{q} \left(\|y_i\|_{q_i} \right)^q \right]^{1/q} \qquad \text{(A-2)}$$

where $p_i \ge 1$, $q_i \ge 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $|x_i|^{p_i}$ and $|y_i|^{q_i}$ are integrable, the L_i are positive constants, and

$$\|x_i\|_{p_i} \triangleq \left[\int_a^b |x_i(t)|^{p_i} dt \right]^{1/p_i} \qquad \text{(A-3a)}$$

$$\|y_i\|_{q_i} \triangleq \left[\int_a^b |y_i(t)|^{q_i} dt \right]^{1/q_i} \qquad \text{(A-3b)}$$

Consider the following integral:

$$\int_a^b \underline{x}(t) \cdot \underline{y}(t)\, dt = \int_a^b \sum_{i=1}^{n} x_i y_i\, dt \qquad \text{(A-4)}$$

Taking magnitudes of both sides of the equation, it is observed that

$$\left| \int_a^b \underline{x} \cdot \underline{y}\, dt \right| \le \sum_{i=1}^{n} \left| \int_a^b x_i\, y_i\, dt \right| \qquad \text{(A-5)}$$

and that equality holds if and only if

$$x_i y_i \ge 0 \qquad (\text{or } x_i\, y_i < 0) \qquad \text{(A-6)}$$

for all i and $a \le t \le b$.

Since the integral on the right hand side of Eq.(A-5) involves scalar quantities, the Hölder's Inequality in Eq.(A-1) is applicable. Therefore, Eq.(A-5) becomes

$$\left| \int_a^b \underline{x} \cdot \underline{y}\, dt \right| \le \sum_{i=1}^{n} \|x_i\|_{p_i} \|y_i\|_{q_i} \qquad \left(\frac{1}{p_i} + \frac{1}{q_i} = 1\right) \qquad \text{(A-7)}$$

where $\|x_i\|_{p_i}$ and $\|y_i\|_{q_i}$ are defined in Eq.(A-3) and p_i and $q_i \ge 1$.

It is well known[44] that Inequality (A-1) is an equality if and only if

$$x(t) = K\, |y(t)|^{q-1}\ \mathrm{SGN}\,[y(t)] \qquad (a \le t \le b) \qquad \text{(A-8)}$$

Making use of this condition and Eq.(A-6), it is seen that (A-7) becomes an equality if and only if

$$x_i(t) = K_i\, |y_i(t)|^{q_i-1}\ \mathrm{SGN}\,[y_i(t)] \qquad (a \le t \le b) \qquad \text{(A-9)}$$

provided that all of the K_i have the same signs.

Now let us define

$$\hat{x}_i \triangleq \frac{\|x_i\|_{p_i}}{L_i} \qquad \text{(A-10a)}$$

$$\hat{y}_i \triangleq L_i \|y_i\|_{q_i} \qquad \text{(A-10b)}$$

where the L_i are positive constants. Since $\hat{x}_i$ and $\hat{y}_i$ are positive

$$\sum_{i=1}^{n} \hat{x}_i \hat{y}_i = \left| \sum_{i=1}^{n} \hat{x}_i \hat{y}_i \right| \qquad \text{(A-11)}$$

The Hölder's Inequality for sums is known to be[50]

$$\left| \sum_{i=1}^{n} \hat{x}_i \hat{y}_i \right| \le \left[\sum_{i=1}^{n} |\hat{x}_i|^p \right]^{1/p} \left[\sum_{i=1}^{n} |\hat{y}_i|^q \right]^{1/q} \qquad \text{(A-12)}$$

where $p \ge 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, equality occurring when

$$\hat{x}_i = \hat{K}\,|\,\hat{y}_i\,|^{q-1} \quad \mathrm{SGN}\,[\,\hat{y}_i\,] \tag{A-13}$$

for all i and $\hat{K} > 0$.

Since $\hat{y}_i$ as defined in Eq.(A-10) is positive, Eq.(A-13) reduces to

$$\hat{x}_i = \hat{K}\,\hat{y}_i^{\,q-1} \tag{A-14}$$

Combining Eqs.(A-12) and (A-7) and using the definitions in Eq.(A-10) we have

$$\left|\int_a^b \underline{x}\cdot\underline{y}\,dt\,\right| \le \left[\sum_{i=1}^{n} L_i^{-p}\left(\|x_i\|_{p_i}\right)^{p}\right]^{1/p}\left[\sum_{i=1}^{n} L_i^{q}\left(\|y_i\|_{q_i}\right)^{q}\right]^{1/q} \tag{A-15}$$

which is the desired form of Hölder's Inequality.

In order for this last relation to be an equality, Eqs.(A-9) and (A-13) must hold and all of the K_i must have the same sign. Using Eqs.(A-10) and (A-14) we find that

$$\hat{x}_i \triangleq L_i^{-1}\|x_i\|_{p_i} = L_i^{-1}\left[\int_a^b |x_i(t)|^{p_i}\,dt\right]^{1/p_i} = \hat{K}|\hat{y}_i|^{q-1} \triangleq \hat{K}L_i^{\,q-1}\left(\|y_i\|_{q_i}\right)^{q-1} \tag{A-16}$$

Substituting Eq.(A-9) into the integral in this last equation and solving for $|K_i|$ gives

$$|\,K_i\,| = \hat{K}\,L_i^{\,q}\left(\|y_i\|_{q_i}\right)^{q-q_i} \qquad (i=1,2,\ldots n) \tag{A-17}$$

Since the K_i can either be all positive or all negative we write

$$K_i = K L_i^{\,q}\left(\|y_i\|_{q_i}\right)^{q-q_i}$$

where K is an arbitrary constant. Therefore, (A-15) is an equality if

$$x_i(t) = K L_i^{\,q}\left(\|y_i\|_{q_i}\right)^{q-q_i}\,|y_i(t)|^{q_i-1}\,\mathrm{SGN}\,[y_i(t)] \quad (i=1,2,\ldots n) \tag{A-18}$$

for $a \le t \le b$.

APPENDIX II - T_{MAX} PROGRAM

```
HNORM[T]=T*T*(T+3.*A)-8.*C1-C2
DISPLAY ['PLEASE TYPE DELTA,UPPER',DIFF,START"]
ACCEPT DELTA,UPPER,DIFF,START
DISPLAY ['ITERATION    LAMBDA    T         S"]
A=0.
K=1
KS=0
KEND=1
KSS=2
GO TO 2
3 A=A-DELTA
33 K=K+1
2 T=START
J=1
C1=A*A*A
C2=4.*(2.-SQRT[2.])
45 TPOS=T
FPOS=HNORM[T]
10 T=T+DELTA
IF (T .GT. UPPER) GO TO 100
FNEG=HNORM[T]
IF ((FPOS*FNEG) .GT. 0.) GO TO 200
30 X=(TPOS*FNEG-T*FPOS)/(FNEG-FPOS)
FX=HNORM[X]
IF (FPOS*FX) 301,400,300
301 T=X
FNEG=FX
FPOS=FPOS/2.
20 TEST=ABS[T-TPOS]
IF (TEST .LE. DIFF) GO TO 400
GO TO 30
300 TPOS=X
FPOS=FX
FNEG=FNEG/2.
GO TO 20
200 FPOS=FNEG
TPOS=T
GO TO 10
100 IF (J .EQ. 2) GO TO 5
GO TO 5
400 CONTINUE
ASLOPE=-(X*X/2.-4.*A*A)/(X*ABS[X/2.+A])
```

```
S=ASLOPE
IF (J .EQ. 2) GO TO 5
T=X+START
J=2
GO TO 45
5 CONTINUE
IF (KS) 21,15,22
22 S2=S
T2=X
A2=A
IF (S1**S2) 16,17,18
18 S1=S2
T1=T2
A1=A2
GO TO 150
16 KS=-1
KSS=1
41 IF (ABS[A1-A2] .LE. DIFF) GO TO 17
GO TO 150
15 S1=S
T1=X
A1=A
KS=2
GO TO 150
150 DISPLAY K,A,X,S
IF (KEND .EQ. 2) GO TO 151
IF (KSS .EQ. 2) GO TO 3
A=(A1+A2)/2.
GO TO 33
17 KEND=2
GO TO 150
21 IF (S*S1) 31,17,32
31 S2=S
T2=X
A2=A
GO TO1
32 S1=S
T1=X
A1=A
GO TO 41
151 DISPLAY ['END OF SEARCH"]
END
```

APPENDIX III - ASCENT PROGRAM

```
HNORM[T]=T*T*(T+3.*A)-8.*C1-C2
DISPLAY ['PLEASE TYPE DELTA,UPPER ,DIFF,START"]
ACCEPT DELTA,UPPER,DIFF,START
DISPLAY ['ITERATION     LAMBDA     T        S"]
A=0.
K=1
KS=0
KEND=1
KSS=2
GO TO 2
3 A=A+DELTA/S
33 K=K+1
2 T=START
J=1
C1=A*A*A
C2=4.*(2.-SQRT[2.])
45 TPOS=T
FPOS=HNORM[T]
10 T=T+DELTA
IF (T .GT. UPPER) GO TO 100
FNEG=HNORM[T]
IF ((FPOS*FNEG) .GT. 0.) GO TO 200
30 X=(TPOS*FNEG-T*FPOS)/(FNEG-FPOS)
FX=HNORM[X]
IF (FPOS*FX) 301,400,300
301 T=X
FNEG=FX
FPOS=FPOS/2.
20 TEST=ABS[T-TPOS]
IF (TEST .LE. DIFF) GO TO 400
GO TO 30
300 TPOS=X
FPOS=FX
FNEG=FNEG/2.
GO TO 20
200 FPOS=FNEG
TPOS=T
GO TO 10
100 IF (J .EQ. 2) GO TO 5
GO TO 5
400 CONTINUE
ASLOPE=-(X*X/2.-4.*A*A)/(X*ABS[X/2..+A])
```

```
S=ASLOPE
IF (J .EQ. 2) GO TO 5
T=X+START
J=2
GO TO 45
5 CONTINUE
IF (KS) 21,15,22
22 S2=S
T2=X
A2=A
IF (S1*S2) 16,17,18
18 S1=S2
T1=T2
A1=A2
GO TO 150
16 KS=-1
KSS=1
41 IF (ABS[A1-A2] .LE. DIFF) GO TO 17
GO TO 150
15 S1=S
T1=X
A1=A
KS=2
GO TO 150
150 DISPLAY K,A,X,S
IF (KEND .EQ. 2) GO TO 151
IF (KSS .EQ. 2) GO TO 3
AT=A
A=(S2*A2-S1*A1+T1-T2)/(S2-S1)
IF (ABS[AT-A] .GT. DIFF) GO TO 33
A=(A1+A2)/2.
GO TO 33
17 KEND=2
GO TO 150
21 IF (S*S1) 31,17,32
31 S2=S
T2=X
A2=A
GO TO 41
32 S1=S
T1=X
A1=A
GO TO 41
151 DISPLAY ['END OF SEARCH"]
END
*
```

APPENDIX IV

```
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*4QLB(100),TMIN(100),YNRMA(100),YNRME(100)
      HNORM(T)=C1*T-C2*(C4*(1.-DEXP(-B*T))-C3*(1.-DEXP(-A*T)))-P
  999 READ(5,78) AINCR,BINCR,DELTA,UPPER,DIFF,START,YD,PL1
      READ(5,77) I5,J5,K5,I6,J6,K6,INTG1
   77 FORMAT(7I5)
   78 FORMAT(8E10.5)
      WRITE(6,79) I5,J5,K5,I6,J6,K6,INTG1
      WRITE(6,76) AINCR,BINCR,START,DELTA,UPPER,DIFF,YD,PL1
   76 FORMAT(/5X,9HA INCR = ,D15.4,5X,9HB INCR = ,D15.4,5X,8HSTART = ,
     2D15.4,5X,8HDELTA = ,D15.4 /5X,8HUPPER = ,D15.4,5X,7HDIFF = ,D15.4,
     35X,5HYD = ,D15.4,5X,6HPL1 = ,D15.4)
   79 FORMAT(/5X,5HI5 = ,I5,5X,5HJ5 = ,I5,5X,5HK5 = ,I5,5X,5HI6 = ,I5,
     25X,5HJ6 = ,I5,5X,5HK6 = ,I5,5X,8HINTG1 = ,I5)
      P=YD/PL1
      DO  5  I=I5,J5,K5
      QI=I
      A=QI/AINCR
      DO  6  J=I6,J6,K6
      JX=J/INTG1
      QJ=J
      B=QJ/BINCR
      QLB(JX)=B
      IF (A .EQ. B) GO TO 13
      GO TO   12
   13 WRITE(6,80) A,B
   80 FORMAT(1H0,'  ROOTS ARE EQUAL FOR A = ",D15.3 ,
     1' AND B = ",D15.3,'.ASTERISKS ARE PRINTED IN OUTPUT ARRAY.')
      GO TO 100
   12 C1=1./(A*B)
      C2=1./(A-B)
      C3=1./(A*A)
      C4=1./(B*B)
      C5=A+B
      C6=C3*(1./A)
      C7=C4*(1./B)
      C8=C3*C4
    2 T=START
      TPOS=T
      FPOS=HNORM(T)
   10 T=T+DELTA
      IF (T .GT. UPPER) GO TO 100
      FNEG=HNORM(T)
      IF ((FPOS*FNEG) .GT. 0.) GO TO 200
   30 X=(TPOS*FNEG  -T*FPOS)/(FNEG-FPOS)
      FX=HNORM(X)
      IF (FPOS*FX) 301,400,300
  301 T=X
      FNEG=FX
      FPOS=FPOS/2.
   20 TEST=DABS(T-TPOS)
      IF (TEST .LE. DIFF) GO TO 400
      GO TO 30
  300 TPOS=X
      FPOS=FX
      FNEG=FNEG/2.
      GO TO 20
  200 FPOS=FNEG
```

```
      TPOS=T
      GO TO 10
  100 TMIN(JX)=-99999999.999
      YNRMA(JX)=-99999.9999
      YNRME(JX)=-99999.9999
      GO TO 6
  400 TMIN(JX)=X
C
C  WE HAVE FOUND T MIN.
C
      E1=1.-DEXP(-A*X)
      E2=1.-DEXP(-B*X)
      E3=DEXP(-2.*A*X)
      E4=DEXP(-2.*B*X)
      E5=DEXP(-C5*X)
      YAREA=PL1*(0.5*C1*X*X - C5*C3*C4*X + C2*(C7*E2-C6*E1))
      F1=C8*X*(C8*C5*C5+X*(X/3.-C5*C1))
      F2=2.*C2*C8*(C3*E1-C4*E2)
      F3=0.5*C2*C2*(C3*C6*(1.-E3)+C4*C7*(1.-E4)-(4.*C8/C5)*(1.-E5))
      F4=2.*C1*C2*(C7*(1.-E2)-C6*(1.-E1)) *X
      YENGY=PL1*DSQRT(F1+F2+F3-F4)
  104 YNRMA(JX)=YAREA
  109 YNRME(JX)=YENGY
    6 CONTINUE
      LL1=I6/INTG1
      LL2=J6/INTG1
      LL3=K6/INTG1
      WRITE(6,7)    A
    7 FORMAT(1H1,62X,4HA = ,D15.4//4(4X,1HB,5X,4HTMIN,3X,6HY AREA,2X,
     27HY ENRGY)/)
      WRITE(6,4) (GLB(JX),TMIN(JX),YNRMA(JX),YNRME(JX),JX=LL1,LL2,LL3)
    4 FORMAT(4(1X,F6.2,1X,F7.4,1X,F7.4,1X,F7.4,1X))
    5 CONTINUE
      GO TO 999
      END
```

APPENDIX V

A = -0.70

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.6185	0.3769	0.4601	-0.20	1.5999	0.3702	0.4556
-0.30	1.5815	0.3636	0.4512	-0.40	1.5633	0.3570	0.4468
-0.50	1.5452	0.3506	0.4424	-0.60	1.5273	0.3442	0.4381
-0.70	*******	*******	*******	-0.80	1.4922	0.3316	0.4293
-0.90	1.4749	0.3255	0.4250	-1.00	1.4579	0.3195	0.4207
-1.10	1.4412	0.3136	0.4166	-1.20	1.4247	0.3078	0.4124
-1.30	1.4084	0.3021	0.4082	-1.40	1.3923	0.2966	0.4041
-1.50	1.3765	0.2910	0.4000	-1.60	1.3610	0.2857	0.3960
-1.70	1.3457	0.2805	0.3920	-1.80	1.3307	0.2754	0.3881
-1.90	1.3160	0.2704	0.3844	-2.00	1.3015	0.2655	0.3806
-2.10	1.2872	0.2607	0.3768	-2.20	1.2732	0.2560	0.3731
-2.30	1.2595	0.2515	0.3695	-2.40	1.2460	0.2470	0.3660
-2.50	1.2328	0.2427	0.3625				

A = -0.80

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.5959	0.3683	0.4543	-0.20	1.5782	0.3620	0.4501
-0.30	1.5608	0.3559	0.4460	-0.40	1.5434	0.3497	0.4418
-0.50	1.5262	0.3436	0.4377	-0.60	1.5091	0.3376	0.4335
-0.70	1.4922	0.3316	0.4293	-0.80	*******	*******	*******
-0.90	1.4589	0.3199	0.4211	-1.00	1.4427	0.3143	0.4171
-1.10	1.4266	0.3087	0.4130	-1.20	1.4107	0.3031	0.4090
-1.30	1.3950	0.2977	0.4050	-1.40	1.3795	0.2923	0.4010
-1.50	1.3643	0.2872	0.3972	-1.60	1.3493	0.2820	0.3933
-1.70	1.3345	0.2770	0.3894	-1.80	1.3200	0.2721	0.3857
-1.90	1.3057	0.2673	0.3820	-2.00	1.2916	0.2625	0.3783
-2.10	1.2777	0.2579	0.3747	-2.20	1.2641	0.2534	0.3711
-2.30	1.2508	0.2490	0.3676	-2.40	1.2376	0.2447	0.3642
-2.50	1.2247	0.2405	0.3608				

WHEN A=B, THE ROOTS ARE EQUAL. ****** IS PRINTED.

A = -0.90

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.5738	0.3599	0.4486	-0.20	1.5570	0.3541	0.4447
-0.30	1.5404	0.3483	0.4408	-0.40	1.5238	0.3426	0.4369
-0.50	1.5074	0.3368	0.4329	-0.60	1.4911	0.3311	0.4289
-0.70	1.4749	0.3255	0.4250	-0.80	1.4589	0.3199	0.4211
-0.90	*******	*******	*******	-1.00	1.4275	0.3091	0.4133
-1.10	1.4120	0.3038	0.4095	-1.20	1.3967	0.2985	0.4056
-1.30	1.3816	0.2933	0.4017	-1.40	1.3668	0.2882	0.3980
-1.50	1.3520	0.2832	0.3942	-1.60	1.3375	0.2783	0.3905
-1.70	1.3233	0.2735	0.3869	-1.80	1.3092	0.2688	0.3833
-1.90	1.2954	0.2642	0.3797	-2.00	1.2817	0.2596	0.3761
-2.10	1.2682	0.2551	0.3726	-2.20	1.2550	0.2508	0.3691
-2.30	1.2420	0.2465	0.3657	-2.40	1.2292	0.2424	0.3624
-2.50	1.2167	0.2383	0.3590				

A = -1.00

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.5521	0.3519	0.4431	-0.20	1.5362	0.3464	0.4394
-0.30	1.5204	0.3410	0.4357	-0.40	1.5046	0.3356	0.4320
-0.50	1.4889	0.3301	0.4282	-0.60	1.4733	0.3248	0.4245
-0.70	1.4579	0.3195	0.4207	-0.80	1.4427	0.3143	0.4171
-0.90	1.4275	0.3091	0.4133	-1.00	*******	*******	*******
-1.10	1.3976	0.2989	0.4059	-1.20	1.3829	0.2938	0.4022
-1.30	1.3684	0.2889	0.3986	-1.40	1.3541	0.2841	0.3950
-1.50	1.3399	0.2793	0.3914	-1.60	1.3258	0.2746	0.3877
-1.70	1.3121	0.2700	0.3843	-1.80	1.2985	0.2655	0.3808
-1.90	1.2850	0.2610	0.3772	-2.00	1.2718	0.2566	0.3738
-2.10	1.2588	0.2523	0.3704	-2.20	1.2459	0.2481	0.3671
-2.30	1.2333	0.2440	0.3638	-2.40	1.2208	0.2400	0.3605
-2.50	1.2086	0.2360	0.3573				

WHEN A=B, THE ROOTS ARE EQUAL. ****** IS PRINTED.

A = -1.10

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.5310	0.3440	0.4376	-0.20	1.5158	0.3389	0.4342
-0.30	1.5007	0.3338	0.4307	-0.40	1.4857	0.3287	0.4271
-0.50	1.4707	0.3236	0.4236	-0.60	1.4559	0.3186	0.4200
-0.70	1.4412	0.3136	0.4166	-0.80	1.4266	0.3087	0.4130
-0.90	1.4120	0.3038	0.4095	-1.00	1.3976	0.2989	0.4059
-1.10	*******	*******	*******	-1.20	1.3692	0.2893	0.3988
-1.30	1.3552	0.2846	0.3953	-1.40	1.3414	0.2800	0.3919
-1.50	1.3277	0.2754	0.3884	-1.60	1.3143	0.2709	0.3850
-1.70	1.3009	0.2665	0.3816	-1.80	1.2877	0.2621	0.3782
-1.90	1.2747	0.2578	0.3748	-2.00	1.2619	0.2536	0.3715
-2.10	1.2493	0.2495	0.3682	-2.20	1.2368	0.2455	0.3650
-2.30	1.2245	0.2415	0.3618	-2.40	1.2124	0.2376	0.3586
-2.50	1.2005	0.2338	0.3555				

A = -1.20

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.5103	0.3363	0.4323	-0.20	1.4958	0.3315	0.4290
-0.30	1.4814	0.3267	0.4257	-0.40	1.4671	0.3220	0.4223
-0.50	1.4529	0.3173	0.4191	-0.60	1.4388	0.3125	0.4157
-0.70	1.4247	0.3078	0.4124	-0.80	1.4107	0.3031	0.4090
-0.90	1.3967	0.2985	0.4056	-1.00	1.3829	0.2938	0.4022
-1.10	1.3692	0.2893	0.3988	-1.20	*******	*******	*******
-1.30	1.3422	0.2803	0.3922	-1.40	1.3289	0.2759	0.3888
-1.50	1.3157	0.2716	0.3855	-1.60	1.3027	0.2673	0.3823
-1.70	1.2898	0.2630	0.3789	-1.80	1.2770	0.2588	0.3757
-1.90	1.2645	0.2547	0.3724	-2.00	1.2520	0.2507	0.3692
-2.10	1.2398	0.2467	0.3661	-2.20	1.2277	0.2428	0.3629
-2.30	1.2158	0.2390	0.3598	-2.40	1.2040	0.2352	0.3568
-2.50	1.1924	0.2315	0.3537				

WHEN A=B, THE ROOTS ARE EQUAL. ****** IS PRINTED.

A = -1.30

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.4899	0.3289	0.4270	-0.20	1.4763	0.3244	0.4239
-0.30	1.4626	0.3199	0.4208	-0.40	1.4490	0.3155	0.4177
-0.50	1.4354	0.3110	0.4146	-0.60	1.4219	0.3066	0.4114
-0.70	1.4084	0.3021	0.4082	-0.80	1.3950	0.2977	0.4050
-0.90	1.3816	0.2933	0.4017	-1.00	1.3684	0.2889	0.3986
-1.10	1.3552	0.2846	0.3953	-1.20	1.3422	0.2803	0.3922
-1.30	*******	*******	*******	-1.40	1.3165	0.2719	0.3858
-1.50	1.3038	0.2677	0.3826	-1.60	1.2911	0.2636	0.3794
-1.70	1.2787	0.2595	0.3762	-1.80	1.2664	0.2555	0.3731
-1.90	1.2542	0.2516	0.3700	-2.00	1.2422	0.2477	0.3669
-2.10	1.2303	0.2439	0.3639	-2.20	1.2186	0.2401	0.3608
-2.30	1.2070	0.2364	0.3578	-2.40	1.1956	0.2328	0.3548
-2.50	1.1843	0.2292	0.3519				

A = -1.40

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.4701	0.3217	0.4218	-0.20	1.4571	0.3175	0.4189
-0.30	1.4441	0.3133	0.4161	-0.40	1.4311	0.3091	0.4131
-0.50	1.4182	0.3049	0.4101	-0.60	1.4052	0.3007	0.4071
-0.70	1.3923	0.2966	0.4041	-0.80	1.3795	0.2923	0.4010
-0.90	1.3668	0.2882	0.3980	-1.00	1.3541	0.2841	0.3950
-1.10	1.3414	0.2800	0.3919	-1.20	1.3289	0.2759	0.3888
-1.30	1.3165	0.2719	0.3858	-1.40	*******	*******	*******
-1.50	1.2918	0.2639	0.3796	-1.60	1.2797	0.2600	0.3766
-1.70	1.2677	0.2561	0.3736	-1.80	1.2558	0.2523	0.3706
-1.90	1.2440	0.2485	0.3676	-2.00	1.2324	0.2447	0.3646
-2.10	1.2209	0.2411	0.3616	-2.20	1.2095	0.2374	0.3587
-2.30	1.1982	0.2339	0.3558	-2.40	1.1871	0.2304	0.3529
-2.50	1.1762	0.2269	0.3501				

WHEN A=B, THE ROOTS ARE EQUAL. ****** IS PRINTED.

A = -1.50

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.4508	0.3147	0.4168	-0.20	1.4384	0.3108	0.4141
-0.30	1.4260	0.3069	0.4114	-0.40	1.4136	0.3029	0.4086
-0.50	1.4012	0.2990	0.4057	-0.60	1.3889	0.2950	0.4029
-0.70	1.3765	0.2910	0.4000	-0.80	1.3643	0.2872	0.3972
-0.90	1.3520	0.2832	0.3942	-1.00	1.3399	0.2793	0.3914
-1.10	1.3277	0.2754	0.3884	-1.20	1.3157	0.2716	0.3855
-1.30	1.3038	0.2677	0.3826	-1.40	1.2918	0.2639	0.3796
-1.50	*******	*******	*******	-1.60	1.2683	0.2564	0.3738
-1.70	1.2567	0.2527	0.3709	-1.80	1.2452	0.2490	0.3680
-1.90	1.2339	0.2454	0.3651	-2.00	1.2226	0.2418	0.3623
-2.10	1.2114	0.2383	0.3594	-2.20	1.2004	0.2348	0.3566
-2.30	1.1895	0.2313	0.3538	-2.40	1.1787	0.2280	0.3510
-2.50	1.1681	0.2247	0.3483				

A = -1.60

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.4319	0.3079	0.4119	-0.20	1.4201	0.3043	0.4093
-0.30	1.4083	0.3006	0.4067	-0.40	1.3965	0.2969	0.4041
-0.50	1.3846	0.2931	0.4014	-0.60	1.3728	0.2894	0.3987
-0.70	1.3610	0.2857	0.3960	-0.80	1.3493	0.2820	0.3933
-0.90	1.3375	0.2783	0.3905	-1.00	1.3258	0.2746	0.3877
-1.10	1.3143	0.2709	0.3850	-1.20	1.3027	0.2673	0.3823
-1.30	1.2911	0.2636	0.3794	-1.40	1.2797	0.2600	0.3766
-1.50	1.2683	0.2564	0.3738	-1.60	*******	*******	*******
-1.70	1.2459	0.2492	0.3682	-1.80	1.2348	0.2457	0.3654
-1.90	1.2237	0.2423	0.3627	-2.00	1.2128	0.2388	0.3599
-2.10	1.2020	0.2354	0.3572	-2.20	1.1913	0.2321	0.3544
-2.30	1.1808	0.2288	0.3517	-2.40	1.1703	0.2256	0.3491
-2.50	1.1600	0.2224	0.3464				

WHEN A=B, THE ROOTS ARE EQUAL. ****** IS PRINTED.

A = -1.70

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.4135	0.3014	0.4071	-0.20	1.4022	0.2979	0.4047
-0.30	1.3909	0.2945	0.4022	-0.40	1.3796	0.2909	0.3997
-0.50	1.3683	0.2875	0.3972	-0.60	1.3570	0.2840	0.3946
-0.70	1.3457	0.2805	0.3920	-0.80	1.3345	0.2770	0.3894
-0.90	1.3233	0.2735	0.3869	-1.00	1.3121	0.2700	0.3843
-1.10	1.3009	0.2665	0.3816	-1.20	1.2898	0.2630	0.3789
-1.30	1.2787	0.2595	0.3762	-1.40	1.2677	0.2561	0.3736
-1.50	1.2567	0.2527	0.3709	-1.60	1.2459	0.2492	0.3682
-1.70	*******	*******	*******	-1.80	1.2243	0.2425	0.3629
-1.90	1.2137	0.2392	0.3602	-2.00	1.2031	0.2359	0.3575
-2.10	1.1927	0.2326	0.3549	-2.20	1.1823	0.2294	0.3523
-2.30	1.1721	0.2263	0.3497	-2.40	1.1619	0.2231	0.3471
-2.50	1.1518	0.2200	0.3445				

A = -1.80

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.3954	0.2950	0.4024	-0.20	1.3846	0.2917	0.4000
-0.30	1.3739	0.2885	0.3977	-0.40	1.3631	0.2853	0.3954
-0.50	1.3523	0.2820	0.3930	-0.60	1.3416	0.2787	0.3906
-0.70	1.3307	0.2754	0.3881	-0.80	1.3200	0.2721	0.3857
-0.90	1.3092	0.2688	0.3833	-1.00	1.2985	0.2655	0.3808
-1.10	1.2877	0.2621	0.3782	-1.20	1.2770	0.2588	0.3757
-1.30	1.2664	0.2555	0.3731	-1.40	1.2558	0.2523	0.3706
-1.50	1.2452	0.2490	0.3680	-1.60	1.2348	0.2457	0.3654
-1.70	1.2243	0.2425	0.3629	-1.80	*******	*******	*******
-1.90	1.2037	0.2361	0.3577	-2.00	1.1935	0.2330	0.3552
-2.10	1.1833	0.2299	0.3526	-2.20	1.1734	0.2268	0.3502
-2.30	1.1634	0.2237	0.3476	-2.40	1.1535	0.2207	0.3451
-2.50	1.1437	0.2177	0.3426				

WHEN A=B, THE ROOTS ARE EQUAL. ****** IS PRINTED.

A = -1.90

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.3777	0.2887	0.3977	-0.20	1.3675	0.2857	0.3955
-0.30	1.3572	0.2827	0.3933	-0.40	1.3469	0.2796	0.3911
-0.50	1.3366	0.2766	0.3889	-0.60	1.3263	0.2735	0.3866
-0.70	1.3160	0.2704	0.3844	-0.80	1.3057	0.2673	0.3820
-0.90	1.2954	0.2642	0.3797	-1.00	1.2850	0.2610	0.3772
-1.10	1.2747	0.2578	0.3748	-1.20	1.2645	0.2547	0.3724
-1.30	1.2542	0.2516	0.3700	-1.40	1.2440	0.2485	0.3676
-1.50	1.2339	0.2454	0.3651	-1.60	1.2237	0.2423	0.3627
-1.70	1.2137	0.2392	0.3602	-1.80	1.2037	0.2361	0.3577
-1.90	*******	*******	*******	-2.00	1.1839	0.2301	0.3528
-2.10	1.1741	0.2271	0.3504	-2.20	1.1644	0.2241	0.3480
-2.30	1.1547	0.2212	0.3455	-2.40	1.1451	0.2183	0.3431
-2.50	1.1356	0.2154	0.3406				

A = -2.00

B	TMIN	Y AREA	Y ENRGY	B	TMIN	Y AREA	Y ENRGY
-0.10	1.3606	0.2828	0.3932	-0.20	1.3508	0.2800	0.3912
-0.30	1.3410	0.2771	0.3891	-0.40	1.3311	0.2742	0.3870
-0.50	1.3213	0.2714	0.3849	-0.60	1.3114	0.2684	0.3828
-0.70	1.3015	0.2655	0.3806	-0.80	1.2916	0.2625	0.3783
-0.90	1.2817	0.2596	0.3761	-1.00	1.2718	0.2566	0.3738
-1.10	1.2619	0.2536	0.3715	-1.20	1.2520	0.2507	0.3692
-1.30	1.2422	0.2477	0.3669	-1.40	1.2324	0.2447	0.3646
-1.50	1.2226	0.2418	0.3623	-1.60	1.2128	0.2388	0.3599
-1.70	1.2031	0.2359	0.3575	-1.80	1.1935	0.2330	0.3552
-1.90	1.1839	0.2301	0.3528	-2.00	*******	*******	*******
-2.10	1.1649	0.2243	0.3482	-2.20	1.1554	0.2215	0.3458
-2.30	1.1461	0.2187	0.3434	-2.40	1.1368	0.2159	0.3411
-2.50	1.1275	0.2130	0.3387				

WHEN A=B, THE ROOTS ARE EQUAL. ****** IS PRINTED.

APPENDIX VI

```
C     THIS PROGRAM WILL FIND THE TIME,AREA AND ENERGY OF BOTH OUTPUTS
C     AND ALSO THE TIME AND VALUE OF THE MAXIMUM VALUE OF EACH OUTPUT
C     ALL AS A FUNCTION OF A AND B.
      DIMENSION QLB(100),TH(100),ALAMBD(100),AY1X1(1500),AY2X1(1500),
     1Y1NRMA(100),Y2NRMA(100),Y1NRME(100),Y2NRME(100),Y1MAG(100),
     2Y2MAG(100),TY1MAG(100),TY2MAG(100),SGN(50),TZERO(50),DELY2(50)
      COMMON R1,R2,D,A1,A2,A3,A4,WD,A5,A6,A7,A8
  999 READ(5,78) INTG1,INTG2,JB,I5,J5,K5,I6,K6,J6ALT,JSWTCH
      READ(5,77)DELTAT,CONSL1,UPPER,EPSLON,EPS1,DIFF,AINCR,BINCR,C1AREA,
     1C2AREA,C1ENGY,C2ENGY,C1MAG,C2MAG
   77 FORMAT(8F10.4)
   78 FORMAT(10I5)
      PRINT 79,DELTAT,CONSL1,UPPER,EPSLON,EPS1,DIFF,AINCR,BINCR,C1AREA,
     1C2AREA,C1ENGY,C2ENGY,C1MAG,C2MAG
      PRINT 76,INTG1,INTG2,JB,I5,J5,K5,I6,K6,J6ALT,JSWTCH
   79 FORMAT(1H1,4X,6HDELTAT,3X,6HCONSL1,4X,5HUPPER,3X,6HEPSLON,4X,
     14HEPS1,5X,4HDIFF,5X,5HAINCR,4X,5HBINCR,3X,6HC1AREA,3X,6HC2AREA,3X,
     26HC1ENGY,3X,6HC2ENGY,4X,5HC1MAG,4X,5HC2MAG/3X,F8.4,1X,2(F8.3,1X),
     33(F8.4,1X),8(F8.3,1X))
   76 FORMAT(//5X,5HINTG1,5X,5HINTG2,5X,2HJB,5X,2HI5,5X,2HJ5,5X,2HK5,
     15X,2HI6,5X,2HK6,5X,5HJ6ALT,5X,6HJSWTCH/2(5X,I5),6(2X,I5),
     25X,I5,5X,I5)
      PRINT 998
  998 FORMAT(/44H DELTAT SETS H IN SIMPSONS RULE INTEGRATION./1X,
     132HCONSL1 IS L1 (INPUT CONSTRAINT)./ 50H UPPER SETS LIMIT ON T IN
     2SEARCH FOR T AND LAMBDA./68H EPSLON IS LARGEST VALUE OF G1 WHICH W
     3ILL STOP SEARCH FOR T, LAMBDA./ 117H EPS1 IS LARGEST VALUE OF THE
     4DIFFERENCE BETWEEN PRESENT AND PREVIOUS VALUES OF BOTH T AND LAMBD
     5A WHICH STOPS SEARCH./1X,121HDIFF IS THE LARGEST DIFFERENCE BETWEE
     6N PRESENT AND PREVIOUS VALUE OF T WHICH WILL SATISFY TERMINAL CONS
     7TRAINT INTEGRATION/56H AINCR AND BINCR HELP SET RANGE AND INCREMEN
     8T ON A AND B)
      PRINT 997
  997 FORMAT(77H IF Y1 AREA AND/OR Y2 AREA ARE LESS THAN C1AREA AND C2AR
     1EA,WE PRINT ASTERISKS/59H SAME FOR Y1 ENERGY AND Y2 ENERGY W.R.T.
     2C1ENGY AND C2ENGY./70H SAME FOR Y1 MAX AND Y2 MAX W.R.T. C1MAG AND
     3 C2MAG (ONLY GREATER THAN)/61H INTG1,INTG2 PERMIT IX AND JX TO STA
     4RT AT 1(W.R.T. I5 AND I6)/34H IF JB = 2,J6=I(SYMMETRY FOR A,B)./ 7
     57H JSWTCH=1 FOR REAL ROOTS IN IMPULSE RESPONSES. ANY OTHER INTEGER
     6 FOR COMPLEX.)
      I66=I6/INTG2
      K66=K6/INTG2
      HA=DELTAT
      CONST=1./CONSL1
      DO 5 I = I5,J5,K5
      IX=I/INTG1
      QI=I
      A=QI/AINCR
      IF (JB .EQ. 2) GO TO 89
      J6=J6ALT
      GO TO 84
   89 J6=I
   84 DO 6 J = I6,J6,K6
      JX=J/INTG2
```

```
      QJ=J
      B=QJ/BINCR
      QLB(JX)=B
      DISCRM=B*B/4.+A
      IF (JSWTCH .EQ. 1) GO TO 86
   85 IF (DISCRM .GE. 0.) GO TO 6
      WC=SQRT(-DISCRM)
C
C   SET JSWTCH TO 1 FOR REAL ROOTS AND ANY CTHER INTEGER FOR CCMPLEX
C
      A5=B/2.
      A6=A5/WC
      A7=-2./A
      A8=1./WC
      GC TO 87
   86 IF (DISCRM .LE. 0.) GO TO 6
      R1=B/2.+SQRT(DISCRM)
      R2=B-R1
      D=R1-R2
      A1=2./A
      A2=R2/D
      A3=R1/R2
      A4=A1/2.
   87 KSS=1
      EA=-0.05
      TZ=0.
      KS=1
      E=-0.1
      TA=0.
      TB=0.
      KUP=0
C
      DFZRC=EPSLON/2.
  497 NECNT=0
  498 ZZ=0.00001
      MZRTST=1
      LZ=1
  499 ZNEG=ZZ
      HZNEG=H(ZZ,E)
  500 ZZ=ZZ+0.1
      IF (ZZ .GT. UPPER) GO TO 550
      HZPOS=H(ZZ,E)
      IF (HZPOS*HZNEG) 501,511,502
  502 HZNEG=HZPOS
      ZNEG=ZZ
      GC TO 500
  550 IF (MZRTST .EQ. 2) GO TO 520
      E=(E+EA)/2.
      IF (NECNT .GT. 50) GO TO 560
      NECNT=NECNT+1
      GC TO 498
  560 PRINT 561,UPPER,A,B
  561 FCRMAT(//5X,29HK(T,E) HAS NO ZERO FOR T .LE.,F10.6,8H AND A =,
     1F10.6,8H AND B = ,F10.6)
      GC TO 6
```

```
  501 ZPOS=ZZ
  503 ZX=(ZNEG*HZPOS-ZPOS*HZNEG)/(HZPOS-HZNEG)
      HZX=H(ZX,E)
      IF (HZX*HZNEG) 504,510,505
  505 ZNEG=ZX
      HZNEG=HZX
      HZPOS=HZPOS/2.
      GO TO 506
  504 ZPOS=ZX
      HZPOS=HZX
      HZNEG=HZNEG/2.
  506  IF (ABS(ZPOS-ZNEG) .LE. DFZRO) GO TO 510
      GO TO 503
  511 ZX=ZZ
  510 LZ=LZ+1
      MZRTST=2
      TZERO(LZ)=ZX
      ZX1=ZX-10.*DFZRO
      MLZ=LZ-1
  515 IF (H(ZX1,E)) 512,513,514
  512 SGN(MLZ)=-1.
      GO TO 516
  513 ZX1=ZX1+DFZRO
      GO TO 515
  514 SGN(MLZ)=1.
  516 ZZ=ZX+0.01
      GO TO 499
  520 TZERO(1)=0.
      TZERO(LZ+1)=UPPER
      IF (H(UPPER,E)) 521,521,523
  521 SGN(LZ)=-1.
      GO TO 524
  523 SGN(LZ)=1.
  524 LDZMAX=LZ-1
      DO 530 LDZ=1,LDZMAX
      LDZZ=LDZ
      TZ1=TZERO(LDZ)
      TZ2=TZERO(LDZ+1)
      MAXZ=((TZ2-TZ1)/(2.*HA) +0.5)
      MAXZI=MAXZ/2
      IF ((MAXZ-2*MAXZI) .EQ. 0) GO TO 551
      MAXZ=MAXZ+1
  551 QMAXZ=MAXZ
      DELY2(LDZ)=(TZ2-TZ1)/(2.*QMAXZ)
  530 CONTINUE
      DELY2(LDZZ+1)=DELY2(LDZZ)
  588 AREA=0.
      KK=1
      G13=G(0.)
      T3=0.
      KPZ=0
  535 KPZ=KPZ+1
      DELKPZ=DELY2(KPZ)
      TUPKPZ=TZERO(KPZ+1) +DELKPZ/4.
      SGNDLZ=SGN(KPZ)*DELKPZ/3.
```

```
   66 T1=T3
      T2=T1+DELKPZ
      T3=T2+DELKPZ
      IF (T2 .GT. TUPKPZ) GO TO 549
      G11=G13
      G12=4.*G(T2)
      G13=G(T3)
      AREA=AREA+(G11+G12+G13)*SGNDLZ
C
      IF (KK .EQ. 1) GO TO 1529
      GPOS=AREA
      IF (ABS(GPOS) .LE. 1.0 E-20) GO TO 1551
      IF (GPOS*GNEG) 50,1551,1552
 1529 KK=2
      GNEG=AREA
      GO TO 66
  549 IF (T2 .GT. UPPER) GO TO 547
      T3=T3-2.*DELKPZ
      GO TO 535
  547 KUP=KUP+1
      IF (KUP .GT. 10) GO TO 546
      E=E-0.25
      GO TO 497
  546 PRINT 548,UPPER,A,B
  548 FORMAT(//5X,20HWE HAVE EXCEEDED T =,F10.6,25H IN MAKING GX = 0 FOR
     1 A =F10.6,8H AND B=,F10.6/49H WE PRINT ASTERISKS IN T-MIN AND -999
     2. ELSEWHERE.    )
      TH(JX)=-9999999.9999
      ALAMBD(JX)=-999.
      Y1NRMA(JX)=-999.
      Y2NRMA(JX)=-999.
      Y1NRME(JX)=-999.
      Y2NRME(JX)=-999.
      Y1MAG(JX)=-999.
      Y2MAG(JX)=-999.
      TY1MAG(JX)=-999.
      TY2MAG(JX)=-999.
      GO TO 6
C
 1552 GNEG=GPOS
      GO TO 66
   50 GT1=GNEG
      GX1=G11
      SGNGX1=SGN(KPZ)
   51 Z1=(T1*GPOS-T3*GT1)/(GPOS-GT1)
      H1=(Z1-T1)/2.
      GZ1=G(Z1)
      GH1=G((H1+T1))
      IF (H((H1+T1),E)) 610,611,612
  610 SGNGX2=-1.
      GO TO 613
  611 SGNGX2=SGNGX1
      GO TO 613
  612 SGNGX2=1.
  613 GX2=4.*GH1*SGNGX2
```

```
      IF (H(Z1,E)) 620,621,622
  620 SGNGX3=-1.
      GO TO 623
  621 SGNGX3=SGNGX2
      GC TO 623
  622 SGNGX3=1.
  623 GX3=GZ1*SGNGX3
      GX=GNEG+(H1/3.)*(GX1+GX2+GX3)
      IF (GNEG*GX) 90,402,91
   90 T3=Z1
      GPOS=GX
      GT1=GT1/2.
      GC TO 52
   91 T1=Z1
      GNEG=GX
      GT1=GX
      GPOS=GPCS/2.
      SGNGX1=SGNGX3
      GX1=GX3
   52 TEST=ABS(T1-T3)
      IF (TEST .LE. DIFF) GO TO 401
      GO TO 51
  401 IF (ABS(GX) .LE. EPS1)  GO TO 402
      GC TO 51
 1551 T1=T3
      GC TO 1403
  402 T1=Z1
C     PARTIAL K-NORM W.R.T. LAMBDA IS ZERO.
C     START K-NCRM=1/L1
C
C
 1403 L=1
      NZILCH =1
      CFHZRO=CIFF/2.
      Y=0.001
      IF (H(Y,E)) 399,390,399
  390 PRINT 389,Y
  389 FCRMAT(//5X,15HK HAS A ZERO AT,F10.6)
      Y=Y+0.001
      T1A=T1+HA/2.
  399 YNEG=Y
      HNEG=H(Y,E)
  400 Y=Y+0.1
      IF (Y .GT. T1A) GO TO 476
  478 HPOS=H(Y,E)
      IF (HNEG*HPOS) 461,471,462
  476 Y=T1-10.*DFHZRO
      NZILCH =2
      GC TO 478
  462 HNEG=HPOS
      IF (NZILCH .EQ. 2) GO TO 420
      YNEG=Y
      GC TO 400
  461 YPOS=Y
  403 YX=(YNEG*HPOS-YPOS*HNEG)/(HPOS-HNEG)
```

```
      HX=H(YX,E)
      IF (HX*HNEG) 404,410,405
  405 YNEG=YX
      HNEG=HX
      HPOS=HPOS/2.
      GO TO 406
  404 YPOS=YX
      HPOS=HX
      HNEG=HNEG/2.
  406 IF (ABS(YPOS-YNEG) .LT. DFHZRO) GO TO 410
      GO TO 403
  471 YX=Y
  410 L=L+1
      TZERO(L)=YX
      YX1=YX-10.*DFHZRO
      ML=L-1
  415 IF (H(YX1,E)) 412,413,414
  412 SGN(ML)=-1.
      GO TO 416
  413 YX1=YX1+DFHZRO
      GO TO 415
  414 SGN(ML)=1.
  416 Y=YX+0.01
      GO TO 399
  420 NMAX=L+1
      TZERO(NMAX)=T1
      TT=T1-10.*DFHZRO
      IF (H(TT,E)) 421,422,423
  421 SGN(L)=-1.
      GO TO 424
  422 IF (H((TT+DFHZRO),E)) 421,421,423
  423 SGN(L)=1.
  424 TZERO(1)=0.
      LCMAX=NMAX-1
      DO  430 LC2=1,LCMAX
      TZB=TZERO(LC2)
      TZA=TZERO(LC2+1)
      MAX=((TZA-TZB)/(2.*HA) +0.5)
      MAXI=MAX/2
      IF ((MAX-2*MAXI) .EQ. 0) GO TO 450
      MAX=MAX+1
  450 QMAX=MAX
      DELY2(LC2)=(TZA-TZB)/(2.*QMAX)
  430 CONTINUE
      Y3=0.
      KQ=0
      KD=0
      FY3=F(0.)
      QPF=0.
  444 KD=KD+1
      KLM=1
      DELKD=DELY2(KD)
      DELKD4=DELKD/4.
  445 Y1=Y3
      Y2=Y1+DELKD
```

```
      Y3=Y2+DELKD
      IF ((Y2-T1) .GT. 0.) GO TO 1181
      IF (KLM .EQ. 2) GO TO 446
      KQ=KQ+1
      SGNDEL=SGN(KQ)*DELY2(KQ)
      TZKQ1=TZERO(KQ)
      TZKQ =TZERO(KQ+1)
      KLM=2
  446 IF (TZKQ .GE. Y2 .AND. TZKQ1 .LE. Y2) GO TO 455
      Y3=Y3-2.*DELKD
      GO TO 444
  455 FY1=FY3
      FY2=4.*F(Y2)
      FY3=F(Y3)
      Q=(FY1+FY2+FY3)*SGNDEL
      QPF=QPF+Q
      GO TO 445
C
 1181 IF (KSS .EQ. 2) GO TO 1575
      SKF=QPF/3. -CONST
      IF (KS .EQ. 1) GO TO 1570
      IF (SKF*SKF2) 1571,443,1574
 1570 KS=2
 1574 SKF2=SKF
      EA=E
      TA=T1
      E=E-0.25
 1581 IF (ABS(E) .GT. UPPER) GO TO 1600
      GO TO 497
 1571 KSS=2
      TB=T1
      EB=E
      SKNEG=SKF2
      SKPOS=SKF
 1572 E=(EA*SKPOS-EB*SKNEG)/(SKPOS-SKNEG)
      GO TO 497
 1575 SKX=QPF/3. -CONST
      IF (SKNEG*SKX)  1576,443,1578
 1576 EB=E
      TB=T1
      SKPOS=SKX
      SKNEG=SKNEG/2.
      GO TO 1580
 1578 EA=E
      TA=T1
      SKNEG=SKX
      SKPOS=SKPOS/2.
 1580 IF (ABS(EA-EB) .LE. EPSLON .AND. ABS(TB-TA) .LE. EPSLON) GO TO 443
      IF (ABS(EA-EB) .LE. EPSLON .AND. ABS(SKX) .LE. EPS1) GO TO 443
      IF (ABS(T1-TZ) .LE. EPSLON .AND. ABS(EA-EB) .LE. EPSLON) GO TO 443
      TZ=T1
      GO TO 1572
 1600 PRINT 1601
 1601 FORMAT(///89H LAMBDA HAS EXCEEDED VALUE OF UPPER IN SEARCH SCHEME
     1-888.888 IS PRINTED IN OUTPUT ARRAY.)
```

```
      Y2MAG(JX)=-8888.888
      TY2MAG(JX)=-888.888
      Y1MAG(JX)=-8888.888
      TY1MAG(JX)=-888.888
      Y2NRME(JX)=-8888.888
      Y1NRME(JX)=-8888.888
      Y2NRMA(JX)=-8888.888
      Y1NRMA(JX)=-8888.888
      ALAMBD(JX)=-888.888
      TH(JX)=-888.888
      GO TO 6
  443 TH(JX)=T1
      ALAMBD(JX)=E
      PRINT 4500,T1,E,SKX,GX
 4500 FORMAT(5X,5HT1 = ,F10.6,5X,5H E = ,F10.6,5X,6HSKX = ,F10.6,5X,
     15HGX = ,F10.7)
C
C     END SEARCH FOR LAMBDA AND T. BEGIN FINDING Y1 AND Y2 AS FUNCTIONS OF TIM
C
      L=1
      DFHZRO=DIFF/2.
      Y=T1
      IF (H(T1,E)) 799,790,799
  790 PRINT 789,T1
  789 FORMAT(//5X,20HK-NORM HAS A ZERO AT,F10.6)
      Y=T1-0.001
  799 YNEG=Y
      HNEG=H(Y,E)
  800 Y=Y-0.1
      IF (Y .LE. 0.) GO TO 820
      HPOS=H(Y,E)
      IF (HNEG*HPOS) 801,811,802
  802 HNEG=HPOS
      YNEG=Y
      GO TO 800
  801 YPOS=Y
  803 YX=(YNEG*HPOS-YPOS*HNEG)/(HPOS-HNEG)
      HX=H(YX,E)
      IF (HX*HNEG) 804,810,805
  805 YNEG=YX
      HNEG=HX
      HPOS=HPOS/2.
      GO TO 806
  804 YPOS=YX
      HPOS=HX
      HNEG=HNEG/2.
  806 IF (ABS(YPOS-YNEG) .LT. DFHZRO) GO TO 810
      GO TO 803
  811 YX=Y
  810 L=L+1
      TZERO(L)=YX
      YX1=YX+10.*DFHZRO
      ML=L-1
  815 IF (H(YX1,E)) 812,813,814
  812 SGN(ML)=-1.
```

```
      GO TO 816
  813 YX1=YX1-DFHZRO
      GO TO 815
  814 SGN(ML)=1.
  816 Y=YX-0.01
      GO TO 799
  820 NMAX=L+1
      TT=TZERO(L)/2.
      TZERO(NMAX)=0.
      IF (H(TT  ,E)) 821,822,823
  821 SGN(L)=-1.
      GO TO 824
  822 IF (H(0.005,E)) 821,821,823
  823 SGN(L)=1.
  824 TZERO(1)=T1
C
C   END FINDING ZEROS OF K(T,E) AND SGN(K(T,E)).
C
      LDMAX=NMAX-1
      DO 830 LD=1,LDMAX
      TZA=TZERO(LD)
      TZB=TZERO(LD+1)
      MAX=((TZA-TZB)/(2.*HA) +0.5)
      MAXI=MAX/2
      IF ((MAX-2*MAXI) .EQ. 0) GO TO 885
      MAX=MAX+1
  885 QMAX=MAX
      DELY2(LD)=(TZA-TZB)/(2.*QMAX)
  830 CONTINUE
      X1=-2.*DELY2(1)
      MNY=0
      KP=0
  835 KP=KP+1
      DELKP=DELY2(KP)
      NMAX1=KP+1
      TNMAX1=TZERO(NMAX1)
      TUPKP=T1-TNMAX1 +DELKP/4.
      T1UP=T1+DELKP/4.
  840 X1=2.*DELKP+X1
      IF (X1 .GT. TUPKP) GO TO 850
      QX11=0.
      QX22=0.
      GY13=F(X1)
      GY23=G(X1)
      Y33=0.
      KD=0
      KQ=0
      MNY=MNY+1
  844 KD=KD+1
      KLM=1
      DELKD=DELY2(KD)
      DELKD4=DELKD/4.
  845 Y11=Y33
      Y22=Y11+DELKD
      Y33=Y22+DELKD
```

```
      TZRTST=T1-Y22
      Y222=X1-Y22
      IF ((Y222+DELKD4) .LT. C. .AND. (Y222-DELKD4) .LT. 0.) GC TC 135
      IF (KLM .EQ. 2) GO TO 846
      KC=KQ+1
      SGNDEL=SGN(KQ)*CELY2(KQ)
      TZKQ=TZERO(KQ)
      TZKQ1=TZERO(KQ+1)
      KLM=2
  846 IF (TZKC .GE. TZRTST .AND. TZKQ1 .LE. TZRTST) GO TO 855
      Y33=Y33-2.*DELKD
      GC TO 844
  850 IF(X1 .LE. T1UP) GO TO 851
      GC TO 860
  851 X1=X1-2.*DELKP
      GC TO 835
  855 TY22=X1-Y22
      TY33=X1-Y33
      GY11=GY13
      GY21=GY23
      GY12=4.*F(TY22)
      GY22=4.*G(TY22)
      GY13=F(TY33)
      GY23=G(TY33)
      C1=(GY11+GY12+GY13)*SGNCEL
      C2=(GY21+GY22+GY23)*SGNCEL
      CX11=QX11+Q1
      CX22=QX22+Q2
      GC TO 845
  135 AY1X1(MNY)=CCNSL1*CX11/6.
      AY2X1(MNY)=CCNSL1*QX22/3.
      GC TO 840
  860 MAXTOT=MNY
C
C   ENC FINDING BOTH OUTPUTS AS FUNCTIONS CF TIME.
C
      LN=0
      NX=-1
      TTEST=0.
      Y1AREA=0.
      Y2AREA=0.
      Y1ENGY=0.
      Y2ENGY=0.
 1200 LN=LN+1
      C2=DELY2(LN)
      TBRKA=T1-TZERO(LN+1) +D2/4.
      TBRKB=T1-TZERO(LN) -C2/4.
      T1T=T1+C2/4.
      Z=2.*D2
      YN11=0.
      YN12=0.
      YN21=0.
      YN22=0.
      NX2TST=1
 1205 TTEST=2.*Z+TTEST
```

```
      IF (TTEST .GT. T1T) GO TO 1220
      IF (TTEST .LE. TBRKA .AND. TTEST .GE. TBRKB) GO TO 1215
      IF ((TTEST-Z) .LE. TBRKA) GO TO 1216
      TTEST=TTEST-2.*Z
 1210 Y1AREA=Y1AREA+(Z/3.)*YN11
      Y2AREA=Y2AREA+(Z/3.)*YN21
      Y1ENGY=Y1ENGY+(Z/3.)*YN12
      Y2ENGY=Y2ENGY+(Z/3.)*YN22
      GO TO 1200
 1216 NX2TST=2
 1215 NX=NX+2
      FZ11=AY1X1(NX)
      FZ21=AY2X1(NX)
      NX1=NX+1
      FZ12=AY1X1(NX1)
      FZ22=AY2X1(NX1)
      IF (NX2TST .EQ. 2) GO TO 1217
      NX2=NX+2
      FZ13=AY1X1(NX2)
      FZ23=AY2X1(NX2)
  243 R11=ABS(FZ11)+4.0*ABS(FZ12)+ABS(FZ13)
      R21=ABS(FZ21)+4.0*ABS(FZ22)+ABS(FZ23)
      R12=FZ11*FZ11+4.0*FZ12*FZ12+FZ13*FZ13
      R22=FZ21*FZ21+4.0*FZ22*FZ22+FZ23*FZ23
  256 YN11=YN11+R11
      YN21=YN21+R21
      YN12=YN12+R12
      YN22=YN22+R22
      IF (NX2TST .EQ. 2) GO TO 1210
      GO TO 1205
 1217 FZ13=0.
      FZ23=0.
      NX=NX1-2
      TTEST =TTEST-Z
      GO TO 243
 1220 Y1NRMA(JX)=Y1AREA +(Z/3.)*YN11
  264 Y2NRMA(JX)=Y2AREA +(Z/3.)*YN21
  271 Y1NRME(JX)=SQRT(Y1ENGY +(Z/3.)*YN12)
  273 Y2NRME(JX)=SQRT(Y2ENGY +(Z/3.)*YN22)
C
C   END FINDING OUTPUT NORMS (AREA AND ENERGY)
C
  348 TA=0.
      TB=0.
      Y1MAX=AY1X1(1)
      Y2MAX=AY2X1(1)
      LN=0
      TTEST=0.
      KLN=1
      DO  310 MA=2,MAXTOT
      IF (KLN .EQ. 2) GO TO 306
  305 LN=LN+1
      TBRKA=T1-TZERO(LN+1)
      TBRKB=T1-TZERO(LN)
      Z=2.*DELY2(LN)
```

```
      KLN=2
  306 TTEST=Z+TTEST
      IF (TTEST .LE. TBRKA .AND. TTEST .GE. TBRKB) GO TO 307
      TTEST=TTEST-Z
      GO TO 305
  307 FM1=AY1X1(MA)
      FM2=AY2X1(MA)
      IF (ABS(Y1MAX) .GT. ABS(FM1)) GO TO 301
      TA=TTEST
      Y1MAX=FM1
  301 IF (ABS(Y2MAX) .GT. ABS(FM2)) GO TO 310
      TB=TTEST
      Y2MAX=FM2
  310 CONTINUE
  302 Y1MAG(JX)=Y1MAX
      TY1MAG(JX)=TA
  304 Y2MAG(JX)=Y2MAX
      TY2MAG(JX)=TB
      PRINT 777,QLB(JX),TH(JX),Y1NRMA(JX),Y2NRMA(JX),Y1NRME(JX),Y2NRME(J
     1X),TY1MAG(JX),Y1MAG(JX),TY 2MAG(JX),Y2MAG(JX),ALAMBD(JX)
  777 FORMAT(/11(1X,F10.3))
    6 CONTINUE
      J66=J6/INTG2
      PRINT 7,A
    7 FORMAT(1H2,62X,4HA = ,F7.2//6X,1HB,6X,5HT-MIN,6X,7HY1 AREA,5X,
     17HY2 AREA,6X,9HY1 ENERGY,5X,9HY2 ENERGY,6X,7HT-Y1MAX,5X,5HY1MAX,
     26X,7HT-Y2MAX,5X,5HY2MAX,5X,6HLAMBDA)
      PRINT4,(QLB(JX),TH(JX),Y1NRMA(JX),Y2NRMA(JX),Y1NRME(JX),Y2NRME(JX)
     1,TY1MAG(JX),Y1MAG(JX),TY2MAG(JX),Y2MAG(JX),ALAMBD(JX),JX=I66,J66,
     2K66)
    4 FORMAT(1(1X,F7.2,2X,F8.3,4X,F9.3,3X,F9.3,5X,F9.3,5X,F9.3,5X,F8.3,
     13X,F9.3,4X,F8.3,2X,F9.3,2X,F8.3))
    5 CONTINUE
      GO TO 999
      END
```

APPENDIX VII

A = 2.00

B	T-MIN	Y1 AREA	Y2 AREA	Y1 ENERGY	Y2 ENERGY
5.00	2.478	0.511	0.500	0.421	0.355
4.00	2.330	0.487	0.500	0.414	0.368
3.00	2.205	0.468	0.500	0.409	0.381
2.00	2.119	0.459	0.500	0.408	0.391
1.00	2.089	0.462	0.500	0.412	0.396
0.60	2.097	0.468	0.500	0.415	0.396
0.30	2.109	0.474	0.500	0.419	0.395
0.00	2.128	0.481	0.500	0.422	0.394
-0.30	2.154	0.491	0.500	0.427	0.392
-0.60	2.187	0.501	0.500	0.432	0.389
-1.00	2.239	0.518	0.500	0.440	0.384
-2.00	2.416	0.569	0.500	0.462	0.369
-3.00	2.643	0.631	0.500	0.487	0.351
-4.00	2.900	0.700	0.500	0.513	0.333
-5.00	3.171	0.771	0.500	0.538	0.317

B	T-Y1MAX	Y1MAX	T-Y2MAX	Y2MAX	LAMBDA
5.00	2.478	0.500	1.710	0.311	-0.372
4.00	2.330	0.500	1.561	0.338	-0.449
3.00	2.205	0.500	1.411	0.365	-0.559
2.00	2.119	0.500	1.292	0.389	-0.717
1.00	2.089	0.500	1.233	0.401	-0.930
0.60	2.097	0.500	1.216	0.402	-1.029
0.30	2.109	0.500	1.202	0.401	-1.108
0.00	2.128	0.500	1.213	0.398	-1.190
-0.30	2.154	0.500	1.206	0.394	-1.275
-0.60	2.187	0.500	1.224	0.389	-1.363
-1.00	2.239	0.500	1.232	0.380	-1.483
-2.00	2.416	0.500	1.305	0.350	-1.792
-3.00	2.643	0.500	1.401	0.315	-2.110
-4.00	2.900	0.500	1.508	0.283	-2.428
-5.00	3.171	0.500	1.617	0.254	-2.741

A = 3.00

B	T-MIN	Y1 AREA	Y2 AREA	Y1 ENERGY	Y2 ENERGY
5.00	2.749	0.579	0.500	0.449	0.334
4.00	2.598	0.553	0.500	0.442	0.346
3.00	2.467	0.533	0.500	0.436	0.357
2.00	2.374	0.522	0.500	0.435	0.367
1.00	2.334	0.523	0.500	0.438	0.372
0.60	2.337	0.528	0.500	0.441	0.373
0.30	2.347	0.534	0.500	0.444	0.372
0.00	2.363	0.540	0.500	0.447	0.371
-0.30	2.387	0.549	0.500	0.452	0.370
-0.60	2.416	0.559	0.500	0.456	0.368
-1.00	2.467	0.575	0.500	0.463	0.364
-2.00	2.635	0.624	0.500	0.484	0.352
-3.00	2.852	0.684	0.500	0.507	0.337
-4.00	3.098	0.750	0.500	0.531	0.321
-5.00	3.360	0.819	0.500	0.554	0.307

B	T-Y1MAX	Y1MAX	T-Y2MAX	Y2MAX	LAMBDA
5.00	2.749	0.500	1.842	0.273	-0.361
4.00	2.598	0.500	1.689	0.295	-0.430
3.00	2.467	0.500	1.530	0.318	-0.527
2.00	2.374	0.500	1.424	0.339	-0.663
1.00	2.334	0.500	1.354	0.352	-0.849
0.60	2.337	0.500	1.332	0.354	-0.937
0.30	2.347	0.500	1.314	0.354	-1.009
0.00	2.363	0.500	1.323	0.352	-1.085
-0.30	2.387	0.500	1.313	0.350	-1.164
-0.60	2.416	0.500	1.329	0.346	-1.246
-1.00	2.467	0.500	1.332	0.339	-1.360
-2.00	2.635	0.500	1.397	0.316	-1.659
-3.00	2.852	0.500	1.483	0.288	-1.968
-4.00	3.098	0.500	1.611	0.261	-2.279
-5.00	3.360	0.500	1.714	0.237	-2.587

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AUTOBIOGRAPHY

Richard David Klafter was born in New York City on August 5, 1936 and received his elementary and high school education at P.S. 117 and Forest Hills High School, respectively. In 1958 he was awarded an S.B. degree in Electrical Engineering from the Massachusetts Institute of Technology where he was active in the school orchestra and was elected to the national honor societies Tau Beta Pi and Eta Kappa Nu. He continued his education at Columbia University receiving an M.S.E.E. degree in 1959 and an E.E. degree in 1963. In the fall of 1959 he joined the staff of the Electrical Engineering Department at The City College of New York as a Lecturer and held this position until 1967 (except for the academic year 1964-65 when he was a National Science Foundation Science Faculty Fellow). He entered the Graduate School of the City University of New York in 1963, passing his doctoral thesis defense in May 1968. Since September 1967 he has been an Assistant Professor of Electrical Engineering at Drexel Institute of Technology in Philadelphia, Pa. During the summer of 1969 Mr. Klafter will be an ASEE-NASA System Design Fellow at Stanford University.

In 1959 Mr. Klafter was married to the former Marcia Balaban. They current live in Willingboro, New Jersey with their two children, Melissa, 7 and Leslie, 9.