8501145

Jiang, Dahe

A STATISTICAL STUDY OF TURBULENT DIFFUSION

City University of New York PH.D. 1984

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

A STATISTICAL STUDY OF TURBULENT DIFFUSION

by

DAHE JIANG

A dissertation submitted to the Graduate Faculty in Engineering in partial fulfillment for the requirements for the degree of Doctor of Philosophy, The City University of New York

1984

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

June 15, 1984
date

C. M. Tchen
Chairman of Examining Committee

June 18, 1984
date

Paul R Karmel
Executive Officer

Professor C. M. Tchen, Chairman

Professor M. K. Kassir

Professor W. J. Pierson

Professor R. L. Varley
Supervisory Committee

The City University of New York

Abstract

A STATISTICAL STUDY OF TURBULENT DIFFUSION

by

Dahe Jiang

Adviser: Professor Chan Mou Tchen

The dispersion of particles in a turbulent fluid flow is studied by means of the transition probability distribution functions. Applying the propagator method, we derive the evolution equations of the transition functions in the form of a series of correlations. Based upon the discussion of the behavior of the series, a low order truncation is made to close the equations. The eddy transport properties are found as integral operators that contain the memory effect.

The formal transport equations are used to investigate the dispersion of fluid particles. The transition function of a single particle and the relative transition function of a pair of particles are found to obey non-linear integro-differential equations. The general properties of these equations agree with observations. In comparison with other theories, our results yield better short time behavior for the one-particle dispersion, and the equation for the relative dispersion of a pair of particles seems more reasonable and more convenient for use in analysis. Based upon the Kolmogoroff spectrum, an analytical proof

is given for the empirical 4/3 power law of relative diffusion. A numerical calculation based upon the von Kármán spectrum agrees with the analysis.

The diffusion of a puff of a passive quantity is studied by means of the dispersion of particles. For the absolute diffusion, the eddy diffusivity is given in tensorial form to account for anisotropy and the mean velocity gradient. With a locally isotropic approximation, the relative diffusion undergoes different stages because of the changes in the relative importance of shear and buoyancy turbulence. A prediction based on the theory and upon an empirical spectrum estimated from turbulent wind data is compared with observations of diffusion made under conditions similar to those when the wind data were obtained.

Acknowledgements

I am greatly indebted to Professor Chan Mou Tchen for suggesting the subject, his continuous guidance and encouragement.

I am grateful to Deans, Professor David H. Cheng and Professor Paul R. Karmel, for their support.

Table of Contents

List of Figures

I INTRODUCTION

The statistical study of turbulent diffusion was pioneered by Taylor[1]. In his work, the variance of the displacement of a particle was expressed in the form of a time integration of the Lagrangian velocity correlation function. A few years later, Richardson[2] introduced the distance-neighbor function to describe the spreading of a puff around its instantaneous center. Richardson discovered the empirical 4/3 power law that, in a time period, the relative eddy diffusivity was proportional to the 4/3 power of the puff size. The work by Taylor and Richardson reveal the two important features of turbulent diffusion. One is the absolute diffusion, referring to the spreading of the puff in a fixed frame, which is related to the dispersion of a single particle. The other is the relative diffusion, referring to the spreading of the puff in a frame moving with its instantaneous center, which is related to the relative dispersion of a pair of particles.

Analytically, the statistical theory involves the following two problems:

(a) Transformation of the Lagrangian velocity correlation function into the Eulerian correlation function.

(b) Investigation of the transition functions for the path-lengths of the particles.

According to Taylor, the eddy diffusivity may be defined as the time integration of the Lagrangian correlation of velocity fluctuations. However, Eulerian correlations are more practical to measure. Therefore, the basic problem in the determination of the eddy diffusivity, following Taylor, is the Lagrangian-Eulerian transformation. In this respect, we may mention the expansion of the Lagrangian velocity correlation into series of Eulerian velocity correlations by Lumley[3,4] (Taylor series expansion) and Phythian[5] (successive iteration), the moment expansion by Deissler[6], and the Wiener-Hermite expansion by Saffman[7]. Corrsin[8,9] introduced the independence hypothesis. These kinematic methods suffer from a lack of the particle path that defines the Lagrangian representation.

From the mathematical standpoint, the trajectories of particles in turbulent motion can be specified by the distributions of their path-lengths, called transition functions. The main task is to establish the governing equations of the transition functions and to study the dispersion of the particles. This method also yields a series of correlations, implying a hierarchy in analogy with the BBGKY hierarchy in the statistical mechanics of many body problems. The closure is usually made by an arbitrary truncation. To develop the governing equations, the propagator method is the most logical in representing a Lagrangian function. In recent years, Kraichnan used mean response functions in DIA (Direct Interaction Approximation) and LHDI (Lagrangian History Direct Interaction Approximation)[10,11], Dupree et al. used a mean coherent response function in their perturbation scheme for the study of Vlasov plasma[12-14]. These mean response functions are found to be related to the mean propagator[15]. Weinstock, Misguich and Balescu, and Tchen used the propagators directly in their theories.[15-21] However, most theories are hypothetically confined to the zeroth order approximation.

The dispersion of marked fluid particles has been investigated by the propagator method. For example, Roberts used DIA and obtained interesting results for the dispersion of a single particle.[22] However, as mentioned by Monin and Yaglom[23], the short time behavior of the resultant one-particle transition function seemed peculiar. Besides, Roberts did not properly describe the relative dispersion of a pair of particles. Kraichnan[24] added the Lagrangian effect in his LHDI and applied it to the relative dispersion. The difficulty in Kraichnan's theory lies in the velocity correlation function, which contains the stochastic path-dynamics. Improvements of Kraichnan's theory were considered by Knobloch[25] and Lundgren.[26] However, Knobloch did not give a compact form for the eddy diffusivity and Lundgren's results are still too complicated for use. In the meanwhile, Misguich and Balescu discussed relative diffusion qualitatively.[27] At present, there is still a great need for a comprehensive investigation of the dispersion of particles.

In this work, a statistical theory of turbulent diffusion is developed based upon the propagator method. Following Tchen[28], we express the operation of the effective propagator in terms of the operations of the exact propagator through an integral equation. Then we develop a new series for the effective propagator which contains only the operations of the mean propagator. This has the purpose of including the effects of both mode-coupling and collective phenomena, in a simplified form. Based upon the discussion of the resulting series, we apply a truncation to establish the formal equations for the transition functions.

For the dispersion of fluid particles, it is found that the trasition functions are governed by non-linear integro-differential equations, and that the eddy transport properties are in the form of integral operators. We compare our equations and analytical results with other theories and observations. The numerical calculations are made on the basis of the von Kármán spectrum.

The results of the dispersion of a single particle and of a pair of particles are applied to the diffusion of a puff of a passive quantity. The absolute diffusion and the relative diffusion are studied based upon the evolution of the mean concentration distribution function and the distance-neighbor function respectively. For a shear turbulence, the eddy diffusivity appears in tensorial form to account for the effects of the anisotropy. For relative diffusion, it is found that the growth of the puff size undergoes different stages depending upon the relative importance of inertia and shear. The calculation based upon an empirical spectrum is compared with the experimental data of the lateral size of a plume.

II THEORY OF TURBULENT TRANSPORT

2.1 Mathematical Description of the Trajectories of Particles

In this work, the turbulent fluid flow is assumed incompressible, statistically stationary and homogeneous. As usual, the molecular diffusivity will be neglected. The basic problem of turbulent diffusion is the dispersion of a single particle and a pair of particles. We use the transition probability distribution functions, or transition functions, to describe the dispersion.

Let the instantaneous position of a specific particle be $\hat{\underset{\sim}{X}}(t)$ and its initial position be $\underset{\sim}{x}_0$. The displacement

$$\hat{\underset{\sim}{\ell}}(\tau) = \hat{\underset{\sim}{X}}(t) - \underset{\sim}{x}_0 \tag{1}$$

is called the path-length of the particle, in the time period $\tau = t - t_0$. Its microscopic distribution can be written as the delta function

$$\hat{P}(\tau, \underset{\sim}{\ell}) = \delta[\underset{\sim}{\ell} - \hat{\underset{\sim}{\ell}}(\tau)] \tag{2}$$

The function $\hat{P}(\tau, \ell)$ is also called the instantaneous transtion function.

The velocity of the particle at time t is

$$\hat{\underset{\sim}{V}}(t) = \frac{d}{dt}\hat{\underset{\sim}{X}}(t) \tag{3}$$

According to (1), the velocity can also be written as

$$\hat{\underset{\sim}{V}}(\tau) = \frac{d}{d\tau}\hat{\underset{\sim}{\ell}}(\tau) \tag{4}$$

The instantaneous transition function is conserved along the trajectory so that

$$[\,\partial_\tau + \hat{\underset{\sim}{v}}(\tau)\cdot\underset{\sim}{\nabla}\,]\,\hat{p}(\tau,\underset{\sim}{\ell}) = 0 \tag{5}$$

or

$$[\,\partial_\tau + \hat{L}(\tau)]\,\hat{p}(\tau,\underset{\sim}{\ell}) = 0 \tag{6}$$

where $\partial_\tau = \frac{\partial}{\partial\tau}$, $\underset{\sim}{\nabla} = \frac{\partial}{\partial\underset{\sim}{\ell}}$, and $\hat{L}(\tau) = \hat{\underset{\sim}{v}}(\tau)\cdot\underset{\sim}{\nabla}$ is a differential operator. In the above equations, we have used the symbol $\sim$ under a letter to represent a vector and the symbol $\hat{(\cdots)}$ to represent the fluctuating quantity. We shall use the symbols $\overline{(\cdots)} = \langle\cdots\rangle$ and $\tilde{(\cdots)}$ to denote the ensemble averaged quantity and the fluctuation, respectively.

The ensemble average of $\hat{P}(\tau,\underset{\sim}{\ell})$, i.e. $\overline{P}(\tau,\underset{\sim}{\ell})$, is briefly called the one-particle transition function. This function defines the probability that the particle will make the transition from the state $(t_o,\underset{\sim}{x}_o)$ to the state $(t,\underset{\sim}{x})$, or in the notation of (5), to be displaced a distance $\underset{\sim}{\ell}$ in the time interval, τ. The transition function is normalized as

$$\int d\underset{\sim}{\ell}\ \overline{P}(\tau,\underset{\sim}{\ell}) = 1$$

where the integral in taken over the whole space.

The trajectories of two particles are shown in Figure 1 The subscripts are used to distinguish the two particles. Similar to (6), we write

$$[\,\partial_\tau + \hat{L}_1(\tau)]\,\hat{P}_1(\tau,\underset{\sim}{\ell}_1) = 0 \tag{7}$$

$$[\,\partial_\tau + \hat{L}_2(\tau)]\,\hat{P}_2(\tau,\underset{\sim}{\ell}_2) = 0 \tag{8}$$

where

$$\hat{L}_1(\tau) = \hat{\underset{\sim}{v}}_1(\tau)\cdot\underset{\sim}{\nabla}_1 = \hat{\underset{\sim}{v}}_1(\tau)\cdot\frac{\partial}{\partial\underset{\sim}{\ell}_1}$$

$$\hat{L}_2(\tau) = \hat{\underset{\sim}{v}}_2(\tau)\cdot\underset{\sim}{\nabla}_2 = \hat{\underset{\sim}{v}}_2(\tau)\cdot\frac{\partial}{\partial\underset{\sim}{\ell}_2}$$

Multiply (7) by $\hat{P}_2(\tau, \underset{\sim}{l}_2)$ and (8) by $\hat{P}_1(\tau, \underset{\sim}{l}_1)$. The addition of the resulting equations yields

$$[\partial_\tau + \hat{L}_1(\tau) + \hat{L}_2(\tau)]\, \hat{P}_1(\tau, \underset{\sim}{l}_1)\hat{P}_2(\tau, \underset{\sim}{l}_2) = 0 \tag{9}$$

Introducing the two-particle instantaneous transition function

$$\hat{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2) = \hat{P}_1(\tau, \underset{\sim}{l}_1)\, \hat{P}_2(\tau, \underset{\sim}{l}_2)$$

we obtain

$$[\partial_\tau + \hat{L}_1(\tau) + \hat{L}_2(\tau)]\, \hat{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2) = 0 \tag{10}$$

The ensemble average of $\hat{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2)$, i.e. $\overline{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2)$, gives the probability that the two particles will make the transition between the two states, $(t_0, \underset{\sim}{x}_{10}, \underset{\sim}{x}_{20})$ and $(t, \underset{\sim}{x}_1, \underset{\sim}{x}_2)$, or be displaced by the distances $\underset{\sim}{l}_1$ and $\underset{\sim}{l}_2$ in the time interval τ. $\overline{P}_{12}$ is called two-particle transition function. It has the following properties:

$$\int d\underset{\sim}{l}_2\; \overline{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2) = \overline{P}_1(\tau, \underset{\sim}{l}_1) \tag{11}$$

$$\iint d\underset{\sim}{l}_1\, d\underset{\sim}{l}_2\; \overline{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2) = 1 \tag{12}$$

$$\lim_{\tau \to \infty} \overline{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2) = \overline{P}_1(\tau, \underset{\sim}{l}_1)\, \overline{P}_2(\tau, \underset{\sim}{l}_2) \tag{13}$$

Note that, for a stationary and homogeneous turbulence, $\overline{P}(\tau, \underset{\sim}{l})$ does not depend on $\underset{\sim}{x}_0$, but $\overline{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2)$ does depend on the initial separation $\underset{\sim}{r}_0 = \underset{\sim}{x}_{20} - \underset{\sim}{x}_{10}$.

According to Figure 1, the transition of the two particles can also be considered as the one between the two states: $(t_0, \underset{\sim}{x}_{10}, \underset{\sim}{r}_0)$ and $(t, \underset{\sim}{x}_1, \underset{\sim}{r})$. Denote the relative displacement as

$$\underset{\sim}{\lambda} = \underset{\sim}{r} - \underset{\sim}{r}_0$$

The two-particle transition function also gives the probability distribution for the first particle to make the displacement $\underset{\sim}{\ell}_1$, and the second particle to make the relative displacement $\underset{\sim}{\lambda}$. The superscript λ denotes the $(\underset{\sim}{\ell}_1, \underset{\sim}{\lambda})$ coordinate system, such that

$$\overline{P}_{12}(\tau, \underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2) = \overline{P}_{12}(\tau, \underset{\sim}{\ell}_1, \underset{\sim}{\ell}_1 + \underset{\sim}{r} - \underset{\sim}{r}_0) = \overline{P}_{12}^{\lambda}(\tau, \underset{\sim}{\ell}_1, \underset{\sim}{\lambda})$$

The relative transition function is defined as

$$\overline{B}(\tau, \underset{\sim}{\lambda}) = \int d\underset{\sim}{\ell}_1 \, \overline{P}_{12}^{\lambda}(\tau, \underset{\sim}{\ell}_1, \underset{\sim}{\lambda}) = \int d\underset{\sim}{\ell}_1 \, \overline{P}_{12}(\tau, \underset{\sim}{\ell}_1, \underset{\sim}{\ell}_1 + \underset{\sim}{\lambda}) \quad (14)$$

It is normalized as

$$\int d\underset{\sim}{\lambda} \, \overline{B}(\tau, \underset{\sim}{\lambda}) = 1$$

Like $\overline{P}_{12}(\tau, \underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2)$, $\overline{B}(\tau, \underset{\sim}{\lambda})$ depends on the initial separation.

The definition and general properties of these transition functions can be found elsewhere[23,29-31].

2.2 Development of Current Theories

Decompose each fluctuating quantity into its ensemble average and fluctuation. Equation (6) can be written as

$$(\partial_\tau + \bar{L})(\bar{P} + \tilde{P}) = -\tilde{L}(\bar{P} + \tilde{P}) \tag{16}$$

Averaging (16) over all realizations, we obtain

$$(\partial_\tau + \bar{L})\bar{P} = -\langle \tilde{L}\tilde{P} \rangle \tag{17}$$

Subtract (17) from (16) to get

$$(\partial_\tau + \bar{L})\tilde{P} = -\tilde{L}\bar{P} - [\tilde{L}\tilde{P} - \langle \tilde{L}\tilde{P} \rangle] \tag{18}$$

The righthand side of (17)

$$\bar{C} \equiv -\langle \tilde{L}\tilde{P} \rangle$$

represents the eddy collision, i.e. the correlation of the fluctuation of the instantaneous transition function with the velocity fluctuation. The determination of $\bar{C}$ leads to the transport equation.

One way to determine $\bar{C}$ is to use the correlation method $\tilde{P}$ may be formally solved by integrating (18) with respect to time. The result is substituted into (19) yielding correlations $\langle \tilde{L}(\tau)\tilde{L}(\tau') \rangle$ and $\langle \tilde{L}(\tau)\tilde{L}(\tau')\tilde{P}(\tau') \rangle$. In the same manner, the determination of $\langle \tilde{L}\tilde{L}\tilde{P} \rangle$ will generate correlations $\langle \tilde{L}\tilde{L}\tilde{L} \rangle$ and $\langle \tilde{L}\tilde{L}\tilde{L}\tilde{P} \rangle$. The sequence continues to generate an infinite chain of correlations for $\tilde{L}$ of increasing order. The velocity fluctuations can be considered as caused by the motion of other fluid particles so that the closure problem is implied in the series of the correlations. Because the correlations for $\tilde{L}$ do not specify the trajectory of the particle, a truncation, say at the fourth order, would not provide a good approximation[6,23].

Recent developments in the studies of the problem of diffusion are based upon the propagator method either implicitly or explicitly. Equation (18) can be written into three different forms:

$$(\partial_\tau + \hat{L})\tilde{p} = -\tilde{L}\bar{p} + \langle \tilde{L}\tilde{p} \rangle \tag{20}$$

$$(\partial_\tau + \bar{L})\tilde{p} = -\tilde{L}\bar{p} - (1-\bar{A})\tilde{L}\tilde{p} \tag{21}$$

$$[\partial_\tau + (1-\bar{A})\hat{L}]\tilde{p} = -\tilde{L}\bar{p} \tag{22}$$

where $(1-\bar{A})$ is the operator counter to the ensemble average operator $\bar{A}$. The exact propagator $\hat{U}(\tau,0)$ is introduced for (20). It satisfies

$$[\partial_\tau + \hat{L}(\tau)]\,\hat{U}(\tau,0) = 0 \qquad (\tau > 0) \tag{23}$$
$$\hat{U}(0,0) = 1$$

The unperturbed propagator $U^o(\tau,0)$ is for (21). It satisfies

$$[\partial_\tau + \bar{L}(\tau)]\,U^o(\tau,0) = 0 \qquad (\tau > 0) \tag{24}$$
$$U^o(0,0) = 1$$

And the effective propagator $\hat{\Lambda}(\tau,0)$ is for (22). It satisfies

$$[\partial_\tau + (1-\bar{A})\hat{L}(\tau)]\,\hat{\Lambda}(\tau,0) = 0 \qquad (\tau > 0) \tag{25}$$
$$\hat{\Lambda}(0,0) = 1$$

The third propagator is in analogy with the one introduced by Weinstock[15]. The propagator method also generates a series of correlations of increasing order. However, it offers a greater hope for a good approximation at a low order truncation, because the propagators, in one way or another, account for the trajectory.

Kraichnan used infinitesimal response functions, which are equivalent to the unperturbed propagator[32]. Because $U^{\circ}(\tau,0)$ refers to the mean velocity not to the true velocity of the particle, the resulting series contains terms which do not decrease fast enough. In DIA and LHDI, Kraichnan intuitively replaced them by mean response functions that are equivalent to the mean propagator[15,32]. The mean propagator is defined by the ensemble average of the exact propagator:

$$\overline{U}(\tau,0) = \langle \hat{U}(\tau,0) \rangle \qquad (26)$$

$$\overline{U}(0,0) = 1$$

The mean propagator has also been used in other work, e.g. implicitly by Dupree et al.[12-14], explicitly by Misguich and Balescu[16-18]. However, most theories are hypothetically restricted to the lowest order:

$$\overline{C} \doteq \overline{C_0} \equiv \int_0^{\tau} d\tau' \langle \hat{L}(\tau)\overline{U}(\tau,\tau')\hat{L}(\tau')\rangle \overline{P}(\tau') \qquad (27)$$

The validity of (27) was discussed by Weinstock[15] who showed that it might be good in the weak coupling and the weak turbulence limits.

The propagators work mathematically like the Green's function. The fluctuation $\tilde{P}$ is formally solved in the three forms using the three propagators:

$$\tilde{p} = -\hat{U}*\tilde{L}\overline{P} + \hat{U}*\langle\tilde{L}\hat{p}\rangle \tag{28}$$

$$\tilde{p} = -U^{o}*\tilde{L}\overline{P} - U^{o}*(1-\overline{A})\tilde{L}\tilde{p} \tag{29}$$

$$\tilde{p} = -\hat{\Lambda}*\tilde{L}\overline{P} \tag{30}$$

For simplicity, the symbol * is used to represent a time convolution (integration), as in

$$\hat{U}*\langle\tilde{L}\tilde{p}\rangle = \int_{0}^{\tau} d\tau' \, \hat{U}(\tau,\tau')\langle\tilde{L}(\tau')\tilde{p}(\tau')\rangle \tag{31}$$

In (31), the operator $\hat{U}(\tau,\tau')$ restricts its operand to be evaluated along the exact trajectory of the particle. The properties of this propagator can be found in the articles by Weinstock,[15] Misguich and Balescu.[16-18]

The substitution of the three solutions of $\tilde{P}$ into (19) yields

$$\overline{C} = \langle\tilde{L}\hat{U}*\tilde{L}\rangle\overline{P} - \langle\tilde{L}\hat{U}\rangle*\langle\tilde{L}\tilde{p}\rangle \tag{32}$$

$$\overline{C} = \langle\tilde{L}U^{o}*\tilde{L}\rangle\overline{P} + \langle\tilde{L}U^{o}*(1-\overline{A})\tilde{L}\tilde{p}\rangle \tag{33}$$

$$\overline{C} = \langle\tilde{L}\hat{\Lambda}*\tilde{L}\rangle\overline{P} \tag{34}$$

repectively, with (34) as the simplest form. However, the determination of the effective propagator encompasses the whole difficulty of the problem. Weinstock[15] derived iteration formulas to relate $\hat{\Lambda}$ to U^{o}, $\hat{U}$, and $\overline{U}$, but the use of the equations is complicated. Knobloch used the unperturbed propagator U^{o}.[25] As described previously, a low order truncation would not provide a good approximation, so that Knobloch did not give a compact form for the transport equation. In the present work, following the considerations by Tchen[28], we chose the exact propagator.

The first term on the righthand side of (32) may be called self-collision because the correlation is along the trajectory. The second term is then called collective collision. Define the fluctuation propagator

$$\tilde{U}(\tau,0) = \hat{U}(\tau,0) - \bar{U}(\tau,0) \tag{35}$$

with

$$\tilde{U}(\tau,0) = 0$$

The self-collision can be divided into two parts:

$$\langle \tilde{L}\hat{U}*\tilde{L}\rangle\bar{P} = \langle \tilde{L}\bar{U}*\tilde{L}\rangle\bar{P} + \langle \tilde{L}\tilde{U}*\tilde{L}\rangle\bar{P} \tag{36}$$

The second term on the righthand side of (36) is called mode-coupling, since it represents the interaction between the fluctuations of the trajectory mode and the fluctuations of the velocity mode. Comparing (32) and (36) with (26), we see that the customary approximation neglects mode-coupling (in analogy with Corrsin's independence hypothesis) and the collective collision.

In a kinetic approach[28], Tchen pointed out that the collective collision can not be neglected. For example, the collective collision contains the long range correlation. It should have a significant contribution to the dispersion when τ is large. On the other hand, mode-coupling is also not negligible when the memory effect is significant. Therefore, a more comprehensive investigation of the eddy collision is to include the collective collsion and mode-coupling in their combined effect on the series for $\bar{C}$. In analogy with the equations given by Tchen[28], we derive equations relating the effective propagator to the exact propagator. The result is then related to the mean propagator. In such a manner, the effects of mode-coupling and the collective collision are combined in the high order terms of the new series.

2.3 Hierarchy from Mode-coupling and Collective collision

To relate the effective propagator to the exact propagator, we derive the equations for the mean propagator. By averaging equation (23), we find that the mean propagator obeys

$$(\partial_\tau + \bar{L})\bar{U} = -\langle \tilde{L}\tilde{U}\rangle \tag{37}$$

The subtraction of (37) from (23) yields

$$(\partial_\tau + \hat{L})\tilde{U} = -\tilde{L}\bar{U} + \langle \tilde{L}\tilde{U}\rangle \tag{38}$$

Similar to (20) and (31), we find

$$\tilde{U} = -\hat{U}*\tilde{L}\bar{U} + \hat{U}*\langle \tilde{L}\tilde{U}\rangle \tag{39}$$

$$\begin{aligned}\bar{H} &\equiv -\langle \tilde{L}\tilde{U}\rangle \\ &= \langle \tilde{L}\hat{U}*\tilde{L}\rangle\bar{U} - \langle \tilde{L}\tilde{U}\rangle*\langle \tilde{L}\tilde{U}\rangle\end{aligned} \tag{40}$$

Denote

$$\bar{\mathbb{C}} \equiv \langle \tilde{L}\hat{U}*\tilde{L}\rangle\bar{P}$$

$$\bar{\mathbb{H}} \equiv \langle \tilde{L}\hat{U}*\tilde{L}\rangle\bar{U}$$

so that (32) and (40) can be written as

$$\bar{C} = \bar{\mathbb{C}} - \bar{H}*\bar{C} \tag{41}$$

$$\bar{H} = \bar{\mathbb{H}} - \bar{H}*\bar{H} \tag{42}$$

These two equations are in analogy with those given by Tchen[28]. Iteratively using (42) and (41), we express $\bar{H}$ in terms of $\bar{\mathbb{H}}$, then $\bar{C}$ in terms of $\bar{\mathbb{H}}$ and $\bar{\mathbb{C}}$. It is found that

$$\overline{C} = \overline{\mathbb{C}} + \sum_{i=1}^{\infty} \frac{(-1)^i 2^i (2i-1)!!}{(i+1)!} (\overline{\mathbb{H}}*)^i \overline{\mathbb{C}} \tag{43}$$

$$= \overline{\mathbb{C}} - \overline{\mathbb{H}}*\overline{\mathbb{C}} + 2\overline{\mathbb{H}}*\overline{\mathbb{H}}*\overline{\mathbb{C}} - 5\overline{\mathbb{H}}*\overline{\mathbb{H}}*\overline{\mathbb{H}}*\overline{\mathbb{C}}$$

$$+ 14\overline{\mathbb{H}}*\overline{\mathbb{H}}*\overline{\mathbb{H}}*\overline{\mathbb{H}}*\overline{\mathbb{C}} - \cdots\cdots$$

At this step, $\overline{C}$ or the operation of the effective propagator $\hat{\Lambda}$, has been expressed as a series of the operations of the exact and the mean propagators.

Next we relate the operations of the exact propagator to the operations of the mean propagator only. To do so, we denote

$$\overline{h}_1 \equiv \langle \tilde{L}\overline{U}*\tilde{L} \rangle$$

$$\overline{h}_2 \equiv \langle \tilde{L}\tilde{U}*\tilde{L} \rangle$$

$$\circ \equiv \overline{U}*$$

$\overline{\mathbb{C}}$ and $\overline{\mathbb{H}}*$ are then written as

$$\overline{\mathbb{C}} = (\overline{h}_1 + \overline{h}_2)\overline{P} \tag{44}$$

$$\overline{\mathbb{H}}* = (\overline{h}_1 + \overline{h}_2)\circ \tag{45}$$

The series (43) becomes

$$\overline{C} = \overline{h}_1\overline{P} + \overline{h}_2\overline{P} + \sum_{i=1}^{\infty} \frac{(-1)^i 2^i (2i-1)!!}{(i+1)!} [(\overline{h}_1 + \overline{h}_2)\circ]^i (\overline{h}_1 + \overline{h}_2)\overline{P} \tag{46}$$

(46) is rearranged into the form

$$\overline{C} = \overline{h}_1\overline{P} + \overline{h}_2\overline{P} + \sum_{i=2}^{\infty} \overline{D}_i\overline{P} \tag{47}$$

where

$$\overline{D}_2 = -\overline{h}_1 \circ \overline{h}_1$$

$$\overline{D}_3 = -\overline{h}_1 \circ (\overline{h}_2 - \overline{h}_1 \circ \overline{h}_1) - (\overline{h}_2 - \overline{h}_1 \circ \overline{h}_1) \circ \overline{h}_1$$

$$\overline{D}_4 = \overline{h}_1\circ\overline{h}_1\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1) + 2\,\overline{h}_1\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1) + (\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ\overline{h}_1\circ\overline{h}_1$$
$$- (\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1)$$

$$\overline{D}_5 = -\overline{h}_1\circ\overline{h}_1\circ\overline{h}_1\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1) - 3\,\overline{h}_1\circ\overline{h}_1\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ\overline{h}_1$$
$$-3\,\overline{h}_1\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ\overline{h}_1\circ\overline{h}_1 - (\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ\overline{h}_1\circ\overline{h}_1\circ\overline{h}_1$$
$$+\overline{h}_1\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1) + (\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ\overline{h}_1\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1) + (\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ(\overline{h}_2-\overline{h}_1\circ\overline{h}_1)\circ\overline{h}_1$$

$$\overline{D}_6 = \cdots\cdots$$

The first term on the righthand side of (47) is

$$\overline{C}_o = \overline{h}_1\overline{P} = \langle\tilde{L}\circ\tilde{L}\rangle\overline{P} = \langle\tilde{L}\,\overline{U}*\tilde{L}\rangle\overline{P} \tag{48}$$

at zero order. The second term represents mode-coupling and the series, $\{\overline{D}_i\overline{P}\}$, represents the collective collision. The subscript i refers to the "order", i.e. the sum of the subscripts of the operators $\overline{h}_1$ and $\overline{h}_2$ contained in each of the terms in $\overline{D}_i\overline{P}$. It is found that all the $\overline{D}_i\overline{P}$ terms have the common operators $\overline{h}_1$ and $(\overline{h}_2-\overline{h}_1\circ\overline{h}_1)$. Such a pattern provides the convenience of expanding $\overline{C}$ into a series containing $\overline{U}$ operators only.

Write (39) as

$$\tilde{U} = -\overline{U}*\tilde{L}\,\overline{U} - \tilde{U}*\tilde{L}\,\overline{U} + \overline{U}*\langle\tilde{L}\tilde{U}\rangle + \tilde{U}*\langle\tilde{L}\tilde{U}\rangle \tag{49}$$

$\overline{h}_2\overline{P}$ is then expanded by repeatedly using (49). The result is substituted into $\overline{D}_i\overline{P}$. The resultant form of the series $\overline{C}$ is rearranged according to the number of times that $\tilde{L}$ occurs in each term. We end up with

$$\overline{C} = \overline{C}_o + \sum_{i=3}^{\infty}\overline{\Delta_i} \tag{50}$$

where

$$\overline{\Delta}_3 = -\langle \tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$

$$\overline{\Delta}_4 = \langle \tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle \tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}$$

$$\begin{aligned}\overline{\Delta}_5 = &-\langle \tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}\\ &+\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\end{aligned}$$

$$\begin{aligned}\overline{\Delta}_6 = &\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}\\ &-\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}\\ &-\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}\\ &-\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}\\ &+2\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} + 2\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\ &-\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\ &+\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\end{aligned}$$

$$\overline{\Delta}_7 = \cdots\cdots$$

The full expressions for the series for $\overline{h_2}\overline{P}$ and $\overline{C}$, up to the eighth order, are given in Appendix 1.

The series $\{\overline{\Delta}_i\}$ is generated by both mode-coupling and the collective collision. It is equivalent to the hierarchy of the $\tilde{L}$ correlations under the operations of $\overline{U}$. Comparing (50) with (34), we see that the operation of the effective propagator $\hat{\Lambda}$ has been expressed in terms of the operations of the mean propagator. We also see that the effects of mode-coupling and the collective collision are combined in the high order terms. It may be noted that Marcuvitz tried to expand a similar series for the Vlasov plasma[33]. However, terms of orders higher than four were not discussed and the expansion was not actually advanced in use.

2.4 Closure

To obtain a good approximation by a low order truncation of the new series (50), we discuss the behavior of the series for different time ranges.

In the short time limit, $\tau \doteq 0$, all the operators $\tilde{L}$ in (50) can be approximated as

$$\hat{L} \doteq \tilde{\underset{\sim}{u}}_o \cdot \underset{\sim}{\nabla} = \sum_{i=1}^{3} \tilde{u}_{oi} \nabla_i \equiv \tilde{u}_{oi} \nabla_i \tag{51}$$

where

$$\tilde{\underset{\sim}{u}}_o \equiv \tilde{\underset{\sim}{V}}(o) \tag{52}$$

is the initial velocity fluctuation of the particle. Here and after, we shall use the summation convention as that in (51) unless where noted. Therefore,

$$\begin{aligned}
\overline{\Delta_3} &\doteq -\langle \tilde{u}_{oi} \tilde{u}_{oj} \tilde{u}_{om} \rangle \nabla_i \overline{U} * \nabla_j' \overline{U} * \nabla_m'' \overline{P} \\
&= -\langle \tilde{u}_{oi} \tilde{u}_{oj} \tilde{u}_{om} \rangle \nabla_i \nabla_j \nabla_m \overline{U} * \overline{U} * \overline{P} \\
\overline{\Delta_4} &\doteq \left[\langle \tilde{u}_{oi} \tilde{u}_{oj} \tilde{u}_{om} \tilde{u}_{on} \rangle - \langle \tilde{u}_{oi} \tilde{u}_{on} \rangle \langle \tilde{u}_{oj} \tilde{u}_{om} \rangle - \langle \tilde{u}_{oi} \tilde{u}_{oj} \rangle \langle \tilde{u}_{om} \tilde{u}_{on} \rangle \right] \\
&\quad \cdot \nabla_i \nabla_j \nabla_m \nabla_n \overline{U} * \overline{U} * \overline{U} * \overline{P}
\end{aligned} \tag{53}$$

......

In deriving (53), we have applied the assumption of homogeneity to transfer all of the spatial derivative operators, ∇_i', to a position in front of the operations of $\overline{U}$. This step is based upon the fact that the operation of $\overline{U}$ implies a spatial integration and $\overline{U}$ itself contains a spatial dependence $\underset{\sim}{\ell} - \underset{\sim}{\ell}'$ (reference to Section 3.1). It is found that all the moments of $\overline{P}(\tau, \underset{\sim}{\ell})$ of odd order are approximately zero and the moments of even order, say the 2nth, are contributed by $\overline{C_o}$ and $\overline{\Delta_{2i}}$ up to i=n. It is also found that, although smaller, the contributions of $\overline{\Delta_{2i}}$ $(2 \leqslant i \leqslant n)$ are of the same order as that contributed by $\overline{C}_0$. In other words, the subsequent terms of $\overline{C}$ decreases slowly when the order increases. Therefore, the approximation

$$\overline{C} \doteq \overline{C}_0 = \langle \tilde{L} \overline{U} * \tilde{L} \rangle \overline{P}$$

may be modified by adding more terms in the short time limit.

When τ is sufficiently large, we make following approximations.

a) The terms in $\{\overline{\Delta}_{2i}\}$ that have any correlations among odd number of $\tilde{L}$ are negligible, e.g.

$$\langle \tilde{L} \circ \tilde{L} \circ \tilde{L} \rangle \overline{P} \doteq 0$$

$$\langle \tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \rangle \circ \tilde{L} \circ \tilde{L} \rangle \overline{P} \doteq 0$$

$$\langle \tilde{L} \circ \tilde{L} \rangle \circ \langle \tilde{L} \circ \tilde{L} \circ \tilde{L} \rangle \overline{P} \doteq 0$$

This approximation is based upon the alternation of the sign during averging and the spatial and the temporal integrations implied by the operations of $\overline{U}$. The correlations are at multiple space-time points. The integrations make them very weak.

b) The terms involving any pair correlation between two $\tilde{L}$, which are separated by a correlation, or correlations, between four or more $\tilde{L}$, are negligible, e.g.

$$\langle \overbrace{\tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \tilde{L} \rangle \circ \tilde{L}} \rangle \overline{P} \doteq 0$$

$$\langle \overbrace{\tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \rangle \circ \langle \tilde{L} \circ \tilde{L} \rangle \circ \tilde{L}} \rangle \overline{P} \doteq 0$$

The symbol $\frown$ denotes the pair correlation. This approximation is made because the pair correlation makes the respective terms much weaker than other terms, e.g.weaker than $\langle\tilde{L}\circ\tilde{L}\rangle\underline{\circ}\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\,\overline{P}$ and $\langle\tilde{L}\circ\tilde{L}\rangle\underline{\circ}\langle\tilde{L}\circ\tilde{L}\rangle\underline{\circ}\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}$, because the underlined time integrations are not confined by the correlation time.

c) As a consequence of a) and b), we assume that

$$\begin{aligned}
&\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} \doteq \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} \qquad (54)\\
&\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} \doteq \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
&\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} \doteq 0
\end{aligned}$$

etc..

By means of these approximations,the series (50) is reduced to

$$\overline{C} \doteq \overline{C}_0 + \sum_{i=2}^{\infty} \overline{\Delta}_{2i}$$

where

$$\begin{aligned}
\overline{\Delta}_4 &= \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
\overline{\Delta}_6 &\doteq \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
&\quad - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
&= \langle\tilde{L}\circ\tilde{L}\circ[\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L} - \langle\tilde{L}\circ\tilde{L}\rangle) - \langle\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L} - \langle\tilde{L}\circ\tilde{L}\rangle)\rangle]\rangle\overline{P} \qquad (55)\\
\overline{\Delta}_8 &\doteq \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
&\quad - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
&\quad - \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
&\quad + \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P}\\
&= \langle\tilde{L}\circ\tilde{L}\circ\{\tilde{L}\circ\tilde{L}\circ[\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L} - \langle\tilde{L}\circ\tilde{L}\rangle) - \langle\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L} - \langle\tilde{L}\circ\tilde{L}\rangle)\rangle]\\
&\quad - \langle\tilde{L}\circ\tilde{L}\circ[\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L} - \langle\tilde{L}\circ\tilde{L}\rangle) - \langle\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L} - \langle\tilde{L}\circ\tilde{L}\rangle)\rangle]\rangle\}\rangle\overline{P}
\end{aligned}$$

Adding both sides of (54) to the expression for $\overline{\Delta_6}$, we find

$$\overline{\Delta_6} \doteq \langle \tilde{L} \circ \tilde{L} \circ (\hat{\Delta}_4 - \overline{\Delta_4}) \rangle = \langle \tilde{L} \circ \tilde{L} \circ \tilde{\Delta}_4 \rangle$$

$$\overline{\Delta_8} \doteq \langle \tilde{L} \circ \tilde{L} \circ (\hat{\Delta}_6 - \overline{\Delta_6}) \rangle = \langle \tilde{L} \circ \tilde{L} \circ \tilde{\Delta}_6 \rangle$$

where $\hat{\Delta}_4$ and $\hat{\Delta}_6$ are defined as

$$\hat{\Delta}_4 \equiv \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \tilde{L}\, \overline{p} - \tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \rangle \circ \tilde{L}\, \overline{p} - \tilde{L} \circ \tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \rangle \overline{p}$$

$$\begin{aligned}\hat{\Delta}_6 \equiv\ & \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \tilde{L}\, \overline{p} - \tilde{L} \circ \tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \tilde{L} \rangle \overline{p} \\ & - \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \rangle \overline{p} + \tilde{L} \circ \tilde{L} \circ \langle \tilde{L} \circ \tilde{L} \rangle \circ \langle \tilde{L} \circ \tilde{L} \rangle \overline{p}\end{aligned}$$

The full expression for the series $\{\overline{\Delta_{2i}}\}$ is very lengthy and complicated. Nevertheless, according to the pattern, it is reasonable to assume that

$$\overline{\Delta_{2i}} \doteq \langle \tilde{L} \circ \tilde{L} \circ \tilde{\Delta}_{2(i-1)} \rangle \qquad (i>2)$$

It may be noted that, among the $\overline{\Delta_{2i}}$, the expression for $\overline{\Delta_4}$ is exact.

To study the behavior of the series $\{\overline{\Delta_{2i}}\}$, we construct another series $\{\overline{\mathbb{\Delta}}_{2i}\}$, (i $\geqslant$ 2):

$$\overline{\mathbb{\Delta}}_4 = \overline{\Delta_4}$$

$$\overline{\mathbb{\Delta}}_{2i} = \langle \tilde{L} \circ \tilde{L} \rangle \circ \overline{\mathbb{\Delta}}_{2(i-1)} \qquad (i>2)$$

The terms of the series $\{\overline{\Delta_{2i}}\}$ decrease faster with increasing order than do the respective terms of the series $\{\overline{\mathbb{\Delta}}_{2i}\}$ for two reasons. First, $\tilde{L} \circ \tilde{L}$ refers to two points but $\tilde{\Delta}_{2(i-1)}$ refers to a number of points. The correlation becomes weaker when that number increases. Such a property is not possessed by the series $\{\overline{\Delta}_{2i}\}$. Secondly, the underlined time integration in $\overline{\Delta}_{2i}$ is restricted by a correlation time but the one in $\overline{\mathbb{\Delta}}_{2i}$ is not.

On the other hand, the behavior of the series $\{\overline{\mathbb{\Delta}}_{2i}\}$ can be compared with that of the series in (43). The norm ratio of two subsequent terms of $\{\overline{\mathbb{\Delta}}_{2i}\}$ is:

$$\mathbb{R}_{\overline{\mathbb{\Delta}}_{2i}} = \frac{\| \langle \tilde{L} \circ \tilde{L} \rangle \circ \overline{\mathbb{\Delta}}_{2i} \|}{\| \overline{\mathbb{\Delta}}_{2i} \|} \tag{56}$$

And the respective ratio of the series (43) is

$$\mathbb{R}_{\overline{\mathbb{H}}} = \frac{2(2i+1)}{(i+2)} \frac{\| (\overline{\mathbb{H}}*)^{i+1} \overline{\mathbb{C}} \|}{\| (\overline{\mathbb{H}}*)^{i} \overline{\mathbb{C}} \|} \tag{57}$$

Recall that

$$\overline{\mathbb{H}}* = \langle \tilde{L} \hat{U}* \tilde{L} \rangle \overline{U}*$$
$$o = \overline{U}*$$

These norm ratios may be estimated in the way similar to that used by Knobloch[34]:

$$\lim_{i \to \infty} \mathbb{R}_{\overline{\mathbb{\Delta}}_{2i}} \doteq \| \langle \tilde{L} \overline{U} * \tilde{L} \rangle \overline{U} * \| \tag{58}$$

$$\lim_{i \to \infty} \mathbb{R}_{\overline{\mathbb{H}}} \doteq 4 \| \langle \tilde{L} \hat{U} * \tilde{L} \rangle \overline{U} * \| \doteq 4 \| \langle \tilde{L} \overline{U} * \tilde{L} \rangle \overline{U} * \| \tag{59}$$

Therefore, the terms of the series for $\{\overline{\mathbb{\Delta}}_{2i}\}$ decrease faster than do the terms in the series given by (43).

According to this discussion, it is shown that a low order truncation of the series provides a better approximation than provided by the truncation of the series in (43), the latter is equivalent to the neglect of the collective collision. Therefore, when τ is large, we have

$$\overline{C} \doteq \overline{C}_0 + \overline{\Delta}_4 \tag{60}$$

with $\overline{\Delta}_4$ as the correction for the zeroth order approximation. In fact, it can be shown that the term $\overline{\Delta}_4$ is significant only when τ is not large. Hence, the zeroth approximation is good for long times.

A complete discussion of the behavior of the series does not seem possible at present. The correlations are not only of increasing order but also at an increasing number of different space-time points. Analytical discussions and experimental investigations of correlations at multiple points are rare. We may mention the discussion made by Monin and Yaglom[35], and the experiments made by Van Atta et al.[36,37]. According to their results, our approximations may work even when τ is not large. Moreover, the low order moments of the transition function are of the practical interest. In view of the behavior of the series $\{\overline{\Delta}_i\}$, it is reasonable to assume that (60) is good for the whole time range.

The condition under which (60) works well is estimated by (59). We have

$$\mathcal{R} \equiv \frac{\sqrt{\langle \hat{\underset{\sim}{u}}^2 \rangle}\, \tau_L}{\mathcal{L}} < \frac{1}{2} \tag{61}$$

where $\langle \hat{\underset{\sim}{u}}^2 \rangle$ is the strength of the turbulence, τ_L is the Lagrangian correlation time scale and $\mathcal{L}$ is a length scale over which $\overline{P}(\tau, \underset{\sim}{\ell})$ has a significant change. (61) is comparable to the condition suggested by Knobloch[34], in which a Eulerian time scale was used.

As mentioned by Monin and Yaglom[35], when one or more points are far from other points, a fourth moment can be approximately written as

$$\langle \hat{A}(\underset{\sim}{x}_1)\hat{A}(\underset{\sim}{x}_2)\hat{A}(\underset{\sim}{x}_3)\hat{A}(\underset{\sim}{x}_4)\rangle \doteq \langle \hat{A}(\underset{\sim}{x}_1)\hat{A}(\underset{\sim}{x}_2)\rangle\langle \hat{A}(\underset{\sim}{x}_3)\hat{A}(\underset{\sim}{x}_4)\rangle$$
$$+\langle \hat{A}(\underset{\sim}{x}_1)\hat{A}(\underset{\sim}{x}_4)\rangle\langle \hat{A}(\underset{\sim}{x}_2)\hat{A}(\underset{\sim}{x}_3)\rangle + \langle \hat{A}(\underset{\sim}{x}_1)\hat{A}(\underset{\sim}{x}_3)\rangle\langle \hat{A}(\underset{\sim}{x}_2)\hat{A}(\underset{\sim}{x}_4)\rangle$$

In our analysis, this picture occurs frequently because of the spatial and the temporal integrations. For an analytical study based upon two-point velocity correlation functions, we further assume that

$$\overline{C} \doteq \overline{C_0} + \overline{\Delta_4} = \overline{C}_0 + \langle \hat{L}\circ\hat{L}\circ\hat{L}\circ\hat{L}\rangle\overline{P} - \langle \hat{L}\circ\langle\hat{L}\circ\hat{L}\rangle\circ\hat{L}\rangle\overline{P} - \langle\hat{L}\circ\hat{L}\rangle\circ\langle\hat{L}\circ\hat{L}\rangle\overline{P}$$
$$\doteq \overline{C}_0 + \langle \hat{L}\circ\hat{L}\circ\hat{L}\circ\hat{L}\rangle\overline{P} \qquad (62)$$

(62) will be used to establish the transport equations.

2.5 Transport Equations

The previous discussion concerns the dispersion of a single particle. It can be readily written

$$(\partial_\tau + \bar{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla})\bar{P} \doteq \underset{\sim}{\nabla}\cdot\underset{\approx}{K}\cdot\{\underset{\sim}{\nabla}'\bar{P}\} \tag{63}$$

where $\bar{\underset{\sim}{u}}$ is the mean velocity of the particle. For a fluid particle, $\bar{\underset{\sim}{u}}$ is the same as the mean velocity of the flow. $\underset{\approx}{K}\{\ \}$ is an integral operator playing the role of the eddy transport property. According to the zeroth order approximation,

$$\underset{\approx}{K}\{\ \} \doteq \langle \hat{\underset{\sim}{v}}\,\bar{U} * \hat{\underset{\sim}{v}} \rangle \tag{64}$$

According to (62),

$$\underset{\approx}{K}\{\ \} \doteq \langle \hat{\underset{\sim}{v}}\,\bar{U} * \hat{\underset{\sim}{v}} \rangle + \langle \hat{\underset{\sim}{v}}\,\bar{U} * \hat{L}\,\bar{U} * \hat{L}\,\bar{U} * \hat{\underset{\sim}{v}} \rangle \tag{65}$$

For the two-particle transition function, equation (11) can be written as

$$[\partial_\tau + \hat{L}_{12}(\tau)]\hat{P}_{12}(\tau, \underset{\sim}{l}_1, \underset{\sim}{l}_2) = 0$$

where

$$\hat{L}_{12}(\tau) \equiv \hat{L}_1(\tau) + \hat{L}_2(\tau)$$

The procedure for obtaining (27) and (62) can by analogy be used to get

$$(\partial_\tau + \bar{L}_{12})\bar{P}_{12} \doteq \bar{C}_0^{12} \tag{66}$$

or

$$(\partial_\tau + \bar{L}_{12})\bar{P}_{12} \doteq \bar{C}_0^{12} + \bar{\Delta}_4^{12} \tag{67}$$

where $\overline{C}_0^{12}$ and $\overline{\Delta}_4^{12}$ are similar to $\overline{C}_0$ and $\overline{\Delta}_4$ but in terms of the operator $\tilde{L}_{12}$ and the two-particle propagator $\overline{U}_{12}$. $\overline{U}_{12}$ is the ensemble average of the two-particle exact propagator $\hat{U}_{12}$, defined by

$$[\partial_\tau + \hat{L}_{12}(\tau)]\,\hat{U}_{12}(\tau,0) = 0 \qquad (\tau>0)$$

$$\hat{U}_{12}(0,0) = 1$$

Equations (66) and (67) can also be written as

$$(\partial_\tau + \underset{\sim}{\overline{u}}_1\cdot\underset{\sim}{\nabla}_1 + \underset{\sim}{\overline{u}}_2\cdot\underset{\sim}{\nabla}_2)\overline{P}_{12} \doteq \sum_{i,j=1,2} \underset{\sim}{\nabla}_i\cdot\underset{\approx}{K}_{ij}\cdot\{\underset{\sim}{\nabla}_j'\,\overline{P}_{12}\} \tag{68}$$

with

$$\underset{\approx}{K}_{ij}\cdot\{\,\} \equiv \langle \tilde{\underset{\sim}{V}}_i\,\overline{U}_{12} * \tilde{\underset{\sim}{V}}_j\cdot\rangle \tag{69}$$

and

$$\underset{\approx}{K}_{ij}\cdot\{\,\} \equiv \langle \hat{\underset{\sim}{V}}_i\,\overline{U}_{12} * \hat{\underset{\sim}{V}}_j\cdot\rangle + \langle \overbrace{\hat{\underset{\sim}{V}}_i\,\overline{U}_{12} * \tilde{L}_{12}\overline{U}_{12} * \tilde{L}_{12}\overline{U} * \hat{\underset{\sim}{V}}_j\cdot}\rangle \tag{70}$$

respectively, playing the role of the transport property. These formulas contain more terms than that in the one-particle case because $\tilde{L}_{12} = \tilde{L}_1 + \tilde{L}_2$.

Transform (68) into the $(\underset{\sim}{\ell}, \underset{\sim}{\lambda})$ system. The transport equation for the relative transition function $\overline{B}(\tau,\underset{\sim}{\lambda})$ can be obtained by the integration with respect to $\underset{\sim}{\ell}$. It is found that

$$\partial_\tau \overline{B} \doteq \underset{\sim}{\nabla}_\lambda \cdot \underset{\approx}{K}^{r}\cdot\{\underset{\sim}{\nabla}_\lambda'\,\overline{B}\} \tag{71}$$

with

$$\underset{\approx}{K}^{r}\{\,\} = \langle \tilde{\underset{\sim}{V}}_1\,\overline{U}_{12} * \hat{\underset{\sim}{V}}_1\rangle - \langle \tilde{\underset{\sim}{V}}_1\,\overline{U}_{12} * \tilde{\underset{\sim}{V}}_2\rangle - \langle \hat{\underset{\sim}{V}}_2\,\overline{U}_{12} * \hat{\underset{\sim}{V}}_1\rangle + \langle \tilde{\underset{\sim}{V}}_2\,\overline{U}_{12} * \hat{\underset{\sim}{V}}_2\rangle \tag{72}$$

as the transport property for the relative dispersion. Note that we have assumed a constant mean velocity and that the initial separation of the two particles is close to zero. In such a case, the two particles move almost together and the moments of the relative displacement are close to zero at short times. Similar to the one-particle case, it is also expected that the correction term $\bar{\Delta}_4^{12}$ contributes to the relative dispersion mostly at short times. By analogy to the one-particle results, one would expect the correction to be small for the second moment. It is, however, essential at short times for the fourth moment. It seems that the neglect of the correction term would not affect the second moment for the relative displacement by very much. A study of the higher order moments may be made based upon our theory. The results would be different from those in other theories because both the zero order equation is different and the higher order correction is obtained .

III DISPERSION OF A SINGLE FLUID PARTICLE

3.1 Formulation of the One-Particle Transition

Equation (63) governs the dispersion of a single particle. To determine the transport property, we relate the mean propagator to the transition function. Since the exact propagator requires its operand to be evaluated along the trajectory, we can write

$$\int_0^{\tau} d\tau'\, \hat{U}(\tau,\tau') \cdots = \int_0^{\tau} d\tau' \int d\underset{\sim}{\ell}'\, \hat{P}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\ell}') \cdots$$

Hence,

$$\int_0^{\tau} d\tau'\, \overline{U}(\tau,\tau') \cdots = \int_0^{\tau} d\tau' \int d\underset{\sim}{\ell}'\, \overline{P}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\ell}') \cdots \tag{73}$$

The properties of the exact propagator and the mean propagator can be found in the articles by Weinstock[15], Misguich and Balescu[16-18]. The relation (73) was also used by Roberts[22]. An explanation of (73) is given in Appendix 2. In such a manner, the transport property $\underset{\approx}{K}\{\}$ is expressed as an operator containing $\hat{\underset{\sim}{v}}$ and $\overline{P}$. The transport equation becomes a non-linear integro-differential equation. According to the zeroth order approximation,

$$(\partial_\tau + \overline{U}_i \nabla_i)\, \overline{P}(\tau,\underset{\sim}{\ell}) \doteq \nabla_i \int_0^{\tau} d\tau' \int d\underset{\sim}{\ell}' \langle \hat{v}_i(\tau)\, \overline{P}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\ell}')\, \hat{v}_j(\tau') \rangle \nabla_j'\, \overline{P}(\tau';\underset{\sim}{\ell}') \tag{74}$$

According to (62),

$$(\partial_\tau + \overline{U}_i \nabla_i)\, \overline{P}(\tau,\underset{\sim}{\ell}) \doteq \nabla_i \int_0^{\tau} d\tau' \int d\underset{\sim}{\ell}' \langle \hat{v}_i(\tau)\, \overline{P}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\ell}')\, \hat{v}_j(\tau') \rangle \nabla_j'\, \overline{P}(\tau';\underset{\sim}{\ell}')$$

$$\begin{aligned}(\partial_\tau + \overline{U}_i \nabla_i)\, \overline{P}(\tau,\underset{\sim}{\ell}) \doteq{}& \nabla_i \int_0^{\tau} d\tau' \int d\underset{\sim}{\ell}' \langle \hat{v}_i(\tau)\, \overline{P}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\ell}')\, \hat{v}_j(\tau') \rangle \nabla_j'\, \overline{P}(\tau';\underset{\sim}{\ell}') \\ &+ \nabla_i \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \iiint d\underset{\sim}{\ell}'\, d\underset{\sim}{\ell}''\, d\underset{\sim}{\ell}''' \\ &\cdot \langle\!\langle \hat{v}_i(\tau)\, \overline{P}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\ell}')\, \hat{v}_j(\tau') \nabla_j'\, \overline{P}(\tau'-\tau'';\underset{\sim}{\ell}'-\underset{\sim}{\ell}'')\, \nabla_m''\, \hat{v}_m(\tau'')\, \overline{P}(\tau''-\tau''';\underset{\sim}{\ell}''-\underset{\sim}{\ell}''')\, \hat{v}_n(\tau''') \rangle\!\rangle \\ &\cdot \nabla_n'''\, \overline{P}(\tau''';\underset{\sim}{\ell}''')\end{aligned}$$

For a fluid particle, its velocity is the same as the local Eulerian velocity, so that

$$\hat{v}_i(\tau) = \hat{u}_i(\tau + t_o, \underset{\sim}{\ell} + \underset{\sim}{x}_o)$$
$$\hat{v}_j(\tau') = \hat{u}_j(\tau' + t_o, \underset{\sim}{\ell}' + \underset{\sim}{x}_o)$$

For stationary and homogeneous turbulence, (74) and (75) become

$$(\partial_\tau + \bar{u}_i \nabla_i)\bar{P}(\tau, \underset{\sim}{\ell}) \doteq \nabla_i \int_0^\tau d\tau' \int d\underset{\sim}{\ell}' R_{ij}(\tau - \tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}')\bar{P}(\tau - \tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}')\nabla_j' \bar{P}(\tau'; \underset{\sim}{\ell}') \tag{76}$$

and

$$\begin{aligned}(\partial_\tau + \bar{u}_i \nabla_i)\bar{P}(\tau, \underset{\sim}{\ell}) \doteq\ & \nabla_i \int_0^\tau d\tau' \int d\underset{\sim}{\ell}' R_{ij}(\tau - \tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}')\bar{P}(\tau - \tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}')\nabla_j' \bar{P}(\tau'; \underset{\sim}{\ell}') \\ & + \nabla_i \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \iiint d\underset{\sim}{\ell}' d\underset{\sim}{\ell}'' d\underset{\sim}{\ell}''' \\ & \cdot R_{im}(\tau - \tau''; \underset{\sim}{\ell} - \underset{\sim}{\ell}'') R_{jn}(\tau' - \tau'''; \underset{\sim}{\ell}' - \underset{\sim}{\ell}''')\bar{P}(\tau - \tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}') \\ & \cdot \nabla_j' \bar{P}(\tau' - \tau''; \underset{\sim}{\ell}' - \underset{\sim}{\ell}'')\nabla_m'' \bar{P}(\tau'' - \tau'''; \underset{\sim}{\ell}'' - \underset{\sim}{\ell}''')\nabla_n''' \bar{P}(\tau'''; \underset{\sim}{\ell}''')\end{aligned} \tag{77}$$

respectively, where

$$R_{ij}(\tau - \tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}') = \langle \hat{u}_i(\tau + t_o, \underset{\sim}{\ell} + \underset{\sim}{x}_o)\hat{u}_j(\tau' + t_o, \underset{\sim}{\ell}' + \underset{\sim}{x}_o)\rangle$$

is the Eulerian velocity correlation function. Equation (76) is exactly the same as that derived by Roberts using DIA[22]. Equation (77) modifies (76).

The moments of the transition function are defined as follows. The first moment

$$\langle \hat{\underset{\sim}{\ell}}(\tau)\rangle = \int d\underset{\sim}{\ell}\, \underset{\sim}{\ell}\, \bar{P}(\tau, \underset{\sim}{\ell}) = \bar{\underset{\sim}{u}}\tau$$

For the present case, the central moments of odd order are all zero and the central moments of even order are:

$$\langle \check{\underset{\sim}{\ell}}^{2n}(\tau)\rangle = \langle [\hat{\underset{\sim}{\ell}}(\tau) - \bar{\underset{\sim}{u}}\tau]^{2n}\rangle = \int d\underset{\sim}{\ell}\, (\underset{\sim}{\ell} - \bar{\underset{\sim}{u}}\tau)^{2n}\bar{P}(\tau, \underset{\sim}{\ell})$$

Therefore,

$$\frac{d}{d\tau}\langle \hat{\underset{\sim}{\ell}}^{2n}(\tau)\rangle = -2n\int d\underset{\sim}{\ell}\,\overline{\underset{\sim}{u}}\cdot(\underset{\sim}{\ell}-\overline{\underset{\sim}{u}}\tau)^{2n-1}\overline{P}(\tau,\underset{\sim}{\ell}) + \int d\underset{\sim}{\ell}(\underset{\sim}{\ell}-\overline{\underset{\sim}{u}}\tau)^{2n}\partial_\tau \overline{P}(\tau,\underset{\sim}{\ell})$$

$$= \int d\underset{\sim}{\ell}(\underset{\sim}{\ell}-\overline{\underset{\sim}{u}}\tau)^{2n}\partial_\tau\overline{P}(\tau,\underset{\sim}{\ell}) \qquad (78)$$

which can be studied by means of the equations (76) and (77).

3.2 Asymptotic Cases

In the short time limit, all the velocity fluctuations are approximately the same as that possessed by the particle initially. For the second central moment, the contribution by Δ_4 is approximately zero. It is found that

$$\frac{d}{d\tau}\langle \hat{\underset{\sim}{\ell}}^2(\tau)\rangle \doteq \int d\underset{\sim}{\ell}\;\underset{\sim}{\ell}^2\,\overline{C_o} \doteq 2\langle \hat{\underset{\sim}{u}}_o^2\rangle\tau \qquad (79)$$

$$\langle \hat{\underset{\sim}{\ell}}^2(\tau)\rangle \doteq \langle \hat{\underset{\sim}{u}}_o^2\rangle\tau^2 \qquad (80)$$

For the fourth moment, we have

$$\frac{d}{d\tau}\langle \hat{\underset{\sim}{\ell}}^4(\tau)\rangle \doteq \int d\underset{\sim}{\ell}\;\underset{\sim}{\ell}^4(\overline{C_o}+\overline{\Delta}_4) \qquad (81)$$

It is found that

$$\int d\underset{\sim}{\ell}\;\underset{\sim}{\ell}^4\,\overline{C_o} \doteq \frac{40}{9}\langle \hat{\underset{\sim}{u}}_o^2\rangle^2\tau^3 \qquad (82)$$

$$\int d\underset{\sim}{\ell}\;\underset{\sim}{\ell}^4\,\overline{\Delta}_4 \doteq \frac{20}{9}\langle \hat{\underset{\sim}{u}}_o^2\rangle^2\tau^3 \qquad (83)$$

Therefore,

$$\frac{d}{d\tau}\langle \hat{\underset{\sim}{\ell}}^4(\tau)\rangle \doteq \frac{20}{3}\langle \hat{\underset{\sim}{u}}_o^2\rangle^2\tau^3 \qquad (84)$$

$$\langle \hat{\underset{\sim}{\ell}}^4(\tau)\rangle \doteq \frac{5}{3}\langle \hat{\underset{\sim}{u}}_o^2\rangle^2\tau^4 \qquad (85)$$

On the other hand, the fourth central moment can be found kinematically as

$$\underset{\sim}{\tilde{\ell}}(\tau) \doteq \underset{\sim}{\tilde{u}}_0 \tau$$

$$\langle \underset{\sim}{\tilde{\ell}}^4(\tau) \rangle \doteq \langle \underset{\sim}{\tilde{u}}_0^4 \rangle \tau^4 \qquad (86)$$

The velocity field may be assumed Gaussian so that

$$\langle \underset{\sim}{\tilde{\ell}}^4(\tau) \rangle \doteq \frac{5}{3} \langle \underset{\sim}{\tilde{u}}_0^2 \rangle^2 \tau^4 \qquad (87)$$

Comparing (87) with (85), we see the necessity of the correction to recover (87), in the short time limit.

In the long time limit, $\overline{\Delta}_4$ is negligible. (76) can be written as

$$(\partial_\tau + \overline{u}_i \nabla_i) \overline{P}(\tau, \underset{\sim}{\ell}) \doteq \nabla_i \int_0^\tau d\tau' \int d\underset{\sim}{\ell}' R_{ij}(\tau'; \underset{\sim}{\ell}') \overline{P}(\tau'; \underset{\sim}{\ell}') \nabla_j \cdot \overline{P}(\tau - \tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}') \qquad (88)$$

Denote τ_c as the corelation time, (88) is approximately

$$(\partial_\tau + \overline{u}_i \nabla_i) \overline{P}(\tau, \underset{\sim}{\ell}) \doteq \int_0^{\tau_c} d\tau' \int d\underset{\sim}{\ell}' R_{ij}(\tau'; \underset{\sim}{\ell}') \overline{P}(\tau'; \underset{\sim}{\ell}') \nabla_i \nabla_j \overline{P}(\tau - \tau_c, \underset{\sim}{\ell})$$

$$\doteq K_{ij}(\infty) \nabla_i \nabla_j \overline{P}(\tau, \underset{\sim}{\ell}) \qquad (89)$$

where $K_{ij}(\infty)$ is the component of the asymptotic eddy diffusivity tensor, defined by

$$\underset{\approx}{K}(\infty) \equiv \int_0^\infty d\tau' \int d\underset{\sim}{\ell}' \underset{\approx}{R}(\tau'; \underset{\sim}{\ell}') \overline{P}(\tau'; \underset{\sim}{\ell}') \qquad (90)$$

(89) indicates that $\overline{P}(\tau, \underset{\sim}{\ell})$ approaches Gaussian in the large time limit.

The discussion for the two asymtotic cases agrees with observations.

3.3 Intermediate Times, Gaussian Approximation

At intermediate times, equation (75) is used to study the moments of $\overline{P}(\tau,\underset{\sim}{\ell})$. For the second central moment, (75) yields

$$\frac{d}{d\tau}\langle \hat{\ell}_i(\tau)\hat{\ell}_j(\tau)\rangle = 2\int_0^\tau d\tau' \int d\underset{\sim}{\ell}' R_{ij}(\tau';\underset{\sim}{\ell}')\overline{P}(\tau';\underset{\sim}{\ell}')$$
$$+2\int_0^\tau d\tau'\int_0^{\tau'} d\tau''\int_0^{\tau''} d\tau''' \iiint d\underset{\sim}{\ell}' d\underset{\sim}{\ell}'' d\underset{\sim}{\ell}''' R_{im}(\tau-\tau'';\underset{\sim}{\ell}'') R_{nj}(\tau'-\tau''';\underset{\sim}{\ell}'')$$
$$\cdot \overline{P}(\tau-\tau';\underset{\sim}{\ell}'-\underset{\sim}{\ell}''+\underset{\sim}{\ell}''')\nabla_n''\overline{P}(\tau'-\tau'';\underset{\sim}{\ell}''-\underset{\sim}{\ell}''')\nabla_m'''\overline{P}(\tau''-\tau''';\underset{\sim}{\ell}''') \tag{91}$$

In the derivation, we have used a coordinate transformation and the normalized $\overline{P}(\tau,\underset{\sim}{\ell})$. By means of Fourier transformation, (91) is turned into

$$\frac{d}{d\tau}\langle \hat{\ell}_i(\tau)\hat{\ell}_j(\tau)\rangle = 2(2\pi)^3\int_0^\tau d\tau' \int d\underset{\sim}{k}\, R_{ij}(\tau';\underset{\sim}{k})\overline{P}(\tau';-\underset{\sim}{k})$$
$$-2(2\pi)^9\int_0^\tau d\tau'\int_0^{\tau'} d\tau''\int_0^{\tau''} d\tau''' \iint d\underset{\sim}{k}\, d\underset{\sim}{k}'\, k_n k_m' R_{im}(\tau-\tau'';\underset{\sim}{k}) R_{nj}(\tau'-\tau''';\underset{\sim}{k}')$$
$$\overline{P}(\tau-\tau';-\underset{\sim}{k})\overline{P}(\tau'-\tau'';-\underset{\sim}{k}-\underset{\sim}{k}')\overline{P}(\tau''-\tau''';-\underset{\sim}{k}') \tag{92}$$

As mentioned in the literature,[22,23,31] the one-particle transition function has been found to be experimentally close to Gaussian for both short time and long time limits. It is also found that a Gaussian approximation for $\overline{P}(\tau,\underset{\sim}{\ell})$ works well for the whole time range. Based upon the Gaussian approximation, the equation for the second central moment is greatly simplified. For the case that the turbulence is also isotropic, the Gaussian form and the corresponding Fourier components of $\overline{P}(\tau,\underset{\sim}{\ell})$ are

$$\overline{P}(\tau,\underset{\sim}{\ell}) = [2\pi\sigma^2(\tau)]^{-3/2}\exp\left[-\frac{(\underset{\sim}{\ell}-\overline{\underset{\sim}{u}}\tau)^2}{2\sigma^2(\tau)}\right]$$

$$\overline{P}(\tau, \underset{\sim}{k}) = (2\pi)^{-3} \exp\left[-i k_i \bar{u}_i \tau - \frac{1}{2} k^2 \sigma^2(\tau)\right] \qquad (94)$$

where $\sigma^2(\tau) = \frac{1}{3}\langle \hat{\underset{\sim}{l}}^2(\tau)\rangle$ is the one-dimensional variance of $\overline{P}(\tau, \underset{\sim}{l})$. The components of the velocity correlation function for isotropic turbulence are of the form

$$R_{ij}(\tau, k) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{E(\tau, k)}{4\pi k^2} \qquad (95)$$

where $E(\tau,k)$ is the three-dimensional spectrum of the turbulent kinetic energy. Substituting (94) and (95) into (92), we obtain

$$\begin{aligned}\frac{d}{d\tau}\sigma^2(\tau) &= \frac{1}{3}\frac{d}{d\tau}\langle \hat{\underset{\sim}{l}}^2(\tau)\rangle \\ &= \frac{4}{3}\int_0^\tau d\tau' \int_0^\infty dk \, \frac{\sin(\bar{u} k \tau')}{\bar{u} k \tau'} E(\tau', k) \exp\left[-\frac{1}{2}k^2\sigma^2(\tau)\right] \\ &\quad - \frac{8}{3}\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \int_0^\infty dk \int_0^\infty dk' \, \frac{\sin[\bar{u} k(\tau-\tau'')] \sin[\bar{u} k'(\tau'-\tau''')]}{\bar{u}^2 k k'(\tau-\tau'')(\tau'-\tau''')} \\ &\quad \cdot \frac{E(\tau-\tau'', k) E(\tau'-\tau''', k')}{k k' \sigma^4(\tau'-\tau'')} \exp\left\{-\frac{1}{2}k^2[\sigma^2(\tau-\tau') + \sigma^2(\tau'-\tau'')] - \frac{1}{2}k'^2[\sigma^2(\tau'-\tau'') + \sigma^2(\tau''-\tau''')]\right\} \\ &\quad \cdot \left\{\sinh[k k' \sigma^2(\tau'-\tau'')] - \frac{3\cosh[k k' \sigma^2(\tau'-\tau'')]}{k k' \sigma^2(\tau'-\tau'')} + \frac{3\sinh[k k' \sigma^2(\tau'-\tau'')]}{k^2 k'^2 \sigma^4(\tau'-\tau'')}\right\}\end{aligned} \qquad (96)$$

When the energy spectrum is known, the variance $\sigma^2(t)$ can be solved from (96). Then the Gaussian solution is obtained.

Denote

$$a \equiv k k' \sigma^2(\tau' - \tau'')$$

The respective factor in the second integral on the right-hand side of (96) is written as

$$f(a) = \frac{1}{a^4}\left[a^2 \sinh(a) - 3a\cosh(a) + 3\sinh(a)\right]$$

It can be proved that

$$\lim_{a\to 0} f(a) = 0 , \qquad \frac{d}{da} f(a) > 0 \qquad (a > 0)$$

Therefore f(a) is a positive function when $\tau'-\tau''>0$. This indicates that a negative contribution is provided by the correction. It may be noted that the correction term actually has only a minor contribution to the second moment. Therefore, we can in practice use the equation

$$\frac{d}{d\tau}\sigma^2(\tau) \doteq \frac{4}{3}\int_0^{\tau} d\tau' \int_0^{\infty} dk \, \frac{\sin(\bar{u}k\tau')}{\bar{u}k\tau'} E(\tau';k)\exp[-\tfrac{1}{2}k^2\sigma^2(\tau')] \tag{97}$$

for the variance.

3.4 Non-Gaussian Behavior

The non-Gaussian behavior of $\bar{P}(\tau,\underset{\sim}{\ell})$ can be described by the moments of order higher than two or the respective cumulants. According to equation (96) or (97), the non-Gaussian behavior of $\bar{P}(\tau,\underset{\sim}{\ell})$ is mainly caused by the memory effect. If the memory effect is neglected, i.e. $\underset{\sim}{\nabla}'\bar{P}(\tau',\underset{\sim}{\ell}')$ replaced by $\underset{\sim}{\nabla}\bar{P}(\tau,\underset{\sim}{\ell})$, the transport property, (64) or (65), becomes the eddy diffusivity instead of an operator. The result is the same as that defined by

$$\underset{\approx}{K}(\tau) = \frac{1}{2}\frac{d}{d\tau}\langle \hat{\underset{\sim}{\ell}}(\tau)\hat{\underset{\sim}{\ell}}(\tau)\rangle = \frac{1}{2}\int d\underset{\sim}{\ell}\,(\underset{\sim}{\ell}-\bar{\underset{\sim}{u}}\tau)(\underset{\sim}{\ell}-\bar{\underset{\sim}{u}}\tau)\,\partial_\tau \bar{P}(\tau,\underset{\sim}{\ell})$$

which can be solved by means of (96) or (97). Accounting for the memory effect, we study the flatness of $\bar{P}(\tau,\underset{\sim}{\ell})$ based upon the behavior of the fourth cumulant defined as

$$C_4 \equiv \langle \hat{\underset{\sim}{\ell}}^4(\tau)\rangle - \frac{5}{3}\langle \hat{\underset{\sim}{\ell}}^2(\tau)\rangle^2$$

According to the previous discussion, the correction term $\bar{\Delta}_4$ is significant when τ is small. For an analytical discussion, we assume a time τ_1. When $\tau<\tau_1$, approximately

$$\bar{\Delta}_4 = \langle \hat{L}\bar{U}*\overbrace{\hat{L}\bar{U}*\hat{L}\bar{U}*}\hat{L}\rangle P \doteq \langle \hat{L}\bar{U}*\hat{L}\rangle\bar{U}*\langle \hat{L}\bar{U}*\hat{L}\rangle\bar{P} \tag{98}$$

Without loss of generality, the mean velocity is taken as zero. Then

$$\frac{d}{d\tau}\langle \hat{\ell}^4(\tau)\rangle = \frac{d}{d\tau}\langle \hat{\ell}_i^2(\tau)\hat{\ell}_j^2(\tau)\rangle = \int d\ell\, \ell_i^2 \ell_j^2 \partial_\tau \overline{P}(\tau,\ell) \doteq \int d\ell\, \ell_i^2 \ell_j^2 (\overline{C_0} + \overline{\Delta_4}) \tag{99}$$

The contribution from $\overline{C}_0$ is

$$\begin{aligned}\int d\ell\, \ell_i^2 \ell_j^2 \overline{C_0} &= \int d\ell\, \ell_i^2 \ell_j^2 \nabla_m \int_0^\tau d\tau' \int d\ell' R_{mn}(\tau-\tau';\ell-\ell')\overline{P}(\tau-\tau';\ell-\ell')\nabla_n' \overline{P}(\tau';\ell') \\ &= \frac{20}{3}\int_0^\tau d\tau' \iint d\ell\, d\ell' [(\ell_i-\ell_i')^2 + 2\ell_i'(\ell_i-\ell_i') + \ell_i'^2] R_{jj}(\tau-\tau';\ell-\ell') \\ &\qquad \cdot \overline{P}(\tau-\tau';\ell-\ell')\overline{P}(\tau';\ell') \\ &= \frac{20}{3}\int_0^\tau d\tau' \iint d\ell\, d\ell' \ell_i^2 R_{jj}(\tau-\tau';\ell)\overline{P}(\tau-\tau';\ell)\overline{P}(\tau';\ell') \\ &\quad + \frac{20}{3}\int_0^\tau d\tau' \iint d\ell\, d\ell' \ell_i'^2 R_{jj}(\tau-\tau';\ell)\overline{P}(\tau-\tau';\ell)\overline{P}(\tau';\ell')\end{aligned} \tag{100}$$

Note that the first term on the righthand side of (100) is smaller than the second, because ℓ_i^2 in the first integral is weighted by both R_{jj} and $\overline{P}$, but $\ell_i'^2$ in the second integral is weighted by $\overline{P}$ only. Therefore,

$$\int d\ell\, \ell_i^2 \ell_j^2 \overline{C_0} \le \frac{40}{3}\int_0^\tau d\tau' \iint d\ell\, d\ell' \ell_i'^2 R_{jj}(\tau-\tau';\ell)\overline{P}(\tau-\tau';\ell)\overline{P}(\tau';\ell')$$

where the equality holds for $\tau \to 0$. Making use of (91), we reduce (100) to

$$\int d\ell\, \ell_i^2 \ell_j^2 \overline{C_0} \le \frac{20}{3}\int_0^\tau d\tau' \langle \hat{\ell}_i^2(\tau')\rangle \frac{d^2}{d\tau^2}\langle \hat{\ell}_j^2(\tau-\tau')\rangle = \frac{20}{3}\int_0^\tau d\tau' \left[\frac{d}{d\tau'}\langle \ell_i^2(\tau')\rangle\right]\frac{d}{d\tau}\langle \ell_j^2(\tau-\tau')\rangle \tag{101}$$

The contribution of $\overline{\Delta_4}$ is

$$\begin{aligned}\int d\ell\, \ell_i^2 \ell_j^2 \overline{\Delta_4} &\doteq \int d\ell\, \ell_i^2 \ell_j^2 \langle \hat{L}\overline{U} * \hat{L}\rangle \overline{U} * \langle \hat{L}\overline{U} * \hat{L}\rangle \overline{P} \\ &= \frac{40}{3}\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \iiiint d\ell\, d\ell'\, d\ell''\, d\ell''' R_{ii}(\tau-\tau';\ell-\ell') R_{jj}(\tau''-\tau''';\ell''-\ell''') \\ &\qquad \cdot \overline{P}(\tau-\tau';\ell-\ell')\overline{P}(\tau'-\tau'';\ell'-\ell'')\overline{P}(\tau''-\tau''';\ell''-\ell''')\overline{P}(\tau''';\ell''') \\ &= \frac{20}{3}\int_0^\tau d\tau'' \left[\frac{d}{d\tau''}\langle \hat{\ell}_j^2(\tau'')\rangle\right]\int_0^{\tau-\tau''} d\tau' \int d\ell\, R_{ii}(\tau-\tau';\ell)\overline{P}(\tau-\tau';\ell) \\ &= \frac{10}{3}\int_0^\tau d\tau'' \left[\frac{d}{d\tau''}\langle \ell_j^2(\tau'')\rangle\right]\frac{d}{d\tau}\langle \hat{\ell}_i^2(\tau-\tau'')\rangle\end{aligned} \tag{102}$$

The addition of (101) and (102) yields

$$\frac{d}{d\tau}\langle \tilde{l}_i^2(\tau)\tilde{l}_j^2(\tau)\rangle \leq 10\int_0^\tau d\tau' \left[\frac{d}{d\tau'}\langle \tilde{l}_i^2(\tau')\rangle\right]\frac{d}{d\tau}\langle \tilde{l}_j^2(\tau-\tau')\rangle$$

The fourth cummmulant of $\overline{P}(\tau,\boldsymbol{l})$ obeys the inequality

$$\frac{d}{d\tau}C_4 = \frac{d}{d\tau}\left[\langle \tilde{l}_i^2(\tau)\tilde{l}_j^2(\tau)\rangle - \frac{5}{3}\langle \tilde{l}_i^2(\tau)\rangle^2\right]$$
$$\leq 10\left\{\int_0^\tau d\tau'\left[\frac{d}{d\tau'}\langle \tilde{l}_i^2(\tau')\rangle\right]\frac{d}{d\tau}\langle \tilde{l}_j^2(\tau-\tau')\rangle - \frac{1}{3}\langle \tilde{l}_i^2(\tau)\rangle\frac{d}{d\tau}\langle \tilde{l}_j^2(\tau)\rangle\right\}$$

When τ is small, approximately

$$\langle \tilde{l}_i^2(\tau)\rangle \doteq \langle \tilde{u}_{0i}^2\rangle\tau^2$$

so that

$$\frac{d}{d\tau}C_4 \leq 10\langle \tilde{u}_{0i}^2\rangle\left[4\int_0^\tau d\tau'\,\tau'(\tau-\tau') - \frac{2}{3}\tau^3\right] = 0.$$

This indicates that the profile of $\overline{P}(\tau,\boldsymbol{l})$ is steeper than that of the Gaussian approximation. The flatness decreases when τ increases from zero. Note that without the correction term, the profile would be predicted even steeper.

When τ is large enough, Δ_4 is negligible. The fourth cummulant changes with time as

$$\frac{d}{d\tau}C_4 \doteq \left[\int d\boldsymbol{l}\, l_i^2 l_j^2\, \overline{C}_0 - \frac{10}{3}\langle \tilde{l}_i^2(\tau)\rangle\frac{d}{d\tau}\langle \tilde{l}_j^2(\tau)\rangle\right]$$
$$= \frac{10}{3}\left\{\int_0^\tau d\tau'\left[\frac{d}{d\tau'}\langle \tilde{l}_i^2(\tau')\rangle\right]\frac{d}{d\tau}\langle \tilde{l}_j^2(\tau-\tau')\rangle - \langle \tilde{l}_i^2(\tau)\rangle\frac{d}{d\tau}\langle \tilde{l}_j^2(\tau)\rangle\right\}$$
$$+ \frac{20}{3}\int_0^\tau d\tau'\int d\boldsymbol{l}\, l_i^2 R_{jj}(\tau',\boldsymbol{l})\,\overline{P}(\tau',\boldsymbol{l})$$
$$= \frac{10}{3}\langle \tilde{l}_i^2(\tau)\rangle\left[\frac{d}{d\tau}\langle \tilde{l}_j^2(\tau-\tau^*)\rangle - \frac{d}{d\tau}\langle \tilde{l}_j^2(\tau)\rangle\right] + \frac{20}{3}\int_0^\tau d\tau'\int d\boldsymbol{l}\, l_i^2 R_{jj}(\tau',\boldsymbol{l})\,\overline{P}(\tau',\boldsymbol{l}) \quad (103)$$

where $0<\tau^*<\tau$. When τ is large enough, the first term on the righthand side of (103) is close to zero, so that

$$\frac{d}{d\tau}C_4 \doteq \frac{20}{3}\int_0^\tau d\tau' \int d\underset{\sim}{\ell}\, \ell_i^2 R_{jj}(\tau';\underset{\sim}{\ell})\,\overline{P}(\tau';\underset{\sim}{\ell}) \quad > 0$$

This indicates that, when τ is large enough, the profile of $\overline{P}(\tau,\underset{\sim}{\ell})$ becomes flatter, and finally it approaches Gaussian in the long time limit.

In principle, the behavior of the flatness for the whole time range may be studied in the following way. The moments of of $\overline{P}(\tau,\underset{\sim}{\ell})$ are related to its Fourier components through

$$\langle \hat{\underset{\sim}{\ell}}^n(\tau)\rangle = (2\pi)^3 i^{-n}\left[\frac{\partial^n}{\partial \underset{\sim}{k}^n}\overline{P}(\tau,\underset{\sim}{k})\right]\Bigg|_{\underset{\sim}{k}\to 0} \tag{104}$$

Therefore

$$\frac{d}{d\tau}\langle \ell_i^2(\tau)\rangle = -(2\pi)^3\left[\frac{\partial^2}{\partial k_i^2}\partial_\tau \overline{P}(\tau,\underset{\sim}{k})\right]\Bigg|_{\underset{\sim}{k}\to 0} \tag{105}$$

$$\frac{d}{d\tau}\langle \ell_i^2(\tau)\ell_j^2(\tau)\rangle = (2\pi)^3\left[\frac{\partial^4}{\partial k_i^2\partial k_j^2}\partial_\tau \overline{P}(\tau,\underset{\sim}{k})\right]\Bigg|_{\underset{\sim}{k}\to 0} \tag{106}$$

The Fourier transform of equation (74) is

$$\begin{aligned}\partial_\tau \overline{P}(\tau,\underset{\sim}{k}) = &-(2\pi)^3 k_i k_j \int_0^\tau d\tau' \int d\underset{\sim}{k}' R_{ij}(\tau';\underset{\sim}{k}')\,\overline{P}(\tau';\underset{\sim}{k}-\underset{\sim}{k}')\,\overline{P}(\tau-\tau';\underset{\sim}{k})\\ &+(2\pi)^9 k_i k_n \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \int_0^{\tau''} d\tau''' \iint d\underset{\sim}{k}' d\underset{\sim}{k}''(k_j - k_j')(k_m - k_m'')\\ &\cdot R_{im}(\tau-\tau'';\underset{\sim}{k}')\,R_{jn}(\tau'-\tau''';\underset{\sim}{k}'')\,\overline{P}(\tau-\tau';\underset{\sim}{k}-\underset{\sim}{k}')\\ &\cdot \overline{P}(\tau'-\tau'';\underset{\sim}{k}-\underset{\sim}{k}'-\underset{\sim}{k}'')\,\overline{P}(\tau''-\tau''';\underset{\sim}{k}-\underset{\sim}{k}'')\,\overline{P}(\tau''';\underset{\sim}{k})\end{aligned} \tag{107}$$

with $\overline{\underset{\sim}{u}}$ taken as zero. Neglecting higher order moments, we can use (105), (106) and (107) to study the evolution of the fourth moment and then the fourth cumulant when the energy spectrum is given.

To find out which property of the turbulent flow affects the non-Gaussian behavior of $\overline{P}(\tau,\underset{\sim}{\ell})$, we simply neglect the correction term. The fourth cumulant then obeys (103). The Fourier transform of (103) leads to

$$\frac{d}{d\tau}C_4 \doteq \frac{10}{3}\left\{\int_0^\tau d\tau'\left[\frac{d}{d\tau'}\langle \hat{l}_i^2(\tau')\rangle\right]\frac{d}{d\tau}\langle \hat{l}_j^2(\tau-\tau')\rangle - \langle \hat{l}_i^2(\tau)\rangle\frac{d}{d\tau}\langle \hat{l}_j^2(\tau)\rangle\right\}$$
$$-\frac{20}{3}(2\pi)^3\int_0^\tau d\tau'\int d\underset{\sim}{k}'\, R_{jj}(\tau';\underset{\sim}{k}')\left[\frac{\partial^2}{\partial k_i^2}\overline{P}(\tau';|\underset{\sim}{k}-\underset{\sim}{k}'|)\right]\Big|_{\underset{\sim}{k}\to 0}$$

Approximately, we have

$$\left[\frac{\partial^2}{\partial k_i^2}\overline{P}(\tau';|\underset{\sim}{k}-\underset{\sim}{k}'|)\right]\Big|_{\underset{\sim}{k}\to 0} \doteq (2\pi)^{-3}\left[-\sigma^2(\tau')\overline{P}(\tau';k')+k'^2\sigma^4(\tau')\overline{P}(\tau';k')\right]$$

which is exact for the Gaussian form. Therefore,

$$\frac{d}{d\tau}C_4 \doteq 10\left\{\int_0^\tau d\tau'\left[\frac{d}{d\tau'}\langle \hat{l}_i^2(\tau')\rangle\right]\frac{d}{d\tau}\langle \hat{l}_j^2(\tau-\tau')\rangle - \frac{1}{3}\langle \hat{l}_i^2(\tau)\rangle\frac{d}{d\tau}\langle \hat{l}_j^2(\tau)\rangle\right\}$$
$$-\frac{20}{3}\int_0^\tau d\tau'\sigma^4(\tau')\int d\underset{\sim}{k}'\, k'^2 R_{jj}(\tau';\underset{\sim}{k}')\overline{P}(\tau';k') \tag{108}$$

Since $\sigma^2(\tau)=\frac{1}{3}\langle \hat{l}_i^2(\tau)\rangle$, the second term on the righthand side of (108) becomes

$$-\frac{40}{27}\int_0^\tau d\tau'\langle \hat{l}_i^2(\tau')\rangle^2\int_0^\infty dk'\,k'^2 E(\tau';k')\overline{P}(\tau';k')$$

The integrand contains the vorticity spectrum. We see that the strength of the vorticity fluctuations plays an important role in the non-Gaussian behavior of $\overline{P}(\tau,\underset{\sim}{l})$. This is in agreement with the comment made by Kraichnan[38].

It may be noted that, with the correction term, the non-Gaussian behavior of $\overline{P}(\tau,\underset{\sim}{l})$ is not as severe as (103) especially when τ approaches zero.

IV RELATIVE DISPERSION OF A PAIR OF FLUID PARTICLES

4.1 Formulation of the Relative Transition

Similar to (74), the two-particle mean propagator is related to the two-particle transition function:

$$\int_0^\tau d\tau' \,\overline{U}_{12}(\tau,\tau') \cdots = \int_0^\tau d\tau' \iint d\boldsymbol{\ell}_1' \, d\boldsymbol{\ell}_2' \; \overline{P}_{12}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1',\boldsymbol{\ell}_2-\boldsymbol{\ell}_2') \cdots$$

or in the $(\boldsymbol{\ell}_1,\boldsymbol{\lambda})$ system:

$$\int_0^\tau d\tau' \,\overline{U}_{12}(\tau,\tau') \cdots = \int_0^\tau d\tau' \iint d\boldsymbol{\ell}_1' \, d\boldsymbol{\lambda}' \; \overline{P}_{12}^{\lambda}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1',\boldsymbol{\lambda}-\boldsymbol{\lambda}') \cdots \qquad (109)$$

The trajectories of the two particles are shown in Figure 1. Assuming that the initial separation of the two particles, $\boldsymbol{r}_0$, is close to zero, we use (68) and (69) to establish the governing equation for the relative transition function $\overline{B}(\tau,\boldsymbol{\lambda})$. In the $(\boldsymbol{\ell}_1,\boldsymbol{\lambda})$ system, the substitution of (109) yields the integro-differential equation

$$\begin{aligned}
(\partial_\tau + \overline{\boldsymbol{u}}\cdot\nabla_1)\overline{P}_{12}^{\lambda}(\tau,\boldsymbol{\ell}_1,\boldsymbol{\lambda})
&\doteq (\nabla_1-\nabla_\lambda)\cdot\int_0^\tau d\tau' \iint d\boldsymbol{\ell}_1' \, d\boldsymbol{\lambda}' \,\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1')\,\overline{P}_{12}^{\lambda}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1',\boldsymbol{\lambda}-\boldsymbol{\lambda}')\cdot(\nabla_1'-\nabla_\lambda')\overline{P}_{12}^{\lambda}(\tau';\boldsymbol{\ell}_1',\boldsymbol{\lambda}') \\
&+(\nabla_1-\nabla_\lambda)\cdot\int_0^\tau d\tau' \iint d\boldsymbol{\ell}_1' \, d\boldsymbol{\lambda}' \,\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1'-\boldsymbol{\lambda}'-\boldsymbol{r}_0)\,\overline{P}_{12}^{\lambda}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1',\boldsymbol{\lambda}-\boldsymbol{\lambda}')\cdot\nabla_\lambda'\overline{P}_{12}^{\lambda}(\tau';\boldsymbol{\ell}_1',\boldsymbol{\lambda}') \\
&+\nabla_\lambda\cdot\int_0^\tau d\tau' \iint d\boldsymbol{\ell}_1' \, d\boldsymbol{\lambda}' \,\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell}_1+\boldsymbol{\lambda}+\boldsymbol{r}_0-\boldsymbol{\ell}_1')\,\overline{P}_{12}^{\lambda}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1',\boldsymbol{\lambda}-\boldsymbol{\lambda}')\cdot(\nabla_1'-\nabla_\lambda')\overline{P}_{12}^{\lambda}(\tau';\boldsymbol{\ell}_1',\boldsymbol{\lambda}') \\
&+\nabla_\lambda\cdot\int_0^\tau d\tau' \iint d\boldsymbol{\ell}_1' \, d\boldsymbol{\lambda}' \,\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell}_1+\boldsymbol{\lambda}-\boldsymbol{\ell}_1'-\boldsymbol{\lambda}')\,\overline{P}_{12}^{\lambda}(\tau-\tau';\boldsymbol{\ell}_1-\boldsymbol{\ell}_1',\boldsymbol{\lambda}-\boldsymbol{\lambda}')\cdot\nabla_\lambda'\overline{P}_{12}^{\lambda}(\tau';\boldsymbol{\ell}_1',\boldsymbol{\lambda}')
\end{aligned} \qquad (110)$$

Integrating (110) with respect to $\boldsymbol{\ell}_1$, we obtain

$$\begin{aligned}
\partial_\tau \overline{B}(\tau,\boldsymbol{\lambda}) &\doteq \nabla_\lambda\cdot\int_0^\tau d\tau' \iint d\boldsymbol{\ell}\, d\boldsymbol{\lambda}' \left[\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell})-\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell}-\boldsymbol{\lambda}'-\boldsymbol{r}_0)\right]\overline{P}_{12}(\tau-\tau';\boldsymbol{\ell},\boldsymbol{\lambda}-\boldsymbol{\lambda}')\cdot\nabla_\lambda'\overline{B}(\tau';\boldsymbol{\lambda}') \\
&+\nabla_\lambda\cdot\int_0^\tau d\tau' \iint d\boldsymbol{\ell}\, d\boldsymbol{\lambda}' \left[\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell}+\boldsymbol{\lambda}-\boldsymbol{\lambda}')-\boldsymbol{R}(\tau-\tau';\boldsymbol{\ell}+\boldsymbol{\lambda}+\boldsymbol{r}_0)\right]\overline{P}_{12}(\tau-\tau';\boldsymbol{\ell},\boldsymbol{\lambda}-\boldsymbol{\lambda}')\cdot\nabla_\lambda'\overline{B}(\tau';\boldsymbol{\lambda}')
\end{aligned} \qquad (111)$$

Coordinate transformations have been used to derive (111).

The second term on the righthand side of (111) can be further transformed into

$$\underset{\sim}{\nabla}_\lambda \cdot \int_0^{\tau} d\tau' \iint d\underset{\sim}{\ell}\, d\underset{\sim}{\lambda}' \left[\underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}+\underset{\sim}{\lambda}'+\underset{\sim}{\lambda}_0) \right] \overline{P}_{12}^{\lambda}(\tau-\tau';\underset{\sim}{\ell},\underset{\sim}{\lambda}'-\underset{\sim}{\lambda}) \cdot \underset{\sim}{\nabla}_\lambda' \overline{B}(\tau';\underset{\sim}{\lambda}') \tag{112}$$

It may be noted that the transition function $\overline{P}_{12}^{\lambda}(\tau-\tau',\underset{\sim}{\ell},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}')$ in (111) and the one in (112) refer to two different transition processes. The latter is equivalent to the former after the interchange of the two particles. However, the two particles are not distinguishable between each other. We cannot distinguish the two processes either. (111) is then written into the symmetric form:

$$\partial_\tau \overline{B}(\tau,\underset{\sim}{\lambda}) = \underset{\sim}{\nabla}_\lambda \cdot \underset{\approx}{K}^{r} \cdot \left\{ \underset{\sim}{\nabla}_\lambda' \overline{B}(\tau';\underset{\sim}{\lambda}') \right\} \tag{113}$$

with

$$\underset{\approx}{K}^{r} \cdot \{\cdot\cdot\} \equiv \int_0^{\tau} d\tau' \iint d\underset{\sim}{\ell}\, d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}+\underset{\sim}{\lambda}'+\underset{\sim}{\lambda}_0) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\lambda}'-\underset{\sim}{\lambda}_0) \right] \overline{P}_{12}^{\lambda}(\tau-\tau';\underset{\sim}{\ell},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \cdots \tag{114}$$

playing the role of the relative transport property.

As mentioned in the literature[22,31], since the one-particle dispersion is dominated by eddies of large scales and the relative dispersion is dominated by eddies of small scales (when the initial separation $\underset{\sim}{\lambda}_0$ and τ are not large), the two-particle transition function in (114) may be approximately written as

$$\overline{P}_{12}^{\lambda}(\tau-\tau',\underset{\sim}{\ell},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \doteq \overline{P}(\tau-\tau';\underset{\sim}{\ell})\overline{B}(\tau-\tau';\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \tag{115}$$

by assuming the two dispersion processes are approximately independent. Therefore,

$$\underset{\approx}{K}\{\cdots\} \doteq \int_0^{\tau} d\tau' \iint d\underset{\sim}{\ell}\, d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(\tau-\tau', \underset{\sim}{\ell}) - \underset{\approx}{R}(\tau-\tau', \underset{\sim}{\ell}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) - \underset{\approx}{R}(\tau-\tau', \underset{\sim}{\ell}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0) \right] \overline{P}(\tau-\tau', \underset{\sim}{\ell}) \overline{B}(\tau-\tau', \underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \cdots \quad (116)$$

When the Eulerian velocity correlation functions are given, in principle, $\overline{P}(\tau, \underset{\sim}{\ell})$ can be solved from (63), then (113) and (116) can be used to solve for $\overline{B}(\tau, \underset{\sim}{\lambda})$.

The two-particle transition function in the transport property accounts for the interaction of the two trajectories. It will be shown that (113) and (114) provide an analytical proof for the empirical 4/3 power law of relative diffusion.

4.2 Asymptotic Cases

The second moment of $\overline{B}(\tau,\underset{\sim}{\lambda})$ is defined as

$$\langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle = \int d\underset{\sim}{\lambda}\, \underset{\sim}{\lambda}\underset{\sim}{\lambda}\, \overline{B}(\tau,\underset{\sim}{\lambda}) \qquad \left(\langle \hat{\underset{\sim}{\lambda}}(\tau)\rangle = 0 \right)$$

According to equation (113), the second moment obeys the equation:

$$\begin{aligned}
\frac{d}{d\tau}\langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle &= \int d\underset{\sim}{\lambda}\, \underset{\sim}{\lambda}\underset{\sim}{\lambda}\, \partial_\tau \overline{B}(\tau,\underset{\sim}{\lambda}) \\
&= \int d\underset{\sim}{\lambda}\, \underset{\sim}{\lambda}\underset{\sim}{\lambda}\, \nabla_\lambda \cdot \int_0^\tau d\tau' \iint d\underset{\sim}{l}\, d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0) \right] \\
&\qquad \cdot \overline{P}_{12}^{\lambda}(\tau-\tau';\underset{\sim}{l},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}')\nabla_\lambda' \overline{B}(\tau,\underset{\sim}{\lambda}') \\
&= 2\int_0^\tau d\tau' \iiint d\underset{\sim}{\lambda}\, d\underset{\sim}{l}\, d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0) \right] \\
&\qquad \cdot \overline{P}_{12}^{\lambda}(\tau-\tau';\underset{\sim}{l},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}')\overline{B}(\tau',\underset{\sim}{\lambda}') \\
&= 2\int_0^\tau d\tau' \iint d\underset{\sim}{l}\, d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0) \right] \\
&\qquad \cdot \overline{P}(\tau-\tau';\underset{\sim}{l})\overline{B}(\tau';\underset{\sim}{\lambda}')
\end{aligned} \tag{117}$$

In the derivation, we have used the property

$$\int d\underset{\sim}{\lambda}\, \overline{P}_{12}^{\lambda}(\tau-\tau';\underset{\sim}{l},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') = \overline{P}(\tau-\tau';\underset{\sim}{l})$$

and incompressibility such that

$$\nabla_j' R_{ij}(\tau-\tau';\underset{\sim}{l}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) = \nabla_j' R_{ij}(\tau-\tau';\underset{\sim}{l}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0) = 0$$

In the short time limit,

$$\begin{aligned}
\frac{d}{d\tau}\langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle &\doteq 2\left[2\underset{\approx}{R}(0,0) - \underset{\approx}{R}(0,\underset{\sim}{r}_0) - \underset{\approx}{R}(0,-\underset{\sim}{r}_0)\right] \int_0^\tau d\tau' \iint d\underset{\sim}{l}\, d\underset{\sim}{\lambda}'\, \overline{P}(\tau-\tau';\underset{\sim}{l})\overline{B}(\tau',\underset{\sim}{\lambda}') \\
&= 2\left[2\underset{\approx}{R}(0,0) - \underset{\approx}{R}(0,\underset{\sim}{r}_0) - \underset{\approx}{R}(0,-\underset{\sim}{r}_0)\right]\tau
\end{aligned} \tag{118}$$

and

$$\langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle \doteq \left[2\underset{\approx}{R}(0,0) - \underset{\approx}{R}(0,\underset{\sim}{r}_0) - \underset{\approx}{R}(0,-\underset{\sim}{r}_0)\right]\tau^2 \tag{119}$$

Since

$$\hat{\underset{\sim}{r}}(\tau) = \hat{\underset{\sim}{\lambda}}(\tau) + \underset{\sim}{r}_0$$

we find

$$\langle \hat{\underset{\sim}{r}}(\tau)\hat{\underset{\sim}{r}}(\tau)\rangle \doteq \underset{\sim}{r}_0\underset{\sim}{r}_0 + [2\underset{\approx}{R}(0,0) - \underset{\approx}{R}(0,\underset{\sim}{r}_0) - \underset{\approx}{R}(0,-\underset{\sim}{r}_0)]\tau^2$$

$$\doteq \underset{\sim}{r}_0 \cdot \underset{\sim}{r}_0 - \underset{\sim}{r}_0\underset{\sim}{r}_0 : \underset{\sim}{\nabla}\underset{\sim}{\nabla}\underset{\approx}{R}(0,0)\tau^2$$

and the variance of the separation

$$\langle \hat{r}_i^2(\tau)\rangle \doteq r_{0_i}^2 \left(1 + \frac{1}{3}\frac{\varepsilon}{\nu}\tau^2\right) \tag{120}$$

where ε is the energy dissipation rate, ν is the molecular viscosity. (119) or (120) states that the two particles move almost together at short times.

In the long time limit, the velocity correlation functions $\underset{\approx}{R}(\tau-\tau', \underset{\sim}{l}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0)$ and $\underset{\approx}{R}(\tau-\tau', \underset{\sim}{l}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0)$ can be neglected, i.e. the two particles move independently. Therefore,

$$\frac{d}{d\tau}\langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle \doteq 4\int_0^\infty d\tau' \int d\underset{\sim}{l}\, \underset{\approx}{R}(\tau';\underset{\sim}{l})\,\overline{P}(\tau';\underset{\sim}{l})$$

$$= 4\,\underset{\approx}{K}(\infty) \tag{121}$$

and

$$\langle \hat{\underset{\sim}{r}}(\tau)\hat{\underset{\sim}{r}}(\tau)\rangle \doteq \langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle \doteq 4\underset{\approx}{K}(\infty)\tau \tag{122}$$

where $\underset{\approx}{K}(\infty)$ is the one-particle asymptotic eddy diffusivity tensor.

4.3 Intermediate Times, the 4/3 Power Law

Write (113) into the integro-differential form:

$$\partial_\tau \overline{B}(\tau,\underset{\sim}{\lambda}) \doteq \underset{\sim}{\nabla}_\lambda \cdot \int_0^\tau d\tau' \iint d\underset{\sim}{\ell}\, d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0)\right]$$
$$\cdot \overline{P}_{12}^{\lambda}(\tau-\tau';\underset{\sim}{\ell},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \cdot \underset{\sim}{\nabla}_\lambda' \overline{B}(\tau',\underset{\sim}{\lambda}') \tag{123}$$

One way to simplify the analysis is to replace $\underset{\sim}{\nabla}_\lambda' \overline{B}(\tau',\lambda')$ by $\underset{\sim}{\nabla}_\lambda \overline{B}(\tau,\underset{\sim}{\lambda})$. This is equivalent to a memory cut off, implying a Markovian process. The resultant equation has a meaning essentially the same as that obtained by Kraichnan[24], based upon LHDI. The transport property becomes a function of time and the instantaneous separation. However, it is well known that the fluctuations of scales comparable to the separation dominate the relative dispersion and the separation itself is strongly affected by the dispersion history. In other words, the initial separation provides a significant influence. Such a cut off might be applied only when the diffusion reaches the stage that the separation is signifinificantly larger than the initial separation. We also note that, using the memory cut off, one cannot recover equation (117) for the second moment of $\overline{B}(\tau,\underset{\sim}{\lambda})$. In our work, the memory cut off is not considered.

There are several models proposed for the form of $\overline{B}(\tau,\underset{\sim}{\lambda})$ in the past years, such as by Richardson[2], Okubo[39], Batchelor[31]. The experiment made by Sullivan[40] illustrated that the Gaussian model suggested by Batchelor, worked better than that of Richardson.

Batchelor suggested the equation

$$\partial_\tau \overline{B}(\tau,\underset{\sim}{\lambda}) = \underset{\approx}{K}^{r}(\tau,\underset{\sim}{r}_0):\nabla_\lambda\nabla_\lambda \overline{B}(\tau,\underset{\sim}{\lambda}) \tag{124}$$

for the relative transition function, where $\underset{\approx}{K}^{r}(\tau,\underset{\sim}{r}_0)$ is the relative eddy diffusivity tensor. Following Taylor, the relative eddy diffusivity can be defined as

$$\underset{\approx}{K}^{r}(\tau,\underset{\sim}{r}_0) \equiv \frac{1}{2}\frac{d}{d\tau}\langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle \tag{125}$$

Making use of (117), we find

$$\begin{aligned}\underset{\approx}{K}^{r}(\tau,\underset{\sim}{r}_0) &= \frac{1}{2}\int d\underset{\sim}{\lambda}\,\underset{\sim}{\lambda}\underset{\sim}{\lambda}\,\partial_\tau \overline{B}(\tau,\underset{\sim}{\lambda}) \\ &\doteq \int_0^\tau d\tau' \iint d\underset{\sim}{l}\,d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}'+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) - \underset{\approx}{R}(\tau-\tau';\underset{\sim}{l}'-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0)\right] \\ &\qquad \cdot \overline{P}(\tau-\tau';\underset{\sim}{l})\overline{B}(\tau';\underset{\sim}{\lambda}')\end{aligned} \tag{126}$$

By means of Fourier transformation, (126) is turned into

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\langle \hat{\underset{\sim}{\lambda}}(\tau)\hat{\underset{\sim}{\lambda}}(\tau)\rangle &= 2\int_0^\tau d\tau' \int d\underset{\sim}{\lambda}'\,\overline{B}(\tau';\underset{\sim}{\lambda}') \int d\underset{\sim}{k}\,\underset{\approx}{R}(\tau-\tau';\underset{\sim}{k})(2\pi)^3\overline{P}(\tau-\tau';-\underset{\sim}{k}) \\ &\qquad \cdot\left\{1-\cos\left[\underset{\sim}{k}\cdot(\underset{\sim}{\lambda}'+\underset{\sim}{r}_0)\right]\right\} \\ &= 2(2\pi)^3\int_0^\tau d\tau' \int d\underset{\sim}{k}\,\underset{\approx}{R}(\tau-\tau';\underset{\sim}{k})\overline{P}(\tau-\tau';-\underset{\sim}{k})\left\{1-R_e\left[(2\pi)^3 e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_0}\overline{B}(\tau';-\underset{\sim}{k})\right]\right\}\end{aligned} \tag{127}$$

where the symbol $R_e[\ ]$ represents the real part.

For the case that the turbulence is also isotropic and the initial separation is small enough, we can write

$$\overline{B}(\tau,\lambda) = \left[2\pi\sigma_\lambda^2(\tau)\right]^{-3/2}\exp\left[-\frac{\lambda^2}{2\sigma_\lambda^2(\tau)}\right]$$

$$\overline{B}(\tau,k) = (2\pi)^{-3}\exp\left[-\frac{1}{2}k^2\sigma_\lambda^2(\tau)\right]$$

Equation (127) is reduced to

$$\frac{d}{d\tau}\sigma_\lambda^2(\tau) \doteq \frac{8}{3}\int_0^\tau d\tau' \int_0^\infty dk\,E(\tau-\tau';k)e^{-\frac{1}{2}k^2\sigma^2(\tau-\tau')}\left[1-\left(1-\frac{1}{6}k^2r_0^2\right)e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau')}\right] \tag{128}$$

Note that, in (128), $\sigma^2(\tau)$ is the variance of $\overline{P}(\tau,\underset{\sim}{l})$ and $\sigma_\lambda^2(\tau)$ is the variance of $\overline{B}(\tau,\underset{\sim}{\lambda})$.

According to (128), the velocity fluctuations of large scales (small k) do not contribute significantly when τ is not large, because of the behavior of the factor

$$\left[1-\left(1-\tfrac{1}{6}k^2 r_o^2\right)e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau')}\right]$$

On the other hand, fluctuations of small scales do not contribute much either, because of the behavior of the factor

$$e^{-\frac{1}{2}k^2\sigma^2(\tau-\tau')}$$

Therefore, when τ is not large, equation (128) automatically shows the dominance of the fluctuations of scales comparable to the separation. Neglecting r_o, we write (128) as

$$\begin{aligned}\frac{d^2}{d\tau^2}\sigma_\lambda^2(\tau) &\doteq \frac{8}{3}\int_0^\infty dk\,E(k)\left[1-e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau)}\right]\\ &\quad+\frac{8}{3}\int_0^\tau d\tau'\int_0^\infty dk\,E(k)\left[\frac{\partial}{\partial\tau}e^{-\frac{1}{2}k^2\sigma^2(\tau-\tau')}\right]\left[1-e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau')}\right]\\ &= \frac{8}{3}\int_0^\infty dk\,E(k)\left[1-e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau)}\right]\\ &\quad-\frac{8}{3}\int_0^\infty dk\,E(k)\left[1-e^{-\frac{1}{2}k^2\sigma^2(\tau)}\right]\left[1-e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau^*)}\right]\end{aligned}$$

where $0<\tau^*<\tau$. The first term on the righthand side dominates. Therefore, based upon the Kolmogoroff spectrum,

$$\frac{d^2}{d\tau^2}\sigma_\lambda^2(\tau) \doteq \frac{8}{3}C\varepsilon^{2/3}\int_0^\infty dk\,k^{-5/3}\left[1-e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau)}\right] \qquad (129)$$

Let

$$\sigma_\lambda^2(\tau)=\alpha\tau^n$$

We find

$$\begin{aligned}\alpha n(n-1)\tau^{n-2} &= \frac{8}{3}C\varepsilon^{2/3}\int_0^\infty dk\,k^{-5/3}\left[1-e^{-\frac{1}{2}\left(k\alpha^{\frac{1}{2}}\tau^{n/2}\right)^2}\right]\\ &= \frac{8}{3}C\varepsilon^{2/3}\alpha^{1/3}\tau^{n/3}\,F\end{aligned}$$

Therefore,

$$
\begin{aligned}
&n = 3 \\
&\alpha = \frac{8}{27}(CF)^{3/2}\varepsilon \\
&\sigma_\lambda^2(\tau) = \frac{8}{27}(CF)^{3/2}\varepsilon\tau^3 \doteq 0.6065\, C^{3/2}\varepsilon\tau^3 \qquad (130) \\
&K^2(\tau) \equiv \frac{1}{3}k_{ii}^2(\tau) = \frac{1}{2}\frac{d}{d\tau}\sigma_\lambda^2(\tau) = (CF)^{1/2}\varepsilon^{1/3}[\sigma_\lambda^2(\tau)]^{2/3} \\
&\langle\hat{\lambda}^2(\tau)\rangle \doteq 3.343\,\varepsilon\tau^3 \qquad (C \doteq 1.5) \qquad (131)
\end{aligned}
$$

where

$$F \equiv \int_0^\infty dk\, k^{-5/3}(1 - e^{-\frac{1}{2}k^2}) = \frac{\sqrt{3}\,\pi}{2^{1/3}\Gamma(\frac{1}{3})} \doteq 1.6121$$

(130) and (131) give the proof of the 4/3 power law.

The analytical proof for the 4/3 power law is based on approximations that the one-particle variance and the velocity fluctuations of scales other than the inertia subrange are unimportant. It will be more convincing to work with a full spectrum and with the one-particle variance as well. As a demonstration, we have calculated numerically the relative dispersion based upon the von Kármán spectrum which has the k^4 behavior at small wave numbers and $k^{-5/3}$ behavior at large wave numbers. The result seems excellent (Section 4.5).

It is also interesting to note that the Gaussian approximation retains the asymptotic behavior of the relative dispersion. In the long time limit, (128) becomes

$$
\begin{aligned}
\frac{d}{d\tau}\sigma_\lambda^2(\tau) &\doteq \frac{8}{3}\int_0^\tau d\tau' \int_0^\infty dk\, E(\tau-\tau'; k)\, e^{-\frac{1}{2}k^2\sigma^2(\tau-\tau')} \\
&= 2\frac{d}{d\tau}\sigma^2(\tau) = 4K(\infty) \equiv \frac{4}{3}k_{ii}(\infty) \qquad (132)
\end{aligned}
$$

In the short time limit,

$$\frac{d}{d\tau}\sigma_\lambda^2(\tau) \doteq \frac{8}{3}\int_0^\tau d\tau' \int_0^\infty dk\, E(k)\left[1-\left(1-\frac{1}{6}k^2 r_0^2\right)e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau')}\right]$$

$$\doteq \frac{8}{3}\int_0^\tau d\tau' \int_0^\infty dk\, E(k)\left[\frac{1}{6}k^2 r_0^2 + \frac{1}{2}k^2\sigma_\lambda^2(\tau')\right]$$

$$= \frac{4}{3}\left[\int_0^\tau d\tau' \sigma_\lambda^2(\tau') + \frac{r_0^2}{3}\tau\right]\int_0^\infty dk\, k^2 E(k)$$

$$= \frac{2}{3}\frac{\varepsilon}{\nu}\left[\int_0^\tau d\tau' \sigma_\lambda^2(\tau') + \frac{r_0^2}{3}\tau\right] \tag{133}$$

where the factor 1/6 comes from the integration over all directions. The solution of (133) is

$$\sigma_\lambda^2(\tau) \doteq \frac{r_0^2}{3}\left[\cosh\left(\sqrt{\frac{2\varepsilon}{3\nu}}\,\tau\right)-1\right]$$

or

$$\langle \hat{r}^2(\tau)\rangle = r_0^2 + 3\sigma_\lambda^2(\tau) \doteq r_0^2\left(1+\frac{\varepsilon}{3\nu}\tau^2\right) \tag{134}$$

which agrees with (120).

Our equation (117) can also be used for models other than the Gaussian approximation. The Fourier transform of (117) is

$$\frac{d}{d\tau}\langle \hat{\lambda}^2(\tau)\rangle = 8\int_0^\tau d\tau' \int d\lambda'\, \bar{B}(\tau',\lambda')\int_0^\infty dk\, E(\tau-\tau';k)\, e^{-\frac{1}{2}k^2\sigma^2(\tau-\tau')}\left[1-\frac{\sin(k\lambda')}{k\lambda'}\right] \tag{135}$$

Neglecting the contribution from $\sigma^2(\tau-\tau')$, we can write

$$\frac{d}{d\tau}\langle \hat{\lambda}^2(\tau)\rangle \doteq 8\int_0^\tau d\tau' \int d\lambda'\, \bar{B}(\tau',\lambda')\int_0^\infty dk\, E(\tau-\tau';k)\left[1-\frac{\sin(k\lambda')}{k\lambda'}\right] \tag{136}$$

For Richardson's model that

$$\bar{B}(\tau,\lambda) = a(\tau)\, e^{-b(\tau)\lambda^{2/3}} \tag{137}$$

$$\partial_\tau \bar{B}(\tau,\lambda) = \lambda^{-2}\frac{\partial}{\partial\lambda}\lambda^2 K^r(\tau,\lambda)\frac{\partial}{\partial\lambda}\bar{B}(\tau,\lambda) \tag{138}$$

with

$$a(\tau) = \frac{2^3}{3\pi^{3/2}\, 7!!}\left[b(\tau)\right]^{9/2}, \qquad b(\tau) = \frac{1}{2}\left[\frac{3\cdot 11\cdot 13}{\sigma_\lambda^2(\tau)}\right]^{1/3}$$

where $\sigma_\lambda^2(\tau)$ is the variance, and the relation between a and b satisfies the normalization, (136) yields

$$\frac{d}{d\tau}\sigma_\lambda^2(\tau) \equiv \frac{1}{3}\frac{d}{d\tau}\langle\hat{\lambda}^2(\tau)\rangle = \frac{8}{3}\int_0^\tau d\tau' a(\tau')\int d\lambda' e^{-b(\tau')\lambda'^{4/3}}\int_0^\infty dk E(k)\left[1-\frac{\sin(k\lambda')}{k\lambda'}\right] \tag{139}$$

With the Kolmogoroff spectrum, (139) becomes

$$\frac{d}{d\tau}\sigma_\lambda^2(\tau) = \frac{8}{3}C\varepsilon^{2/3}\int_0^\tau d\tau' a(\tau')\int d\lambda' \lambda'^{2/3} e^{-b(\tau')\lambda'^{4/3}}\int_0^\infty dk' k'^{-5/3}\left[1-\frac{\sin(k')}{k'}\right] \tag{140}$$

where $k' = k\lambda'$, and

$$F \equiv \int_0^\infty dk\, k^{-5/3}\left[1-\frac{\sin(k)}{k}\right] = -\frac{3}{10}\Gamma\left(-\frac{2}{3}\right) \doteq 1.2055 \tag{141}$$

With the definition of $a(\tau)$ and $b(\tau)$, we find

$$\begin{aligned}\frac{d}{d\tau}\sigma_\lambda^2(\tau) &= \frac{8}{3}\cdot 12\pi C\varepsilon^{2/3}F\int_0^\tau d\tau' a(\tau')\int_0^\infty d\Omega\, \Omega^{10} e^{-b(\tau')\Omega^2} \\ &= 24(3\cdot 11\cdot 13)^{-1/3}C\varepsilon^{2/3}F\int_0^\tau d\tau'\left[\sigma_\lambda^2(\tau')\right]^{1/3}\end{aligned} \tag{142}$$

Finally,

$$\sigma_\lambda^2(\tau) \doteq 0.5112\, C^{3/2}\varepsilon\tau^3 \doteq 0.9392\,\varepsilon\tau^3 \qquad (C \doteq 1.5)$$

$$\langle\hat{\lambda}^2(\tau)\rangle \doteq 2.82\,\varepsilon\tau^3 \tag{143}$$

$$K^{\Omega}(\tau,\lambda) \doteq 0.5843\,\varepsilon^{1/3}\lambda^{4/3} \tag{144}$$

For Okubo's model that

$$\bar{B}(\tau,\lambda) = a(\tau)e^{-b(\tau)\lambda^{4/3}} \tag{145}$$

$$\partial_\tau \bar{B}(\tau,\lambda) = \lambda^{-2}\frac{\partial}{\partial\lambda}\lambda^2 K^{\Omega}(\tau,\lambda)\frac{\partial}{\partial\lambda}\bar{B}(\tau,\lambda) \tag{146}$$

with

$$a(\tau) = \frac{[b(\tau)]^{9/4}}{3\pi\,\Gamma(9/4)} \doteq 0.09365\,[b(\tau)]^{9/4}$$

$$b(\tau) = \left[\frac{20\,\Gamma(1/4)}{77\,\Gamma(3/4)}\right]^{-2/3} [\sigma_\lambda^2(\tau)]^{-2/3} \doteq 1.1919\,[\sigma_\lambda^2(\tau)]^{-2/3} \quad (147)$$

for the normalization, where $\sigma_\lambda^2(\tau)$ is the variance. The same procedure of deriving (143) and (144) yields

$$\sigma_\lambda^2(\tau) \doteq 1.0683\,\varepsilon\tau^3 \quad (148)$$

$$\langle \hat{\lambda}^2(\tau) \rangle \doteq 3.205\,\varepsilon\tau^3 \quad (149)$$

$$K^2(\tau,\lambda) \doteq 0.4384\,\varepsilon^{2/3}\lambda^{2/3}\,\tau \quad (150)$$

Both Richardson's and Okubo's models are non-Gaussian. However, according to Sullivan's experiment, Richardson's model is worse than the Gaussian approximation. We have calculated the one-dimensional profile for Okubo's model. It is found that

$$\bar{B}(\tau,\lambda_1) \doteq 0.5034\,[\sigma_\lambda^2(\tau)]^{-1/2}\left[\Gamma\left(\tfrac{3}{2}\right) - \int_0^{1.1919\left[\frac{\lambda_1^2}{\sigma_\lambda^2(\tau)}\right]^{2/3}} dx\, x^{1/2} e^{-x}\right]$$

$$= 0.5034\,\sigma_\lambda^{-1}(\tau)\,\gamma\left\{\frac{3}{2},\ 1.1919\left[\frac{\lambda_1^2}{\sigma_\lambda^2(\tau)}\right]^{2/3}\right\}$$

which seems better than the Gaussian approximation in comparison with Sullivan's experiment (Figure 2).

It should be noted that the curve which Sullivan drew for Richardson's model was in error. The correct one-dimensional expression is

$$\bar{B}(\tau,\lambda_1) = \frac{8}{7!!\sqrt{\pi}}\, b^{3/2}\,[\sigma_\lambda^2(\tau)]^{-1/2}\, e^{-b\left[\frac{\lambda^2}{\sigma_\lambda^2(\tau)}\right]^{1/3}} \left\{ b^2\left[\frac{\lambda^2}{\sigma_\lambda^2(\tau)}\right]^{2/3} + 2b\left[\frac{\lambda^2}{\sigma_\lambda^2(\tau)}\right]^{1/3} + 2 \right\}$$

with $b = (3\cdot 11\cdot 13)^{1/3}/2$. It is plotted in Figure 2 as the darker dot-dashed line.

4.4 Galilean Invariance

A random Galilean transformation is produced by adding to each realization of the velocity field a spacially and temporally constant independent random velocity with zero mean. Such a transformation does not affect the relative motion of the fluid particles. Therefore, an approximate theory of the relative dispersion should be consistent with the Galilean invariance.

Let the added random velocity be $\tilde{\underset{\sim}{w}}$. We use the superscripts o and n to denote the coordinates and the transition functions in the old and new systems respectively. In the new system, equation (123) is written as

$$\partial_\tau \bar{B}^n(\tau,\underset{\sim}{\lambda}) = \underset{\sim}{\nabla}_\lambda \cdot \int_0^\tau d\tau' \iint d\underset{\sim}{\ell}\, d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}^n(\tau-\tau';\underset{\sim}{\ell}) - \underset{\approx}{R}^n(\tau-\tau';\underset{\sim}{\ell}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_0) - \underset{\approx}{R}^n(\tau-\tau';\underset{\sim}{\ell}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_0)\right]$$
$$\cdot \bar{P}_{12}^{\lambda n}(\tau-\tau';\underset{\sim}{\ell},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \cdot \underset{\sim}{\nabla}'_\lambda \bar{B}^n(\tau',\underset{\sim}{\lambda}') \qquad (151)$$

where the initial separation $\underset{\sim}{r}_0$ in the new system is assumed as the same as that in the old system. The Eulerian velocity correlation functions in the new system are related to that in the old system, e.g.

$$\underset{\approx}{R}^n(\tau-\tau';\underset{\sim}{\ell}) = \langle \tilde{\underset{\sim}{u}}^o[\tau-\tau';\underset{\sim}{\ell}+\tilde{\underset{\sim}{w}}\cdot(\tau-\tau')]\,\tilde{\underset{\sim}{u}}^o(0,0)\rangle + \langle \tilde{\underset{\sim}{w}}\tilde{\underset{\sim}{w}}\rangle \qquad (152)$$

However, it should be noted that the equality

$$\underset{\approx}{R}^n(\tau-\tau';\underset{\sim}{\ell}) = \underset{\approx}{R}^o(\tau-\tau';\underset{\sim}{\ell}) + \langle \tilde{\underset{\sim}{w}}\tilde{\underset{\sim}{w}}\rangle \qquad (153)$$

does not hold because of the random displacement $\tilde{\underset{\sim}{w}}\cdot(\tau-\tau')$. Since the transformation adds a constant velocity to an individual realization, the transformation may be considered as made by the movement of the observer. In the old system, the observer does not move, but in the new system the observer moves with a constant velocity in an individual realization, which is random in different realizations. Recalling that the velocities of the particles should be taken along the trajectories in the integration, one may consider that the corresponding velocities in the new system along the new trajectories are essentially the same as that in the old system along the old trajectories except the added velocity. Therefore, the righthand side of (151) can be expressed in terms of the velocity correlation functions of the old system and $\langle \tilde{\underset{\sim}{w}}\tilde{\underset{\sim}{w}} \rangle$ by replacing the superscript of the two-particle transition function:

$$\partial_\tau \overline{B}^n(\tau,\underset{\sim}{\lambda}) = \underset{\sim}{\nabla}_\lambda \cdot \int_0^\tau d\tau' \iint d\underset{\sim}{l}\, d\underset{\sim}{\lambda}' \Big[2\underset{\approx}{R}^o(\tau-\tau';\underset{\sim}{l}) + 2\langle \tilde{\underset{\sim}{w}}\tilde{\underset{\sim}{w}} \rangle - \underset{\approx}{R}^o(\tau-\tau';\underset{\sim}{l}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_o) - \langle \tilde{\underset{\sim}{w}}\tilde{\underset{\sim}{w}} \rangle$$

$$- \underset{\approx}{R}^o(\tau-\tau';\underset{\sim}{l}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_o) - \langle \tilde{\underset{\sim}{w}}\tilde{\underset{\sim}{w}} \rangle \Big] \overline{P}_{12}^{\lambda o}(\tau-\tau';\underset{\sim}{l},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \cdot \underset{\sim}{\nabla}'_\lambda \overline{B}^n(\tau';\underset{\sim}{\lambda}')$$

$$= \underset{\sim}{\nabla}_\lambda \cdot \int_0^\tau d\tau' \iint d\underset{\sim}{l}\, d\underset{\sim}{\lambda}' \Big[2\underset{\approx}{R}^o(\tau-\tau';\underset{\sim}{l}) - \underset{\approx}{R}^o(\tau-\tau';\underset{\sim}{l}+\underset{\sim}{\lambda}'+\underset{\sim}{r}_o) - \underset{\approx}{R}^o(\tau-\tau';\underset{\sim}{l}-\underset{\sim}{\lambda}'-\underset{\sim}{r}_o) \Big]$$

$$\cdot \overline{P}_{12}^{\lambda o}(\tau-\tau';\underset{\sim}{l},\underset{\sim}{\lambda}-\underset{\sim}{\lambda}') \underset{\sim}{\nabla}'_\lambda \overline{B}^n(\tau';\underset{\sim}{\lambda}') \quad (154)$$

This equation is the same as the one in the old system. Therefore the Galilean invariance has been retained.

In Kraichnan's LHDI, the Galilean invariance was kept by intuitively interchanging the variables in respective integrals . In the recent article by Lundgren, the invariance was considered by the argument that the one-particle transition process should not appear in the relative transport property. Such an argument is rather doubtful. In fact, the one-particle transition process is intrinsically involved. For example, according to Taylor, the two-time two-particle Lagrangian velocity correlation function $\langle \tilde{\underset{\sim}{v}}_1(t)\tilde{\underset{\sim}{v}}_2(t')\rangle$ plays an important part in the relative dispersion. Suppose $t>t'$. The displacement of particle one in $t-t'$ for a forward transition does not concern the transition of the second particle. The one-particle transition process is thus implied.

4.5 Comparison With Other Theories

It is worthwhile to compare our equation (123), or (116), and the relative eddy diffusivity (125) with that of other theories. In recent years, analytical studies on the same problem were made by Roberts[22], Kraichnan[24], Knobloch[25], Lundgren[26], and Mikkelsen[41]. For convenience, alterations are made in the notation and symbols.

Roberts applied DIA and obtained an integro-differential equation for the relative transition function. The equation is similar to our equation (123). However, instead of the two-particle transition for the relative transport property, there is a product of two one-particle transition functions. As the consequence, by the memory cut off, the relative eddy diffusivity in the equation

$$\partial_\tau \overline{B}(\tau,\underset{\sim}{\lambda}) = \underset{\sim}{\nabla}_\lambda \cdot \underset{\approx}{K}^{R}(\tau,\underset{\sim}{\lambda}) \cdot \underset{\sim}{\nabla}_\lambda \overline{B}(\tau,\underset{\sim}{\lambda}) \tag{156}$$

was found as

$$\underset{\approx}{K}^{R}(\tau,\underset{\sim}{\lambda}) = 2\int_0^{\tau} d\tau' \int d\underset{\sim}{\ell}\, [\underset{\approx}{R}(\tau',\underset{\sim}{\ell}) - \underset{\approx}{R}(\tau',\underset{\sim}{\lambda}+\underset{\sim}{\lambda}_0-\underset{\sim}{\ell})]\, \overline{P}(\tau',\underset{\sim}{\ell}) \tag{157}$$

(157) is dominated by fluctuations of large scales, so that the 4/3 power law was not recovered.

Kraichnan applied LHDI and derived an equation which is the same as (156), with the relative eddy diffusivity as

$$\underset{\approx}{K}^{R}(\tau,\underset{\sim}{\lambda}) = 2\int_0^{\tau} d\tau' \, [\langle \underset{\sim}{\tilde{u}}(\tau,\underset{\sim}{\ell})\underset{\sim}{\tilde{u}}(\tau,\underset{\sim}{\ell}|\tau')\rangle - \langle \underset{\sim}{\tilde{u}}(\tau,\underset{\sim}{\ell}+\underset{\sim}{\lambda})\underset{\sim}{\tilde{u}}(\tau,\underset{\sim}{\ell}|\tau')\rangle] \tag{158}$$

The velocity correlation functions in (158) are of the Lagrangian-Eulerian hybrid type. For example,

$$\langle \tilde{u}(\tau, \ell+\lambda)\tilde{u}(\tau, \ell|\tau')\rangle$$

is the correlation between a Eulerian velocity fluctuation at the space-time point $(\tau, \ell+\lambda)$ and the Lagrangian velocity fluctuation of a particle at time τ', which would be located at the position ℓ at τ. Such a velocity correlation function is difficult to treat in practice.

Knobloch noted the complexity of the velocity correlation functions in Kraichnan's formula. The unperturbed propagator U^o was used to relate the velocity correlation functions to the Eulerian. As described previously, the unperturbed propagator does not provide a well behaved series for the transport equation. Therefore, Knobloch did not give a compact form for the relative eddy diffusivity.

Lundgren independently derived approximate formulas (156) and (158). The main effort was devoted to express the velocity correlation functions in (158) in terms of one-time Eulerian velocity correlation functions, and to get rid of the one-particle dispersion process in the eddy diffusivity for the Galilean invariance. Besides the fact that (158) implies the one-particle dispersion process, his result is still too complicated to use:

$$K^{r}(\tau, \lambda) = \int_0^\tau d\tau' \int d\ell \, [R(\tau, \ell) - R(\tau, \ell+\lambda)] \, Q(\tau, \ell|\tau') \qquad (159)$$

$$Q(\tau, \ell|\tau') \equiv -\frac{1}{4\pi} \nabla_\ell \cdot \int d\ell' \left[\nabla_{\ell'} \cdot \frac{1}{|\ell'|}\right] \overline{B}(\tau-\tau', \ell-\ell')$$

Mikkelsen studied the relative diffusion of a Gaussian puff. The puff was assumed strictly Gaussian in any individual realization. Despite the strong constraint, the relative eddy diffusivity has an expression similar to ours:

$$\frac{d}{d\tau}\sigma_\lambda^2(\tau) = \frac{8}{3}\int_0^\infty dk\, E(k)\left[1 - e^{-\frac{1}{2}k^2\sigma_\lambda^2(\tau)}\right]\int_0^\tau d\tau'\, \rho_L(\tau') \tag{160}$$

where

$$\rho_L(\tau) = \frac{\langle \tilde{V}_1(\tau)\,\tilde{V}_1(0)\rangle}{\langle \tilde{V}_1^2\rangle}$$

is the Lagrangian velocity correlation coefficient. Equation (160) does not have a time convolution as in (128). Also, the one-particle Lagrangian velocity correlation coefficient is to be found empirically.

4.6 Dispersion in Turbulence With the Von Kármán Spectrum

To illustrate the dispersion of fluid particles, a numerical calculation is made with the von Kármán spectrum:

$$E(k) = \Lambda k_e^4 \frac{(k/k_e)^4}{[1+(k/k_e)^2]^{17/6}} \tag{161}$$

where Λ is the Loitsianski integral, k_e has the dimension of a wave number. This spectrum has the advantage that both the empirical behavior, k^4 at small wave numbers and $k^{-5/3}$ in the inertia subrange are included.

For simplicity, the variables are non-dimensionized by

$$\tau'' \equiv (\Lambda k_e^7)^{1/2}\tau \ , \quad k'' \equiv k/k_e \ , \quad \sigma^{2''}(\tau'') \equiv k_e^2\sigma^2(\tau)$$

$$\sigma_\Lambda^{2''}(\tau'') \equiv k_e^2\sigma_\Lambda^2(\tau) \ , \quad K_{ii}''(\tau'') \equiv k_e^2 K_{ii}(\tau)/(\Lambda k_e^7)^{1/2}$$

but the superscript " will be dropped hereafter.

The calculation is made upon the Gaussian approximation. The mean velocity is assumed zero. For the one-particle dispersion process, we use the equation

$$\frac{d}{d\tau}\sigma^2(\tau) = \frac{4}{3}\int_0^\tau d\tau' \int_0^\infty dk \frac{k^4}{(1+k^2)^{17/6}} \exp\left[-\tfrac{1}{2}k^2\sigma^2(\tau')\right] \tag{162}$$

to calculate the variance $\sigma^2(\tau)$. Then the eddy diffusivity is calculated according to

$$K(\tau) = \frac{1}{3}K_{ii}(\tau) = \frac{1}{2}\frac{d}{d\tau}\sigma^2(\tau) \tag{163}$$

For the relative dispersion, we use the non-dimensionized equation (128):

$$\frac{d}{d\tau}\sigma_r^2(\tau) = \frac{8}{3}\int_0^{\tau} d\tau' \int_0^{\infty} dk \frac{k^4}{(1+k^2)^{17/6}} e^{-\frac{1}{2}k^2\sigma^2(\tau-\tau')}\left[1 - e^{-\frac{1}{2}k^2\sigma_r^2(\tau')}\right] \tag{164}$$

Note that the initial separation of the two particles has been assumed close to zero, so that

$$\sigma_r^2(\tau) \doteq \sigma_x^2(\tau)$$

The relative eddy diffusivity is calculated according to

$$K^r(\tau) \equiv \frac{1}{3} K_{ii}^r(\tau) = \frac{1}{2}\frac{d}{d\tau}\sigma_r^2(\tau) \tag{165}$$

The results are shown in Figure 4 and 5. The general properties of the dispersion processes agree with previous analyses. In particular, the growth of the variance of the separation, $\sigma_r^2(\tau)$, shows a τ^3 slope, which is consistent with the 4/3 power law and is what we expected.

V LARGRANGIAN-EULERIAN TRANSFORMATION

Following Taylor, the eddy diffusivity may be defined as the time derivative of the second central moment of the displacement of a particle, which can be expressed as the time integration of the Lagrangian velocity correlation function. For practical reasons, efforts were made to relate the Lagragian correlation to the Eulereian correlations. For a fluid particle, the transformation is usually made by a delta function:

$$\langle \tilde{\underset{\sim}{v}}(t)\hat{\underset{\sim}{v}}(t_0)\rangle = \int d\underset{\sim}{x}' \langle \tilde{\underset{\sim}{u}}(t,\underset{\sim}{x})\,\delta[\underset{\sim}{x}'-\hat{\underset{\sim}{x}}(t_0)]\,\tilde{\underset{\sim}{u}}(t_0,\underset{\sim}{x}')\rangle \tag{166}$$

Making use of (2), we can write it as

$$\begin{aligned}\langle \tilde{\underset{\sim}{v}}(t)\hat{\underset{\sim}{v}}(t_0)\rangle &= \int d\underset{\sim}{x}' \langle \tilde{\underset{\sim}{u}}(t,\underset{\sim}{x})\,\hat{P}(t-t_0,\underset{\sim}{x}-\underset{\sim}{x}')\,\tilde{\underset{\sim}{u}}(t_0,\underset{\sim}{x}')\rangle \\ &= \int d\underset{\sim}{x}' \langle \tilde{\underset{\sim}{u}}(t,\underset{\sim}{x})\,\overline{P}(t-t_0,\underset{\sim}{x}-\underset{\sim}{x}')\,\tilde{\underset{\sim}{u}}(t_0,\underset{\sim}{x}')\rangle \\ &\quad + \int d\underset{\sim}{x}' \langle \tilde{\underset{\sim}{u}}(t,\underset{\sim}{x})\,\tilde{P}(t-t_0,\underset{\sim}{x}-\underset{\sim}{x}')\,\tilde{\underset{\sim}{u}}(t_0,\underset{\sim}{x}')\rangle \\ &= \langle \tilde{\underset{\sim}{u}}\,\overline{P}\,\vartriangle \hat{\underset{\sim}{u}}\rangle + \langle \tilde{\underset{\sim}{u}}\,\tilde{P}\,\vartriangle \hat{\underset{\sim}{u}}\rangle\end{aligned} \tag{167}$$

where $\vartriangle$ denotes the spatial integration. The neglect of the second term on the righthand side of (167) is known as Corrsin's independence approximation. This approximation has been applied not only to the Lagrangian-Eulerian transformation , but also to the neglect of the mode-coupling in diffusion equations[22,26]. Weinstock[43] discussed the validity of the independence approximation. However, only the correlations to third order was taken into account. In deriving transition equations, mode-coupling can not be neglected without considering the effect of the collective phenomena. It is necessary to study more carefully the validity of the independence approximation and to explain that it does not apply to the diffusion equations. Besides, it is obvious that the Lagrangian-Eulerian transformation requires the determination of the transition function. The results in the previous sections may be applied to the problem.

In (167), the second integral also represents mode-coupling, which provides the error term for the independence approximation. To study the validity, we write (28) as

$$\tilde{p} = -\bar{U}*\tilde{L}\bar{P} - \tilde{U}*\tilde{L}\bar{P} + \bar{U}*\langle\tilde{L}\tilde{p}\rangle + \tilde{U}*\langle\tilde{L}\tilde{p}\rangle \tag{168}$$

The procedure of expending $\bar{h}_2\bar{P} = \langle\tilde{L}\tilde{U}*\tilde{L}\rangle\bar{P}$ can be used analogously to expand the integrand of the second integral on the righthand side of (167):

$$\bar{\mathbf{e}} \equiv \langle\tilde{\mathbf{u}}\tilde{p}\triangle\tilde{\mathbf{u}}\rangle \tag{169}$$

The result is

$$\bar{\mathbf{e}} = \sum_{i=3}^{\infty} \bar{\mathbf{e}}_i \tag{170}$$

with

$$\bar{\mathbf{e}}_3 = -\langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle$$

$$\bar{\mathbf{e}}_4 = \langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle - \langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle$$

$$\begin{aligned}\bar{\mathbf{e}}_5 = &-\langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle + \langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle \\ &+ \langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle + \langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{e}}_6 = &\langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle - \langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle \\ &-\langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\rangle\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle - \langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle \\ &-\langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{U}*\tilde{L}\bar{P}\triangle\tilde{\mathbf{u}}\rangle - \langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle \\ &-\langle\tilde{\mathbf{u}}\bar{U}*\tilde{L}\bar{U}*\tilde{L}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle \\ &+2\langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle + \langle\tilde{\mathbf{u}}\bar{U}*\langle\tilde{L}\bar{U}*\langle\tilde{L}\bar{U}*\tilde{L}\rangle\bar{U}*\tilde{L}\rangle\bar{P}\triangle\tilde{\mathbf{u}}\rangle\end{aligned}$$

$$\bar{\mathbf{e}}_7 = \cdots\cdots$$

The series $\{\Delta_i\}$ differs from $\{\underset{\approx}{e}_i\}$ in that the latter lacks the respective terms for the collective collision. However, this difference is actually trivial. All the absent terms are zero for homogeneous turbulence. For example,

$$\begin{aligned}\langle \hat{\underset{\sim}{u}} \bar{U} * \hat{L} \rangle \bar{U} * \langle \hat{L} \bar{P} \cdot \hat{\underset{\sim}{u}} \rangle &= \langle \hat{\underset{\sim}{u}} \bar{U} * \hat{L} \rangle \bar{U} * \underset{\sim}{\nabla}'' \cdot \int d\underset{\sim}{x}''' \langle \hat{\underset{\sim}{u}}(t'', \underset{\sim}{x}'') \bar{P}(t''-t_0, \underset{\sim}{x}''-\underset{\sim}{x}''') \hat{\underset{\sim}{u}}(t_0, \underset{\sim}{x}''') \rangle \\ &= \langle \hat{\underset{\sim}{u}} \bar{U} * \hat{L} \rangle \bar{U} * \underset{\sim}{\nabla}'' \cdot \int d\underset{\sim}{x}''' \underset{\approx}{R}(t''-t_0, \underset{\sim}{x}''-\underset{\sim}{x}''') \bar{P}(t''-t_0, \underset{\sim}{x}''-\underset{\sim}{x}''') \\ &= \langle \hat{\underset{\sim}{u}} \bar{U} * \hat{L} \rangle \bar{U} * \underset{\sim}{\nabla}'' \cdot \int d\underset{\sim}{x}''' \underset{\approx}{R}(t''-t_0, \underset{\sim}{x}''') \bar{P}(t''-t_0, \underset{\sim}{x}''') \\ &= 0 \end{aligned} \tag{171}$$

In fact, all the terms having the form

$$\langle \hat{\underset{\sim}{u}} \bar{U} * \hat{L} \cdots \hat{L} \rangle \bar{U} * \langle \hat{L} \bar{U} * \hat{L} \cdots \hat{L} \bar{P} \cdot \hat{\underset{\sim}{u}} \rangle \tag{172}$$

are zero because of homogeneity. These terms coprrespond to the collective phenomena. Therefore, we may treat the series $\{\underset{\approx}{e}_i\}$ in a way similar to that for the series $\{\hat{\Delta}_i\}$.

In the short time limit, $t-t_0 \doteq 0$, the contribution of $\overline{\underset{\approx}{e}}$ is negligible. All $\overline{\underset{\approx}{e}}_i$ terms are approximately zero. For example,

$$\langle \tilde{u}_i \bar{U} * \hat{L} \bar{P} \cdot \tilde{u}_j \rangle \doteq \langle \tilde{u}_{1i} \bar{U} * \tilde{u}_{1m} \nabla_m' \bar{P} \cdot \tilde{u}_{1j} \rangle$$

where $\tilde{\underset{\sim}{u}}_1 = \tilde{\underset{\sim}{u}}(t, \underset{\sim}{x})$. It is shown that

$$\begin{aligned}\langle \tilde{u}_i \bar{U} * \hat{L} \bar{P} \cdot \tilde{u}_j \rangle &\doteq \langle \tilde{u}_{1i} \tilde{u}_{1m} \tilde{u}_{1j} \rangle \int_{t_0}^{t} dt' \iint d\underset{\sim}{x}' d\underset{\sim}{x}'' \bar{P}(t-t', \underset{\sim}{x}-\underset{\sim}{x}') \nabla_m' \bar{P}(t'-t_0, \underset{\sim}{x}'-\underset{\sim}{x}'') \\ &= \langle \tilde{u}_{1i} \tilde{u}_{1m} \tilde{u}_{1j} \rangle \int_{t_0}^{t} dt' \iint d\underset{\sim}{x}' d\underset{\sim}{x}'' \bar{P}(t-t', \underset{\sim}{x}-\underset{\sim}{x}') \nabla_m'' \bar{P}(t'-t_0, \underset{\sim}{x}'') \\ &= 0 \end{aligned} \tag{173}$$

In the long time limit, the approximations a), b) and c) in Section 2.4 can be applied analogously, i. e.

$$\overline{\underset{\approx}{e}} \doteq \overline{\underset{\approx}{e}}_a \doteq \langle \overbrace{\hat{\underset{\sim}{u}} \bar{U} * \underbrace{\hat{L} \bar{U} * \hat{L}}} \bar{P} \cdot \hat{\underset{\sim}{u}} \rangle \tag{174}$$

Similar to the truncation made for the $\{\Delta_i\}$ series, (174) may be considered as a good approximation for the whole time range, such that

$$\langle \hat{\underset{\sim}{v}}(t)\hat{\underset{\sim}{v}}(t_0)\rangle \doteq \int d\underset{\sim}{x}'\underset{\approx}{R}(t-t_0,\underset{\sim}{x}-\underset{\sim}{x}')\overline{P}(t-t_0,\underset{\sim}{x}-\underset{\sim}{x}') + \overline{\underset{\approx}{e}}_4 \qquad (175)$$

And the integral

$$\overline{\underset{\approx}{e}}_4 \doteq \langle \hat{\underset{\sim}{u}}\,\overline{U}*\hat{L}\overline{U}*\hat{L}\overline{P}\triangle\hat{\underset{\sim}{u}}\rangle \qquad (176)$$

is taken as the correction of the independence approximation.

According to (145), the correction is negligible in the short time limit. It is also negligible in the long time limit. The operator in (176), $\overline{U}*\hat{L}\overline{U}*\hat{L}$, may be estimated as having the order

$$\mathcal{R}^2 = \frac{\langle \hat{\underset{\sim}{u}}^2\rangle \tau_L^2}{\mathcal{L}^2}$$

where $\mathcal{R}$ is the parameter introduced in Section 2.4. Because of the structure of the pair correlations in (176), we have

$$\overline{\underset{\approx}{e}}_4 \ll \mathcal{R}^2\langle \hat{\underset{\sim}{u}}\,\overline{P}\triangle\hat{\underset{\sim}{u}}\rangle < \int d\underset{\sim}{x}'\langle \hat{\underset{\sim}{u}}(t,\underset{\sim}{x})\overline{P}(t-t_0,\underset{\sim}{x}-\underset{\sim}{x}')\hat{\underset{\sim}{u}}(t_0,\underset{\sim}{x}')\rangle \qquad (177)$$

in the long time limit.

To estimate the correction in the whole time range, we write

$$\begin{aligned}\overline{e}_{4ij} &= \langle \hat{u}_i\overline{U}*\hat{L}\overline{U}*\hat{L}\overline{P}\triangle\hat{u}_j\rangle \\ &= \int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\iiint d\underset{\sim}{x}'d\underset{\sim}{x}''d\underset{\sim}{x}''' \\ &\quad\cdot\langle \hat{u}_i(t,\underset{\sim}{x})\overline{P}(t-t',\underset{\sim}{x}-\underset{\sim}{x}')\nabla_m'\hat{u}_m(t',\underset{\sim}{x}')\overline{P}(t'-t'',\underset{\sim}{x}'-\underset{\sim}{x}'')\hat{u}_n(t'',\underset{\sim}{x}'')\overline{P}(t''-t_0,\underset{\sim}{x}''-\underset{\sim}{x}''')\hat{u}_j(t_0,\underset{\sim}{x}''')\rangle \\ &= \int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\iiint d\underset{\sim}{x}'d\underset{\sim}{x}''d\underset{\sim}{x}''' \\ &\quad\cdot R_{in}(t-t'',\underset{\sim}{x}-\underset{\sim}{x}'')R_{mj}(t'-t_0,\underset{\sim}{x}'-\underset{\sim}{x}''')\overline{P}(t-t',\underset{\sim}{x}-\underset{\sim}{x}')\nabla_m'\overline{P}(t'-t'',\underset{\sim}{x}'-\underset{\sim}{x}'')\nabla_n''\overline{P}(t''-t_0,\underset{\sim}{x}''-\underset{\sim}{x}''')\end{aligned} \qquad (178)$$

After coordinate transformations, it is found that

$$\overline{e}_{4ij} = \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \iiint d\underset{\sim}{x}'\, d\underset{\sim}{x}''\, d\underset{\sim}{x}'''$$
$$\cdot R_{in}(t-t'; \underset{\sim}{x}') R_{mj}(t''-t_0, \underset{\sim}{x}''+\underset{\sim}{x}''') \overline{P}(t-t'; \underset{\sim}{x}-\underset{\sim}{x}') \nabla''_m \overline{P}(t'-t''; \underset{\sim}{x}'-\underset{\sim}{x}'') \nabla'''_n \overline{P}(t''-t_0, \underset{\sim}{x}''') \tag{179}$$

The Fourier transformation yields

$$\overline{e}_{4ij} = -(2\pi)^9 \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \iint d\underset{\sim}{k}\, d\underset{\sim}{k}'$$
$$\cdot k_m k_n' R_{in}(t-t'; \underset{\sim}{k}) R_{mj}(t''-t_0, \underset{\sim}{k}') \overline{P}(t-t'; -\underset{\sim}{k}) \overline{P}(t'-t''; -\underset{\sim}{k}-\underset{\sim}{k}') \overline{P}(t''-t_0, -\underset{\sim}{k}') \tag{180}$$

For the case that the turbulence is also isotropic and P is approximately Gaussian, we have

$$\overline{e}_4 \equiv \frac{1}{3}\overline{e}_{4ii} = -\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \int_0^\infty dk \int_0^\infty dk' \; \frac{2E(t-t'; k)E(t''-t_0, k')}{k k' \sigma^4(t'-t'')}$$
$$\cdot \exp\left\{-\frac{1}{2}k^2[\sigma^2(t-t') + \sigma^2(t'-t'')] - \frac{1}{2}k'^2[\sigma^2(t'-t'') + \sigma^2(t''-t_0)]\right\}$$
$$\cdot \left\{\sinh[k k' \sigma^2(t'-t'')] - \frac{3\cosh[k k' \sigma^2(t'-t'')]}{k k' \sigma^2(t'-t'')} + \frac{3\sinh[k k' \sigma^2(t'-t'')]}{k^2 k'^2 \sigma^4(t'-t'')}\right\} \tag{181}$$

(180) and (181) are equivalent to the respective terms of the equation for the one-particle transition function. Similarly, it also has a negative contribution to the Lagrangian velocity correlation function. The magnitude of

may be found with a given spectrum. Based upon the spectrum used by Roberts[22] and Kraichnan[38]:

$$E(t, k) = 2v_1^2 k^4 (\sqrt{\pi}\, k_0^5)^{-1} \exp\left[-k^2(k_0^{-2} + \frac{1}{2}v_1^2 t^2)\right] \tag{182}$$

where $v_1^2 = \frac{1}{3}\langle \tilde{\underset{\sim}{u}}^2 \rangle$, and k_0^{-1} is an outer length scale. The calculation shows that the maximum magnitude of the correction is only five per cent of that of the main term. It seems that Corrsin's approximation works fairly well in the whole time range.

Instead of $\overline{\underset{\approx}{e}}_4$, Weinstock discussed the validity of the independence approximation based upon

$$\overline{\underset{\approx}{e}}_3 = -\langle \hat{\underset{\sim}{u}} \overline{U} * \hat{L} \overline{P}^{\Delta} \hat{\underset{\sim}{u}} \rangle$$

However, the magnitude of $\overline{\underset{\approx}{e}}_4$ is generally larger than $\overline{\underset{\approx}{e}}_3$.[36]

It should be noted that mode-coupling in the equation of the transition function is different from that in the Lagrangian-Eulerian transformation. In the former, mode-coupling $\langle \hat{L} \tilde{U} * \hat{L} \rangle$ in

$$\overline{h}_2 \overline{P} = \langle \hat{L} \hat{U} * \hat{L} \rangle \overline{P} = \underset{\sim}{\nabla} \cdot \langle \hat{\underset{\sim}{v}} \hat{U} * \hat{\underset{\sim}{v}} \rangle \cdot \underset{\sim}{\nabla}' \overline{P}$$

is an operator acting on $\underset{\sim}{\nabla}'\overline{P}$, in the latter, it is simply a physical quantity. If we write

$$\underset{\sim}{\nabla} \cdot \langle \hat{\underset{\sim}{v}} \hat{U} * \hat{\underset{\sim}{v}} \rangle \cdot \underset{\sim}{\nabla}' \overline{P} = \underset{\sim}{\nabla} \cdot \langle \hat{\underset{\sim}{v}} \hat{U} * \hat{\underset{\sim}{v}} \cdot \underset{\sim}{\nabla}' \overline{P} \rangle$$

the coupling may be considered as between $\hat{U}$ and $\hat{\underset{\sim}{v}} \cdot \underset{\sim}{\nabla}'\overline{P}$. Therefore, the mode-coupling in the two cases has different physical meanings. One should be cautious in applying the independence approximation to cases different from the Lagrangian-Eulerian transformation.

VI DIFFUSION OF A PUFF

6.1 Characteristics and Features of a Puff

Consider a puff of a passive quantity released in a turbulent fluid flow at an instant. By passive, we mean that the quantity is source-free, non-reactive, and its spreading does not affect the turbulent velocity field.

Let the normalized concentration distribution of the puff be $\hat{\psi}(t,\mathbf{x})$. At t_o, it is known as $\psi(t_o,\mathbf{x})$. When $t>t_o$, $\hat{\psi}(t,x)$ fluctuats because of the random motion of the fluid flow. The most important characteristics of $\hat{\psi}(t,\mathbf{x})$ are

Nomalization $\quad \int d\mathbf{x}\, \hat{\psi}(t,\mathbf{x}) = 1 \quad (183)$

Instantaneous centroid $\quad \int d\mathbf{x}\, \mathbf{x}\, \hat{\psi}(t,\mathbf{x}) = \hat{\mathbf{C}}(t) \quad (184)$

Instantaneous second central moment

$$\int d\mathbf{x}\,[\mathbf{x}-\hat{\mathbf{C}}(t)][\mathbf{x}-\hat{\mathbf{C}}(t)]\,\hat{\psi}(t,\mathbf{x}) = \hat{\mathbf{\Sigma}}^{\Omega}(t) \quad (185)$$

where the superscript Ω means that $\hat{\mathbf{\Sigma}}^{\Omega}(t)$ is relative to $\hat{\mathbf{C}}(t)$, the instantaneous centroid. The ensemble averaged concentration distribution is

$$\overline{\psi}(t,\mathbf{x}) = \langle \hat{\psi}(t,\mathbf{x}) \rangle \quad (186)$$

which similarly has the characteristics

Normalization $\quad \int d\mathbf{x}\, \overline{\psi}(t,\mathbf{x}) = 1 \quad (187)$

Ensemble averaged centro $\int d\mathbf{x}\, \overline{\psi}(t,\mathbf{x}) = 1 \ldots) = \overline{\mathbf{C}}(t) \quad (188)$

Ensemble averaged second central moment

$$\int d\mathbf{x}\,[\mathbf{x}-\overline{\mathbf{C}}(t)][\mathbf{x}-\overline{\mathbf{C}}(t)]\,\overline{\psi}(t,\mathbf{x}) = \overline{\mathbf{\Sigma}}^{a}(t) \quad (189)$$

where the superscript a means that $\overline{\underset{\approx}{\Sigma}}^{a}(t)$ is relative to $\overline{\underset{\sim}{C}}(t)$, the ensemble averaged centroid.

Note that the ensemble average of (185) also defines a mean second central moment

$$\overline{\underset{\approx}{\Sigma}}^{r}(t) = \langle \hat{\underset{\approx}{\Sigma}}^{r}(t) \rangle \tag{190}$$

$\overline{\underset{\sim}{C}}(t)$ represents the mean position of the puff in a fixed frame. Relative to $\overline{\underset{\sim}{C}}(t)$, $\overline{\underset{\approx}{\Sigma}}^{a}(t)$ describes the spreading of the puff in the fixed frame, so that it refers to the absolute diffusion. On the other hand, $\hat{\underset{\sim}{C}}(t)$ represents the instantaneous position of the puff. Relative to $\hat{\underset{\sim}{C}}(t)$, $\overline{\underset{\approx}{\Sigma}}^{r}(t)$ describes the spreading of the puff in a frame moving with the instantaneous centroid. Therefore, $\overline{\underset{\approx}{\Sigma}}^{r}(t)$ refers to the relative diffusion.

The traces of the mean second central moments are used to define the puff sizes:

$$\sigma_{pa}^{2}(t) = \frac{1}{3} \overline{\Sigma}_{ii}^{a}(t) \tag{191}$$

$$\sigma_{pr}^{2}(t) = \frac{1}{3} \overline{\Sigma}_{ii}^{r}(t) \tag{192}$$

$\sigma_{pa}(t)$ is called the absolute size which specifies the envelop of the trajectory of the instantaneous centroid.

$\sigma_{pr}(t)$ is called the relative size, specifying the mixing of the quantity with the fluid. It should be noted that the puff size, used in practice, customarily refers to the relative size. In the following, we shall simply refer to the relative size as the puff size. The meanings of the two sizes are shown in Figure 5.

Write

$$\underset{\approx}{\Sigma}^{a}(t)=\int d\underset{\sim}{x}\,[\underset{\sim}{x}-\bar{\underset{\sim}{\xi}}(t)][\underset{\sim}{x}-\bar{\underset{\sim}{\xi}}(t)]\,\bar{\psi}(t,\underset{\sim}{x})=\int d\underset{\sim}{x}\,\underset{\sim}{x}\underset{\sim}{x}\,\bar{\psi}(t,\underset{\sim}{x})-\bar{\underset{\sim}{\xi}}(t)\bar{\underset{\sim}{\xi}}(t) \qquad (193)$$

and

$$\begin{aligned}\underset{\approx}{\Sigma}^{r}(t) &= \left\langle \int d\underset{\sim}{x}\,[\underset{\sim}{x}-\hat{\underset{\sim}{\xi}}(t)][\underset{\sim}{x}-\hat{\underset{\sim}{\xi}}(t)]\,\hat{\psi}(t,\underset{\sim}{x})\right\rangle \\ &= \int d\underset{\sim}{x}\,\underset{\sim}{x}\underset{\sim}{x}\,\bar{\psi}(t,\underset{\sim}{x}) - \langle \hat{\underset{\sim}{\xi}}(t)\hat{\underset{\sim}{\xi}}(t)\rangle \\ &= \underset{\approx}{\Sigma}^{a}(t) - \langle \hat{\underset{\sim}{\xi}}(t)\hat{\underset{\sim}{\xi}}(t)\rangle \end{aligned} \qquad (194)$$

Therefore, the (relative) puff size is generally smaller than the absolute size.

Richardson introduced a distance-neighbor function to specify the relative diffusion. The definition is

$$q(t,\underset{\sim}{r})=\int d\underset{\sim}{x}\,\langle \hat{\psi}(t,\underset{\sim}{x})\,\hat{\psi}(t,\underset{\sim}{x}+\underset{\sim}{r})\rangle$$

Rewrite (194) as

$$\begin{aligned} 2\underset{\approx}{\Sigma}^{r}(t) &= \iint d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\,(\underset{\sim}{x}_1\underset{\sim}{x}_1+\underset{\sim}{x}_2\underset{\sim}{x}_2)\langle \hat{\psi}(t,\underset{\sim}{x}_1)\,\hat{\psi}(t,\underset{\sim}{x}_2)\rangle \\ &\quad - \iint d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\,(\underset{\sim}{x}_1\underset{\sim}{x}_2+\underset{\sim}{x}_2\underset{\sim}{x}_1)\langle \hat{\psi}(t,\underset{\sim}{x}_1)\,\hat{\psi}(t,\underset{\sim}{x}_2)\rangle \\ &= \iint d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\,(\underset{\sim}{x}_2-\underset{\sim}{x}_1)(\underset{\sim}{x}_2-\underset{\sim}{x}_1)\,\langle \hat{\psi}(t,\underset{\sim}{x}_1)\,\hat{\psi}(t,\underset{\sim}{x}_2)\rangle \\ &= \iint d\underset{\sim}{x}_1\,d\underset{\sim}{r}\,\underset{\sim}{r}\underset{\sim}{r}\,\langle \hat{\psi}(t,\underset{\sim}{x}_1)\,\hat{\psi}(t,\underset{\sim}{x}_1+\underset{\sim}{r})\rangle \\ &= \int d\underset{\sim}{r}\,\underset{\sim}{r}\underset{\sim}{r}\,q(t,\underset{\sim}{r}) \end{aligned} \qquad (195)$$

Therefore, the square of the puff size is one half of the variance of the distance-neighbor function:

$$\begin{aligned}\sigma_{pr}^{2}(t) &= \frac{1}{2}\sigma_{r}^{2}(t) \\ &= \frac{1}{2}\cdot\frac{1}{3}\langle \hat{r}^{2}(t)\rangle = \frac{1}{6}\int d\underset{\sim}{r}\,r^{2}\,q(t,\underset{\sim}{r}) \end{aligned} \qquad (196)$$

6.2 Absolute Diffusion and Relative Diffusion

According to (188) and (189), both the mean position of the puff and the spreading in the fixed frame are determined by the mean concentration distribution $\overline{\psi}(t,\underset{\sim}{x})$. The evolution of $\overline{\psi}(t,\underset{\sim}{x})$ is thus called absolute diffusion. With the molecular diffusivity neglected, the quantity can be considered as tagged onto the fluid particles, which were in the region where the quantity was released. Then the evolution of $\overline{\psi}(t,\underset{\sim}{x})$ can be considered as the result of the dispersion of those "marked" fluid particles.

Using the instantaneous transition function for a single particle, we can write

$$\hat{\psi}(t,\underset{\sim}{x}) = \int d\underset{\sim}{x}_o\, \hat{P}(t-t_o, \underset{\sim}{x}-\underset{\sim}{x}_o)\,\psi(t_o,\underset{\sim}{x}_o)$$

$\psi(t_o, x)$ is assumed known, so that

$$\overline{\psi}(t,\underset{\sim}{x}) = \int d\underset{\sim}{x}_o\, \overline{P}(t-t_o, \underset{\sim}{x}-\underset{\sim}{x}_o)\,\psi(t_o,\underset{\sim}{x}_o) \tag{197}$$

In (197), for a homoneneous turbulence, the transition function $\overline{P}(t-t_o, \underset{\sim}{x}-\underset{\sim}{x}_o)$ depends spatially on the displacement $\underset{\sim}{x}-\underset{\sim}{x}_o$ only. Hence, the governing equation of $\overline{P}$, (63) can be applied to get

$$(\partial_t + \overline{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}_x)\overline{\psi}(t,\underset{\sim}{x}) \doteq \underset{\sim}{\nabla}_x\cdot\underset{\approx}{K}\cdot\{\underset{\sim}{\nabla}_x'\overline{\psi}(t',\underset{\sim}{x}')\} \tag{198}$$

where $\underset{\approx}{K}\{\cdots\}$ is defined by (64) or (65). Practically, one may use the Gaussian approximation, so that

$$(\partial_t + \overline{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla}_x)\overline{\psi}(t,\underset{\sim}{x}) \doteq \underset{\approx}{K}(t) : \underset{\sim}{\nabla}_x\underset{\sim}{\nabla}_x\,\overline{\psi}(t,\underset{\sim}{x}) \tag{199}$$

The eddy diffusivity tensor is to be solved from

$$\underset{\sim}{\overline{K}}(\tau) = \frac{1}{2}\frac{d}{d\tau}\langle \underset{\sim}{\tilde{\ell}}(\tau)\underset{\sim}{\tilde{\ell}}(\tau)\rangle = \int_0^{\tau} d\tau' \int d\underset{\sim}{\ell}' \underset{\sim}{R}(\tau';\underset{\sim}{\ell}')\overline{P}(\tau';\underset{\sim}{\ell}') \qquad (200)$$

The distance-neighbor function describes the spreading of the puff arround its instantaneous centroid. The evolution of the distance-neighbor function is called the relative diffusion. Similar to (197), we write

$$\hat{\psi}(t,\underset{\sim}{x}_1)\hat{\psi}(t,\underset{\sim}{x}_2) = \iint d\underset{\sim}{x}_{10} d\underset{\sim}{x}_{20}\, \hat{P}_1(t-t_0,\underset{\sim}{x}_1-\underset{\sim}{x}_{10})\hat{P}_2(t-t_0,\underset{\sim}{x}_2-\underset{\sim}{x}_{20})\psi(t_0,\underset{\sim}{x}_{10})\psi(t_0,\underset{\sim}{x}_{20})$$

and

$$\langle \hat{\psi}(t,\underset{\sim}{x}_1)\hat{\psi}(t,\underset{\sim}{x}_2)\rangle = \iint d\underset{\sim}{x}_{10} d\underset{\sim}{x}_{20}\, \overline{P}_{12}(t-t_0,\underset{\sim}{x}_1-\underset{\sim}{x}_{10},\underset{\sim}{x}_2-\underset{\sim}{x}_{20})\psi(t_0,\underset{\sim}{x}_{10})\psi(t_0,\underset{\sim}{x}_{20}) \qquad (201)$$

According to Figure 1, the coordinates have the relations

$$\underset{\sim}{x}_2 = \underset{\sim}{x}_1 + \underset{\sim}{r}\,; \quad \underset{\sim}{x}_{20} = \underset{\sim}{x}_{10} + \underset{\sim}{r}_0\,, \quad \underset{\sim}{\lambda} = \underset{\sim}{r} - \underset{\sim}{r}_0$$

Therefore, the distance-neighbor function can be expressed as

$$\begin{aligned} g(t,\underset{\sim}{r}) &= \int d\underset{\sim}{x}_1 \langle \hat{\psi}(t,\underset{\sim}{x}_1)\hat{\psi}(t,\underset{\sim}{x}_1+\underset{\sim}{r})\rangle \\ &= \iiint d\underset{\sim}{x}_1 d\underset{\sim}{x}_{10} d\underset{\sim}{r}_0 \overline{P}_{12}(t-t_0,\underset{\sim}{x}_1-\underset{\sim}{x}_{10},\underset{\sim}{x}_1-\underset{\sim}{x}_{10}+\underset{\sim}{r}-\underset{\sim}{r}_0)\psi(t_0,\underset{\sim}{x}_{10})\psi(t_0,\underset{\sim}{x}_{10}+\underset{\sim}{r}_0) \\ &= \int d\underset{\sim}{r}_0 \overline{B}(t-t_0,\underset{\sim}{r}-\underset{\sim}{r}_0)\, g(t_0,\underset{\sim}{r}_0) \end{aligned} \qquad (202)$$

In analogy with the equation for $\overline{\psi}(t,\underset{\sim}{x})$, one might use the equation for the relative transition function to establish the governing equation for $g(t,\underset{\sim}{r})$. However, it is not so straight forward. There is an important difference between the transition functions $\overline{P}$ abd $\overline{B}$. Even for stationary and homogeneous turbulence, $\overline{B}$ depends on the initial separation of the two particles.

As a compromise, we assume first that the puff was originally released from a point source at t_{oo}. Secondly, we assume that, in the time period $t_o - t_{oo}$, the puff size increases to

$$\sigma_{pr}^2(t_o) = \frac{1}{6} r_o^2$$

Practically, we use Batchelor's model. Then, equations (124) and (126) are applied to obtain

$$\partial_t g(t, \underset{\sim}{r}) = \underset{\approx}{K}^r(t, t_{oo}) : \underset{\sim}{\nabla}_r \underset{\sim}{\nabla}_r \, g(t, \underset{\sim}{r}) \tag{203}$$

with

$$\underset{\approx}{K}^r(t, t_{oo}) = \int_{t_{oo}}^{t} dt' \iint d\underset{\sim}{l}' d\underset{\sim}{\lambda}' \left[2\underset{\approx}{R}(t-t', \underset{\sim}{l}') - \underset{\approx}{R}(t-t', \underset{\sim}{l}'+\underset{\sim}{\lambda}') - \underset{\approx}{R}(t-t', \underset{\sim}{l}'-\underset{\sim}{\lambda}') \right] \bar{P}(t-t', \underset{\sim}{l}') \bar{B}(t'-t_{oo}, \underset{\sim}{\lambda}') \tag{204}$$

The relative eddy diffusivity is solved from

$$\underset{\approx}{K}^r(t, t_{oo}) = \frac{1}{2} \frac{d}{dt} \langle \hat{\underset{\sim}{\lambda}}(t-t_{oo}) \hat{\underset{\sim}{\lambda}}(t-t_{oo}) \rangle \tag{205}$$

$$\langle \hat{\underset{\sim}{\lambda}}(t_o-t_{oo}) \hat{\underset{\sim}{\lambda}}(t_o-t_{oo}) \rangle = \underset{\sim}{r}_o \underset{\sim}{r}_o$$

And the puff size:

$$\sigma_{pr}^2(t) = \frac{1}{2} \sigma_\lambda^2(t-t_{oo}) = \frac{1}{6} \langle \hat{\underset{\sim}{\lambda}}^2(t-t_{oo}) \rangle \tag{206}$$

with the condition

$$\sigma_{pr}^2(t_o) = \frac{1}{6} r_o^2 \tag{207}$$

6.3 Diffusion in a Homogeneous Shear Turbulence

In the previous sections, the mean velocity of the turbulent flow is assumed constant or zero. Now we consider that the mean velocity has a constant spacial gradient. Let the mean velocity be

$$\bar{\underset{\sim}{u}} = \Gamma x_3 \underset{\sim}{e}_1 \tag{208}$$

where $\underset{\sim}{e}_1$ is a unit vector. We shall use $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ for the other two unit vectors for the Cartesian coordinate system. The evolution of the mean concentration distribution of a puff is governed by the equation

$$(\partial_t + \Gamma x_3 \nabla_{x_1}) \bar{\Psi}(t,\underset{\sim}{x}) = \underset{\approx}{K}(t) : \underset{\sim}{\nabla}_x \underset{\sim}{\nabla}_x \bar{\Psi}(t,\underset{\sim}{x}) \tag{209}$$

The eddy diffusivity tensor is to be solved from the equation for the one-particle transitin function. The Gaussian form of $\bar{P}(\tau,\underset{\sim}{l})$ in the anisotropic case is

$$\bar{P}(\tau,\underset{\sim}{l}) = \frac{(2\pi)^{-3/2}}{\sigma_1\sigma_2\sigma_3\sqrt{1-\beta^2}} \exp\left\{-\frac{1}{2(1-\beta^2)}\left[\frac{l_1^2}{\sigma_1^2} - \frac{2\beta l_1 l_3(1-\Gamma\tau)}{\sigma_1\sigma_3} + \frac{l_2^2}{\sigma_2^2} + \frac{l_3^2(1-\Gamma\tau)^2}{\sigma_3^2}\right]\right\} \tag{210}$$

where σ_i, (i=1,2,3), are the variances and β is the coefficient defined as

$$\beta \equiv \sqrt{\frac{\langle \tilde{l}_1(\tau)\tilde{l}_3(\tau)\rangle}{\sigma_1(\tau)\sigma_3(\tau)}} \tag{211}$$

All σ_i and β are functions of time. In (210), we have assumed that there is no correlation between the displacement fluctuations along $\underset{\sim}{e}_2$ direction and either of the other two directions. β accounts for the correlation of the fluctuations along $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_3$ directions, caused by the interaction of the mean wind and its gradient. The Fourier components of $\overline{P}(\tau, \underset{\sim}{\ell})$ are

$$\overline{P}(\tau, \underset{\sim}{k}) = (2\pi)^{-3} f(\tau, k_1\sigma_1, k_2\sigma_2, k_3\sigma_3, \beta, \Gamma)$$
$$= (2\pi)^{-3} \frac{\sqrt{1-\beta^2}}{1-\Gamma\tau} \exp\left\{-\frac{1}{2}(1-\beta^2)(k_1^2\sigma_1^2 + k_2^2\sigma_2^2) - \frac{1}{2}\left[k_3\sigma_3 + k_1\sigma_1\beta(1-\Gamma\tau)\right]^2\right\} \tag{212}$$

The substitution of (212) into (92) yields

$$\frac{d}{d\tau}\sigma_1^2(\tau) = 2\int_0^\tau d\tau' \int d\underset{\sim}{k}\, R_{11}(\tau', \underset{\sim}{k}) f(\tau', k_1\sigma_1, k_2\sigma_2, k_3\sigma_3, \beta, \Gamma)$$
$$\frac{d}{d\tau}\sigma_2^2(\tau) = 2\int_0^\tau d\tau' \int d\underset{\sim}{k}\, R_{22}(\tau', \underset{\sim}{k}) f(\tau', k_1\sigma_1, k_2\sigma_2, k_3\sigma_3, \beta, \Gamma)$$
$$\frac{d}{d\tau}\sigma_3^2(\tau) = 2\int_0^\tau d\tau' \int d\underset{\sim}{k}\, R_{33}(\tau', \underset{\sim}{k}) f(\tau', k_1\sigma_1, k_2\sigma_2, k_3\sigma_3, \beta, \Gamma)$$
$$\frac{d}{d\tau}\beta^2(\tau) = -\frac{1}{2}\left(\frac{1}{\sigma_1^2}\frac{d}{d\tau}\sigma_1^2 + \frac{1}{\sigma_3^2}\frac{d}{d\tau}\sigma_3^2\right)$$
$$+ \frac{2}{\sigma_1\sigma_3}\int_0^\tau d\tau' \int d\underset{\sim}{k}\, R_{13}(\tau', \underset{\sim}{k}) f(\tau', k_1\sigma_1, k_2\sigma_2, k_3\sigma_3, \beta, \Gamma) \tag{213}$$

The components of the eddy diffusivity tensor may be calculated by

$$K_{ii}(t) = \frac{1}{2}\frac{d}{dt}\sigma_i^2(t), \quad i = 1, 2, 3 \quad (\textit{no summation})$$
$$K_{13}(t) = K_{31}(t) = \frac{1}{2}\sigma_1\sigma_3\frac{d}{dt}\beta^2 + \frac{\beta^2}{4}\left(\frac{\sigma_3}{\sigma_1}\frac{d}{dt}\sigma_1^2 + \frac{\sigma_1}{\sigma_3}\frac{d}{dt}\sigma_3^2\right) \tag{214}$$

Because of the anisotropy, the principle axes of the diffusivity tensor are inclined. Denote the unit vectors of these axes as $\underset{\sim}{\mathfrak{e}}_1$, $\underset{\sim}{\mathfrak{e}}_2$ and $\underset{\sim}{\mathfrak{e}}_3$. It is found that

$$\underset{\sim}{\mathfrak{e}}_1 = [K_{13}(t)]^{-1}\underset{\sim}{e}_1 + 2\left\{[K_{11}(t) - K_{33}(t)] + \sqrt{[K_{11}(t) - K_{33}(t)]^2 + 4K_{13}^2(t)}\right\}^{-1}\underset{\sim}{e}_3$$
$$\underset{\sim}{\mathfrak{e}}_2 = \underset{\sim}{e}_2$$
$$\underset{\sim}{\mathfrak{e}}_3 = 2\left\{[K_{11}(t) - K_{33}(t)] + \sqrt{[K_{11}(t) - K_{33}(t)]^2 + 4K_{13}^2(t)}\right\}^{-1}\underset{\sim}{e}_1 - [K_{13}(t)]^{-1}\underset{\sim}{e}_3 \tag{215}$$

When the initial puff size is small enough, the relative diffusion is dominated by velocity fluctuations of small scales. It is well known that small scale eddies are close to isotropic in a shear turbulence. On the other hand, in the time period of interest, the absolute size is considerably smaller than the outer scale of the turbulence. Therefore, the effect of anisotropy does not have a significant influence on the relative diffusion. Approximately, we can calculate the variance of $\overline{P}(\tau, \ell)$ from the isotropic equation (97), in which the energy spectrum is everaged over all directions. We may study the growth of the puff size based upon the equation:

$$\frac{d}{d\tau}\sigma_{\ell}^{2}(t) \doteq \frac{8}{3}\int_{t_0}^{t}dt'\int_{0}^{\infty}dk\, E_{is}(t',k)\, e^{-\frac{1}{2}k^2\sigma_{is}^2(t')}\left[1-e^{-\frac{1}{2}k^2\sigma_{\ell}^2(t-t')}\right] \tag{216}$$

Note that, in (216), $\sigma_{\ell}^{2}(t)$ is the variance of the separation of a pair of fluid particles, related to the puff size as

$$\sigma_{p\ell}^{2}(t) = \frac{1}{2}\sigma_{\ell}^{2}(t) \tag{217}$$

For a shear turbulence, the energy spectrum appears in the form

$$E_{is}(t,k) \doteq A_n k^{-n}$$

where n=5/3 is respective to the inertia subrange and n=1 or 3 is respective to the shear subrange. A_n is presented to connect these subranges. As described previously, (216) is dominated by the eddies of scales comparable to the puff size. Therefore we may calculte the growth of the puff size according to

$$\frac{d}{dt}\sigma_{p\lambda}^{2}(t) \doteq \frac{4}{3}\int_0^t dt'\int_0^\infty dk\, A_n k^{-n}[1-e^{-k^2\sigma_{p\lambda}^2(t')}]$$
$$= \frac{4}{3}A_n\int_0^t dt'\,[\sigma_{p\lambda}^2(t')]^{\frac{(n-1)}{2}}\int_0^\infty dk'\,k'^{-n}(1-e^{-k'^2})$$
$$\equiv \frac{4}{3}A_n F_n\int_0^t dt'\,[\sigma_{p\lambda}^2(t')]^{\frac{(n-1)}{2}} \qquad (219)$$

The solution is

$$\sigma_{p\lambda}^2(t) = \left[\frac{A_n F_n}{3(n+1)}\right]^{\frac{2}{3-n}}\left[\frac{(3-n)t}{2}\right]^{\frac{4}{3-n}} + B_n \qquad (n\neq 3)$$

$$\sigma_{p\lambda}^2(t) = B_3\exp\left[-\sqrt{\frac{8}{3}A_3F_3}\; t\right] \qquad (n=3)$$

When n=5/3, the variance is proportional to t^3, which corresponds to the 4/3 power law of relative diffusion. These results are comparable with that given by Misguich and Balescu, and Mikkelsen.

It is interesting to consider the case that the shear turbulence is inhomogenous, such as in the atmosphere or in the ocean. Although the formal theory might be extended to the inhomogeneous case as did by Roberts,[22] an analytical study is not practical at present. To reduce the difficulty, the locally homogeneous approximation is suggested. The inhomogeneity may be considered that the energy spectrum has a parameter representing the position. Then the discussion on the absolute diffusion and the relative diffusion of a puff may apply.

Recently, RISØ Laboratory of Denmark performed an experiment for the diffusion of a smoke plume in the atmospheric boundary layer.[44] The velocity spectra in three directions and the lateral size of the plume were measured.

The theoretical prediction of the growth of the lateral size was based upon the equation derived by Mikkelsen:[41,44]

$$\frac{d}{dt}\sigma_{p\lambda}^2(t) = \langle \tilde{v}^2 \rangle \left\{ 1 - \int_{-\infty}^{\infty} d\xi \, \frac{\rho_v(\xi)}{\sqrt{\pi}\,\sigma_{p\lambda}(t)} \exp\left[-\frac{\xi^2}{4\sigma_{p\lambda}^2(t)} \right] \right\} \cdot \frac{t_L}{1 + t_L/t} \qquad (220)$$

$$\rho_v(\xi) = a + b \log_{10} \xi \qquad (221)$$

However, instead of the measured spectra, the velocity correlation was estimated by (221) and the Lagrangian time scale t_L is estimated to produce the best fit for the experimental data. The spectra were measured at one hight, 10 meters. Our equation requires the information how the spectrum varies with height, so that it is not directly applicable. Nevertheless, we have used equations (97) and (216) to calculate the growth of a puff size based upon the v-component of the spectra. The nondimensionized spectrum is:

$$E'(k') = \begin{cases} 79.35\, k'^{4} \exp(-5k') & (0.0 \leq k' < 1.1429) \\ 0.6665\, k'^{-3} & (1.1429 \leq k' < 2.8571) \\ 0.08165\, k'^{-1} & (2.8571 \leq k' < 71.429) \\ 1.4056\, k'^{-5/3} & (71.429 \leq k' < 1079.1) \\ 326.61\, k'^{-2.447} & (1079.1 \leq k' < 28571) \end{cases}$$

with

$$E'(k') = \frac{E(k)\,k_0}{\frac{1}{2}\langle \tilde{v}^2 \rangle}, \quad k' = k/k_0, \quad k_0 = 0.0035\, m^{-1}, \quad \frac{1}{2}\langle \tilde{v}^2 \rangle \doteq 0.6736\, (m^2/s^2)$$

It is found that, with the full spetrum and the one-particle variance, the growth of the puff size shows a $t^{3/2}$ behavior. Such a behavior agrees with the previous discussion and the observations[45]. The v-component spectrum and the one we used in calculation are shown in Figure 6. The results of the calculation are shown in Figure 7. To compare our calculation with the experimental data, we match the curve for the absolute puff size at the point x= 200m. In Figure 8, the circles and the dots are the experimental points of the absolute size and the lateral size of the plume respectively. Curves 1 and 2 are based upon the dimensionless calculation. It should be noted that curve 2 is respective to the initial size Ω_0=0.02m. However, according to Mikkelsen[44], the initial size is approximately 0.25m. By means of the treatment stated in Section 6.2, curve 2 should be modified by a distance over which the plume increases its lateral size to 0.26m. Curve 3 is the result of such a transformation. We see that curves 1 and 3 agree well with the experimental data. Mikkelsen's theoretical prediction of the lateral size of the plume is shown as the dashed curve. In the prediction, the velocity correlation function was estimated empirically as

$$\rho_v(\xi) = \begin{cases} \text{Const.} & (\xi < 2.5m) \\ 1.05 - 0.20 \log_{10}\xi & (\xi \geq 2.5m) \end{cases}$$

and the constant was chosen for a best fit of the lower part of the curve. The absolute size was not predicted by Mikkelsen.

VII SUMMARY

In this work, a statistical study is made for the dispersion of particles in a turbulent fluid flow. The transition functions are introduced to describe the probability distribution for the displacement of the particles. The governing equations for the transition equations appear in series form as correlations, which implies the closure problem. The exact propagator physically specifies the trajectory which characterises the transport process, and mathematically keeps the nonlinearity which is implied in the problem. Therefore, the propagator method has the advantage over other methods in that the resulting series decreases faster. By means of the mean propagator, the effects of both mode-coupling and the collective phenomena are included. The transport equations established in such a way apply to strong turbulence.

The transport equations are developed to analyse the dispersion of fluid particles. The absolute dispersion of a single particle and the relative dispersion of a pair of particles are found to obey nonlinear integro-differential equations. The general properties of our results on dispersion agree with observations. Also, our equations are more appropriate than those of other theories. We properly describe the short time behavior of one-particle dispersion. For the relative dispersion, it is shown that our equation works well with a full spectrum, while other theories have the difficulty that the velocity correlation function is in a complicated, Lagrangian-Eulerian hybrid form.

The results for the dispersion of fluid particles are applied to describe the diffusion of a puff of a passive quantity. The absolute diffusion is related to the one-particle dispersion and the relative diffusion is related to the relative dispersion of a pair of particles. The eddy diffusivities appear in tensorial form to account for anisotropy. The diffusion in shear and buoyancy turbulence is also discussed, including the spectra k^{-1} and k^{-3}. Based upon an experimental spectrum, the results of calcuation agree with observations. More application depends on the knowledge how the velocity correlation functions or the spectra vary with position.

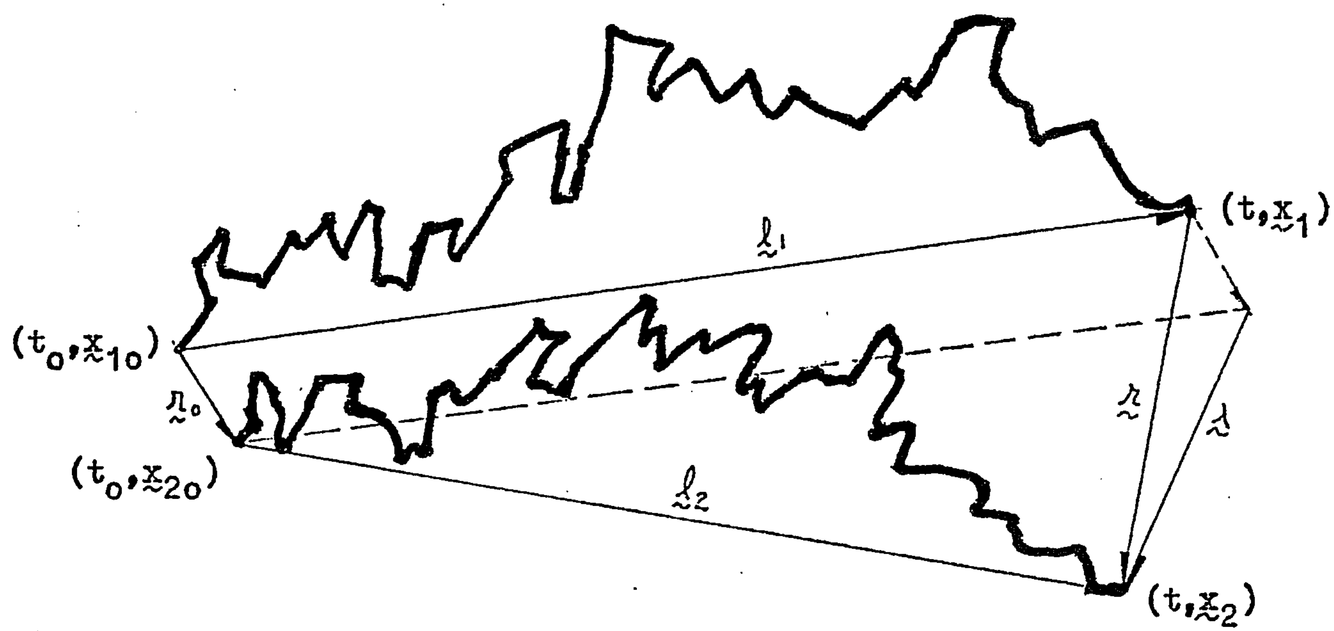


Figure 1. Trajectories of a pair of particles

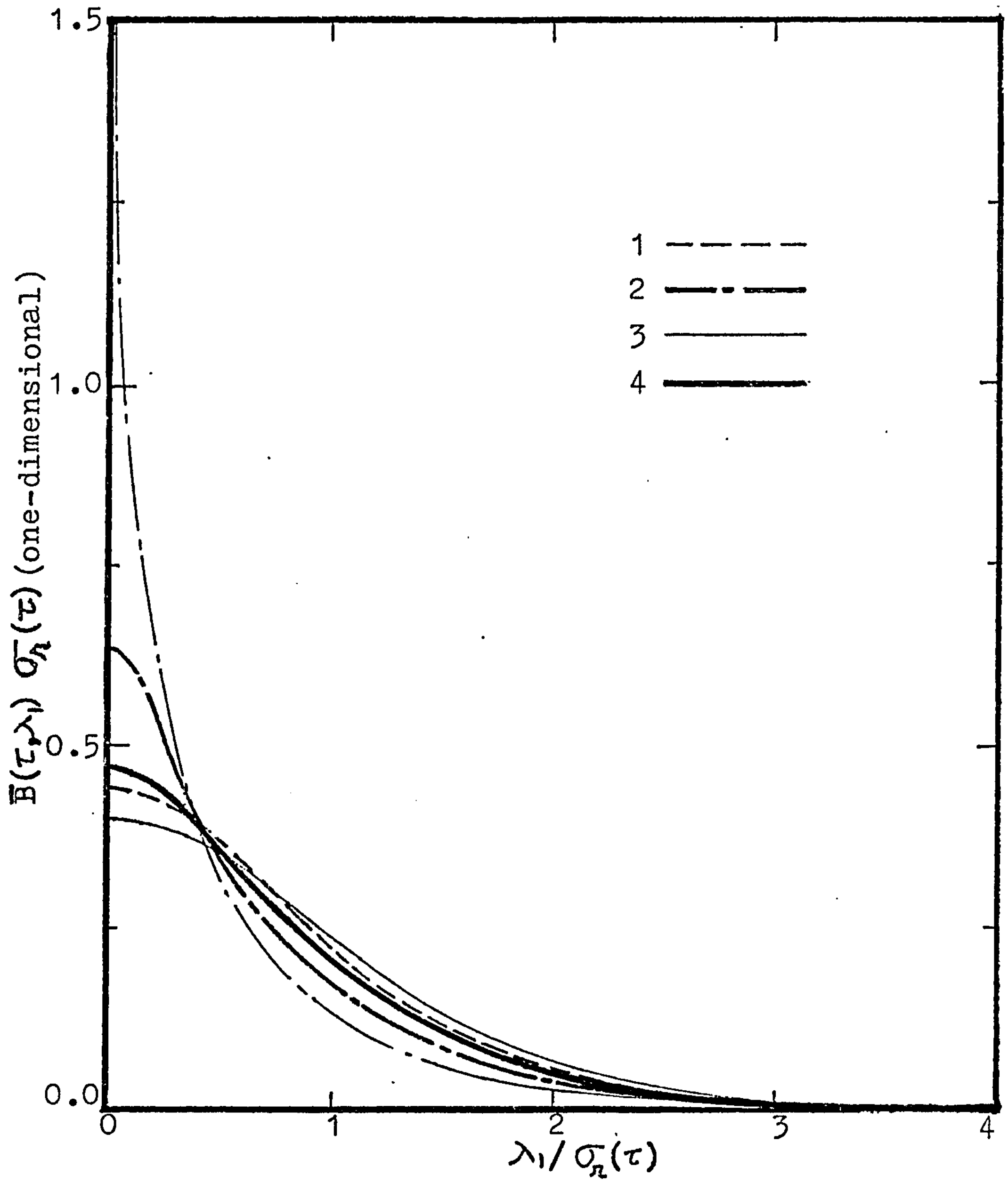


Figure 2. Profiles of $\overline{B}(\tau,\lambda_1)$
1-Experimental, by Sullivan; 2-Richardson's model
3-Batchelor's model; 4-Okubo's model

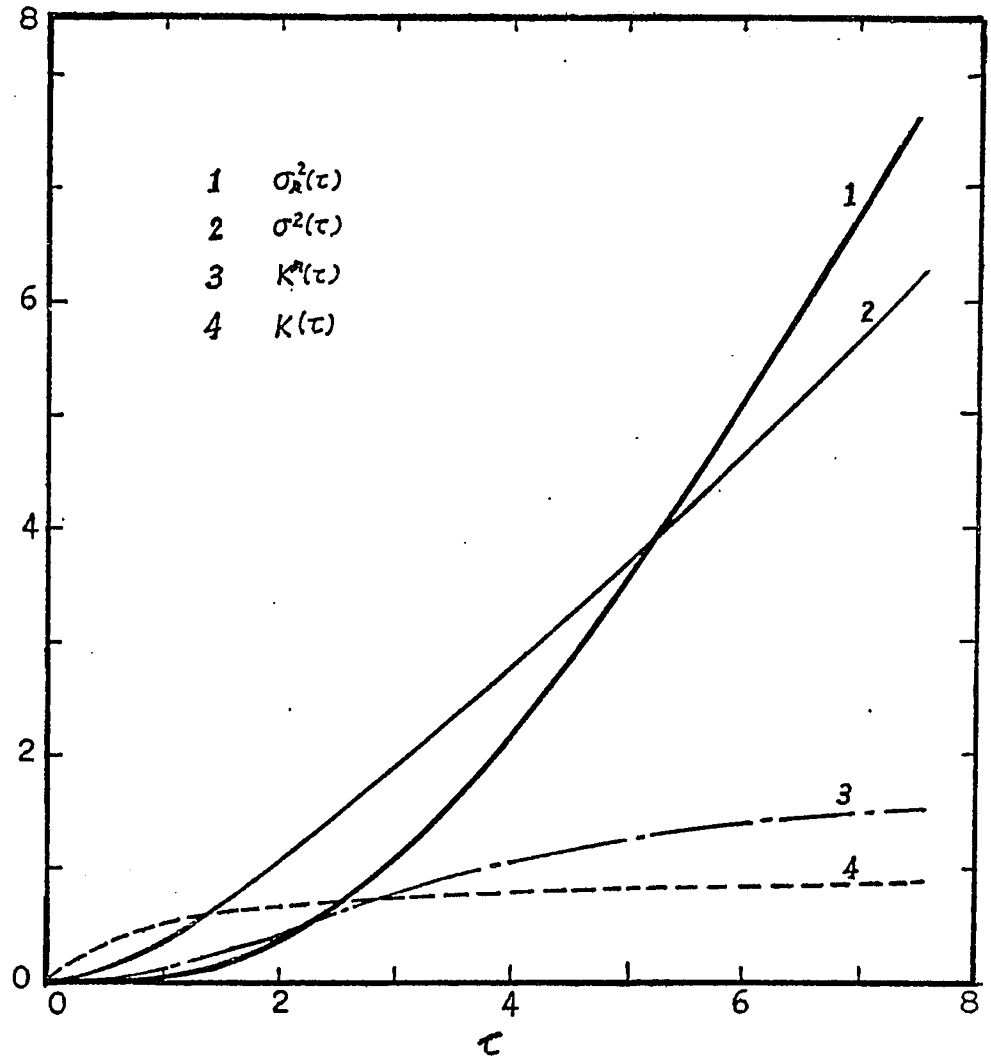


Figure 3. Dispersion of fluid particles, based upon the von Kármán spectrum

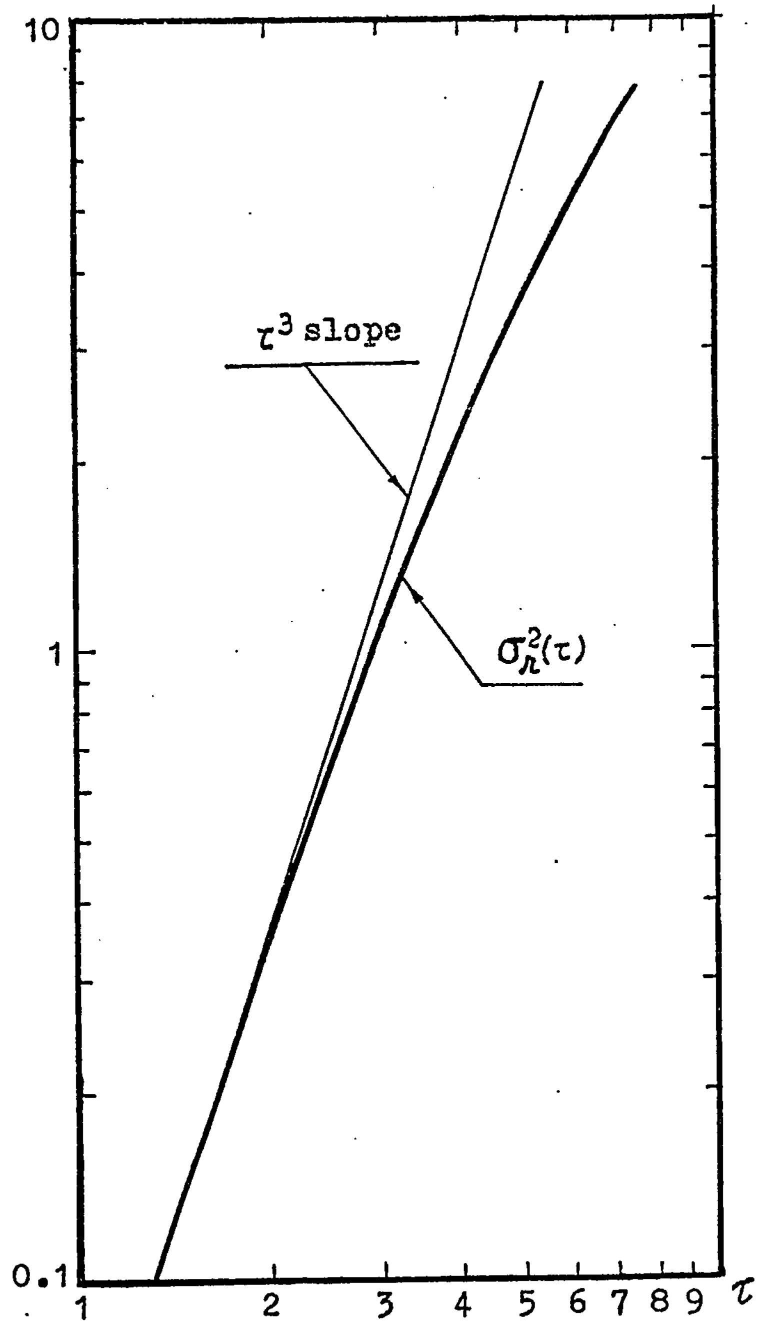


Figure 4. Relative dispersion, the τ^3 behavior

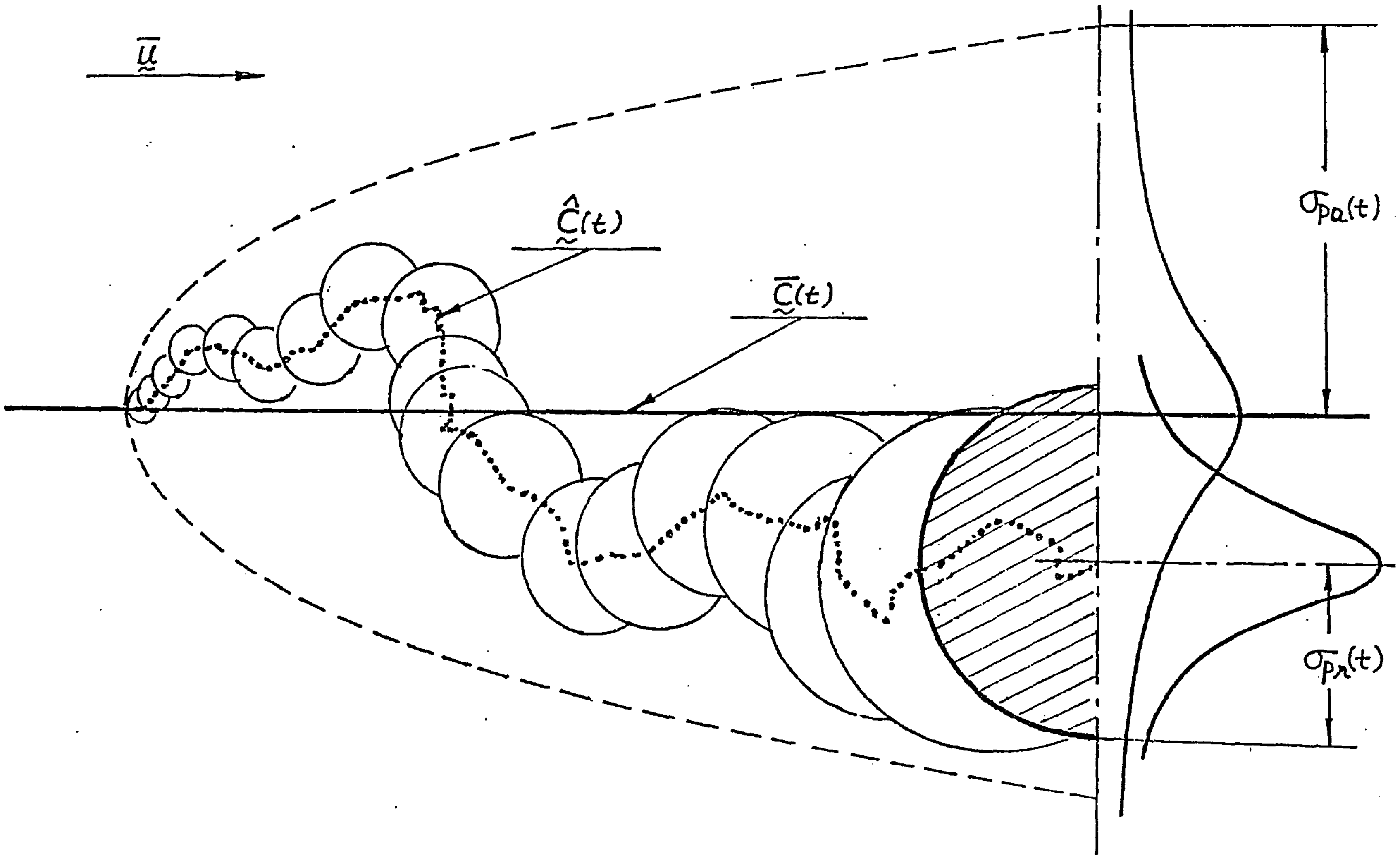
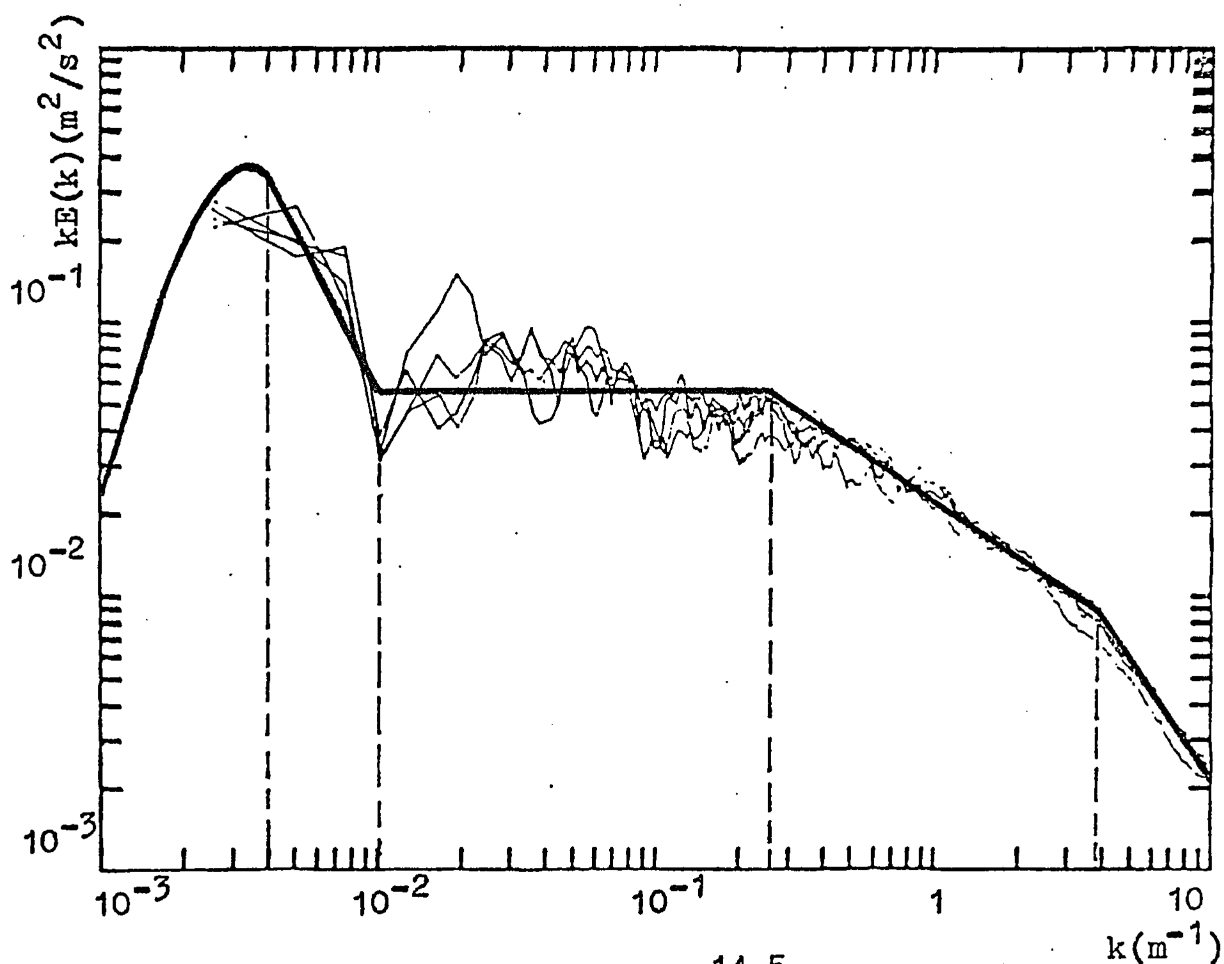


Figure 5 Diffusion of a puff,
the absolute size and the relative size



—— : $kE(k) = \begin{cases} 1.0177\times10^{14}k^{5}\exp(-5000k/3.5) & (0 \leq k < 0.004) \\ 5.5\times10^{-6}k^{-2} & (0.004 \leq k < 0.01) \\ 5.5\times10^{-2} & (0.01 \leq k < 0.25) \\ 0.02183k^{-2/3} & (0.25 \leq k < 3.777) \\ 0.06155k^{-1.447} & (3.777 \leq k < 100) \end{cases}$

: experimental v-spectrum, by RISØ
$\bar{u} = 5.93$ m/s, measuring hight: 10 m

Figure 6. Velocity spectrum, v-component

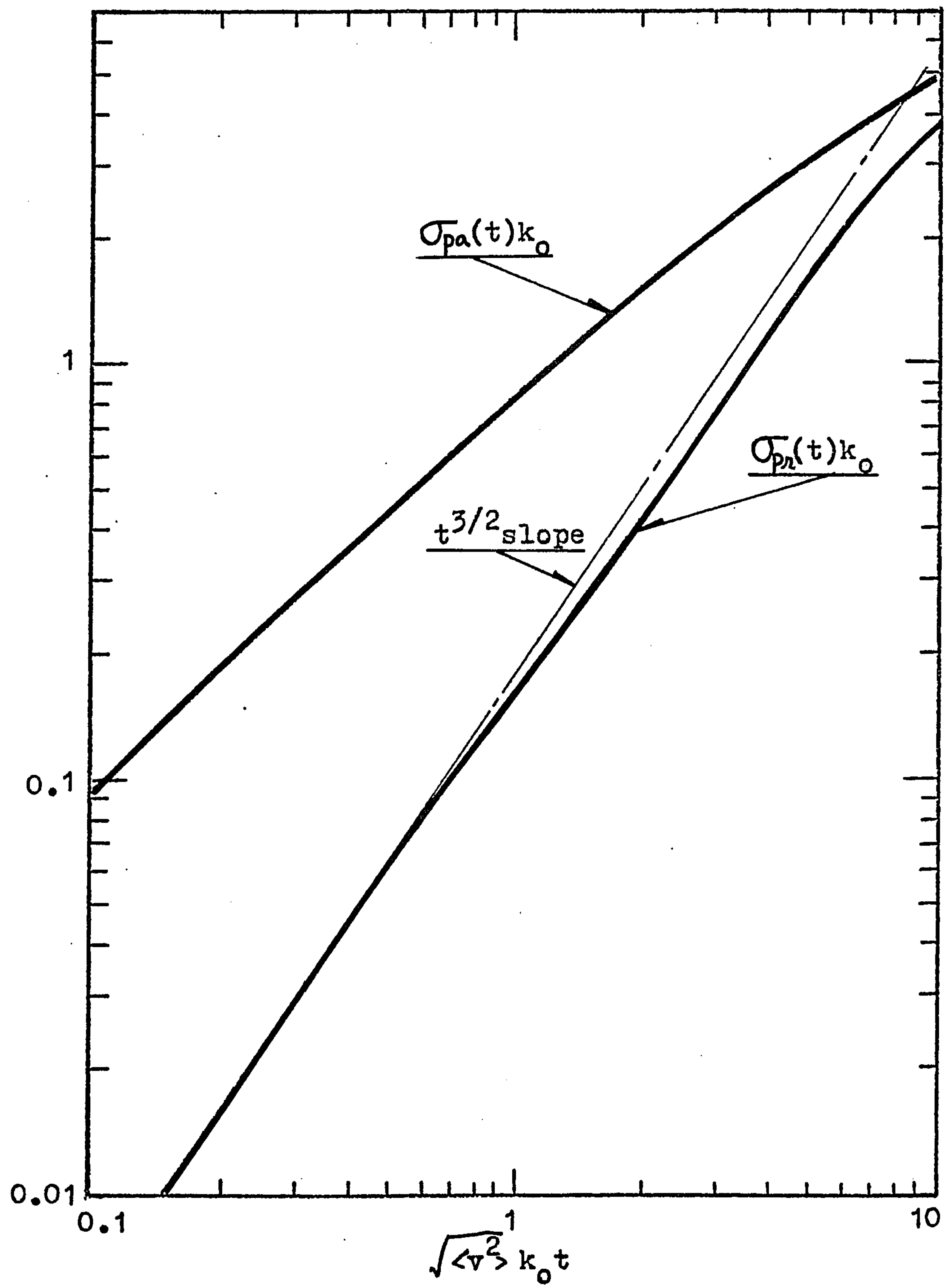


Figure 7. Puff size, calculated with the empirical spectrum

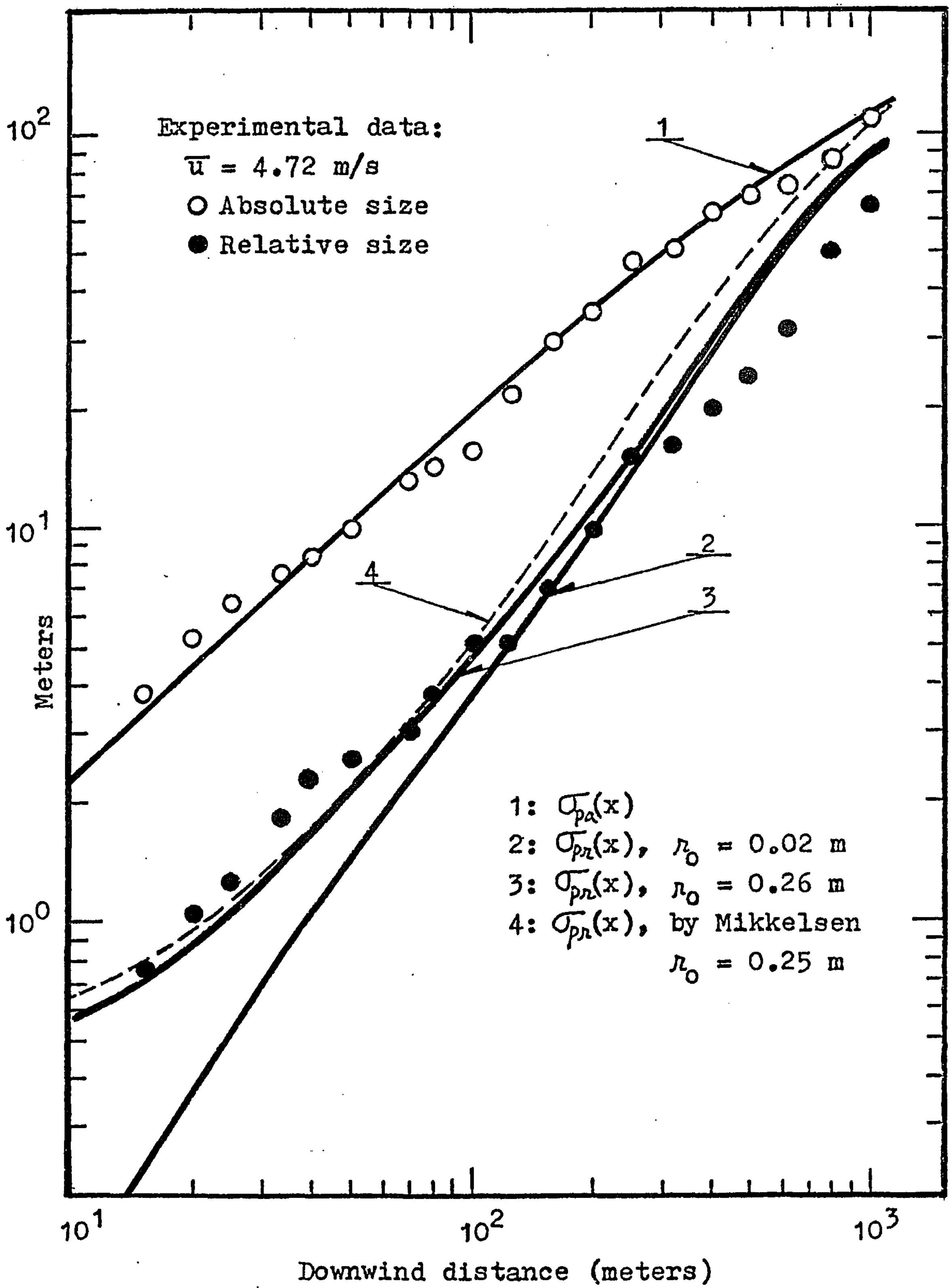


Figure 8. Growth of the lateral size of a plume

Appendix 1

The expansion of h P is made by the repeated substitution of (49):

$$\tilde{U} = -\overline{U}*\tilde{L}\overline{U} - \tilde{U}*\tilde{L}\overline{U} + \overline{U}*\langle\tilde{L}\tilde{U}\rangle + \tilde{U}*\langle\tilde{L}\tilde{U}\rangle$$

The result is

$$\overline{h_2}\,\overline{P} = \langle\tilde{L}\tilde{U}*\tilde{L}\rangle\overline{P}$$

$$= -\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$

$$+\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$

5th

$$-\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$+\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$

6th

$$+\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$
$$-\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$-\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} + 2\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$+\langle\tilde{L}\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$

7th

$$-\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$
$$+\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$+\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$
$$+\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$
$$+\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$-2\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - 2\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$-2\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - 2\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$-\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$
$$-\langle\tilde{L}\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$
$$-\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$

$$+\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$
$$-\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P}$$
$$-\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$

$$
8th\left\{
\begin{aligned}
&-\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}\rangle \overline{P}-\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}o\tilde{L}\rangle \overline{P}\\
&-\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}-\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&-\langle \tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}o\tilde{L}\rangle \overline{P}-\langle \tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}\\
&-\langle \tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}-\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}\\
&-\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}-\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+2\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}o\tilde{L}\rangle \overline{P}+2\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}\\
&+2\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+2\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}+2\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+2\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+2\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}+2\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+2\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+2\langle \tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}o\tilde{L}o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}+\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&+\langle \tilde{L}o\tilde{L}o\tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&-5\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}-2\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&-2\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}-2\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}\\
&-\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\langle \tilde{L}o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle o\tilde{L}\rangle \overline{P}
\end{aligned}
\right.
$$

where

$$o \equiv \overline{U}*$$

The expansion is very lengthy and complicated.

Since $\overline{D_i P}$ terms have the common operators $\overline{h_1}$ and $(\overline{h_2}-\overline{h_1}\circ\overline{h_1})$, the $\{\overline{\Delta_{2i}}\}$ series can be obtained by the substitution of the expression of $\overline{h_2 P}$. Making use of the three approximations in Section 2.4, we find that

$$
\begin{aligned}
\overline{\Delta_4} &= \langle \tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle \tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P} \\
&= \langle \tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L}-\langle\tilde{L}\circ\tilde{L}\rangle)\rangle\overline{P} - \langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} \\
\overline{\Delta_6} &= \langle \tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P} \\
&\qquad - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P} \\
&= \langle \tilde{L}\circ\tilde{L}\circ[\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L}-\langle\tilde{L}\circ\tilde{L}\rangle) - \langle\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L}-\langle\tilde{L}\circ\tilde{L}\rangle)\rangle]\rangle\overline{P} \\
\overline{\Delta_8} &= \langle \tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P} \\
&\qquad - \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P} \\
&\qquad - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} + \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P} \\
&\qquad + \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\tilde{L}\rangle\overline{P} - \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\tilde{L}\rangle\overline{P} \\
&= \langle \tilde{L}\circ\tilde{L}\circ\{\tilde{L}\circ\tilde{L}\circ[\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L}-\langle\tilde{L}\circ\tilde{L}\rangle) - \langle\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L}-\langle\tilde{L}\circ\tilde{L}\rangle)\rangle] \\
&\qquad - \langle\tilde{L}\circ\tilde{L}\circ[\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L}-\langle\tilde{L}\circ\tilde{L}\rangle) - \langle\tilde{L}\circ\tilde{L}\circ(\tilde{L}\circ\tilde{L}-\langle\tilde{L}\circ\tilde{L}\rangle)\rangle]\rangle\}\rangle\overline{P} \\
&= \langle\tilde{L}\circ\tilde{L}\circ(\hat{\Delta}_6-\overline{\Delta_6})\rangle
\end{aligned}
$$

Approximately,

$$\langle\tilde{L}\circ\tilde{L}\circ\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P} \doteq \langle\tilde{L}\circ\tilde{L}\rangle\circ\langle\tilde{L}\circ\langle\tilde{L}\circ\tilde{L}\rangle\circ\tilde{L}\rangle\overline{P}$$

Therefore,

$$\overline{\Delta_6} \doteq \langle\tilde{L}\circ\tilde{L}\circ(\hat{\Delta}_4-\overline{\Delta_4})\rangle$$

Appendix 2

The exact propagator is introduced to find the formal solution of the equation

$$[\partial_\tau + \hat{L}(\tau)]\tilde{A}(\tau, \underset{\sim}{\ell}) = \tilde{B}(\tau, \underset{\sim}{\ell}) \tag{A-1}$$

such that

$$\tilde{A}(\tau, \underset{\sim}{\ell}) = \hat{U}(\tau, o)\tilde{A}(o, \underset{\sim}{\ell}) + \int_o^\tau d\tau' \hat{U}(\tau, \tau')\tilde{B}(\tau') \tag{A-2}$$

where the $\underset{\sim}{\ell}'$ dependence in the integrand is implied. The first term on the righthand side of (A-2) represents the effect of the initial condition and is usually neglected, so that

$$\tilde{A}(\tau, \underset{\sim}{\ell}) \doteq \int_o^\tau d\tau' \hat{U}(\tau, \tau')\tilde{B}(\tau') \tag{A-3}$$

Recalling

$$\hat{L}(\tau) = \underset{\sim}{\hat{v}}(\tau)\cdot\underset{\sim}{\nabla} = \underset{\sim}{\hat{v}}(\tau)\cdot\frac{\partial}{\partial\underset{\sim}{\ell}}$$

we transform (A-1) into the Fourier space

$$[\partial_\tau + i\underset{\sim}{k}\cdot\underset{\sim}{\hat{v}}(\tau)]\tilde{A}(\tau, \underset{\sim}{k}) = \tilde{B}(\tau, \underset{\sim}{k})$$

Its solution is

$$\begin{aligned}\tilde{A}(\tau, \underset{\sim}{k}) &= e^{-\int_o^\tau d\tau' i\underset{\sim}{k}\cdot\underset{\sim}{\hat{v}}(\tau')}\tilde{A}(o, \underset{\sim}{k}) + \int_o^\tau d\tau' e^{-\int_{\tau'}^{\tau} d\tau'' i\underset{\sim}{k}\cdot\underset{\sim}{\hat{v}}(\tau'')}\tilde{B}(\tau'; \underset{\sim}{k}) \\ &= e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\hat{\ell}}(\tau)}\tilde{A}(o, \underset{\sim}{k}) + \int_o^\tau d\tau' e^{-i\underset{\sim}{k}\cdot[\underset{\sim}{\hat{\ell}}(\tau) - \underset{\sim}{\hat{\ell}}(\tau')]}\tilde{B}(\tau'; \underset{\sim}{k})\end{aligned} \tag{A-4}$$

The first term on the righthand side refers to the initial condition and is customarily neglected in the present case. The second term involves a convolution so that

$$\begin{aligned}\tilde{A}(\tau, \underset{\sim}{\ell}) &\doteq \int_o^\tau d\tau' \int d\underset{\sim}{\ell}'\, \delta\{(\underset{\sim}{\ell} - \underset{\sim}{\ell}') - [\underset{\sim}{\hat{\ell}}(\tau) - \underset{\sim}{\hat{\ell}}(\tau')]\}\tilde{B}(\tau'; \underset{\sim}{\ell}') \\ &= \int_o^\tau d\tau' \int d\underset{\sim}{\ell}'\, \hat{P}(\tau-\tau', \underset{\sim}{\ell} - \underset{\sim}{\ell}')\tilde{B}(\tau', \underset{\sim}{\ell}')\end{aligned} \tag{A-5}$$

Comparing (A-5) with (A-3), we see the equivalence:

$$\int_o^\tau d\tau' \hat{U}(\tau, \tau')\cdots = \int_o^\tau d\tau' \int d\underset{\sim}{\ell}'\, \hat{P}(\tau-\tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}')\cdots$$

$$\int_o^\tau d\tau' \overline{U}(\tau, \tau')\cdots = \int_o^\tau d\tau' \int d\underset{\sim}{\ell}'\, \overline{P}(\tau-\tau'; \underset{\sim}{\ell} - \underset{\sim}{\ell}')\cdots$$

Bibliography

1
G.I. Taylor, Diffusion by continuous movements, Proc. Lond. Math. Soc., 20, 196-211(1921)

2
L.F. Richardson, Atmospheric diffusion on a distance-neighbor graph, Proc. Roy. Soc., A 110, 709-737 (1926)

3
J.L. Lumley, The mathematical nature of the problem of relating Lagrangian and Eulerian statistical functions in turbulence, Mecanique de la Turbulence, Paris, Ed. CNRS, 17-26(1962)

4
J.L. Lumley, An approach to the Eulerian-Lagrangian problem, J. Math. Phys., 3, No. 2, 309-312(1962)

5
R. Phythian, Some variation methods in the theory of turbulent diffusion, J. Fluid Mech., 53, Pt. 3, 469-480 (1972)

6
R.G. Deissler, Analysis of multipoint-multitime correlations and diffusion in decaying homogeneous turbulence, Nat. Aeronaut. Space Adm., Tech. Rep. R-96 (1961)

7
P.G. Saffman, Application of the Wiener-Hermite expansion to the diffusion of a passive scalar in a homogeneous turbulent flow, Phys. Fluids, 12, No. 9, 1786-1798(1969)

8
S. Corrsin, Progress report on some turbulent diffusion research, Advances in Geophysics, Vol. 6, 161-163, (1959)

9
S. Corrsin, Theories of turbulent dispersion, Mecanique de la Turbulence, Paris, Ed. CNRS, 27-52(1962)

10
R.H. Kraichnan, Decay of isotropic turbulence in the direct interaction approximation, Phys. Fluids, 7, No. 7, 1030-1048(1964)

11
R.H. Kraichnan, Lagrangian-history closure approximation for turbulence, Phys. Fluids, 8, No. 4, 575-598 (1965)

12
T.H. Dupree, A perturbation theory for strong turbulence, Phys. Fluids, 9, 1773-1782(1966)

13
T.H. Dupree, Theory of phase space density granuation in plasma, Phys. Fluids, 15, No. 2, 334-344(1972)

14
T. Boutros-Ghali and T.H. Dupree, Theory of two-point correlation in a Vlasov plasma, Phys. Fluids, 24, No. 10, 1839-1858(1981)

15
J. Weinstock, Formulation of a statistical theory of strong plasma turbulence, Phys. Fluids, 12, No. 5, 1045-1058(1969)

16
J.H. Misguich and R. Balescu, Renormalized quasi-linear approximation of plasma turbulence. Part 1: Modification of the Weinstock weak coupling limit, J. Plasma Phys., 13, Pt. 3, 385-417(1975)

17
J.H. Misguich and R. Balescu, Kinetic description of average trajectories in turbulent plasma, Plasma Phys., 19, 611-625(1977)

18
J.H. Misguich and R. Balescu, Asymptotic propagators and trajectories in plasma turbulence theory, Plasma Phys., 21, 749-779(1979)

19
C.M. Tchen, Kinetic basis of cascade transfer in turbulence, in Theory and Modeling of Atomospheric Turbulence, Vol. 1, NASA Contractor Report 3787, A1-A43 (1984)

20
C.M. Tchen, Kinetic theory of turbulent transfer with double memory loss, in Theory and Modeling of Atmospheric Turbulence, Vol. 1, NASA Contractor Report 3787, B1-B36, (1984)

21
J.H. Misguich and C.M. Tchen, Equivalent methods for describing quasi-linear turbulent trajectories, Report EUR-CEA-FC-1157(1982)

22
P.H. Roberts, Analytical theory of turbulent diffusion, J. Fluid Mech., 11, No. 2, 257-283(1961)

23
A.S. Monin and A.M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. 2, translated and edited by J.L. Lumley, MIT Press, Chap. 8, Section 24(1975)

24
R.H. Kraichnan, Dispersion of particle pairs in homogeneous turbulence, Phys. Fluids, 9, No. 10, 1937-1943 (1966)

25
E. Knobloch, The diffusion of scalar and vector fields by homogeneous stationary turbulence, J. Fluid Mech., 83, Pt. 1, 129-140(1977)

26
T.S. Lundgren, Turbulent pair dispersion and scalar diffusion, J. Fluid Mech., 111, 25-57(1981)

27
J.H. Misguich and R. Balescu, On relative spacial diffusion in plasma and fluid turbulence: clumps, Richardson's law and intrinsic stochasticity, Plasma Phys., 24, 3(1982)

28
C.M. Tchen and J.H. Misguich, A group kinetic theory of turbulent collective collision, to appear in Theory and Modeling of Atmospheric Turbulence, Vol. 2, NASA Contractor Report

29
C.M. Tchen, Mean value and correlation problems connected with the motion of small particles suspended in a turbulent fluid, Doct. Dissertation, Delft, Netherlands(1947)

30
G.K. Batchelor, Diffusion in a field of homogeneous turbulence, Austr. J. Sci. Res., A2, No. 4, 437-450 (1949)

31
G.K. Batchelor, Diffusion in a field of homogeneous turbulence. II: The relative motion of particles, Proc Cambr. Phil. Soc., No. 2, 345-362(1952)

32
D.C. Leslie, Developments in The Theory of Turbulence, Clarendon, Oxford(1973)

33
N. Marcuvitz, On the theory of plasma turnulence, J. Math Phys., Vol. 15, No. 6, 870-879(1974)

34
E. Knobloch, Turbulent diffusion of magnetic fields, Astrophysics. J., 225, 1050-1057(1978)

35
A.S. Monin and A.M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. 1, translated and edited by J.L. Lumley, MIT Press, p230(1971)

36
C.W. Van Atta and W.Y. Chen, Correlation measurements in grid turbulence using harmonic analysis, J. Fluid Mech., 34, Pt. 3, 497-515(1968)

37
C.W. Van Atta and T.T. Yeh, Some measurements of multipoint time correlations in grid turbulence, J. Fluid Mech., 41, Pt. 1, 169-178

38
R.H. Kraichnan, Diffusion by a random velocity field, Phys. Fluids, 13, No. 1, 22-31(1970)

39
A. Okubo, A review of theoretical models for turbulent diffusion in the sea, J. Oceanogr. Soc. Japan, 20th Anniver., Volume, 286-320(1962)

40
P.J. Sullivan, Some data on the distance-neighbour function for the relative diffusion, J. Fluid Mech., 47, 601(1971)

41
T. Mikkelsen, A statistical theory on the turbulent diffusion of Gaussian puffs, RISØ Report, RISØ-R-475, Denmark(1982)
42
P.G. Saffman, An approximate calculation of the Lagrangian auto correlation coefficient for stationary homogeneous turbulence, Appl. Scient. Res., All, No. 3, 245-255(1963)
43
J. Weinstock, Lagrangian-Eulerian relation and the independence approximation, Phys. Fluids, 19, No. 11, 1702-1711(1976)
44
T. Mikkelsen, The Borris field experiment: Observations of smoke diffusion in the surface layer over homogeneous terrain, RISØ Report, RISØ-R-479, Denmark, (1983)
45
A.S. Monin and A.M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. 2, translated and edited by J.L. Lumley, MIT Press, p564(1975)