Aspects of Supercompactness, HOD and Set Theoretic Geology

by

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Abstract

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In this thesis, we study HOD, primarily in the context of large cardinals and GCH. Chapter 1 contains our introductory comments and preliminary remarks. In Chapter 2, we extend a property of HOD-supercompactness due to Sargsyan to various models of set theory containing supercompact cardinals. In doing so, we develop a new method for coding sets while preserving GCH. In Chapter 3, we extend this alternative method of coding. This allows us to produce models of $V =$ HOD and GCH in the presence of large cardinals (including supercompact cardinals). In the remaining chapters, we use this coding to extend a variety of earlier results. In Chapter 4, we generalize theorems about the Ground Axiom to models with supercompact cardinals that satisfy GCH. In Chapter 5, we extend results in set theoretic geology to models that satisfy GCH. Finally, in Chapter 6, we use the coding to produce

a model of the Wholeness Axiom, $V = \textsc{hod}$ and GCH.

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Contents

1 Introduction and preliminaries **1**

2 Extensions of the Sargsyan property **5**

2.1 The Sargsyan property and the class of supercompact cardinals 5

2.2 Obtaining the Sargsyan property over a ground model without the GCH via set forcing 14

2.3 The Sargsyan property in an inner model of V 16

2.4 The Sargsyan property in a model that preserves GCH 18

3 $V = \text{HOD}$**,** supercompacts, and GCH 24

4 The Ground Axiom **31**

4.1 A model of $\text{GA} + V = \text{HOD} + \text{GCH}$ containing a nontrivial class of supercompact cardinals 32

4.2 The class of supercompact cardinals, $V = \text{HOD}$, GCH and $\neg\text{GA}$ 34

4.3 A model for GA $+ V \neq$ HOD containing a proper class of measurable cardinals in which GCH holds 36

5 Results in Set Theoretic Geology 40

5.1 V is the mantle, generic mantle, generic HOD, and the HOD with GCH and a proper class of measurable cardinals 41

5.2 V is the mantle, generic mantle, generic HOD, but not HOD with GCH and a proper class of measurable cardinals 48

6 The Wholeness Axiom 54

Bibliography 57

Chapter 1

Introduction and preliminaries

The model HOD has long been of interest to set theorists. Philosophically, large cardinal consistency with an inner model such as HOD can be seen as further verification of the existence of large cardinals. See McAloon [McA71], Jech [Jec03], and Myhill and Scott [MS71] for an additional discussion of HOD.

In the hierarchy of large cardinals, measurable cardinals are of particular importance. Although initially pursued in the context of the measure problem, measurability was isolated from the ideas of Lebesgue measure by Ulam. In the 1960s, the study of large cardinals took a different turn with the creation of forcing by Paul Cohen and Dana Scott's result that the existence of a measurable cardinal implies $V \neq L$ (see [Jec03] for a discussion of both topics), which stimulated the investigation of elementary embeddings of inner models. In this setting, measurable cardinals came to have a central role in the discovery of large cardinals, with the result "*There exists an elementary embedding* $j : V \longrightarrow M$ *for some inner model* M *iff there is a measurable cardinal*" as a prototype for further generalizations. In particular, taking the concept of elementary embedding as basic, Solovay and Reinhardt independently generalized the concept of measurability (and strong compactness), and formulated the concept of a supercompact cardinal [SRK78]. Supercompact cardinals are extremely important in the large cardinal hierarchy, because although they are much larger than measurable cardinals in both size and consistency strength, their definition is fairly uncomplicated. This allows supercompact cardinals to figure as an upper bound of many forcing constructions involving large cardinals.

Achieving the relative consistency of supercompact cardinals with models of $V = \text{HOD}$ is relevant for the general study of inner models with large cardinals, in particular to the extension of many other results previously only achievable in relatively small extensions of L. This idea will be one of the focuses later on in this dissertation.

We will take this opportunity to mention some preliminary material that will be used throughout this dissertation. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta], [\alpha, \beta), (\alpha, \beta]$, and (α, β) are as in standard interval notation. When forcing, $q \leq p$ will mean that q *is stronger than* p. For G, V-generic over a poset $\mathbb{P}$, we will use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We will heavily abuse notation and confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V. If κ is a cardinal and $\mathbb{P}$ is a partial ordering, $\mathbb{P}$ is *κ-closed* if for every $\delta \leq \kappa$, given every sequence $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$ such that $\beta < \gamma < \delta$ implies $p_\gamma \leq p_\beta$ (a decreasing chain of length less than or equal to δ), there is some $p \in \mathbb{P}$ (a lower bound to this chain) such that $p \leq p_\alpha$ for all $\alpha < \delta$. $\mathbb{P}$ is *κ-directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$, there is a lower bound $p \in \mathbb{P}$. $\mathbb{P}$ is *κ-strategically closed* if in the two person game in which the players construct a decreasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages, player II has a strategy which ensures the game can always be continued.

We also take this opportunity to discuss a generalization of Hamkins' Gap Forcing Theorem [Ham99], [Ham01a] (as it is stated in [ACH07]), as its results are used extensively throughout this dissertation. A forcing notion $\mathbb{P}$ (and the forcing extensions to which it gives rise) *admits a closure point at* δ if it factors as $\mathbb{Q} * \dot{\mathbb{R}}$, where $\mathbb{Q}$ is nontrivial, $|\mathbb{Q}| \leq \delta$, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-

strategically closed". Our arguments will rely on the following consequence of the main result of [Ham03].

Theorem 1 ([Ham03])**.** *If* $V \subseteq V[G]$ *admits a closure point at* δ *and* $j : V[G] \to M[j(G)]$ *is an ultrapower embedding in* $V[G]$ *with* $\delta < \mathrm{cp}(j)$*, then* $j \restriction V : V \longrightarrow M$ *is a definable class in* V*.*

This theorem follows from [Ham03, Theorem 3, Corollary 14]. If $j : V[G] \to M[j(G)]$ witnesses the λ-supercompactness of κ in $V[G]$, then by [Ham03, Corollary 4], the restriction $j \restriction V : V \to M$ witnesses the λ-supercompactness of κ in V. This theorem clearly can be applied to measurability embeddings as well, which gives us the result that if our forcing exhibits the closure point property at a sufficiently small cardinal, we can infer that the measurable cardinals and supercompact cardinals of the forcing extension already existed in the ground model.

We mention that we are assuming complete familiarity with the notions of measurability, strong compactness, and supercompactness. Interested readers may consult [Jec03] for further details. We note only that all elementary embeddings witnessing the λ-supercompactness of κ are presumed to come from some fine, κ-complete, normal ultrafilter μ over $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$.

Chapter 2

Extensions of the Sargsyan property

In this chapter, we will extend an earlier result due to Sargsyan [Sar08] relating to HOD and supercompact cardinals.

2.1 The Sargsyan property and the class of supercompact cardinals

As mentioned in [Sar08], at a set theory seminar at Berkeley in 2005, Woodin asked if it were possible to construct a model of set theory in which κ is supercompact, but not HOD-supercompact. We will extend the following recent result of Sargsyan [Sar08], which answers Woodin's question with the following theorem:

Theorem 2. [Sar08] *Suppose* $V \models \mathrm{ZFC} + \mathrm{GCH} +$ "*κ is a supercompact cardinal.*" *Then there is a forcing extension of V in which κ is supercompact, but* *not* HOD*-supercompact.*

Note that the cardinal κ is HOD-*supercompact* iff κ is supercompact and for all strong limit cardinals λ, there exists an embedding $j : V \longrightarrow M$, such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$, and $j(\mathrm{HOD}) \cap V_\lambda = \mathrm{HOD} \cap V_\lambda$. We follow Sarsyan's convention in using $j(N)$, where N is a proper class, to mean $j(N) = \bigcup_{\alpha < \mathrm{ORD}} j(V_\alpha^N)$. Since $N = \mathrm{HOD}$ is a definable class, $j \restriction \mathrm{HOD} : \mathrm{HOD} \to j(\mathrm{HOD})$ is fully elementary. We will say that a supercompact cardinal κ, as in Theorem 2, satisfies the *Sargsyan property.*

There are a number of natural questions that arise as a result of this theorem. These are as follows:

1. *Can this result be extended to the class of supercompact cardinals, K, assuming K has more than one member?*

2. *Can this result be obtained by doing set forcing over a model of* ZFC *that does not satisfy* GCH*?*

3. *If* $V \models \mathrm{ZFC} + \mathrm{GCH} +$ "*κ is a supercompact cardinal*", *for any ordinal $\lambda > \kappa$, is there an inner model $M \subseteq V$ such that there is a supercompact cardinal $\kappa^* > \lambda$ which satisfies the Sargsyan property?*

4. *Can we produce an inner model as in Question 3 if we start with a model, V, that does not satisfy* GCH*?*

5. *Can the Sargsyan property hold in a forcing extension which still satisfies* GCH*?*

The following theorem answers Question 1 in the affirmative:

Theorem 3. *Let $V \models$ ZFC. Assume $K \subseteq V$, the class of supercompact cardinals, is proper. Then there is proper class partial ordering $\mathbb{P} \subseteq V$ such that:*

(1) $V^{\mathbb{P}} \models$ ZFC+ *"If $\kappa \in K$, then κ is a supercompact cardinal whose supercompactness is indestructible under κ-directed closed forcing."*

(2) $V^{\mathbb{P}} \models$ *"K is the class of supercompact cardinals."*

(3) $V^{\mathbb{P}} \models$ *"The only strongly compact cardinals are the elements of K or their measurable limit points."*

(4) $V^{\mathbb{P}} \models$ *"If $\kappa \in K$, then κ is supercompact, but not* HOD*-supercompact."*

(5) $V^{\mathbb{P}} \models$ *"If δ is a measurable limit point of K, then the strong compactness of δ is indestructible under δ-directed closed forcing which doesn't change $P(\delta)$."*

The model $V^{\mathbb{P}}$ of Theorem 3 is an example of a model as in Apter [Apt98], but with the additional feature that no supercompact cardinal κ is HOD-supercompact. Notice that in the hypotheses for Theorem 3 , K is explicitly taken as being a proper class. If K is a set, the forcing $\mathbb{P}$ used will be identical to when K is a proper class. If there exists a measurable cardinal in $V^{\mathbb{P}}$ greater than the supremum of K, then our desired model will be $V^{\mathbb{P}}$ truncated at the least measurable cardinal greater than $\sup(K)$. Otherwise, $V^{\mathbb{P}}$ will be our desired model.

Let $\gamma < \kappa$ be such that γ and κ are regular cardinals. For Theorem 3, we use the standard notion of forcing $\mathbb{P}_{\gamma,\kappa}$ for adding a non-reflecting stationary set of ordinals of cofinality γ to κ. Specifically, $\mathbb{P}_{\gamma,\kappa} = \{p :$ For some $\alpha < \kappa$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, such that $\beta \in S_p$ implies $\beta > \gamma$ and $\mathrm{cof}(\beta) = \gamma\}$, ordered by $q \leq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. See Apter [Apt98] for more details. We also use $\mathrm{Add}(\gamma, \kappa)$, the standard partial ordering for adding κ many Cohen subsets of γ.

The following partial ordering, used by Sargsyan in [Sar08], is designed to code sets of ordinals into the continuum function and hence, into HOD.

Definition 4. *Suppose $\kappa < \lambda$ are cardinals and A is a subset of κ. Let λ_α be the $(\alpha + 1)^{\mathrm{st}}$ successor cardinal strictly greater than λ. Let* $\mathbb{S}_{\kappa,\lambda}(A) = \prod_{\alpha \in A} \mathrm{Add}(\lambda_\alpha, \lambda_\alpha^{++})$.

If GCH holds in V then in $V^{\mathbb{S}_{\kappa,\lambda}(A)}$, $\alpha \in A \leftrightarrow 2^{\lambda_\alpha} = \lambda_\alpha^{++}$. This implies that in $V^{\mathbb{S}_{\kappa,\lambda}(A)}$, $A \in \mathrm{HOD}$.

We turn now to the proof of Theorem 3.

Proof. Without loss of generality, we can assume that V is a model of GCH. Let f be a universal Laver function for K, as in Apter [Apt98]. Let $S = \{\delta : \delta$ is a measurable cardinal and $f''\delta \subseteq V_\delta\}$.

The partial ordering $\mathbb{P}$ used in the proof of Theorem 3 will be a reverse Easton support proper class iteration $\langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle : \alpha \in \mathrm{ORD}\rangle$. At every stage α, an ordinal ρ_α will be chosen. We start with $\mathbb{P}_0 = \mathrm{Add}(\omega, 1)$ and $\rho_0 = 0$. At limit ordinals γ, $\rho_\gamma = \sup_{\beta<\gamma} \rho_\beta$. For $\kappa \in K$, $\dot{\mathbb{Q}}_\kappa$ will be a term for $\mathrm{Add}(\kappa, 1)$. Otherwise, $\dot{\mathbb{Q}}_\alpha$ will be a term for trivial forcing and $\rho_{\alpha+1} = \rho_\alpha$, unless the following conditions hold:

(i) $\alpha \in S$.

(ii) $\forall \beta < \alpha$, $\rho_\beta < \alpha$ and $f(\alpha) = \langle \dot{\mathbb{Q}}, \sigma \rangle$, where $\Vdash_{\mathbb{P}_\alpha * \dot{\mathrm{Add}}(\alpha,1)}$ "$\dot{\mathbb{Q}}$ is α-directed closed" and σ is a regular cardinal $> \max\{\alpha, |\mathrm{TC}(\dot{\mathbb{Q}})|\}$.

Under these circumstances, let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathrm{Add}}(\alpha, 1) * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\gamma_\alpha,\sigma^+} * \mathbb{S}_{\alpha,\sigma^+}(\dot{X})$ and $\rho_{\alpha+1} = (\beth(\sigma^+))^+$, where $\beth(\sigma^+)$ is the least beth fixed point $> \sigma^+$, $\dot{X}$ is the name of the generic subset of α added by $\mathrm{Add}(\alpha, 1)$, $\mathbb{S}_{\alpha,\sigma^+}(\dot{X})$ is a term for the forcing in Definition 4 and γ_α is the least regular cardinal $\geq$ sup

$(\{\kappa \in K : \kappa < \alpha\}) =_{df} \lambda_\alpha$ (where if there are no supercompacts less than α, $\lambda_\alpha = \omega$).

Lemma 4.1. $V^{\mathbb{P}} \models$ ZFC + *"If $\kappa \in K$, then κ is a supercompact cardinal whose supercompactness is indestructible under κ-directed closed forcing."*

Proof. It suffices to show that κ can be made indestructibly supercompact by forcing with $\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1)$, since the tail forcing is κ-directed closed. Let $G * g \subseteq \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1)$ be V-generic. Now we want to show that $V[G][g] \models$ "κ is supercompact" and $V[G][g] \models$ "κ's supercompactness is indestructible by κ-directed closed forcing."

Fix any $\mathbb{Q} \in V[G][g]$ which is κ-directed closed. Fix any $\dot{\mathbb{Q}}$, a name for $\mathbb{Q}$, for which $1 \Vdash$ "$\dot{\mathbb{Q}}$ is κ-directed closed." Let $g^* \subseteq \mathbb{Q}$ be $V[G][g]$-generic. We want to show $V[G][g][g^*] \models$ "κ is λ-supercompact" for arbitrary limit cardinals $\lambda > \kappa$. Fix any $\lambda > \kappa$, λ a limit cardinal such that $\dot{\mathbb{Q}} \in H_\lambda$ and let $\theta >> 2^{\lambda^{<\kappa}}$ be a regular cardinal. Fix in V a θ-supercompactness embedding $j : V \longrightarrow M$ such that $j(f)(\kappa) = \langle \dot{\mathbb{Q}}, \theta \rangle$. We have that $j(\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1))$ $= \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1) * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\gamma_\kappa, \theta^+} * \mathbb{S}_{\kappa, \theta^+}(\dot{X}) * \dot{\mathbb{P}}_{tail} * \dot{\mathrm{Add}}(j(\kappa), 1)$, with $\dot{X}$ being the name for the generic subset added by $\mathrm{Add}(\kappa, 1)$. Force to add $G^* \subseteq \mathbb{P}_{\gamma_\kappa, \theta^+} * \mathbb{S}_{\kappa, \theta^+}(\dot{X}) * \dot{\mathbb{P}}_{tail}$ a $V[G][g][g^*]$-generic. Then j lifts to $j : V[G] \longrightarrow M[j(G)]$ in $V[G][g][g^*][G^*]$ and $j(G) = G * g * g^* * G^*$. In order to complete the proof of the lemma we need to lift again. We can, if $j''(g * g^*) \in M[j(G)]$. It is by the usual argument, since $g * g^* \in M[j(G)]$ and $M[j(G)]^\theta \subseteq M[j(G)]$ in $V[G][g][g^*][G^*]$. Since $j''(g * g^*) \in M[j(G)]$, there is a master condition p^* below $j''(g * g^*)$. This is because $j(\mathrm{Add}(\kappa, 1) * \dot{\mathbb{Q}})$ is $j(\kappa)$-directed closed, so we can force to add an H which is $V[G][g][g^*][G^*]$-generic such that $p^* \in H$. Now we can lift j to $j : V[G][g][g^*] \longrightarrow M[j(G)][H]$ in $V[G][g][g^*][G^*][H]$. Since we chose θ to be sufficiently large, $G^* * H$ is generic over a partial ordering which does not add any ultrafilters over $(P_\kappa(\lambda))^{V[G][g][g^*]}$. We thus have that $V[G][g][g^*] \models$ "κ is λ-supercompact". □

Lemma 4.2. *K is the class of supercompact cardinals in $V^{\mathbb{P}}$*

Proof. Since we have just shown that for $\kappa \in K$, κ remains supercompact, it suffices to show that no new supercompact cardinals are created by $\mathbb{P}$. Since $\mathbb{P}$ admits a closure point at ω, by an application of Theorem 1, no new supercompacts were created by $\mathbb{P}$. □

Lemma 4.3. $V^{\mathbb{P}} \models$ *"The only strongly compact cardinals are the elements of K or their measurable limit points."*

Proof. The proof of this lemma uses ideas from the proof of Lemma 3 in Apter [Apt98]. If δ is a strongly compact cardinal in $V^{\mathbb{P}}$ which is neither an element of K nor a measurable limit point of K, then $\delta \in (\gamma, \gamma')$, where

γ is the least regular cardinal $\geq \sup(\{\kappa \in K : \kappa < \delta\})$ and γ' is the least element of $K > \delta$. By the same arguments as in Lemma 3 of Apter [Apt98], in $V^{\mathbb{P}}$, unboundedly many cardinals in γ' in the open interval (γ, γ'), contain non-reflecting stationary sets of ordinals of cofinality γ. By Theorem 4.8 of [SRK78] and the succeeding remarks, if ζ contains a non-reflecting stationary set of ordinals of cofinality γ, then there are no strongly compact cardinals in the interval $(\gamma, \zeta]$. Thus, the last two sentences immediately imply that $V^{\mathbb{P}} \models$ "No cardinal in the interval (γ, γ') is strongly compact". This proves Lemma 4.3.

□

Lemma 4.4. $V^{\mathbb{P}} \models$ *"If $\kappa \in K$, then κ is not* HOD*-supercompact."*

Proof. This proof follows closely the proof of Lemma 2.2 in Sargsyan [Sar08]. Let $G \subseteq \mathbb{P}$ be V-generic. Factor $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{tail}$, where $\mathbb{P}_\kappa$ is the forcing up to stage κ, $\dot{\mathbb{Q}}_\kappa = \dot{\mathrm{A}}\mathrm{dd}(\kappa, 1)$ and $\dot{\mathbb{P}}_{tail}$ is a term for the forcing beyond κ. Let $G = G_\kappa * g * G_{tail}$ be the corresponding generics. Assume κ is HOD-supercompact in $V[G] = W$ and let $\text{HOD} = \text{HOD}^W$. Fix a strong limit cardinal λ such that $\text{HOD}^{W_\lambda} = \text{HOD} \cap W_\lambda$. Let $j : W \longrightarrow M$ be a λ-supercompactness embedding such that $j(\text{HOD}) \cap W_\lambda = \text{HOD} \cap W_\lambda = \text{HOD}^{W_\lambda}$. By Theorem 1, $i = j \restriction V$ is definable in V and j is the lift of i. Let $N = j(V) = \bigcup_{\alpha < \text{ORD}} i(V_\alpha)$.

If H is the N-generic for $i(\mathbb{P})$, then $M = N[H] = N[j(G)]$. We also have that $H \cap \mathbb{P}_\kappa = G_\kappa$. Let g' be the generic for $\mathrm{Add}(\kappa, 1)$ given by H. Then in $N[H] = M$, g' is ordinal definable. But because κ is HOD-supercompact, $j(\text{HOD}) \cap W_\lambda = \text{HOD} \cap W_\lambda = \text{HOD}^{W_\lambda}$. This implies $g' \in \text{HOD}^{W_\lambda}$. Thus g' is ordinal definable in $W_\lambda = V_\lambda^{V[G]} = V_\lambda^{V[G_\kappa][g][G_{tail}]}$. Since $\mathbb{P}_{tail}$ is κ-closed, g' could not have been added by $\mathbb{P}_{tail}$. So g' had to have been added over $V_\lambda[G_\kappa]$, and more particularly, g' is added over $V_\lambda[G_\kappa]$ by homogeneous forcing. This fact, along with g' being ordinal definable in W_λ, implies that g' is in $V_\lambda[G_\kappa]$. This is impossible, as g' is a $V[G_\kappa]$-generic object for $\mathrm{Add}(\kappa, 1)$. Therefore κ is not HOD-supercompact. □

Lemma 4.5. $V^{\mathbb{P}} \models$ *"If δ is a measurable limit point of K, then the strong compactness of δ is indestructible under δ-directed closed forcing which doesn't change $P(\delta)$".*

Proof. The proof is the same as Lemma 4 in Apter [Apt98]. Namely, let $\mathbb{Q} \in V^{\mathbb{P}}$ be such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is δ-directed closed and forcing with $\dot{\mathbb{Q}}$ doesn't change $P(\delta)$". By Lemma 4.1, if $\gamma \in K$, $\gamma < \delta$, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "γ is supercompact". Thus, by the choice of $\mathbb{Q}$, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "δ is a measurable cardinal", so $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "δ is a measurable limit of supercompact cardinals and hence is strongly compact".

□

Lemmas 4.1-4.5 complete the proof Theorem 3. $\square$

2.2 Obtaining the Sargsyan property over a ground model without the GCH via set forcing

Thus far we have been working with a ground model which satisfies GCH. This brings us now to Question 2: Can the Sargsyan theorem be obtained by doing set forcing over a model of ZFC that does not satisfy GCH? We answer this question in the affirmative with the following theorem:

Theorem 5. *Let $V \models$ ZFC + "κ is a supercompact cardinal." Then there is a set forcing extension of V in which κ is supercompact, but not HOD-supercompact.*

Proof. Let f be a Laver function for κ. We now have the following definition (which only makes sense if $f(\alpha)$ is an ordinal).

Definition 6. *Assume α is a cardinal. Let $\mathbb{B}_\gamma(\alpha)$ be the forcing to make GCH hold at α many consecutive cardinals starting at γ, where γ is the least strong limit cardinal $> f(\alpha)$.*

Let $S = \{\delta : \delta \text{ is a measurable cardinal, } f''\delta \subseteq \delta \text{ and } f(\delta) > \delta\}$. Our partial ordering $\mathbb{P}$ will be a length $\kappa + 1$ reverse Easton support iteration with $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1)$. Let $\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle : \alpha < \kappa\rangle$ with $\mathbb{P}_0 = \mathrm{Add}(\omega, 1)$. Then $\dot{\mathbb{Q}}_\alpha$ will be a term for trivial forcing unless $\alpha \in S$. If $\alpha \in S$, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Add}}(\alpha, 1) * \dot{\mathbb{B}}_\gamma(\alpha) * \mathbb{S}_{\alpha,\gamma}(\dot{X})$, where $\dot{X}$ is the name of the generic subset of α added by $\mathrm{Add}(\alpha, 1)$ and $\mathbb{S}_{\alpha,\gamma}(\dot{X})$ is a term for the forcing defined above in Definition 4 making GCH fail above γ. Let $G \subseteq \mathbb{P}_\kappa$ be V-generic and let g be $V[G]$-generic for $(\mathrm{Add}(\kappa, 1))^{V[G]}$.

Lemma 6.1. $V[G][g] \models$ "*κ is supercompact.*"

Proof. Let $\lambda > \kappa$ be an arbitrary strong limit cardinal and let $j : V \longrightarrow M$ be a λ-supercompactness embedding with $j(f)(\kappa) = \lambda$. Since $\kappa \in j(S)$, the stage κ forcing in $M^{\mathbb{P}_\kappa}$ is nontrivial. We have that $j(\mathbb{P}) = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1) * \dot{\mathbb{B}}_\gamma(\kappa) * \mathbb{S}_{\kappa,\gamma}(\dot{X}) * \dot{\mathbb{P}}_{tail}$, with $\dot{X}$ being the name for the generic subset added by $\mathrm{Add}(\kappa, 1)$, and $\dot{\mathbb{P}}_{tail}$ a term for the remainder of the forcing up to and including stage $j(\kappa)$. Since the first stage of nontrivial forcing in $\mathbb{B}_\gamma(\kappa) * \mathbb{S}_{\kappa,\gamma}(\dot{X})$ is beyond $\gamma > \lambda$, we may write $j(\mathbb{P}) = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1) * \dot{\mathbb{P}}_{tail}$, where the first stage of nontrivial forcing in $\dot{\mathbb{P}}_{tail}$ is greater than λ. Standard arguments (see the proofs of Lemma 4.1 and Lemma 2.1 of [Sar08]) show that for any cardinal $\delta < \lambda$, $V[G][g] \models$ "κ is δ-supercompact". Since λ was arbitrary, this completes the proof of Lemma 6.1. □

Lemma 6.2. $V[G][g] \models$ "*κ is not* HOD*-supercompact.*"

Proof. Since $\mathbb{P}$ admits a closure point at ω, Sargsyan's argument of Lemma 2.2 in [Sar08](which was generalized in Lemma 4.4 above) shows that κ is not HOD-supercompact in $V[G][g]$. $\square$

Lemmas 6.1-6.2 prove Theorem 5. $\square$

2.3 The Sargsyan property in an inner model of V

Section 2.3 answers Questions 3 and 4. We begin with Question 3, i.e., if $V \models$ ZFC+GCH+ "κ is a supercompact cardinal", then for any ordinal $\lambda > \kappa$, is there an inner model $M \subseteq V$ such that there is a supercompact cardinal $\kappa^* > \lambda$ which satisfies the Sargsyan property? Note that λ is not particularly relevant to the question, i.e., we essentially will show that for any model that contains a supercompact cardinal with the Sargsyan property, then for every λ we can find an inner model with the desired property, just by iterating the ultrapower for an arbitrary number of steps.

This leads us to the following theorem:

Theorem 7. *Let $V \models$ ZFC + GCH + "κ is supercompact." For any ordinal $\lambda > \kappa$, there exists an $M \subseteq V$, M an inner model of V such that:*

- *$M \models$ "$\kappa^* > \lambda$ and κ^* is supercompact."*

- $M \models$ "*κ^* is supercompact, but not* HOD*-supercompact.*"

Proof. Note that the Sargsyan partial ordering $\mathbb{P}$ is *friendly* with respect to the supercompact cardinal κ, i.e., for every $\delta < \kappa$, there is a final segment of $\mathbb{P}$ which is δ-directed closed. Therefore, Theorem 7 follows by forthcoming results of Apter, Gitman and Hamkins. □

We turn now to Question 4, i.e., can we produce an inner model as in Question 3 if we start with a model that does not satisfy GCH? This leads us to the following theorem:

Theorem 8. *Let $V \models$ ZFC + "κ is supercompact." For any ordinal $\lambda > \kappa$, there exists an $M \subseteq V$, M an inner model of V such that:*

- $M \models$ "*$\kappa^* > \lambda$ and κ^* is supercompact.*"
- $M \models$ "*κ^* is supercompact, but not HOD supercompact.*"

Proof. Let $\mathbb{P}$ be the partial ordering of Theorem 5. Since $\mathbb{P}$ is friendly, again by forthcoming work of Apter, Gitman and Hamkins, it suffices to find a $\lambda^* > \lambda$ and an elementary embedding $j : V \longrightarrow M$ witnessing the λ-supercompactness of κ such that $|j(2^\kappa)| = (\lambda^*)^+$. So let $\lambda^* > \lambda$ be an arbitrary singular strong limit cardinal of cofinality κ. Then $|j(2^\kappa)| = (2^\kappa)^{[\lambda^*]^{<\kappa}} = (2^\kappa)^{\lambda^*} = (\lambda^*)^+$. The fact that $[\lambda^*]^{<\kappa} = \lambda^*$ follows since λ^* is a singular strong limit cardinal of cofinality κ. The fact that $(2^\kappa)^{\lambda^*} = (\lambda^*)^+$ follows by Solovay's theorem [Sol74] that GCH holds at any singular strong limit cardinal above a strongly compact cardinal, together with the fact that since λ^* is a strong limit cardinal, $2^\kappa < \lambda^*$. □

It should be noted that the work that was cited and adapted in this section consists of very new and unpublished results, and the applications will only be more secure once that work is finalized as expected.

2.4 The Sargsyan property in a model that preserves GCH

The key to Sargsyan's proof is the coding of generic subsets into HOD, but his coding requires significant violations of GCH. This brings us to Question 5, i.e., can the Sargsyan property hold in a forcing extension which still satisfies GCH?

The following theorem answers Question 5 in the affirmative.

Theorem 9. *Let* $V \models$ ZFC + GCH + "*κ is a supercompact cardinal.*" *Then there is a forcing extension* $V^{\mathbb{P}}$ *such that* $V^{\mathbb{P}} \models$ GCH + "*κ is supercompact, but not* HOD*-supercompact.*"

Before proving Theorem 9, we present some relevant definitions and lemmas.

Definition 10. *Let* σ *be a cardinal. Then* $\mathrm{Coll}(\sigma^+, \sigma^{++})$ *is the standard Lévy collapse of* σ^{++} *to* σ^+*.*

Definition 11. *Let* $\alpha < \gamma$*. Let* $M_{\alpha,\gamma}$ *be the reverse Easton support iteration using* $\mathrm{Add}(\delta^+, 1)$ *at every inaccessible cardinal* $\delta \in (\alpha, \gamma)$ *and trivial forcing at all other stages.*

Standard arguments show that this forcing preserves GCH. In addition, standard arguments (see the first paragraph of the proof of the main theorem of [ACH07]) together with Theorem 1 show that no new measurable cardinals are created, all ground model measurable cardinals are preserved, and every measurable cardinal δ in (α, γ) carries the maximum number of normal measures, i.e., $2^{2^\delta} = \delta^{++}$.

Definition 12. *Let* σ *be a cardinal. Let* γ_σ *be the supremum of the first* σ *many measurable cardinals beyond* σ*. Suppose that* $A \subseteq \sigma$*. For every* $\alpha \in A$*, let* σ_α *be the* $(\alpha + 1)^{\mathrm{st}}$ *measurable cardinal beyond* σ *and then let* $\mathbb{C}\mathrm{oll}_{\sigma,\gamma_\sigma}(\mathrm{A})$ *be the reverse Easton support iteration which forces with* $\mathrm{Coll}(\sigma_\alpha^+, \sigma_\alpha^{++})$ *for every* $\alpha \in A$*, and does trivial forcing otherwise. Now we define* $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A) = \mathrm{Add}(\sigma^+, 1) * \dot{\mathbb{C}}\mathrm{oll}_{\sigma,\gamma_\sigma}(A)$*.*

Let $\bar{V} = V^{M_{\sigma,\gamma_\sigma}}$ be our ground model.

Lemma 12.1. *Suppose* $A \subseteq \sigma, A \in \bar{V}$. *Forcing over* $\bar{V}$ *with* $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)$ *will give us the required number of normal measures in our forcing extension, i.e., in* $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)}$

(i) *if* $\alpha \in A$, σ_α *carries fewer than the maximum number of normal measures, i.e.,* σ_α^+ *many normal measures.*

(ii) *if* $\alpha \notin A$, σ_α *carries the maximum number of normal measures, i.e.,* $2^{2^{\sigma_\alpha}} = \sigma_\alpha^{++}$ *many normal measures.*

Proof. We begin by observing that standard arguments show that GCH is preserved to $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)}$. Also, by Theorem 1, no new measurable cardinals in the open interval (σ, γ_σ) are created by forcing with $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)$. Since our proof will show that all $\bar{V}$-measurable cardinals in (σ, γ_σ) are preserved when forcing with $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)$, we will write σ_α without fear of ambiguity.

(i) If $\alpha \in A$, then we force with $\mathrm{Coll}(\sigma_\alpha^+, \sigma_\alpha^{++})$ at stage σ_α. Let $\mathbb{P}_0$ be the forcing up until stage σ_α, let $\mathbb{P}_1 = \mathrm{Coll}(\sigma_\alpha^+, \sigma_\alpha^{++})$ be the forcing at stage σ_α, and let $\mathbb{P}_2$ be the forcing beyond stage σ_α. Since $|\mathbb{P}_0| < \sigma_\alpha$, and the next forcing is $\mathrm{Coll}(\sigma_\alpha^+, \sigma_\alpha^{++})$, the arguments of [ACH07] hold (Main Theorem, paragraph 2). Namely, $\bar{V}^{\mathbb{P}_0 * \dot{\mathbb{P}}_1} \models$ "σ_α carries σ_α^+ many normal measures." Additionally, any nontrivial forcing beyond σ_α will be sufficiently closed so as not to add any new normal measures to σ_α.

So in $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)} = \bar{V}^{\mathbb{P}_0 * \dot{\mathbb{P}}_1 * \dot{\mathbb{P}}_2}$, there are exactly σ_α^+ many normal measures on σ_α, as desired.

(ii) If $\alpha \notin A$, then no nontrivial forcing occurs at σ_α. Any nontrivial forcing which occurs below σ_α will be small with respect to it, and will not affect the number of normal measures on σ_α. In addition, any nontrivial forcing beyond σ_α will be sufficiently closed so as not to add any new normal measures to σ_α. So in $\bar{V}^{\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(A)} = \bar{V}^{\mathbb{P}_0 * \dot{\mathbb{P}}_1 * \dot{\mathbb{P}}_2}$, there are exactly $\sigma_\alpha^{++} = 2^{2^{\sigma_\alpha}}$ many normal measures on σ_α, as desired.

□

We now can prove Theorem 9.

Proof. Let f be a Laver function for κ. Let $S = \{\delta : \delta$ is a measurable limit of measurable cardinals, $f''\delta \subseteq \delta$ and $f(\delta) > \delta\}$. Our partial ordering $\mathbb{P}$ will be a length $\kappa + 1$ reverse Easton support iteration with $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1)$. Let $\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle : \alpha < \kappa\rangle$ with $\mathbb{P}_0 = \mathrm{Add}(\omega, 1)$. Then $\dot{\mathbb{Q}}_\alpha$ will be a term for trivial forcing, unless $\alpha \in S$. If $\alpha \in S$, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Add}}(\alpha, 1) * \dot{M}_{\alpha^*,\gamma_{\alpha^*}} * \bar{\mathbb{S}}_{\alpha^*,\gamma_{\alpha^*}}(\dot{X})$, where $\dot{X}$ is the name of the generic subset of α added by $\mathrm{Add}(\alpha, 1)$, α^* is the least measurable cardinal greater than $f(\alpha)$ and γ_{α^*} is the supremum of the first α^* many measurable cardinals beyond α^*. Thus, forcing with $\mathbb{Q}_\alpha$ introduces a new subset X of α and codes X beyond $f(\alpha)$ by first forcing to blow up the number of normal measures on the first α^* many consecutive measurable cardinals beyond $f(\alpha)$ and then forcing to reduce the number of normal measures according to the subset added to α. So $\eta \in X \leftrightarrow \sigma_\eta$ carries σ_η^+ many normal measures.

Let $G \subseteq \mathbb{P}_\kappa$ be V-generic, and let g be $V[G]$-generic for $(\mathrm{Add}(\kappa, 1))^{V[G]}$. Standard arguments show that $V[G][g] \models$ GCH.

We follow the proof of Sargsyan.

Lemma 12.2. $V[G][g] \models$ "*κ is supercompact.*"

Proof. Let $\lambda > \kappa$ be an arbitrary strong limit cardinal, and let $j : V \longrightarrow M$ be a λ-supercompactness embedding with $j(f)(\kappa) = \lambda$. Since $\kappa \in j(S)$, the stage κ forcing in $M^{\mathbb{P}_\kappa}$ is nontrivial, and we have that $j(\mathbb{P}) = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1) * \dot{M}_{\kappa^*,\gamma_{\kappa^*}} * \bar{\dot{\mathbb{S}}}_{\kappa^*,\gamma_{\kappa^*}}(\dot{X}) * \mathbb{P}_{tail}$, with $\dot{X}$ the name for the generic subset added by $\mathrm{Add}(\kappa, 1)$, κ^* the least measurable cardinal greater than λ in M and $\dot{\mathbb{P}}_{tail}$ a term for the forcing defined in the half-open interval $(\gamma_{\kappa^*}, j(\kappa)]$. Since the first stage of nontrivial forcing in $M_{\kappa^*,\gamma_{\kappa^*}} * \dot{\bar{\mathbb{S}}}_{\kappa^*,\gamma_{\kappa^*}}(\dot{X})$ is beyond λ, we may write $j(\mathbb{P})$ as $\mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1) * \dot{\mathbb{P}}_{tail}$, where the first stage of nontrivial forcing in $\dot{\mathbb{P}}_{tail}$ is beyond λ. As in Lemma 6.1, we can show that for any cardinal $\gamma < \lambda$, $V[G][g] \models$ "κ is γ-supercompact". Since λ was arbitrary, this completes the proof of the Lemma 12.2. $\square$

Lemma 12.3. $V[G][g] \models$ "*κ is not* HOD*-supercompact.*"

Proof. Since $\mathbb{P}$ admits a closure point at ω, Sargsyan's argument of Lemma 2.2 in [Sar08] once again shows that κ is not HOD-supercompact in $V[G][g]$.

$\square$

Lemmas 12.2-12.3 prove Theorem 9. $\square$

Chapter 3

Getting a model of $V =$ HOD with a proper class of supercompact cardinals that satisfies GCH

The standard method of forcing $V =$ HOD while preserving large cardinals, using the continuum function as an oracle, involves forcing significant failures of the GCH. Recently, a method for forcing $V =$ HOD while preserving certain large cardinals and GCH was developed by Brooke-Taylor [BT09]. His method involves using whether $\diamondsuit^*$ holds as an oracle, and his proofs do not indicate how to preserve supercompactness in general. In this chapter, we will describe an alternative method of forcing $V =$ HOD while preserving the GCH and a proper class of supercompact cardinals. This method will be an extension of the coding introduced in Section 2.4.

Before stating the main theorem of this chapter, we define a building block of our forcing, which we call the lottery sum after Hamkins [Ham00]. Specifically, the *lottery sum* of a collection A of posets is defined as $\oplus A = \{\langle \mathbb{Q}, p\rangle : \mathbb{Q} \in A \text{ and } p \in \mathbb{Q}\} \cup \{\mathbf{1}\}$, ordered with $\mathbf{1}$ above everything and $\langle \mathbb{Q}, p\rangle \leq \langle \mathbb{Q}', q\rangle$ when $\mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} q$. Since all compatible conditions must be in the same $\mathbb{Q}$, the forcing effectively holds a lottery of all the posets in A, and the generic chooses the "winning" poset $\mathbb{Q}$ and then forces with it.

We now state formally the main theorem of this chapter.

Theorem 13. *Let $V \models$ ZFC + GCH + "There is a (proper) class of supercompact cardinals K." Then there is a class forcing $\mathbb{Q} \subseteq V$ such that $V^{\mathbb{Q}} \models$ ZFC + GCH + $V =$ HOD + "K is the class of supercompact cardinals."*

Proof. Let $\mathbb{M}$ be the reverse Easton support class iteration using $\mathrm{Add}(\delta^+, 1)$ at every inaccessible cardinal δ and trivial forcing at all other stages. Let $\bar{V} = V^{\mathbb{M}}$. As in Section 2.4, $\bar{V} \models$ GCH+ "Every measurable cardinal carries the maximum number of normal measures." In addition, standard arguments show that for every $\kappa \in K$, $\bar{V} \models$ "κ is supercompact." Therefore, as in Lemma 4.2, $\bar{V} \models$ "K is the class of supercompact cardinals."

We now work in $\bar{V}$. Let $\mathbb{C}_\sigma = \mathrm{Coll}(\sigma^+, \sigma^{++})$. Let $\mathbb{B}_\sigma = \{\emptyset\}$. Let $\mathbb{P}$ be the reverse Easton support class iteration defined as follows. Let $\mathbb{P} =$

$\mathrm{Add}(\omega, 1) * \langle \dot{\mathbb{Q}}_\sigma : \sigma \in \textsc{ord} \rangle$. Then $\dot{\mathbb{Q}}_\sigma$ will be a term for trivial forcing unless $\sigma \in \bar{V}$ is a "successor" measurable cardinal, i.e., σ is not a measurable limit of measurable cardinals in $\bar{V}$. At a nontrivial stage of forcing σ, we let $\dot{\mathbb{Q}}_\sigma$ be a term for the lottery sum between the collapse forcing $\mathbb{C}_\sigma$ and trivial forcing, i.e., $\dot{\mathbb{Q}}_\sigma = \oplus\{\dot{\mathbb{C}}_\sigma, \dot{\mathbb{B}}_\sigma\} = \oplus\{\dot{\mathrm{C}}\mathrm{oll}(\sigma^+, \sigma^{++}), \{\emptyset\}\}$. We let the "generic decide" which "bit" of information will be coded.

Lemma 13.1. *In $\bar{V}^{\mathbb{P}}$, every set of ordinals is ordinal definable, i.e., $\bar{V}^{\mathbb{P}} \models V = \textsc{hod}$.*

Proof. First of all, we need a lemma to ensure that our coding is the same in $\bar{V}^{\mathbb{P}}$ as in $\bar{V}$.

Lemma 13.2. $\bar{V}^{\mathbb{P}} \models$ *"δ is a successor measurable cardinal"* $\leftrightarrow$ $\bar{V} \models$ *"δ is a successor measurable cardinal."*

Proof. Since $\mathbb{P}$ admits a closure point at ω, by Theorem 1, no new measurable cardinals were created by $\mathbb{P}$.

(i) $\Longleftarrow$ Let δ be a successor measurable cardinal in $\bar{V}$. Let $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$, where $\mathbb{P}_\delta$ is the forcing up to stage δ. Since δ is a successor measurable cardinal in $\bar{V}$, the forcing below stage δ is small forcing with respect to δ. It will therefore not affect the measurability of δ, and the remnant of the forcing is sufficiently closed so as not to affect δ's measurability. So δ is measurable in $\bar{V}^{\mathbb{P}}$.

Suppose that δ is now a measurable limit of measurable cardinals in $\bar{V}^{\mathbb{P}}$. If that were the case, then many new measurable cardinals would have to have been created by $\mathbb{P}$. As before, by Theorem 1, no new measurable cardinals were created. So successor measurable cardinals are preserved to $\bar{V}^{\mathbb{P}}$. In particular, δ is a successor measurable cardinal in $\bar{V}^{\mathbb{P}}$.

(ii) $\Longrightarrow$ Let δ be a successor measurable cardinal in $\bar{V}^{\mathbb{P}}$. Since no new measurable cardinals were created by $\mathbb{P}$, we know that δ was measurable in $\bar{V}$. Suppose that δ were a measurable limit of measurable cardinals in $\bar{V}$. In particular, it was a measurable limit of successor measurable cardinals. But the successor measurable cardinals are preserved by $\mathbb{P}$ (see (i)), so δ would remain a measurable limit of measurable cardinals in $\bar{V}^{\mathbb{P}}$. This is a contradiction, so δ was a successor measurable cardinal in $\bar{V}$.

□

We return to the proof of Lemma 13.1. Now it will suffice to show that $\forall A \subseteq \textsc{ord}$, $A \in \bar{V}^{\mathbb{P}}$, $D_A = \{p \in \mathbb{P} : p \Vdash \text{“}\dot{A} \text{ is coded”}\}$ is dense in $\mathbb{P}$. Fix some $A \in \bar{V}^{\mathbb{P}}$. There exists a δ such that $A \in \bar{V}^{\mathbb{P}_\delta}$. Fix any $p \in \mathbb{P}$ and an ordinal ρ such that $p \Vdash$ "$\dot{A} \subseteq \rho$". Since $\mathbb{P}$ uses Easton support, support$(p) \subseteq \gamma$ for some γ. Let σ be the least measurable cardinal $> \max(\gamma, \delta, \rho)$. Since there exists a proper class of supercompact cardinals, in particular σ exists, and there are ρ many measurable cardinals above σ. Then for every $\alpha \in A$, as in Theorem 9, let σ_α be the $(\alpha + 1)^{\text{st}}$ measurable cardinal beyond σ. So now we can extend p by going to σ and then if $\alpha \in A$, at stage σ_α, extending p to a condition which forces that $\mathbb{C}_{\sigma_\alpha}$ will be chosen. Therefore p can be extended to code A for any $A \in \bar{V}^{\mathbb{P}}$, $A \subseteq \text{ORD}$. Therefore $\bar{V}^{\mathbb{P}} \models V = \text{HOD}$. □

Lemma 13.3. $\bar{V}^{\mathbb{P}} \models$ "*K is the class of supercompact cardinals.*"

Proof. Again, since $\mathbb{P}$ admits a closure point at ω, by an application of Theorem 1, no new supercompacts were created by $\mathbb{P}$. So it suffices to show that all supercompacts in $\bar{V}$ are preserved to $\bar{V}^{\mathbb{P}}$.

Let $\kappa \in K$. Now it remains to show that κ is supercompact in $\bar{V}^{\mathbb{P}}$.

Let $G \subseteq \mathbb{P}$ be $\bar{V}$-generic. Let $\lambda > \kappa$ be such that λ has cofinality κ and is a limit of measurable cardinals. Note that $\lambda^{<\kappa} = \lambda$. Let $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda} * \dot{\mathbb{P}}_{tail}$ and let $G = G_\kappa * G_{\kappa,\lambda} * G_{tail}$, where $\mathbb{P}_\kappa$ is the forcing up to stage κ, $\dot{\mathbb{P}}_{\kappa,\lambda}$ is a term for the forcing from κ to λ, and $\dot{\mathbb{P}}_{tail}$ is a term for the forcing beyond λ. $\mathbb{P}_{tail}$ will be sufficiently closed so that by the choice of λ, $\bar{V}^{\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda}} \models$

"κ is λ-supercompact" $\leftrightarrow$ $\bar{V}^{\mathbb{P}} \models$ "κ is λ-supercompact". Since λ may be chosen to be arbitrarily large, it will suffice to show that $\bar{V}^{\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda}} \models$ "κ is λ-supercompact".

To this end, let $j : \bar{V} \longrightarrow M$ be a 2^λ-supercompactness embedding for κ. Since M and $\bar{V}$ agree up to 2^λ, it follows that up to and including stage λ, this forcing is the same in M as it is in $\bar{V}$ and that we may factor $j(\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda})$ as $\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda} * \dot{\mathbb{P}}_{\lambda,j(\kappa)} * j(\dot{\mathbb{P}}_{\kappa,\lambda})$, where $\dot{\mathbb{P}}_{\lambda,j(\kappa)}$ is a term for the forcing defined in the open interval $(\lambda, j(\kappa))$ and $j(\dot{\mathbb{P}}_{\kappa,\lambda})$ is a term for the forcing defined in the closed interval $[j(\kappa), j(\lambda)]$. Standard arguments (see [Lav78] and the proof of Lemma 4.1) once again show that $\bar{V}^{\mathbb{P}_\kappa * \dot{\mathbb{P}}_{\kappa,\lambda}} \models$ "κ is λ-supercompact". This proves Lemma 13.3. □

Standard arguments show that $\bar{V}^{\mathbb{P}} \models$ GCH. With $\mathbb{Q} = \mathbb{M} * \dot{\mathbb{P}}$, Lemmas 13.1-13.3 prove Theorem 13. □

We conclude Chapter 3 by observing that if $V \models$ "The only strongly compact cardinals are the members of K or their measurable limit points", then $\bar{V}^{\mathbb{P}} \models$ "The only strongly compact cardinals are the members of K or their measurable limit points." This follows by unpublished work of Hamkins and Reitz [HR]. In addition, Theorem 13 can be proven using the same methods as given in this chapter if the class of supercompact cardinals K is a set, assuming there is a proper class of measurable cardinals in V. In this case, standard arguments in conjuction with Lemma 13.2 yield that $\bar{V}^{\mathbb{P}} \models$ "There is a proper class of measurable cardinals." Finally, the method used to force $V =$ HOD is due to Sy Friedman and was told to us by Joel Hamkins.

Chapter 4

The Ground Axiom, large cardinals, and GCH

Reitz [Rei06] and Hamkins [Ham05] introduced the idea of the Ground Axiom.

Definition 14. *The* Ground Axiom *(*GA*) is the assertion that the universe of sets V is not a forcing extension of any inner model $W \subseteq V$ by nontrivial set forcing $\mathbb{P} \in W$.*

We begin this chapter by producing a model of set theory which satisfies $V =$ HOD+GA+GCH + "There exists a proper class of measurable cardinals." This theorem follows by Reitz's work [Rei06], [Rei07]. We provide here an alternative method of proving this theorem. Using this new technique, we produce a model of ZFC which satisfies $V =$ HOD + GA + GCH+ "There exists a nontrivial class of supercompact cardinals." This theorem does not follow from Reitz's work. We will use this new method developed to extend results of Reitz [Rei06], [Rei07], and recent results of Hamkins, Reitz, and Woodin [HRW08].

4.1 A model of $\mathrm{GA}+V=\mathrm{HOD}+\mathrm{GCH}$ containing a nontrivial class of supercompact cardinals

Let $V \models \mathrm{ZFC} + \mathrm{GCH} +$ "There exists a proper class of measurable cardinals."

Definition 15. *For any cardinal σ and $\alpha < \sigma$, let σ_α be the $(\alpha+1)^{\mathrm{st}}$ measurable cardinal beyond σ. The* Measurable Cardinals Coding Axiom (MCA) *is the assertion that for every cardinal δ, and for every $A \subseteq \delta$, $\exists \sigma > \delta$ such that for every $\alpha < \delta$, $\alpha \in A \leftrightarrow \sigma_\alpha$ carries σ_α^+ many normal measures.*

Note that MCA implies the existence of a proper class of measurable cardinals.

Theorem 16. *The* MCA *implies the* GA*.*

Proof. We follow Reitz's proof of Theorem 9 [Rei06] and Theorem 10 [Rei07]. Suppose $V \models \mathrm{MCA}$. Suppose further that V is a set forcing extension of an inner model $V = W[h]$, where h is W-generic for some poset $\mathbb{Q} \in W$. For $\sigma > |\mathbb{Q}|$, the models W and V will agree on the properties "σ is a measurable cardinal" and "σ carries σ^+ many normal measures." Every set of ordinals A in V is coded into the "number of normal measures" function of V. The claim is that one such code for A must appear above $|\mathbb{Q}|$. If $A \subseteq \delta$ and $|\mathbb{Q}| = \gamma$, let $\gamma^* = \max(\gamma, \delta)$. Then by the MCA, there exists a $\sigma > \gamma^*$ such that A is coded on the δ measurable cardinals beyond σ. Thus A is coded into the "number of normal measures" function of V above $|\mathbb{Q}|$, and so the code appears also in W. Thus $A \in W$, and so every set of ordinals of V is also in W. This shows that $V = W$, and so the forcing $\mathbb{Q}$ was trivial. Thus $V \models$ GA. $\square$

Theorem 17. *If $V \models$ ZFC + GCH + "There exists a proper class of measurable cardinals", then there is a forcing extension by class forcing which satisfies* ZFC + GCH + GA.

Proof. The basic idea is to use forcing to code every set in the universe by controlling the number of normal measures on successor measurable cardinals. Let $\mathbb{P}$ be the two step reverse Easton support class iteration defined as in Theorem 13. Then by use of Lemma 13.1 and its proof and Lemma 13.2, $V^{\mathbb{P}} \models$ MCA. Hence, by Theorem 16, $V^{\mathbb{P}} \models$ GA. $\square$

As GCH and successor measurable cardinals are preserved by $\mathbb{P}$, an immediate consequence is:

Corollary 18. *If* $V \models$ ZFC+GCH+ *"There exists a proper class of measurable cardinals", then there is a forcing extension by class forcing which satisfies* ZFC + GCH + GA+ *"There exists a proper class of measurable cardinals".*

Now, using our previous efforts, this result extends to a model which contains a nontrivial class of supercompact cardinals.

Theorem 19. *If* $V \models$ ZFC + *"K is a nontrivial class of supercompact cardinals" + "There exists a proper class of measurable cardinals", then there is a forcing extension by class forcing which satisfies* ZFC + GCH + GA + *"K is the class of supercompact cardinals" + "There exists a proper class of measurable cardinals".*

Proof. The model constructed in Theorem 13 is the witnessing model. □

4.2 The class of supercompact cardinals, $V =$ HOD, GCH and ¬GA

In Theorem 17 [Rei06] and Theorem 18 [Rei07], by forcing over an arbitrary model V of ZFC, Reitz produces a model of $V =$ HOD in which GA fails. His proof, however, requires forcing significant failures of GCH. Using MCA, we produce a model of ZFC + GCH containing a nontrivial class of supercompact cardinals satisfying $V =$ HOD + ¬GA.

Theorem 20. *If $V \models$ ZFC + GCH + "K is a nontrivial class of supercompact cardinals" + "There is a proper class of measurable cardinals", then there is a class forcing extension $V[G][H]$ satisfying* ZFC + GCH + *"K is the class of supercompact cardinals" + "There exists a proper class of measurable cardinals"* $+\neg$GA $+ V =$ HOD.

Proof. Force $V =$ HOD as before using the class forcing iteration $\mathbb{P}$, as in Theorem 19. Let $G \subseteq \mathbb{P}$ be V-generic. Then $V[G] \models$ ZFC + GCH + $V =$ HOD + "K is the class of supercompact cardinals" + "There exists a proper class of measurable cardinals". Following Reitz [Rei06] [Rei07], the idea is to do set forcing to obtain an extension $V[G][H]$ in which H is coded as described before in Theorem 9. Now working in $V[G]$, let σ be some regular cardinal less than the least supercompact cardinal. Let $\mathbb{Q} = \mathrm{Add}(\sigma, 1)$, and let $\dot{X}$ be the name of the subset added by $\mathbb{Q}$. As described in Definition 12, let $\bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(X) = \mathrm{Add}(\sigma^+, 1) * \dot{\mathbb{C}}\mathrm{oll}_{\sigma,\gamma_\sigma}(X)$ be the forcing that codes X. Then let $H \subseteq \bar{\mathbb{S}}_{\sigma,\gamma_\sigma}(X)$ be $V[G]$-generic. Standard arguments show that $V[G][H] \models$ GCH. Since H was added by set forcing, by the Lévy-Solovay results [LS67], $V[G]$ and $V[G][H]$ agree on measurable cardinals and the number of normal measures they carry, above the size of the forcing that added H. Also, since this forcing is small with respect to the least, and therefore all, supercompact cardinals, by the results of [LS67], K remains the class of supercompact cardinals in $V[G][H]$. Thus every set in $V[G]$ remains coded in $V[G][H]$, and so every set in $V[G]$ is ordinal definable in $V[G][H]$. Since every set in $V[G][H]$ is definable from H together with a name from $V[G]$, it follows that every set is ordinal definable in $V[G][H]$. So $V[G][H] \models V = \mathrm{HOD}$. Let $V^* = V[G][H]$ and let $W^* = V[G]$. Then $V^* = W^*[H]$, where H is added by set forcing. Thus $V[G][H] \models \neg\mathrm{GA}$. $\square$

4.3 A model for GA + $V \neq$ HOD containing a proper class of measurable cardinals in which GCH holds

In their recent paper [HRW08], Hamkins, Reitz and Woodin produce a model for GA + $V \neq$ HOD + GCH by forcing over L. Their construction easily adapts to producing a model of GA + $V \neq$ HOD + GCH by forcing over a canonical inner model containing a proper class of measurable cardinals. In addition, they produce a model of GA + $V \neq$ HOD + ¬GCH by forcing over an arbitrary model of ZFC. We modify their latter construction to produce a model of GA + $V \neq$ HOD + GCH by forcing over an arbitrary model of ZFC + GCH containing a proper class of measurable cardinals. A corollary of our construction will be the production of a model of ZFC + GA + $V \neq$ HOD + GCH containing a nontrivial class of supercompact cardinals, assuming our ground model contains a nontrivial class of supercompact cardinals together with a proper class of measurable cardinals. This will be the main result of this section.

Theorem 21. *Suppose* $V \models$ ZFC + GCH+ *"There exists a proper class of measurable cardinals." Then there is a class forcing extension* $V^{\mathbb{Q}} \models$ ZFC + GCH+ *"There exists a proper class of measurable cardinals"* +GA+$V \neq$ HOD.

Proof. Let $V \models$ ZFC + GCH+"There exists a proper class of measurable cardinals." Using the two step class forcing $\mathbb{P}$ again, as described previously in Theorem 13, we let $V^{\mathbb{P}} = \bar{V}$. Note that as in the proof of Theorem 17, $\bar{V} \models$ MCA and hence, by Theorem 16, $\bar{V} \models$ GA. Also, note that since $\bar{V} \models$ MCA, $\bar{V}$ contains a proper class of measurable cardinals. Then working in $\bar{V}$, let $\mathbb{P}^*$ be the proper class reverse Easton iteration defined as follows. Begin by forcing with $\mathrm{Add}(\omega, 1)$. If γ is a successor measurable cardinal, then force with $\mathrm{Add}(\gamma^{+17}, 1)$; otherwise, do trivial forcing. Let G be $\bar{V}$-generic over $\mathbb{P}^*$. Since $\mathbb{P}^*$ is nontrivial and almost homogeneous as well as GCH preserving, $\bar{V}[G] \models V \neq$ HOD + GCH. In addition, the identical argument as in Lemma 13.2 can be carried out to show that the successor measurable cardinals are preserved in $\bar{V}[G]$. Now it remains to show that $\bar{V}[G] \models$ GA.

Suppose that it does not, i.e., $\bar{V}[G] = W[h]$ where h is a set which is W-generic for some forcing notion $\mathbb{Q} \in W$. Since this is set forcing over W, W

and $W[h]$ will agree with $\bar{V}[G]$ about the number of normal measures carried on measurable cardinals above the size of $|\mathbb{Q}|$. Since every set of ordinals in $\bar{V}$ is coded above $|\mathbb{Q}|$, it follows that every set of ordinals in $\bar{V}$ is coded in W and therefore $\bar{V} \subseteq W$. Following the proof of Theorem 2 in [HRW], we factor the forcing $\mathbb{P}^* = \mathbb{P}_1 * \dot{\mathbb{P}}_2$ and $\bar{V}[G] = \bar{V}[G_1][G_2]$ at κ, where κ is a successor measurable cardinal above $|\mathbb{Q}|$. Since κ is a successor measurable cardinal, the forcing at stage κ is $\mathrm{Add}(\kappa^{+17}, 1)$. Also, since the forcing beyond κ is at least κ-closed and $W[h] = \bar{V}[G]$, $(2^{<\kappa})^{W[h]} = (2^{<\kappa})^{\bar{V}[G]} = (2^{<\kappa})^{\bar{V}[G_1]}$. Since $G_1 \in \bar{V}[G]$, $G_1 \in W[h]$. Therefore, there is a $\mathbb{Q}$-name $\dot{G}_1 \in W$ such that $G_1 = (\dot{G}_1)_h$, and we can find a name $\dot{G}_1$ of hereditary size κ. Thus, we can find an $A \subseteq \kappa$, $A \in W$ such that $(2^{<\kappa})^W$ and $\dot{G}_1$ are coded by A. We can conclude $(2^{<\kappa})^W = (2^{<\kappa})^{\bar{V}[A]}$, because $(2^{<\kappa})^W \supseteq (2^{<\kappa})^{\bar{V}[A]}$ since $A \in W$ and $\bar{V} \subseteq W$. Conversely, $(2^{<\kappa})^W \subseteq (2^{<\kappa})^{\bar{V}[A]}$ since A codes $(2^{<\kappa})^W$. Since $\mathbb{Q}$ can be coded as a bounded subset of κ in W, $\mathbb{Q} \in \bar{V}[A]$. Since W and $\bar{V}[A]$ agree on the dense subsets of $\mathbb{Q}$, they also agree that $h \subseteq \mathbb{Q}$ is $\bar{V}[A]$-generic. Thus, we can consider $\bar{V}[A][h]$. Since A and h are subsets of κ in $\bar{V}[G_1][G_2]$ and $\mathbb{P}_2$ is κ-closed, it follows that A and h are in $\bar{V}[G_1]$, and therefore $\bar{V}[A][h] \subseteq \bar{V}[G_1]$. Conversely, $\bar{V}[A][h] \supseteq \bar{V}[G_1]$ because G_1 is in $\bar{V}[A][h]$, since $\dot{G}_1 \in \bar{V}[A]$ and $G_1 = (\dot{G}_1)_h$. So now we have $\bar{V}[A][h] = \bar{V}[G_1]$.

What we want to show is that $W[h] = \bar{V}[A][h] = \bar{V}[G_1]$, which will be a contradiction, by showing $W = \bar{V}[A]$. We show that $W = \bar{V}[A]$ exactly as at the end of the proof of Theorem 1 of [HRW] (here $\bar{V}$ will play the role of L). By setting $\mathbb{Q} = \mathbb{P} * \dot{\mathbb{P}}^*$, this completes the proof of Theorem 21. $\square$

Corollary 22. *Suppose* $V \models \text{ZFC} + \text{GCH} +$ *"K is a nontrivial class of supercompact cardinals" + "There exists a proper class of measurable cardinals." Then there is a class forcing extension $V^{\mathbb{Q}}$ that preserves K as the class of supercompact cardinals such that* $V^{\mathbb{Q}} \models \text{ZFC} + \text{GCH}+$ *"There exists a proper class of measurable cardinals"* $+\text{GA} + V \neq \text{HOD}$.

Proof. Let $\mathbb{Q} = \mathbb{P} * \dot{\mathbb{P}}^*$ be as in the proof of Theorem 21, with $\bar{V} = V^{\mathbb{P}}$. As in the proof of Theorem 13, $\bar{V} \models$ "K is the class of supercompact cardinals". It remains to show that $\bar{V}^{\mathbb{P}^*} \models$ "K is the class of supercompact cardinals". No new supercompacts were created by $\mathbb{P}^*$, by an application of Theorem 1. Conversely, fix $\kappa \in K$, and let $\lambda > \kappa$ be a singular strong limit cardinal of cofinality κ which is a limit of measurable cardinals. Since $\mathbb{P}^*$ is successively more and more closed, the identical lifting argument as in Lemma 13.3 can be applied here to show that κ will remain supercompact in $\bar{V}^{\mathbb{P}^*}$. $\square$

Chapter 5

Results in Set Theoretic Geology, but while preserving GCH

A very recent paper by Fuchs, Hamkins and Reitz [FHR] introduced the idea of *set theoretic geology.* It codifies some of the underlying ideas of forcing by describing a meta view of how set theoretic models and their set forcing extensions relate.

We describe some of the important concepts and definitions.

Definition 23. *A class W is a* ground of V *if W is a transitive class model of* ZFC *and V is obtained by set forcing over W, that is, if there is some forcing notion $\mathbb{P} \in W$ and a W-generic filter $G \subseteq \mathbb{P}$ such that $V = W[G]$.*

Definition 24. *The* mantle *of a model of set theory is the intersection of all of its ground models.*

Definition 25. *The* generic mantle *of a model of set theory V is the intersection of all ground models of all set forcing extensions of V.*

Definition 26. *The* generic HOD of V, *denoted* gHOD*, is the intersection of all the* HOD*s of all set forcing extensions of V. That is, $x \in$* gHOD *if and only if $x \in \text{HOD}^{V[G]}$ for all set forcing extensions $V[G]$.*

In this chapter, using variations of the coding introduced in Theorem 13, we will extend some of the theorems in [FHR] to models that satisfy GCH and contain a proper class of measurable cardinals. We note that as [FHR] is as of yet unpublished, the final version may have slightly different numbering for the theorems that will be extended. We also note that the idea for the following theorems are stated informally in [FHR] using the $\diamondsuit^*$ coding of Brooke-Taylor [BT09]. However, we provide an alternative method of proof.

5.1 V is the mantle, generic mantle, generic HOD, and the HOD with GCH and a proper class of measurable cardinals

We begin with the following theorem, which generalizes Theorem 50 of [FHR].

Theorem 27. *Every model V of* ZFC + GCH + *"There exists a proper class of measurable cardinals" has a class forcing extension $V[G]$ satisfying* GCH *and containing a proper class of measurable cardinals in which V is the mantle,* *the generic mantle, the generic* HOD *and the* HOD. *In other words,* $V = \mathrm{M}^{\mathrm{V[G]}} = \mathrm{gM}^{\mathrm{V[G]}} = \mathrm{gHOD}^{\mathrm{V[G]}} = \mathrm{HOD}^{\mathrm{V[G]}}$ *and* $V[G] \models$ GCH+ *"There exists a proper class of measurable cardinals."*

Proof. Let V be as in the hypotheses of Theorem 27. Following the proof of Theorem 50 of [FHR], quoting verbatim when applicable, the strategy is to perform class forcing so that the various forcings will keep the mantles and the HODs balanced. Sets in V will be coded while still preserving GCH, which ensures that every set in V will be in the mantle, the generic mantle, the HOD and the generic HOD. The forcing maintains certain factor and homogeneity properties so that no additional sets are added to the mantles or HODs.

Turning now to the proof of Theorem 27, let $V^* = V^{\mathbb{M}}$ be as in Theorem 13. Then $V^* \models$ GCH+ "There exists a proper class of measurable cardinals" +"If σ is measurable, then σ carries the maximum number of normal measures." Let $\bar{V} = (V^*)^{\mathrm{Add}(\omega,1)}$. Note that by the results of [LS67], V^* and $\bar{V}$ have the same measurable cardinals and $\bar{V} \models$ GCH+ "If σ is measurable, then σ carries the maximum number of normal measures." We will define $\mathbb{Q}_\sigma$ in $\bar{V}$ when σ is a "successor" measurable cardinal as in Theorem 13. Note that $\mathbb{Q}_\sigma$ is σ-closed and has size $2^{2^\sigma} = \sigma^{++}$ (since GCH holds), which clearly is strictly less than the next stage of forcing at the next successor measurable cardinal.

Following the proof in [FHR] closely, let $\mathbb{P}$ be defined in $\bar{V}$ as the class forcing product $\mathbb{P} = \prod_\sigma \mathbb{Q}_\sigma$, with set support. That is, conditions in $\mathbb{P}$ are set functions p, with $\mathrm{dom}(p) \subseteq$ORD and $p(\sigma) \in \mathbb{Q}_\sigma$, ordered by end extension of the domain and strengthening in each coordinate. (In particular, we do not use Easton support, which would not work here, because it would create new unwanted definable subsets of the inaccessible cardinals.) Suppose that $G \subseteq \mathbb{P}$ is $\bar{V}$-generic and consider the model $\bar{V}[G]$. The class forcing $\mathbb{P}$ factors at every successor measurable cardinal σ as $\mathbb{P}_\sigma \times \mathbb{P}^\sigma$, where $\mathbb{P}_\sigma = \prod_{\sigma' < \sigma} \mathbb{Q}_\sigma$ and $\mathbb{P}^\sigma = \prod_{\sigma' \geq \sigma} \mathbb{Q}_\sigma$. Again the products have set support, which for $\mathbb{P}_\sigma$ means full support.

When σ is a successor measurable cardinal, we claim that $|\mathbb{P}_\sigma| < \sigma$. Let γ be the maximum measurable cardinal $< \sigma$. For a successor measurable cardinal $\delta \leq \gamma$, $|\mathbb{Q}_\delta| \leq \gamma^{++}$. Thus, $|\prod_{\delta \leq \gamma} \mathbb{Q}_\delta| \leq |\mathbb{Q}_\gamma| = \gamma^{++} < \sigma$. Combining this with the fact that the tail forcing $\mathbb{P}^\sigma$ is σ-closed, it follows that every set in $\bar{V}[G]$ is added by some large enough initial factor $\mathbb{P}_\sigma$.

To verify that the various stages of forcing do not interfere with each other, we factor our forcing at any successor measurable cardinal σ as $\mathbb{P} = \mathbb{P}_\sigma \times \mathbb{Q}_\sigma \times \mathbb{P}'$. Any nontrivial forcing which occurs in $\mathbb{P}_\sigma$ will have size less than σ, and will therefore not affect the number of normal measures on σ, and any nontrivial forcing in $\mathbb{P}'$ will be sufficiently closed so as not to add any new normal measures to σ. As for $\mathbb{Q}_\sigma$, it will have size σ^{++}, so it will not affect the number of normal measures on the next successor measurable cardinal and beyond. Additionally $\mathbb{Q}_\sigma$ is σ-closed, so the number of normal measures on measurable cardinals below σ will not be affected.

We also need to ensure that the class of successor measurable cardinals is preserved in $\bar{V}^{\mathbb{P}}$. We can carry out the proof of Lemma 13.2, replacing the iterations by products, since forcing over V^* introduces a closure point at ω, and V^* and $\bar{V}$ contain the same measurable cardinals. Thus, the class of successor measurable cardinals is preserved by $\mathbb{P}$.

The claim is that every set of ordinals in $\bar{V}$ is coded into the "number of normal measures" pattern of $\bar{V}[G]$. Suppose that x is a set of ordinals in $\bar{V}$ and p is any condition in $\mathbb{P}$. Choose a regular cardinal β and an ordinal τ with $x \subseteq \beta$ and $\mathrm{dom}(p) \subseteq \tau$. Let σ be the least measurable cardinal $> \max(\beta, \tau)$. Since there exists a proper class of measurable cardinals in $\bar{V}$, in particular there are β many measurable cardinals above σ. Then for every $\alpha \in x$, let σ_α be the $(\alpha+1)^{\mathrm{st}}$ measurable cardinal beyond σ and let γ_β be the supremum of the first β many measurable cardinals beyond σ. So now, since x is a set and $\mathbb{P}$ uses set support, we may extend p to a stronger condition $q \leq p$ with dom(q)= $\mathrm{dom}(p) \cup [\sigma, \gamma_\beta)$, where q opts on the interval $[\sigma, \gamma_\beta)$ to force the number of normal measures on σ_α to be collapsed or not according to whether $\alpha \in x$. That is, we build q so that for every $\alpha < \beta$, if $\alpha \in x$, then $q(\sigma_\alpha)$ opts to collapse the number of normal measures on σ_α. If $\alpha \notin x$, then $q(\sigma_\alpha)$ opts for trivial forcing. Since we have argued that there is no interference between the levels of coding, the condition q forces that the number of normal measures pattern in $\bar{V}[G]$ for those values of σ_α is exactly the same as the pattern of x on β. Thus, it is dense that x is coded in this way, and so generically every set in $\bar{V}$ will be coded into the number of normal measures pattern of $\bar{V}[G]$. A simple padding argument shows that this implies that every set in $\bar{V}$ is in fact coded unboundedly often into the number of normal measures pattern of $\bar{V}[G]$ on the successor measurable cardinals. Since the class of successor measurable cardinals is definable in $\bar{V}[G]$, we may conclude immediately that every set in $\bar{V}$ is ordinal definable in $\bar{V}[G]$.

For the generic HOD, we consider the set forcing extensions of $\bar{V}[G]$. Suppose that $\bar{V}[G][h]$ is obtained by further forcing $h \subseteq \mathbb{Q} \in \bar{V}[G]$. Since the number of normal measures on measurable cardinals of $\bar{V}[G][h]$ and $\bar{V}[G]$ agree above $|\mathbb{Q}|$, it follows that $\bar{V}[G]$ and $\bar{V}[G][h]$ have the same successor measurable cardinals and the same number of normal measures pattern above $|\mathbb{Q}|$. This implies that a tail segment of the successor measurable cardinals remains definable in $\bar{V}[G][h]$, and the number of normal measures pattern on this segment is the same in $\bar{V}[G][h]$ as in $\bar{V}[G]$. Since every set of ordinals x in $\bar{V}$ was coded unboundedly into this pattern, we conclude that every set in $\bar{V}$ remains ordinal definable in the extension $\bar{V}[G][h]$. Since the forcing $\mathbb{Q}$ was arbitrary, we conclude $\bar{V} \subseteq \mathrm{gHOD}^{\bar{V}[G]}$, and consequently also $\bar{V} \subseteq \mathrm{gHOD}^{\bar{V}[G]} \subseteq \mathrm{gM}^{\bar{V}[G]} \subseteq \mathrm{M}^{\bar{V}[G]}$.

The remaining arguments follow closely those in [FHR] and are included here for completeness. To prove the theorem, we need to argue that conversely, the mantle $\mathrm{M}^{\bar{V}[G]} \subseteq \bar{V}$. For a successor measurable cardinal σ, factor the forcing at σ as $\mathbb{P} = \mathbb{P}_\sigma \times \mathbb{P}^\sigma$. The generic filter G similarly factors as $G_\sigma \times G^\sigma$. Since $\mathbb{P}_\sigma$ is set forcing in $\bar{V}$, it follows that the tail extension $\bar{V}[G^\sigma]$ is a ground of $\bar{V}[G]$, and so the mantle of $\bar{V}[G]$ is contained within every $\bar{V}[G^\sigma]$. Since any name mentions only a set number of conditions in $\mathbb{P}$, it follows that every set $y \in \bar{V}[G]$ is in $\bar{V}[G_\sigma]$ for some σ. Since G_σ and G^σ are mutually generic, it follows by Lemma 49 in [FHR] that $\bar{V}[G_\sigma] \cap \bar{V}[G^\sigma] = \bar{V}$. Thus, any object y in the mantle of $\bar{V}[G]$ must be in $\bar{V}$. Altogether, we have established $\bar{V} \subseteq \mathrm{gHOD}^{\bar{V}[G]} \subseteq \mathrm{gM}^{\bar{V}[G]} \subseteq \mathrm{M}^{\bar{V}[G]} \subseteq \bar{V}$, and so all these are equal, as we claimed.

Finally, we consider $\mathrm{HOD}^{\bar{V}[G]}$.

The argument is identical to the one in [FHR], as the subposets used in the forcing $\mathbb{P}$, the forcing to collapse cardinals, have the same "latent homogeneity" as the forcing used in [FHR], i.e., the forcing to collapse or preserve GCH. Therefore, the identical argument will go through, and for completeness we include it here. We have shown that every set x in $\bar{V}$ is coded unboundedly often and therefore $\bar{V} \subseteq \text{HOD}^{\bar{V}[G]}$, and it remains for us to prove the converse inclusion. For this, in order to control $\text{HOD}^{\bar{V}[G]}$, one would expect to appeal to the homogeneity properties of the forcing $\mathbb{P}$. Unfortunately, the forcing $\mathbb{P}$ is not almost homogeneous, because different conditions can make fundamentally different choices in the lotteries about how the forcing will proceed. Nevertheless, we claim that there is sufficient latent homogeneity in the forcing for the argument to succeed. In order to show $\text{HOD}^{\bar{V}[G]} \subseteq \bar{V}$, let us first show that $\text{HOD}^{\bar{V}[G]}$ is contained in every tail extension $\bar{V}[G^\sigma]$. Consider $\bar{V}[G]$ as a forcing extension of the tail extension $\bar{V}[G^\sigma]$ by the initial forcing $G_\sigma \subseteq \mathbb{P}_\sigma$. Although $\mathbb{P}_\sigma$ is not almost homogenous, it is densely almost homogeneous, meaning that there is a dense set of conditions q such that $\mathbb{P}_\sigma \restriction q$ is almost homogeneous. The point is simply that because we have used full support, rather than Easton support, we may extend any condition in $\mathbb{P}_\sigma$ to a condition with full support up to σ. Furthermore, we may extend to a condition that makes a definite selection in each of the lotteries before stage σ as to which of the two posets should be used in that coordinate. Since each of these individual posets is almost homogenous, $\mathbb{P}_\sigma \restriction q$ is the full product of almost homogeneous forcing, and consequently is itself almost homogeneous. Thus, by genericity, there is some $q \in G_\sigma$ such that $\mathbb{P}_\sigma \restriction q$ is almost homogenous. It follows that every ordinal definable set of ordinals added by this forcing is definable in the ground model $\bar{V}[G^\sigma]$ from ordinal parameters and the poset $\mathbb{P}_\sigma \restriction q$ used as an additional parameter. So we have proved that $\text{HOD}^{\bar{V}[G]}$ is included in every tail extension $\bar{V}[G^\sigma]$. Thus, $\text{HOD}^{\bar{V}[G]}$ is contained within the intersection of the tail extensions $\bar{V}[G^\sigma]$, which we previously argued was $\bar{V}$. So $\text{HOD}^{\bar{V}[G]} \subseteq \bar{V}$, and we have proved all the desired equalities $\bar{V} = \text{M}^{\bar{V}[G]} = \text{gM}^{\bar{V}[G]} = \text{gHOD}^{\bar{V}[G]} = \text{HOD}^{\bar{V}[G]}$. Since $\bar{V}[G] \models \text{GCH}+$ "There exists a proper class of measurable cardinals", this concludes the proof of Theorem 27. □

5.2 *V* is the mantle, generic mantle, generic HOD, but not HOD with GCH and a proper class of measurable cardinals

We conclude this chapter with the following theorem, which generalizes Theorem 51 of [FHR].

Theorem 28. *If $V \models \text{ZFC}+\text{GCH}+$ "There exists a proper class of measurable cardinals", then there is a class forcing extension $V[G]$ satisfying* GCH *and containing a proper class of measurable cardinals in which V is the mantle,* *the generic mantle, the generic* HOD*, but $V[G]$ is the* HOD*. In other words, $V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]}$ but $\mathrm{HOD}^{V[G]} = V[G]$ and $V[G] \models$* GCH+ *"There exists a proper class of measurable cardinals".*

Proof. Let V be as in the hypotheses of Theorem 28. Let V be generically extended to $\bar{V}$ as in Theorem 27. This will preserve GCH and ensure that all measurable cardinals σ carry the maximum number of normal measures.

As in the proof of Theorem 27, we follow the proof of the corresponding theorem in [FHR] (Theorem 51), quoting verbatim when applicable. In the current theorem, we want to keep the mantles and the generic HOD low, while allowing $\mathrm{HOD}^{\bar{V}[G]}$ to expand to $\bar{V}[G]$. Pushing the classes up at least to $\bar{V}$, the strategy will be once again to force that every set of ordinals in $\bar{V}$ is coded unboundedly often into the "number of normal measures" pattern of $\bar{V}[G]$. However, if we use the iterated forcing from Theorem 13, then $\bar{V}[G] \models$ GA. Consequently, the mantle of $\bar{V}[G]$ will be equal to $\bar{V}[G]$ and therefore the generic mantle and generic HOD will also be equal to $\bar{V}[G]$. Hence, the altered strategy will be to code every set of ordinals in $\bar{V}$ unboundedly often, whereas every new set of ordinals will be coded only boundedly often. The subtle effect is that the new sets become ordinal definable, but only temporarily so, for further forcing can erase the bounded coding and make them drop out of HOD.

Definition 29. *Let* $f : \omega \times \text{ORD} \longrightarrow \text{ORD}$ *be the function defined inductively as follows: Let* $f(\langle 0, 0\rangle) = f(0)_0 = \kappa_0 = \aleph_1$. *Fix* α, *and assume* $f(\langle n, \alpha\rangle) = f(n)_\alpha$ *is defined. Then for successor stages* $n+1$, *let* $f(\langle n+1, \alpha\rangle) = f(n+1)_\alpha$ *be the least measurable cardinal* $>$ *the supremum of the first* $f(n)_\alpha$ *many measurable cardinals beyond* $f(n)_\alpha$. *For* $\alpha > 0$, *let*

$$\lambda_\alpha = \sup_{n<\omega, \beta<\alpha} f(n)_\beta.$$

Now let $f(\langle 0, \alpha\rangle) = f(0)_\alpha = \kappa_\alpha = \lambda_\alpha^+$.

Let δ be a cardinal and let $\bar{\mathbb{S}}_{\delta,\gamma_\delta}(A)$ be the forcing that codes $A \subseteq \delta$ as previously defined in Definition 12. We now describe the self encoding forcing $\mathbb{Q}_\alpha$ at κ_α. This is the forcing iteration $\mathbb{Q}_\alpha$ of length ω that begins by adding a Cohen subset of κ_α, and then proceeds in each subsequent stage to code the generic filter from the prior stage into the number of normal measures pattern on the next block of measurable cardinals as determined by the function f. All coding will take place in the interval $I_\alpha = [\kappa_\alpha, \lambda_{\alpha+1})$. More specifically, the forcing begins at stage 0 with $\mathbb{Q}_{0,\alpha} = \text{Add}(\kappa_\alpha, 1)$ adding a Cohen subset $g_{0,\alpha} \subseteq \kappa_\alpha$. Then at stage 1, we let $\mathbb{Q}_{1,\alpha} = \bar{\mathbb{S}}_{f(1)_\alpha, \gamma_{f(1)_\alpha}}(g_{0,\alpha})$, i.e., $\mathbb{Q}_{1,\alpha}$ codes $g_{0,\alpha}$. In general, dropping the α for the moment, $\mathbb{Q}_n = \bar{\mathbb{S}}_{f(n),\gamma_{f(n)}}(g_{n-1})$, so the n^{th} stage of forcing codes the subset added by the previous stage of forcing and the function f ensures that the blocks of coding are disjoint. We also can see that the entire iteration $\mathbb{Q}_\alpha$ has size $\lambda_{\alpha+1}^\omega$. The end result is that if $G(\alpha) \subseteq \mathbb{Q}_\alpha$ is $\bar{V}$-generic, then $G(\alpha)$ is coded explicitly on the number of normal measures pattern in $\bar{V}[G(\alpha)]$ on the interval $I_\alpha = [\kappa_\alpha, \lambda_{\alpha+1})$. Additionally $\mathbb{Q}_\alpha$ is κ_α-closed and will not affect the number of normal measures on measurable cardinals below κ_α or the number of normal measures on measurable cardinals above $\lambda_{\alpha+1}^\omega$.

Now let $\mathbb{P} = \prod_\alpha \mathbb{Q}_\alpha$ be the set support class product of these posets. Routine arguments show that $\bar{V}^{\mathbb{P}} \models \text{ZFC} + \text{GCH} +$ "There exists a proper class of measurable cardinals". The remaining arguments follow from those in [FHR], with some minor changes. For any ordinal α, factor this forcing as $\mathbb{P}_\alpha \times \mathbb{P}^\alpha$. Here $\mathbb{P}_\alpha = \prod_{\beta<\alpha} \mathbb{Q}_\beta$ and $\mathbb{P}^\alpha = \prod_{\beta\geq\alpha} \mathbb{Q}_\beta$, again using set support in these products (which for $\mathbb{P}_\alpha$ means full support). Let $G \subseteq \mathbb{P}$ be $\bar{V}$-generic. For any ordinal $\alpha > 0$, we may factor the forcing as $\mathbb{P}_\alpha \times \mathbb{Q}_\alpha \times \mathbb{P}^{\alpha+1}$, with corresponding factorization of G into $G_\alpha \times G(\alpha) \times G^{\alpha+1}$. The usual arguments now show that every set in $\bar{V}[G]$ is added by some stage $\bar{V}[G_\alpha]$. Because the final factor $\mathbb{P}^{\alpha+1}$ is $\kappa_{\alpha+1}$-closed and the initial factor $\mathbb{P}_\alpha$ has size less than the least measurable cardinal greater than κ_α, neither of these affects the number of normal measures for measurable cardinals in the interval $I_\alpha = [\kappa_\alpha, \lambda_{\alpha+1})$.

Now it remains to observe that $\bar{V}[G]$ codes the sets of ordinals in $\bar{V}$ unboundedly and the new sets of ordinals only boundedly so. The argument is lifted directly from [FHR], with minor changes where applicable. If x is any set of ordinals in $\bar{V}$, then in any coordinate stage α of forcing above $\sup(x)$, it is dense for x to appear as an interval in the generic object added at the first stage of forcing in $\mathbb{Q}_\alpha$. The subsequent stages of forcing in $\mathbb{Q}_\alpha$ will therefore have the effect of coding x into the number of normal measures pattern in I_α. Thus, the set x is coded into the number of normal measures pattern of $\bar{V}[G]$, and is consequently ordinal definable. Since x is coded unboundedly often in this way, x will remain ordinal definable in any set forcing extension of $\bar{V}[G]$, because any such extension $\bar{V}[G][h]$ has the same number of normal measures on measurable cardinals as $\bar{V}[G]$ above the size of the forcing for which h is generic. Thus, x is ordinal definable in any such $\bar{V}[G][h]$, and so $\bar{V} \subseteq \text{gHOD}^{\bar{V}[G]}$. The fact that $\bar{V} = \text{M}^{\bar{V}[G]} = \text{gM}^{\bar{V}[G]} = \text{gHOD}^{\bar{V}[G]}$ now follows as in the proof of Theorem 27.

It remains to compute $\text{HOD}^{\bar{V}[G]}$. We have observed that every set in $\bar{V}[G]$ is added by some initial factor forcing $\mathbb{P}_\alpha$ for some ordinal α, and therefore every object in $\bar{V}[G]$ has the form τ_{G_α}, for some ordinal α and some $\mathbb{P}_\alpha$-name τ in $\bar{V}$. Since we have already established that $\bar{V} \subseteq \text{HOD}^{\bar{V}[G]}$, it follows that the name τ is ordinal definable in $\bar{V}[G]$. Furthermore, for every $\beta < \alpha$, the generic filter $G(\beta) \subseteq \mathbb{Q}_\beta$ added at stage β is coded onto the number of normal measures on measurable cardinals in the interval I_β, and hence

$G_\alpha = \prod_{\beta<\alpha} G(\beta)$ is ordinal definable in $\bar{V}[G]$. So τ_{G_α} is ordinal definable in $\bar{V}[G]$ and so $\textsc{hod}^{\bar{V}[G]} = \bar{V}[G]$, as desired. $\square$

Chapter 6

The Wholeness Axiom with GCH and $V = \text{HOD}$

Using earlier constructions, we can extend a result of Hamkins [Ham01b] involving the Wholeness Axiom and V=HOD. The Wholeness Axioms, proposed by Paul Corazza [Cor00], are intended as a slight weakening of Kunen's famous inconsistency result [Kun71] of the nonexistence of a nontrivial elementary embedding from the universe to itself. The Wholeness Axioms are formalized in the language $\{\in, \mathbf{j}\}$, augmenting the usual language of set theory $\{\in\}$ with an additional unary function symbol **j** to represent the embedding. We begin with the following definitions.

Definition 30. *The Wholeness Axiom* WA_n*, where* n *is among* $0, 1, ..., \infty$*, consists of the following:*

1. *(Elementarity) All instances of* $\varphi(x) \leftrightarrow \varphi(\boldsymbol{j}(\mathrm{x}))$ *for* φ *in the language*

$\{\in\}$.

2. *(Separation) All instances of the* $\sum_n$*-Separation Axioms for formulae in the full language* $\{\in, \boldsymbol{j}\}$.

3. *(Nontriviality) The axiom* $\exists x\, \boldsymbol{j}(\mathrm{x}) \neq \mathrm{x}$.

The following theorem generalizes the Main Theorem of [Ham01b].

Theorem 31. *If the Wholeness Axiom* WA$_0$ *is itself consistent, then it is consistent with* $V =$ HOD + GCH.

Proof. Suppose that $V \models$ WA$_0$. As in [Ham01b], every model of one of the Wholeness Axioms has the form $\langle V, \in, j\rangle$, where $\langle V, \in\rangle$ satisfies ZFC and $j : V \longrightarrow V$ is a nontrivial amenable elementary embedding.

We now have the following easy lemma.

Lemma 31.1. *If* $V \models$ WA$_0$*, then* $V \models$*"There exists a proper class of measurable cardinals."*

Proof. Let $j : V \longrightarrow V$ be the witnessing embedding, with critical sequence $\{\kappa_n : n \in \omega\}$, defined by $\kappa_0 = \kappa = \mathrm{cp(j)}$ and $\kappa_{n+1} = j(\kappa_n)$. Since $j \restriction V_\kappa$ is the identity function, it follows that $V_\kappa \prec V_{\kappa_1}$. As in [Ham01b], by iteratively applying j to this fact, one easily concludes that

$$V_{\kappa_0} \prec V_{\kappa_1} \prec V_{\kappa_2} \prec \cdots \prec V.$$

By [Ham01b], $V \models$ "κ is supercompact." Hence, $V_\kappa \models$ "There exists a proper class of measurable cardinals." But since $V_\kappa \prec V$, $V \models$ "There exists a proper class of measurable cardinals." □

Again, as in [Ham01b], we force with the usual reverse Easton support class iteration that forces the GCH. Thus, at any cardinal γ, we force with $\mathrm{Add}(\gamma^+, 1)$. By [Ham01b], the resulting forcing extension $\bar{V}$ satisfies WA_0 and the GCH. Furthermore, as in [ACH07], this forcing ensures that every measurable cardinal σ carries the maximum number of normal measures.

We define now in $\bar{V}$ the poset $\mathbb{P}_\kappa$ used in the construction of the witnessing model for Theorem 31. $\mathbb{P}_\kappa$ is a reverse Easton support κ-iteration, defined as $\mathbb{P}_\kappa = \mathrm{Add}(\omega, 1) * \langle \dot{\mathbb{Q}}_\sigma : \sigma < \kappa \rangle$. Here, $\dot{\mathbb{Q}}_\sigma$ is a term for trivial forcing unless σ is a "successor" measurable cardinal. When this is true, $\dot{\mathbb{Q}}_\sigma$ is a term for the poset defined as in Theorem 13. As we have observed previously, this iteration preserves GCH. Our earlier arguments also show that this iteration forces V=HOD in $\bar{V}_\kappa^{\mathbb{P}_\kappa}$. The same lifting arguments as given in [Ham01b] (literally presented unchanged) now show that $\bar{V}_\kappa^{\mathbb{P}_\kappa} \models \mathrm{WA}_0$. □

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