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ON MORDELL'S EQUATION $y^2 + k = x^3$, with $k = \pm 2^n 3^m$

by

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INTRODUCTION

The diophantine equation

(1) $y^2 + k = x^3$, $k \neq 0$ has, for each value of k ,

a finite number of solutions. Mordell's proof [21] of this theorem, however, does not give a bound on $|x|$. Baker [1a] has recently shown that any integral solution of (1) satisfies

$$\max(|x|,|y|) < \exp(10^{10} |k|^{10^4}) .$$

This result, theoretically at least, gives the solution of (1) for any value of k .

[6, Ch. XX, pp.533-539], [12], [18] and [19] contain excellent histories and bibliographies of the results prior to 1916, 1950, 1966 and 1968, respectively.

It is in recognition of Mordell's outstanding research on equation (1) that I have used his name in the title of this paper.

In this thesis, I will solve

(2) $y^2 + k = x^3$, $k = \pm 2^n 3^m$ for all values of k .

(2) has previously been solved for a finite number of values of k and I will use some of these results, which are listed in Appendix II. (Appendix II will henceforth be identified as [A].) I have been able to verify these solutions except for the two most recent results, Finkelstein and London ($k = 18$) [10] and Coghlan and Stephens ($k = 18$; 72; 288 and 648) [4], both of which use computer searches to prove that a given pair of units in a semi-real quartic field is fundamental. I

will therefore use these results only when solving (2) for $k = 18$; 72; 288 and 648. In particular, the solution of (2) for negative k will not depend on [10] and [4].

I will conclude this paper with the solution of $u^3 + Bv^3 = C$, where $(B,C) = (3^s, 2^r)$ or $(2^s, 3^r)$, $0 \le s \le 2$, $r \ge 0$.

SECTION 1.

PRELIMINARIES

Z and Q will denote the integers and rationals, respectively. All other Roman letters will represent elements of Z and, in particular, p will be a positive prime of Z .

Definitions: If $d(\neq 1)$ is square-free, then Γ_d = ring of integers (over Z) of $Q(d^{\frac{1}{2}})$. If $d(\neq 1)$ is cube-free, then Ω_d = ring of integers (over Z) of $Q(d^{\frac{1}{3}})$.

By [2, p.132, Theorem 1],

(1) $$\Gamma_d = \{a+bd^{\frac{1}{2}} \mid a,b \in Z\} \quad \text{if} \quad d \equiv 2 \text{ or } 3 \bmod 4 ,$$

(2) $$\Gamma_d = \{(a+bd^{\frac{1}{2}})/2 \mid a,b \in Z , \; a \equiv b \bmod 2\} \quad \text{if} \quad d \equiv 1 \bmod 4 .$$

By [15, p.105],

(3) $$\Omega_d = \{a+bd^{\frac{1}{3}} + cd^{\frac{2}{3}} \mid a,b,c \in Z\} \quad \text{for} \quad d = 2,3 \text{ or } 6 ,$$

(4) $$\Omega_{12} = \{a+b(12)^{\frac{1}{3}} + c(12)^{\frac{2}{3}}/2 \mid a,b,c \in Z\} .$$

By [11, p.213], Γ_d $(d = -1, \pm 2, \pm 3 \text{ or } 6)$ is Euclidean and is therefore a unique factorization domain (U.F.D.). The class number of Ω_d $(d = 2,3,6 \text{ or } 12)$ is 1 by [2, p.427, Table 6], and therefore Ω_d is a U.F.D. Λ will represent either Γ_d $(d = -1, \pm 2, \pm 3 \text{ or } 6)$, Ω_d $(d = 2,3,6 \text{ or } 12)$ or Z , and hence Λ is a U.F.D.

From [33, p.132, (13)], the norms (over Q) in $Q(d^{\frac{1}{2}})$ and $Q(d^{\frac{1}{3}})$ are given respectively by:

(5) $$N(\alpha+\beta d^{\frac{1}{2}}) = \alpha^2 - d\beta^2 ,$$

(6) $N(\alpha+\beta d^{\frac{1}{3}} + \gamma d^{\frac{2}{3}}) = \alpha^3 + d\beta^3 + d^2\gamma^3 - 3d\alpha\beta\gamma$,

where $\alpha,\beta,\gamma \in Q$.

If $d < 0$ and $d \neq -1$ or -3 , then by (1), (2) and (5), the units of Γ_d are ± 1 . Similarly, the units of Γ_{-1} are ± 1 and $\pm (-1)^{\frac{1}{2}}$ and the units of Γ_{-3} are ± 1 and $(\pm 1 \pm (-3)^{\frac{1}{2}})/2$. Let $\epsilon = 1$ for Γ_d ($d < 0$, $d \neq -1, -3$) and Z, $\epsilon = (-1)^{\frac{1}{2}}$ for Γ_{-1} , and $\epsilon = (1 + (-3)^{\frac{1}{2}})/2$ for Γ_{-3} .

Since Γ_d ($d > 0$) and Ω_d are subrings of the reals, the only roots of unity they contain are ± 1 . It follows from [15,p.112, Dirichlet's Unit Theorem] that the positive units of each of these rings form an infinite cyclic group. The generator ϵ ($0 < \epsilon < 1$) of this group is called the fundamental unit of the ring.

Therefore the units of each of the rings Z, Γ_d and Ω_d are of the form $\pm\epsilon^r$ ($r \in Z$) for the corresponding ϵ .

By [2 ,p.422 and 5 ,p.304] and the above,we have:

Table 1

Ring	ϵ	ϵ^{-1}
Γ_{-1}	$(-1)^{\frac{1}{2}}$	$(-1)^{\frac{1}{2}}$
$\Gamma_{-2},\Gamma_{-6},Z$	1	1
Γ_{-3}	$(1+(-3)^{\frac{1}{2}})/2$	$(1-(-3)^{\frac{1}{2}})/2$
Γ_2	$-1+2^{\frac{1}{2}}$	$1+2^{\frac{1}{2}}$
Γ_3	$2-3^{\frac{1}{2}}$	$2+3^{\frac{1}{2}}$
Γ_6	$5-2\cdot 6^{\frac{1}{2}}$	$5+2\cdot 6^{\frac{1}{2}}$
Ω_2	$-1+2^{\frac{1}{3}}$	$1+2^{\frac{1}{3}}+2^{\frac{2}{3}}$
Ω_3	$-2+3^{\frac{2}{3}}$	$4+3\cdot 3^{\frac{1}{3}}+2\cdot 3^{\frac{2}{3}}$
Ω_6	$1-6\cdot 6^{\frac{1}{3}}+3\cdot 6^{\frac{2}{3}}$	$109+60\cdot 6^{\frac{1}{3}}+33\cdot 6^{\frac{2}{3}}$
Ω_{12}	$1+3(12)^{\frac{1}{3}}-3(12)^{\frac{2}{3}}/2$	$55+24(12)^{\frac{1}{3}}+21(12)^{\frac{2}{3}}/2$

Lower case Greek letters will represent elements of Λ .

Definitions: $(\alpha,\beta)_\Lambda$ is the G.C.D. of α and β in Λ . $\alpha|_\Lambda\beta$ is read as: α divides β in Λ . The subscript Λ in the above notations will be omitted when $\Lambda = Z$. It will be clear from the text whether (a,b) represents an ordered pair or the G.C.D. of a and b . (a,b,c), (a,b,c,d) etc. will be ordered triples, quadruples etc. of integers.

We will use the following simple results repeatedly:

Lemma 1.1: If $(a,b) = 1$, then $(a,b)_\Lambda = 1$.

Proof: There are integers e and f such that $ea + fb = 1$.

Lemma 1.2: If $\varphi^2 = \alpha\beta$ and $(\alpha,\beta)_\Lambda = 1$, then $\alpha = \mu\gamma^2$, where $\mu = \pm 1$ or $\mu = \pm\epsilon$. Of course, if Λ is real and $\alpha > 0$, then $\mu = 1$ or ϵ.

Proof: Since Λ is a U.F.D., $\alpha = \pm\epsilon^s\psi^2$. If $s = 2t$, let $\mu = \pm 1$, $\gamma = \epsilon^t\psi$. If $s = 2t + 1$, let $\mu = \pm\epsilon$, $\gamma = \epsilon^t\psi$.

Lemma 1.3: If $\varphi^3 = \alpha\beta$ and $(\alpha,\beta)_\Lambda = 1$, then $\alpha = \mu\gamma^3$, where $\mu = 1$, ϵ or ϵ^{-1} .

Proof: $\alpha = \pm\epsilon^s\psi^3 = \epsilon^s(\pm\,\psi)^3$. Now use $s \equiv 0,1$ or $-1 \bmod 3$.

Lemma 1.4: If δ is a prime of Λ, $(\alpha,\beta)_\Lambda = \delta^s$, $\delta^t|_\Lambda\alpha$ and $\delta^{t+1}\nmid_\Lambda\alpha$, then $(\alpha/\delta^t,\beta)_\Lambda = 1$.

Proof: $(\alpha/\delta^t,\beta)_\Lambda = \delta^r$ and $\delta\nmid_\Lambda \alpha/\delta^t$.

Lemma 1.5: $(\alpha + \beta,\alpha^2 - \alpha\beta + \beta^2)_\Lambda|_\Lambda 3\beta^2$.

Proof: $(\alpha^2 - \alpha\beta + \beta^2) + (2\beta - \alpha)(\alpha + \beta) = 3\beta^2$.

Lemma 1.6: If Λ is a real ring and $\alpha \neq 0$ or $\beta \neq 0$, then

$\alpha^2 \pm \alpha\beta + \beta^2 > 0$.

Proof: $4(\alpha^2 \pm \alpha\beta + \beta^2) = (2\alpha \pm \beta)^2 + 3\beta^2$.

Lemma 1.7: If $\sum_{j=1}^{n} p^{r_j} a_j = 0$, where $(a_j, p) = 1$, $r_j \geq 0$ and $n \geq 1$, then $\min\{r_j\}$ is taken on for at least two values of j . If $p = 2$ and $n = 3$, then $\min\{r_j\}$ occurs at exactly two values of j .

Proof: The first statement is obvious. If $p = 2$ and $n = 3$, then $r_1 = r_2 = r_3$ would imply $a_1 + a_2 + a_3 = 0$. But $a_1 + a_2 + a_3$ is odd.

Lemma 1.8: If $as^2 + bs + c = 0$, then $\text{Disc}(s) = b^2 - 4ac$ is a square in Z .

Proof: $b^2 - 4ac = (2as + b)^2$.

Lemma 1.9: $x \equiv 1 \bmod 3 \Leftrightarrow x^3 \equiv 1 \bmod 9$.

$x \equiv -1 \bmod 3 \Leftrightarrow x^3 \equiv -1 \bmod 9$.

Proof: If $x \equiv 1 \bmod 3$, then $x \equiv 1, 4$ or $-2 \bmod 9$. Thus $x^3 \equiv 1 \bmod 9$. Similarly, if $x \equiv -1 \bmod 3$, then $x^3 \equiv -1 \bmod 9$.

If $x^3 \equiv 1 \bmod 9$, then since $(x,3) = 1$, $x \equiv \pm 1 \bmod 3$. But if $x \equiv -1 \bmod 3$, then $x^3 \equiv -1 \not\equiv 1 \bmod 9$. Hence $x \equiv 1 \bmod 3$. Similarly, if $x^3 \equiv -1 \bmod 9$, then $x \equiv -1 \bmod 3$.

Lemma 1.10: If $x^3 \equiv 1 \bmod 3$, then $x^3 \equiv 1 \bmod 9$. If $x^3 \equiv -1 \bmod 3$, then $x^3 \equiv -1 \bmod 9$.

Proof: If $x^3 \equiv 1 \bmod 3$, then $x \equiv 1 \bmod 3$ and therefore by Lemma 1.9, $x^3 \equiv 1 \bmod 9$. The second statement follows similarly.

If $ay^2 \pm 2^n 3^m = bx^3$, then since $a, b, x, y \in Z$, $2^n 3^m \in Z$ and hence $m \geq 0$ and $n \geq 0$. Also, we may obviously assume that $y \geq 0$.

In Sections 2, 3, 4, 5, 12, 13 and 15 both the equations $ay^2 + 2^n3^m = bx^3$ and $ay^2 - 2^n3^m = bx^3$ will be solved simultaneously. The first of these will be referred to as the positive case and the second as the negative case. On those lines preceded by an asterisk, the upper sign of all the symbols $\pm$ and $\mp$ (also called double signs) refer to the positive case and the lower sign to the negative case. In addition, the same is true for those double signs with an asterisk placed above them. Otherwise, the symbol $\pm$ has its usual meaning of plus or minus.

[A] will refer to Appendix II. Theorem numbers will refer to the section in which the theorem appears.

SECTION 2.

$y^2 \pm 2^n = x^3$, x odd

Proposition 2.1: $y^2 + 2^{3k} = x^3$, x odd $\Rightarrow$ $(k,x,y) = (0,1,0)$.

$y^2 - 2^{3k} = x^3$, x odd $\Rightarrow$ $(k,x,y) = (0,-1,0),(1, 1,3)$

or $(3,-7,13)$.

Proof: If $k = 0$ or 1 , the solutions are listed in [A].

Assume $k > 1$. Now ,

(*) $y^2 = x^3 \mp 2^{3k} = ab$, where $a = x \mp 2^k$ and $b = x^2 \pm 2^k x + 2^{2k}$.

Therefore a and b are both odd. By Lemma 1.6, $b > 0$ and thus, since $a = y^2/b$, $a > 0$. By Lemma 1.5, $(a,b)|3\cdot 2^{2k}$ and hence $(a,b) = 1$ or 3 .

Suppose first that $(a,b) = 3$. Then $3|y$ and $(y/3)^2 = (a/3)(b/3)$. Therefore by Lemma 1.2, $a = 3u^2$ and $b = 3v^2$. Hence v is odd . Eliminating x from the latter two equations, we obtain

$$3u^4 \pm 3\cdot 2^k u^2 + (2^{2k} - v^2) = 0 .$$

Thus by Lemma 1.8,

$$\mathrm{Disc}(u^2) = 9\cdot 2^{2k} - 4\cdot 3(2^{2k} - v^2) = d^2 .$$

Hence $d = 2D$ and $3v^2 - D^2 = 3\cdot 2^{2k-2}$. Since $k > 1$ and v is odd, D is odd. Therefore $3v^2 - D^2 \equiv 2 \bmod 4$. But, since $k \geq 2$, $3\cdot 2^{2k-2} \equiv 0 \bmod 4$.

Hence $(a,b) = 1$. Thus $a = u^2$ and $b = v^2$, implying that $(uv,2) = 1$. Eliminating x , we obtain

(1) (*) $u^4 \pm 3\cdot 2^k u^2 + (3\cdot 2^{2k} - v^2) = 0$.

By Lemma 1.8, $-3\cdot 2^{2k} + 12v^2 = d^2$. Thus $d = 2D$ and

$$(2) \qquad v^2 - D^2 = 3\cdot 2^{2k-2} .$$

Since $k \geq 2$ and v is odd, D is odd. Also if $3|D$, then $3|v$, which implies, by (2), that $3^2|3\cdot 2^{2k-2}$. Hence $(D,3) = 1$ and thus $(v,3) = 1$. Therefore $D \equiv \pm v \bmod 3$. Let $V = \pm v$, such that $D \equiv V \bmod 3$. Now by (2),

$$(V + D)(V - D) = 3\cdot 2^{2k-2} .$$

Thus $V - D = \pm 3\cdot 2^s$ and $V + D = \pm 2^t$, where $t + s = 2k - 2$ and the same sign holds in both equations. Since V and D are odd, $s \geq 1$ and $t \geq 1$. Hence

$$(2a) \qquad D = \pm(2^{t-1} - 3\cdot 2^{s-1}) .$$

Since D is odd, either $(t > 1$ and $s = 1)$ or $(t = 1$ and $s > 1)$.

Solving (1) for u^2,

$$(3) \qquad u^2 = \overset{(*)}{\mp} 3\cdot 2^{k-1} \pm D .$$

Suppose first that $t > 1$ and $s = 1$. Hence $t = 2k - 3$. By (2a), $D = \pm(2^{2k-4} - 3)$. Thus $k > 2$ and by (3),

$$u^2 = \mp 3\cdot 2^{k-1} \pm (2^{2k-4} - 3) .$$

If $k > 3$, then $u^2 \equiv \pm 3 \bmod 8$. Therefore $k = 3$ and hence $u^2 = \pm 12 \pm 1$, which is impossible.

Thus $t = 1$ and $s > 1$, which implies that $s = 2k - 3$ and by (2a), $D = \pm(1 - 3\cdot 2^{2k-4})$. Hence $k > 2$ and by (3), (3a) $u^2 \overset{(*)}{=} \mp 3\cdot 2^{k-1} \pm (1 - 3\cdot 2^{2k-4})$.

The second minus sign cannot hold modulo 3 .

If the first minus sign held, then $u^2 < 0$. Therefore there are no solutions in the positive case, and in the negative case

(3b) $u^2 = 3(2^{k-1} - 2^{2k-4}) + 1$.

If $k > 3$, then $2k - 4 > k - 1$ and by (3b), $u^2 < 0$. Hence $k = 3$ and thus $u^2 = 1$. Since $a = u^2$, $x = u^2 - 2^k = -7$. Hence $y^2 = x^3 + 2^{3k} = 169$ and therefore $y = 13$.

<u>Proposition 2.2</u>: $y^2 + 2^{3k+1} = x^3$, x odd $\Rightarrow (k,x,y) = (0,3,5)$.

$y^2 - 2^{3k+1} = x^3$, x odd $\Rightarrow (k,x,y) = (0,-1,1)$ or $(2,17,71)$.

<u>Proof</u>: Suppose $3|y$. Then $(x,3) = 1$ and by Lemma 1.9, $x^3 \equiv \pm 1 \bmod 9$. $2^{3k+1} = 2\cdot 8^k \equiv \pm 2 \bmod 9$. Thus $0 \equiv y^2 = x^3 \pm 2^{3k+1} \equiv \pm 1 \pm 2 \bmod 9$, which is a contradiction. Hence $(y,3) = 1$.

[A] gives the solutions for $k = 0$; so we may assume that $k > 0$. Let $\Lambda = \Omega_2$ and $\theta = 2^{\frac{1}{3}}$. Now, y is odd and

(*) $y^2 = x^3 \mp 2^{3k+1} = \alpha\beta$, where $\alpha = x \mp 2^k\theta$ and $\beta = x^2 \pm 2^k\theta x + (2^k\theta)^2$.

Since by Lemma 1.6, $\beta > 0$, we have $\alpha = y^2/\beta > 0$. By Lemma 1.5, $(\alpha,\beta)_\Lambda |_\Lambda 3(2^k\theta)^2$. But $(y,6) = 1$, and therefore by Lemma 1.1, $(y,6)_\Lambda = 1$. Thus, since $\alpha\beta |_\Lambda y^2$, $(\alpha,\beta)_\Lambda = 1$. Therefore Lemma 1.2 and (3), Section 1 give

(*) $x \mp 2^k\theta = \alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $-1+\theta$. We may assume that $c \geq 0$, since $\alpha = \mu(-a - b\theta - c\theta^2)^2$.

If $\mu = -1+\theta$, we have (by comparing the coefficients of $1, \theta$ and θ^2):

$$x = -(a^2 + 4bc) + 2(b^2 + 2ac) (\Rightarrow (a,2) = 1) ,$$

$$(*) \quad \mp 2^k = -2(c^2 + ab) + (a^2 + 4bc) (\Rightarrow 2 | a , \text{ since } k > 0) .$$

Therefore we have a contradiction.

Hence $\mu = 1$, which gives:

$$(4) \qquad x = a^2 + 4bc \ (\Rightarrow a \text{ odd}),$$

$$(5) \quad (*) \quad \mp 2^{k-1} = c^2 + ab ,$$

$$(6) \qquad 0 = b^2 + 2ac \ (\Rightarrow b \text{ even}) .$$

If $b = 0$, then by (6), $c = 0$, which contradicts (5). Thus $b \neq 0$, which implies that $b = 2^s B$, where $s \geq 1$ and B is odd. From (6), $c = 2^{2s-1}C$, where C is odd. Also $C > 0$, since $c \geq 0$. Again by (6),

$$(7) \qquad 0 = B^2 + aC$$

and from (5),

$$(8) \quad (*) \quad \mp 2^{k-1} = 2^{4s-2}C + 2^s aB .$$

If $p | C$, then by (7), $p | B$ (p being a positive prime of Z) . Thus by (8), $p | 2^{k-1}$. But C is odd and therefore $C = 1$. Hence $a = -B^2$ and by (8),

$$(9) \quad (*) \quad \mp 2^{k-1} = 2^{4s-2} - 2^s B^3 .$$

Now, since $s \geq 1$, $4s - 2 > s$. Hence by Lemma 1.7, $k - 1 = s$, and we have by (9),

(*) $$(\mp 1)^3 + B^3 = 2(2^{s-1})^3 .$$

[3 ,pp.70-72] gives

(*) $$B = \mp 1 = 2^{s-1} .$$

The positive case does not hold. In the negative case, $B = 1$ and $s = 1$. Hence $k = 2$ and $a = -1$. Also, $c = 2^{2s-1}C = 2$ and $b = 2^s B = 2$. By (4), $x = 17$ and therefore $y = 71$.

Proposition 2.3: $y^2 + 2^{3k+2} = x^3$, x odd $\Rightarrow (k,x,y) = (0,5,11)$.
$y^2 - 2^{3k+2} = x^3$, x odd has no solutions.

Proof: If $3|y$, then $(x,3) = 1$ and hence $x^3 \equiv \pm 1 \bmod 9$. But $2^{3k+2} \equiv \pm 4$ and $y^2 \equiv 0 \bmod 9$. Thus $(y,3) = 1$.

The solutions for $k = 0$ are given by [A].

Assume $k > 0$. Let $\Lambda = \Omega_2$ and $\theta = 2^{\frac{1}{3}}$. Now, y is odd and

(*) $$y^2 = x^3 \mp 2^{3k+2} = \alpha\beta, \text{ where } \alpha = x \mp 2^k\theta^2 \text{ and}$$

(*) $$\beta = x^2 \pm 2^k\theta^2 x + (2^k\theta^2)^2 .$$

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda 3\cdot 2^{2k+1}\cdot\theta$. Thus, since $(y,6) = 1$, $(\alpha,\beta)_\Lambda = 1$. As in Proposition 2.2, $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $-1 + \theta$ and $b \geq 0$.

If $\mu = -1 + \theta$, we have:

$$x = -(a^2 + 4bc) + 2(b^2 + 2ac) \quad (\Rightarrow a \text{ is odd}) ,$$
$$0 = -2(c^2 + ab) + (a^2 + 4bc) \quad (\Rightarrow a \text{ is even}) .$$

Thus $\mu = 1$, which gives:

(10) $x = a^2 + 4bc$ ($\Rightarrow$ a odd) ,

(11) $0 = c^2 + ab$,

(12) (*) $\mp 2^k = b^2 + 2ac$ ($\Rightarrow$ b is even, since $k > 0$) .

Hence by (11), c is even. If $c = 0$, then by (11), $b = 0$, which contradicts (10). Thus $c \neq 0$, implying that $c = 2^s C$, where $s \geq 1$ and C is odd. By (11), $b = 2^{2s} B$, where B is odd. Therefore $B > 0$. Again by (11), $0 = C^2 + aB$ and by (12),

(*) $\mp 2^k = 2^{4s} B^2 + 2^{s+1} aC$.

As in Proposition 2.2, $B = 1$. Therefore $a = -C^2$ and

(*) $\mp 2^k = 2^{4s} - 2^{s+1} C^3$.

Also, $4s > s + 1$. Thus $k = s + 1$ and

(*) $(\mp 1)^3 + C^3 = 4(2^{s-1})^3$.

But this equation has no solutions by [3, p.70-72].

Propositions 2.1, 2.2 and 2.3 prove

<u>Theorem 2</u>: $y^2 + 2^n = x^3$, x odd $\Leftrightarrow$ $(n,x,y) = (0,1,0),(1,3,5)$ or $(2,5,11)$.

$y^2 - 2^n = x^3$, x odd $\Leftrightarrow$ $(n,x,y) = (0,-1,0),(1,-1,1),(3,1,3),$ $(7,17,71)$, or $(9,-7,13)$.

SECTION 3.

$3y^2 \pm 2^n = x^3$, x odd

Proposition 3.1: $3y^2 + 2^{3k} = x^3$, x odd $\Rightarrow (k,x,y) = (1,11,21)$.
$3y^2 - 2^{3k} = x^3$, x odd has no solutions.

Proof: (*) $(9y)^2 \pm 27\cdot 2^{3k} = (3x)^3$. [A] lists the solutions for $k = 0$ and $k = 1$.

Assume $k > 1$. Thus y is odd. Now,

(*) $3y^2 = ab$, where $a = x \mp 2^k$ and $b = x^2 \pm 2^k x + 2^{2k}$.

Hence b is odd. As in Proposition 2.1, $a > 0$ and $(a,b) | 3\cdot 2^{2k}$. Therefore $(a,b) = 1$ or 3. But

(*) $a = x \mp 2^k \equiv x^3 \mp 2^{3k} = 3y^2 \equiv 0 \bmod 3$. Hence

(*) $x \equiv \pm 2^k \bmod 3$, which implies that

(*) $b = x^2 \pm 2^k x + 2^{2k} \equiv 2^{2k} + 2^{2k} + 2^{2k} \equiv 0 \bmod 3$.

Thus $(a,b) = 3$. Therefore $3^2 | ab = 3y^2$, implying that $3 | y$. Hence $3(y/3)^2 = (a/3)(b/3)$. Thus $(a/3,b/3) = (r^2,3s^2)$ or $(3r^2,s^2)$.

If $(a/3,b/3) = (r^2,3s^2)$, then eliminating x, we obtain

(*) $3(r^4 \pm 2^k r^2 - s^2) = 2^{2k}$, which cannot hold. Thus $(a/3,b/3) = (3r^2,s^2)$. Hence s is odd. Eliminating x,

(*) $27r^4 \pm 9\cdot 2^k r^2 + (2^{2k} - s^2) = 0$.

Therefore $\mathrm{Disc}(r^2) = 108s^2 - 27\cdot 2^{2k} = d^2$. $3^3\, 2^2 | d^2$, since $k > 1$. Thus $d = 18D$ and $s^2 - 3D^2 = 2^{2k-2}$. Since $k > 1$, D is odd and therefore $s^2 - 3D^2 \equiv -2 \bmod 8$. But $2^{2k-2} \equiv 0 \bmod 4$.

Proposition 3.2: $3y^2 + 2^{3k+1} = x^3$, x odd has no solutions.

$$3y^2 - 2^{3k+1} = x^3 \Rightarrow (k,x,y) = (0,1,1) \text{ or } (4,1915,48383) .$$

Proof: If $k = 0$, then

(*) $(9y)^2 \pm 54 = (3x)^3$, and we obtain the solution from [A].

Assume $k > 0$. Let $\Lambda = \Omega_2$ and $\theta = 2^{\frac{1}{3}}$. $(x,3) = 1$ and $(y,2) = 1$. Thus $x^3 \equiv \pm 1 \bmod 9$ and $2^{3k+1} \equiv \pm 2 \bmod 9$. Hence $3y^2 \equiv \pm 1 \pm 2 \bmod 9$, implying that $(y,3) = 1$.

(*) $3y^2 = x^3 \mp 2^{3k+1} = \alpha\beta$, where $\alpha = x \mp 2^k\theta$ and $\beta = x^2 \pm 2^k\theta x + (2^k\theta)^2$.

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda 3(2^k\theta)^2$. Since $(3y^2,2) = 1$, $(\alpha,\beta)_\Lambda |_\Lambda 3 = (1 + \theta)^3(-1 + \theta)$. By (6), Section 1, $N(1 + \theta) = 3$ and $N(-1 + \theta) = 1$. Thus $1 + \theta$ is a prime of Λ, and $-1 + \theta$ is a unit of Λ. Also,

(*) $x \pm 2^k \equiv x^3 \pm 2^{3k} \equiv x^3 \mp 2^{3k+1} = 3y^2 \equiv 0 \bmod 3$ and

(*) $\alpha = (1 + \theta)[(x \pm 2^k)/3 \mp 2^k - (x \pm 2^k)/3 \cdot \theta + (x \pm 2^k)/3 \cdot \theta^2]$.

Hence $1 + \theta |_\Lambda \alpha$. If $(1 + \theta)^2 |_\Lambda \alpha$, then

(*) $9 = N(1 + \theta)^2 | N(\alpha) = x^3 \mp 2^{3k+1} = 3y^2$,

which contradicts $(y,3) = 1$. Therefore $(1 + \theta)^2 \nmid_\Lambda \alpha$ and, by Lemma 1.4, $(\alpha/(1 + \theta),\beta)_\Lambda = 1$.

Now, $(1 + \theta)^2y^2 = \alpha/(1 + \theta) \cdot \beta/(-1 + \theta)$. Thus $\alpha/(1 + \theta) = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $-1 + \theta$, and we may assume that $a - b \geq 0$.

If $\mu = 1$, then:

$x = (a^2 + 4bc) + 2(b^2 + 2ac)$ ($\Rightarrow$ a is odd),

(*) $\mp 2^k = (a^2 + 4bc) + 2(c^2 + ab)$ ($\Rightarrow$ a is even).

Thus $\mu = -1 + \theta$, and we have:

(2) $x = -(a^2 + 4bc) + 4(c^2 + ab)$ ($\Rightarrow$ a is odd) ,

(3) (*) $\mp 2^{k-1} = -(c^2 + ab) + (b^2 + 2ac)$,

(4) $0 = (a^2 + 4bc) - (b^2 + 2ac)$.

Hence by (4), b is odd and

(5) $2c(a - 2b) = a^2 - b^2 \equiv 0 \bmod 8$.

Therefore $4|c$, which implies by (3), that $2|2^{k-1}$. Hence $k > 1$. Multiplying (3) by $4(a - 2b)^2$ and using (5), we obtain

(6) (*) $\mp 2^{k+1}(a - 2b)^2 = 3(a - b)(a^3 - 3a^2b + 3ab^2 - 5b^3)$.

Let $g = (a,b)$, which is defined since $a \neq 0$ $(g > 0)$. Thus $(g,2) = 1$. Let $A = a/g$ and $B = b/g$. Hence $(AB,2) = 1$, $(A,B) = 1$ and $A - B = (a - b)/g \geq 0$. (6) gives

(7) (*) $\mp 2^{k+1}(A - 2B)^2 = 3g^2(A - B)(A^3 - 3A^2B + 3AB^2 - 5B^3)$.

Therefore $g^2|(A - 2B)^2$, implying that $g|A - 2B$. Also by (7), $3|A - 2B$ and therefore $3|A \Leftrightarrow 3|B$. Thus, since $(A,B) = 1$, we have $(A,3) = (B,3) = 1$. Now, $A - B = (A - 2B) + B$ and hence $(A - B,3) = 1$. $(A - 2B,A - B) = 1$; for if $d|(A - 2B,A - B)$, then $d|B$ and therefore $d|A$.

If $d|(A - 2B,A + B)$, then $d|3B$. But $(d,B)|(A + B,B) = 1$. Thus $d|3$. Also, $A + B \equiv A - 2B \equiv 0 \bmod 3$. Hence $(A - 2B,A + B) = 3$. From (5), $2c(A - 2B)/3 = g(A - B) \cdot (A + B)/3$. Thus $(A - 2B)/3|g$. Hence $(A - 2B)/g|3$ (since $g|A - 2B$) and therefore $(A - 2B)/g = \pm 1$ or ± 3 .

If $A - 2B = \pm g$, then by (7), $3 | 2^{k+1}$. Hence $A - 2B = \pm 3g$, and (7) gives

(8) (*) $\mp 3 \cdot 2^{k+1} = (A - B)(A^3 - 3A^2B + 3AB^2 - 5B^3)$.

Therefore, since $A - B \geq 0$, $(A - B, 3) = 1$ and $(AB, 2) = 1$, $A - B = 2^r$, $r \geq 1$. (8) gives

(*) $\mp 3(2^{k+1-r}) = (A - B)^3 - 4B^3 = 2^{3r} - 4B^3$.

Since $3r > 2$, we have by Lemma 1.7, $k + 1 - r = 2$. Thus

(8a) (*) $2^{3r-2} \pm 3 = B^3$.

If $r = 2t$, then by (8a),

(*) $(2^{3t-1})^2 \pm 3 = B^3$.

By [A], there are no solutions in either the positive or negative case (since $2^{3t-1} \neq 2$).

Thus $r = 2t + 1$, and by (8a),

(*) $(2^{3t+2})^2 \pm 24 = (2B)^3$.

By [A], there are no solutions in the positive case, and in the negative case $(2B, 2^{3t+2}) = (2,4)$ or $(10,32)$.

If $(2B, 2^{3t+2}) = (2,4)$, then $t = 0$, $r = 1$ and $B = 1$. But $A = B + 2^r = 3$, contradicting $(A,3) = 1$. Therefore $(2B, 2^{3t+2}) = (10,32)$. Hence $t = 1$ and $B = 5$. This gives $r = 3$, $A = 3$ and, since $2 = k + 1 - r$, $k = 4$. Now, $\pm 3g = A - 2B = 3$. Since $g > 0$, $g = 1$, and hence $a = gA = 13$ and $b = gB = 5$. From (5), $c = 24$, and thus by (2), $x = 1915$. Finally, $y^2 = 2340914689$ and therefore $y = 48383$.

<u>Proposition 3.3</u>: $3y^2 + 2^{3k+2} = x^3$, x odd has no solutions.

$3y^2 - 2^{3k+2} = x^3$, x odd $\Rightarrow (k,x,y) = (0,-1,1)$ or $(2,11,23)$.

<u>Proof</u>: $(y,2) = 1$ and $(x,3) = 1$. Thus $x^3 \equiv \pm 1 \bmod 9$. Also, $2^{3k+2} \equiv \pm 4 \bmod 9$. Hence $3y^2 \equiv \pm 1 \pm 4 \bmod 9$, implying that $(y,3) = 1$.

Let $\Lambda = \Omega_2$ and $\theta = 2^{\frac{1}{3}}$. Now ,

(*) $3y^2 = \alpha\beta$, where $\alpha = x \mp 2^k\theta^2$ and $\beta = x \pm 2^k\theta^2 x + (2^k\theta^2)^2$.

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda 3(2^k\theta^2)^2$. Thus, since $(y,2) = 1$, $(\alpha,\beta)_\Lambda |_\Lambda 3 = (1+\theta)^3(-1+\theta)$. Now,

(*) $x \mp 2^k \equiv x^3 \mp 2^{3k} \equiv x^3 \mp 2^{3k+2} = 3y^2 \equiv 0 \bmod 3$ and

(*) $\alpha = (1+\theta)[x - 2(x \mp 2^k)/3 + (-x + 2\cdot(x \mp 2^k)/3)\theta + (x \mp 2^k)/3\cdot\theta^2]$.

Hence $1 + \theta |_\Lambda \alpha$. If $(1+\theta)^2 |_\Lambda \alpha$, then $9 = N(1+\theta)^2 | N(\alpha) = 3y^2$, which contradicts $(y,3) = 1$. As in Proposition 3.2, $\alpha/(1+\theta) = \mu(a + b\theta + c\theta)^2$, where $\mu = 1$ or $-1+\theta$ and $c \geq 0$.

If $\mu = 1$, then:

$x = (a^2 + 4bc) + 2(b^2 + 2ac)$ ($\Rightarrow a$ is odd) ,

$0 = (a^2 + 4bc) + 2(c^2 + ab)$ ($\Rightarrow a$ is even).

Hence $\mu = -1 + \theta$, which gives:

(9) $x = -(a^2 + 4bc) + 4(c^2 + ab)$ ($\Rightarrow a$ is odd) ,

(10) $0 = -(c^2 + ab) + (b^2 + 2ac)$,

(11)(*) $\mp 2^k = (a^2 + 4bc) - (b^2 + 2ac)$.

Now, since a is odd, $b(b - a)$ is even and hence by (10), c is even. (10) also gives

(12) $$a(b - 2c) = b^2 - c^2 .$$

Multiplying (11) by $(b - 2c)^2$ and using (12),

(13)(*) $$\mp 2^k(b - 2c)^2 = -3c(c^3 - 6bc^2 + 6b^2c - 2b^3) .$$

If $c = 0$, then by (13), $b = 0$. Hence by (9) and (11),

(*) $x = -a^2 = \pm 2^k$. Thus $k = 0$, $x = \pm 1$, which cannot hold in the positive case and implies $y = 1$ in the negative case.

Assume $c \neq 0$. Let $g = (b,c) > 0$, $B = b/g$, $C = c/g$. Hence $C > 0$, $(B,C) = 1$ and by (13),

(14)(*) $$\mp 2^k(B - 2C)^2 = -3g^2C(C^3 - 6BC^2 + 6B^2C - 2B^3) .$$

Thus $g^2 | 2^k(B - 2C)^2$.

If $k = 0$, then $g^2 | (B - 2C)^2$. If $k > 0$, then by (11), b is odd and hence $(g,2) = 1$. Therefore $g^2 | (B - 2C)^2$ in either case, which implies that $g | B - 2C$.

From (14), $3 | B - 2C$. As in Proposition 3.2, we have $(B,3) = (C,3) = 1$, $(B - 2C, B - C) = 1$ and $(B - 2C, B + C) = 3$. (12) gives

(15) $$a(B - 2C)/3 = g(B - C) \cdot (B + C)/3 .$$

As in Proposition 3.2, $B - 2C = \pm g$ or $\pm 3g$.

If $B - 2C = \pm g$ then by (14), $3 | 2^k$. Thus

(16) $$B - 2C = \pm 3g ,$$

and (14) gives

(17)(*) $\mp 3\cdot 2^k = -C(C^3 - 6BC^2 + 6B^2C - 2B^3)$.

Since $C > 0$ and $(C,3) = 1$, $C = 2^r$, where $r \le k$.

If $k = 0$, then $r = 0$ and hence $C = 1$. By (17), $(B - 1)^3 = 1$ or -2, implying $B = 2$. (17) then gives $0 = B - 2C = \pm 3g$, which is a contradiction. Therefore $k > 0$ and thus by (11), $(b,2) = 1$. Hence $(B,2) = 1$ and $(g,2) = 1$. Since c is even, $2 | c/g = C$. Thus $r \geq 1$.

Now by (17),

(18)(*) $\mp 3\cdot 2^{k-r} = 2B^3 - 3\cdot 2^{r+1}B^2 + 3\cdot 2^{2r+1}B - 2^{3r}$.

Since $r \geq 1$ and $(B,2) = 1$, Lemma 1.7 gives $k - r = 1$, and hence by (18),

(19)(*) $\mp 3 = (B - 2^r)^3 + 2^{3r-1}$.

If $r = 2t$, then by (19),

(*) $(2^{3t+1})^2 \pm 24 = (2(2^r - B))^3$.

By [A], there are no solutions to this equation since $2^{3t+1} \neq 4$ or 32. Therefore $r = 2t + 1$, implying, by (19), that

(*) $(2^{3t+1})^2 \pm 3 = (2^r - B)^3$.

By [A], there are no solutions in the positive case, and in the negative case, $2^{3t+1} = 2$ and $2^r - B = 1$. Thus $t = 0$, $r = 2t + 1 = 1$, $B = 2^r - 1 = 1$, $C = 2^r = 2$, $k = r + 1 = 2$ and by (16), $\pm 3g = 3$. Hence $g = 1$, $b = gB = 1$, $c = gC = 2$ and by (12), $a = 1$. By (9), $x = 11$ and hence $y = 23$.

Propositions 3.1, 3.2 and 3.3 prove

Theorem 3: $3y^2 + 2^n = x^3$, x odd $\Leftrightarrow (n,x,y) = (3,11,21)$.

$3y^2 - 2^n = x^3$, x odd $\Leftrightarrow (n,x,y) = (1,1,1),(2,-1,1),(8,11,23)$ or $(13,1915,48383)$.

SECTION 4.

$y^2 \pm 3\cdot 2^n = x^3$, x odd.

Remark: $(x,3) = 1$; for if $3|x$, then $3|y$, which together imply that $9|y^2 - x^3 = 3\cdot 2^n$. Similarly, $(y,3) = 1$.

Proposition 4.1: $y^2 + 3\cdot 2^{3k} = x^3$, x odd has no solutions.

$y^2 - 3\cdot 2^{3k} = x^3$, x odd $\Rightarrow (k,x,y) = (0,1,2),(1,1,5),(3,25,131)$ or $(4,-23,11)$.

Proof: The solutions for $k = 0$ and $k = 1$ are given in [A].

Assume $k > 1$. Therefore y is odd. Let $\Lambda = \Omega_3$ and $\theta = 3^{\frac{1}{3}}$.

Now,

(*) $y^2 = \alpha\beta$, where $\alpha = x \mp 2^k\theta$ and $\beta = x^2 \pm 2^k\theta x - (2^k\theta)^2$.

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda|_\Lambda\ 3\cdot 2^{2k}\theta^2$. Hence $(\alpha,\beta)_\Lambda = 1$.

Thus $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $-2 + \theta^2$ and $c \geq 0$.

If $\mu = -2 + \theta^2$, then:

(1) $x = -2(a^2 + 6bc) + 3(3c^2 + 2ab)$ ($\Rightarrow c$ is odd) ,

(1a) (*) $\mp 2^k = -2(3c^2 + 2ab) + 3(b^2 + 2ac)$ ($\Rightarrow b$ is even) ,

$0 = -2(b^2 + 2ac) + (a^2 + 6bc)$ ($\Rightarrow a$ is even) .

Since $k \geq 2$, (1a) gives $0 \equiv -6c^2 \bmod 4$. It follows that c is even, contradicting (1).

Hence $\mu = 1$, implying that:

(2) $x = a^2 + 6bc$ ($\Rightarrow a$ is odd) ,

(3) (*) $\mp 2^k = 3c^2 + 2ab$,

(4) $0 = b^2 + 2ac$ ($\Rightarrow b$ is even) .

If $b = 0$, then by (4), $c = 0$, which contradicts (3). Thus $b = 2^r B$, where B is odd and $r \geq 1$. By (4), $c = 2^{2r-1}C$, where C is odd. $C > 0$, since $c \geq 0$. Using (4) again, $0 = B^2 + aC$ and from (3),

(*) $\mp 2^k = 3 \cdot 2^{4r-2}C^2 + 2^{r+1}aB$.

As in Proposition 2.2, $C = 1$. Therefore $a = -B^2$ and

(4a)(*) $\mp 2^k = 3 \cdot 2^{4r-2} - 2^{r+1}B^3$.

If $r > 1$, then $4r - 2 > r + 1$. Hence $r + 1 = k$ and by (4a),

(4b)(*) $B^3 = \pm 1 + 3 \cdot 2^{3r-3} \equiv \pm 1 \bmod 3$.

Therefore by Lemma 1.10,

(*) $B^3 \equiv \pm 1 \bmod 9$, which implies by (4b), that $9 \mid 3 \cdot 2^{3r-3}$.

Thus $r = 1$, implying that $2 = 4r - 2 = r + 1$. Hence by (4a), $k > 2$ and

(*) $3(1)^2 \pm 2^{k-2} = B^3$.

It follows from Theorem 3 that this equation has no solutions in the positive case,and in the negative case $(B,k) = (1,3)$ or $(-1,4)$.

If $(B,k) = (1,3)$, then $b = 2^r B = 2$, $c = 2^{2r-1}C = 2$ and by (4), $a = -1$. By (2), $x = 25$ and hence $y = 131$.

If $(B,k) = (-1,4)$, then $b = -2$, $c = 2$ and by (4), $a = -1$. Thus by (2), $x = -23$, which implies $y = 11$.

<u>Proposition 4.2</u>: $y^2 + 3 \cdot 2^{3k+1} = x^3$, x odd has no solutions.

$y^2 - 3 \cdot 2^{3k+1} = x^3$, x odd $\Rightarrow (k,x,y) = (1,1,7)$.

Proof: $(y,2) = 1$. Let $\Lambda = \Omega_6$ and $\theta = 6^{\frac{1}{3}}$. Now,

(*) $y^2 = \alpha\beta$, where $\alpha = x \mp 2^k\theta$ and $\beta = x^2 - 2^k\theta x + (2^k\theta)^2$.

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda\ 3 \cdot 2^{2k}\theta^2$. Since $(y,6) = 1$, $(\alpha,\beta)_\Lambda = 1$. Thus $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $1 - 6\theta + 3\theta^2$ and $c \geq 0$.

If $\mu = 1$, then:

(5) $x = a^2 + 12bc$ ($\Rightarrow$ a is odd),

(6)(*) $\mp 2^k = 6c^2 + 2ab$ ($\Rightarrow k \geq 1$),

(7) $0 = b^2 + 2ac$ ($\Rightarrow$ b is even).

If $b = 0$, then by (7), $c = 0$, contradicting (6). Hence $b = 2^rB$, where B is odd and $r \geq 1$. From (7), $c = 2^{2r-1}C$, where C is odd. Therefore $C > 0$ and $0 = B^2 + aC$. From (6),

(*) $\mp 2^{k-1} = 3 \cdot 2^{4r-2}C^2 + 2^r aB$.

As in Proposition 2.2, $C = 1$, implying that $a = -B^2$ and hence

(*) $\mp 2^{k-1} = 3 \cdot 2^{4r-2} - 2^r B^3$.

Now, $4r - 2 > r$. Hence $k - 1 = r$, which implies that

(*) $\mp 1 + B^3 = 3 \cdot 2^{3r-2}$.

As in Proposition 4.1, this cannot hold.

Hence $\mu = 1 - 6\theta + 3\theta^2$, yielding:

(8) $x = (a^2 + 12bc) - 36(b^2 + 2ac) + 18(6c^2 + 2ab)$ ($\Rightarrow$ a is odd),

(9)(*) $\mp 2^k = (6c^2 + 2ab) - 6(a^2 + 12bc) + 18(b^2 + 2ac)$ ($\Rightarrow k \geq 1$),

(10) $0 = 3(a^2 + 12bc) - 6(6c^2 + 2ab) + (b^2 + 2ac)$.

By (10), b is odd. Thus $3a^2 + b^2 \equiv 4 \bmod 8$ and hence by (10), $4 | 2ac$. Therefore c is even. It follows by (9), that

$$(10a)(*) \quad \mp 2^{k-1} = (3c^2 + ab) - 3(a^2 + 12bc) + 9(b^2 + 2ac) .$$

Now, $ab - 3a^2 + 9b^2$ is odd and hence by (10a), 2^{k-1} is odd. Therefore $k = 1$ and our equations become

$$(*) \qquad y^2 \pm 48 = x^3, \quad x \text{ odd}.$$ [A] lists the solutions.

<u>Proposition 4.3</u>: $y^2 + 3\cdot 2^{3k+2} = x^3$, x odd has no solutions.

$y^2 - 3\cdot 2^{3k+2} = x^3$, x odd $\Rightarrow (k,x,y) = (0,13,47)$.

<u>Proof</u>: $(y,2) = 1$. If $k = 0$, then $y^2 \pm 12 = x^3$, and we may obtain the solution from [A].

Assume $k > 0$, and let $\Lambda = \Omega_{12}$, $\theta = (12)^{\frac{1}{3}}$. Now,

$$(*) \quad y^2 = \alpha\beta , \text{ where } \alpha = x \mp 2^k\theta \text{ and } \beta = x^2 \pm 2^k\theta x + (2^k\theta)^2 .$$

$\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda 3\cdot 2^{2k}\theta^2$. Hence $(\alpha,\beta)_\Lambda = 1$. Thus, by (4) of Section 1, $\alpha = \mu(a + b\theta + c\theta^2/2)^2$, where $\mu = 1$ or $1 + 3\theta - 3\theta^2/2$ and $c \geq 0$.

If $\mu = 1 + 3\theta - 3\theta^2/2$, then:

$$x = (a^2 + 12bc) - 18(3c^2 + 2ab) + 18(2b^2 + 2ac) \qquad (\Rightarrow a \text{ is odd}),$$
$$0 = -3(a^2 + 12bc) + 6(3c^2 + 2ab) + (2b^2 + 2ac) \qquad (\Rightarrow a \text{ is even}) .$$

Therefore $\mu = 1$, which gives:

$$x = a^2 + 12bc \qquad (\Rightarrow a \text{ is odd}) ,$$
$$(11)(*) \quad \mp 2^k = 3c^2 + 2ab \qquad (\Rightarrow c \text{ is even, since } k \geq 1) ,$$
$$(12) \qquad 0 = b^2 + ac .$$

By (12), b is even. If $b = 0$, then by (12), $c = 0$, which contradicts (11). Hence $b = 2^r B$, where $r \geq 1$ and $(B,2) = 1$. Thus $c = 2^{2r} C$, $(C,2) = 1$ and therefore $C > 0$. Hence $0 = B^2 + aC$ and, by (11),

$$(*) \qquad \mp 2^k = 3 \cdot 2^{4r} C^2 + 2^{r+1} aB .$$

As in Proposition 4.1, $C = 1$ and $a = -B^2$. Thus

$$(*) \qquad \mp 2^k = 3 \cdot 2^{4r} - 2^{r+1} B^3 .$$

Now, $4r > r + 1$ and therefore $k = r + 1$. Hence

$$(*) \qquad \mp 1 + B^3 = 3 \cdot 2^{3r-1},$$ which, as in Proposition 4.1, cannot hold.

By Propositions 4.1, 4.2 and 4.3,we have

<u>Theorem 4</u>: $y^2 + 3 \cdot 2^n = x^3$, x odd has no solutions.

$y^2 - 3 \cdot 2^n = x^3$, x odd $\Leftrightarrow$ $(n,x,y) = (0,1,2),(2,13,47),(3,1,5),(4,1,7),(9,25,131)$ or $(12,-23,11)$.

SECTION 5.

$y^2 \pm 3^2 \cdot 2^n = x^3$, $(x,6) = 1$

Proposition 5.1: $y^2 + 3^2 \cdot 2^{3k} = x^3$, $(x,6) = 1$ has no solutions.

$y^2 - 3^2 \cdot 2^{3k} = x^3$, $(x,6) = 1 \Rightarrow (k,x,y) = (5,73,827)$.

Proof: By [A], there are no solutions for $k = 0$ with $(x,6) = 1$.

Assume $k > 0$. Hence $(y,6) = 1$. Let $\Lambda = \Omega_3$ and $\theta = 3^{\frac{1}{3}}$.

Now,

(*) $y^2 = \alpha\beta$, where $\alpha = x \mp 2^k\theta^2$ and $\beta = x^2 \pm 2^k\theta^2 x + (2^k\theta^2)^2$.

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda 3\cdot(2^k\theta^2)^2$. Thus $(\alpha,\beta)_\Lambda = 1$ and hence $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $-2 + \theta^2$ and $b \geq 0$.

If $\mu = -2 + \theta^2$, then:

(1a) $x = -2(a^2 + 6bc) + 3(3c^2 + 2ab)$ ($\Rightarrow c$ is odd) ,

(1b) $0 = -2(3c^2 + 2ab) + 3(b^2 + 2ac)$ ($\Rightarrow b$ is even) ,

(*) $\mp 2^k = (a^2 + 6bc) - 2(b^2 + 2ac)$ ($\Rightarrow a$ is even) .

Since a is even, it follows from (1a) that $-6c^2 \equiv 0 \bmod 4$. Thus c is even, contradicting (1a).

Therefore $\mu = 1$, which gives:

(2) $x = a^2 + 6bc$ ($\Rightarrow (a,6) = 1$) ,

(3) $0 = 3c^2 + 2ab$ ($\Rightarrow c$ is even),

(4) (*) $\mp 2^k = b^2 + 2ac$ ($\Rightarrow b$ is even) .

Thus from (3), $3|b$. If $c = 0$ then by (3), $b = 0$, which contradicts (4). Hence $c = 2^r C$, where $r \geq 1$ and $(C,2) = 1$. Therefore by (3),

$b = 3\cdot 2^{2r-1}B$, B odd, which implies that $B > 0$. From (3) and (4),

$0 = C^2 + aB$ and

(*) $\qquad \mp 2^k = 9\cdot 2^{4r-2}B^2 + 2^{r+1}aC$.

As in Proposition 2.2, $B = 1$. Thus $a = -C^2$ and

(*) $\qquad \mp 2^k = 9\cdot 2^{4r-2} - 2^{r+1}C^3$.

If $r > 1$, then $4r - 2 > r + 1$ and therefore $k = r + 1$. Hence

(*) $\qquad 1 = 9(\mp 2^{r-1})^3 + (\pm C)^3$.

It follows from [15, p.112, Delone-Nagell Theorem] that $1 = 9u^3 + v^3$, $u \neq 0$ has exactly one solution, $(u,v) = (1,-2)$. Hence $C = \pm 2$. But C is odd. Therefore $r = 1$, implying that $4r - 2 = r + 1 = 2 < k$.

Hence

(*) $\qquad 3^2 \pm 2^{k-2} = C^3$.

By Theorem 2, there are no solutions in the positive case, and in the negative case, $k = 5$ and $C = 1$. Thus $c = 2$, $b = 6$ and $a = -1$, which implies, by (2), that $x = 73$. Therefore $y = 827$.

<u>Proposition 5.2</u>: $y^2 + 3^2\cdot 2^{3k+1} = x^3$, $(x,6) = 1$ has no solutions.

$y^2 - 3^2\cdot 2^{3k+1} = x^3$, $(x,6) = 1 \Rightarrow (k,x,y) = (0,7,19)$.

<u>Proof</u>: R. Finkelstein and H. London have recently shown in [10] that the only solution of $y^2 + 18 = x^3$ is $x = y = 3$. [A] gives the solution of $y^2 - 18 = x^3$.

We may therefore assume that $k > 0$. Let $\Lambda = \Omega_{12}$ and $\theta = (12)^{\frac{1}{3}}$.

Now , $(y,6) = 1$ and

(*) $\quad y^2 = \alpha\beta$, where $\alpha = x \mp 2^{k-1}\theta^2$ and $\beta = x^2 \pm 2^{k-1}\theta^2 x + (2^{k-1}\theta^2)^2$.

As above, $\alpha > 0$ and $(\alpha,\beta)_\Lambda \mid_\Lambda 3\cdot(2^{k-1}\theta^2)^2$. Thus $(\alpha,\beta)_\Lambda = 1$, which implies that $\alpha = \mu(a + b\theta + c\theta^2/2)^2$, where $\mu = 1$ or $1 + 3\theta - 3\theta^2/2$ and $b \geq 0$.

If $\mu = 1 + 3\theta - 3\theta^2/2$, then:

$$x = (a^2 + 12bc) + 18(2b^2 + 2ac) - 18(3c^2 + 2ab) \quad (\Rightarrow a \text{ is odd}),$$

$$(*)\ \mp 2^k = -3(a^2 + 12bc) + 6(3c^2 + 2ab) + (2b^2 + 2ac) \quad (\Rightarrow a \text{ is even}).$$

Hence $\mu = 1$, implying that

(5) $$x = a^2 + 12bc \qquad (\Rightarrow (a,6) = 1),$$

(6) $$0 = 3c^2 + 2ab \qquad (\Rightarrow c \text{ is even}),$$

(7)(*) $$\mp 2^{k-1} = b^2 + ac \quad .$$

By (6), $3|b$. If $c = 0$, then by (6), $b = 0$, contradicting (7). Hence $c = 2^r C$, $r \geq 1$, C odd and therefore by (6), $b = 3\cdot 2^{2r-1}B$, B odd. Therefore $B > 0$, $0 = C^2 + aB$ and

(*) $$\mp 2^{k-1} = 9\cdot 2^{4r-2}B^2 + 2^r aC \, .$$ As in Proposition 2.2, $B = 1$ and $a = -C^2$. Thus

(*) $$\mp 2^{k-1} = 9\cdot 2^{4r-2} - 2^r C^3 \, .$$ Since $4r - 2 > r$, $k - 1 = r$. Hence

(7a)(*) $$\mp 1 = 9\cdot 2^{3r-2} - C^3 \, .$$

If $r = 2t$, then by (7a),

(*) $$(3\cdot 2^{3t-2})^2 \pm 1 = C^3 \, .$$

By [A], the positive case cannot hold, and in the negative case $C = 2$. But C is odd. Thus $r = 2t + 1$, implying by (7a),

(*) $$(3\cdot 2^{3t+2})^2 \pm 8 = (2C)^3 \, .$$

Again by [A], the positive case cannot hold. In the negative case we obtain $3 \cdot 2^{3t+2} = 3$ or 312, which cannot hold.

Lemma 5.3: $y^2 + 3^2 \cdot 2^5 = x^3$, $(x,6) = 1$ has no solutions.

Proof: $(y,6) = 1$. Let $\Lambda = \Gamma_{-2}$ and $\theta = (-2)^{\frac{1}{2}}$. Then $x^3 = \alpha\bar{\alpha}$, where $\alpha = y + 12\theta$. Now, $(\alpha,\bar{\alpha})_\Lambda |_\Lambda \alpha - \bar{\alpha} = 24\theta$ and, since $(x,6) = 1$, $(\alpha,\bar{\alpha})_\Lambda = 1$. Hence by Lemma 1.3, $\alpha = (a + b\theta)^3$, implying that

(7a) $\quad 12 = 3a^2b - 2b^3 \quad (\Rightarrow 3|b)$.

But, since $3|b$, (7a) implies that $9|12$.

We now solve $y^2 - 3^2 \cdot 2^5 = x^3$, $(x,6) = 1$. Let $F(x) = x^3 + 3x + 2 \in Q[x]$. $F(x)$ is irreducible over Q and since $F'(x) = 3x^2 + 3 > 0$, F has one real root θ. Let $\Pi = Z[\theta] = \{a + b\theta + c\theta^2 | a,b,c \in Z\}$. Since Π is a real ring, it contains only two roots of unity, ± 1. Also, Π is an order [2, p.88] of the algebraic number field $Q(\theta)$ and therefore, by Dirichlet's Unit Theorem [2, p.112], the positive units of Π form an infinite cyclic group. Let ϵ_0 be the fundamental unit of Π, $0 < \epsilon_0 < 1$.

By [5, p.114], Π is the ring of integers of $Q(\theta)$ (this is easily checked directly by the usual methods) and, since $1, \theta, \theta^2$ is an integral basis for Π, the discriminant of $Q(\theta)$ $(= \Delta)$ is -216 (see [2, p.404] and [33, p.83]).

Artin's inequality [1, p.70] is

(8) $\quad |\Delta| < 4\epsilon_0^{-3} + 24$.

Hence, by (8), $\epsilon_0^{-3} > 45$ and therefore $\epsilon_0^{-1} > 4$.

Now, $F(-1) < 0$ and $F(0) > 0$. Hence $-1 < \theta < 0$. Also, $(1 + \theta - \theta^2)(17 - 3\theta + 5\theta^2) = 1$. Thus $\epsilon = 1 + \theta - \theta^2$ is a unit of Π and $\epsilon^{-1} = 17 - 3\theta + 5\theta^2$. It follows that $17 < \epsilon^{-1} < 25$ and therefore $0 < \epsilon < 1$. Hence $\epsilon = \epsilon_0^s$, where $s \geq 1$. Therefore $25 > \epsilon^{-1} = \epsilon_0^{-s} > 4^s$ and thus $s \leq 2$.

Suppose $s = 2$. Hence $\epsilon = \epsilon_0^2$ and since $\epsilon_0 = a + b\theta + c\theta^2$, we obtain, by comparing the coefficients of θ, $1 = 2ab - 2c^2 - 6bc$. This obvious contradiction proves that $s = 1$ and hence $\epsilon = \epsilon_0$ is the fundamental unit of Π.

Let $\epsilon^t = a_t + b_t\theta + c_t\theta^2 \in \Pi$ for all $t \in Z$.

Lemma 5.4: If $n \geq 1$, then

$$(9) \qquad \epsilon^{3^n} = (1 + 3^n x_n) + 3^n y_n \theta + 3^n z_n \theta^2 ,$$

with $y_n \equiv -1$, $z_n \equiv 1 \mod 3$.

Proof: $\epsilon^3 = 25 + 33\theta - 15\theta^2$. Hence Lemma 5.4 holds for $n = 1$, with $x_1 = 8$, $y_1 = 11$ and $z_1 = -5$.

Assume the lemma for some $n \geq 1$. Cubing (9) and using $3n \geq n + 2$ and $2n + 1 \geq n + 2$, $a_{3^{n+1}} \equiv 1 + 3^{n+1}x_n$, $b_{3^{n+1}} \equiv 3^{n+1}y_n$, $c_{3^{n+1}} \equiv 3^{n+1}z_n \mod 3^{n+2}$. Therefore $y_{n+1} \equiv y_n$, $z_{n+1} \equiv z_n \mod 3$.

Corollary 5.5: If $n \geq 1$, $b_{2(3^n)} \equiv 2y_n 3^n$, $c_{2(3^n)} \equiv 2z_n 3^n \mod 3^{n+1}$.

Proof: Square (9) and use $2n \geq n + 1$.

Corollary 5.6: If $n \geq 1$, $b_{t+k3^n} \equiv b_t$, $c_{t+k3^n} \equiv c_t \mod 3^n$.

Proof: $\varepsilon^{t+3^n} = \varepsilon^t\varepsilon^{3^n} = (a_t + b_t\theta + c_t\theta^2)(1 + 3^n x_n + 3^n y_n\theta + 3^n z_n\theta^2)$.

Hence the corollary holds for $k = 1$. By induction, it holds for all $k \geq 0$. If $k < 0$, then $b_t = b_{(t+k3^n)+(-k)3^n} \equiv b_{t+k3^n} \bmod 3^n$.

Corollary 5.7: If $n \geq 1$ and $b_t + 2c_t \equiv 0 \bmod 3^n$, then $t \equiv 0 \bmod 3^n$.

Proof: Suppose the corollary is false. Choose the minimal $n \geq 1$ such that for some t , $b_t + 2c_t \equiv 0 \bmod 3^n$ and $t \not\equiv 0 \bmod 3^n$. There is a k such that $0 < t - k3^n < 3^n$. Now, $t - k3^n \not\equiv 0 \bmod 3^n$ and by Corollary 5.6, $b_{t-k3^n} + 2c_{t-k3^n} \equiv b_t + 2c_t \equiv 0 \bmod 3^n$. Hence we may assume $0 < t < 3^n$.

If $n = 1$, $t = 1$ or 2 . But $b_1 + 2c_1 = -1 \not\equiv 0 \bmod 3$ and $b_2 + 2c_2 = -2 \not\equiv 0 \bmod 3$.

Therefore $n > 1$ and thus $n - 1 \geq 1$. Obviously, $b_t + 2c_t \equiv 0 \bmod 3^{n-1}$. Hence by the minimality of n , $t \equiv 0 \bmod 3^{n-1}$. Therefore $t = 3^{n-1}$ or $2\cdot3^{n-1}$. But, by Lemma 5.4, $b_{3^{n-1}} + 2c_{3^{n-1}} = (y_{n-1} + 2z_{n-1})3^{n-1} \not\equiv 0 \bmod 3^n$ (since $y_{n-1} + 2z_{n-1} \equiv 1 \bmod 3$) and, by Corollary 5.5, $b_{2\cdot3^{n-1}} + 2c_{2\cdot3^{n-1}} \equiv 2(y_{n-1} + 2z_{n-1})3^{n-1} \not\equiv 0 \bmod 3^n$.

Corollary 5.8: $12 = a^3 + 3ab^2 - 2b^3 \Rightarrow a = 2, \quad b = 1$.

Proof: Suppose

(10) $\quad 12 = a^3 + 3ab^2 - 2b^3$. Hence $a(a^2 + 3b^2)$ is even.

If a is odd, then $a^2 + 3b^2$ is even and therefore b is odd. Hence $a^2 + 3b^2 \equiv 0 \bmod 4$ and by (10), $4|-2b^3$, implying that b is even.

Thus a is even. Let $a = 2A$, which implies, by (10),

(11) $6 = 4A^3 + 3Ab^2 - b^3$.

If b is even, then $4|6$. Thus b is odd. If A is even, then by (11), b is even and hence A is odd. Also by (11), $A - b \equiv A^3 - b^3 = 6 - 3A^3 - 3Ab^2 \equiv 0 \bmod 3$. Therefore $6|A - b$.

Now ,

(12) $(2 + \theta)\mu = a + b\theta$, where

(12a) $\mu = (A - b)/6 + A - (A - b)/3 \cdot\theta + (A - b)/6 \cdot \theta^2$.

By [33, p.132, (13)], the norm (over Q) of elements of Π is

$$N(a + b\theta + c\theta^2) = a^3 - 2b^3 + 4c^3 + 6abc - 6a^2c + 9ac^2 + 3ab^2 - 6bc^2.$$

Hence $N(2 + \theta) = 12$, $N(a + b\theta) = a^3 + 3ab^2 - 2b^3 = 12$ and therefore by (12), $N(\mu) = 1$. By [2, p.89, Theorem 4], μ is a unit of Π and therefore $\mu = \pm \epsilon^t$. Comparing this with (12a), $b_t + 2c_t = 0$. By Corollary 5.8, $t \equiv 0 \bmod 3^n$ for all $n \geq 1$. Hence $t = 0$, which implies that $\mu = \pm 1$. Since $N(\mu) = 1$, $\mu = 1$ and by (12), $a = 2$ and $b = 1$.

<u>Corollary 5.9</u>: $y^2 - 3^2 2^5 = x^3$, $(x,6) = 1 \Rightarrow x = 1$, $y = 17$.

<u>Proof</u>: $(y,6) = 1$. Let $\Lambda = \Gamma_2$ and $\theta = 2^{\frac{1}{2}}$. $\alpha\bar{\alpha} = x^3$, where $\alpha = y + 12\theta$. Now, $(\alpha,\bar{\alpha})_\Lambda |_\Lambda \alpha - \bar{\alpha} = 24\theta$. Thus $(\alpha,\bar{\alpha})_\Lambda = 1$. Hence by Lemma 1.3 and Table 1,

$$\alpha = \mu(a + b2^{\frac{1}{2}})^3, \text{ where } \mu = 1 \text{ or } \pm 1 + \theta \ .$$

If $\mu = 1$, then:

$$y = a^3 + 6ab^2 \qquad (\Rightarrow (a,6) = 1) ,$$

(12b) $12 = 3a^2b + 2b^3 = b(3a^2 + 2b^2) \quad (\Rightarrow 3|b)$.

Hence, since $3a^2 + 2b^2$ is odd, $4|b$, and thus by (12b), $b = \pm 12$.
By (12b), $3a^2 + 2b^2 = \pm 1$, which is impossible.

If $\mu = -1 + \theta$, then :

(12c) $$y = -(a^3 + 6ab^2) + 2(3a^2b + 2b^3),$$

$$12 = (a^3 + 6ab^2) - (3a^2b + 2b^3) = (a - b)^3 + 3(a-b)(-b)^2 - 2(-b)^3 .$$

Thus by Corollary 5.8, $a - b = 2$ and $-b = 1$. Hence $a = 1$, $b = -1$ and (12c) implies that $y = -17$.

Therefore $\mu = 1 + \theta$, yielding:

$$y = (a^3 + 6ab^2) + 2(3a^2b + 2b^3),$$

$$12 = (a^3 + 6ab^2) + (3a^2b + 2b^3) = (a+b)^3 + 3(a+b)b^2 - 2b^3 .$$

Hence, by Corollary 5.8, $a + b = 2$ and $b = 1$. Therefore $a = 1$ and thus $y = 17$. Hence $x = 1$.

<u>Proposition 5.10</u>: $y^2 + 3^2\, 2^{3k+2} = x^3$, $(x,6) = 1$ has no solutions.

$y^2 - 3^2\, 2^{3k+2} = x^3$, $(x,6) = 1 \Rightarrow (k,x,y) = (1,1,17)$.

<u>Proof</u>: [A] shows that there are no solutions for $k = 0$, with $(x,6) = 1$. Lemma 5.3 and Corollary 5.9 give the solution for $k = 1$.

Assume $k > 1$. Let $\Lambda = \Omega_6$ and $\theta = 6^{\frac{1}{3}}$. Now, $(y,6) = 1$ and

(*) $y^2 = \alpha\beta$, where $\alpha = x \mp 2^k\theta^2$ and $\beta = x^2 \pm 2^k\theta^2 x + (2^k\theta^2)^2$.

As above, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda\, 3\cdot(2^k\theta^2)^2$. Hence $(\alpha,\beta)_\Lambda = 1$ and $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $1 - 6\theta + 3\theta^2$ and $b \geq 0$.

If $\mu = 1 - 6\theta + 3\theta^2$, then:

$$x = (a^2 + 12bc) - 36(b^2 + 2ac) + 18(6c^2 + 2ab) \quad (\Rightarrow a \text{ is odd}),$$

(13) $$0 = (3c^2 + ab) - 3(a^2 + 12bc) + 9(b^2 + 2ac),$$

(14)(*) $\mp 2^k = 3(a^2 + 12bc) - 6(6c^2 + 2ab) + (b^2 + 2ac)$.

Hence, by (14), b is odd and therefore $b^2 + 3a^2 \equiv 4 \bmod 8$. Again by (14) (since $k \geq 2$), $4|2ac$ and hence c is even. It follows from (13) that $ab - 3a^2 + 9b^2$ is even, contradicting $(ab,2) = 1$.

Thus $\mu = 1$, which gives:

$x = a^2 + 12bc$ $(\Rightarrow (a,6) = 1)$,

(15) $0 = 3c^2 + ab$,

(16)(*) $\mp 2^k = b^2 + 2ac$ $(\Rightarrow b$ is even$)$.

By (15), c is even and $3|b$. If $c = 0$, then by (15), $b = 0$, which contradicts (16). Hence $c = 2^r C$, $r \geq 1$, C odd, implying by (15), that $b = 3\cdot 2^{2r}B$, B odd. Thus $B > 0$ and from (15) and (16), $0 = C^2 + aB$ and

(*) $\mp 2^k = 9\cdot 2^{4r}B^2 + 2^{r+1}aC$.

As in Proposition 2.2, $B = 1$ and $a = -C^2$. Therefore

(*) $\mp 2^k = 9\cdot 2^{4r} - 2^{r+1}C^3$.

Since $4r > r + 1$, $k = r + 1$ and

(17)(*) $9\cdot 2^{3r-1} \pm 1 = C^3$.

If $r = 2t$, then by (17),

(*) $(3\cdot 2^{3t+1})^2 \pm 8 = (2C)^3$, which has no solutions by [A].

If $r = 2t + 1$, then

(*) $(3\cdot 2^{3t+1})^2 \pm 1 = C^3$, and this has no solutions by [A], since C is odd.

Propositions 5.1, 5.2 and 5.10 prove

<u>Theorem 5</u>: $y^2 + 3^2 2^n = x^3$, $(x,6) = 1$ has no solutions.

$y^2 - 3^2 2^n = x^3$, $(x,6) = 1 \Leftrightarrow (n,x,y) = (1,7,9),(5,1,17)$ or $(15,73,827)$.

SECTION 6.

$2y^2 + 1 = 3^m; \ y^2 + 2^n = 3^m$

Remark: Corresponding to the cases $m = 3t, 3t+1$ or $3t+2$, $2y^2 + 1 = 3^m$ can be transformed into: $(4y)^2 + 8 = (2\cdot 3^t)^3$, $(12y)^2 + 72 = (2\cdot 3^{t+1})^3$ or $(36y)^2 + 648 = (2\cdot 3^{t+2})^3$, respectively. These are special cases of $Y^2 + 8 = X^3$, $Y^2 + 72 = X^3$ and $Y^2 + 648 = X^3$. The solution of the first equation is listed in [A] and the latter two equations are solved in [4].

The difficult case of $y^2 + 2^n = 3^m$ is $y^2 + 2 = 3^m$ (see Theorem 6b). As above, this equation can be divided into special cases of $Y^2 + 2 = X^3$, $Y^2 + 18 = X^3$ and $Y^2 + 162 = X^3$. The solution of the first equation is given in [A], the second is solved in [10] and the third is solved in [12].

Since, for reasons discussed in the introduction, I wish to avoid the use of [4] and [10] in solving $y^2 - 2^n 3^m = x^3$, I shall solve $2y^2 + 1 = 3^m$ and $y^2 + 2 = 3^m$ directly.

Let $\Lambda = \Gamma_{-2}$, $\theta = (-2)^{\frac{1}{2}}$ and $\delta = 1 + \theta \in \Lambda$. Hence $\delta^t = a_t + b_t\theta$ for $t \geq 0$.

Lemma 6.1: If $n \geq 3$, then

(1) $\delta^{2^n} = (1 + 2^{n+1}x_n) + 2^n y_n \theta$, where $x_n \equiv 1$, $y_n \equiv -1 \bmod 4$.

Proof: $\delta^8 = 17 + 56\theta$. Thus $x_3 = 1$ and $y_3 = 7$.

Assume the lemma for some $n \geq 3$. Squaring (1) and using $n \geq 3$:

$$x_{n+1} = x_n + 2^n x_n^2 - 2^{n-1} y_n^2 \equiv x_n \bmod 4 ,$$

$$y_{n+1} = y_n(1 + 2^{n+1}x_n) \equiv y_n \bmod 4 .$$

Corollary 6.2: If $n \geq 3$ and $t \geq 0$, then:

$$b_{t+2^n} \equiv b_t + 2^n \bmod 2^{n+1} ,$$

$$a_{t+2^n} \equiv a_t + 2^{n+1} \bmod 2^{n+2} \quad \text{if } t \text{ is even,}$$

$$a_{t+2^n} \equiv a_t \bmod 2^{n+2} \quad \text{if } t \text{ is odd.}$$

Proof: Since $a_0 = 1$ and $b_0 = 0$, Corollary 6.2 holds for $t = 0$ by Lemma 6.1. Assume that it holds for some $t \geq 0$. Now,

$$a_{t+1} + b_{t+1}\theta = \delta^{t+1} = \delta^t.\delta = (a_t + b_t\theta)(1 + \theta) .$$

Hence $a_{t+1} = a_t - 2b_t$ and $b_{t+1} = a_t + b_t$.

If t is even, then by the induction hypothesis,

$$a_{t+1+2^n} + b_{t+1+2^n}\theta = \delta^{t+1+2^n} = \delta^{t+2^n}\delta = \left[(a_t+2^{n+1}+2^{n+2}s)+(b_t+2^n+2^{n+1}r)\theta\right](1+\theta).$$

Hence $a_{t+1+2^n} \equiv a_t - 2b_t = a_{t+1} \bmod 2^{n+2}$ and $b_{t+1+2^n} \equiv a_t+b_t+2^n = b_{t+1} + 2^n \bmod 2^{n+1}$.

If t is odd, then by the induction hypothesis,

$$\delta^{t+1+2^n} = \delta^{t+2^n}\delta = \left[(a_t+2^{n+2}s)+(b_t+2^n+2^{n+1}r)\theta\right](1+\theta) .$$

Thus $a_{t+1+2^n} \equiv a_t - 2b_t - 2^{n+1} \equiv a_{t+1} + 2^{n+1} \bmod 2^{n+2}$ and $b_{t+1+2^n} \equiv a_t + b_t + 2^n = b_{t+1} + 2^n \bmod 2^{n+1}$.

Corollary 6.3: If $n \geq 3$, t odd, $0 \leq t < 2^n$ and $a_t \equiv \pm 1 \bmod 2^{n+2}$, then $t = 1$ or 5 .

Proof: Suppose Corollary 6.3 is false. Choose the minimal $n \geq 3$ such that there is a $t \in Z$, with t odd, $0 \leq t < 2^n$, $a_t \equiv \pm 1 \bmod 2^{n+2}$ and

$t \neq 1$ or 5 .

If $n = 3$, then $t = 3$ or 7 . But $a_3 = -5 \not\equiv \pm 1 \bmod 2^5$ and $a_7 = 43 \not\equiv \pm 1 \bmod 2^5$. Thus $n > 3$ and therefore $n - 1 \geq 3$.

Obviously, $a_t \equiv \pm 1 \bmod 2^{n+1}$. If $0 \leq t < 2^{n-1}$, then by the minimality of n, $t = 1$ or 5 . Hence $2^{n-1} \leq t < 2^n$ and therefore $0 \leq t - 2^{n-1} < 2^{n-1}$. Also, $t - 2^{n-1}$ is odd. By Corollary 6.2 (for $n - 1$),

$$\pm 1 \equiv a_t = a_{(t-2^{n-1})+2^{n-1}} \equiv a_{t-2^{n-1}} \bmod 2^{n+1} .$$

Again using the minimality of n, $t - 2^{n-1} = 1$ or 5 .

If $t = 2^{n-1} + 1$, then by Lemma 6.1,

$$\delta^t = \delta\,\delta^{2^{n-1}} = (1 + \theta)(1 + 2^n x_{n-1} + 2^{n-1} y_{n-1}\theta) .$$

Therefore $1 + 2^n(x_{n-1} - y_{n-1}) = a_t \equiv \pm 1 \bmod 2^{n+2}$. Since $n > 3$, the minus sign cannot hold and thus $x_{n-1} - y_{n-1} \equiv 0 \bmod 4$. But, by Lemma 6.1, $x_{n-1} - y_{n-1} \equiv 2 \bmod 4$.

If $t = 2^{n-1} + 5$, then

$$\delta^t = \delta^5\,\delta^{2^{n-1}} = (1 - 11\theta)(1 + 2^n x_{n-1} + 2^{n-1} y_n \theta) .$$

Thus $1 + 2^n(x_{n-1} + 11y_{n-1}) = a_t \equiv \pm 1 \bmod 2^{n+2}$. It follows, as above, that $x_{n-1} + 11y_{n-1} \equiv 0 \bmod 4$. But $x_{n-1} + 11y_{n-1} \equiv 1 - 11 \equiv 2 \bmod 4$.

Corollary 6.4: If $n \geq 3$, t even, $0 \leq t < 2^n$ and $a_t \equiv \pm 1 \bmod 2^{n+1}$, then $t = 0$ or 2 .

Proof: Suppose Corollary 6.4 is false. Choose the minimal $n \geq 3$ such that there is a $t \in Z$, with t even, $0 \leq t < 2^n$, $a_t \equiv \pm 1 \bmod 2^{n+1}$ and $t \neq 0$ or 2 .

If $n = 3$, then $t = 4$ or 6. But $a_4 = -7$ and $a_6 = 23$, neither of which is congruent to ± 1 modulo 2^4. Therefore $n > 3$ and hence $n - 1 \geq 3$.

Obviously, $a_t \equiv \pm 1 \bmod 2^n$. If $0 \leq t < 2^{n-1}$, then by the minimality of n, $t = 0$ or 2. Therefore $2^{n-1} \leq t < 2^n$ and thus $0 \leq t - 2^{n-1} < 2^{n-1}$. Also, $t - 2^{n-1}$ is even. By Corollary 6.2, $a_t = a_{(t-2^{n-1})+2^{n-1}} \equiv a_{t-2^{n-1}} + 2^n \bmod 2^{n+1}$. Hence $a_{t-2^{n-1}} \equiv a_t \equiv$

$\pm 1 \bmod 2^n$. By the minimality of n, $t - 2^{n-1} = 0$ or 2.

If $t = 2^{n-1}$, then by Lemma 6.1, $a_t = 1 + 2^n x_{n-1} \not\equiv \pm 1 \bmod 2^{n+1}$ (since $n > 3$ and x_{n-1} is odd).

If $t = 2^{n-1} + 2$, then

$\delta^t = \delta^2 \, \delta^{2^{n-1}} = (-1 + 2\theta)(1 + 2^n x_{n-1} + 2^{n-1} y_{n-1}\theta)$.

Thus as above,

$$a_t = -1 - 2^n x_{n-1} - 2^{n+1} y_{n-1} \not\equiv \pm 1 \bmod 2^{n+1}.$$

<u>Theorem 6a</u>: $2y^2 + 1 = 3^m \Rightarrow m = 0, 1, 2$ or 5.

<u>Proof</u>: $m \geq 0$, since $2y^2 + 1 \in Z$.

By (5) of Section 1, $N(\delta) = N(\bar{\delta}) = 3$, and therefore δ and $\bar{\delta}$ are primes in Λ.

(2) $\qquad \alpha\bar{\alpha} = 3^m = \delta^m \bar{\delta}^m$, where $\alpha = 1 + y\theta$.

Also, $(\alpha,\bar{\alpha})_\Lambda |_\Lambda \alpha + \bar{\alpha} = 2$ and $(\alpha,\bar{\alpha})_\Lambda |_\Lambda \alpha\bar{\alpha} = 1 + 2y^2$. Since $(1 + 2y^2, 2) = 1$, $(\alpha,\bar{\alpha})_\Lambda = 1$. Therefore, by (2), if $\delta |_\Lambda \alpha$, then $\delta^m |_\Lambda \alpha$ and if $\bar{\delta} |_\Lambda \alpha$, then $\bar{\delta}^m |_\Lambda \alpha$. Hence, since the only units of Λ are ± 1, $\alpha = \pm 1, \pm \delta^m, \pm \bar{\delta}^m$ or $\pm \delta^m \bar{\delta}^m (= \pm 3^m)$.

If $\alpha = \pm 1$ or $\pm 3^m$, then by (2), $y = 0$ and thus $m = 0$.

If $\alpha = \pm \delta^m$ or $\pm \bar{\delta}^m$, then $\delta^m = \pm \alpha$ or $\pm \bar{\alpha}$, implying by (2), that $a_m = \pm 1$. Now, there is an $n \geq 3$ such that $0 \leq m < 2^n$ (e.g., $n = m + 3$). Thus by Corollaries 6.3 and 6.4, $m = 0,1,2$ or 5.

We now solve $y^2 + 2^n = 3^m$.

Corollary 6.5 (to Lemma 6.1): If $b_t \equiv \pm 1 \bmod 2^n$ for some $n \geq 3$ and $0 \leq t < 2^n$, then $t = 1$ or 3.

Proof: Suppose Corollary 6.5 is false. Choose the minimal $n \geq 3$ such that there is a t, with $b_t \equiv \pm 1 \bmod 2^n$, $0 \leq t < 2^n$ and $t \neq 1$ or 3.

If $n = 3$, then $t = 0,2,4,5,6$ or 7. But $b_0 = 0$, $b_2 = 2$, $b_4 = 4$, $b_5 = -11$, $b_6 = -10$ and $b_7 = 13$. Thus $n > 3$ and therefore $n-1 \geq 3$.

Obviously, $b_t \equiv \pm 1 \bmod 2^{n-1}$. If $0 \leq t < 2^{n-1}$, then by the minimality of n, $t = 1$ or 3. Hence $2^{n-1} \leq t < 2^n$, and thus $0 \leq t - 2^{n-1} < 2^{n-1}$. Now by Corollary 6.2,

$$b_t = b_{(t-2^{n-1})+2^{n-1}} \equiv b_{t-2^{n-1}} + 2^{n-1} \bmod 2^n .$$

Hence $b_{t-2^{n-1}} \equiv b_t \equiv \pm 1 \bmod 2^{n-1}$. Therefore by the minimality of n, $t - 2^{n-1} = 1$ or 3.

If $t = 1 + 2^{n-1}$, then

$$\delta^t = \delta \cdot \delta^{2^{n-1}} = (1 + \theta)(1 + 2^n x_{n-1} + 2^{n-1} y_{n-1} \theta) .$$

Thus $\pm 1 \equiv b_t \equiv 1 + 2^{n-1} y_{n-1} \bmod 2^n$. Since $n > 3$, the minus cannot hold. Hence $2^{n-1} y_{n-1} \equiv 0 \bmod 2^n$. But by Lemma 6.1, y_{n-1} is odd.

If $t = 3 + 2^{n-1}$, then

$$\delta^t = \delta^3 \delta^{2^{n-1}} = (-5 + \theta)(1 + 2^n x_{n-1} + 2^{n-1} y_{n-1} \theta) .$$

Therefore $\pm 1 \equiv b_t \equiv 1 - 5y_{n-1}2^{n-1} \bmod 2^n$, and we have the same contradiction as above.

Proposition 6.6: $y^2 + 2 = 3^m \Rightarrow m = 1$ or 3 .

Proof: Since $y^2 + 2 \in Z$, $m \geq 0$. Now,

(3) $\beta\bar{\beta} = 3^m = \delta^m \bar{\delta}^m$, where $\beta = y + \theta$.

Also, $(\beta,\bar{\beta})_\Lambda |_\Lambda (\beta-\bar{\beta})^2 = -8$ and $(\beta,\bar{\beta})_\Lambda |_\Lambda \beta\bar{\beta} = 3^m$. Therefore, since $(3^m,8) = 1$, $(\beta,\bar{\beta})_\Lambda = 1$. As in Theorem 6a, $\beta = \pm 1, \pm \delta^m, \pm \bar{\delta}^m$ or $\pm 3^m$, But by (3), $\beta \neq \pm 1$ or $\pm 3^m$. Thus, as in Theorem 6a, $\delta^m = \pm \beta$ or $\pm \bar{\beta}$. Therefore $b_m = \pm 1$. There is an $n \geq 3$ such that $0 \leq m < 2^n$ and thus, by Corollary 6.5, $m = 1$ or 3 .

Theorem 6b: $y^2 + 2^n = 3^m \Rightarrow (n,m,|y|) = (0,0,0),(1,1,1),(1,3,5),(3,2,1)$ or $(5,4,7)$.

Proof: If $m < 0$, then $y^2 < 1$ and thus $y = 0$. Hence $2^n = 3^m$, which implies that $m = 0$. Hence $m \geq 0$ and therefore, since $2^n \in Z$, $n \geq 0$.

If $m = 0$, then $2^n \leq 1$. Thus $n = 0$ and hence $y = 0$.

Assume $m \geq 1$. Therefore $(y,3) = 1$. If $n = 0$, then $3^m = y^2 + 1 \equiv 2 \bmod 3$, which is impossible. If $n = 1$, then Proposition 6.6 gives: $m = 1$ or 3, and hence $|y| = 1$ or 5, respectively.

Assume $n \geq 2$ (and $m \geq 1$) . Thus $(y,2) = 1$ and therefore $y^2 \equiv 1 \bmod 4$. Also, $2^n \equiv 0 \bmod 4$. Hence $3^m \equiv 1 \bmod 4$, implying that $m = 2s$. Now, $(|y| + 3^s)(|y| - 3^s) = y^2 - 3^m = -2^n$. Thus

(4) $|y| + 3^s = 2^r$ and $|y| - 3^s = -2^{n-r}$.

Since y is odd, $r \geq 1$ and $n - r \geq 1$. Therefore $|y| = 2^{r-1} - 2^{n-r-1}$.

Since $|y| > 0$, $n - r - 1 = 0$, by Lemma 1.7. Hence by (4),

(5) $$3^s = 2^{r-1} + 1 .$$

By (5), $r = 1$ is impossible and $r = 2$ yields: $s = 1$, $n = r + 1 = 3$, $m = 2s = 2$, $y^2 = 3^m - 2^n = 1$.

Assume $r > 2$. Thus $2^{r-1} \equiv 0 \bmod 4$ and hence by (5), $3^s \equiv 1 \bmod 4$. Therefore $s = 2t$. By (5), $(3^t+1)(3^t-1) = 2^{r-1}$. Hence $3^t + 1$ and $3^t - 1$ are both powers of 2 and their difference is 2 . Obviously, $3^t + 1 = 4$ and therefore $t = 1$. Hence $s = 2$ and by (5), $r = 4$. Thus $m = 2s = 4$ and $n = r + 1 = 5$. Finally, $y^2 = 3^m - 2^n = 49$.

SECTION 7.

$y^2 - 2^n 3^m = x^3, \ (x,6) = 1$

Proposition 7.1: $y^2 - 2^{2k+1} 3^{2v+1} = x^3, \ (x,6) = 1 \Rightarrow (k,v,x,y) =$ $(0,2,-5,19), (0,4,19,215), (1,0,1,5), (1,3,-13,73), (2,5,-47,2359)$ or $(4,0,25,131)$.

Proof: The solutions for $v = 0$ are given by Theorem 4.

Assume $v > 0$ and let $\Lambda = \Gamma_6$, $\theta = 6^{\frac{1}{2}}$. Now, $(y,6) = 1$ and $\alpha\bar{\alpha} = x^3$, where $\alpha = y + 2^k 3^v \theta$. $(\alpha,\bar{\alpha})_\Lambda |_\Lambda \ \alpha - \bar{\alpha} = 2^{k+1} 3^v \theta$ and since $(x,6) = 1$, $(\alpha,\bar{\alpha})_\Lambda = 1$. Hence, by Lemma 1.3, $\alpha = \mu(a+b\theta)^3$, where $\mu = 1$ or $5 \pm 2\theta$.

If $\mu = 5 \pm 2\theta$:

$$y = 5(a^3 + 18ab^2) \pm 12(3a^2 b + 6b^3) \qquad (\Rightarrow (a,3) = 1),$$

$$2^k 3^v = 5(3a^2 b + 6b^3) \pm 2(a^3 + 18ab^2) \qquad (\Rightarrow 3|a, \text{ since } v \geq 1).$$

Therefore $\mu = 1$, which gives:

(1) $\quad y = a^3 + 18ab^2 \quad (\Rightarrow (a,6) = 1$ and, since $y \geq 0$, $a \geq 0)$,

(2) $\quad 2^k 3^{v-1} = b(a^2 + 2b^2)$.

Since $a^2 + 2b^2$ is positive and odd, (2) implies that $a^2 + 2b^2 = 3^r$ and thus $b = 2^k 3^{v-1-r}$. Hence

(2a) $\quad a^2 + 2^{2k+1} 3^{2(v-1-r)} = 3^r$.

If $v-1-r > 0$, then since $(a,3) = 1$, $r = 0$. By (2a), $a^2 + 2^{2k+1} 3^{2(v-1)} = 1$, which cannot hold.

Therefore $v-1-r = 0$ and by (2a), $a^2 + 2^{2k+1} = 3^{v-1}$. Since $a \geq 0$, it follows from Theorem 6b that $(k,v,a) = (0,2,1), (0,4,5), (1,3,1)$ or $(2,5,7)$.

Using $b = 2^k$ and (1), we obtain the given solutions with $v > 0$.

<u>Proposition 7.2</u>: $y^2 - 2^{2k}3^{2v+1} = x^3$, $(x,6) = 1 \Rightarrow (k,v,x,y) = (0,0,1,2)$, $(1,0,13,47), (2,0,1,7)$ or $(6,0,-23,11)$.

<u>Proof</u>: If $v = 0$, we obtain the solution from Theorem 4.

Assume $v > 0$. Let $\Lambda = \Gamma_3$ and $\theta = 3^{\frac{1}{2}}$. Now, $(y,3) = 1$ and $\alpha\bar{\alpha} = x^3$, where $\alpha = y + 2^k3^v\theta$. As in Proposition 7.1, $(\alpha,\bar{\alpha})_\Lambda = 1$ and therefore $\alpha = \mu(a + b\theta)^3$, where $\mu = 1$ or $2 \pm \theta$.

If $\mu = 2 \pm \theta$, then:

$$y = 2(a^3 + 9ab^2) \pm 3(3a^2b + 3b^3) \qquad (\Rightarrow (a,3) = 1),$$

$$2^k3^v = \pm(a^3 + 9ab^2) + 2(3a^2b + 3b^3) \qquad (\Rightarrow 3|a).$$

Therefore $\mu = 1$, which implies that:

(3) $$y = a^3 + 9ab^2 \qquad (\Rightarrow (a,3) = 1),$$

(4) $$2^k3^{v-1} = b(a^2 + b^2).$$

If $k = 0$, then by (4), $b = 3^r = 2^k3^r$.

If $k > 0$, then since $y^2 = x^3 + 2^{2k}3^{2v+1}$ and x is odd, y is odd. Rewriting (3) as $y = a(a^2 + b^2) + 8ab^2$, we see that $a^2 + b^2$ is odd, and hence by (4), $b = 2^k3^r$ in either case. Thus (4) implies that

(4a) $$3^{v-1-r} = a^2 + 2^{2k}3^{2r}.$$

If $v-1-r = 0$, then (4a) implies that $a = 0$, contradicting $(a,3) = 1$. Hence by Lemma 1.7, $r = 0$, and we obtain $3^{v-1} = a^2 + 2^{2k}$. But $a^2 + 2^{2k} \equiv 2 \bmod 3$ and $3^{v-1} \equiv 1$ or $0 \bmod 3$.

<u>Proposition 7.3</u>: $y^2 - 2^{2k}3^{2v} = x^3$, $(x,6) = 1 \Rightarrow (k,v,x,y) = (0,0,-1,0)$.

<u>Proof</u>: Theorem 2 solves the case $v = 0$.

Assume $v > 0$. Now, $ab = x^3$, where $a = y + 2^k3^v$ and $b = y - 2^k3^v$. $(a,b) | a-b = 2^{k+1}3^v$ and therefore $(a,b) = 1$. Thus $a = s^3$ and $b = t^3$, implying that $st = x$ and therefore $(st,6) = 1$.

$2^{k+1}3^v = a - b = s^3 - t^3 = (s - t)(s^2 + st + t^2)$. Since $s^2 + st + t^2$ is positive and odd, $s - t = 2^{k+1}3^r$ and $s^2 + st + t^2 = 3^{v-r}$. Eliminating s from these two equations,we obtain

$$(5) \qquad 3^{v-r} = 3t^2 + 2^{k+1}3^{r+1}t + 2^{2k+2}3^{2r} .$$

If $r = 0$, then since $v > 0$, $3|2^{2k+2}$. Thus $r \geq 1$, which implies by (5) and Lemma 1.7, that $v - r = 1$. (5) now gives

$$t^2 + 2^{k+1}3^rt + (2^{2k+2}3^{2r-1} - 1) = 0 .$$

Therefore by Lemma 1.8,

$$\text{Disc}(t) = 4 - 2^{2k+2}\, 3^{2r-1} = d^2 ,$$

which is impossible,since $k \geq 0$ and $r \geq 1$.

<u>Proposition 7.4</u>: $y^2 - 2^{2k+1}3^{2v} = x^3$, $(x,6) = 1 \Rightarrow (k,v,x,y) = (0,0,-1,1)$, $(0,1,7,19),(1,0,1,3),(2,1,1,17),(3,0,17,71),(4,0,-7,13)$ or $(7,1,73,827)$.

<u>Proof</u>: The solutions for $v = 0$ and $v = 1$ are obtained from Theorems 2 and 5.

Assume $v > 1$. Hence $(y,6) = 1$. Let $\Lambda = \Gamma_2$ and $\theta = 2^{\frac{1}{2}}$. Now, $\alpha\bar{\alpha} = x^3$, where $\alpha = y + 2^k3^v\theta$. As in Proposition 7.1, $(\alpha,\bar{\alpha}) = 1$, and hence $\alpha = \mu(a + b\theta)^3$, where $\mu = 1$ or $\pm 1 + \theta$.

If $\mu = 1$, we obtain:

$$y = a^3 + 6ab^2 \qquad (\Rightarrow (a,6) = 1),$$

$$2^k3^v = 3a^2b + 2b^3 = b(3a^2 + 2b^2) \qquad (\Rightarrow 3|b, \text{ since } v > 1) .$$

Since a is odd, $b = 2^k 3^r$, $r \geq 1$ and therefore $3^{v-r} = 3a^2 + 2^{2k+1}3^{2r}$. But since $2r > 1$, Lemma 1.7 yields $v-r = 1$. Hence $1 = a^2 + 2^{2k+1}3^{2r-1}$, which cannot hold.

Therefore $\mu = \pm 1 + \theta$, yielding:

(6) $$y = \pm (a^3 + 6ab^2) + 2(3a^2b + 2b^3),$$

(7) $$2^k 3^v = (a^3 + 6ab^2) \pm (3a^2b + 2b^3).$$

If $3|b$, then by (7), $3|a$ and hence by (6), $3|y$, which is false. Hence $(b,3) = 1$, implying by (7), that $(a,3) = 1$.

Let $B = \pm b$; (7) becomes

(8) $$2^k 3^v = a^3 + 6aB^2 + 3a^2B + 2B^3.$$

Hence $3|a^3 + 2B^3$ and therefore $a - B \equiv a^3 - B^3 \equiv 0 \bmod 3$. By Lemma 1.9, $a^3 \equiv B^3 \bmod 9$. Now, since $a^3 \equiv \pm 1 \bmod 9$, $a^3 + 2B^3 \equiv \pm 3 \bmod 9$. Also, $6aB^2 + 3a^2B = 3aB(2B + a) \equiv 0 \bmod 9$. Therefore by (8), $2^k 3^v \equiv \pm 3 \bmod 9$. But this contradicts $v \geq 2$.

Propositions 7.1, 7.2, 7.3 and 7.4 prove

Theorem 7: $y^2 - 2^n 3^m = x^3$, $(x,6) = 1 \Leftrightarrow (n,m,x,y) = (0,0,-1,0), (0,1,1,2), (1,0,-1,1), (1,2,7,19), (1,5,-5,19), (1,9,19,215), (2,1,13,47), (3,0,1,3), (3,1,1,5), (3,7,-13,73), (4,1,1,7), (5,2,1,17), (5,11,-47,2359), (7,0,17,71), (9,0,-7,13), (9,1,25,131), (12,1,-23,11)$ or $(15,2,73,827)$.

SECTION 8.

$y^2 - 3^m = 2^j x^3, \ (x,3) = 1$

Proposition 8.1: $y^2 - 3^{2v} = x^3, \ (x,3) = 1 \Rightarrow (v,x,y) = (0,-1,0),(0,2,3),$ $(1,-2,1)$ or $(1,40,253)$.

Proof: [A] lists the solutions for $v = 0$ and $v = 1$, so that we may assume $v \geq 2$.

If x is odd, then $(x,6) = 1$, and by Proposition 7.3, $v = 0$, which contradicts $v \geq 2$. Hence x is even and therefore y is odd. Let $Y = \pm y$, such that $Y \equiv 3^v \bmod 4$.

(1) $ab = (x/2)^3$, where $a = (Y + 3^v)/2$ and $b = (Y - 3^v)/4$.

Now, $(a,b) | a - 2b = 3^v$. But $(x,3) = 1$ and therefore $(a,b) = 1$. Thus $a = s^3$ and $b = t^3$, implying by (1), that $x = 2st$ and hence $(st,3) = 1$. Therefore by Lemma 1.9, $3^v = a - 2b = s^3 - 2t^3 \equiv \pm 1 \pm 2 \bmod 9$, which is a contradiction of $v \geq 2$.

Proposition 8.2: $y^2 - 3^{2v+1} = x^3, \ (x,3) = 1 \Rightarrow (v,x,y) = (0,1,2)$.

Proof: If x is odd, then by Proposition 7.2, $(v,x,y) = (0,1,2)$ and therefore we may assume that x is even. Hence y is odd, implying that $y^2 \equiv 1 \bmod 8$. Also, $3^{2v+1} = 3 \cdot 9^v \equiv 3 \bmod 8$. Therefore $0 \equiv x^3 = y^2 - 3^{2v+1} \equiv -2 \bmod 8$, which is an obvious contradiction.

Propositions 8.1 and 8.2 prove

Theorem 8a: $y^2 - 3^m = x^3, \ (x,3) = 1 \Leftrightarrow (m,x,y) = (0,-1,0),(0,2,3),(1,1,2),$ $(2,-2,1)$ or $(2,40,253)$.

Corollary 8.3: $a^2 - 2^n 3^m = 1 \Rightarrow (n,m,|a|) = (0,1,2),(3,0,3),(3,1,5),(4,1,7)$ or $(5,2,17)$.

Proof: If $n = m = 0$, then $a^2 = 2$.

If $n = 0$ and $m > 0$, then $(a,3) = 1$, and by Theorem 8a, $(m,|a|) = (1,2)$.

If $n > 0$ and $m = 0$, then a is odd, and by Theorem 2, $(n,|a|) = (3,3)$.

Finally, if $n > 0$ and $m > 0$, then $(a,6) = 1$, and by Theorem 7, $(n,m,|a|) = (3,1,5),(4,1,7)$ or $(5,2,17)$.

Proposition 8.4: $y^2 - 3^{2v} = 2x^3$, $(x,3) = 1 \Rightarrow (v,x,y) = (1,2,5)$.

Proof: $(2y)^2 - 4\cdot 3^{2v} = (2x)^3$, and [A] lists the solutions for $v = 0$ and $v = 1$. Hence we may assume that $v \geq 2$.

Now, y is odd and therefore $2x^3 = y^2 - (3^v)^2 \equiv 0 \bmod 8$. Hence x is even. Let $Y = \pm y$, such that $Y \equiv 3^v \bmod 4$. Therefore $Y + 3^v \equiv 2\cdot 3^v \not\equiv 0 \bmod 4$. Also, $(Y + 3^v)(Y - 3^v) = 16(x/2)^3$ and therefore $Y - 3^v \equiv 0 \bmod 8$. Hence

(2) $\quad ab = (x/2)^3$, where $a = (Y + 3^v)/2$ and $b = (Y - 3^v)/8$.

$(a,b)\,|\,a - 4b = 3^v$ and therefore $(a,b) = 1$. Thus by (2), $a = r^3$ and $b = s^3$, implying that $x = 2rs$ and therefore $(rs,3) = 1$. Hence $3^v = a - 4b = r^3 - 4s^3 \equiv \pm 1 \pm 4 \bmod 9$, which contradicts $v \geq 2$.

Proposition 8.5: $y^2 - 3^{2v+1} = 2x^3$, $(x,3) = 1 \Rightarrow (v,x,y) = (0,-1,1)$ or $(1,-1,5)$.

Proof: Obviously , $(y,6) = 1$. Thus $2x^3 = y^2 - 3^{2v+1} \equiv 1 - 3 \equiv 2 \bmod 4$, and therefore x is odd.

If $v = 0$, then $(2y)^2 - 12 = (2x)^3$, and we obtain the solution from [A].

We may therefore assume that $v > 0$. Let $\Lambda = \Gamma_3$ and $\theta = 3^{\frac{1}{2}}$. Now,

$$(1 + \theta)\left[(3^{v+1} - y)/2 + (y - 3^v)/2 \cdot \theta\right] = y + 3^v\theta .$$

Therefore if we define $\alpha = (y + 3^v\theta)/(1 + \theta)$, $\alpha \in \Lambda$. Thus $\bar{\alpha} = (y - 3^v\theta)/(1 - \theta) \in \Lambda$.

Also, $\alpha\bar{\alpha} = (-x)^3$ and $(\alpha,\bar{\alpha})_\Lambda |_\Lambda (1 + \theta)\alpha - (1 - \theta)\bar{\alpha} = 2\cdot 3^v\theta$. Since $(x,6) = 1$, $(\alpha,\bar{\alpha})_\Lambda = 1$. Therefore $\alpha = \mu(a + b\theta)^3$, where $\mu = 1$ or $2 \pm \theta$. Hence

$$y + 3^v\theta = \psi(a + b\theta)^3, \text{ where } \psi = \pm 1 + \theta \text{ or } 5 + 3\theta .$$

If $\psi = \pm 1 + \theta$, then:

$$y = \pm(a^3 + 9ab^2) + 3(3a^2b + 3b^3) \qquad (\Rightarrow (a,3) = 1) ,$$

$$3^v = (a^3 + 9ab^2) \pm (3a^2b + 3b^3) \qquad (\Rightarrow 3|a, \text{since } v > 0) .$$

Therefore $\psi = 5 + 3\theta$, which implies:

(3) $$y = 5(a^3 + 9ab^2) + 9(3a^2b + 3b^3) \qquad (\Rightarrow (a,3) = 1) ,$$

(4) $$3^{v-1} = (a + b)(a^2 + 4ab + 5b^2) .$$

If $3|b$, then $a^2 + 4ab + 5b^2 \equiv a^2 \not\equiv 0 \bmod 3$, and if $(b,3) = 1$, then $a^2 + 4ab + 5b^2 \equiv 1 \pm 4 + 5 \not\equiv 0 \bmod 3$. Thus $a^2 + 4ab + 5b^2 \not\equiv 0 \bmod 3$. Also, $a^2 + 4ab + 5b^2 = (a + 2b)^2 + b^2 \geq 0$. Therefore by (4), $(a + 2b)^2 + b^2 = 1$. Hence either $(a + 2b = \pm 1$ and $b = 0)$ or $(a + 2b = 0$ and $b = \pm 1)$. Since by (4), $a + b > 0$, either $(a = 1$ and $b = 0)$ or $(a = 2$ and $b = -1)$. In either case, (4) implies that $v = 1$ and by (3), $y = \pm 5$. Therefore $x = -1$.

Propositions 8.4 and 8.5 prove

Theorem 8b: $y^2 - 3^m = 2x^3$, $(x,3) = 1 \Leftrightarrow (m,x,y) = (1,-1,1),(2,2,5)$ or $(3,-1,5)$.

Proposition 8.6: $y^2 - 3^{2v} = 4x^3$, $(x,3) = 1 \Rightarrow (v,x,y) = (2,-2,7)$.

Proof: If $v = 0$, $(4y)^2 - 16 = (4x)^3$ and by [A], there are no solutions with $(x,3) = 1$.

Hence we may assume that $v > 0$ and therefore $(y,6) = 1$.

(5) $\quad ab = x^3$, where $a = (y + 3^v)/2$ and $b = (y - 3^v)/2$.

$(a,b) | a - b = 3^v$ and therefore $(a,b) = 1$. Thus by (5), $a = s^3$ and $b = t^3$, which implies that $x = st$ and hence $(st,3) = 1$. Also,

(6) $\quad 3^v = a - b = s^3 - t^3 = (s - t)(s^2 + st + t^2)$.

Now, $s - t \equiv s^3 - t^3 = 3^v \equiv 0 \bmod 3$ and $s^2 + st + t^2 > 0$, which together imply, by (6), that $s - t = 3^r$, $r \geq 1$. Hence by (6), $3^{v-r} = s^2 + st + t^2$. Eliminating s from the last two equations, we obtain

(7) $\quad 3t^2 + 3^{r+1}t + 3^{2r} = 3^{v-r}$.

Therefore, since $r + 1 \geq 2$ and $2r \geq 2$, Lemma 1.7 gives $v - r = 1$. By (7), $(2t + 3^r)^2 + 3^{2r-1} = 4$. It follows that $r = 1$ and $2t + 3^r = \pm 1$. Thus $v = r + 1 = 2$ and $t = -1$ or -2 .

If $t = -1$, then $y = 3^v + 2t^3 = 7$ and if $t = -2$, then $y = -7$. Therefore $x = -2$.

Theorem 8c: $y^2 - 3^m = 4x^3$, $(x,3) = 1 \Leftrightarrow (m,x,y) = (4,-2,7)$.

Proof: y is odd. If m is odd, then $3^m \equiv 3 \bmod 8$, and therefore

$4x^3 = y^2 - 3^m \equiv -2 \bmod 8$, which cannot hold. Hence $m = 2v$, and the proof is complete by Proposition 8.6.

SECTION 9.

$y^2 + 2^n 3^m = x^3, \ (x,6) = 1$

The following theorem is due to Hemer [12] (cf. [18,p.11, Theorem 11]).

Theorem 9.1: Let k be square-free and $k \neq 1$. If the class number of Γ_k is not divisible by three and if each prime factor of $2f$ does not factor into two distinct prime ideals in Γ_k, then all the integral solutions of the equation $y^2 - kf^2 = x^3$ are given by the solutions of the equation

$$\pm y + f \cdot k^{\frac{1}{2}} = \mu \alpha^3 ,$$

where $\alpha \in \Gamma_k$ and $\mu = 1$ or ϵ(= the fundamental unit of Γ_k). *

We shall be using this theorem in the special case $k = -6$ and $f = 2^r 3^s$; for, by [2,p.425], the class number of Γ_{-6} is 2 and by [34,p.235], 2 and 3 are both squares of prime ideals in Γ_{-6}. In this case $\mu = 1$, since $\epsilon = 1$ (see Table 1).

Lemma 9.2: $a^2 - 2^n 3^m = -1 \Rightarrow (n,m,|a|) = (0,0,0)$ or $(1,0,1)$.

Proof: Since $2^n 3^m \in Z$, $n \geq 0$ and $m \geq 0$. $m \geq 1$ cannot hold modulo 3, and $n \geq 2$ cannot hold modulo 4.

Proposition 9.3: $y^2 + 2^{2k+1} 3^{2v+1} = x^3$, $(x,6) = 1 \Rightarrow (k,v,x,y) = (0,1,7,17)$ or $(2,2,1153,39151)$.

Proof: Let $\theta = (-6)^{\frac{1}{2}}$. $y^2 + 2^{2k+1} 3^{2v+1} = x^3$ can be expressed in the form

*The proof of Theorem 9.1 is given in Appendix I.

(1) $$y^2 - (-6)(2^k3^v)^2 = x^3 .$$

By Theorem 9.1 , all the integral solutions of (1) are contained in the solutions of the equation

$$\pm y + 2^k3^v\theta = (a + b\theta)^3 .$$

Therefore,

(2) $\pm y = a^3 - 18ab^2$ ($\Rightarrow$ $(a,6) = 1$,since by (1), $(y,6) = 1$),

(3) $2^k3^v = 3b(a^2 - 2b^2)$ ($\Rightarrow$ $v \geq 1$) .

Since $(a,6) = 1$, $(a^2 - 2b^2,6) = 1$ and hence by (3), $b = \pm 2^k3^{v-1}$ and $a^2 - 2b^2 = \pm 1$. Thus

$$a^2 - 2^{2k+1}3^{2(v-1)} = \pm 1 .$$

Since $(a,3) = 1$, Corollary 8.3 and Lemma 9.2 imply $(k,v,|a|) = (0,1,1)$ or $(2,2,17)$. Therefore $b = \pm 1$ or ± 12 (respectively), implying, by (2), that $y = 17$ or 39151 (respectively). Therefore $x = 7$ or 1153 (respectively).

<u>Lemma 9.4</u>: $2^n - 1 = 3^m \Rightarrow (n,m) = (1,0)$ or $(2,1)$.

<u>Proof</u>: $2^n - 1 = 3^m > 0$. Therefore $n > 0$ and hence,since $3^m = 2^n-1 \in Z$, $m \geq 0$.

If $m = 0$, then $n = 1$, so we may assume that $m > 0$. Thus $2^n \equiv 1 \bmod 3$ and therefore $n = 2k$. Hence

$$(2^k + 1)(2^k - 1) = 3^m ,$$

which implies that

(4) $$2^k + 1 = 3^r \quad \text{and} \quad 2^k - 1 = 3^{m-r} \ .$$

Therefore $2 = 3^r - 3^{m-r}$. By (4), $r \neq 0$ and thus $m - r = 0$. Again by (4), $k = 1$ and therefore $n = 2k = 2$.

Proposition 9.5: $y^2 + 2^{2k}3^{2v+1} = x^3$, $(x,6) = 1 \Rightarrow (k,v,x,y) = (0,2,7,10)$, $(1,2,13,35)$ or $(2,3,73,595)$.

Proof: Obviously, $(y,3) = 1$, and if $k > 0$, then y is odd.

Let $\Lambda = \Gamma_{-3}$ and $\theta = (-3)^{\frac{1}{2}}$. Now, $\alpha\bar{\alpha} = x^3$, where $\alpha = y + 2^k3^v\theta$. $(\alpha,\bar{\alpha})_\Lambda |_\Lambda\ \alpha - \bar{\alpha} = 2^{k+1}3^v\theta$ and therefore $(\alpha,\bar{\alpha})_\Lambda = 1$. Hence

(5) $$\alpha = \mu\Big((a + b\theta)/2\Big)^3 ,$$

where $\mu = 1$ or $(\pm 1 + \theta)/2$ and $a \equiv b \bmod 2$.

We divide the proof into three cases:

Case 1: $\mu = (\pm 1 + \theta)/2$. Hence by (5),

$$16y = \pm(a^3 - 9ab^2) - 3(3a^2b - 3b^3) \qquad (\Rightarrow (a,3) = 1) ,$$

(6) $$2^{k+4}3^v = (a^3 - 9ab^2) \pm (3a^2b - 3b^3) \ .$$

Since $(a,3) = 1$, (6) yields $v = 0$, and by Theorem 4, there are no solutions in this case.

Case 2: $\mu = 1$ and $a = 2A$, $b = 2B$. Hence

(7) $$y = A(A^2 - 9B^2) \qquad (\Rightarrow (A,3) = 1),$$

(8) $$2^k3^v = 3B(A^2 - B^2) \qquad (\Rightarrow v \geq 1) \ .$$

If $k = 0$, then by (8), $A^2 - B^2$ is odd. If $k > 0$, then since y is odd, (7) implies that $A^2 - 9B^2$ is odd and therefore $A^2 - B^2$ is odd for all values of k .

(8) yields

$$(9) \qquad \pm 2^k 3^{v-1} = |B|(A^2 - B^2) = |B|(|A| + |B|)(|A| - |B|) .$$

Note: The double signs in the remainder of Case 2 will depend on the double sign in (9).

Since $A^2 - B^2$ is odd, (9) implies

$$(10) \qquad |B| = 2^k 3^r, |A| + |B| = 3^s \text{ and } |A| - |B| = \pm 3^t ,$$

where $r + s + t = v - 1$. Therefore

$$(11) \qquad 2|A| = 3^s \pm 3^t \text{ and } 2|B| = 3^s \mp 3^t .$$

Since $(A,3) = 1$, $s = 0$ or $t = 0$. But if $s = 0$ then by (10), $|A| = 0$ or $|B| = 0$, both of which are impossible by (7) and (10). Hence $s > 0$ and $t = 0$, which by (11), implies that $(|B|,3) = 1$ and hence $r = 0$. Thus by (10), $s = v - 1$ and by (11),

$$2^{k+1} \pm 1 = 3^{v-1} \text{ (since by (10), } |B| = 2^k) .$$

If the upper sign holds, then $1^2 + 2^{k+1} = 3^{v-1}$ and by Theorem 6b, $(k,v) = (0,2)$ or $(2,3)$. These yield $|B| = 2^k = 1$ or 4 (respectively) and by (10), $|A| = 2$ or 5 (respectively). By (7) and $x^3 = y^2 + 2^{2k}3^{2v+1}$, $(k,v,x,y) = (0,2,7,10)$ or $(2,3,73,595)$.

If the lower sign holds, then by Lemma 9.4, $(k,v) = (0,1)$ or $(1,2)$. But $v = 1$ implies that $s = v - 1 = 0$. Thus $(k,v) = (1,2)$ and hence $s = 1$. By (10), $|B| = 2$ and $|A| = 1$. By (7), $y = 35$ and thus $x = 13$.

Case 3: $\mu = 1$ and ab odd. By (5),

$$(12) \qquad 8y = a(a^2 - 9b^2) \qquad (\Rightarrow (a,3) = 1)\ ,$$

$$2^{k+3}3^v = 3b(a^2 - b^2) \qquad (\Rightarrow v \geq 1)\ .$$

Therefore

$$(13) \qquad \pm\, 2^{k+3}3^{v-1} = |b|(a^2 - b^2) = |b|(|a| + |b|)(|a| - |b|)\ .$$

Suppose first that $(b,3) = 1$. Then by (13), $|b| = 1$ and, since $a^2 - b^2 = a^2 - 1 \geq 0$, the upper sign holds in (13). Hence by (13), $a^2 - 1 = 2^{k+3}3^{v-1}$. Since $k + 3 \geq 3$ and $(a,3) = 1$, we have, by Corollary 8.3, $(k,v,|a|) = (0,2,5),(1,2,7)$ or $(2,3,17)$. By (12), $y = 10,\ 35$ or 595 (respectively). These solutions were obtained in Case 2.

<u>Note</u>: All double signs in the remainder of Case 3 will depend on that in (13).

Now suppose that $3|b$. Then $(a^2 - b^2,3) = 1$. By (13),

$$(14) \qquad |b| = 3^{v-1},\ |a| + |b| = 2^s \text{ and } |a| - |b| = \pm\, 2^t,$$

where $s + t = k + 3$. Hence, since $3|b$, $v \geq 2$ and

$$(14a) \qquad 2|b| = 2^s \mp 2^t\ .$$

Therefore $s = 1$ or $t = 1$.

If $s = 1$, then by (14), $|a| + |b| = 2$, which contradicts the assumptions that $3|b$ and b is odd.

Therefore $t = 1$ and by (14), $s = k + 2$. By (14) and (14a), $2^{k+1} \mp 1 = 3^{v-1}$. If the upper sign holds, then since $v \geq 2$, Lemma 9.4 yields $(k,v) = (1,2)$. By (14) and (12), $y = 35$.

If the lower sign holds, then by Theorem 6b, $(k,v) = (0,2)$ or $(2,3)$.

By (14) and (12), $y = 10$ or 595, respectively. These solutions were obtained in Case 2.

Lemma 9.6: $y^2 - 2^n = 3^m \Rightarrow (n,m,|y|) = (0,1,2),(3,0,3)$ or $(4,2,5)$.

Proof: If $n < 0$, then since $3^m = y^2 - 2^n$, $3^m \notin Z$ and therefore $m < 0$. Hence $y^2 = 2^n + 3^m \leq 2^{-1} + 3^{-1} < 1$ and therefore $y = 0$. But then $-2^n = 3^m$, which is impossible. Thus $n \geq 0$ and therefore $m \geq 0$.

If $m = 0$, then $y^2 - 2^n = 1^3$. By Theorem 2, $(n,|y|) = (3,3)$.

If $m > 0$, then $(y,3) = 1$. Therefore $2^n \equiv 1 \bmod 3$ and hence $n = 2r$. Thus

$$(|y| + 2^r)(|y| - 2^r) = 3^m .$$

Therefore $|y| + 2^r = 3^s$ and $|y| - 2^r = 3^{m-s}$. Hence $s > m - s$ and $2^{r+1} = 3^s - 3^{m-s}$. Thus $m - s = 0$ and $1^2 + 2^{r+1} = 3^m$.

By Theorem 6b, $(r,m) = (0,1)$ or $(2,2)$. Hence $|y| = 3^m - 2^r = 2$ or 5 (respectively), and $n = 2r = 0$ or 4 (respectively).

Proposition 9.7: $y^2 + 2^{2k}3^{2v} = x^3$, $(x,6) = 1 \Rightarrow (k,v,x,y) = (0,0,1,0)$, $(0,2,13,46),(1,0,5,11)$ or $(2,2,193,2681)$.

Proof: If $v = 0$, we obtain the solution from Theorem 2. Thus we may assume $v > 0$ and therefore $(y,3) = 1$. We note that if y is even, $k = 0$.

Let $\Lambda = \Gamma_{-1}$ and $\theta = (-1)^{\frac{1}{2}}$. $\alpha\bar{\alpha} = x^3$, where $\alpha = y + 2^k 3^v \theta$. $(\alpha,\bar{\alpha})_\Lambda |_\Lambda\ \alpha - \bar{\alpha} = 2^{k+1}3^v\theta$. Thus $(\alpha,\bar{\alpha})_\Lambda = 1$. Also, $\theta = (-\theta)^3$ is the fundamental unit of Γ_{-1} and therefore $\alpha = (a + b\theta)^3$. Hence

$$(15) \qquad y = a(a^2 - 3b^2) \qquad\qquad (\Rightarrow (a,3) = 1) ,$$

$$2^k 3^v = 3a^2 b - b^3 \qquad (\Rightarrow 3 | b \text{, since } v > 0) .$$

Thus $b = 3B$ and $2^k 3^v = 9B(a^2 - 3B^2)$. Therefore $v \geq 2$ and

(16) $$2^k 3^{v-2} = B(a^2 - 3B^2) .$$

Now, $(a^2 - 3B^2, 3) = 1$. It follows by (16), that

(17) $$a^2 - 3B^2 = \pm 2^r \text{ and } B = \pm 2^{k-r} 3^{v-2} ,$$

which implies that

(18) $$a^2 - 2^{2(k-r)} \cdot 3^{2v-3} = \pm 2^r .$$

If a is odd, then by (18), $r = 0$ or $k - r = 0$. If a is even, then by (15), y is even and so $k = 0$. Then by (17), $r = 0$. Thus $r = 0$ or $k - r = 0$ in all cases.

Suppose first that $r = 0$. Therefore by (18),

(19) $$a^2 - 2^{2k} 3^{2v-3} = \pm 1 .$$

Since $2v - 3 \geq 1$, the lower sign cannot hold in (19) modulo 3. Hence since $(a,3) = 1$, we have by Corollary 8.3 , $(k,v,|a|) = (0,2,2)$ or $(2,2,7)$. By (17), $b = 3B = \pm 2^{k-r} 3^{v-1} = \pm 3$ or ± 12 (respectively) and therefore by (15), $y = 46$ or 2681 (respectively). Hence $x = 13$ or 193 (respectively).

Now suppose that $k - r = 0$. By (18),

(20) $$a^2 - 3^{2v-3} = \pm 2^k .$$

If the plus sign holds in (20), then by Lemma 9.6, $(k,v,|a|) = (0,2,2)$ and by (17), $b = 3B = \pm 3$. Hence by (15), $y = 46$ and thus $x = 13$.

If the minus sign holds in (20), then by Theorem 6b, $(k,v,|a|) = (1,2,1)$ or $(1,3,5)$. By (17), $b = 3B = \pm 3$ or ± 9. Therefore by (15), y is even. But this contradicts $k = 1$.

Proposition 9.8: $y^2 + 2^{2k+1}3^{2v} = x^3$, $(x,6) = 1 \Rightarrow (k,v,x,y) = (1,2,97,955)$.

Proof: If $v = 0$, we obtain the solution from Theorem 2 and therefore we may assume that $v > 0$. Hence $(y,6) = 1$.

Let $\Lambda = \Gamma_{-2}$ and $\theta = (-2)^{\frac{1}{2}}$. Now, $\alpha\bar{\alpha} = x^3$, where $\alpha = y + 2^k3^v\theta$. $(\alpha,\bar{\alpha})_\Lambda |_\Lambda\ \alpha - \bar{\alpha} = 2^{k+1}3^v\theta$ and hence $(\alpha,\bar{\alpha})_\Lambda = 1$. Therefore $\alpha = (a + b\theta)^3$, which yields:

(21) $$y = a(a^2 - 6b^2) \qquad (\Rightarrow (a,6) = 1),$$

(22) $$2^k3^v = 3a^2b - 2b^3 = b(3a^2 - 2b^2) \quad (\Rightarrow 3|b).$$

By (21), $3a^2 - 2b^2$ is odd and therefore by (22),

(23) $$b = \pm 2^k3^r \quad \text{and} \quad 3a^2 - 2b^2 = \pm 3^{v-r},$$

where $r \geq 1$. Hence $3a^2 - 2^{2k+1}3^{2r} = \pm 3^{v-r}$. Since $2r \geq 2$, $v - r = 1$ and thus

(24) $$a^2 - 2^{2k+1}3^{2r-1} = \pm 1.$$

Since $(a,3) = 1$ and $2r - 1 \geq 1$, the minus sign is impossible in (24). By Corollary 8.3 , $(k,r,|a|) = (1,1,5)$. Hence $v = r + 1 = 2$ and by (23), $b = \pm 6$. By (21), $y = 955$ and therefore $x = 97$.

Propositions 9.3, 9.5, 9.7 and 9.8 prove

<u>Theorem 9</u>: $y^2 + 2^n 3^m = x^3$, $(x,6) = 1 \Leftrightarrow (n,m,x,y) = (0,0,1,0),(0,4,13,46),$ $(0,5,7,10),(1,3,7,17),(2,0,5,11),(2,5,13,35),(3,4,97,955),(4,4,193,2681),$ $(4,7,73,595)$ or $(5,5,1153,39151)$.

SECTION 10.

$2y^2 - 3^m = x^3, \ (x,3) = 1$

Proposition 10.1: $2y^2 - 3^{2v+1} = x^3$, $(x,3) = 1 \Rightarrow (v,x,y) = (0,-1,1)$, $(0,5,8), (0,4079,184211), (2,-1,11), (3,5,34)$ or $(6,239,2761)$.

Proof: If $v = 0$, $(4y)^2 - 24 = (2x)^3$, and the solutions to this equation are given in [A].

Assume $v > 0$ and let $\Lambda = \Gamma_6$ and $\theta = 6^{\frac{1}{2}}$. Now, $(y,3) = 1$ and $\alpha\bar{\alpha} = 2x^3$, where $\alpha = 2y + 3^v\theta$. Further,

$$(2 + \theta)[3^{v+1} - 2y + (y - 3^v)\theta] = \alpha$$

and therefore $(2 + \theta)|_\Lambda \alpha$. Taking conjugates, $2 - \theta|_\Lambda \bar{\alpha}$.

(1) $$\alpha/(2 + \theta) \cdot \bar{\alpha}/(2 - \theta) = \alpha\bar{\alpha}/-2 = (-x)^3 .$$

$(\alpha/(2 + \theta), \bar{\alpha}/(2 - \theta))_\Lambda |_\Lambda \alpha - \bar{\alpha} = 2\cdot 3^v\theta$ and, since $(x,6) = 1$, $(\alpha/(2 + \theta), \bar{\alpha}/(2 - \theta))_\Lambda = 1$. Therefore by (1), Lemma 1.3 and Table 1, $\alpha/(2 + \theta) = \mu(a + b\theta)^3$, where $\mu = 1$ or $5 \pm 2\theta$. Hence $\alpha = \delta(a+b\theta)^3$, where $\delta = \pm 2 + \theta$ or $22 + 9\theta$.

If $\delta = \pm 2 + \theta$, then:

$$y = \pm (a^3 + 18ab^2) + 3(3a^2b + 6b^3) \qquad (\Rightarrow (a,3) = 1),$$

$$3^v = (a^3 + 18ab^2) \pm 2(3a^2b + 6b^3) \qquad (\Rightarrow 3|a) .$$

Thus $\delta = 22 + 9\theta$ and therefore:

(1a) $$y = 11(a^3 + 18ab^2) + 27(3a^2b + 6b^3) \qquad (\Rightarrow (a,3) = 1),$$

(2) $$3^{v-1} = (a + 2b)(3a^2 + 16ab + 22b^2) .$$

Since by (2), $3a^2 + 16ab + 22b^2 \neq 0$, $3a^2 + 16ab + 22b^2 = 3(a + 8b/3)^2 + 2b^2/3 > 0$.

If $a \equiv b \bmod 3$, then since $(a,3) = 1$, $3a^2 + 16ab + 22b^2 \equiv 16 + 22 \equiv 2 \bmod 3$. But then, by (2), $3a^2 + 16ab + 22b^3 = 1 \not\equiv 2 \bmod 3$, so we have a contradiction. Therefore $a \not\equiv b \bmod 3$ and hence $a + 2b \not\equiv 0 \bmod 3$. By (2), $a + 2b = 1$ and thus $3^{v-1} = 3a^2 + 16ab + 22b^2$. Hence

(3) $\quad a = 1 - 2b$ and $3^{v-1} = 2(b + 1)^2 + 1$.

By Theorem 6a, $v = 1,2,3$ or 6 .

If $v = 1$, then by (3), $b = -1$ and $a = 3$, contradicting $(a,3) = 1$.

If $v = 2$, then by (3), $b + 1 = \pm 1$. Hence $b = 0$ or -2 and by (3), $a = 1$ or 5 , respectively. These give, by (1a), $y = \pm 11$, which implies that $x = -1$.

If $v = 3$, then as above, $y = \pm 34$ and hence $x = 5$.

If $v = 6$, then as above, $y = \pm 2761$ and thus $x = 239$.

<u>Proposition 10.2</u>: $2y^2 - 3^{2v} = x^3$, $(x,3) = 1 \Rightarrow (v,x,y) = (0,-1,0)$, $(0,1,1)$, $(0,23,78)$ or $(1,-1,2)$.

<u>Proof</u>: $(4y)^2 - 8\cdot 3^{2v} = (2x)^3$. The solutions for $v = 0$ and $v = 1$ are given in [A].

Assume $v > 1$ and let $\Lambda = \Gamma_2$, $\theta = 2^{\frac{1}{2}}$. Now, $(y,3) = 1$ and $\alpha\bar{\alpha} = (-x)^3$, where $\alpha = 3^v + y\theta$. $(\alpha,\bar{\alpha})_\Lambda |_\Lambda \ \alpha + \bar{\alpha} = 2\cdot 3^v$ and therefore, since $(x,6) = 1$, $(\alpha,\bar{\alpha})_\Lambda = 1$. Thus $\alpha = \mu(a + b\theta)^3$, where $\mu = 1$ or $\pm 1 + \theta$.

If $\mu = 1$, then:

$$(4) \qquad 3^v = a^3 + 6ab^2 = a(a^2 + 6b^2) \qquad (\Rightarrow 3|a) ,$$

$$y = 3a^2b + 2b^3 \qquad (\Rightarrow (3,b) = 1) .$$

From (4), $a = 3^r$, where $r \geq 1$, and using (4) again, $3^{v-r} = 3^{2r} + 6b^2$. Since $2r > 1$ and $(b,3) = 1$, Lemma 1.7 implies that $v - r = 1$. Hence $1 = 3^{2r-1} + 2b^2$, which is impossible.

If $\mu = \pm 1 + \theta$, then:

$$(5) \qquad 3^v = \pm (a^3 + 6ab^2) + 2(3a^2b + 2b^3) ,$$

$$(6) \qquad y = (a^3 + 6ab^2) \pm (3a^2b + 2b^3) .$$

Let $A = \pm a$ (corresponding to the double sign in (5)). By (5),

$$(7) \qquad 3^v = A^3 + 4b^3 + 6Ab(A + b) .$$

If $3|A$, then by (7), $3|b$. But then by (6), $3|y$. Hence $(A,3) = 1$ and by (7), $(b,3) = 1$. Also, from (7), $A + b \equiv A^3 + 4b^3 \equiv 0 \bmod 3$. Using (7) again we have, since $v \geq 2$, $A^3 + 4b^3 \equiv 0 \bmod 9$. But $A^3 + 4b^3 \equiv \pm 1 \pm 4 \bmod 9$.

Propositions 10.1 and 10.2 prove

<u>Theorem 10</u>: $2y^2 - 3^m = x^3$, $(x,3) = 1 \Leftrightarrow (m,x,y) = (0,-1,0),(0,1,1),$ $(0,23,78),(1,-1,1),(1,5,8),(1,4079,184211),(2,-1,2),(5,-1,11),(7,5,34)$ or $(13,239,2761)$.

SECTION 11.

$2y^2 + 3^m = x^3,\ (x,3) = 1$

Proposition 11.1: $2y^2 + 3^{2v+1} = x^3$, $(x,3) = 1 \Rightarrow (v,x,y) = (1,5,7)$ or $(2,35,146)$.

Proof: Obviously, $(y,3) = 1$. Let $\theta = (-6)^{\frac{1}{2}}$. Now, $2y^2 + 3^{2v+1} = x^3$ implies that

$$(4y)^2 - (-6)(2\cdot 3^v)^2 = (2x)^3 .$$

By the remark following Theorem 9.1, $\pm 4y + 2\cdot 3^v\theta = (a + b\theta)^3$. Hence

(1) $$\pm 4y = a^3 - 18ab^2 \qquad (\Rightarrow a \text{ is even and } (a,3) = 1),$$

(2) $$2\cdot 3^v = 3a^2b - 6b^3 \qquad (\Rightarrow v \geq 1) .$$

Thus $a = 2A$ and by (2),

(3) $$3^{v-1} = b(2A^2 - b^2) .$$

By (1), $(A,3) = 1$ and therefore $2A^2 - b^2 \equiv \pm 1 \bmod 3$. Hence by (3), $b = \pm 3^{v-1}$ and $2A^2 - b^2 = \pm 1$. Therefore $2A^2 - 3^{2(v-1)} = \pm 1 = (\pm 1)^3$. Since $(a,3) = 1$, Theorem 10 implies that $(v,|A|) = (1,1)$ or $(2,2)$.

If $(v,|A|) = (1,1)$, then $b = \pm 1$, $a = 2A = \pm 2$, and by (1), $y = 7$. Hence $x = 5$.

If $(v,|A|) = (2,2)$, then $b = \pm 3$, $a = \pm 4$, $y = 146$ and therefore $x = 35$.

Proposition 11.2: $2y^2 + 3^{2v} = x^3$, $(x,3) = 1 \Rightarrow (v,x,y) = (0,1,0), (2,11,25)$ or $(4,971,21395)$.

Proof: If $v = 0$, then $(4y)^2 + 8 = (2x)^3$. Hence by [A], $(x,y) = (1,0)$.

Assume $v > 0$. Therefore $(y,3) = 1$. We also note that x is odd. Let $\Lambda = \Gamma_{-2}$ and $\theta = (-2)^{\frac{1}{2}}$. $\alpha\bar{\alpha} = x^3$, where $\alpha = 3^v + y\theta$. $(\alpha,\bar{\alpha})_\Lambda |_\Lambda\ \alpha + \bar{\alpha} = 2\cdot 3^v$. Since $(x,6) = 1$, $(\alpha,\bar{\alpha})_\Lambda = 1$. Hence $\alpha = (a + b\theta)^3$, and it follows that:

(4) $\qquad 3^v = a^3 - 6ab^2 \qquad (\Rightarrow 3|a),$

(5) $\qquad y = b(3a^2 - 2b^2) \qquad (\Rightarrow (b,3) = 1)$.

By (4), $a = 3A$ and therefore $v \geq 2$. Hence

$$3^{v-2} = A(3A^2 - 2b^2) .$$

Since $(b,3) = 1$, $3A^2 - 2b^2 = \pm 1$ and $A = \pm 3^{v-2}$. Thus $2b^2 - 3^{2v-3} = \mp 1 = (\mp 1)^3$ and by Theorem 10, $(v,|b|) = (2,1)$ or $(4,11)$.

If $(v,|b|) = (2,1)$, $a = 3A = \pm 3$ and by (5), $y = 25$. Thus $x = 11$.

If $(v,|b|) = (4,11)$, $a = \pm 27$, $y = 21395$ and therefore $x = 971$.

Propositions 11.1 and 11.2 prove

<u>Theorem 11</u>: $2y^2 + 3^m = x^3$, $(x,3) = 1 \Leftrightarrow (m,x,y) = (0,1,0),(3,5,7),$ $(4,11,25),(5,35,146)$ or $(8,971,21395)$.

SECTION 12.

$y^2 \pm 2^n = 3x^3$, x odd.

Note that $(y,3) = 1$.

Lemma 12.1: If $2^s \equiv \pm 1 \bmod 9$, then $3|s$.

Proof: $2^{2s} \equiv 1 \bmod 9$ and the order of 2 modulo 9 is 6 . Hence $6|2s$.

Lemma 12.2: $2^n + 1 = 3x^3 \Rightarrow n = 1$.

Remark: Since $2^n \equiv -1 \bmod 3$, $n = 2r + 1$, implying that $(3\cdot 2^{r+2})^2 + 72 = (3x)^3$. The solution is therefore given by [4] but, in accordance with the Introduction , I will prove this lemma directly.

Proof: As was shown above, $n = 2r + 1 > 0$. Hence x is odd. We show first that $3|r$.

Let $\Lambda = \Gamma_{-2}$, $\theta = (-2)^{\frac{1}{2}}$. Now,

(1) $\alpha\bar{\alpha} = 3x^3 = (1 + \theta)(1 - \theta)\, x^3$, where $\alpha = 1 + (-2)^r\theta$. Further,

$$(\alpha,\bar{\alpha})_\Lambda |_\Lambda\ \alpha + \bar{\alpha} = 2$$

and therefore $(\alpha,\bar{\alpha})_\Lambda = 1$ (since $(3x^3,2) = 1$). By the definition of α , $1 + \theta |_\Lambda \alpha$ and therefore $1 - \theta |_\Lambda \bar{\alpha}$. Hence by (1),

$$\alpha/(1+\theta) \cdot \bar{\alpha}/(1-\theta) = x^3 .$$

Since $(\alpha,\bar{\alpha})_\Lambda = 1$ and the only units of Λ are ± 1 , $\alpha = (1 + \theta)(a + b\theta)^3$. Comparing coefficients:

(2) $$1 = a^3 + 4b^3 - 6ab(a + b) ,$$

(3) $$\pm 2^r = a^3 - 2b^3 + 3ab(a - 2b) .$$

By (2),

(4) $$a + b \equiv a^3 + 4b^3 \equiv 1 \bmod 3 .$$

If $a \equiv -1 \bmod 3$, then by (4), $b \equiv -1 \bmod 3$ and hence $ab(a + b) \equiv 1 \bmod 3$. Thus by (2) and Lemma 1.9, $1 \equiv -1 -4 -6 = -11 \bmod 9$, which is a contradiction.

If $a \equiv 0 \bmod 3$, then by (4), $b \equiv 1 \bmod 3$ and by (2), $1 \equiv 4 \bmod 9$.

Hence $a \equiv 1 \bmod 3$ and by (4), $b \equiv 0 \bmod 3$. By (3), $\pm 2^r \equiv 1 \bmod 9$. By Lemma 12.1, $3|r$.

Let $r = 3t$. Thus $n = 6t + 1$ and $3x^3 + 2(-2^{2t})^3 = 1$. Nagell [31,p.251] has shown that $Au^3 + Bv^3 = 1$ has at most one solution in non-zero integers u and v. Hence $x = 1$ and $-2^{2t} = -1$. Therefore $t = 0$ and $n = 6t + 1 = 1$.

Lemma 12.3: $2^n - 1 = 3x^3 \Rightarrow n = 0$ or 2.

Proof: $2^n \equiv 1 \bmod 3$ and therefore $n = 2r$. Hence $(3 \cdot 2^r)^2 - 9 = (3x)^3$. By [A], $r = 0$ or 1 and thus $n = 0$ or 2.

Proposition 12.4: $y^2 + 2^{3k} = 3x^3$, x odd $\Rightarrow (k,x,y) = (3,11,59)$.

$y^2 - 2^{3k} = 3x^3$, x odd $\Rightarrow (k,x,y) = (0,1,2)$ or $(4,-5,61)$.

Proof: If $k = 0$,

(*) $$(3y)^2 \pm 9 = (3x)^3, \text{ and the solutions are given in [A].}$$

Assume $k > 0$. Hence $(y,6) = 1$. Let $\Lambda = \Omega_3$ and $\theta = 3^{\frac{1}{3}}$.

Now,

(*) $$y^2 = \alpha\beta, \text{ where } \alpha = \mp 2^k + x\theta \text{ and } \beta = 2^{2k} \pm 2^k x\theta + x^2\theta^2 .$$

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda | 3\cdot 2^{2k}$. Therefore $(\alpha,\beta)_\Lambda = 1$ and hence $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $-2 + \theta^2$ and $a \geq 0$.

If $\mu = -2 + \theta^2$, then:

$$(*) \qquad \mp 2^k = -2(a^2 + 6bc) + 3(3c^2 + 2ab) \qquad (\Rightarrow c \text{ is even}),$$

$$x = -2(3c^2 + 2ab) + 3(b^2 + 2ac) \qquad (\Rightarrow b \text{ is odd}),$$

$$(5) \qquad 0 = -2(b^2 + 2ac) + (a^2 + 6bc) \qquad (\Rightarrow a \text{ is even}) .$$

Thus by (5), $0 \equiv -2 \bmod 4$, so we have a contradiction.

Hence $\mu = 1$, which gives:

$$(6)(*) \qquad \mp 2^k = a^2 + 6bc ,$$

$$(7) \qquad x = 3c^2 + 2ab \qquad (\Rightarrow c \text{ is odd}),$$

$$(8) \qquad 0 = b^2 + 2ac \qquad (\Rightarrow b \text{ is even}) .$$

If $b = 0$, then by (8), $a = 0$, contradicting (6). Hence $b = 2^r B$, $r \geq 1$, $(B,2) = 1$. Thus by (8), $a = 2^{2r-1}A$, $(A,2) = 1$ and therefore $A > 0$. By (8) and (6), $0 = B^2 + Ac$ and

$$(*) \qquad \mp 2^k = 2^{4r-2}A^2 + 3\cdot 2^{r+1}Bc .$$

As in Proposition 2.2, $A = 1$. Thus $c = -B^2$ and

$$(8a)(*) \qquad \mp 2^k = 2^{4r-2} - 3\cdot 2^{r+1}B^3 .$$

If $r > 1$, then $4r - 2 > r + 1$ and hence $k = r + 1$. Therefore

$$(*) \qquad 2^{3r-3} \pm 1 = 3B^3 .$$

By Lemmas 12.2 and 12.3, there are no solutions in the positive case , and in the negative case $r = 1$, contradicting $r > 1$.

Hence $r = 1$ and $2 = r + 1 = 4r - 2 < k$. Therefore by (8a),

(*) $$2^{k-2} \pm 1 = 3(\pm B)^3 .$$

In the positive case, Lemma 12.2 gives $k = 3$ and thus $B = 1$. Hence $a = 2^{2r-1}A = 2$, $b = 2^r B = 2$ and $c = -B^2 = -1$. By (7), $x = 11$ and therefore $y = 59$.

In the negative case, Lemma 12.3 gives $k = 4$ (since $k > 2$) and hence $B = -1$. Thus $a = 2$, $b = -2$ and $c = -1$. By (7), $x = -5$ and therefore $y = 61$.

Proposition 12.5: $y^2 + 2^{3k+1} = 3x^3$, x odd $\Rightarrow (k,x,y) = (0,1,1)$ or $(2,19,143)$.

$y^2 - 2^{3k+1} = 3x^3$, x odd has no solutions.

Proof: If $k = 0$, then

(*) $$(3y)^2 \pm 18 = (3x)^3,$$ and the solutions are given in [A].

Assume $k > 0$. Let $\Lambda = \Omega_{12}$ and $\theta = (12)^{\frac{1}{3}}$. Now, $(y,6) = 1$ and

(9)(*) $$9y^2 = \alpha\beta, \text{ where } \alpha = 3x \mp 2^{k-1}\theta^2 \text{ and } \beta = 9x^2 \pm 3\cdot 2^{k-1}x\theta^2 + (2^{k-1}\theta^2)^2 .$$

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda\ 3\cdot(2^{k-1}\theta^2)^2 = 9\cdot 2^{2k+2}\theta$. Hence $(\alpha,\beta)_\Lambda |_\Lambda\ 9\cdot 2^{2k+2}\theta^3 = 3^3\cdot 2^{2k+4}$. Since $(9y^2,2) = 1$, $(\alpha,\beta)_\Lambda |_\Lambda\ 3^3$.

Now, $3 = \epsilon\delta^3$, where $\epsilon = 1 + 3\theta - 3\theta^2/2$ and $\delta = 3 + \theta + \theta^2/2$. By Table 1, ϵ is a unit of Λ and, since $N(\delta) = 3$ (by (6) of Section 1), δ is a prime of Λ. Hence $(\alpha,\beta)_\Lambda = \delta^s$, $s \ge 0$.

(*) $\alpha = \delta^2[3x \mp 2^{k+1} + (x \pm 2^{k+1})\,\theta - (x \pm 2^{k-1})\,\theta^2]$

and therefore $\delta^2 |_{\Lambda}\, \alpha$. If $\delta^3 |_{\Lambda}\, \alpha$, then

$$27 = N(\delta^3)\,|N(\alpha) = (3x)^3 + (18)\cdot 2^{3k} = 9y^2 \quad .$$

Thus $\delta^3 \nmid_{\Lambda} \alpha$ and by Lemma 1.4 , $(\alpha/\delta^2, \beta)_{\Lambda} = 1$. By (9), $(\epsilon\delta^2 y)^2 = (\alpha/\delta^2)\cdot\beta$. Therefore $\alpha/\delta^2 = \mu\gamma^2$, where $\mu = 1$ or ϵ and $\gamma \in \Lambda$. Since $\delta\gamma \in \Lambda$, (4) of Section 1 implies $\alpha = \mu(a + b\theta + c\theta^2/2)^2$, where we may choose $b \geq 0$.

If $\mu = \epsilon = 1 + 3\theta - 3\theta^2/2$, then:

$3x = (a^2 + 12bc) + 18(2b^2 + 2ac) - 18(3c^2 + 2ab)$ ($\Rightarrow$ a is odd),

(*) $\mp 2^k = -3(a^2 + 12bc) + 6(3c^2 + 2ab) + (2b^2 + 2ac)$ ($\Rightarrow$ a is even, since $k \geq 1$) .

Hence $\mu = 1$, and therefore:

(10) $3x = a^2 + 12bc$ ($\Rightarrow$ a is odd and $3|a$),

(11) $0 = 3c^2 + 2ab$ ($\Rightarrow$ c is even),

(12) (*) $\mp 2^{k-1} = b^2 + ac$.

If $c = 0$, then by (11), $b = 0$, which contradicts (12). Thus $c = 2^r C$, C odd, $r \geq 1$ and therefore by (11), $b = 2^{2r-1}B$, B odd. Hence $B > 0$. By (10), $a = 3A$ and from (11) and (12), we obtain $0 = C^2 + AB$ and

(*) $\mp 2^{k-1} = 2^{4r-2}B^2 + 3\cdot 2^r AC$.

As in Proposition 2.2, $B = 1$. Therefore $A = -C^2$ and

(*) $\mp 2^{k-1} = 2^{4r-2} - 3\cdot 2^r C^3$.

Since $4r - 2 > r$, $k - 1 = r$. Hence

(*) $$2^{3r-2} \pm 1 = 3C^3 .$$

By Lemmas 12.3 and 12.2, there are no solutions in the negative case, and in the positive case $r = 1$, $C = 1$, $A = -C^2 = -1$, $a = 3A = -3$, $b = 2^{2r-1}B = 2$, $c = 2^rC = 2$, $k = r + 1 = 2$ and by (10), $x = 19$. Therefore $y = 143$.

Proposition 12.6: $y^2 + 2^{3k+2} = 3x^3$, x odd $\Rightarrow (k,x,y) = (1,3,7)$.
$y^2 - 2^{3k+2} = 3x^3$, x odd $\Rightarrow (k,x,y) = (0,-1,1)$ or $(2,35,359)$.

Proof: $(y,6) = 1$. Calculating modulo 3, we see that $k = 0$ is impossible in the positive case and $k = 1$ is impossible in the negative case.

If $k = 1$ in the positive case, then $(3y)^2 + 288 = (3x)^3$ and from [4], $(x,y) = (3,7)$.

If $k = 0$ in the negative case, then $(3y)^2 - 36 = (3x)^3$ and from [A], $(x,y) = (-1,1)$.

Thus we may assume $k > 1$. Let $\Lambda = \Omega_6$ and $\theta = 6^{\frac{1}{3}}$. Now,

(*) $$9y^2 = \alpha\beta, \text{ where } \alpha = 3x \mp 2^k\theta^2 \text{ and } \beta = 9x^2 \pm 3x2^k\theta^2 + (2^k\theta^2)^2 .$$

As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda 3\cdot(2^k\theta^2)^2 = 9\cdot 2^{2k+1}\theta$. As in Proposition 12.5, $(\alpha,\beta)_\Lambda|_\Lambda 3^3$. $3 = \epsilon\delta^3$, where $\epsilon = 1 - 6\theta + 3\theta^2$ is the fundamental unit of Λ (see Table 1) and $\delta = 3 + 2\theta + \theta^2$ is a prime of Λ (since $N(\delta) = 3$). Now,

(*) $$\alpha = \delta^2[3x \mp 2^{k+2} + (2x \pm 2^{k+2})\theta - (2x \pm 2^k)\theta^2] .$$

If $\delta^3|_\Lambda \alpha$, then $27 = N(\delta^3)|N(\alpha) = 9y^2$, which contradicts $(y,3) = 1$. As in Proposition 12.5, $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or ϵ and $b \geq 0$.

If $\mu = \varepsilon = 1 - 6\theta + 3\theta^2$, then:

$$3x = (a^2 + 12bc) - 36(b^2 + 2ac) + 18(6c^2 + 2ab) \qquad (\Rightarrow \text{ a is odd}),$$

$$(13) \quad 0 = (3c^2 + ab) - 3(a^2 + 12bc) + 9(b^2 + 2ac) ,$$

$$(14)(*) \quad \mp 2^k = 3(a^2 + 12bc) - 6(6c^2 + 2ab) + (b^2 + 2ac) .$$

Since $k > 1$, (14) implies that b is odd. Thus $3a^2 + b^2 \equiv 0 \bmod 4$. It follows from (14) that $0 \equiv 2ac \bmod 4$ and therefore c is even. Therefore by (13), $ab - 3a^2 + 9b^2$ is even, but it is also odd.

Hence $\mu = 1$, implying that:

$$(15) \qquad 3x = a^2 + 12bc \qquad (\Rightarrow \ a \text{ is odd and } 3|a) ,$$

$$(16) \qquad 0 = 3c^2 + ab ,$$

$$(17)(*) \quad \mp 2^k = b^2 + 2ac \qquad (\Rightarrow b \text{ is even, since } k > 1).$$

Hence by (16), c is even. If $c = 0$, then by (16), $b = 0$, which contradicts (17). Thus $c = 2^rC$, C odd, $r \geq 1$. By (16), $b = 2^{2r}B$, B odd, which implies that $B > 0$. By (15), $a = 3A$. From (16) and (17), $0 = C^2 + AB$ and

$$(*) \qquad \mp 2^k = 2^{4r}B^2 + 3 \cdot 2^{r+1}AC .$$

As in Proposition 2.2, $B = 1$ and therefore $A = -C^2$. Hence

$$(*) \qquad \mp 2^k = 2^{4r} - 3 \cdot 2^{r+1}C^3 .$$

Since $4r > r + 1$, $k = r + 1$. Therefore

$$(*) \qquad 2^{3r-1} \pm 1 = 3C^3 .$$

It follows from Lemmas 12.2 and 12.3 that there are no solutions in the positive case, and in the negative case $r = 1$, $C = 1$, $A = -C^2 = -1$, $a = 3A = -3$, $b = 2^{2r}B = 4$, $c = 2^rC = 2$, $k = r + 1 = 2$ and by (15), $x = 35$. Therefore $y = 359$.

Propositions 12.4, 12.5 and 12.6 prove

Theorem 12: $y^2 + 2^n = 3x^3$, x odd $\Leftrightarrow$ $(n,x,y) = (1,1,1),(5,3,7),(7,19,143)$ or $(9,11,59)$.

$y^2 - 2^n = 3x^3$, x odd $\Leftrightarrow$ $(n,x,y) = (0,1,2),(2,-1,1),(8,35,359)$ or $(12,-5,61)$.

SECTION 13.

$y^2 \pm 2^n = 9x^3$, x odd

Note that $(y,3) = 1$.

Lemma 13.1: If $2^n + 1 = 9x^3$, then $n = 3$. If $2^n - 1 = 9x^3$, then $n = 0$.

Proof: $2^n \equiv \pm 1 \bmod 9$ and thus, by Lemma 12.1, $n = 3k$. If $2^n + 1 = 9x^3$, then $1 = (-2^k)^3 + 9x^3$. The Delone-Nagell theorem yields $(-2^k,x) = (1,0)$ or $(-2,1)$. The first of these is obviously impossible and the second gives $k = 1$. Therefore $n = 3$.

If $2^n - 1 = 9x^3$, then $1 = (2^k)^3 + 9x^3$ and, as above, $(2^k,x) = (1,0)$ or $(-2,1)$. Hence $k = 0$ and thus $n = 0$.

Lemma 13.2: If $y^2 + 2^n = 9x^3$, x odd, then $n \equiv 0 \bmod 3$.

Proof: $2^n \equiv -y^2 \equiv -1 \bmod 3$. Hence $n = 2r + 1$. It will now be shown that $2^r \equiv \pm 2 \bmod 9$. This will complete the proof of Lemma 13.2, since then $2^{r-1} \equiv \pm 1 \bmod 9$ and thus, by Lemma 12.1, $r \equiv 1 \bmod 3$. Hence $n = 2r + 1 \equiv 0 \bmod 3$.

Let $\Lambda = \Gamma_{-2}$ and $\theta = (-2)^{\frac{1}{2}}$.

(1) $\alpha\beta = -9x^3$, where $\alpha = y + 2^r\theta$ and $\beta = -\bar{\alpha} = -y + 2^r\theta$. Since $(\alpha,\beta)_\Lambda |_\Lambda\ \alpha + \beta = 2^{r+1}\theta$ and $(9x^3,2) = 1$, $(\alpha,\beta)_\Lambda = 1$.

(2) $9 = (1 + \theta)^2(1 - \theta)^2$ and, since $N(1 \pm \theta) = 3$, $1 \pm \theta$ are primes of Λ . Since ± 1 are the only units of Λ, $1 + \theta$ and $1 - \theta$ are not associates in Λ . Therefore if both divided α or both divided β in Λ, then $3 = (1 + \theta)(1 - \theta)$ would divide α or β in Λ. But this contradicts $(y,3) = 1$. Hence by (1) and (2), $(1 + \theta)^2 |_\Lambda\ \alpha =$

$y + 2^r\theta$ or $(1+\theta)^2 |_\Lambda\ \beta = -y + 2^r\theta$. Let $Y = \pm y$, such that $(1+\theta)^2 |_\Lambda\ Y + 2^r\theta$. Taking conjugates $(1-\theta)^2 |_\Lambda\ Y - 2^r\theta$. By (1) and (2),

(3) $$(Y + 2^r\theta)/(1+\theta)^2 \cdot (Y - 2^r\theta)/(1-\theta)^2 = x^3 .$$

Further, $(Y + 2^r\theta, Y - 2^r\theta)_\Lambda = (\alpha,\beta)_\Lambda = 1$. Hence it follows, from (3), that $Y + 2^r\theta = (1+\theta)^2(a + b\theta)^3$. Comparison of coefficients yields:

(4) $$Y = (8b^3 - a^3) + 6ab(b - 2a) ,$$

(5) $$2^r = 2(a^3 + b^3) - 3ab(b + 4a) .$$

(4) yields, since $(y,3) = 1$,

(6) $$a + b \equiv a^3 + b^3 \equiv a^3 - 8b^3 \equiv -Y \equiv \pm 1 \bmod 3 .$$

Note: The double signs in the remainder of this lemma will depend on the double sign in (6).

If $3|a$, then by (6), $b \equiv \pm 1 \bmod 3$ and thus by (5) and Lemma 1.9, $2^r \equiv \pm 2 \bmod 9$. Similarly, we obtain this result if $3|b$.

If $(ab,3) = 1$, then by (6), $a \equiv b \equiv \mp 1 \bmod 3$. Hence $ab(b + 4a) \equiv \pm 1 \bmod 3$ and by Lemma 1.9, $a^3 + b^3 \equiv \mp 2 \bmod 9$. (5) gives $2^r \equiv \mp 4 \mp 3 \equiv \pm 2 \bmod 9$.

Theorem 13a: $y^2 + 2^n = 9x^3$, x odd $\Rightarrow (n,x,y) = (3,1,1), (3,201,8549), (9,73,1871)$ or $(15,17,107)$.

Proof: By Lemma 13.2, $n = 3k$. Calculating modulo 3, we see that $k = 0$ is impossible.

If $k = 1$, $(9y)^2 + 648 = (9x)^3$, x odd, and by [A], $(x,y) = (1,1)$ or $(201,8549)$.

Assume $k > 1$. Hence y is odd. Let $\Lambda = \Omega_3$ and $\theta = 3^{\frac{1}{3}}$. Now, $y^2 = \alpha\beta$, where $\alpha = x\theta^2 - 2^k$ and $\beta = (x\theta^2)^2 + 2^k x\theta^2 + 2^{2k}$. As in Proposition 2.2, $\alpha > 0$ and $(\alpha,\beta)_\Lambda |_\Lambda \, 3\cdot 2^{2k}$. Since $(y,6) = 1$, $(\alpha,\beta)_\Lambda = 1$. Hence $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $-2 + \theta^2$ and $a \geq 0$.

If $\mu = -2 + \theta^2$, then:

(7) $$-2^k = -2(a^2 + 6bc) + 3(3c^2 + 2ab) \qquad (\Rightarrow c \text{ is even}),$$
$$0 = -2(3c^2 + 2ab) + 3(b^2 + 2ac) \qquad (\Rightarrow b \text{ is even}),$$
$$x = -2(b^2 + 2ac) + (a^2 + 6bc) \qquad (\Rightarrow a \text{ is odd}) .$$

Since $k \geq 2$, (7) implies that $0 \equiv -2 \bmod 4$, which is a contradiction. Hence $\mu = 1$, implying:

(8) $$-2^k = a^2 + 6bc \qquad (\Rightarrow a \text{ is even and } (a,3) = 1),$$

(9) $$0 = 3c^2 + 2ab \qquad (\Rightarrow c \text{ is even}),$$

(10) $$x = b^2 + 2ac \qquad (\Rightarrow b \text{ is odd}).$$

By (8) and (9), $b = 3B$. If $c = 0$ then, by (9) and (10), $a = 0$, which contradicts (8). Hence $c = 2^t C$, C odd, $t \geq 1$. By (9), $a = 2^{2t-1}A$, A odd, and therefore $A > 0$. (8) and (9) yield: $0 = C^2 + AB$ and $-2^k = 2^{4t-2}A^2 + 9\cdot 2^{t+1}BC$. As in Proposition 2.2, $A = 1$ and therefore $B = -C^2$. Hence

(11) $$-2^k = 2^{4t-2} - 9\cdot 2^{t+1}C^3 .$$

If $t = 1$, $2 = 4t - 2 = t + 1$. Hence $k > 2$ and by (11), $2^{k-2} + 1 = 9C^3$. By Lemma 13.1, $k = 5$ and thus $C = 1$. Therefore, $n = 3k = 15$, $B = -C^2 = -1$, $a = 2^{2t-1}A = 2$, $b = 3B = -3$, $c = 2^t C = 2$

and by (10), $x = 17$. Hence $y = 107$.

Now assume that $t > 1$. Thus $4t - 2 > t + 1$ and by (11), $k = t + 1$. Therefore $2^{3t-3} + 1 = 9C^3$. It follows from Lemma 13.1, that $t = 2$ and hence $C = 1$. Thus $k = t + 1 = 3$, $n = 9$, $B = -1$, $a = 8$, $b = -3$, $c = 4$ and by (10), $x = 73$. Therefore $y = 1871$.

<u>Lemma 13.3</u>: If $y^2 - 2^n = 9x^3$, x odd, then $n \equiv 1 \bmod 3$.

<u>Proof</u>: $2^n \equiv y^2 \equiv 1 \bmod 3$ and hence $n = 2r$. Let $Y = \pm y$, such that $Y \equiv 2^r \bmod 3$. Now,

(12) $$(Y + 2^r)(Y - 2^r) = 9x^3 \quad \text{and} \quad (Y + 2^r, Y - 2^r) \mid 2^{r+1} .$$

Since $(9x^3, 2) = 1$, $(Y + 2^r, Y - 2^r) = 1$. Hence, since $3 \mid Y - 2^r$, (12) yields $9 \mid Y - 2^r$. By (12), $Y + 2^r = u^3$ and $Y - 2^r = 9v^3$. Therefore $2^{r+1} = u^3 - 9v^3$ and hence $(u,3) = 1$. It follows from Lemma 1.9 that $2^{r+1} \equiv u^3 \equiv \pm 1 \bmod 9$. By Lemma 12.1, $r \equiv -1 \bmod 3$. Hence $n = 2r \equiv 1 \bmod 3$.

<u>Theorem 13b</u>: $y^2 - 2^n = 9x^3$, x odd $\Rightarrow (n,x,y) = (4,1,5)$.

<u>Proof</u>: By Lemma 13.3, $n = 3k + 1$. Since $3k + 1 \neq 0$, y is odd.

Calculating modulo 3, we see that $k = 0$ is impossible. If $k = 1$, then $(9y)^2 - (48)(27) = (9x)^3$, x odd and by [A], $(x,y) = (1,5)$. Hence we may assume that $k > 1$. Let $\Lambda = \Omega_6$ and $\theta = 6^{\frac{1}{3}}$.

(13) $$4y^2 = \alpha\beta, \text{ where } \alpha = x\theta^2 + 2^{k+1} \text{ and } \beta = (x\theta^2)^2 - 2^{k+1}x\theta^2 + 2^{2(k+1)} .$$

As above, $\alpha > 0$ and $(\alpha,\beta)_\Lambda \mid_\Lambda 3 \cdot 2^{2(k+1)}$. Since $(4y^2,3) = 1$, $(\alpha,\beta)_\Lambda \mid 2^{2k+2}$. Now, $2 = \epsilon^{-1}\delta^3$, where $\epsilon^{-1} = 109 + 60\,\theta + 33\,\theta^2$ is a unit of Λ (see Table 1) and $\delta = 2 - \theta$ is a prime of Λ (since $N(\delta) = 2$).

$$\alpha = \delta^2[33x + 20\cdot 2^k + (18x + 11\cdot 2^k)\theta + (10x + 6\cdot 2^k)\theta^2] .$$

Therefore $\delta^2|_\Lambda \alpha$. If $\delta^3|_\Lambda \alpha$, then $8 = N(\delta^3)|N(\alpha) = (2^{k+1})^3 + 36x^3 = 4y^2$. But y is odd. Hence by Lemma 1.4, $(\alpha/\delta^2,\beta)_\Lambda = 1$. By (13), $(\epsilon^{-1}\delta^2 y)^2 = (\alpha/\delta^2)\cdot\beta$. As in Proposition 12.5, $\alpha = \mu(a + b\theta + c\theta^2)^2$, where $\mu = 1$ or $\epsilon(= 1 - 6\theta + 3\theta^2)$ and $a \geq 0$.

If $\mu = \epsilon$, then:

(14) $\quad 2^{k+1} = (a^2 + 12bc) - 36(b^2 + 2ac) + 18(6c^2 + 2ab) \quad (\Rightarrow a \text{ is even}),$

(15) $\quad 0 = -3(a^2 + 12bc) + (3c^2 + ab) + 9(b^2 + 2ac) ,$

(16) $\quad x = 3(a^2 + 12bc) - 6(6c^2 + 2ac) + (b^2 + 2ac) .$

Since a is even, (16) implies that b is odd. Thus by (15), c is odd. Let $a = 2A$. (14) and (15) yield:

(17) $\quad 2^{k-1} = (A^2 + 3bc) - 9(b^2 + 4Ac) + 9(3c^2 + 2Ab) ,$

(18) $\quad 0 = -3(4A^2 + 12bc) + (3c^2 + 2Ab) + 9(b^2 + 4Ac) .$

Since $3c^2 + 9b^2 \equiv 0 \bmod 4$, it follows from (18) that $2Ab \equiv 0 \bmod 4$. Hence A is even. But, since $3bc - 9b^2 + 27c^2$ is odd, (17) implies that 2^{k-1} is odd, which contradicts $k \geq 2$.

If $\mu = 1$, we obtain:

(19) $\quad 2^{k+1} = a^2 + 12bc \quad (\Rightarrow a \text{ is even and } (a,3) = 1) ,$

(20) $\quad 0 = 3c^2 + ab ,$

(21) $\quad x = b^2 + 2ac \quad (\Rightarrow b \text{ is odd}) .$

By (20), $3|b$ and c is even. If $c = 0$ then by (20), $b = 0$ (since

$a \neq 0$) which contradicts (21). Therefore $c = 2^t C$, C odd, $t \geq 1$. By (20), $a = 2^{2t}A$, A odd, and therefore $A > 0$. Let $b = 3B$. By (19) and (20), $0 = C^2 + AB$ and

$$2^{k+1} = 2^{4t}A^2 + 9\cdot 2^{t+2}BC \quad .$$

As above, $A = 1$. Therefore $B = -C^2$ and $2^{k+1} = 2^{4t} - 9\cdot 2^{t+2}C^3$. Now, $4t > t + 2$. Thus $k + 1 = t + 2$ and $2^{3t-2} - 1 = 9C^3$. There are no solutions by Lemma 13.1.

SECTION 14 .

$y^2 + 3^m = 2^j x^3, \ (x,3) = 1$

Theorem 14a: $y^2 + 3^m = x^3$, $(x,3) = 1 \Rightarrow (m,x,y) = (0,1,0),(4,13,46)$ or $(5,7,10)$.

Proof: If x is odd, we obtain the solution from Theorem 9. Hence we may assume that x is even and therefore y is odd. Thus $3^m = x^3 - y^2 \equiv -1 \bmod 8$. But $3^m \equiv 1$ or $3 \bmod 8$.

Theorem 14b: $y^2 + 3^m = 2x^3$, $(x,3) = 1 \Rightarrow (m,x,y) = (0,1,1),(4,5,13)$ or $(6,53,545)$.

Proof: y is odd. If m were odd, $2x^3 = y^2 + 3^m \equiv 4 \bmod 8$ and therefore $x^3 \equiv 2 \bmod 4$. But this is impossible and thus $m = 2v$.

Thus $y^2 + 3^m = 2x^3$ can be transformed into

(1) $$(2y)^2 - (-1)(2\cdot 3^v)^2 = (2x)^3 .$$

Now, the class number of Γ_{-1} is 1 and by [34,p.235, Theorem 6.2.1], 2 is the square of a prime ideal and 3 is a prime ideal in Γ_{-1}. Thus Theorem 9.1 can be applied to (1), to obtain (where $\theta = (-1)^{\frac{1}{2}}$):

$$\pm 2y + 2\cdot 3^v\theta = \mu(a + b\theta)^3 ,$$

where $\mu = 1$ or θ . Since $\theta = (-\theta)^3$, we may assume that $\mu = 1$ and therefore:

(2) $$\pm 2y = a^3 - 3ab^2 ,$$

(3) $$2\cdot 3^v = 3a^2b - b^3 .$$

If b were even, then by (2), a would be even and by (2) again,

$4 | \pm 2y$. But y is odd and therefore b is odd.

If $v = 0$, then by (3), $2 = b(3a^2 - b^2)$, and we find that $(b, |a|) = (1,1)$. Therefore by (2), $|y| = 1$ and, since $m = 2v = 0$, $x = 1$.

If $v > 0$, then by (1), $(y,3) = 1$ (since $(x,3) = 1$). Hence by (2), $(a,3) = 1$ and by (3), $b = 3B$. Substituting in (3) we obtain:

(4) $2 \cdot 3^{v-2} = B(a^2 - 3B^2)$. Since B is odd and $(a^2 - 3B^2, 3) = 1$, $B = \pm 3^{v-2}$ and $a^2 - 3B^2 = \pm 2$.

Therefore, since the upper sign cannot hold in the latter equation modulo 3, $a^2 + 2 = 3^{2v-3}$. By Theorem 6b, $(v, |a|) = (2,1)$ or $(3,5)$ and therefore $|b| = 3|B| = 3^{v-1} = 3$ or 9 , respectively. By (2), $(m, |y|) = (4,13)$ or $(6,545)$. Hence $x = 5$ or 53, respectively.

<u>Theorem 14c</u>: $y^2 + 3^m = 4x^3$, $(x,3) = 1 \Rightarrow (m,x,y) = (1,1,1)$ or $(1,7,37)$.

<u>Proof</u>: y is odd. If m were even, then $4x^3 = y^2 + 3^m \equiv 2 \bmod 8$. Hence $m = 2v + 1$. Therefore $4x^3 = y^2 + 3^m \equiv 4 \bmod 8$ and hence x is odd. Further, $(y,3) = 1$, since $m > 0$.

Let $\Lambda = \Gamma_{-3}$ and $\theta = (-3)^{\frac{1}{2}}$. By (2) of Section 1, $\alpha = (y + 3^v\theta)/2 \in \Lambda$ and $\bar{\alpha} \in \Lambda$. Now, $\alpha\bar{\alpha} = x^3$ and $(\alpha,\bar{\alpha})_\Lambda |_\Lambda \ \alpha - \bar{\alpha} = 3^v\theta$. Since $(x,3) = 1$, $(\alpha,\bar{\alpha})_\Lambda = 1$. Hence $\alpha = \mu[(a + b\theta)/2]^3$, where $a \equiv b \bmod 2$ and $\mu = 1$ or $(\pm 1 + \theta)/2$.

If $\mu = 1$, then $4 \cdot 3^{v-1} = b(a^2 - b^2)$. But, whether a and b are both even or both odd, $8 | b(a^2 - b^2)$. Hence $\mu = (\pm 1 + \theta)/2$, which yields:

$$8y = \pm (a^3 - 9ab^2) - 9(a^2b - b^3) \qquad (\Rightarrow (a,3) = 1) ,$$

$$(5) \qquad 8\cdot 3^{v} = (a^3 - 9ab^2) \pm 3(a^2b - b^3) \quad .$$

Since $(a,3) = 1$, (5) implies that $v = 0$. Therefore $m = 1$ and $(4y)^2 + 48 = (4x)^3$. By [A], $(x,y) = (1,1)$ or $(7,37)$.

$y^2 \pm 2^n 3^m = x^3$

Theorem 15: All the solutions of $y^2 \pm 2^n 3^m = x^3$ are given by the following two tables, where $x = 2^r 3^t a$ and $y = 2^s 3^u b$.

Explanation of Tables 2 and 3: If $a = 0$ (or $b = 0$) the values of r and t (respectively s and u) are irrelevant.

n and m are given modulo 6 in the tables and, of course, they are non-negative. The restrictions on n and m arise since x and y are integers.

The solutions are numbered for reference in the proof and [# k] will refer to solutions number k in the tables.

Table 2

Solutions of $y^2 + 2^n 3^m = x^3$

n ≡ (mod 6)	m ≡ (mod 6)	Restrictions on n & m	3r	3t	a	2s	2u	b	Solution number
0	0		n	m	1	-	-	0	1
0	3		n	m	1	-	-	0	2
0	4		n	m-4	13	n+2	m-4	23	3
0	5		n	m-5	7	n+2	m-5	5	4
1	0		n-1	m+3	1	n-1	m	5	5
1	2		n-1	m+1	1	n-1	m	1	6
1	2	n≥7	n-7	m+1	19	n-7	m	143	7
1	3		n-1	m-3	7	n-1	m-3	17	8

n ≡ (mod 6)	m ≡ (mod 6)	Restrictions on n & m	3r	3t	a	2s	2u	b	Solution number
2	0		n+1	m	1	n	m	1	9
2	0		n-2	m	5	n-2	m	11	10
2	0	m≥6	n+1	m-6	53	n	m-6	545	11
2	4		n+1	m-4	5	n	m-4	13	12
2	5		n-2	m-5	13	n-2	m-5	35	13
3	0		n	m	1	-	-	0	14
3	2		n	m+1	1	n+1	m	1	15
3	2	n≥9	n-9	m+1	11	n-9	m	59	16
3	2	m≥8	n	m-8	971	n+1	m-8	21395	17
3	3		n	m	1	-	-	0	18
3	3		n	m-3	5	n+1	m-3	7	19
3	3		n-3	m	11	n-3	m+3	7	20
3	4		n-3	m+2	1	n-3	m	1	21
3	4		n	m+2	1	n+3	m	1	22
3	4		n	m+5	1	n+1	m	11	23
3	4		n	m-4	11	n+1	m-4	25	24
3	4	n≥15	n-15	m+2	17	n-15	m	107	25
3	4		n-3	m+5	67	n-3	n	8549	26
3	4	n≥9	n-9	m+2	73	n-9	m	1871	27
3	4		n-3	m-4	97	n-3	m-4	955	28
3	5		n	m-5	35	n+3	m-5	73	29
4	1		n+2	m-1	1	n	m-1	1	30
4	1		n+2	m-1	7	n	m-1	37	31
4	1	m≥7	n-4	m-7	73	n-4	m-7	595	32
4	3		n+2	m	1	n	m+1	1	33
4	4		n-4	m-4	193	n-4	m-4	2681	34
5	2		n-5	m+4	1	n-5	m	7	35
5	5		n-5	m-5	1153	n-5	m-5	39151	36

Table 3

Solutions of $y^2 - 2^n 3^m = x^3$

n ≡ (mod 6)	m ≡ (mod 6)	Restrictions on n & m	3r	3t	a	2s	2u	b	Solution number
0	0		n	m	-1	-	-	0	37
0	0		-	-	0	n	m	1	38
0	0		n+3	m	1	n	m+2	1	39
0	1	n≥12	n-12	m-1	-23	n-12	m-1	11	40
0	1		n	m-1	1	n+2	m-1	1	41
0	2	n≥12	n-12	m+1	-5	n-12	m	61	42
0	2		n+3	m-2	-1	n	m-2	1	43
0	2		-	-	0	n	m	1	44
0	2		n	m+1	1	n+2	m	1	45
0	2		n+3	m+1	1	n	m	5	46
0	2		n+9	m-2	5	n	m-2	253	47
0	3		n	m	-1	-	-	0	48
0	4		-	-	0	n	m	1	49
1	0		n-1	m	-1	n-1	m	1	50
1	0	n≥7	n-7	m	17	n-7	m	71	51
1	2		n-1	m-2	7	n-1	m-2	19	52
1	3		n-1	m	1	n-1	m+1	1	53
1	3	m≥9	n-1	m-9	19	n-1	m-9	215	54
1	3	n≥13	n-13	m	1915	n-13	m+1	48383	55
1	5		n-1	m-5	-5	n-1	m-5	19	56
2	0		-	-	0	n	m	1	57
2	1		n+1	m-1	-1	n	m-1	1	58
2	1		n-2	m-1	13	n-2	m-1	47	59
2	2		n-2	m+1	-1	n-2	m	1	60
2	2		-	-	0	n	m	1	61
2	2		n+4	m-2	1	n	m-2	5	62
2	2		n+4	m+1	1	n	m	7	63

n ≡ (mod 6)	m ≡ (mod 6)	Restrictions on n & m	3r	3t	a	2s	2u	b	Solution number
2	2	n≥8	n-8	m+1	35	n-8	m	359	64
2	3		n-2	m	-1	n-2	m+1	1	65
2	3		n+1	m-3	-1	n	m-3	5	66
2	3		n+1	m	1	n	m+1	1	67
2	3	n≥8	n-8	m	11	n-8	m+1	23	68
2	3		n+1	m	61	n	m+1	389	69
2	4		-	-	0	n	m	1	70
3	0	n≥9	n-9	m	-7	n-9	m	13	71
3	0		n	m	-1	-	-	0	72
3	0		n-3	m	1	n-3	m+2	1	73
3	0		n	m	1	n+1	m	1	74
3	0		n	m	23	n+3	m+2	13	75
3	1	m≥7	n-3	m-7	-13	n-3	m-7	73	76
3	1		n	m-1	-1	n+1	m-1	1	77
3	1		n-3	m-1	1	n-3	m-1	5	78
3	1		n	m-1	5	n+7	m-1	1	79
3	1	m≥7	n	m-7	5	n+3	m-7	17	80
3	1	n≥9	n-9	m-1	25	n-9	m-1	131	81
3	1	m≥13	n	m-13	239	n+1	m-13	2761	82
3	1		n	m-1	4079	n+1	m-1	184211	83
3	2		n	m-2	-1	n+3	m-2	1	84
3	2	n≥15	n-15	m-2	73	n-15	m-2	827	85
3	3		n	m	-1	-	-	0	86
3	5		n	m-5	-1	n+1	m-5	11	87
4	0		-	-	0	n	m	1	88
4	1		n-4	m-1	1	n-4	m-1	7	89
4	2		-	-	0	n	m	1	90
4	4		n+5	m-4	-1	n	m-4	7	91
4	4		-	-	0	n	m	1	92
4	4		n-4	m+2	1	n-4	m	5	93
4	4		n+5	m+2	1	n	m	17	94
5	2		n-5	m-2	1	n-5	m-2	17	95
5	5	m≥11	n-5	m-11	-47	n-5	m-11	2359	96

Proof: By direct but laborious calculation, the above can be shown to be solutions.

Suppose now that

$$(1)(*) \qquad y^2 \pm 2^n 3^m = x^3 .$$

If $x = 0$, then there are no solutions in the positive case, and in the negative case n and m are even. Hence n and m are congruent to 0, 2 or 4 modulo 6 and this leads to the solutions: [#38], [#44], [#49], [#57], [#61], [#70], [#88], [#90] and [#92].

If $y = 0$, then n and m are congruent to 0 or 3 modulo 6 and this yields the solutions: [#1], [#2], [#14], [#18], [#37], [#48], [#72] and [#86].

Suppose now that $xy \neq 0$. Therefore $x = 2^r 3^t a$ and $y = 2^s 3^u b$, where $(ab,6) = 1$. By (1),

$$(1a)(*) \qquad 2^{2s} 3^{2u} b^2 \pm 2^n 3^m = 2^{3r} 3^{3t} a^3 .$$

It follows from Lemma 1.7 that one of the following three possibilities holds:

(2_1) $\qquad 2s = 3r < n$,

(2_2) $\qquad 2s = n < 3r$ or

(2_3) $\qquad 3r = n < 2s$.

Again using Lemma 1. 7 , either:

(3_1) $\qquad 2u = 3t \leq m$,

(3_2) $\qquad 2u = m \leq 3t$ or

(3_3) $\qquad 3t = m \leq 2u$.

If (2_1) holds, $2s = 3r = 6d$ and if (3_1) holds, $2u = 3t = 6e$. We divide the proof into the nine cases: (2_i) and (3_j).

<u>(2_1) and (3_1)</u>: By (1a),

$$(*) \qquad b^2 \pm 2^{n-6d} 3^{m-6e} = a^3 .$$

Since $(ab,6) = 1$, Theorem 9 implies in the positive case: $(n - 6d, m - 6e, a, b) = (1,3,7,17)$ [#8], $(2,0,5,11)$ [#10], $(2,5,13,35)$[#13], $(3,4,97,955)$[#28], $(4,4,193,2681)$[#34], $(4,7,73,595)$[#32] or $(5,5,1153,39151)$[#36],and Theorem 7 implies in the negative case: $(n - 6d, m - 6e, a, b) = (1,0,-1,1)$[#50], $(1,2,7,19)$[#52], $(1,5,-5,19)$[#56], $(1,9,19,215)$[#54], $(2,1,13,47)$[#59], $(3,1,1,5)$[#78], $(3,7,-13,73)$[#76], $(4,1,1,7)$[#89], $(5,2,1,17)$[#95], $(5,11,-47,2359)$[#96], $(7,0,17,71)$[#51], $(9,0,-7,13)$[#71], $(9,1,25,131)$[#81], $(12,1,-23,11)$[#40] or $(15,2,73,827)$[#85].

<u>(2_1) and (3_2)</u>: By (1a),

$$(4)(*) \qquad b^2 - 2^{n-6d} = 3^{3t-m} a^3 .$$

Since $m = 2u$ is even, we may further divide this case into three sub-cases:

<u>$m = 6g$</u>: Therefore by (4),

$$(*) \qquad b^2 \pm 2^{n-6d} = (3^{t-2g} a)^3 .$$

It follows from Theorem 2 that in the positive case: $(n - 6d, 3^{t-2g}a, b) = (1,3,5)$[#5] or $(2,5,11)$[#10], and in the negative case: $(n - 6d, 3^{t-2g}a, b) = (1,-1,1)$[#50], $(7,17,71)$[#51] or $(9,-7,13)$[#71].

<u>$m = 6g + 2$</u>: By (4),

(5)(*) $$b^2 \pm 2^{n-6d} = 3(3^{t-2g-1}a)^3 .$$

If $t - 2g - 1 < 0$, then $3(t - 2g - 1) + 1 < 0$, which contradicts (5) (since $(a,3) = 1$). Hence $3^{t-2g-1}a \in Z$. We shall use analogous results for some other cases in this section.

Theorem 12 implies in the positive case: $(n - 6d, 3^{t-2g-1}a, b) =$ (1,1,1)[#6], (5,3,7)[#35], (7,19,143)[#7] or (9,11,59)[#16], and in the negative case: $(n - 6d, 3^{t-2g-1}a, b) =$ (2,-1,1)[#60], (8,35,359)[#64] or (12,-5,61)[#42].

<u>$m = 6g - 2$</u>: By (4),

(*) $$b^2 \pm 2^{n-6d} = 9(3^{t-2g}a)^3 .$$

In the positive case, Theorem 13a implies that $(n - 6d, 3^{t-2g}a, b) =$ (3,1,1)[#21], (3,201,8549)[#26], (9,73,1871)[#27] or (15,17,107)[#25], and in the negative case, it follows from Theorem 13b that $(n - 6d, 3^{t-2g}a, b) =$ (4,1,5)[#93].

<u>(2_1) and (3_3)</u>: By (1a),

(6)(*) $$3^{2u-m}b^2 \pm 2^{n-6d} = a^3 .$$

Using $m = 3t$, we divide this case into two further sub-cases:

<u>$m = 6g$</u>: By (6),

(*) $$(3^{u-3g}b)^2 \pm 2^{n-6d} = a^3 .$$

By Theorem 2, in the positive case $(n - 6d, a, 3^{u-3g}b) =$ (2,5,11)[#10], and in the negative case : $(n - 6d, a, 3^{u-3g}b) = (1,-1,1)$ [#50], $(3,1,3)$ [#73], $(7,17,71)$ [#51] or $(9,-7,13)$ [#71].

<u>$m = 6g + 3$</u>: By (6),

$$(*) \qquad 3(3^{u-3g-2}b)^2 \pm 2^{n-6d} = a^3 .$$

By Theorem 3, in the positive case $(n - 6d, a, 3^{u-3g-2}b) = (3,11,21)$ [#20], and in the negative case : $(n - 6d, a, 3^{u-3g-2}b) = (1,1,1)$ [#53], $(2,-1,1)$ [#65], $(8,11,23)$ [#68] or $(13,1915,48383)$ [#55].

<u>$(2_2$ and (3_1)</u>: By (1a),

$$(7)(*) \qquad b^2 \pm 3^{m-6e} = 2^{3r-n}a^3 .$$

Since $n = 2s$ is even, this case resolves into three sub-cases:

<u>$n = 6f$</u>: By (7),

$$(*) \qquad b^2 \pm 3^{m-6e} = (2^{r-2f}a)^3 .$$

Since b is odd, there are no solutions in the positive case by Theorem 14a. In the negative case, Theorem 8a implies that $(m - 6e, 2^{r-2f}a, b) = (2,-2,1)$ [#43] or $(2,40,253)$ [#47].

<u>$n = 6f + 2$</u>: By (7),

$$(*) \qquad b^2 \pm 3^{m-6e} = 2(2^{r-2f-1}a)^3 .$$

In the positive case, Theorem 14b yields : $(m - 6e, 2^{r-2f-1}a, b) = (0,1,1)$ [#9], $(4,5,13)$ [#12] or $(6,53,545)$ [#11]. In the negative case, Theorem 8b implies that $(m - 6e, 2^{r-2f-1}a, b) = (1,-1,1)$ [#58], $(2,2,5)$ [#62] or $(3,-1,5)$ [#66].

n = 6f - 2: By (7),

(*) $$b^2 \pm 3^{m-6e} = 4(2^{r-2f}a)^3 .$$

Theorem 14c gives, in the positive case, $(m - 6e, 2^{r-2f}a, b) = (1,1,1)$ [#30] or $(1,7,37)$ [#31]. Theorem 8c yields, in the negative case, $(m - 6e, 2^{r-2f}a, b) = (4,-2,7)$ [#91].

(2_2) and (3_2): By (1a),

(8)(*) $$b^2 \pm 1 = 2^{3r-n}3^{3t-m}a^3 .$$

In the positive case we have, since $b^2 + 1 \equiv 2 \bmod 8$ and $b^2 + 1 \equiv 2 \bmod 3$, $m = 3t$ and $3r - n = 1$. Hence $(2b)^2 + 4 = (2a)^3$ and, by [A], $a = b = 1$. Since $n = 2s$ and $m = 2u$ are even, $n \equiv 2 \bmod 6$ and $m \equiv 0 \bmod 6$. This is solution [#9].

Hence we may assume the negative case in (8). We divide the proof into nine sub-cases:

n = 6f and m = 6g: By (8), $b^2 - 1 = (2^{r-2f}3^{t-2g}a)^3$.

There are no solutions, with $(b,6) = 1$, by [A]

n = 6f and m = 6g + 2: $(3b)^2 - 9 = (2^{r-2f}3^{t-2g}a)^3$, and by [A], $(2^{r-2f}3^{t-2g}a, 3b) = (6,15)$ [#46].

n = 6f and m = 6g - 2: $(9b)^2 - 81 = (2^{r-2f}3^{t-2g+2}a)^3$, and by [A], there are no solutions.

n = 6f + 2 and m = 6g: $(2b)^2 - 4 = (2^{r-2f}3^{t-2g}a)^3$, and by [A], there are no solutions.

n = 6f + 2 and m = 6g + 2: $(6b)^2 - 36 = (2^{r-2f}3^{t-2g}a)^3$.

[A] implies that $(2^{r-2f}3^{t-2g}a, 6b) = (12,42)$ [#63].

<u>n = 6f + 2 and m = 6g - 2</u>: $(18b)^2 - (18)^2 = (2^{r-2f}3^{t-2g+2}a)^3$, and there are no solutions by [A].

<u>n = 6f - 2 and m = 6g</u>: $(4b)^2 - 16 = (2^{r-2f+2}3^{t-2g}a)^3$. By [A], there are no solutions.

<u>n = 6f - 2 and m = 6g + 2</u>: $(12b)^2 - (12)^2 = (2^{r-2f+2}3^{t-2g}a)^3$, and by [A], there are no solutions.

<u>n = 6f - 2 and m = 6g - 2</u>: $(36b)^2 - (48)(27) = (2^{r-2f+2}3^{t-2g+2}a)^3$. By [A], $(2^{r-2f+2}3^{t-2g+2}a, 36b) = (72,612)$ [#94].

<u>(2_2) and (3_3)</u>: By (1a),

(9)(*) $$3^{2u-m}b^2 \pm 1 = 2^{3r-n}a^3 .$$

First, we consider the positive case, with m even. Hence $3^{2u-m}b^2 + 1 \equiv 2 \bmod 8$ and therefore $3r - n = 1$. Since $n = 2s$ and $m = 3t$, $n = 6f + 2$ and $m = 6g$. By (9), $(2\cdot 3^{u-3g}b)^2 + 4 = (2^{r-2f}a)^3$ and by [A], $(2^{r-2f}a, 2\cdot 3^{u-3g}b) = (2,2)$ [#9].

Now, consider the positive case, with m odd. Hence $3^{2u-m}b^2 + 1 \equiv 4 \bmod 8$, and by (9), $3r - n = 2$. Since $n = 2s$ and $m = 3t$, $n = 6f - 2$ and $m = 6g + 3$. By (9), $(4\cdot 3^{u-3g}b)^2 + (16)(27) = (3\cdot 2^{r-2f+2}a)^3$ and by [A], $(3\cdot 2^{r-2f+2}a, 4\cdot 3^{u-3g}b) = (12,36)$ [#33].

Next, consider the negative case, with m odd. Since $3^{2u-m}b^2 - 1 \equiv 2 \bmod 8$, $3r - n = 1$. $n = 2s$ and $m = 3t$ imply that $n = 6f + 2$ and $m = 6g + 3$. By (9), $(2\cdot 3^{u-3g}b)^2 - 4\cdot(27) = (3\cdot 2^{r-2f}a)^3$. By [A], $(3\cdot 2^{r-2f}a, 2\cdot 3^{u-3g}b) = (6,18)$ [#67] or $(366,7002)$ [#69].

Finally, there remains only the negative case, with m even.

Hence $m = 6g$ and, since n is even, we have the following three sub-cases:

<u>$n = 6f$</u>: $(3^{u-3g}b)^2 - 1 = (2^{r-2f}a)^3$, and by [A], $(2^{r-2f}a, 3^{u-3g}b) = (2,3)$ [#39].

<u>$n = 6f + 2$</u>: $(2\cdot 3^{u-3g}b)^2 - 4 = (2^{r-2f}a)^3$ and there are no solutions by [A].

<u>$n = 6f - 2$</u>: $(4\cdot 3^{u-3g}b)^2 - 16 = (2^{r-2f+2}a)^3$, and by [A], there are no solutions.

<u>(2_3) and (3_1)</u>: By (1a),

$$(10)(*) \qquad 2^{2s-n}b^2 \pm 3^{m-6e} = a^3 .$$

Since $n = 3r \equiv 0 \bmod 3$, we have two sub-cases:

<u>$n = 6f$</u>: $(*)$ $(2^{s-3f}b)^2 \pm 3^{m-6e} = a^3$.

In the positive case, Theorem 14a implies that $(m - 6e, a, 2^{s-3f}b) = (4,13,46)$ [#3] or $(5,7,10)$ [#4], and in the negative case, Theorem 8a yields $(m - 6e, a, 2^{s-3f}b) = (1,1,2)$ [#41].

<u>$n = 6f + 3$</u>: $(*)$ $2(2^{s-3f-2}b)^2 \pm 3^{m-6e} = a^3$.

In the positive case, it follows from Theorem 11 that $(m - 6e, a, 2^{s-3f-2}b) = (3,5,7)$ [#19], $(4,11,25)$ [#24], $(5,35,146)$ [#29] or $(8,971,21395)$ [#17]. In the negative case, Theorem 10 implies that $(m - 6e, a, 2^{s-3f-2}b) = (0,1,1)$ [#74], $(1,-1,1)$ [#77], $(1,5,8)$ [#79], $(1,4079,184211)$ [#83], $(2,-1,2)$ [#84], $(5,-1,11)$ [#87], $(7,5,34)$ [#80] or $(13,239,2761)$ [#82].

<u>(2_3) and (3_2)</u>: By (1a),

(11) (*) $2^{2s-n}b^2 \pm 1 = 3^{3t-m}a^3$.

Since $n = 3r$ and $m = 2s$, we divide the proof into six sub-cases:

<u>n = 6f and m = 6g</u>: By (11), (*) $(2^{s-3f}b)^2 \pm 1 = (3^{t-2g}a)^3$, and by [A], there are no solutions.

<u>n = 6f and m = 6g + 2</u>: (*) $(3\cdot 2^{s-3f}b)^2 \pm 9 = (3^{t-2g}a)^3$. By [A], there are no solutions in the positive case, and in the negative case $(3^{t-2g}a, 3\cdot 2^{s-3f}b) = (3,6)$ [#45].

<u>n = 6f and m = 6g - 2:</u> (*) $(9\cdot 2^{s-3f}b)^2 \pm 81 = (3^{t-2g+2}a)^3$. By [A], there are no solutions.

<u>n = 6f + 3 and m = 6g</u>: (*) $(2^{s-3f}b)^2 \pm 8 = (2\cdot 3^{t-2g}a)^3$. By [A], there are no solutions in the positive case, and in the negative case $(2\cdot 3^{t-2g}a, 2^{s-3f}b) = (2,4)$ [#74].

<u>n = 6f + 3 and m = 6g + 2</u>: (*) $(3\cdot 2^{s-3f}b)^2 \pm 72 = (2\cdot 3^{t-2g}a)^3$. By [A], there are no solutions in the negative case . By [4], in the positive case $(2\cdot 3^{t-2g}a, 3\cdot 2^{s-3f}b) = (6,12)$ [#15].

<u>n = 6f + 3 and m = 6g - 2</u>: (*) $(9\cdot 2^{s-3f}b)^2 \pm 24\cdot 27 = (2\cdot 3^{t-2g+2}a)^3$. By [A], there are no solutions in the negative case . By [4], in the positive case $(2\cdot 3^{t-2g+2}a, 9\cdot 2^{s-3f}b) = (18,72)$ [#22] or $(54,396)$ [#23].

<u>(2_3) and (3_3)</u>: By (1a),

(12) (*) $2^{2s-n}3^{2u-m}b^2 \pm 1 = a^3$.

Since $n = 3r$ and $m = 3t$ we have the following four sub-cases:

<u>n = 6f and m = 6g</u>: By (12), (*) $(2^{s-3f}3^{u-3g}b)^2 \pm 1 = a^3$, and by [A], there are no solutions.

n = 6f and m = 6g + 3: (*) $(2^{s-3f}3^{u-3g}b)^2 \pm 27 = (3a)^3$, and by [A], there are no solutions.

n = 6f + 3 and m = 6g: (*) $(2^{s-3f}3^{u-3g}b)^2 \pm 8 = (2a)^3$. By [A], there are no solutions in the positive case, and in the negative case $(2a, 2^{s-3f}3^{u-3g}b) = (2,4)$ [#74] or (46,312) [#75].

n = 6f + 3 and m = 6f + 3: (*) $(2^{s-3f}3^{u-3g}b)^2 \pm 8\cdot 27 = (6a)^3$.
By [A], there are no solutions.

SECTION 16.

$u^3 + Bv^3 = C$

A simple consequence of the solution of $y^2 - 2^n 3^m = x^3$ is given by the following theorem.

Theorem 16: $u^3 + Bv^3 = C$, where $(B,C) = (3^s, 2^r)$ or $(2^s, 3^r)$, $0 \le s \le 2$, $r \ge 0$, has the following integral solutions:

Table 4

B	C	(u^3,v^3) $r \equiv 0 \bmod 3$	(u^3,v^3) $r \equiv 1 \bmod 3$	(u^3,v^3) $r \equiv 2 \bmod 3$
1	2^r	$(0,2^r),(2^r,0)$	$(2^{r-1},2^{r-1})$	————
3	2^r	$(2^r,0)$	$(-2^{r-1},2^{r-1}),(5^3 2^{r-7},2^{r-7})$	$(2^{r-2},2^{r-2})$, $(-7^3 2^{r-5},5^3 2^{r-5})$
9	2^r	$(-2^{r-3},2^{r-3})$, $(2^r,0),(-2^{r+3},2^r)$	————	————
1	3^r	$(0,3^r),(3^r,0)$	————	$(3^{r-2},2^3 3^{r-2})$, $(2^3 3^{r-2},3^{r-2})$
2	3^r	$(3^r,0)$	$(3^{r-1},3^{r-1})$, $(-5^3 3^{r-1},4^3 3^{r-1})$	————
4	3^r	$(3^r,0)$	$(-3^{r-1},3^{r-1})$	————

Proof: Let $Y = u^3 - Bv^3$. Then

(1) $$Y + C = 2u^3 \quad \text{and} \quad Y - C = -2Bv^3 .$$ Therefore

(2) $$(4BY)^2 - 2^4B^2C^2 = (-4Buv)^3 .$$

Since B and C are powers of 2 and 3 , we can solve (2) by using Theorem 15. Of course, some of the equations are special cases of equations whose solutions are well-known (e.g., $B = 2$, $C = 3^r$, $r = 3t$ gives $u^3 + (-3^t)^3 = 2(-v)^3$) (cf. [32]).

We solve a typical case: $B = 3$, $C = 2^r$, $r \equiv 2 \bmod 3$. $2^4B^2C^2 = 2^{2r+4}3^2$ and $2r + 4 \equiv 2 \bmod 6$. By (2) and Table 3, $\pm 12Y = 2^{r+1}\cdot 3$, $2^{r+2}\cdot 3$, $5\cdot 2^{r+2}$, $7\cdot 2^{r+2}\cdot 3$ or $359\cdot 2^{r-2}\cdot 3$. Using (1), these give the solutions in Table 4.

APPENDIX I.

Proof of Theorem 9.1

We note that if $f = x = 0$, $y = 0$ and $y + f\cdot k^{\frac{1}{2}} = 1\cdot 0^3$. Therefore we may assume that $f \neq 0$ or $x \neq 0$.

We first prove the theorem under the assumption that (f,x^3) is cube-free.

Let $[\gamma]$ be the principal ideal generated by $\gamma \in \Gamma_k$. Now,
(#) $[x]^3 = [\gamma][\bar{\gamma}]$, where $\gamma = y + f\cdot k^{\frac{1}{2}}$.

Suppose that a prime ideal, say Ψ, divides both $[\gamma]$ and $[\bar{\gamma}]$. We will show that Ψ^3 divides both $[\gamma]$ and $[\bar{\gamma}]$ and also, that Ψ^4 does not divide either $[\gamma]$ or $[\bar{\gamma}]$.

This will complete the proof (with (f, x^3) cube-free), since then the G.C.D. of $[\gamma]$ and $[\bar{\gamma}]$ is of the form $\Psi_1^3 \cdot \Psi_2^3 \cdot \ldots \cdot \Psi_t^3$, where $t \geq 0$ and the Ψ_i are distinct primes. Hence the G.C.D. is a cube. Since we have unique factorization of ideals, $[\gamma] = \Xi^3$ by (#). Thus Ξ^3 is equivalent to $[1]^3$. Since the class number is not divisible by 3, Ξ is equivalent to $[1]$ and hence $\Xi = [\alpha]$ (see [15, pp.81-85]). Hence $[\gamma] = [\alpha^3]$ and therefore $\gamma = \mu\,\alpha^3$, where μ is a unit of Γ_k. As in Lemma 1.3, we may choose $\mu = 1$, ϵ or ϵ^{-1}, where ϵ is the fundamental unit of Γ_k. If $\mu = \epsilon^{-1}$, then since $\epsilon^{-1} = \bar{\epsilon}$, $-\bar{\gamma} = \epsilon(-\bar{\alpha})^3$ and hence the theorem (with (f,x^3) cube-free).

By [15, p.72], there is a unique rational prime, say p, such that Ψ divides $[p]$. Hence $N(\Psi)\,|\,N[p] = p^2$ and therefore $N(\Psi) = p$ or p^2. By (#), Ψ divides $[x]^3$ and hence $p\,|_Z\, N(\Psi)\,|_Z\, N[x^3] = x^6$. Thus $p\,|\,x$. We divide the proof into three cases.

Case 1: $p \neq 2$. Ψ divides $[2][y] = [\gamma + \bar{\gamma}]$. Therefore Ψ divides $[y]$ and, as for x , $p|y$. Hence $p^2|y^2 - x^3 = f^2 \cdot k$. Since k is square-free, $p|f$. Thus $p^r|f$, $p^{r+1}\nmid f$, where $r = 1$ or 2 (since (f,x^3) is cube-free). If $r = 2$, then $p^3|x^3 + f^2k = y^2$, $p^2|y$, $p^4|x^3$ and hence $p^2|x$. Thus $p^r|x$ and $p^r|y$ for $r = 1$ or 2 . Further,

$$(y/p^r)^2 - k(f/p^r)^2 = p^r(x/p^r)^3 .$$

If $(p,k) = 1$, then since $(f/p^r,p) = 1$ and Δ = discriminant of $Q(k^{\frac{1}{2}}) = k$ or $4k$, Δ is a quadratic residue modulo p and $p\nmid\Delta$. By [34, p.235], p decomposes into two distinct prime ideals, contradicting $p|f$. Hence $p|_Z k|_Z\Delta$ and by [34], $[p] = \Psi^2$. Also, $p^3|x^3 + kf^2 = y^2$. Therefore $p^2|y$ and $\Psi^6 = [p]^3$ divides $[y]^2$. Hence Ψ^3 divides $[y]$. Since $\Psi^2 = [p]$ divides both $[k^{\frac{1}{2}}]^2$ and $[f]$, Ψ divides $[k^{\frac{1}{2}}]$ and hence Ψ^3 divides $[f \cdot k^{\frac{1}{2}}]$. Therefore Ψ^3 divides both $[\gamma]$ and $[\bar{\gamma}]$.

Suppose Ψ^4 divides either $[\gamma]$ or $[\bar{\gamma}]$. Then Ψ^7 divides $[\gamma][\bar{\gamma}] = [x]^3$ and therefore Ψ^3 divides $[x]$. Hence $[p]^4 = \Psi^8$ divides $[x]^3$ and therefore $p^2|x$. Since $p^2|y$, $p^4|f^2 \cdot k$ and, since k is square-free, $p^2|f$. Since $p|k$, $p^5|x^3 + f^2 \cdot k = y^2$, $p^3|y$, $p^6|f^2 \cdot k$ and finally, $p^3|f$, a contradiction.

Case 2: $p = 2$, $2|\Delta$. By [34], $k \equiv 2$ or $3 \bmod 4$ and $\Psi^2 = [2]$. Since $2 = p|x$, Ψ^6 divides $[x]^3 = [\gamma][\bar{\gamma}]$.

If Ψ^4 $(= [2^2])$ divides $[\gamma]$, $2^2|_{\Gamma_k} \gamma$ and since $k \not\equiv 1 \bmod 4$, $2^2|y$ and $2^2|f$. Hence $2^4|x^3$, $2^2|x$ and thus $\Psi^{12} = [2^6]$ divides $[x^3] = [\gamma][\bar{\gamma}]$. Therefore $[2^3] = \Psi^6$ divides either $[\gamma]$ or $[\bar{\gamma}]$

which, as above, gives $2^3 | f$, a contradiction. Hence Ψ^4 does not divide $[\gamma]$ and, similarly, Ψ^4 does not divide $[\bar{\gamma}]$. Since Ψ^6 divides $[\gamma][\bar{\gamma}]$, Ψ^3 divides both $[\gamma]$ and $[\bar{\gamma}]$.

<u>Case 3</u>: $p = 2$, Δ odd. Hence, since 2 does not decompose into two distinct prime ideals, $\Delta = k \equiv 5 \bmod 8$ and $\Psi = [2]$ (by [34]). Since $2 = p | x$, $[2]^3$ divides $[x]^3 = [\gamma][\bar{\gamma}]$. Hence $[2^2]$ divides either $[\gamma]$ or $[\bar{\gamma}]$ and therefore $2^2 |_{\Gamma_k} \alpha$ or $\bar{\alpha}$. By (2), Section 1, y and f are even.

Suppose f/2 is odd. Then, since $(y/2)^2 - k(f/2)^2 = 2(x/2)^3$, y/2 is odd. But then $4 \equiv 1 - 5 \equiv 2(x/2)^3 \bmod 8$, an impossibility.

Hence $2^2 | f$ and thus $2^3 | y^2$, yielding $2^2 | y$. If $2^3 | y$, then $2^6 | f^2 \cdot k$, implying that $2^3 | f$, a contradiction. Hence y/4 and f/4 are both odd and thus $2 |_{\Gamma_k} \gamma/4$ and $\bar{\gamma}/4$. Therefore $\Psi^3 = [2]^3$ divides both $[\gamma]$ and $[\bar{\gamma}]$. If Ψ^4 divides either $[\gamma]$ or $[\bar{\gamma}]$, then $2^4 |_{\Gamma_k} \gamma$ or $\bar{\gamma}$, implying that $2^3 | f$, a contradiction.

Now, let d be the largest integer whose cube divides (f,x^3) . We have just shown that Theorem 9.1 holds when $d = 1$. Assume it holds for all $d < d_0$ $(d_0 \geq 2)$ and let d_0 be the largest integer whose cube divides (f,x^3) . Let p be a positive prime dividing d_0 . Hence $p^3 | (f,x^3)$ and therefore $p^3 | y^2$. Thus $p^2 | y$, $p^4 | x^3$, $p^2 | x$, $p^6 | y^2$ and hence $p^3 | y$. Letting $F = f/p^3$, $X = x/p^2$ and $Y = y/p^3$, we have $Y^2 - kF^2 = X^3$. Let D be the largest integer whose cube divides (F,X^3) . Hence $(pD)^3 | f$ and $x^3 (= p^6 X^3)$ and therefore $pD \leq d_0$. Since $p \geq 2$, $D < d_0$. Hence, by the induction hypothesis, there is an $\alpha \in \Gamma_k$ such that

$$\pm Y + F \cdot k^{\frac{1}{2}} = \mu \alpha^3 ,$$

where $\mu = 1$ or ϵ . Multiplying by p^3 , $\pm y + f \cdot k^{\frac{1}{2}} = \mu(p\alpha)^3$ and $p\alpha \in \Gamma_k$. Hence the theorem holds for $d = d_0$ and by induction, for all d .

APPENDIX II

Some Previously Solved Cases

This appendix contains the previously solved cases of $y^2 + k = x^3$, $k = \pm 2^n 3^m$ which are used in this paper. Except for the cases $k = 18$; 72; 288 and 648 which are solved in [4] and [10], these were obtained from [12], [13] and [14] (cf. [6], [20], [10a] (for $k = 412$), [16] (for $k = 48$) and [10]).

Table 5.

k	(x,y)
1	(1,0)
2	(3,5)
3	no solutions
4	(2,2),(5,11)
8	(2,0)
9	no solutions
12	no solutions
18	(3,3)
24	no solutions
27	(3,0)
48	(4,4),(28,148)
54	(7,17)
72	(6,12)
81	(13,46)
216	(6,0),(10,28),(33,189)
288	(9,21)
412	(12,36)
648	(9,9),(18,27),(22,100),(54,396),(97,955),(1809,76941)

Table 6.

k	(x,y)
-1	(-1,0),(0,1),(2,3)
-2	(-1,1)
-3	(1,2)
-4	(0,2)
-8	(-2,0),(1,3),(2,4),(46,312)
-9	(-2,1),(0,3),(3,6),(6,15),(40,253)
-12	(-2,2),(13,47)
-16	(0,4)
-18	(7,19)
-24	(-2,4),(1,5),(10,32),(8158,736844)
-27	(-3,0)
-36	(-3,3),(0,6),(4,10),(12,42)
-48	(1,7)
-54	(3,9)
-72	(-2,8)
-81	(0,9)
-108	(-3,9),(-2,10),(6,18),(366,7002)
-144	(0,12)
-216	(-6,0)
-324	(0,18)
-648	no solutions
-1296	(-8,28),(0,36),(9,45),(72,612)

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